MODELLING THE VERTICAL FLOW OF GROUNDWATER IN A SATURATED-UNSATURATED MEDIUM

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BLOEMFONTEIN

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DECLARATION

I, Rendani Vele Makahane, hereby declare that the dissertation hereby submitted by me to the Institute for Groundwater Studies in the Faculty of Natural and Agricultural Sciences at the University of the Free State, in fulfilment of the degree Doctor of Philosophy, is my own independent work and I have not previously submitted it for a qualification at another institution of higher education. In addition, I declare that all sources cited have been acknowledged by means of a list of references.

I furthermore cede copyright of the dissertation and its contents in favour of the University of the Free State.

In addition, the following papers have been submitted to Springers:

1. Makahane R.V., & Atangana A., 2020. Analysis of the existing model for the vertical flow of groundwater in the saturated-unsaturated zone.
2. Makahane R.V., & Atangana A., 2020. New model of the saturated-unsaturated groundwater flow with power law and scale-invariant mean square displacement.
3. Makahane R.V., & Atangana A., 2020. New model of the 1-d unsaturated-saturated groundwater flow with crossover from usual to confined flow mean square displacement.
4. Makahane R.V., & Atangana A., 2020. New model of the 1-d unsaturated-saturated groundwater flow with crossover from usual to sub- flow mean square displacement.
5. Makahane R.V., & Atangana A., 2020. New model of the 1-d saturated-unsaturated groundwater flow using the fractal fractional derivative.
6. Makahane R.V., & Atangana A., 2020. Application of the fractional-stochastic approach to the saturated-unsaturated zone model.
7. Makahane R.V., & Atangana A., 2020. Transfer function of the Sumudu, Laplace transforms and their application to groundwater.

Rendani Vele Makahane

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DEDICATION

A special dedication to my nephews and nieces, may this motivate you to do it a little extra when it comes to educating yourself. To my late nephew Zwothe Makahane, this is yours baby.

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To God be the glory

ABSTRACT

The classical equation developed to replicate the vertical groundwater flow through the saturated-unsaturated zone is based on the concept of rate of change. While a lot of success has been achieved using this concept, researchers discovered that only classical mechanical homogenous problems with no memory could be depicted using these mathematical tools. Therefore, a need to formulate new mathematical tools that can be used to depict non-conventional behaviour was urgent as some physical problems were not understandable. For this reason, we introduced to the saturated-unsaturated groundwater flow equation differential operators that are based on the power-law kernel, exponential decay kernel and Mittag-Leffler function to develop new models. These operators are the Caputo, Riemann-Liouville, Caputo-Fabrizio and the Atangana-Baleanu. Some of these operators are local and unable to model problems that display fractal behaviours. To account for this, we used the recently introduced fractal-fractional approach to develop more models. However, the above differential operators are based on sets of constant parameters/coefficients and largely ignore random fluctuations that are likely to occur in nature. To account for randomness, a stochastic approach is used. However, this approach is unable to capture fading memory and long-range behaviour. To help us capture problems with randomness, fading memory and long-range behavior we used the recently established stochastic-fractional approach to generate more models that give us a better understanding of the complex flow. The numerical solutions and stability analysis for all the models are detailed. We also suggested a new approach for modelling groundwater flow based on the Sumudu and Laplace transform. The new models presented in this thesis will be relevant to the majority if not all problems associated with unsaturated-saturated groundwater flow. For this reason, we have not focused on any specific problem/study area, however, theoretical parameters and assumptions are presented to illustrate the effectiveness of the new models, in which readers can make possible modifications to parameters and assumptions to match their specific groundwater problem/study area.

Keywords: Groundwater, saturated-unsaturated zone, Caputo, Caputo-Fabrizio, Atangana-Baleanu, fractal-fractional, stochastic-fractional, transfer function, Sumudu transform, Laplace transform.

LIST OF GREEK NOTATIONS

α	Alpha	ω	Omega
β	Beta	π	Pi
Δ	Delta	ψ	Psi
ϵ	Epsilon variant	τ	Tau
Γ	Gamma	θ	Theta
Λ	Lambda	σ	Sigma

LIST OF ABBREVIATIONS AND NOTATIONS

ψ_b	Air entry suction pressure head	1-d	One dimension
AB	Atangana-Baleanu	θ_0	residual water content
ABC	Atangana-Baleanu-Caputo	K_s	saturated hydraulic conductivity
CF	Caputo-Fabrizio	θ_s	saturated water content
FFR	Fractal Fractional Riemann Liouville	θ	soil water content
f	Function	S_s	specific storage of the soil
λ	pore size distribution index	t	Time
ψ	pressure head	z	Vertical direction

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CHAPTER 1
INTRODUCTION

1.1. BACKGROUND STUDY

With the rapid expansion of the population, agriculture, and industries, there is a substantial increase in the usage of groundwater resources to supplement surface water supplies. Due to this profound effect, groundwater development should be undertaken only after careful planning, which includes analyzing, understanding, and modeling the groundwater movement. Therefore, it is desired to have a model that predicts the flow of water through the saturated zone in both confined and unconfined aquifers and for circulation in the unsaturated zone. Instead of dealing with the complexity of the real world, certain assumptions can be introduced in the form of a mathematical model consisting of equations which connect parameters and variables relevant to the physical process (Pelka, 1983). For this study, the physical process under consideration takes place in the geological formation above an aquifer, and in an aquifer, this area is referred to as the saturated-unsaturated zone. An aquifer is a natural system and has a high degree of variability (Peter *et al*., 2011). Based on its structure, hydraulic performance, texture, geological formation and the mobility of water, three main types of aquifers are identified; unconfined, confined and semi-confined/leaky (Christiansen & Hamblin, 2014). A confined aquifer has an aquitard at its top and its base; the groundwater here is under pressure. An unconfined aquifer consists of an unsaturated medium and a saturated medium separated by a water table. In a semi-confined aquifer, one boundary is an aquiclude, and the other is an aquitard or both boundaries are aquitards, (Gushman & Tartakovsky, 2016; Pelka, 1983). Aquifers occur in a myriad of the geological formation of either unconsolidated and consolidated bedrock or both. Unconsolidated geological materials are made of porous media such as gravel, sand, silts, and clay; in this type of formation, groundwater is stored and travels in pore spaces between the particles. In consolidated bedrock formations, the water is stored and travels in fractures, fissures, and joints in the rocks, this is called secondary porosity (Uhl *et al*., 2009). The process of groundwater movement is a difficult one due to the presence of two distinct regions, namely, unsaturated and saturated zone. In the next few paragraphs, the essentials of the theory of subsurface water flow in the saturated and unsaturated area are discussed.

1.1.1 Theory of groundwater flow

The vadose zone, also referred to as the unsaturated zone is found immediately below the surface and above the groundwater level, and it is essentially a two-phase flow of two immiscible fluids containing both water and air in the pores (Freeze, 1971; Kresic, 2007). The unsaturated zone is further divided with respect to the occurrence and circulation of water into the uppermost zone of soil water, gravitational water, and the capillary fringe (Kresic, 2007). Flow in this zone is determined by the negative hydrostatic potential and the gravitational potential owing to capillary suction (Allepalli & Govindaraju, 2009). As water seeps into the ground, it substitutes the soil moisture deficit, and the excess water moves down by the force of gravity, which then builds up the groundwater (Gushman & Tartakovsky, 2016). Below the water table, groundwater moves both vertically and laterally, and water constantly occupies all pore spaces and/or rock fractures; this zone is referred to as the groundwater zone or saturated zone. In the saturated zone, movement is overseen by the total energy head or the piezometric head, made up of the positive pressure potentials and the elevation head (Allepalli & Govindaraju, 2009).

The phenomena by which water is absorbed or released from storage by the soil differ significantly in the saturated zone and in the unsaturated zone. In the unsaturated zone, water is released from storage primarily through a process called desaturation, although some distortion of the soil's framework may occur. In the saturated area, water is taken into or released from storage because of the distortion of the soil's framework and a consequent change in the porosity of the soil. In dealing with the unsaturated flow, soil physicists generally tend to ignore the deformation characteristics of the soil skeleton, while the soil engineers and hydrogeologists concentrating on the saturated zone, frequently neglect the effects of desaturation (Narasimhan, 1975). The water in the saturated area is enough to be withdrawn for drinking, irrigation or other uses; therefore, this zone is referred to as an aquifer (Mahajan, 2008).

When solving a natural problem, for instance, the movement of water through a porous media, knowledge on aquifer properties is essential (Fitts, 1990). Aquifer properties that govern the groundwater flow include porosity (n), permeability, storativity (S), hydraulic conductivity (K), hydraulic head (h), hydraulic gradient (i), transmissivity (T) and specific yield (Sy). Porosity is an essential property of rocks that allow the storage and flow of water underground. It directly influences the permeability, the hydraulic conductivity of rocks, therefore the velocity of groundwater, and other fluids that may be present (Mahajan, 2008). When modelling the movement of groundwater, the content of soil water and the hydraulic conductivity are essential parameters. The hydraulic conductivity of a porous material is subject to the shape, connectedness, and size of the filled pores. It is shallow at low and moderate water content, and it increases non-linearly to its saturated value as water content increases (Hu *et al.*, 2015). The hydraulic conductivity under saturated conditions is determined by the soil grain size (porosity); meanwhile, it is determined by porosity and degree of saturation in the unsaturated zone. Using the properties mentioned above, mathematical models for groundwater flow can be generated.

1.1.2 Equation of motion

The fundamental equation of groundwater flow developed initially for only saturated conditions was first established in 1856 by Henri Darcy (Hsu *et al.,* 2002). Laplace's equation governs the Darcian macroscopic theory for the hydraulic head (Tritscher *et al.,* 2001). Darcy's law states that for a given type of porous medium, the rate of flow q is proportional to the change in head h and inversely proportional to the distance of z, calling the proportionality constant K the hydraulic conductivity resulting to the following equation (Wang & Anderson, 1982):

$$q = -K\frac{\partial h}{\partial \mathrm{z}} \tag{1.1}$$

Darcy's law is applied to most groundwater flow. However, it is not applicable if the medium is too irregular or if the flow velocity is too high in a medium with large pores (Fitts, 1990).

1.1.3 Equation of continuity

The conservation law, as applied to fluid transfer, is also known as an equation of continuity. This law states that under transient flow, the rate at which saturation changes in a closed system is the same as the rate of change of the total amount of fluxes in and out of the system (Pelka, 1983). The equation of continuity that describes the transient movement of water in a vertical cross-section of a saturated-unsaturated aquifer under isothermal conditions and in the absence of sources and sinks is given by (Shahraiyni and Ashtiani, 2008; Hsu *et al.*, 2002):

$$\frac{\partial \theta}{\partial t} + \nabla q = 0 \tag{1.2}$$

where ∇ is the gradient operator $\partial/\partial z$; z is the vertical direction. θ is the volumetric content of soil-water (L^0) defined as nS_w, n is the medium's porosity (L^0), S_w is the level of volumetric fluid saturation, $0 \leq S_w \leq 1$ measured in (L^0); q is specific discharge (LT^{-1}); t is time (T). Richards was perhaps the first person to express the equation of continuity for transient soil-water flow in a formula of a partial differential, parabolic equation in the soil physics literature. Familiarly known as Richard's equation, it can be written as:

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial\psi}{\partial z} + 1\right)\right] \tag{1.3}$$

where K represents the hydraulic conductivity, θ represents the volumetric water content, ψ represents the pressure head, z represents the vertical direction, and time is represented by t. The Richards' equation numerical solution has a disadvantage of being computationally expensive and unpredictable since it is not guaranteed that a solver will converge for a certain set of constitutive soil relations (Tocci *et al.*, 1997). Therefore, this method cannot be applied where there is a high risk of non-convergence. Another disadvantage is over-emphasising of the role of capillary and being overly simplistic (Gray & Hassanizadeh, 1991).

Water movement through a porous media has been of great interest to humankind since early history (Nassehzadeh-Tabrizi *et al.*, 1978). The study of water flow in the vadose /unsaturated zone and the saturated zone has frequently been accomplished by separate equations, applied distinctly on each zone. In most cases, the saturated zone is found in the area of interest, and it can fluctuate depending on how the water table moves between the unsaturated and saturated zone. In this case, Richards' equation alone cannot govern the flow in both zones (Cheng & Gulliksson, 2003). To model a flow region where both saturated and unsaturated regimes coexist, one should combine the physical features governing the nature of each of these crucial phenomena (Narasimhan, 1975). Up until recently, very little attention seems to have been paid to the expansion of the combined saturated-unsaturated model, considering the factors particular to each regime of flow.

1.1.4 Principal equation of flow in porous media

Most of the groundwater flow models cover only either the saturated or unsaturated part of an aquifer, treating both parts separately. The flow of groundwater through the saturated-unsaturated zone is tightly coupled, and this coupling is vital in addressing problems of flow. It becomes evident that groundwater flow in the saturated and unsaturated must be viewed as a whole and should be described by a sole mathematical model. By considering the assumptions that air pressure remains constant, i.e. zero in the unsaturated zone and there is negligible flow of water vapour, the flow of water through the saturated-unsaturated media can be mathematically represented by the combination the generalized Darcy's Law and the continuity equation (List and Radu, 2015; Pelka, 1983; Zimmerman & Bodvarsson, 1989). The equation includes the hydraulic properties of the soil which are a function of the suction head of the soil and therefore is non-linear (Allepalli and Govindaraju, 1996). The 1-d saturated-unsaturated groundwater flow equation is given below:

$$[S_S S_a(\psi) + C(\psi)]\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left[K_z(\psi)\left(\frac{\partial \psi}{\partial z} - 1\right)\right] \tag{1.4}$$

where: ψ = pressure head, n= porosity, S_S = specific storage of the soil, S_a = saturation of the aqueous phase= $\frac{\theta}{n}$, $C(\psi)$ = capillary capacity of soil = $d\theta/d\psi$, $q(t)$ = groundwater flux =$-K_z(\psi)\left(\frac{\partial \psi}{\partial z} - 1\right)$, K_z= hydraulic conductivity, z = vertical co-ordinate.

1.2. PROBLEM STATEMENT

The mathematical model in equation (1.4) was constructed to replicate the vertical groundwater flow through the saturated-unsaturated zone. The geological structures of a saturated zone are different from those of unsaturated zone thus; the groundwater flows from one geological formation to another with different scales. In several already published materials, it was argued that the differential operator based on the concept of rate of change applied in classical mechanics by Sir Newton could not accurately replicate the movement of materials in different scales. This is due to the fact that the differential operator is local, thus cannot reproduce the long-range behavior of a flow in a fracture or cannot describe the flow from one scale to another. It is also confirmed that this mathematical tool "derivative (rate of change)" does not produce a mean square displacement with crossover behavior. In this light, it is strongly argued that the above mathematical model will not be able to replicate the vertical movement of groundwater flow in the saturated-unsaturated zone. We strongly support this argument due to the fact that the geological formation within which this process takes place is not homogeneous, and the above model does not in any way incorporate into its mathematical formulation the effect of heterogeneity, neither the scaling factor of the aquifer. For this study, we will use the newly established differential operators based on the exponential decay kernel, power-law kernel and the Mittag-Leffler function to develop new models. New models will also be derived using the recently introduced fractal-fractional and the fractional-stochastic approach. A new approach in groundwater modelling using the Sumudu and Laplace transform is also suggested.

1.3. AIMS AND OBJECTIVES

1.3.1 Aims

This study aims to model the vertical flow of groundwater in a saturated-unsaturated medium.

1.3.2 Objectives

- Analyse existing models for vertical flow in saturated-unsaturated zone
- Develop a new model with power-law and fractal properties
- Develop a new model with crossover from usual to confined flow mean square displacement
- Develop a new model with crossover from usual to sub-flow mean square displacement
- Apply the fractional-stochastic approach to the 1-d saturated-unsaturated zone model
- Sumudu, Laplace transforms and their application to groundwater
- Numerical simulations and interpretation

1.4. RESEARCH OUTLINE

- Chapter 1: Introduction
- Chapter 2: Literature review
- Chapter 3: Analysis of an existing model of vertical flow in saturated-unsaturated zone
- Chapter 4: New model with power-law and fractal properties
- Chapter 5: New model with crossover from usual to confined flow mean square displacement
- Chapter 6: New model with crossover from usual to sub-flow mean square displacement
- Chapter 7: Application of the fractional-stochastic approach to saturated-unsaturated zone model
- Chapter 8: Sumudu, Laplace transforms and their application to groundwater
- Chapter 9: Numerical simulations, interpretation, and conclusion

1.5. RESEARCH FRAMEWORK

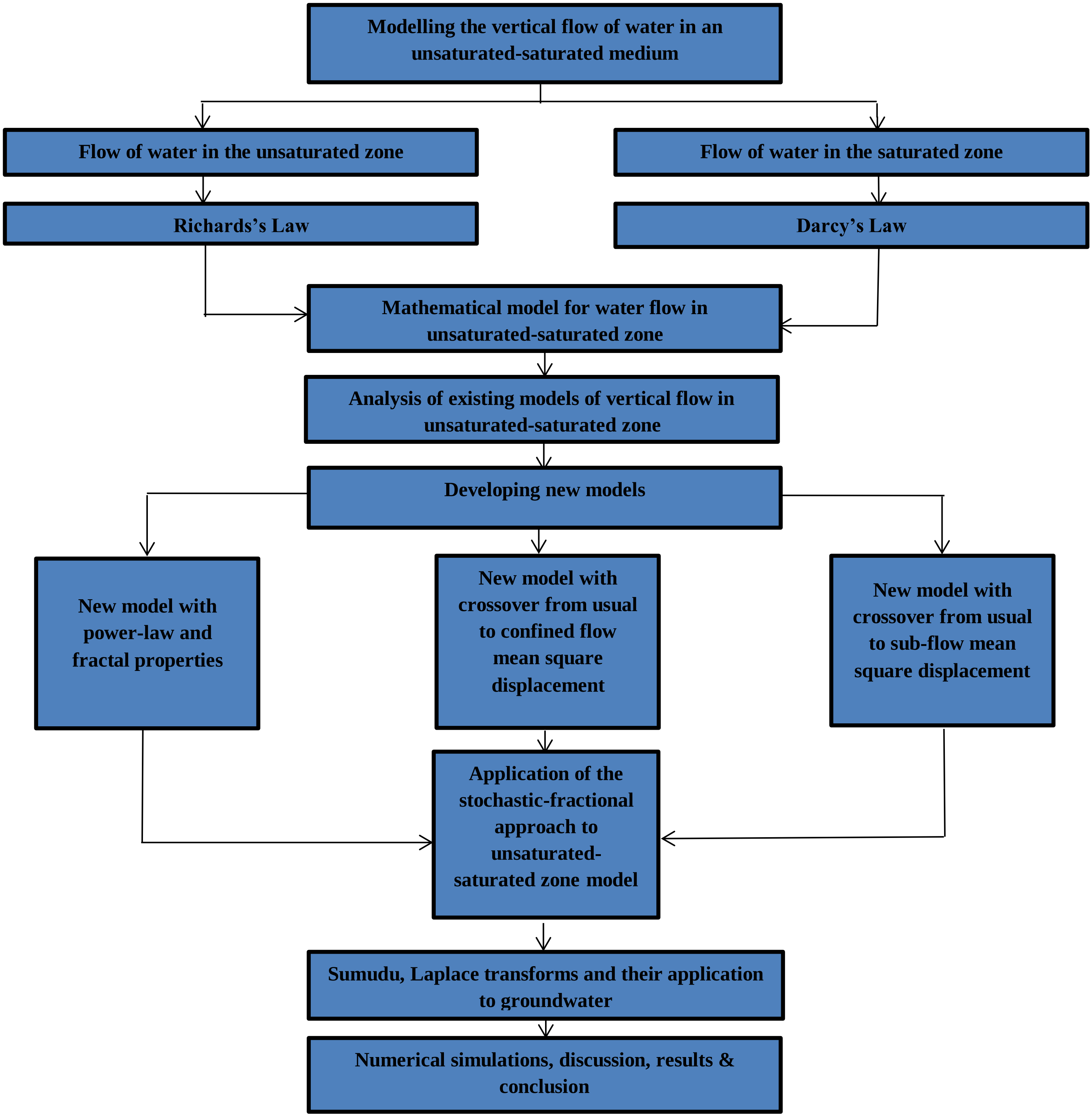

Figure 1: Schematic representation of the research framework

CHAPTER 2
LITERATURE REVIEW ON FRACTIONAL CALCULUS AND STOCHASTIC APPROACH

2.1. INTRODUCTION

To capture heterogeneities observed by mankind in their day-to-day life, mathematicians count on mathematical models to produce analytical and numerical solutions. To solve equations analytically, integral transforms are used; meanwhile, fractional derivatives are used to solve equations numerically. Usually, these models are built with constant coefficients; however, it was discovered by statisticians that changing a constant coefficient into a distribution assistance with capturing processes with statistical settings (Atangana & Araz, 2019). This section serves as a brief propaedeutic chapter for understanding the methods, results, and conclusions presented in this thesis.

2.2. FRACTIONAL CALCULUS

History indicates that Isaac Newton and Gottfried Wilhelm Leibniz discovered calculus autonomously in the 17th century. In his discovery of calculus, G. W. Leibniz introduced the idea of a symbolic technique and used the symbol $d^n y/dx^n = D^n y$ for the n^{th} derivative, where n is a non-negative integer (Debnath, 2004). In 1695, Guillaume de L'Hospital wrote to G. W. Leibniz and asked him what will the result be if $n = 1/2$. Leibniz responded using the following words "An apparent paradox, which one-day useful consequences will be drawn." Those are the words that marked the beginning of fractional calculus (Loverro, 2004; Nishimoto, 1991). Fractional calculus is an arena of mathematics that raises out of the traditional descriptions of the calculus derivative and also integral operators of non-integer order (Petras, 2011; Dalir & Bashour, 2010). It can be used to model engineering and physical processes, which are described best using fractional differential equations. However, it is important to note that in many cases, standard mathematical models of integer-order derivative, together with non-linear models, do not work efficiently. Lately, fractional calculus has been used in many other disciplines such as economics, chemistry, continuum mechanics, physics, biology, electromagnetics, image processing etc. (Atangana & Secer, 2013; Dalir & Bashour, 2010). Researchers such as Laplace, Fourier, Euler are one of the few that experimented with fractional calculus and mathematical outcomes. While using their methodology and notation, many found definitions that are appropriate for the concept of a non-integer order derivative or integral (Loverro, 2004; Shukla & Sapra, 2019). Such definitions include integrals and derivatives of, Caputo, Riemann-Liouville, Caputo-Fabrizio, and the recently introduced Atangana-Baleanu. The fractional calculus can be understood precisely by knowing some simple mathematical definitions like Mittag- Leffler function, Laplace transforms, Beta function and Gamma function. These are discussed in the following subsections.

2.2.1 Gamma function

The gamma function (Γ) appears to be a crucial function of fractional calculus and is the extension of the factorial function for the non-negative integers with its argument shifted down by 1, to real and complex numbers (Loverro, 2004; Shukla & Sapra, 2019). The gamma function can be represented as:

$$\Gamma(z) = (z-1)! \tag{2.1}$$

If the real part of the complex number z is greater than zero, then the integral is given as:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{2.2}$$

2.2.2 Beta function

Euler and Legendre are the first people that studied the Beta function/Euler integral. For all real number $x > 0$ and $y > 0$ the Beta function is defined as (Loverro, 2004):

$$\beta(y,z) = \int_0^1 t^{y-1}(1-t)^{z-1} \tag{2.3}$$

The main property of this function is its relationship to the gamma function:

$$\beta(y,z) = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)} \tag{2.3.1}$$

The above relationship is significant for fractional calculus (Shukla & Sapro, 2019).

2.2.3 Mittag-Leffler function

This function depends on complex parameters α and β. It was named after Gosta Mittag-Leffler. If the real parts of α are strictly positive, then the function is defined by the series below (Kowankar & Gangal, 1996; Mainardi, 2010):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.4}$$

In the concept of integer-order differential equations, the exponential function e^z plays an imperative part, such as finding the solution to $y^1 + y = 0$. In the same way, the above Mittag-Leffler function is very vital for the fractional-order calculus, and it has an important role in finding solutions to the non-integer order differential equations (Shukla & Sapra, 2019).

2.2.4 Laplace Transform

Pierre-Simon Laplace is an astronomer and mathematician who introduced the Laplace transform. The Laplace transform is a compelling method and it is used to solve convolution and differential equations. This integral transform allows one to turn complicated equations into simple ones (Luchko, 2019; Zachary & Tseng, 2008). Suppose you have differential equation $g(t)$, if the Laplace transform is used on an equation, it will convert it into an algebraic one $G(s)$. Using the inverse Laplace transform, the solution $Y(s)$ can be mapped back to the original domain $y(t)$ (Bryant, 2008; Sontakke & Shaikh, 2015). This procedure is represented by the diagram below (La Rosa, n.d):

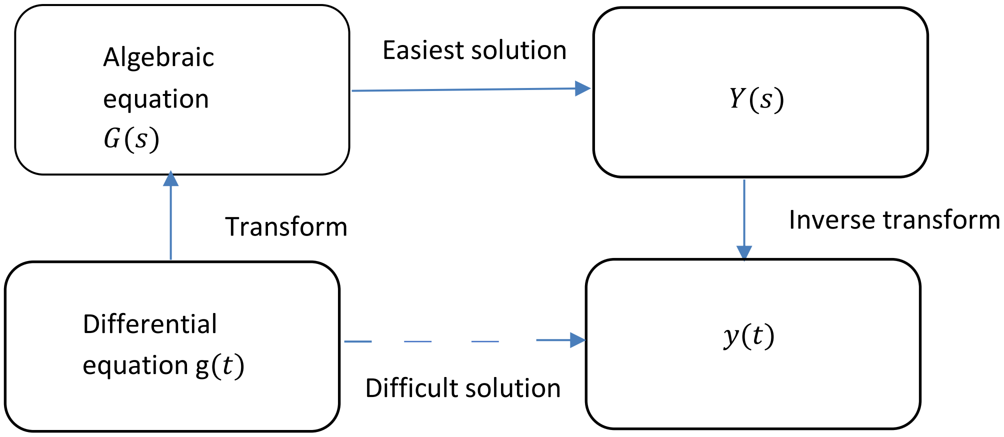

Figure 2: Integral transforms (La Rosa, n.d)

There are two main advantages of the Laplace transform. Firstly, problems are solved more directly; there is no need for finding a general solution of an initial value problem, and nonhomogeneous ordinary differential equations are not required to convert into homogeneous ODEs. Secondly, the use of a unit step function and Dirac's delta makes the method prevailing for complications with inputs (Shukla & Sapra, 2019). The formal definitions of the Laplace transform and convolution are presented in the following chapter.

2.2.4.1 Transfer function and block diagram

Another way in which the Laplace transform is used in modelling complex processes is by making use of a transfer function and representing it using a block diagram. A transfer function of a time-invariant, linear system is a proportion of the output variable (response function), $Z(s) = \mathcal{L}\{z(t)\}$ to the input variable also referred to as adriving function $Y(s) = \mathcal{L}\{y(t)\}$, of the Laplace transform, with all zero initial conditions (Zwart, 2004). The transfer functions together with Bode diagrams, it can be used to describe the relationship between the cause and effect throughout a system (Aziz, 2010). By taking the Laplace transform of a differential equation, the equation is then transformed into:

$$G(s) \equiv \frac{Z(s)}{Y(s)} \tag{2.5}$$

where, $G(s)$ is a transfer function. Linking the output variable to the input variable can be designated schematically by the use of a block diagram, as shown below:

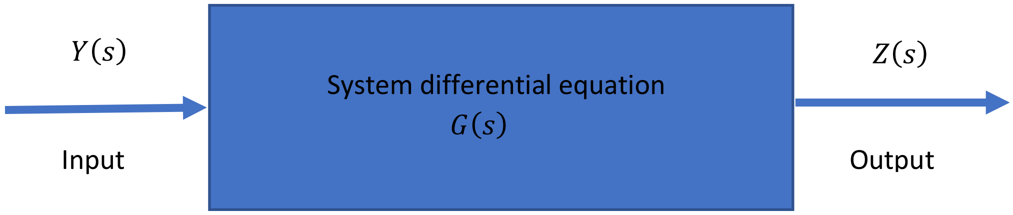

Figure 3: Representation of a transfer function (HELM, 2008)

2.2.5 Definitions, advantages, and disadvantages of the fractional derivatives and integrals

Nowadays, the fractional derivatives that are used are built based on three types of dominant calculus; the exponential decay law, Mittag-Leffler function, and power law kernel. The fractional-order derivative is not certainly exceptional; with that said, there are some accepted and standard definitions in the literature (Matlob & Jamali, 2017). In this section, we mention these definitions and also highlight their advantages and disadvantages. It is also crucial to note that despite all these disadvantages, the fractional derivatives and integrals have been used in various fields with a lot of success. The mathematical representation of these definitions is provided in chapters that follow.

The Riemann-Liouville fractional derivative is built using the power law kernel (Alqahtani, 2016). With the R-L fractional derivative, a random function doesn’t have to be constant at its origin, and it doesn’t have to be differentiable (Atangana, 2018b). This derivative has particular limitations when modelling practical events with fractional differential equations. R-L fractional derivative is a non-local operator and the derivative of its constant is not zero (Atangana & Bildik, 2013; Atangana, 2017). Additionally, if a random function at the origin is continuous, then the origin of its fractional derivative will have singularity. These disadvantages lessen the application of this derivative (Atangana & Secer, 2013).

In 1967 an alternative fractional derivative also founded on the power law was presented by Caputo. It has been pointed out by many researchers that fractional derivatives based on the power-law are powerful and appropriate mathematical tools that can accurately describe the flow in a media that is elastic (Alqahtani, 2016). The greatest advantage of a Caputo derivative is its ability to incorporate the traditional boundary and initial conditions in the construction of problems. Additionally, the derivative of its constant is zero, therefore the Caputo is much appreciated to describe natural problems compared to the Riemann-Liouville (Atangana & Secer, 2013; Alqahtani, 2016). However, the Caputo fractional derivative is local and demand sophisticated conditions of regularity for differentiability, this reduces its field application (Atangana & Bildik, 2013; Atangana, 2018b).

Caputo and Fabrizio recently suggested a new derivative in the field of fractional calculus. This derivative is built using the exponential decay law and it is collectively known as the Caputo-Fabrizio fractional derivative. Some benefits of this derivative over the Caputo and R-L are that it can be used where exponential law is noticed in nature, the derivative is free from singularity (Caputo & Fabrizio, 2015). In addition, this derivative can describe heterogeneities of materials and configurations with different scales (Atangana & Alqahtani, 2016). However, issues were pointed out contrary to this derivative. Firstly, the kernel is local, secondly the anti-derivative related to it is merely just an average of the function and it's integral (Atangana, 2018a). Therefore, the operator is rather just a filter than a fractional derivative. Nevertheless, many works with great success were achieved using this new derivative. This derivative is used in groundwater studies, diffusion models, mechanical engineering, thermal science and others (Alqahtani, 2016).

In 2016, Atangana and Baleanu presented a non-local derivative built using the generalized Mittag-Leffler function with no singular kernel and satisfies all problems pointed out contrary to the Caputo-Fabrizio fractional derivative (Alqahtani, 2016). This new approach can depict material heterogeneity and structures with diverse scales (Atangana, 2018a). Due to its Mittag-Leffler memory, the Atangana-Baleanu derivative is able to differentiate between dynamical systems taking place in diverse scales without steady-state (Atangana & Gomez-Aguilar, 2017). Mittag-Leffler function is a good filter compared to the power-law and exponential functions, making the Atangana–Baleanu fractional derivative a prevailing mathematical tool to model complex practical problems (Atangana, 2018a).

2.3. STOCHASTIC APPROACH

There are two types of modeling that are generally used to solve groundwater problems; Deterministic and stochastic. Deterministic models suppose that all the input parameters are known with certainty, and subsequently, each parameter has a deterministic value. Nevertheless, this hypothesis is not always true owing to heterogeneities associated with hydrogeological parameters. Hence, deterministic modelling can result in weighty errors in the forecast, and thus, the purpose of numerical modelling cannot be accomplished (Baalousha & Kongeter, 2006). A possible method to enumerate uncertainty owing to the spatial heterogeneity of underground systems is to view heterogeneity in a stochastic sense (Chang & Yeh, 2010). The stochastic approach has been used heavily in mathematical biology. Birth and death processes, for instance, were originally used to study the growth of populations over time (Bressloff, 2014). A stochastic process is a sequence of a random variable $[X(t)]$whose value undergo a change over time due to probabilistic laws. The properties of a stochastic process must be established from a single time series. It is assumed that many stochastic processes are Markovian. The property of a first-order Markov process is that the dependence of forthcoming values of the process on previous values rest on only the present values and not on previous observations or values. Therefore, the present value summaries the state of the process. As a result, the present value of the process is usually called the state. This also makes sense when referring to the state of the level of an aquifer. Another assumption is that the stochastic process is stationary, meaning that the probability dispersal of the process will not change with time. The statistics used when describing the distribution of a continuous-state stationary stochastic process are the sample variance, the sample mean and several autocorrelations. To estimate the mean and the variance the following equations are used:

$$\hat{\mu}_x = \bar{x} = \frac{1}{\Gamma}\sum_{t=0}^{T} X_t \tag{2.6}$$

and

$$\delta^2{}_x = \frac{1}{\Gamma}\sum_{t=0}^{T} (x_t - \bar{x})^2 \tag{2.7}$$

The mean and the variance are both independent of time t. However, stochastic processes are not always stationary, so statistical methods that depend on the stationary assumptions do not apply and the problems become more complicated (Loucks *et al.*, 2005).

CHAPTER 3

ANALYSIS OF THE EXISTING MODEL OF VERTICAL FLOW IN THE SATURATED-UNSATURATED ZONE

3.1. INTRODUCTION

Instead of dealing with the complex physical reality, a mathematical model consisting of a system of equations that connects the parameters and variables, relevant to their physical meaning is built by abstracting the process and introducing certain assumptions and simplifications. For this study, a transient 1-d vertical flow equation will be analyzed based on the physics of water transfer between the unsaturated and saturated zones. This area is limited by the soil surface and by the lower impervious boundary of the aquifer.

3.2. MATHEMATICAL MODEL

The unsaturated and saturated zone can be regarded as only one continuum. What links the saturated and unsaturated zone is that the solutions of the equation governing the flow in the unsaturated zone determine the recharge and storage information used to solve the equation governing the saturated flow (Pikul *et al.*, 1974). The equation which is used for both zones is a combination of the continuity law and Darcy's law. Therefore, by implementing suitable values of parameters, equation (1.5) can represent the Richards' equation which governs flow in the unsaturated zone and the groundwater flow equation in the saturated zone. The resulting equation for a saturated flow is linear and it is non-linear for the unsaturated flow (Nishigaki & Kono, 1980; Cheng & Gulliksson 2003). Wherever flow is saturated ($\psi > 0$ and$p > 0$), modelling is easier as $\theta$and Kremains constant for a specified soil textural class (Allepalli & Gavindraraju, 1996; Kresic, 2006). For flow in the unsaturated zone ($\psi \leq 0$ and$p \leq 0$), K depends strongly on θ. As θ decreases, larger pores empty first, the result is not only that fewer pores are filled to conduct water, but the remaining pores have a smaller radius and higher resistance to flow. As a result, K decreases intensely as θ decreases (Nimmo *et al.*, 2009). Due to this complexity, empirical relationships are essential to model unsaturated conductivity and compute flow in the vadose zone (Cattaneo *et al.*, 2015). The empirical solution scheme used for this study is that of Brook and Corey (1964).

3.3. BROOK AND COREY MODEL

Soil hydraulic functions refer to the hydraulic conductivity function, $K(\psi)$ and the soil water content function, $\theta(\psi)$ that are required to explain the movement of water. Numerous functions have been suggested to empirically define the soil hydraulic properties. One of the popular models is the equations of Brook and Corey (Allepalli & Gavindraraju, 1996):

$$K(\psi) = K_s \left(\frac{\psi_b}{\psi}\right)^{2+3\lambda} \qquad \psi \leq 0 \qquad (3.1)$$

$$K(\psi) = K_s \qquad \psi > 0 \qquad (3.2)$$

$$\theta(\psi) = \theta_0 + (\theta_s - \theta_0)\left(\frac{\psi_b}{\psi}\right)^{\lambda} \qquad \psi \leq 0 \qquad (3.3)$$

where: K_s= saturated hydraulic conductivity, ψ = pressure head, ψ_b= air entry suction pressure head, $\theta(\psi)$ = soil water content, θ_0= residual water content, θ_s= saturated water content, λ = pore size distribution index.

3.4. PARAMETERS FOR SATURATED-UNSATURATED ZONE

The functional relationship between ψ and θ is called the water retention curve and it is shown in figure 4 below. Incorporated in this figure are the soil hydraulic functions ($K(\psi)$ and $\theta(\psi)$) showing how they differ from unsaturated to the saturated zone. This figure is modified after Nishigaki and Kono, 1980.

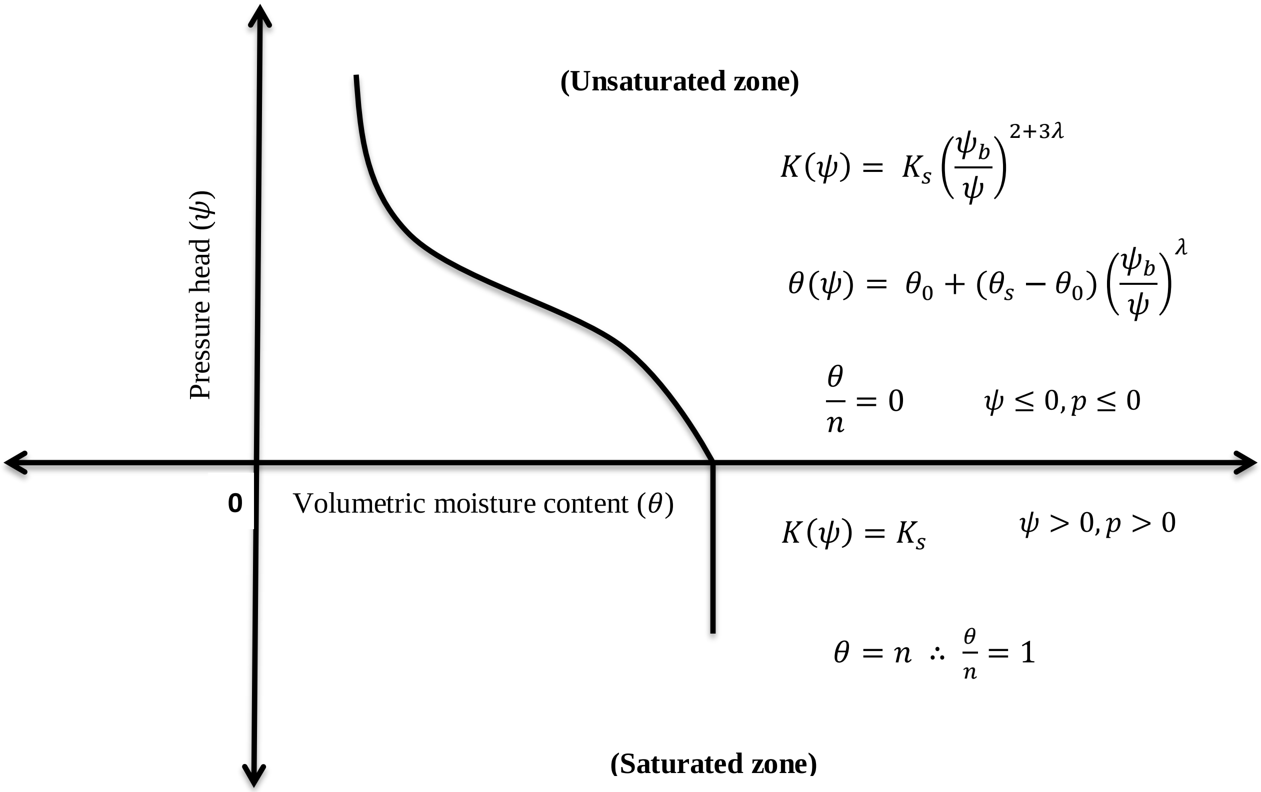

Figure 4: Relationship between the pressure head and volumetric water content for saturated-unsaturated flow (modified after Nishigaki and Kono, 1980).

3.5. ANALYTICAL TOOLS FOR SATURATED-UNSATURATED POROUS MEDIA

Now that the flow in the unsaturated and saturated zone has been distinguished, classical equations suitable for each zone will be derived below using powerful methods that result in explicit forms of a solution. In the case of the saturated zone, analytical and numerical solutions will be derived using integral transform, the method of separation of variables and numerical schemes respectively. The conditions under which the numerical method used converges will be derived and presented in detail. Due to the complexity of the unsaturated model, we relied only on the numerical method to derive an approximate solution. Some numerical simulations will be depicted for different values of the used input parameters. To familiarize readers with these methods, below we will discuss the definition and some properties of the Fourier Transform and Laplace Transform that will be useful for understanding this section.

3.5.1 Definition and properties of the Laplace Transform

Definition: If the Laplace transform of $f(t)$, represented by $F(s)$ or $\mathcal{L}\{f(t)\}$ is defined for $[0,\infty)$, then the Laplace integral is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\ dt \tag{3.4}$$

The Laplace transform has a number of properties. We state few definitions of these properties in two ways, first in words and then in symbols just like we did for the above defined Laplace transform.

The first very useful property is the *linearity* of the Laplace transform:

$\mathcal{L}$ is a linear operator. This means that of any two functions f and g for which the Laplace is defined and two constants a and $b \in$ **R** we have:

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\} \tag{3.4.1}$$

If $\mathcal{L}\{f\} = F(s)$ and $\mathcal{L}\{g\} = G(s)$ then we have:

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{F(s)\} + b\mathcal{L}\{G(s)\} \tag{3.4.1.1}$$

Laplace transform of a first derivative. If $\mathcal{L}\{f\} = F(s)$ then

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) \tag{3.4.2}$$

Laplace transform of a second derivative. If $\mathcal{L}\{f\} = F(s)$ then:

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0) \tag{3.4.3}$$

Time scaling. Let $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \tag{3.4.4}$$

For any$a > 0$

To prove this property, we start with replacing some variables in the definition of the Laplace transform; we use the change of variable $at = \tau$, from where$dt = d\tau/a$. Note that for $a > 0$ the limits of integration will not change:

$$\mathcal{L}\{f(\tau)\} = \frac{1}{a}\int_0^{\infty} f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F\left(\frac{s}{a}\right) \tag{3.4.4.1}$$

Note: if we allow any sign for $a \neq 0$ then

$$\mathcal{L}\{f(at)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right) \tag{3.4.4.2}$$

Differentiation of the frequency (multiplication by t): Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

To prove this property, we differentiate either side of the Laplace transform with respect to s to get:

$$F'(s) = \int_0^\infty (-t)e^{-st}f(t)\ dt \tag{3.4.5}$$

Time delay: Let$\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{f(t)\} = \ e^{-eT}F(s) \tag{3.4.6}$$

The property *time delay* implies that the function $f(t)$ in the Laplace transform is delayed by T seconds and "zero padded" up to$T$. Therefore, $f(t) = f(t-T)$:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t-T)\,dt \tag{3.4.6.1}$$

Let: $t = \tau + T$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-s(\tau+T)}f(\tau)\ dt \tag{3.4.6.2}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-sT}F(s) \tag{3.4.6.3}$$

Convolution: The convolution assists in solving integral equations. If we let $F(s)$ and $G(s)$ denote the transforms of $f(t)$ and$g(t)$, respectively, then the inverse of the product $F(s)$and $G(s)$ is given by the function$h(t) = (f * g)(t)$. It is called the convolution of $f(t)$and $g(t)$ and can be regarded as a generalised product of $f(t)$ and$g(t)$ (Mathews & Howell, 2012):

$$h(t) = (f * g)(t) = \int_0^t f(\tau)\ g(t-\tau)d\tau \tag{3.4.7}$$

The integral representation of the Laplace transform of a convolution is therefore given by:

Let: $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then,

$$\mathcal{L}[f(t) * g(t); s] = \mathcal{L}[F(s)]\mathcal{L}[G(s)] \tag{3.4.7.1}$$

3.5.2 Definitions and properties of the Fourier Transform

Properties of the Fourier transform include linearity, time delay, time scaling, duality, correlation, time derivative, convolution, etc. However, we will only give the definition of the Fourier Transform.

Definitions: The Fourier transform of a function f is usually represented by adding a circumflex $\hat{f}$. There are many forms of stating the Fourier transform of an integrable function$F: R \to C$. In our case, the following definition will be used:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt \tag{3.5}$$

3.5.3 Definition of the inverse Fourier Transform

This transform has an important role in the application of the Fourier transform. Ideally, in many situations, the Fourier transform is applied first, performing some simplification, then apply the inverse Fourier transform. The definition of the inverse Fourier is given by:

$$\mathcal{F}^{-1}\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega \tag{3.6}$$

3.6. SATURATED FLOW EQUATION

For flow in the saturated zone, we now know that the volumetric water content is equal to the porosity, we can, therefore, cancel them out of the equation. The hydraulic conductivity in the saturated zone is constant and the capillary capacity becomes zero, therefore the equation for vertical groundwater flow in the saturated zone is given as (Maslouhi *et al.*, 2009):

$$(S_S)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left[K_s\left(\frac{\partial \psi}{\partial z} - 1\right)\right] \tag{3.7}$$

Since the hydraulic conductivity of the saturated zone is constant, the above equation can be further simplified into:

$$\frac{\partial\psi}{\partial t}=\frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2} \tag{3.8}$$

In the next few pages, we attempt to find the analytical solution, numerical solution, and the stability of the solutions generated for equation (3.8).

3.6.1 Analytical solution using the Integral Transform

The above equation for the saturated zone (3.8) can be solved analytically using an integral transform, for instance, the Fourier transform or Laplace, for this case we will make use of both the Laplace and Fourier transform to provide an exact solution. Thus, applying the Laplace transform on either side of equation (3.8) we obtain:

$$\mathcal{L}\left(\frac{\partial\psi}{\partial t}\right)=\mathcal{L}\left(\frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2}\right) \tag{3.9}$$

Applying the Laplace transform of a derivative into equation (3.9) we obtain the following with s being considered as Laplace variable:

$$s\tilde{\psi}(z,s)-\psi(z,0)=\left(\frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2}\right)(z,s) \tag{3.9.1}$$

To eliminate the space component in order to obtain an algebraic, we use the Fourier transform to the above equation:

$$\mathcal{F}\big(s\tilde{\psi}\,(z,s)-\psi(z,0)\,\big)=\frac{K_s}{S_s}\,\mathcal{F}\left(\frac{\partial^2\tilde{\psi}\,(z,s)}{\partial z^2}\right) \tag{3.9.2}$$

Using the properties of the Fourier transform presented earlier, the right-hand side of the equation above can further be simplified into a linear form as follows:

$$s\tilde{\psi}_{\mathcal{F}}(\omega,s)-\psi_{\mathcal{F}}(\omega,0)=\frac{K_s}{S_s}(i\omega)^2\tilde{\psi}_{\mathcal{F}}(\omega,s) \tag{3.9.3}$$

The above equation can be factorised to obtain the equation below:

$$\left(s-\frac{K_s}{S_s}(i\omega)^2\right)\tilde{\psi}_{\mathcal{F}}(\omega,s)=\psi_{\mathcal{F}}(\omega,0) \tag{3.9.4}$$

We further simplify the above equation into:

$$\tilde{\psi}_{\mathcal{F}}(\omega, s) = \frac{\psi_{\mathcal{F}}(\omega, 0)}{\left(s - \frac{K_s}{S_s}(i\omega)^2\right)} \tag{3.9.5}$$

And then:

$$\tilde{\psi}_{\mathcal{F}}(\omega, s) = \frac{\psi_{\mathcal{F}}(\omega, 0)}{\left(\frac{S_s}{K_s}s + (\omega)^2\right)\frac{K_s}{S_s}} \tag{3.9.6}$$

The above formula is the multiplication of two Fourier transform of two functions; thus, one can use the convolution theorem to obtain the inverse transform as follow:

$$\mathcal{F}^{-1}\big(\psi_{\mathcal{F}}(\omega, 0)\big) = \psi(z, 0) \tag{3.9.7}$$

$$\psi(z, s) = \mathcal{F}^{-1}\left(\frac{1}{\frac{K_s}{S_s}\left(\omega^2 + \left(\sqrt{\frac{S_s}{K_s}s}\right)^2\right)}\right) \tag{3.9.8}$$

$$\psi(z, s) = \mathcal{F}^{-1}\left(\frac{2\sqrt{\frac{S_s}{K_s}s}}{\frac{K_s}{S_s}2\sqrt{\frac{S_s}{K_s}s}\left(\omega^2 + \left(\sqrt{\frac{S_s}{K_s}s}\right)^2\right)}\right) \tag{3.9.9}$$

$$\psi(z, s) = \frac{1}{\frac{K_s}{S_s}2\sqrt{\frac{S_s}{K_s}s}\; exp\left(-\sqrt{\frac{S_s}{K_s}s}(z)\right)} \tag{3.9.10}$$

By integrating the above equation, we obtain:

$$\psi(z, s) = \frac{1}{\frac{K_s}{S_s}2\sqrt{\frac{S_s}{K_s}s}}\int_0^z \psi(\lambda, 0)\, exp\left[-\sqrt{\frac{S_s}{K_s}s}(z - \lambda)\right] d\lambda \tag{3.9.11}$$

To obtain the exact solution of the saturated zone, one can use the inverse Laplace transform on both sides of equation (3.9.11), such that:

$$\psi(z,s) = \mathcal{L}^{-1}\big(\psi(z,s)\big) \tag{3.9.12}$$

3.6.2 Analytical solution using the method of separation of variables

Using the method of separation of variable, we can now obtain an analytical solution of the above equation (3.8). This method is used to solve a wide variety of linear and homogeneous differential equations, such as the saturated groundwater flow equation. The boundary conditions are linear and homogenous, such that:

$$\psi\big|_{z=0} = \psi\big|_{z=L} = 0 \tag{3.10}$$

The dependence of ψ on z and t can be represented as a product of function z and function t, such that:

$$\psi(z,t) = F(z)G(t) \tag{3.10.1}$$

Substitution ψ back into equation (3.8) using the product rule we obtain:

$$\frac{\partial(F(z)G(t)\,)}{\partial t} = \frac{K_s}{S_s}\,\frac{\partial^2(F(z)G(t)\,)}{\partial z^2} \tag{3.10.2}$$

Since the left-hand side is dependent only on t and the right-hand side is dependent only on z, we can separate the variables as follows:

$$F(z)\frac{S_s}{K_s}\frac{dG(t)}{dt} = G(t)\,\frac{d^2F(z)}{dz^2} \tag{3.10.3}$$

Variables can be further separated to obtain the following:

$$\frac{\frac{S_s}{K_s}\frac{dG(t)}{dt}}{G(t)} = \frac{\frac{d^2F(z)}{dz^2}}{F(z)} \tag{3.10.4}$$

The RHS and the LHS of equation (3.10.4) can be a function of z and t respectively, if they both equate to a constant valueα, such that:

$$\frac{S_s}{K_s}\,\frac{dG(t)}{dt} = \alpha G(t) \tag{3.10.4.1}$$

and:

$$\frac{d^2F(z)}{dz^2} = \alpha F(z) \tag{3.10.4.2}$$

Now we try to obtain a solution that is not zero satisfying the boundary conditions. For our first attempt, we assume that for the constant $\alpha < 0$ real numbers B, C exist such that:

$$F(z) = Be^{\sqrt{-\alpha z}} + Ce^{-\sqrt{-\alpha z}} \tag{3.10.5}$$

From the boundary conditions given in equation (3.10), we get:

$$F(0) = 0 = F(L) \tag{3.10.5.1}$$

$B = 0 = C$ implies that ψ is identically 0, therefore, we reject this case. In the second assumption, we assume that when $\lambda = 0$ real numbers B, C exist such that:

$$F(z) = B(z) + C \tag{3.10.6}$$

The same conclusion that ψ is identically 0 can be made, and again we reject this case. The last case is when$\alpha > 0$. Then there exists the real number A, B, C such that:

$$G(t) = Ae^{-\alpha\frac{S_s}{K_s}t} \tag{3.10.7}$$

And

$$F(z) = B\, sin\left(\sqrt{\alpha}z\right) + C\, cos\left(\sqrt{\alpha}z\right) \tag{3.10.8}$$

From the above information $C = 0$ and that for some positive integer$n$:

$$\sqrt{\alpha} = n\frac{\pi}{L} \tag{3.10.9}$$

Therefore, a general solution can be given as:

$$\psi(z,t) = \sum_{n=1}^{\infty} D_n\, sin\frac{n\pi z}{L}\; exp\left(-\frac{n^2\pi^2\frac{S_s}{K_s}t}{L^2}\right) \tag{3.10.10}$$

where D_n is a coefficient determined by the initial condition, given the following initial condition:

$$\psi\,\big|_{t=0} = v(z) \tag{3.10.10.1}$$

We obtain:

$$v(z) = \sum_{n=1}^{\infty} D_n \, sin\frac{n\pi z}{L} \tag{3.10.10.2}$$

Multiplying both sides with $sin\frac{n\pi z}{L}$ and integrating over $[0, L]$ results into:

$$D_n = \frac{2}{L}\int_0^L v(z)\, sin\frac{n\pi z}{L} dx \tag{3.10.10.3}$$

Hence, the complete solution for equation (3.8) is given by:

$$\psi(z,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L v(z)\, sin\frac{n\pi z}{L} dx\right) sin\frac{n\pi z}{L}\ exp\left(-\frac{n^2\pi^2\frac{S_s}{K_s}t}{L^2}\right) \tag{3.10.10.4}$$

3.6.3 Numerical solution

The analytical solutions of our model are limited to less complexities, such as the assumption of homogeneity, isotropy, simple initial condition, and simple geological formation. However, natural systems can have a more complex geological formation, a complex initial condition and they can be heterogeneous and anisotropic. Such complexities require numerical solutions. Depending on how ψ_t is approximated, we have three basic finite difference schemes: Crank-Nicolson, explicit, implicit and schemes. Using these schemes, we attempt to find numerical solutions for the 1-dimension saturated groundwater flow equation (3.8).

3.6.3.1 Numerical solution using forward Euler method (FTCS)

Appling the explicit forward Euler method on the equation (3.8), we obtain the following numerical solution:

$$\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} = \frac{K_s}{S_s}\left[\frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2}\right] \tag{3.11}$$

3.6.3.2 Numerical solution using backward Euler method (BTCS)

We obtain another numerical solution by applying the implicit backward Euler method on equation (3.8):

$$\frac{\psi_i^{n+1} - \psi_i{}^n}{\Delta t} = \frac{K_s}{S_s}\left[\frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{(\Delta z)^2}\right] \tag{3.12}$$

3.7.3.3 Numerical solution using the Crank-Nicolson method

Applying the finite difference spatial discretization on equation (3.8), the implicit Crank-Nicolson discretization is as follows:

$$\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} = \frac{K_s}{S_s}\left[\frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{2(\Delta z)^2} + \frac{\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n}}{(2\Delta z)^2}\right] \tag{3.13}$$

3.6.4 Numerical stability analysis

In physical problems such as groundwater flow, stability analysis of an equation is very important. A procedure called von Neumann stability analysis based on the Fourier series is used to analyse the stability of finite difference schemes. The stability of finite difference schemes is linked to numerical errors. A scheme is said to be von Neumann stable if its amplification factor is less or equal to 1. Accuracy, however, requires that the amplification factor be as close to 1 as possible. In this section, the numerical solutions provided in equations 3.11-3.13 for the 1-d saturated groundwater flow equation (3.8) are subjected under von Neumann stability analysis. A detailed procedure for each solution is given below.

3.7.4.1. Stability analysis of a forward Euler method (FTCS)

Let us recall on the forward Euler method (3.11) which can also be written as:

$$\psi_i^{n+1} = \psi_i^n\,(1 - 2\alpha) + \alpha(\psi_{i+1}^n + \psi_{i-1}^n) \tag{3.14}$$

where $\alpha = \frac{K_s}{S_s}\left(\frac{\Delta t}{(\Delta z)^2}\right)$

We consider a harmonic initial perturbation:

$$\psi_i^0 = e^{ik_m z} \tag{3.14.1}$$

Which with time evolves as:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{3.14.1.1}$$

To analysis the stability of this scheme, let us find the amplification factor $\sigma$by inserting equation (3.14.1.1) into equation (3.14):

$$\sigma^{n+1}e^{ik_m\Delta z} = \sigma^n e^{ik_m z}(1-2\alpha) + \alpha\left(\sigma^n e^{ik_m(\mathrm{z}+\Delta\mathrm{z})} + \sigma^n e^{ik_m(\mathrm{z}-\Delta\mathrm{z})}\right) \qquad (3.14.2)$$

We can simplify equation (3.14.2) by pulling out the common factor, such that:

$$\sigma^{n+1}e^{ik_m\Delta z} = \sigma^n e^{ik_m z}\left[1-2\alpha+\alpha\left(e^{ik_m\Delta z}+e^{-ik_m\Delta z}\right)\right] \qquad (3.14.3)$$

Further simplification of the above equation results into the following:

$$\sigma^{n+1}e^{ik_m\Delta z} = \sigma^n e^{ik_m z}\left[1+2\alpha\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right] \qquad (3.14.4)$$

Considering the definition for a hyperbolic cosine, we can rewrite the above equation (3.14.4) as:

$$\sigma^{n+1}e^{ik_m\Delta z} = \sigma^n e^{ik_m z}[1+2\alpha(\cos(k_m\Delta z)-1)] \qquad (3.14.5)$$

Using the double angle identity forcos, equation (3.14.5) is written such that:

$$\sigma^{n+1}e^{ik_m\Delta z} = \sigma^n e^{ik_m\Delta z}\left[1+2\alpha\left(1-2sin^2\left(\frac{k_m\Delta z}{2}\right)-1\right)\right] \qquad (3.14.6)$$

$$\sigma^{n+1}e^{ik_m\Delta z} = \sigma^n e^{ik_m\Delta z}\left[1-4\alpha sin^2\frac{k_m\Delta z}{2}\right] \qquad (3.14.6.1)$$

Dividing both sides by$\sigma^n e^{ik_m\Delta z}$ we get:

$$\sigma = 1-4\alpha\, sin^2\frac{k_m\Delta z}{2} \qquad (3.14.6.2)$$

The above equation gives us the amplification factorσ for the forward Euler method, and the stability condition is written as:

$$\left|1-4\alpha\, sin^2\frac{k_m\Delta z}{2}\right| \le 1 \qquad (3.14.7)$$

However, if $k_m\Delta z = \pi \Rightarrow \sigma = 1-4\alpha$. The stability of the above schema (3.14) for all $k_m\Delta z$ will only hold if$\alpha \le \frac{1}{2}$. Therefore, the forward Euler method (3.14) is conditionally stable or von Neumann unstable as applied to the 1-d saturated groundwater flow equation (3.8).

3.7.4.2. Stability analysis of a backward Euler method (BTCS)

For the above backward Euler method (3.12), we can obtain ${-\psi_i}^n$ from solving a system of linear equations:

$$-\psi_i^n = \alpha\psi_{i+1}^{n+1} - (1+2\alpha)\psi_i^{n+1} + \alpha\psi_{i-1}^{n+1} \tag{3.15}$$

where $\alpha = \frac{K_s}{S_s}\left(\frac{\Delta t}{(\Delta z)^2}\right)$

If we consider a harmonic initial perturbation (3.14.1.2) we can find the amplification factor σ for this scheme by inserting it into equation (3.15) such that:

$$-\sigma^n e^{ik_i\Delta z} = \alpha\sigma^{n+1}e^{ik_m z} - (1+2\alpha)\sigma^{n+1}e^{ik_m(\mathrm{z}+\Delta\mathrm{z})} + \alpha\sigma^{n+1}e^{ik_m(\mathrm{z}-\Delta\mathrm{z})} \tag{3.15.1}$$

We can simplify equation (3.15.1) by pulling out the common factor, such that:

$$-\sigma^n e^{ik_m z} = \sigma^{n+1}e^{ik_m z}\left[-(1+2\alpha) + \alpha\left(e^{ik_m\Delta z} + e^{-ik_m\Delta z}\right)\right] \tag{3.15.2}$$

Further simplification of the above equation results into the following:

$$-\sigma^n e^{ik_m z} = \sigma^{n+1}e^{ik_m z}\left[-(1+2\alpha) + 2\alpha\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}\right)\right] \tag{3.15.3}$$

Considering the definition for a hyperbolic cosine, we can rewrite the above equation (3.15.3) as:

$$-\sigma^n e^{ik_m z} = \sigma^{n+1}e^{ik_m z}[-1 - 2\alpha + 2\alpha\cos(k_m\Delta z)] \tag{3.15.4}$$

Using the double angle identity forcos, equation (3.15.4) is written such that:

$$\sigma^n e^{ik_m z} = \sigma^{n+1}e^{ik_m z}\left[1 + 2\alpha - 2\alpha - 4\alpha sin^2\left(\frac{k_m\Delta z}{2}\right)\right] \tag{3.15.5}$$

$$\sigma^n e^{ik_m z} = \sigma^{n+1}e^{ik_m z}\left[1 + 4\alpha sin^2\frac{k_m\Delta z}{2}\right] \tag{3.15.5.1}$$

Dividing both sides by$\sigma^n e^{ik_m z}$:

$$\sigma = \left(1 + 4\alpha\, sin^2\frac{k_m\Delta z}{2}\right)^{-1} \tag{3.15.5.2}$$

The above equation gives us the amplification factorσ for the backward Euler method and the stability condition is written as:

$$\left|\left(1 + 4\alpha\, sin^2 \frac{k_m \Delta z}{2}\right)^{-1}\right| < 1 \tag{3.15.6}$$

This equation is unconditionally stable because σ will always be less than 1.

3.7.4.3. Stability analysis of the Crank-Nicolson method

So far, we have considered the forward Euler scheme and the backward Euler scheme for the 1-dimension saturated groundwater flow equation. The forward Euler scheme is an explicit method, therefore easier to implement. However, the results obtained for the forward Euler scheme indicates that it is only stability under the condition that $\alpha \leq \frac{1}{2}$. The backward Euler scheme, on the other hand, is an implicit method and is unconditionally stable; however, it requires more arithmetic operations to find values at a certain time step (Grigoryan, 2012). It is also essential to note that the two schemes use different sets of points in the computation of$\psi_i{}^{n+1}$. The accuracy and the stability of the two schemes can be improved by developing a single implicit scheme, which will be the combination of the two schemes with different weights. With this method, a larger set of points can be used to compute the same values (Narasimhan, 2011; Grigoryan, 2012). Let's consider the numerical solution obtained in equation (3.13), which can be written as:

$$\psi_i^{n+1} - \psi_i^n = \alpha[(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) + (\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)] \tag{3.16}$$

Substituting equation (3.14.1.1) into equation (3.16) which leads to:

$$\begin{aligned}\sigma^{n+1}e^{ik_iz} - \sigma^n e^{ik_iz} = \alpha\big[\big(\sigma^{n+1}e^{ik_m(\mathrm{z}+\Delta \mathrm{z})} - 2\sigma^{n+1}e^{ik_mz} + \sigma^{n+1}e^{ik_m(\mathrm{z}-\Delta \mathrm{z})}\ \big) + \\ \big(\sigma^n e^{ik_m(\mathrm{z}+\Delta \mathrm{z})} - 2\sigma^n e^{ik_mz} + \sigma^n e^{ik_m(\mathrm{z}-\Delta \mathrm{z})}\big)\big]\end{aligned} \tag{3.16.1}$$

We can simplify the above equation by pulling out the common factor, to get:

$$\begin{aligned}&\sigma^n e^{ik_iz}(\sigma - 1) \\ &\qquad = \alpha\big[\sigma^n e^{ik_mz}\sigma\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\ \big) \\ &\qquad + \sigma^n e^{ik_mz}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big]\end{aligned} \tag{3.16.2}$$

Solving forσ, we obtain the following growth factor:

$$\sigma = \frac{1 - 2\alpha(1 - cosk_m\Delta z)}{1 + 2\alpha(1 - cosk_m\Delta z)} \tag{3.16.3}$$

Then the stability condition for the scheme is given by:

$$0 < \left|\frac{1 - 2\alpha(1 - cosk_m\Delta z)}{1 + 2\alpha(1 - cosk_m\Delta z)}\right| < 1 \tag{3.16.4}$$

This scheme was developed in 1947 by J. Crank and Nicolson, and it is illustrated as the average of the backward and forward Euler scheme. The denominator of equation (3.16.3) will always be greater than the numerator since α and $1 - cosk\Delta z$ are positive. This also means that the value of σ is less than 1; therefore, the Crank-Nicolson method for the 1-d saturated groundwater flow equation (3.8) is unconditionally stable.

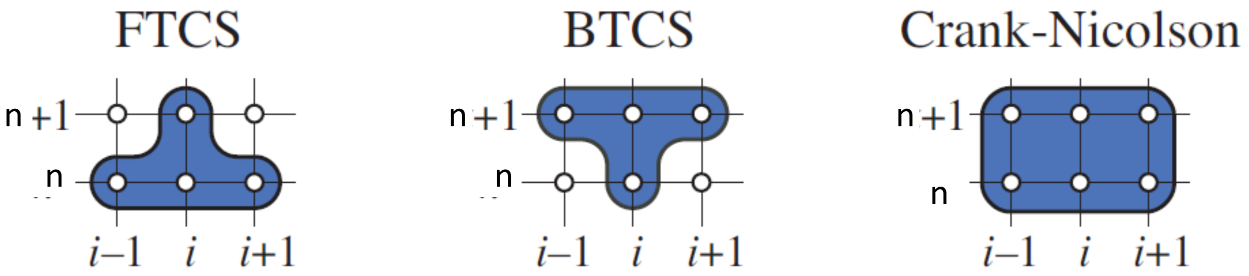

Figure 5: Comparison between computational molecules for the 3 schemes (Recktenwald, 2011).

The above figure 4 illustrates the difference between the computational molecules of the three schemes, it is clear that the Crank-Nicolson scheme is the most stable and accurate of the three schemes. This is because it requires the simultaneous calculation of ψ at all nodes on the $n + 1$ mesh line.

3.7. UNSATURATED FLOW EQUATION

In heterogeneous soils, water content is uneven across layer boundaries due to exceptional unsaturated capillary head relations in the different soil layers (Assouline, 2013). The complexity of the unsaturated flow can then be illustrated by the equation below:

$$\left(\frac{\partial\theta}{\partial\psi}\right)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K_z(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right)\right] \tag{3.17}$$

The capillary head (ψ) is continuous, and can be represented by an equation with ψ as the dependent variable and assess the moisture content in terms of ψ, $\theta = \theta(\psi)$ (Farthing & Ogden, 2017).

$$\left(\frac{\partial\theta(\psi)}{\partial\psi}\right)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K_z(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right)\right] \tag{3.17.1}$$

$$\frac{\partial\theta(\psi)}{\partial t} = \frac{\partial}{\partial z}\left[K_z(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right)\right] \tag{3.17.2}$$

The above equation (3.17.2) consists of the soil hydraulic functions ($K(\psi)$ and $\theta(\psi)$). We will use the popular soil hydraulic property equations (3.1 and 3.3) of Brook and Corey (1956) to describe these functions. The above equation can be written as:

$$\frac{\partial\left(\theta_0 + (\theta_s - \theta_0)\left(\frac{\psi_b}{\psi}\right)^{\lambda}\right)}{\partial t} = \frac{\partial}{\partial z}\left[K_s\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\left(\frac{\partial\psi}{\partial z} - 1\right)\right] \tag{3.17.3}$$

$$\frac{\partial\theta_0}{\partial t} + \ (\theta_s - \theta_0)\frac{\partial}{\partial t}\left(\frac{\psi_b}{\psi}\right)^{\lambda} = \frac{\partial}{\partial z}\left[K_s\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial\psi}{\partial z} - K_s\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\right] \tag{3.17.4}$$

$$(\theta_s - \theta_0)\psi_b^{\lambda}\frac{\partial}{\partial t}\left(\frac{1}{\psi}\right)^{\lambda} = K_s\frac{\partial}{\partial z}\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial\psi}{\partial z} + K_s\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial^2\psi}{\partial z^2} - K_s\frac{\partial}{\partial z}\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda} \tag{3.17.5}$$

$$\begin{aligned}&(\theta_s - \theta_0)\psi_b^{\lambda}\left(-\lambda\psi'\psi^{-\lambda-1}\right)\\&\qquad = K_s\frac{\partial}{\partial z}\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial\psi}{\partial z} + K_s\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial^2\psi}{\partial z^2} - K_s\frac{\partial}{\partial z}\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\end{aligned} \tag{3.17.6}$$

The above equation can be further simplified into:

$$\begin{aligned}&-(\theta_s - \theta_0)\psi_b^{\lambda}\frac{\partial\psi}{\partial t}\frac{\lambda}{\psi_i^{\lambda+1}}\\&\qquad = K_s\frac{\partial}{\partial z}\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial\psi}{\partial z} + K_s\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\frac{\partial^2\psi}{\partial z^2} - K_s\frac{\partial}{\partial z}\left(\frac{\psi_b}{\psi}\right)^{2+3\lambda}\end{aligned} \tag{3.17.7}$$

The above equation is nonlinear and cannot be handled analytically; thus; we depend on the numerical methods to offer us the numerical solution. This will be done in the next section.

3.7.1 The discretized version of the unsaturated groundwater flow model

The method of discretization transfers models, continuous functions, equations into distinct duplicates. It is usually the initial step in making continuous functions, models and equations suitable for numerical evaluations. This section aims to provide a numerical solution to the non-linear partial differential equation representing the dynamical system underlying the flow of subsurface water in an unsaturated zone. To achieve this, we substitute the intervals [0, T] to $0 = t_0 < t_1 < t_2 < t_3 < t_4 <\ldots\ldots\ldots\ldots<t_n$ and the intervals [0, R] to $0 = z_0 < z_1 < z_2 < z_3 < z_4 <\ldots\ldots\ldots\ldots<z_m$. Also, we recall that:

$$\frac{\partial\psi}{\partial t}(z_i, t_{n+1}) = \frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta t} \tag{3.18}$$

And the second derivative gives us the following:

$$\frac{\partial^2\psi}{\partial z^2}(z_i, t_{n+1}) = \frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{(\Delta z)^2} \tag{3.19}$$

By substituting the above equation 3.18 and 3.19 into equation (3.17.7), the groundwater flow equation representing the unsaturated flow can be written in the discrete form given below:

$$\begin{aligned} -(\theta_s - \theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}\frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta t} &= K_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta z} + K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \\ &\times\frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{(\Delta z)^2} - K_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z} \end{aligned} \tag{3.20}$$

The above unsaturated zone equation (3.20) is highly non-linear; therefore, we cannot perform a stability check.

In this chapter, the 1-d saturated-unsaturated groundwater flow equation was analysed, and the analytical and numerical solutions are detailed. However, this equation has its limitations as already indicated in chapter 1. To overcome these limitations, we attempt to modify the classical equations using the newly established differential operators based on the power-law kernel, exponential decay kernel, and Mittag-Leffler function. New equations are also derived using fractal-fractional derivatives, different numerical schemes, and the recently introduced fractional-stochastic approach. A new method of groundwater analysis is introduced using transforms. This is done in the chapters that follow.

CHAPTER 4

NEW MODEL WITH POWER-LAW AND FRACTAL PROPERTIES

4.1. INTRODUCTION

In this chapter, we will make use of a differential operator based on the power-law kernel which has scale-invariant and mean-square displacement properties to develop a new model. The power law describes the relationship between two variables, which indicates that a respective change in one variable will result in a respective proportional change in the other, irrespective of the initial size of the variables (NECSI, 2011). The power-law was first used in economics a hundred years ago. Its behaviour was later observed in many areas such as biology, computer science, physics, demography, information theory language, astronomy, amongst others, and has been successfully applied (Pinto *et al.*, 2001). It has received lots of consideration due to its ability to describe material moving in singular media and its long-range memory (Ghoshdastidar and Dukkipati, 2013). In diffusion, power-law is very important because it reveals the underlying regularities in the properties of a system; it is also strongly believed that the power-law can model nature and its complexities (Atangana & Gomez-Aguilar, 2018). One attribute of the power law is its scale-invariant, which is defined as a feature of laws or objects that do not change if the scale of length or other variables are multiplied by a common factor, thus represents universality. The mean-square displacement measures the derivation of the particle's position with respect to time. It can also determine if the spreading of a particle is merely due to diffusion or if there is a force of advective contributing to the spreading. The mean square displacement is useful because it can model the movement of water in a closed system. To accommodate readers that are not used to these differential and integral operators, we present below the definitions and their properties.

4.2. DEFINITION AND PROPERTIES OF CAPUTO FRACTIONAL OPERATOR

The Caputo fractional operator is suitable for the study of differential equations of fractional order. It was introduced by M. Caputo in 1967. Using this derivative has an advantage that it doesn't only consider simply interpreted initial conditions, but also the derivative of a constant is zero, meaning that this fractional derivative is bounded (Sontakke & Shaikh, 2015). Suppose that $\alpha > 0, t > \alpha, \alpha, t \in R$ For: $n - 1 < \alpha < n$, then the Caputo fractional derivative of is given as:

$$ {}_{0}^{C}D_{*}^{\alpha} f(t) = \frac{1}{\Gamma\,(n-\alpha)} \int_{\alpha}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \qquad (4.1)$$

Properties of Caputo Fractional Differential operators

Representation: Let $n-1<\alpha<n, n \in R$ and $f(t)$ be such that $D_*^\alpha f(t)$ exist. Then:

$$D_*^\alpha f(t)=I^{n-\alpha}D^n f(t) \tag{4.1.1}$$

Therefore, the Caputo operator is equal to $(n-\alpha)$ fold integration after n^{th} order differentiation.

Linearity: Let $n-1<\alpha<n, n \in N, \alpha, \lambda \in C$ and functions $f(t)$ and $g(t)$be such that both $D^\alpha f(t)$ and $D_*^\alpha g(t)$ exist. This fractional derivative is a linear operator, i.e.

$$D_*^\alpha\big(\lambda f(t)+g(t)\big)=\lambda D_*^\alpha f(t)+D_*^\alpha g(t) \tag{4.1.2}$$

Interpolation: Let $n-1<\alpha<n, n \in N, \alpha \in R$ and $f(t)$ be such that $D_*^\alpha f(t)$ exist. Then the following properties hold for the Caputo operator.

$$\operatorname*{Lim}_{\alpha\to n} D_*^\alpha f(t)=f^{(n)}(t), \tag{4.1.3}$$

$$\operatorname*{Lim}_{\alpha\to n-1} D_*^\alpha f(t)=f^{(n-1)}(t)-f^{(n-1)}(0) \tag{4.1.3.1}$$

Non-commutation: Let $n-1<\alpha<n, m, n \in N, \alpha \in R$ and the function $f(t)$ is such that $D_*^\alpha f(t)$ exist. Then in general

$$D_*^\alpha D^\alpha f(t)=D_*^{\alpha+m} f(t)\neq D^m D_*^\alpha f(t) \tag{4.1.4}$$

4.3. NUMERICAL SOLUTION FOR THE SATURATED-UNSATURATED ZONE USING THE CAPUTO FRACTIONAL DERIVATIVE

Looking at the convolution integral equation of the Laplace mentioned in chapter 3, we can define the Caputo fractional derivative for $0<\alpha\leq 1$ as:

$${}_0^C D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{d}{d\tau}f(\tau)(t-\tau)^{-\alpha}d\tau \tag{4.2}$$

$${}_0^C D_t^\alpha f(t)=\frac{d}{dt}f(t)*\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.2.1}$$

In this section, the Caputo equation (4.1) will be discretised to make it suitable for numerical evaluation. To achieve this we substitute the interval [0, T] to $0 = t_0 < t_1 < t_2 < t_3 < t_4$ <……………$<t_n$. The Caputo fractional differential operator can be discretised as follows:

$$ {}_{0}^{C}D_t^{\alpha}\psi(t_n) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\psi'(t)(t_n-\tau)^{-\alpha}d\tau \tag{4.3}$$

Using the numerical approximation of the first derivative, the above is converted to:

$$ {}_{0}^{C}D_t^{\alpha}\psi(t_n) = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}(t_n-\tau)^{-\alpha}d\tau \tag{4.3.1}$$

Since the term inside the integral $\frac{\psi^{n+1}-\psi^{n}}{\Delta t}$does not depend on the parameter τ; it can be placed out of the integral to obtain:

$$ {}_{0}^{C}D_t^{\alpha}\psi(t_n) = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\int_{t_j}^{t_{j+1}}(t_n-\tau)^{-\alpha}d\tau \tag{4.3.2}$$

To easily integrate this equation, we will integrate by substituting$t_n-\tau=y$, and then $d\tau=-dy$. We further substitute τ with the lower and upper boundary in this manner $t_n-(t_{j+1})$ and $t_n-(t_j)$ which gives us the following:

$$ {}_{0}^{C}D_t^{\alpha}\psi(t_n) = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\int_{t_n-t_j}^{t_n-t_{j+1}}(y)^{-\alpha}(-dy) \tag{4.3.3}$$

Now that the equation has been integrated, we can substitute $(\Delta t_n-\Delta t_j)$ and $(\Delta t_n-\Delta t_{(j+1)})$ back into the equation to yield the following:

$$ {}_{0}^{C}D_t^{\alpha}\psi(t_n) = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\Delta t^{1-\alpha}\left\{\frac{(n-j)^{1-\alpha}-(n-j+1)^{1-\alpha}}{1-\alpha}\right\} \tag{4.3.4}$$

Finally, we obtained:

$$ {}_{0}^{C}D_t^{\alpha}\psi(t_n) = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n}\left(\psi^{j+1}-\psi^{j}\right)\left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right) \tag{4.3.5}$$

With the benefits and application of the power law, we now use the above discretized Caputo fractional derivative to convert the classical model derived in chapter 3 to a model with the waiting time distribution and scale-invariant mean square displacement. To achieve this, we replaced the dt in the classical models derived in chapter 3 with the Caputo fractional derivative as shown below.

4.3.1 New models for the saturated zone with the Caputo fractional derivative

By replacing dt in classical equation derived for modelling the saturated zone we obtain the following equation:

$$ {}_{0}^{C}D_{t}^{\alpha}\psi(t_n) = \frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2} \tag{4.4}$$

4.3.1.1 New model of the saturated zone using the Caputo fractional derivative and the explicit forward Euler method

Appling the explicit forward Euler method on the equation (4.4), we obtain the following numerical solution:

$$\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\sum_{j=0}^{n}\left(\psi_i^{j+1}-\psi_i^{j}\right)\left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right) = \frac{K_s}{S_s}\left[\frac{\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}}{(\Delta z)^2}\right] \tag{4.5}$$

Before we can perform stability analysis for the above explicit forward Euler method, we need to simplify the equation. To do that we use the following parameters:

$$a = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)},$$

$$b = \frac{K_s}{S_s(\Delta z)^2}$$

$$R_{n,j}^{\alpha} = \left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right)$$

Now, we can rewrite the equation with parameter:

$$\sum_{j=0}^{n}\left(\psi_i^{j+1}-\psi_i^{j}\right)aR_{n,j}^{\alpha} = b(\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}) \tag{4.5.1}$$

The above equation is simplified into:

$$(\psi_i^{n+1} - \psi_i^n) a R_{n,n}^{\alpha} + \sum_{j=0}^{n-1} \left(\psi_i^{j+1} - \psi_i^j\right) a R_{n,j}^{\alpha} \; = b(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) \qquad (4.5.2)$$

Now we have this equation for a numerical scheme:

$$\psi_i^{n+1} a R_{n,n}^{\alpha} = \psi_i^n \left(a R_{n,n}^{\alpha} - 2b\right) + b(\psi_{i+1}{}^{n} + \psi_{i-1}{}^{n}) - \sum_{j=0}^{n-1} \left(\psi_i^{j+1} - \psi_i^j\right) a R_{n,j}^{\alpha} \qquad (4.5.3)$$

For stability analysis, we assume that:

$$\psi_i^n = \sigma^n e^{i k_m z} \qquad (4.5.4)$$

Substituting the above into equation (4.5.3), we obtain the following:

$$\begin{aligned} &\sigma^{n+1} e^{i k_m z} a R_{n,n}^{\alpha} \\ &\qquad = \sigma^n e^{i k_m z} \left(a R_{n,n}^{\alpha} - 2b\right) + b e^{i k_m z} \left(\sigma^n e^{i k_m \Delta z} + \sigma^n e^{i k_m \Delta z}\right) \\ &\qquad - \sum_{j=0}^{n-1} \left(\sigma^{j+1} - \sigma^j\right) e^{i k_m z} a R_{n,j}^{\alpha} \end{aligned} \qquad (4.5.5)$$

If we simplify the above equation, we obtain the following:

$$\sigma^{n+1} a R_{n,n}^{\alpha} = \sigma^n \left[a R_{n,n}^{\alpha} - 2b + b\left(e^{i k_m \Delta z} + e^{-i k_m \Delta z}\right)\right] - \sum_{j=0}^{n-1} \left(\sigma^{j+1} - \sigma^j\right) a R_{n,j}^{\alpha} \qquad (4.5.6)$$

$$\sigma^{n+1} a R_{n,n}^{\alpha} = \sigma^n \left[a R_{n,n}^{\alpha} + 2b\left(\frac{e^{i k_m \Delta z} + e^{-i k_m \Delta z}}{2} - 1\right)\right] - \sum_{j=0}^{n-1} \left(\sigma^{j+1} - \sigma^j\right) a R_{n,j}^{\alpha} \qquad (4.5.7)$$

$$\sigma^{n+1} a R_{n,n}^{\alpha} = \sigma^n \left[a R_{n,n}^{\alpha} + 2b(\cos(k_m \Delta z) - 1)\right] - \sum_{j=0}^{n-1} \left(\sigma^{j+1} - \sigma^j\right) a R_{n,j}^{\alpha} \qquad (4.5.8)$$

Using the double angle for $cos,$ we can simplify the equation into:

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} + 2b\left(1 - 2\sin^2\left(\frac{k_m \Delta z}{2}\right) - 1\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (4.5.9)$$

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} - 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (4.5.10)$$

If $n = 0$ the above equation becomes:

$$a\, \sigma^{1} R_{0,n}^{\alpha} = \sigma^{0}\left[aR_{0,n}^{\alpha} - 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)\right] \qquad (4.5.11)$$

$$\left|\frac{\sigma^{1}}{\sigma^{0}}\right| < 1 \rightarrow \left|\frac{\sigma_{1}}{\sigma^{0}}\right| = \left|\frac{aR_{0,n}^{\alpha} - 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)}{aR_{0,n}^{\alpha}}\right| < 1 \qquad (4.5.12)$$

$$aR_{0,n}^{\alpha} - 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) < aR_{0,n}^{\alpha} \qquad (4.5.13)$$

$$0 < 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \qquad (4.5.14)$$

$$0 < 4\frac{K_s}{S_s(\Delta z)^2}\sin^2\left(\frac{k_m \Delta z}{2}\right) \qquad (4.5.15)$$

We assume that $\forall\, n > 0$:

$$\left|\frac{\sigma^{n}}{\sigma^{0}}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^{0}}\right| < 1$$

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} - 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (4.5.16)$$

$$\sigma^{n+1} = \frac{\sigma^{n}\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right)\right]}{aR_{n,n}^{\alpha}} - \frac{\sum_{j=0}^{n-1}\sigma^{n-j}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}} \tag{4.5.17}$$

$$|\,\sigma^{n+1}| \leq |\sigma^{n}|\left|\frac{\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right)\right]}{aR_{n,n}^{\alpha}}\right| - \frac{\sum_{j=0}^{n-1}\left|\sigma^{n-j}\right|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}} \tag{4.5.18}$$

Using the recursive hypothesis:

$$|\sigma^{n+1}| < |\sigma^{0}|\left[\frac{aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right)}{aR_{n,n}^{\alpha}}\right] + \frac{\sum_{j=0}^{n-1}|\sigma^{0}|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}} \tag{4.5.19}$$

$$|\sigma^{n+1}| < |\sigma^{0}|\left[\frac{\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right)\right]}{aR_{n,n}^{\alpha}} + \frac{\sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}}\right] \tag{4.5.19.1}$$

$$\left|\frac{\sigma^{n+1}}{\sigma^{0}}\right| < 1$$

$$\left|\frac{aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right) + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}}\right| < 1 \tag{4.5.20}$$

$$aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right) + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < aR_{n,n}^{\alpha} \tag{4.5.21}$$

The stability condition is given as:

$$\sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < 4b\sin^{2}\left(\frac{k_m\Delta z}{2}\right) \tag{4.5.22}$$

4.3.1.2. A new model of the saturated zone using the Caputo fractional derivative and the implicit backward Euler method

We obtain another numerical solution by applying the implicit backward Euler method on equation (4.4):

$$\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\sum_{j=0}^{n}\left(\psi_i^{n+1}-\psi_i^{n}\right)\left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right)=\frac{K_s}{S_s}\left[\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{(\Delta z)^2}\right] \quad (4.6)$$

Before we can perform stability analysis for the above explicit forward Euler method, we need to simplify the equation. To do that we use the following parameters:

$$a=\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)},$$

$$b=\frac{K_s}{S_s(\Delta z)^2}$$

$$R_{n,j}^{\alpha}=\left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right)$$

Now we can rewrite the equation with parameter:

$$\sum_{j=0}^{n}\left(\psi_i^{n+1}-\psi_i^{n}\right)aR_{n,j}^{\alpha}=b\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right) \quad (4.6.1)$$

$$\left(\psi_i^{n+1}-\psi_i^{n}\right)aR_{n,n}^{\alpha}+\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^{j}\right)aR_{n,j}^{\alpha}\ =b\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right) \quad (4.6.2)$$

Now we have the above equation as a numerical scheme, which can be further simplified as:

$$(\psi_i^{n+1}-\psi_i^{n})aR_{n,n}^{\alpha}=b(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})-\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^{j}\right)aR_{n,j}^{\alpha} \quad (4.6.3)$$

$$-\psi_i^{n}aR_{n,n}^{\alpha}=-\psi_i^{n+1}\left(aR_{n,j}^{\alpha}+2b\right)+b(\psi_{i+1}^{n+1}+\psi_{i-1}^{n+1})-\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^{j}\right)aR_{n,j}^{\alpha} \quad (4.6.4)$$

For stability analysis, we assume that:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{4.6.5}$$

Substituting the above into equation (4.6.4), we obtain the following:

$$\begin{aligned}
-\sigma^n e^{ik_m z} aR_{n,n}^{\alpha} \\
&= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b\right) + b\left(\sigma^{n+1} e^{ik_m(z+\Delta z)} + \sigma^{n+1} e^{ik_m(z-\Delta z)}\right) \\
&\quad - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}
\end{aligned} \tag{4.6.6}$$

$$\begin{aligned}
-\sigma^n e^{ik_m z} aR_{n,n}^{\alpha} \\
&= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b\right) + b\sigma^{n+1} e^{ik_m z}\left(e^{ik_m \Delta z} + e^{-ik_m \Delta z}\right) \\
&\quad - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}
\end{aligned} \tag{4.6.7}$$

$$\begin{aligned}
-\sigma^n e^{ik_m z} aR_{n,n}^{\alpha} \\
&= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - b\left(e^{ik_m \Delta z} + e^{-ik_m \Delta z}\right)\right) \\
&\quad - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}
\end{aligned} \tag{4.6.8}$$

$$\begin{aligned}
-\sigma^n e^{ikmz} aR_{n,n}^{\alpha} \\
&= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - 2b\left(\frac{e^{ik_m \Delta z} + e^{-ik_m \Delta z}}{2}\right)\right) \\
&\quad - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}
\end{aligned} \tag{4.6.9}$$

$$-\sigma^n e^{ik_m z} aR_{n,n}^{\alpha} = -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - 2b(\cos(k_m \Delta z)\right) - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha} \tag{4.6.10}$$

$$-\sigma^n e^{ik_m z} aR_{n,n}^{\alpha} = -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - 2b\left(1 - 2\sin^2\left(\frac{k_m \Delta z}{2}\right)\right)\right) - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha} \tag{4.6.11}$$

$$-\sigma^n aR_{n,n}^{\alpha} = -\sigma^{n+1}\left(aR_{n,n}^{\alpha} + 4\mathrm{b}\sin^2\left(\frac{k_m \Delta z}{2}\right)\right) - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \tag{4.6.12}$$

If $n = 0$ then the above equation becomes:

$$\sigma^1\left(aR_{0,n}^{\alpha} + 4\mathrm{b}\sin^2\left(\frac{k_m \Delta z}{2}\right)\right) = \sigma^0 aR_{0,n}^{\alpha} \tag{4.6.13}$$

$$\left|\frac{\sigma^1}{\sigma^0}\right| < 1 \rightarrow \left|\frac{\sigma^1}{\sigma^0}\right| = \left|\frac{aR_{0,n}^{\alpha}}{aR_{0,n}^{\alpha} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)}\right| < 1 \tag{4.6.14}$$

$$aR_{0,n}^{\alpha} < aR_{0,n}^{\alpha} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{4.6.15}$$

$$0 < 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{4.6.16}$$

We assume that $\forall\, n > 0$:

$$\left|\frac{\sigma^n}{\sigma^0}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^0}\right| < 1$$

$$\sigma^{n+1}\left(aR_{n,n}^{\alpha} + 4\mathrm{b}\sin^2\left(\frac{k_m\Delta z}{2}\right)\right) + \sum_{j=0}^{n-1}\left(\sigma_{j+1} - \sigma_j\right)aR_{n,j}^{\alpha} = \sigma^n aR_{n,n}^{\alpha} \tag{4.6.17}$$

$$\sigma^{n+1} = \frac{\sigma^n aR_{n,n}^{\alpha}}{\left[aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)\right]} - \frac{\sum_{j=0}^{n-1}\sigma^{n-j}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)} \tag{4.6.18}$$

$$|\sigma^{n+1}| \leq |\sigma^n|\left|\frac{aR_{n,n}^{\alpha}}{\left[aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)\right]}\right| - \frac{\sum_{j=0}^{n-1}\left|\sigma^{n-j}\right|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)} \tag{4.6.19}$$

Using the recursive hypothesis:

$$|\sigma^{n+1}| < |\sigma^0|\left|\frac{aR_{n,n}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)}\right| + \frac{\sum_{j=0}^{n-1}|\sigma^0|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)} \tag{4.6.20}$$

$$|\sigma^{n+1}| < |\sigma^0|\left[\frac{aR_{n,n}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)} + \frac{\sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)}\right] \tag{4.6.21}$$

$$\frac{|\sigma^{n+1}|}{|\sigma^0|} < 1$$

$$\left|\frac{aR_{n,n}^{\alpha} + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)}\right| < 1 \tag{4.6.22}$$

$$aR_{n,n}^{\alpha} + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m\Delta z}{2}\right) \tag{4.6.23}$$

The stability condition is given as:

$$\sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < 4b\sin^2\left(\frac{k_m\Delta z}{2}\right) \tag{4.6.24}$$

4.3.1.3. A new model of the saturated zone using the Caputo fractional derivative and the implicit Crank-Nicolson method

Applying the finite difference spatial discretization on equation (4.4), the implicit Crank-Nicolson discretization is as follows:

$$\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\left(\psi_i^{n+1}-\psi_i^n\right)((n-j)^{\alpha}-(n-j+1)^{\alpha})$$
$$=\frac{K_s}{S_s}\left[\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{2(\Delta z)^2}+\frac{\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}}{2(\Delta z)^2}\right] \tag{4.7}$$

Before we can perform stability analysis for the above implicit Crank-Nicolson method, we need to simplify the equation. For simplicity let us put:

$$a=\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)},$$

$$c=\frac{K_s}{S_s 2(\Delta z)^2},$$

$$R_{n,j}^{\alpha}=((n-j)^{\alpha}-(n-j+1)^{\alpha})$$

Now we can rewrite the equation as:

$$aR_{n,j}^{\alpha}=c[(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})+(\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n})] \tag{4.7.1}$$

$$\sum_{j=0}^{n}(\psi_i^{n+1}-\psi_i^n)aR_{n,j}^{\alpha}=c\left[\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right)+(\psi_{i+1}^{n}-2\psi_i^{n}\ +\psi_{i-1}^{n})\right] \tag{4.7.2}$$

$$(\psi_i^{n+1}-\psi_i^n)aR_{n,n}^{\alpha}+\sum_{j=0}^{n-1}(\psi_i^{j+1}-\psi_i^{j})aR_{n,j}^{\alpha}$$
$$=c\left[\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right)+(\psi_{i+1}^{n}-2\ \psi_i^{n}+\psi_{i-1}^{n})\right] \tag{4.7.3}$$

$$\begin{aligned}\psi_i^{n+1} aR_{n,n}^{\alpha} \ &= c\left(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}\right) + c\left(\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n}\right) + \psi_i^{n} aR_{n,n}^{\alpha} \\ &\quad - \sum_{j=0}^{n-1}\left(\psi_i^{j+1} - \psi_i^{j}\right) aR_{n,j}^{\alpha}\end{aligned} \tag{4.7.4}$$

For stability analysis, we assume that:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{4.7.5}$$

$$\begin{aligned}&\sigma^{n+1} e^{ik_m z} aR_{n,n}^{\alpha} \\ &\quad = c\left(\sigma^{n+1} e^{ik_m(z+\Delta z)} - 2\sigma^{n+1} e^{ik_m z} + \sigma^{n+1} e^{ik_m(z-\Delta z)}\right) \\ &\quad + c\left(\sigma^{n} e^{ik_m(z+\Delta z)} - 2\sigma^{n} e^{ik_m z} + \sigma^{n} e^{ik_m(z-\Delta z)}\right) \\ &\quad + \sigma^n e^{ik_m z} aR_{n,n}^{\alpha} - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}\end{aligned} \tag{4.7.6}$$

$$\begin{aligned}&\sigma^{n+1} e^{ik_m z} aR_{n,n}^{\alpha} - c\sigma^{n+1} e^{ik_m z}\left(e^{ik_m \Delta z} - 2 + e^{-ik_m \Delta z}\right) \\ &\quad = c\ \sigma^n e^{ik_m z}\left(e^{ik_m \Delta z} + \sigma_n e^{-ik_m \Delta z}\right) + \ \sigma^n e^{ik_m z}\left(aR_{n,n}^{\alpha} - 2c\right) \\ &\quad - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}\end{aligned} \tag{4.7.7}$$

$$\begin{aligned}&\sigma^{n+1} aR_{n,n}^{\alpha} - c\sigma^{n+1}\left(e^{ik_m \Delta z} - 2 + e^{-ik_m \Delta z}\right) \\ &\quad = c\ \sigma^n\left(e^{ik_m \Delta z} + e^{-ik_m \Delta z}\right) + \ \sigma^n\left(aR_{n,n}^{\alpha} - 2c\right) \\ &\quad - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha}\end{aligned} \tag{4.7.8}$$

$$\sigma^{n+1}\left(aR_{n,n}^{\alpha}-2c\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right)$$
$$=\sigma^{n}\left(aR_{n,n}^{\alpha}+2c\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right)-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)aR_{n,j}^{\alpha} \quad (4.7.9)$$

$$\sigma^{n+1}\left(aR_{n,n}^{\alpha}-2c(cos(k_m\Delta z)-1)\right)$$
$$=\sigma^{n}\left(aR_{n,n}^{\alpha}+2c(cos(k_m\Delta z)-1)\right)-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)aR_{n,j}^{\alpha} \quad (4.7.10)$$

If $n=0$ the above equation becomes:

$$\sigma^{1}\left(aR_{n,n}^{\alpha}-2c(cos(k_m\Delta z)-1)\right)=\sigma^{0}\left(aR_{n,n}^{\alpha}+2c(cos(k_m\Delta z)-1)\right) \quad (4.7.11)$$

$$\left|\frac{\sigma^1}{\sigma^0}\right|<1\rightarrow\ \left|\frac{\sigma^1}{\sigma^0}\right|=\left|\frac{aR_{n,n}^{\alpha}+2c(cos(k_m\Delta z)-1)}{aR_{n,n}^{\alpha}-2c(cos(k_m\Delta z)-1)}\right|<1 \quad (4.7.12)$$

$$aR_{n,n}^{\alpha}+2c(cos(ik_m\Delta z)-1)<aR_{n,n}^{\alpha}-2c(cos(k_m\Delta z)-1) \quad (4.7.13)$$

$$0<-4c(cos(k_m\Delta z)-1) \quad (4.7.14)$$

$$4ccos(k_m\Delta z)<4c \quad (4.7.15)$$

$$cos(k_m\Delta z)<1 \quad (4.7.16)$$

We can write 1 as $cos2\pi$ such that:

$$cos(k_m\Delta z)<cos2\pi \quad (4.7.16.1)$$

$$ik_m\Delta z<2\pi \quad (4.7.16.2)$$

Therefore:

$$k_m<\frac{2\pi}{\Delta z} \quad (4.7.16.3)$$

We assume that $\forall\, n>0$:

$$\left|\frac{\sigma^n}{\sigma^0}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^0}\right| < 1$$

$$\sigma^{n+1}\left(aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)\right)$$

$$= \sigma^n\left(aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)\right) - \sum_{j=0}^{n-1}\sigma^{n-j}aR_{n,j}^{\alpha} \tag{4.7.17}$$

$$\sigma^{n+1} = \sigma^n\left[\frac{aR_{n,n}^{\alpha} + 2c(cos(ik_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right] - \frac{\sum_{j=0}^{n-1}\sigma^{n-j}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} \tag{4.7.18}$$

$$|\,\sigma^{n+1}| \leq |\sigma^n|\left|\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right| + \frac{\sum_{j=0}^{n-1}\left|\sigma^{n-j}\right|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} \tag{4.7.19}$$

Using the recursive hypothesis:

$$|\,\sigma^{n+1}| < |\sigma^0|\left|\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right| + \frac{\sum_{j=0}^{n-1}|\sigma^0|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} \tag{4.7.20}$$

$$|\,\sigma^{n+1}| < |\sigma^0|\left[\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} + \frac{\sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right] \tag{4.7.21}$$

$$\left|\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1) + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right| < 1 \tag{4.7.22}$$

$$aR_{n,n}^{\alpha} + 2c(cos(km\Delta z) - 1) + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1) \tag{4.7.23}$$

$$\sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < -4c(cos(k_m\Delta z) - 1) \tag{4.7.24}$$

The stability condition is given as:

$$4ccos(k_m\Delta z) + \sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < 4c \tag{4.7.25}$$

The above stability condition can be simplified into:

$$4ccos(k_m\Delta z) < 4c - \sum_{j=0}^{n-1} aR_{n,j}^{\alpha} \tag{4.7.26}$$

$$k_m\Delta z < cos^{-1}\left(1 - \frac{\sum_{j=0}^{n-1} aR_{n,j}^{\alpha}}{4c}\right) \tag{4.7.27}$$

4.3.2 New model of the unsaturated zone using the Caputo fractional derivative

By replacing the dt of the classical equation derived in chapter 3 for modelling the unsaturated zone with the Caputo fractional derivative, we obtain the following equation:

$$\begin{aligned}
-\left((\theta_s-\theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}\right)\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\left(\psi_i^{n+1}-\psi_i^{n}\right)\left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right)& \\
= K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\frac{(\psi_{i+1}^{n+1}-\psi_i^{n+1})}{\Delta z}+K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}& \\
\times\frac{(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})}{(\Delta z)^2}-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}&
\end{aligned} \tag{4.8}$$

To make further simplification on the above equation, let us say:

$$\mathscr{p} = -(\theta_s-\theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}$$

$$\mathscr{a} = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\ ,$$

$$\mathscr{b} = \frac{K_s}{\Delta z} \quad ,$$

$$c = \frac{1}{\Delta z} \quad ,$$

$$d = \frac{1}{(\Delta z)^2}$$

$$R_{n,j}^{\alpha} = \left((n-j)^{\alpha} - (n-j+1)^{\alpha}\right)$$

Now we rewrite the equation as:

$$\begin{aligned} \mathscr{p} \sum_{j=0}^{n} (\psi_i^{n+1} - \psi_i^{n}) a R_{n,j}^{\alpha} \\ = \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right) c(\,\psi_{i+1}^{n+1} - \psi_i^{n+1}) \\ + K_s \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \times d(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) \\ - \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right) \end{aligned} \qquad (4.8.1)$$

The above numerical scheme is highly non-linear; therefore, it is not possible to perform a stability analysis.

4.4. NEW MODELS OF THE SATURATED-UNSATURATED ZONE USING THE FRACTAL- RL FRACTIONAL DERIVATIVE

The Euclidean model is usually satisfactory when describing patterns with a simple and pure structure (Gaddis & Zyda, 1986). However, many patterns in nature are so irregular and fragmented. The existence of such complexity challenges us to study these patterns since Euclid geometry cannot explain them (Mandelbrot, 1982). Responding to this challenge, there exists an unusual derivative in the field of applied mathematics and mathematics called a fractal derivative where in the variable is scaled according to t^{a}. This derivative was suggested to model real-world problems such as heterogeneity and fracture network in a system that is not appropriate for classical physical law such as Darcy’s, Fick’s and Fourier’s law (Atangana, 2017; Allwright & Atangana, 2018). Unlike the usual Euclidean structures, magnifying fractal results in the resolution of more details. Fractal structures are found everywhere in nature, such as clouds, coastlines, trees, etc. Their properties include scale-invariant, an infinite amount of details, self-similarity, etc. (Liu et al., 2003). As our dependence on groundwater resources increases, modelling and prediction of subsurface flow processes continue to be an important topic for hydrogeological research (Sivakumar et al., 2005). Moreover, it is very vital to note that there is no model that can completely describe a natural process, however at a given scale, if the model is accurate it can describe the process with enough precision (Gaddis & Zyda, 1986). Above, we've made use of the Caputo fractional derivative to generate new equations that can model groundwater flow in the saturated-unsaturated zone. However, this fractional derivative including others with the power-law kernel are local operators and therefore, are not very much appropriate for modelling complex natural problems, for instance, natural problems that display fractal behaviours (Atangana, 2017). For the past few years, the idea of non-local operators has attracted more researchers owing to its ability to model more complex natural problems using mathematical equations. For this section, we use the new concept of differentiation that was introduced by Atangana in 2017, it is the combination of the concept of fractal derivative and fractional differentiation which takes into account the memory, fractal effect, non-locality, and elasticity. We will introduce the fractal fractional derivative into our saturated-unsaturated classical equation by replacing the dt with the derivative and introducing the fractal operator to the equation. We use the fractal fractional derivative in Riemann-Liouville sense:

$$ {}^{FFR}_{\ \ \ 0}D_t^{\alpha,\beta}\psi(z,t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \psi(z,\tau)(t-\tau)^{-\alpha}d\tau.\frac{1}{\beta t^{\beta-1}} \tag{4.9}$$

For us to make use of the above fractional derivative, we will have to discretise it first. Using the first-order approximation:

$$\frac{dF}{dt} = \frac{F(t+h)-F(h)}{\Delta t}$$

$$\frac{dF}{dt} = \frac{F(t_{n+1})-F(t_n)}{\Delta t}$$

The solution for $F(t_{n+1})$ is given in detail below:

$$F(t_{n+1}) = \frac{1}{(1-\alpha)}\int_0^{t_{n+1}} \psi(z_i,\tau)(t_{n+1}-\tau)^{-\alpha}d\tau\frac{1}{\beta t_{n+1}^{\beta-1}} \tag{4.9.1}$$

$$F(t_{n+1}) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n} \psi_i^j \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{-\alpha} d\tau \frac{1}{\beta t_{n+1}^{\beta-1}} \qquad (4.9.1.1)$$

For us to easily integrate the equation, the following substitutions will be done:

$$d\tau = -dy$$

$$t_{n+1} - \tau = y$$

$$lower\ boundary = t_{n+1} - (t_{j+1})$$

$$upper\ boundary = t_{n+1} - t_j$$

$$F(t_{n+1}) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n} \psi_i^j \int_{t_j}^{t_{j+1}} (y)^{-\alpha} d\tau \frac{1}{\beta t_{n+1}^{\beta-1}}$$

$$F(t_{n+1}) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n} \psi_i^j \int_{t_{j+1}}^{t_j} \frac{(y)^{1-\alpha}}{1-\alpha} d\tau \frac{1}{\beta t_{n+1}^{\beta-1}}$$

Now that the equation has been integrated, we can substitute $(\Delta t_{n+1} - \Delta t_j)$ and $(\Delta t_{n+1} - \Delta t_{(j+1)})$ back into the equation to yield the following:

$$F(t_{n+1}) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n} \psi_i^j \frac{\left[\left(\Delta t_{n+1} - \Delta t_j\right) - \left(\Delta t_{n+1} - \Delta t_{(j+1)}\right)\right]^{1-\alpha}}{1-\alpha} \frac{1}{\beta t_{n+1}^{\beta-1}} \qquad (4.9.1.2)$$

$$F(t_{n+1}) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n} \psi_i^j \, (\Delta t)^{1-\alpha} \frac{((n+1-j)^{1-\alpha} - (n-j)^{1-\alpha})}{1-\alpha} \, \frac{1}{\beta t_{n+1}^{\beta-1}} \qquad (4.9.1.3)$$

$$F(t_{n+1}) = \frac{(\Delta t)^{1-\alpha}}{(2-\alpha)} \sum_{j=0}^{n} \psi_i^j \, ((n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}) \frac{1}{\beta t_{n+1}^{\beta-1}} \qquad (4.9.1.4)$$

The solution for $F(t_n)$ is given in detail below:

$$F(t_n) = \frac{1}{(1-\alpha)} \int_0^{t_n} \psi(z_i, \tau)(t_n - \tau)^{-\alpha} d\tau \frac{1}{\beta t_n^{\beta-1}} \qquad (4.9.2)$$

$$F(t_n) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n-1} \psi_i^j \int_{t_j}^{t_{j+1}} (t_n - \tau)^{-\alpha} d\tau \frac{1}{\beta t_n^{\beta-1}} \qquad (4.9.2.1)$$

For us to easily integrate the equation, the following substitutions will be done:

$$d\tau = -dy$$

$$t_n - \tau = y$$

$$lower\ boundary = t_n - \left(t_{j+1}\right)$$

$$upper\ boundary = t_n - t_j$$

$$F(t_n) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n-1} \psi_i^j \int_{t_j}^{t_{j+1}} (y)^{-\alpha} d\tau \frac{1}{\beta t_n^{\beta-1}}$$

$$F(t_n) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n-1} \psi_i^j \int_{t_{j+1}}^{t_j} \frac{(y)^{1-\alpha}}{1-\alpha} d\tau \frac{1}{\beta t_n^{\beta-1}}$$

Now that the equation has been integrated, we can substitute $(\Delta t_n - \Delta t_j)$ and $(\Delta t_n - \Delta t_{(j+1)})$ back into the equation to yield the following:

$$F(t_n) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n-1} \psi_i^j \frac{\left[\left(\Delta t_n - \Delta t_j\right) - \left(\Delta t_n - \Delta t_{(j+1)}\right)\right]^{1-\alpha}}{1-\alpha} \frac{1}{\beta t_n^{\beta-1}} \quad (4.9.2.2)$$

$$F(t_n) = \frac{1}{(1-\alpha)} \sum_{j=0}^{n-1} \psi_i^j (\Delta t)^{1-\alpha} \frac{((n+1-j)^{1-\alpha} - (n-j)^{1-\alpha})}{1-\alpha} \frac{1}{\beta t_n^{\beta-1}} \quad (4.9.2.3)$$

$$F(t_n) = \frac{(\Delta t)^{1-\alpha}}{(2-\alpha)} \sum_{j=0}^{n-1} \psi_i^j ((n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}) \frac{1}{\beta t_n^{\beta-1}} \quad (4.9.2.4)$$

Now we can substitute the solutions for $F(t_{n+1})$ and $F(t_n)$ back into equation (4.9), such that:

$$\begin{aligned} &{}^{FFR}_{\ \ 0}D_t^{\alpha,\beta} \psi(z,t) \\ &= \frac{(\Delta t)^{-\alpha}}{(2-\alpha)} \left[\sum_{j=0}^{n} \psi_i^j ((n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}) \frac{1}{\beta t_{n+1}^{\beta-1}} \right. \\ &\left. - \sum_{j=0}^{n-1} \psi_i^j ((n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}) \frac{1}{\beta t_{n-1}^{\beta-1}} \right] \end{aligned} \quad (4.9.2.5)$$

4.4.1 New models of the saturated zone using the fractal-fractional derivative

Now we can introduce the fractal derivative into our classical 1-d saturated equation, such that:

$$ {}_{0}^{RL}D_t^{\alpha}\psi(z,t)=\frac{K_s}{S_s}\beta t^{\beta-1}\ \frac{\partial^2\psi}{\partial z^2} \tag{4.10}$$

4.4.1.1. A new model of the saturated zone using the fractal-fractional derivative and the explicit forward Euler method

Appling the explicit forward Euler method on the equation (4.10), we obtain the following numerical solution:

$$\begin{aligned}\frac{(\Delta t)^{-\alpha}}{(2-\alpha)}\Bigg[\sum_{j=0}^{n}\psi_i^j\ ((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha})\\ -\sum_{j=0}^{n-1}\psi_i^j\ ((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha})\Bigg]\\ =\frac{K_s}{S_s}\beta t_n^{\beta-1}\ \left[\frac{\psi_{i+1}^n-2\,\psi_i^n+\psi_{i-1}^n}{(\Delta z)^2}\right]\end{aligned} \tag{4.10.1}$$

Before we can perform stability analysis for the above explicit forward Euler method, we need to simplify the equation. To do that we make use the following parameters:

$$a=\frac{(\Delta t)^{-\alpha}}{(2-\alpha)}$$

$$b=\frac{K_s}{(\Delta z)^2 S_s}\beta t_n^{\beta-1}$$

$$\delta_{\alpha,j}^{1,n}=((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha})$$

$$\delta_{\alpha,j}^{2,n}=((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha})$$

Now we can rewrite the equation with parameter:

$$a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{1,n}-a\sum_{j=0}^{n-1}\psi_i^j\delta_{\alpha,j}^{2,n}=b(\psi_{i+1}^n-2\,\psi_i^n+\psi_{i-1}^n) \tag{4.10.1.2}$$

$$a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{1,n} = b(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) + a\sum_{j=0}^{n-1}\psi_i^j\delta_{\alpha,j}^{2,n} \qquad (4.10.1.3)$$

4.4.1.2. A new model of the saturated zone using the fractal-fractional derivative using the implicit backward Euler method

We obtain another numerical solution by applying the implicit backward Euler method on equation (4.10):

$$\frac{(\Delta t)^{-\alpha}}{(2-\alpha)}\left[\sum_{j=0}^{n}\psi_i^j\left((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha}\right) - \sum_{j=0}^{n-1}\psi_i^j\left((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha}\right)\right] = \frac{K_s}{S_s}\beta t_n^{\beta-1}\left[\frac{\psi_{i+1}^{n+1}-2\,\psi_i^{n+1}+\psi_{i-1}^{n+1}}{(\Delta z)^2}\right] \qquad (4.10.2)$$

To simplify the above equation, we make use of the following parameters:

$$a = \frac{(\Delta t)^{-\alpha}}{(2-\alpha)}$$

$$b = \frac{K_s}{S_s(\Delta z)^2}\beta t_n^{\beta-1}$$

$$\delta_{\alpha,j}^{1,n} = \left((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha}\right)$$

$$\delta_{\alpha,j}^{2,n} = \left((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha}\right)$$

Now we can rewrite the equation with parameter:

$$a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{1,n} - a\sum_{j=0}^{n-1}\psi_i^j\delta_{\alpha,j}^{2,n} = b(\psi_{i+1}^{n+1} - 2\,\psi_i^{n+1} + \psi_{i-1}^{n+1}) \qquad (4.10.2.1)$$

$$a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{1,n} = b(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) + a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{2,n} \qquad (4.10.2.2)$$

4.4.1.3. A new model of the saturated zone using the fractal derivative using the implicit Crank-Nicolson method

Applying the finite difference spatial discretization on equation (4.10), the implicit Crank-Nicolson discretization is as follows:

$$\frac{(\Delta t)^{1-\alpha}}{(2-\alpha)}\sum_{j=0}^{n}\psi_i^j\,[((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha})-((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha})]$$

$$=\frac{K_s}{S_s}\beta t^{\beta-1}\left[\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{2(\Delta z)^2}+\frac{\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}}{2(\Delta z)^2}\right] \quad (4.10.3)$$

$$\frac{(\Delta t)^{-\alpha}}{(2-\alpha)}\left[\sum_{j=0}^{n}\psi_i^j\,((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha}) - \sum_{j=0}^{n-1}\psi_i^j\,((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha})\right]$$

$$=\frac{K_s}{2(\Delta z)^2 S_s}\beta t_n^{\beta-1}\,[(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}) + (\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n})] \quad (4.10.3.1)$$

For simplicity, we will make use of the following parameters:

$$a=\frac{(\Delta t)^{-\alpha}}{(2-\alpha)}$$

$$b=\frac{K_s}{2(\Delta z)^2 S_s}\beta t_n^{\beta-1}$$

$$\delta_{\alpha,j}^{1,n}=((n+1-j)^{1-\alpha}-(n-j)^{1-\alpha})$$

$$\delta_{\alpha,j}^{2,n}=((n-j)^{1-\alpha}-(n-j-1)^{1-\alpha})$$

Such that:

$$a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{1,n}-a\sum_{j=0}^{n-1}\psi_i^j\delta_{\alpha,j}^{2,n}=b[(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})+(\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n})] \quad (4.10.3.2)$$

$$a\sum_{j=0}^{n}\psi_i^j\delta_{\alpha,j}^{1,n} = b[(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) + (\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n})] + a\sum_{j=0}^{n-1}\psi_i^j\delta_{\alpha,j}^{2,n} \quad (4.10.3.3)$$

4.4.2 New model of the unsaturated zone using the fractal-fractional derivative

Now, using the fractal fractional derivative, we provide the solution for the unsaturated zone. By replacing dt in classical equation (3.20) derived for modelling the unsaturated zone, we obtain the following equation with the R-L derivative and fractal operator:

$$\begin{aligned}
&\frac{(\Delta t)^{-\alpha}}{(2-\alpha)}\left[\sum_{j=0}^{n}\psi_i^j\left((n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}\right)\right.\\
&\left.-\sum_{j=0}^{n-1}\psi_i^j\left((n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\right)\right]\\
&= \frac{1}{-(\theta_s-\theta_0)\psi_b^{\lambda}\dfrac{\lambda}{\psi_i^{\lambda+1}}}\beta t_n^{\beta-1}\left[K_s\frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\frac{(\psi_{i+1}^{n+1} - \psi_i^{n+1})}{\Delta z}\right.\\
&+ K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \times \frac{(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1})}{(\Delta z)^2}\\
&\left.- K_s\,\frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\right] \qquad (4.11)
\end{aligned}$$

For further simplification on the above equation, let us say:

$$\mathcal{p} = -(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}$$

$$\mathcal{a} = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}$$

$$\mathcal{b} = \frac{K_s}{\Delta z}$$

$$\mathcal{B} = \beta t_n^{\beta-1}$$

$$c = \frac{1}{\Delta z}$$

$$d = K_s$$

$$e = \frac{1}{(\Delta z)^2}$$

$$R_{n,j}^{\alpha} = \left((n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}\right) - \left((n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}\right)$$

Now we rewrite the equation as:

$$\begin{aligned} aR_{n,j}^{\alpha} = \frac{1}{\mathscr{p}}\mathcal{B}&\left(\mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right) \times c(\,\psi_{i+1}^{n+1} - \psi_i^{n+1})\right) \\ &+ d\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \times e(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) \\ &- \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right) \qquad (4.11.1) \end{aligned}$$

The above numerical scheme is highly non-linear; therefore, it is not possible to perform a stability analysis.

4.5. NUMERICAL SOLUTION OF THE NEW SATURATED-UNSATURATED GROUNDWATER FLOW MODEL USING THE TOUFIK AND ATANGANA NUMERICAL SCHEME

The Adams-Bashforth method has been recognised in many publications as a powerful scheme when solving non-linear equations; however, it might not be effective for fractional differential equations that possess a non-local and non-singular kernel. In 2017, Toufik and Atangana introduced a new numerical scheme for non-linear fractional differential equations of a fractional derivative with the non-local and non-singular kernel. This scheme extends the limitations of the Adams-Bashforth method (Toufik & Atangana,2017). Now, we are going to develop new numerical solutions for the saturated-unsaturated groundwater equation using the new numerical scheme.

4.5.1 New model of the saturated zone using the Toufik and Atangana numerical scheme

For the saturated zone, we have:

$$ {}_{0}^{C}D^{\alpha}\psi(z,t) = \frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2} \tag{4.12}$$

where

$$ {}_{0}^{C}D^{\alpha}\psi(z,t) = F\big(z,t,\psi(z,\alpha)\big)$$

Then we have the numerical scheme as:

$$\psi(z_i,t) - \psi(z_i,0) = \frac{1}{\Gamma(\alpha)}\int_0^t F\big(z_i,\tau,\psi(z_i,\tau)\big)(t-\tau)^{\alpha-1}d\tau \tag{4.12.1}$$

At point $t_{n+1}, n = 0,1,2,3,4 \dots\dots\dots\dots$

$$\psi(z_i,t_{n+1}) - \psi(z_i,0) = \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} F\big(z_i,\tau,\psi(z_i,\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau \tag{4.12.2}$$

$$\begin{aligned}\psi(z_i,t_{n+1}) - \psi(z_i,0) &\\ = \frac{(\Delta t)^{\alpha}}{\Gamma(2+\alpha)}\sum_{j=0}^{n}\Big[&F\big(z_i,t_j,\psi_i^j\big)\big((n-j+1)^{\alpha}(n-j+2+\alpha)\\ &-(n-j)^{\alpha}(n-j+2+2\alpha)\big)\\ &-F\big(z_i,t_{j-1},\psi_i^{j-1}\big)\big((n-j+1)^{\alpha+1}-(n-j)^{\alpha}(n-j+1+\alpha)\big)\Big]\end{aligned} \tag{4.12.3}$$

where:

$$F\big(z_i,t_j,\psi_i^j\big) = \frac{K_s}{S_s}\frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{(\Delta z)^2} \tag{4.12.3.1}$$

$$F\big(z_i,t_{j-1},\psi_i^{j-1}\big) = \frac{K_s}{S_s}\frac{\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}}{(\Delta z)^2} \tag{4.12.3.2}$$

The final numerical solution for the saturated groundwater flow equation is given as:

$$\begin{aligned}\psi(z_i,t_{n+1})-\psi(z_i,0) &= \frac{(\Delta t)^{\alpha}}{\Gamma(2+\alpha)}\sum_{j=0}^{n}\Bigg[\frac{K_s}{S_s}\frac{\psi_{i+1}^{j}-2\psi_i^{j}+\psi_{i-1}^{j}}{(\Delta z)^2}\big((n-j+1)^{\alpha}(n-j+2\\ &\quad+\alpha)-(n-j)^{\alpha}(n-j+2+2\alpha)\big)\\ &\quad-\frac{K_s}{S_s}\frac{\psi_{i+1}^{j-1}-2\psi_i^{j-1}+\psi_{i-1}^{j-1}}{(\Delta z)^2}\big((n-j+1)^{\alpha+1}\\ &\quad-(n-j)^{\alpha}(n-j+1+\alpha)\big)\Bigg] \qquad (4.12.4)\end{aligned}$$

4.5.2 New model of the unsaturated zone using the Toufik and Atangana numerical scheme

Now, using the new numerical scheme the new model for the unsaturated zone is given by the following equation:

$$\begin{aligned}\left(-(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}\right){}_0^{C}D^{\alpha}\psi(z,t) &= K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\frac{(\psi_{i+1}^{n+1}-\psi_i^{n+1})}{\Delta z}\\ &\quad+K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})}{(\Delta z)^2}\\ &\quad-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z} \qquad (4.13)\end{aligned}$$

where

$${}_0^{c}D^{\alpha}\psi(z,t)=F\big(z,t,\psi(z,\alpha)\big)$$

Then we have the numerical scheme as:

$$\psi(z_i,t)-\psi(z_i,0)=\frac{1}{\Gamma(\alpha)}\int_0^t F\big(z_i,\tau,\psi(z_i,\tau)\big)(t-\tau)^{\alpha-1}d\tau \qquad (4.13.1)$$

At point $t_{n+1}, n=0,1,2,3,4\ldots\ldots\ldots\ldots$

$$\psi(z_i, t_{n+1}) - \psi(z_i, 0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F\big(z_i, \tau, \psi(z_i, \tau)\big)(t_{n+1} - \tau)^{\alpha-1} d\tau \tag{4.13.2}$$

$$\begin{aligned} &\psi(z_i, t_{n+1}) - \psi(z_i, 0) \\ &\qquad = \frac{(\Delta t)^\alpha}{\Gamma(2+\alpha)} \sum_{j=0}^{n} \Big[F\big(z_i, t_j, \psi_i^j\big)\big((n-j+1)^\alpha (n-j+2+\alpha) \\ &\qquad - (n-j)^\alpha (n-j+2+2\alpha)\big) \\ &\qquad - F\big(z_i, t_{j-1}, \psi_i^{j-1}\big)\big((n-j+1)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha)\big)\Big] \end{aligned} \tag{4.13.3}$$

where:

$$\begin{aligned} F\big(z_i, t_j, \psi_i^j\big) = \frac{1}{-(\theta_s - \theta_0)\psi_b^\lambda \dfrac{\lambda}{\psi_i^{\lambda+1}}} &\left[K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \right)}{\Delta z} \frac{\big(\psi_{i+1}^j - \psi_i^j\big)}{\Delta z} \right. \\ &+ K_s \left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \times \frac{\big(\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j\big)}{(\Delta z)^2} \\ &\left. - K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \right)}{\Delta z} \right] \end{aligned} \tag{4.13.4}$$

$$
\begin{aligned}
&F\left(z_i, t_{j-1}, \psi_i^{j-1}\right) \\
&= \frac{1}{-(\theta_s - \theta_0)\psi_b^{\lambda}\dfrac{\lambda}{\psi_i^{\lambda+1}}}\left[K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)}{\Delta z} \frac{\left(\psi_{i+1}^{j-1} - \psi_i^{j-1}\right)}{\Delta z}\right. \\
&+ K_s \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda} \times \frac{\left(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\right)}{(\Delta z)^2} \\
&\left. - K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)}{\Delta z}\right]
\end{aligned}
\tag{4.13.5}
$$

The final numerical solution for the unsaturated groundwater flow equation is given as:

$$
\begin{aligned}
&\left(\psi(z_i,t_{n+1})-\psi(z_i,0)\right)\\
&=\frac{(\Delta t)^{\alpha}}{\Gamma(2+\alpha)}\sum_{j=0}^{n}\Bigg[\frac{1}{-(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}}\Bigg(K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\right)}{\Delta z}\frac{\left(\psi_{i+1}^{j}-\psi_i^{j}\right)}{\Delta z}\\
&+K_s\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\times\frac{\left(\psi_{i+1}^{j}-2\psi_i^{j}+\psi_{i-1}^{j}\right)}{(\Delta z)^2}\\
&-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\right)}{\Delta z}\Bigg)\Big((n-j+1)^{\alpha}(n-j+2+\alpha)\\
&-(n-j)^{\alpha}(n-j+2+2\alpha)\Big)\\
&-\frac{1}{-(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}}\Bigg(K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)}{\Delta z}\frac{\left(\psi_{i+1}^{j-1}-\psi_i^{j-1}\right)}{\Delta z}\\
&+K_s\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\times\frac{\left(\psi_{i+1}^{j-1}-2\psi_i^{j-1}+\psi_{i-1}^{j-1}\right)}{(\Delta z)^2}\\
&-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)}{\Delta z}\Bigg)\Big((n-j+1)^{\alpha+1}\\
&-(n-j)^{\alpha}(n-j+1+\alpha)\Big)\Bigg]
\end{aligned}
\tag{4.13.6}
$$

For simplicity of the above equation let:

$$\mathscr{p}=-(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}$$

$$\mathscr{a}=\frac{(\Delta t)^{\alpha}}{\Gamma\,(2-\alpha)}$$

$$\mathscr{b} = \frac{K_s}{\Delta z}$$

$$c = \frac{1}{\Delta z}$$

$$\mathscr{d} = K_s$$

$$e = \frac{1}{(\Delta z)^2}$$

$$f = \big((n-j+1)^{\alpha}(n-j+2+\alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\big)$$

$$g = \big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)$$

$$\begin{aligned}
&\big(\psi(z_i, t_{n+1}) - \psi(z_i, 0)\big) \\
&\quad = \mathscr{a} \sum_{j=0}^{n} \Bigg[\frac{1}{\mathscr{p}} \Bigg(\mathscr{b} \Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_{i}^{j}}\right)^{2+3\lambda} \Bigg) c\big(\psi_{i+1}^{j} - \psi_{i}^{j}\big) \\
&\quad + \mathscr{d} \left(\frac{\psi_b}{\psi_{i}^{j}}\right)^{2+3\lambda} \times e\big(\psi_{i+1}^{j} - 2\psi_{i}^{j} + \psi_{i-1}^{j}\big) \\
&\quad - \mathscr{b} \Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_{i}^{j}}\right)^{2+3\lambda} \Bigg) \Bigg) f \\
&\quad - \frac{1}{\mathscr{p}} \Bigg(\mathscr{b} \Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_{i}^{j-1}}\right)^{2+3\lambda} \Bigg) c\big(\psi_{i+1}^{j-1} - \psi_{i}^{j-1}\big) \\
&\quad + \mathscr{d} \left(\frac{\psi_b}{\psi_{i}^{j-1}}\right)^{2+3\lambda} \times e\big(\psi_{i+1}^{j-1} - 2\psi_{i}^{j-1} + \psi_{i-1}^{j-1}\big) \\
&\quad - \mathscr{b} \Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_{i}^{j-1}}\right)^{2+3\lambda} \Bigg) \Bigg) g \Bigg]
\end{aligned} \tag{4.13.7}$$

CHAPTER 5

NEW MODEL WITH CROSSOVER FROM USUAL TO CONFINED FLOW MEAN SQUARE DISPLACEMENT

5.1. INTRODUCTION

A system is subject to exponential decay if the quantity declines at the rate relative to its current value. Mathematically, this process can be expressed by $\frac{dN}{dt} = -\lambda N$, where N is the quantity and λ the exponential decay constant. The solution to this equation is $N(t) = N_0 e^{-\lambda t}$ .Where $N(t)$ is the quantity at time$t$, and $N_0 = N(0)$ is the initial quantity (Leike, 2002). In recent years, many researchers have been modelling natural problems in their particular fields using fractional-order derivatives due to their efficiency in describing such problems. These derivatives include the well-known Riemann-Liouville and the recently proposed Caputo-Fabrizio derivatives. The definition presented by Riemann-Liouville and the first Caputo version has a disadvantage of its singularity. With this disadvantage, the two fractional operators cannot fully describe the memory effect (Atangana and Alkahtani, 2015). To provide the full explanation of memory, Fabrizio and Caputo proposed a new fractional operator built on the exponential decay law with no singular kernel. This fractional operator is based on the convolution of a first-order derivative and the exponential function. An advantage of this derivative is that it can be used when exponential law is observed in nature (Gomez-Aguilar et al., 2017; Alkahtani, 2016; Atanackovic et al., 2018). A straightforward example of a real-world process that follows the exponential decay law is the cooling of hot water. The speed at which hot water cools is proportional to the change in temperature between the cooling body and the environment; therefore, if the temperature difference is higher than the cooling follows an exponential decay. Another distribution of theoretical and practical importance is Poisson. The Poisson distribution allows us to find probabilities for random points in time for a process, for example, modelling a number of cars that pass a certain place in a given period (Rice, 2007). The relation between these two distributions is that, if the Poisson provides a suitable description of events occurring per interval of time, then the exponential will describe the length of time between events (Cooper, 2005). Groundwater flow problems can sometimes be explained using both these distributions. Other properties that can be observed in groundwater flow problems are the fading memory and the fatigue effect. The fading memory states that the probability of something happening in the future has no relation to whether or not it has occurred in the past, for example, should anything cause a temporary pause in groundwater flow, whatever happens after the break has no relation to whether or not it has happened before the break. On the other hand, the fatigue effect would explain why groundwater cannot flow endlessly; at some point, it would stop, be it due to no pressure or force exerted on it. Both these properties, exponential decay law, and Poisson distribution can be observed in groundwater flow problems and are necessary to incorporate in a mathematical groundwater flow model. Therefore, in this chapter, we will generate new vertical saturated-unsaturated groundwater flow equations by replacing the ordinary time derivative on the original equation with the Caputo-Fabrizio fractional-order derivative. With that said, useful information on the derivative will be presented to inform readers that are not familiar with it.

5.2. THE CAPUTO-FABRIZIO FRACTIONAL ORDER DERIVATIVE

Singularity is the main problem in the first definition of Caputo. To avoid this problem, Fabrizio and Caputo presented a fractional derivative without singularity. The definition of this derivative is given as follows (Atangana and Alkahtani, 2015; Atangana and Baleanu, 2016):

Definition: let $f \in H^1(a,b), b > a, \alpha \in [0,1]$ then, the new Caputo fractional derivative is:

$$D_t^{\alpha}\big(f(t)\big) = \frac{M(\alpha)}{1-\alpha}\int_b^t f'(\tau)exp\left[-\alpha\frac{t-\tau}{1-\alpha}\right]d\tau \tag{5.1}$$

Where $M(\alpha)$ denotes a normalization function obeying$M(0) = M(1) = 1$. However, should the function not belong to $H^1(a,b)$ the derivative has the form:

$$D_t^{\alpha}\big(f(t)\big) = \frac{\alpha M(\alpha)}{1-\alpha}\int_b^t \big(f(t) - f(\tau)\big)exp\left[-\alpha\frac{t-\tau}{1-\alpha}\right]d\tau \tag{5.2}$$

If$\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty], \alpha = \frac{1}{1+\sigma} \in [0,1]$, then equation (5.4) above assumes the following form:

$$D_t^{\sigma}\big(f(t)\big) = \frac{N(\sigma)}{\sigma}\int_b^t f'(\tau)exp\left[-\frac{t-\tau}{\sigma}\right]d\tau \qquad N(0) = N(\infty) = 1 \tag{5.3}$$

5.3. NUMERICAL SOLUTIONS FOR THE SATURATED-UNSATURATED ZONE USING THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

In this section, we apply the Caputo-Fabrizio fractional derivative to our classical saturated-unsaturated equation (derived in chapter 3) to generate new numerical solutions for modelling 1-d groundwater flow in the saturated-unsaturated media. We begin by generating a discretized version of the Caputo-Fabrizio fractional derivative. For time t_n the Caputo-Fabrizio fractional derivative is written as:

$$ {}_{0}^{CF}D_t^{\alpha}\big(\psi(t)\big) = \frac{M(\alpha)}{1-\alpha}\int_0^{t_n}\psi'(\tau)exp\left[-\alpha\frac{t_n-\tau}{1-\alpha}\right]d\tau \tag{5.4} $$

If $n \geq 0$, equation 4.6 becomes:

$$ {}_{0}^{CF}D_t^{\alpha}\psi(t_n) = \frac{M(\alpha)}{1-\ \alpha}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}exp\ \left[-\frac{\alpha}{1-\ \alpha}\ (t_n-\ \tau)\right]d\tau \tag{5.4.1} $$

$$ {}_{0}^{CF}D_t^{\alpha}\psi(t_n) = \frac{M(\alpha)}{1-\ \alpha}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\int_{t_j}^{t_{j+1}}exp\ \left[-\frac{\alpha}{1-\ \alpha}\ (t_n-\ \tau)\right]d\tau \tag{5.4.1.1} $$

To easily integrate this equation, we will integrate by substituting $t_n - \tau = y$, $d\tau = -dy$ and $\lambda = \frac{\alpha}{1-\ \alpha}$We further substitute τ with the lower and upper boundary in this manner $t_n - (t_{j+1})$ and $t_n - (t_j)$ which gives us the following:

$$ {}_{0}^{CF}D_t^{\alpha}\psi(t_n) = \frac{M(\alpha)}{1-\ \alpha}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\int_{t_n-t_{j+1}}^{t_n-\ t_j}exp\ [-\lambda\ y]\,dy \tag{5.4.2} $$

$$ {}_{0}^{CF}D_t^{\alpha}\psi(t_n) = \frac{M(\alpha)}{1-\ \alpha}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\left[-\frac{1}{\lambda}exp(-\lambda y)\ \Big|_{t_n-\ t_{j+1}}^{t_n-\ t_j}\right] \tag{5.4.3} $$

Now that the equation has been integrated, we can substitute $(\Delta t_n - \Delta t_j)$ and $(\Delta t_n - \Delta t_{(j+1)})$ back into the equation to yield the following:

$$ \begin{aligned} {}_{0}^{CF}D_t^{\alpha}\psi(t_n) = \frac{M(\alpha)}{1-\ \alpha}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\Delta t}\ &\left[-\frac{1}{\lambda}\ exp\Big(-\lambda\big(t_n-t_j\big)\Big)\right. \\ &\left.+\frac{1}{\lambda}\ exp\Big(-\lambda\big(t_n-t_{j+1}\big)\Big)\right] \end{aligned} \tag{5.4.4} $$

Our final discretized version of the Caputo-Fabrizio fractional derivative is given by:

$$ {}^{CF}_{0}D_t^{\alpha}\psi(t_n) = \frac{M(\alpha)}{1-\ \alpha}\sum_{j=0}^{n}\frac{\psi^{j+1}-\psi^{j}}{\lambda\Delta t}\left[exp\big(-\lambda\Delta t(n-j-1)\big) - exp\ \big(-\lambda\Delta t(n-j)\big)\right] \quad (5.4.5) $$

The Caputo-Fabrizio fractional integral of order α of a function $\psi(x,t)$ is given as:

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z,t) = \frac{1-\alpha}{M(\alpha)}\psi(z,t) + \frac{\alpha}{M(\alpha)}\int_0^t \psi(z,\tau)d\tau \quad (5.5) $$

At point (z_i, t_n) we have,

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z_i,t_n) = \frac{1-\alpha}{M(\alpha)}\psi(z_i,t_n) + \frac{\alpha}{M(\alpha)}\int_0^{t_n} \psi(z_i,\tau)d\tau \quad (5.5.1) $$

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z_i,t_n) = \frac{1-\alpha}{M(\alpha)}\psi_i^n + \frac{\alpha}{M(\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \psi_i^j d\tau \quad (5.5.2) $$

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z_i,t_n) = \frac{1-\alpha}{M(\alpha)}\psi_i^n + \frac{\alpha}{M(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\int_{t_j}^{t_{j+1}} d\tau \quad (5.5.3) $$

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z_i,t_n) = \frac{1-\alpha}{M(\alpha)}\psi_i^n + \frac{\alpha}{M(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\int_{t_j}^{t_{j+1}} d\tau \quad (5.5.4) $$

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z_i,t_n) = \frac{1-\alpha}{M(\alpha)}\psi_i^n + \frac{\alpha}{M(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\left(t_{j+1}-t_j\right) \quad (5.5.5) $$

$$ {}^{CF}_{0}I_t^{\alpha}\psi(z_i,t_n) = \frac{1-\alpha}{M(\alpha)}\psi_i^n + \frac{\alpha}{M(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\Delta t \quad (5.5.6) $$

For our numerical solution, we replace the time derivative on our saturated equation (3.8) with the Caputo-Fabrizio fractional derivative, such that:

$$ {}^{CF}_{0}D^{\alpha}\psi(z,t) = \frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2} \quad (5.6) $$

where:

$$ {}_{0}^{CF}D^{\alpha}\psi(z,t) = f\big(z,t,\psi(z,\alpha)\big) \tag{5.6.1}$$

We apply the Caputo-Fabrizio integral to the above equation:

$$\psi(z,t) = \psi(z,0) + \frac{1-\alpha}{M(\alpha)} f\big(z,t,\psi(z,t)\big) + \frac{\alpha}{M(\alpha)} \int_0^t f\big(z,\tau\ \ \psi(z,\tau)\big) d\tau \tag{5.6.2}$$

When $t = t_{n+1}$Now we have the following equation:

$$\psi(z_i,t_{n+1}) = \psi(z_i,0) + \frac{1-\alpha}{M(\alpha)} f\big(z_i,t_n,\psi(z_i,t_n)\big) + \frac{\alpha}{M(\alpha)} \int_0^{t_{n+1}} f\big(z_i,\tau\ \ \psi(z_i,\tau)\big) d\tau \tag{5.6.3}$$

When $t = t_n$:

$$\psi(z_i,t_n) = \psi(z_i,0) + \frac{1-\alpha}{M(\alpha)} f\big(z_i,t_{n-1},\psi(z_i,t_{n-1})\big) + \frac{\alpha}{M(\alpha)} \int_0^{t_n} f\big(z_i,\tau\ \ \psi(z_i,\tau)\big)\, d\tau \tag{5.6.4}$$

Subtracting the above solution for t_n from t_{n+1} , we obtain the following:

$$\begin{aligned}\psi_i^{n+1} - \psi_i^n &= \frac{1-\alpha}{M(\alpha)} [f(z_i,t_n,\psi_i^n) - f(z_i,t_{n-1},\psi_i^{n-1})] \\ &\quad + \frac{\alpha}{M(\alpha)} \int_{t_n}^{t_{n+1}} f\big(z_i,\tau,\psi(x_i,\tau)\big) d\tau\end{aligned} \tag{5.6.5}$$

$$\begin{aligned}\psi_i^{n+1} &= \psi_i^n + \frac{1-\alpha}{M(\alpha)} [f(z_i,t_n,\psi_i^n) - f(z_i,t_{n-1},\psi_i^{n-1})] \\ &\quad + \frac{\alpha}{M(\alpha)} \left[\frac{3}{2}\Delta t f(z_i,t_n,\psi_i^n) - \frac{\Delta t}{2} f(z_i,t_{n-1},\psi_i^{n-1})\right]\end{aligned} \tag{5.6.6}$$

where:

$$F\big(z_i,t_j,\psi_i^n\big) = \frac{K_s}{S_s} \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} \tag{5.6.7}$$

$$F\left(z_i, t_{j-1}, \psi_i^{n-1}\right) = \frac{K_s}{S_s}\frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2} \tag{5.6.8}$$

We substitute the above equation (5.6.7) and (5.6.8) into equation (5.6.6) to obtain the final numerical solution for the saturated groundwater flow equation:

$$\begin{aligned}\psi_i^{n+1} = \psi_i^n &+ \frac{1-\alpha}{M(\alpha)}\left[\frac{K_s}{S_s}\frac{\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n}}{(\Delta z)^2} - \frac{K_s}{S_s}\frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2}\right] \\ &+ \frac{\alpha}{M(\alpha)}\left[\frac{3\Delta t}{2}\left(\frac{K_s}{S_s}\frac{\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n}}{(\Delta z)^2}\right)\right. \\ &\left.- \frac{\Delta t}{2}\left(\frac{K_s}{S_s}\frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2}\right)\right]\end{aligned} \tag{5.6.9}$$

To check the stability of the above numerical solution, we assume a Fourier expansion in space of:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{5.6.10}$$

We also assume the following parameters:

$$a = \frac{1-\alpha}{M(\alpha)}$$

$$b = \frac{\alpha}{M(\alpha)}$$

$$c = \frac{K_s}{S_s(\Delta z)^2}$$

$$d = \frac{\Delta t}{2}$$

Then our equation becomes:

$$\begin{aligned}\sigma^{n+1}e^{ik_m z} = \sigma^n e^{ik_m z} &+ a\left[c\left(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\right)\right. \\ &\left.- c\left(\sigma^{n-1} e^{ik_m(z+\Delta z)} - 2\sigma^{n-1} e^{ik_m z} + \sigma^{n-1} e^{ik_m(z-\Delta z)}\right)\right] \\ &+ b\left[3dc\left(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\right)\right. \\ &\left.- dc\left(\sigma^{n-1} e^{ik_m(z+\Delta z)} - 2\sigma^{n-1} e^{ik_m z} + \sigma^{n-1} e^{ik_m(z-\Delta z)}\right)\right]\end{aligned} \tag{5.6.11}$$

$$\begin{aligned}\sigma^{n+1}e^{ik_m z} = \sigma^n e^{ik_m z} &\\ + a\big[\sigma^n e^{ik_m z} c\big(e^{ik_m \Delta z} - 2 + e^{-ik_m \Delta z}\big) &\\ - \sigma^{n-1} e^{ik_m z} c\big(e^{ik_m \Delta z} - 2 + e^{-ik_m \Delta z}\big)\big] &\\ + b\big[3\sigma^n e^{ik_m z} dc\big(e^{ik_m \Delta z} - 2 + e^{-ik_m \Delta z}\big) &\\ - \sigma^{n-1} e^{ik_m z} dc\big(e^{ik_m \Delta z} - 2 + e^{-ik_m \Delta z}\big)\big] &\end{aligned} \tag{5.6.12}$$

$$\begin{aligned}\sigma^{n+1} = \sigma^n + a[\sigma^n c(2cos(k_m \Delta z) - 2) - \sigma^{n-1} c(2cos(k_m \Delta z) - 2)] &\\ + b[3\sigma^n dc(2cos(k_m \Delta z) - 2) - \sigma^{n-1} dc(2cos(k_m \Delta z) - 2)] &\end{aligned} \tag{5.6.13}$$

$$\begin{aligned}\frac{\sigma^{n+1}}{\sigma^n} = a\left[2c(cos(k_m \Delta z) - 1) - 2c\frac{\sigma^{n-1}}{\sigma^n}(cos(k_m \Delta z) - 1)\right] &\\ + b\left[3dc(2cos(k_m \Delta z) - 2) - dc\frac{\sigma^{n-1}}{\sigma^n}(2cos(k_m \Delta z) - 2)\right] &\end{aligned} \tag{5.6.14}$$

$$\frac{\sigma^{n+1}}{\sigma^n} = 2ac(cos(k_m \Delta z) - 1)\left[1 - \frac{\sigma^{n-1}}{\sigma^n}\right] + 2bdc(cos(k_m \Delta z) - 1)\left[3 - \frac{\sigma^{n-1}}{\sigma^n}\right] \tag{5.6.15}$$

For $n = 0$, the equation becomes:

$$\left|\frac{\sigma^1}{\sigma^0}\right| = |2ac(cos(k_m \Delta z) - 1) + 6bdc(cos(k_m \Delta z) - 1)| \tag{5.6.16}$$

The stability is given by:

$$|2ac(cos(k_m \Delta z) - 1) + 6bdc(cos(k_m \Delta z) - 1)| < 1 \tag{5.6.17}$$

Now we assume that $\forall\, n > 0$:

$$\left|\frac{\sigma^n}{\sigma^0}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^0}\right| < 1$$

The equation then becomes:

$$\begin{aligned}\sigma^{n+1} = \sigma^n &+ a[\sigma^n 2c(cos(k_m\Delta z) - 1) - \sigma^{n-1}2c(cos(k_m\Delta z) - 1)] \\ &+ b[\sigma^n 3dc(2cos(k_m\Delta z) - 2) - \sigma^{n-1}dc(2cos(k_m\Delta z) - 2)]\end{aligned} \tag{5.6.18}$$

Which implies:

$$\begin{aligned}|\sigma^{n+1}| = |\sigma^n|\big(1 + 2ac(cos(k_m\Delta z) - 1)\big) &- |\sigma^{n-1}|2ac(cos(k_m\Delta z) - 1) \\ + |\sigma^n|3bdc(2cos(k_m\Delta z) - 2) &- |\sigma^{n-1}|bdc(2cos(k_m\Delta z) - 2)\end{aligned} \tag{5.6.19}$$

Using the recursive hypothesis, the equation becomes:

$$\begin{aligned}|\sigma^{n+1}| < |\sigma^0|\big(1 + 2ac(cos(k_m\Delta z) - 1)\big) &- |\sigma^0|2ac(cos(k_m\Delta z) - 1) \\ + |\sigma^0|3bdc(2cos(k_m\Delta z) - 2) &- |\sigma^0|bdc(2cos(k_m\Delta z) - 2)\end{aligned} \tag{5.6.20}$$

$$\begin{aligned}|\sigma^{n+1}| < |\sigma^0|\big[\big(1 + 2ac(cos(k_m\Delta z) - 1)\big) &- 2ac(cos(k_m\Delta z) - 1) \\ + 3bdc(2cos(k_m\Delta z) - 2) &- bdc(2cos(k_m\Delta z) - 2)\big]\end{aligned} \tag{5.6.21}$$

$$|\sigma^{n+1}| < |\sigma^0|[1 + 4bdc(cos(k_m\Delta z) - 1)] \tag{5.6.22}$$

$$1 + 4bdc(cos(k_m\Delta z) - 1) < 1 \tag{5.6.23}$$

$$4bdccos(k_m\Delta z) < 4bdc \tag{5.6.24}$$

The stability condition is given by:

$$cos(k_m\Delta z) < 1 \tag{5.6.25}$$

For our numerical solution, we replace the time derivative on our unsaturated equation (3.20) with the Caputo-Fabrizio fractional derivative, such that:

$$\left(-(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}\right){}^{CF}_{\ 0}D^{\alpha}\psi(z,t)=K_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n+1}-\psi_i^{n+1}}{\Delta z}$$
$$+K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{(\Delta z)^2}$$
$$-K_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\tag{5.7}$$

Which can also be written as:

$${}^{CF}_{\ 0}D^{\alpha}\psi(z,t)=\frac{1}{-(\theta_s-\theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}}\left(K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\frac{\left(\psi_{i+1}^{n+1}-\psi_i^{n+1}\right)}{\Delta z}\right.$$
$$+K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right)}{(\Delta z)^2}$$
$$\left.-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\right)\tag{5.7.1}$$

where

$${}^{CF}_{\ 0}D^{\alpha}\psi(z,t)=f\big(z,t,\psi(z,\alpha)\big)\tag{5.7.2}$$

We apply the Caputo-Fabrizio integral to the above equation:

$$\psi(z,t)=\psi(z,0)+\frac{1-\alpha}{M(\alpha)}f\big(z,t,\psi(z,t)\big)+\frac{\alpha}{M(\alpha)}\int_0^t f\big(z,\tau\ \ \psi(z,\tau)\big)d\tau\tag{5.7.3}$$

When $t=t_{n+1}$Now we have the following equation:

$$\psi(z_i, t_{n+1}) = \psi(z_i, 0) + \frac{1-\alpha}{M(\alpha)} f\left(z_i, t_j, \psi\left(z_i, t_j\right)\right) + \frac{\alpha}{M(\alpha)} \int_0^{t_{n+1}} f\left(z_i, \tau\ \ \psi(z_i, \tau)\right) d\tau \tag{5.7.4}$$

When $t = t_n$:

$$\psi(z_i, t_n) = \psi(z_i, 0) + \frac{1-\alpha}{M(\alpha)} f\left(z_i, t_{n-1}, \psi(z_i, t_{n-1})\right) + \frac{\alpha}{M(\alpha)} \int_0^{t_n} f\left(z_i, \tau\ \ \psi(z_i, \tau)\right) d\tau \tag{5.7.5}$$

Subtracting the above solution for t_n from t_{n+1}, we obtain the following:

$$\begin{aligned} \psi_i^{n+1} - \psi_i^n &= \frac{1-\alpha}{M(\alpha)} [f(z_i, t_n, \psi_i^n) - f(z_i, t_{n-1}, \psi_i^{n-1})] \\ &\quad + \frac{\alpha}{M(\alpha)} \int_{t_n}^{t_{n+1}} f\left(z_i, \tau, \psi(x_i, \tau)\right) d\tau \end{aligned} \tag{5.7.6}$$

$$\begin{aligned} \psi_i^{n+1} - \psi_i^n &= + \frac{1-\alpha}{M(\alpha)} [f(z_i, t_n, \psi_i^n) - f(z_i, t_{n-1}, \psi_i^{n-1})] \\ &\quad + \frac{\alpha}{M(\alpha)} \left[\frac{3}{2} \Delta t f(z_i, t_n, \psi_i^n) - \frac{\Delta t}{2} f(z_i, t_{n-1}, \psi_i^{n-1})\right] \end{aligned} \tag{5.7.7}$$

where:

$$\begin{aligned} F(z_i, t_n, \psi_i^n) = \frac{1}{-(\theta_s - \theta_0)\psi_b^\lambda \frac{\lambda}{\psi_i^{\lambda+1}}} &\left(K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\right)}{\Delta z} \frac{(\psi_{i+1}^n - \psi_i^n)}{\Delta z} \right. \\ &+ K_s \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \times \frac{(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)}{(\Delta z)^2} \\ &\left. - K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\right)}{\Delta z} \right) \end{aligned} \tag{5.7.8}$$

$$F(z_i, t_{n-1}, \psi_i^{n-1})$$

$$= \frac{1}{-(\theta_s - \theta_0)\psi_b^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}}\left(K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}\right)}{\Delta z} \frac{(\psi_{i+1}^{n-1} - \psi_i^{n-1})}{\Delta z} + K_s \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda} \times \frac{(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1})}{(\Delta z)^2} - K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}\right)}{\Delta z} \right) \tag{5.7.9}$$

We substitute the above equation (5.7.8) and (5.7.9) into equation (5.7.7) to obtain the final numerical solution for the saturated groundwater flow equation:

$$\psi_i^{n+1} = \psi_i^n + \frac{1-\alpha}{M(\alpha)} \left[\frac{1}{-(\theta_s - \theta_0)\psi_b^\lambda \frac{\lambda}{\psi_i^{\lambda+1}}} \left(K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n} \right)^{2+3\lambda} \right)}{\Delta z} \frac{(\psi_{i+1}^n - \psi_i^n)}{\Delta z} \right. \right.$$

$$\left. + K_s \left(\frac{\psi_b}{\psi_i^n} \right)^{2+3\lambda} \times \frac{(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)}{(\Delta z)^2} - K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n} \right)^{2+3\lambda} \right)}{\Delta z} \right)$$

$$- \frac{1}{-(\theta_s - \theta_0)\psi_b^\lambda \frac{\lambda}{\psi_i^{\lambda+1}}} \left(K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}} \right)^{2+3\lambda} \right)}{\Delta z} \frac{(\psi_{i+1}^{n-1} - \psi_i^{n-1})}{\Delta z} \right.$$

$$+ K_s \left(\frac{\psi_b}{\psi_i^{n-1}} \right)^{2+3\lambda} \times \frac{(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1})}{(\Delta z)^2}$$

$$\left. \left. - K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{j-1}} \right)^{2+3\lambda} \right)}{\Delta z} \right) \right]$$

$$+ \frac{\alpha}{M(\alpha)} \left[\frac{3\Delta t}{2} \frac{1}{-(\theta_s - \theta_0)\psi_b^\lambda \frac{\lambda}{\psi_i^{\lambda+1}}} \left(K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n} \right)^{2+3\lambda} \right)}{\Delta z} \frac{(\psi_{i+1}^n - \psi_i^n)}{\Delta z} \right. \right.$$

$$\left. + K_s \left(\frac{\psi_b}{\psi_i^n} \right)^{2+3\lambda} \times \frac{(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)}{(\Delta z)^2} - K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n} \right)^{2+3\lambda} \right)}{\Delta z} \right)$$

$$- \frac{\Delta t}{2} \frac{1}{-(\theta_s - \theta_0)\psi_b^\lambda \frac{\lambda}{\psi_i^{\lambda+1}}} \left(K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}} \right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}} \right)^{2+3\lambda} \right)}{\Delta z} \frac{(\psi_{i+1}^{n-1} - \psi_i^{n-1})}{\Delta z} \right.$$

$$+ K_s \left(\frac{\psi_b}{\psi_i^{n-1}} \right)^{2+3\lambda} \times \frac{(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1})}{(\Delta z)^2}$$

$$- K_s \frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}\right)}{\Delta z} \Bigg)\Bigg)\Bigg] \tag{5.7.10}$$

Now, let:

$$\mathscr{p} = -(\theta_s - \theta_0)\psi_b^{\lambda} \ \frac{\lambda}{\psi^{\lambda+1}}$$

$$\mathscr{a} = \frac{1-\alpha}{M(\alpha)}$$

$$\mathscr{b} = \frac{\alpha}{M(\alpha)}$$

$$\mathscr{c} = \frac{K_s}{\Delta z}$$

$$\mathscr{d} = \frac{1}{\Delta z}$$

$$\mathscr{e} = \frac{1}{(\Delta z)^2}$$

$$\mathscr{f} = \frac{3\Delta t}{2}$$

$$\mathscr{g} = \frac{\Delta t}{2}$$

Using the above substitutions, equation (5.7.10) becomes:

$$
\begin{aligned}
\psi_i^{n+1} = \psi_i^n + \mathscr{a}\Bigg[& \frac{1}{\mathscr{p}}\Bigg(c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^n}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^n}\bigg)^{2+3\lambda}\Bigg) \mathscr{d}(\psi_{i+1}^n - \psi_i^n) + K_s\bigg(\frac{\psi_b}{\psi_i^n}\bigg)^{2+3\lambda} \\
& \times e(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) - c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^n}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^n}\bigg)^{2+3\lambda}\Bigg)\Bigg) \\
& - \Bigg(c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^{n-1}}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^{n-1}}\bigg)^{2+3\lambda}\Bigg) \mathscr{d}(\psi_{i+1}^{n-1} - \psi_i^{n-1}) + K_s\bigg(\frac{\psi_b}{\psi_i^{n-1}}\bigg)^{2+3\lambda} \\
& \times e(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}) - c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^{j-1}}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^{j-1}}\bigg)^{2+3\lambda}\Bigg)\Bigg)\Bigg] \\
& + \mathscr{b}\Bigg[\mathscr{f}\frac{1}{\mathscr{p}}\Bigg(c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^n}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^n}\bigg)^{2+3\lambda}\Bigg) \mathscr{d}(\psi_{i+1}^n - \psi_i^n) + K_s\bigg(\frac{\psi_b}{\psi_i^n}\bigg)^{2+3\lambda} \\
& \times e(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) - c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^n}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^n}\bigg)^{2+3\lambda}\Bigg)\Bigg) \\
& - \mathscr{g}\frac{1}{\mathscr{p}}\Bigg(c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^{n-1}}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^{n-1}}\bigg)^{2+3\lambda}\Bigg) \mathscr{d}(\psi_{i+1}^{n-1} - \psi_i^{n-1}) + K_s\bigg(\frac{\psi_b}{\psi_i^{n-1}}\bigg)^{2+3\lambda} \\
& \times e(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}) \\
& - c\Bigg(\bigg(\frac{\psi_b}{\psi_{i+1}^{n-1}}\bigg)^{2+3\lambda} - \bigg(\frac{\psi_b}{\psi_i^{n-1}}\bigg)^{2+3\lambda}\Bigg)\Bigg)\Bigg] \qquad (5.7.11)
\end{aligned}
$$

CHAPTER 6

NEW MODEL WITH CROSSOVER FROM USUAL TO SUB- FLOW MEAN SQUARE DISPLACEMENT

6.1. INTRODUCTION

In 1903, Gosta Mittag-Leffler introduced the Mittag-Leffler function, which became very prominent only in the previous two decades as a result of its application in natural problems, for example, in engineering, earth science, and biology. This function responds to the conventional questions of complex analysis, in particular, to portray the procedure of the analytic continuation of power-law series outside the disc of their convergence (Gomez-Aguilar & Atangana, 2019). The ordinary and generalized Mittag-Leffler functions interpolate between a purely exponential and power law. In diffusion, Mittag-Leffler is essential because it can capture non-locality and avoid singularity. It is also crucial for the description of long-time behavior. It can be used as a waiting time distribution as well as the first passage time distribution for renewal processes. Properties of Mittag-Leffler include usual differentiation and integration, Euler transforms, Laplace transforms, Whittaker transforms, generalized, and hypergeometric series, among others. Recently, a non-local kernel with non-singularity was suggested by Atangana and Baleanu. This new kernel has an advantage that a usual initial condition can be obtained when using the Laplace transform unlike with the Riemann-Liouville. Furthermore, since this kernel has no singularity it can depict a full memory (Alkahtani et al., 2017). The kernel is more natural, and it is the first to show the cross-over behavior between the stretched exponential law for earlier time and power-law for the latter time (Gomez-Aguilar & Atangana, 2019). Below, we recall the definitions of the new kernel.

6.2. A-B DERIVATIVE WITH FRACTIONAL ORDER

In this section, we illustrate definitions of the fractional derivatives with no singular and non-local kernel. Atangana and Baleanu introduced these definitions in their work published in a thermal science journal.

Definition 1: Let $f \in H'(a,b), b > a, \alpha \in [0,1]$ and not necessarily differentiable then, the definition of the fractional derivative (A-B fractional derivative in Riemann-Liouville sense) is given as:

$$ {}^{ABR}_{\ a}D_t^{\alpha}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(\tau)\, E_\alpha\left[-\alpha\frac{(t-\tau)^\alpha}{1-\alpha}\right] d\tau \qquad (6.1) $$

Definition 2: Let $f \in H'(a,b), b > a, \alpha \in [0,1]$ then, the definition of the recently introduced AB fractional derivative in Caputo sense is given as:

$$ {}^{ABC}_{\ a}D_t^{\alpha}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(\tau)\, E_\alpha\left[-\alpha\frac{(t-\tau)^\alpha}{1-\alpha}\right] d\tau \qquad (6.2) $$

Definition 3: The fractional integral associate to the new fractional derivative with the non-local kernel (A-B fractional integral) is defined as:

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(f(t)\big) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(\tau)(t-\tau)^{\alpha-1}\, d\tau \qquad (6.3) $$

Numerical solution of the integral

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(\psi(z,t)\big) = \frac{1-\alpha}{B(\alpha)}\psi(z,t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t \psi(z,\tau)(t-\tau)^{\alpha-1}\, d\tau \qquad (6.3.1) $$

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi(z_i,t_n) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^{t_n} \psi(z_i,\tau)(t_n-\tau)^{\alpha-1}\, d\tau \qquad (6.3.2) $$

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^{t_n} \psi_i^j(t_n-\tau)^{\alpha-1}\, d\tau \qquad (6.3.3) $$

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\int_{t_j}^{t_{j+1}} (t_n-\tau)^{\alpha-1}\, d\tau \qquad (6.3.4) $$

Let $y = t_n - \tau$, $y = t_n - t_j$, $y = t_n - t_{j+1}$ and $dy = -d\tau$, therefore:

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\int_{t_n-t_j}^{t_n-t_{j+1}} (y)^{\alpha-1}\, (-dy) \qquad (6.3.5) $$

$$ {}^{AB}_{\ a}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\int_{t_n-t_{j+1}}^{t_n-t_j} (y)^{\alpha-1}\, dy \qquad (6.3.6) $$

$$ {}_{a}^{AB}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\,[(y)^{\alpha-1}dy] \tag{6.3.7} $$

$$ {}_{a}^{AB}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\left[\left(t_n-t_j\right)^{\alpha-1} - \left(t_n-t_{j+1}\right)^{\alpha-1}\right] \tag{6.3.8} $$

Using $t_n = n\Delta t$ and $t_{j+1} = (j+1)\Delta t$ we can write the above equation as:

$$ {}_{a}^{AB}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\,\Delta t^{\alpha-1}[(n-j)^{\alpha-1} - (n-j-1)^{\alpha-1}] \tag{6.3.9} $$

The final equation can be written as:

$$ {}_{a}^{AB}I_t^{\alpha}\big(\psi(z_i,t_n)\big) = \frac{1-\alpha}{B(\alpha)}\psi_i^n + \frac{\alpha\Delta t^{\alpha}}{B(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\psi_i^j\,[(n-j)^{\alpha} - (n-j-1)^{\alpha}] \tag{6.3.10} $$

6.3. NUMERICAL SOLUTION OF THE SATURATED-UNSATURATED GROUNDWATER FLOW EQUATION USING THE A-B FACTIONAL DERIVATIVE

Now, let's consider the model using the non-local and non-singular kernel. To do this, the time derivative of the original model is replaced with the time-fractional derivative built using the Mittag-Leffler function to derive the numerical solution of the new model. Therefore, the saturated equation is written as follows:

$$ {}_{0}^{AB}D^{\alpha}\psi(z,t) = \frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2} \tag{6.4} $$

where

$$ {}_{0}^{AB}D^{\alpha}\psi(z,t) = f\big(z,t,\psi(z,\alpha)\big) \tag{6.4.1} $$

Then the AB integral is written as:

$$ \begin{aligned} &\psi(z_i,t) - \psi(z_i,0) \\ &\qquad = \frac{(1-\alpha)}{B(\alpha)} f\big(z,t,\psi(z,t)\big) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f\big(z_i,\tau,\psi(z_i,\tau)\big)\,(t-\tau)^{\alpha-1}d\tau \end{aligned} \tag{6.4.2} $$

At point $t_{n+1}, n = 0, 1, 2, 3, 4 \dots \dots \dots \dots$

$$\psi(z_i, t_{n+1}) - \psi(z_i, 0) = \frac{(1-\alpha)}{\mathrm{B}(\alpha)} f\big(z, t, \psi(z,t)\big) + \frac{\alpha}{\mathrm{B}(\alpha) \times \Gamma(\alpha)} \int_0^{t_{n+1}} f\big(z_i, \tau, \psi(z_i, \tau)\big)(t_{n+1} - \tau)^{\alpha-1} d\tau \tag{6.4.3}$$

$$\begin{aligned} \psi(z_i, t_{n+1}) - \psi(z_i, 0) &= \frac{(1-\alpha)}{\mathrm{B}(\alpha)} f\big(z, t, \psi(z,t)\big) \\ &+ \frac{\alpha(\Delta t)^\alpha}{\mathrm{B}(\alpha) \times \Gamma(2+\alpha)} \sum_{j=0}^{n} \Big[f\big(z_i, t_j, \psi_i^j\big)\big((n-j+1)^\alpha (n-j+2+\alpha) \\ &- (n-j)^\alpha (n-j+2+2\alpha)\big) \\ &- f\big(z_i, t_{j-1}, \psi_i^{j-1}\big)\big((n-j+1)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha)\big)\Big] \end{aligned} \tag{6.4.4}$$

where:

$$F\big(z_i, t_j, \psi_i^j\big) = \frac{K_s}{S_s} \frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{(\Delta z)^2} \tag{6.4.4.1}$$

$$F\big(z_i, t_{j-1}, \psi_i^{j-1}\big) = \frac{K_s}{S_s} \frac{\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}}{(\Delta z)^2} \tag{6.4.4.2}$$

The final solution for the saturated groundwater flow equation is given as:

$$
\begin{aligned}
&\psi(z_i, t_{n+1}) - \psi(z_i, 0) \\
&\qquad = \frac{(1-\alpha)}{\mathrm{B}(\alpha)}\left(\frac{K_s}{S_s}\frac{\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n}}{(\Delta z)^2}\right) \\
&\qquad + \frac{\alpha(\Delta t)^{\alpha}}{\mathrm{B}(\alpha)\Gamma(2+\alpha)}\sum_{j=0}^{n}\left[\frac{K_s}{S_s}\frac{\psi_{i+1}^{j} - 2\psi_i^{j} + \psi_{i-1}^{j}}{(\Delta z)^2}\big((n-j+1)^{\alpha}(n-j+2\right. \\
&\qquad + \alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\big) \\
&\qquad - \frac{K_s}{S_s}\frac{\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}}{(\Delta z)^2}\big((n-j+1)^{\alpha+1} \\
&\qquad \left. - (n-j)^{\alpha}(n-j+1+\alpha)\big)\right] \qquad (6.4.5)
\end{aligned}
$$

We also assume the following parameters:

$$a = \frac{(1-\alpha)}{\mathrm{B}(\alpha)}$$

$$b = \frac{K_s}{S_s(\Delta z)^2}$$

$$c = \frac{\alpha(\Delta t)^{\alpha}}{\mathrm{B}(\alpha)\Gamma(2+\alpha)}$$

$$R_{n.j}^{\alpha} = \big((n-j+1)^{\alpha}(n-j+2+\alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\big)$$

$$\delta_{n,j}^{\alpha} = \big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)$$

Then our equation is written as:

$$
\begin{aligned}
\psi_i^{n+1} - \psi_i^{n} &= a[b(\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n})] \\
&+ c\sum_{j=0}^{n}\big[b\big(\psi_{i+1}^{j} - 2\psi_i^{j} + \psi_{i-1}^{j}\big)R_{n,j}^{\alpha} - b\big(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\big)\delta_{n,j}^{\alpha}\big] \qquad (6.4.6)
\end{aligned}
$$

$$
\begin{aligned}
\psi_i^{n+1} - \psi_i^{n} &= a[b(\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n})] \\
&+ c\big[b(\psi_{i+1}^{n} - 2\psi_i^{n} + \psi_{i-1}^{n})R_{n,n}^{\alpha} - b(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1})\delta_{n,n}^{\alpha}\big] \\
&+ c\sum_{j=0}^{n-1}\big[b\big(\psi_{i+1}^{j} - 2\psi_i^{j} + \psi_{i-1}^{j}\big)R_{n,j}^{\alpha} - b\big(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\big)\delta_{n,j}^{\alpha}\big] \qquad (6.4.7)
\end{aligned}
$$

To check the stability of the above numerical solution, we assume a Fourier expansion in the space of:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{6.4.8}$$

Using the above equation, equation 6.4.7 becomes:

$$\begin{aligned}
&\sigma^{n+1} e^{ik_m z} - \sigma^n e^{ik_m z} \\
&\quad = a\big[b\big(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\big)\big] \\
&\quad + c\big[b\big(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\big)R_{n,n}^{\alpha} \\
&\quad - b\big(\sigma^{n-1} e^{ik_m(z+\Delta z)} - 2\sigma^{n-1} e^{ik_m z} + \sigma^{n-1} e^{ik_m(z-\Delta z)}\big)\delta_{n,n}^{\alpha}\big] \\
&\quad + c\sum_{j=0}^{n-1}\big[b\big(\sigma^j e^{ik_m(z+\Delta z)} - 2\sigma^j e^{ik_m z} + \sigma^j e^{ik_m(z-\Delta z)}\big)R_{n,j}^{\alpha} \\
&\quad - b\big(\sigma^{j-1} e^{ik_m(z+\Delta z)} - 2\sigma^{j-1} e^{ik_m z} + \sigma^{j-1} e^{ik_m(z-\Delta z)}\big)\delta_{n,j}^{\alpha}\big]
\end{aligned} \tag{6.4.9}$$

$$\begin{aligned}
&e^{ik_m z}(\sigma^{n+1} - \sigma^n) \\
&\quad = a\big[b\sigma^n e^{ik_m z}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big] \\
&\quad + c\big[R_{n,n}^{\alpha} b\sigma^n e^{ik_m z}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big) \\
&\quad - \delta_{n,n}^{\alpha} b\sigma^{n-1} e^{ik_m z}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big] \\
&\quad + c\sum_{j=0}^{n-1}\big[R_{n,j}^{\alpha} b\sigma^j e^{ik_m z}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big) \\
&\quad - \delta_{n,j}^{\alpha} b\sigma^{j-1} e^{ik_m z}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big]
\end{aligned} \tag{6.4.10}$$

$$\begin{aligned}
\sigma^{n+1} - \sigma^n &= a\big[b\sigma^n\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big] \\
&+ c\big[R_{n,n}^{\alpha} b\sigma^n\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big) \\
&- \delta_{n,n}^{\alpha} b\sigma^{n-1}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big] \\
&+ c\sum_{j=0}^{n-1}\big[R_{n,j}^{\alpha} b\sigma^j\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big) \\
&- \delta_{n,j}^{\alpha} b\sigma^{j-1}\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big]
\end{aligned} \tag{6.4.11}$$

$$\begin{aligned}\sigma^{n+1}-\sigma^{n} = {} & 2ab\sigma^{n}\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right) \\ & + c\left[2R_{n,n}^{\alpha}b\sigma^{n}\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right) \right. \\ & \left. - 2\delta_{n,n}^{\alpha}b\sigma^{n-1}\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right] \\ & + c\sum_{j=0}^{n-1}\left[2R_{n,j}^{\alpha}b\sigma^{j}\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right) \right. \\ & \left. - 2\delta_{n,j}^{\alpha}b\sigma^{j-1}\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right] \qquad (6.4.12)\end{aligned}$$

Considering the definition for a hyperbolic cosine, we can rewrite the above equation (6.4.12) as:

$$\begin{aligned}\sigma^{n+1}-\sigma^{n} = {} & 2ab\sigma^{n}(cos(k_m\Delta z)-1) \\ & + c\left[2R_{n,n}^{\alpha}b\sigma^{n}(cos(k_m\Delta z)-1)-2\delta_{n,n}^{\alpha}b\sigma^{n-1}(cos(k_m\Delta z)-1)\right] \\ & + c\sum_{j=0}^{n-1}\left[2R_{n,j}^{\alpha}b\sigma^{j}(cos(k_m\Delta z)-1)\right. \\ & \left. - 2\delta_{n,j}^{\alpha}b\sigma^{j-1}(cos(k_m\Delta z)-1)\right] \qquad (6.4.13)\end{aligned}$$

If n=0 the equation above becomes:

$$\sigma^{1}-\sigma^{0} = 2ab\sigma^{0}(cos(k_m\Delta z)-1)+c\left[2R_{n,n}^{\alpha}b\sigma^{0}(cos(k_m\Delta z)-1)\right] \qquad (6.4.14)$$

$$\sigma^{1} = \sigma^{0}\left[1+2ab(cos(k_m\Delta z)-1)+R_{n,n}^{\alpha}cb(cos(k_m\Delta z)-1)\right] \qquad (6.4.15)$$

$$\left|\frac{\sigma^{1}}{\sigma^{0}}\right|<1$$

$$\left|\frac{1}{1+2ab(cos(k_m\Delta z)-1)+R_{n,n}^{\alpha}cb(cos(k_m\Delta z)-1)}\right|<1 \qquad (6.4.16)$$

The denominator of the above equation will always be greater than the numerator; therefore, the A-B fractional derivative for the 1-d saturated groundwater flow equation is unconditionally stable.

Now we assume $\forall n>0$ to prove that:

$$\left|\frac{\sigma^{n}}{\sigma^{0}}\right|<1$$

$$\begin{aligned}\sigma^{n+1} = \sigma^{n}\big[1 + 2ab(cos(k_m\Delta z) - 1) + R^{\alpha}_{n,n}cb(cos(k_m\Delta z) - 1)\big] \\ - 2\delta^{\alpha}_{n,n}cb\sigma^{n-1}(cos(k_m\Delta z) - 1) \\ + c\sum_{j=0}^{n-1}\big[2R^{\alpha}_{n,j}b\sigma^{j}(cos(k_m\Delta z) - 1) \\ - 2\delta^{\alpha}_{n,j}b\sigma^{j-1}(cos(k_m\Delta z) - 1)\big]\end{aligned} \tag{6.4.17}$$

We let:

$$A = \big[1 + 2ab(cos(k_m\Delta z) - 1) + R^{\alpha}_{n,n}cb(cos(k_m\Delta z) - 1)\big]$$

$$B = 2\delta^{\alpha}_{n,n}cb(cos(k_m\Delta z) - 1)$$

$$A_1 = 2R^{\alpha}_{n,j}b(cos(k_m\Delta z) - 1)$$

$$A_2 = 2\delta^{\alpha}_{n,j}b(cos(k_m\Delta z) - 1)$$

Then equation (6.4.17) becomes:

$$\sigma^{n+1} = \sigma^{n}A - \sigma^{n-1}B + c\sum_{j=0}^{n-1}\big[\sigma\sigma^{j}{}_{j}A_1 - \sigma^{j-1}A_2\big] \tag{6.4.18}$$

$$|\sigma^{n+1}| \le |\sigma^{n}||A| - |\sigma^{n-1}||B| + c\sum_{j=0}^{n-1}\big[\left|\sigma^{j}\right||A_1| - \left|\sigma^{j-1}\right||A_2|\big] \tag{6.4.19}$$

$$|\sigma^{n+1}| \le |\sigma^{0}|\left(|A| - |B| + c\sum_{j=0}^{n-1}(|A_1| - |A_2|)\right) \tag{6.4.20}$$

$$\frac{|\sigma^{n+1}|}{|\sigma^{0}|} \le |A| - |B| + c\sum_{j=0}^{n-1}(|A_1| - |A_2|) \tag{6.4.21}$$

The stability condition is therefore given by:

$$\left|\frac{1}{|A| - |B| + c\sum_{j=0}^{n-1}(|A_1| - |A_2|)}\right| \le 1 \tag{6.4.22}$$

Now, using the AB fractional derivative the new model for the unsaturated zone is given by the following equation:

$$\begin{aligned}
&{}^{ABC}_{\ \ \ 0}D^{\alpha}\psi(z,t)\left(-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}\right)\\
&\qquad = K_s\,\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}\,\frac{(\psi_{i+1}^{n+1}-\psi_i^{n+1})}{\Delta z}\\
&\qquad + K_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})}{(\Delta z)^2}\\
&\qquad - K_s\,\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\right)}{\Delta z}
\end{aligned} \tag{6.5}$$

where

$${}^{ABC}_{\ \ \ 0}D^{\alpha}\psi(z,t)=F\big(z,t,\psi(z,\alpha)\big) \tag{6.5.1}$$

Then the AB integral is given as:

$$\begin{aligned}
&\psi(z_i,t)-\psi(z_i,0)\\
&\qquad =\frac{(1-\alpha)}{B(\alpha)}f\big(z,t,\psi(z,t)\big)\\
&\qquad +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f\big(z_i,\tau,\psi(z_i,\tau)\big)(t-\tau)^{\alpha-1}d\tau
\end{aligned} \tag{6.5.2}$$

At point $t_{n+1}, n=0,1,2,3,4\ldots\ldots\ldots\ldots$

$$\begin{aligned}
&\psi(z_i,t_{n+1})-\psi(z_i,0)\\
&\qquad =\frac{(1-\alpha)}{B(\alpha)}f\big(z,t,\psi(z,t)\big)\\
&\qquad +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_{n+1}} F\big(z_i,\tau,\psi(z_i,\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau
\end{aligned} \tag{6.5.3}$$

$$
\begin{aligned}
&\psi(z_i, t_{n+1}) - \psi(z_i, 0) \\
&\qquad = \frac{(1-\alpha)}{B(\alpha)} f\big(z, t, \psi(z,t)\big) \\
&\qquad + \frac{(\Delta t)^{\alpha}}{B(\alpha)\Gamma(2+\alpha)} \sum_{j=0}^{n} \Big[F\big(z_i, t_j, \psi_i^j\big)\big((n-j+1)^{\alpha}(n-j+2+\alpha) \\
&\qquad - (n-j)^{\alpha}(n-j+2+2\alpha)\big) \\
&\qquad - F\big(z_i, t_{j-1}, \psi_i^{j-1}\big)\big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)\Big]
\end{aligned}
\tag{6.5.4}
$$

where:

$$
\begin{aligned}
F\big(z_i, t_j, \psi_i^j\big) = \frac{1}{-(\theta_s - \theta_0){\psi_b}^{\lambda}\ \dfrac{\lambda}{\psi_i^{\lambda+1}}} &\left(K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \right)}{\Delta z} \frac{\big(\psi_{i+1}^j - \psi_i^j\big)}{\Delta z} \right. \\
&+ K_s \left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \times \frac{\big(\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j\big)}{(\Delta z)^2} \\
&\left. - K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \right)}{\Delta z} \right)
\end{aligned}
\tag{6.5.4.1}
$$

$$
\begin{aligned}
&F\big(z_i, t_{j-1}, \psi_i^{j-1}\big) \\
&= \frac{1}{-(\theta_s - \theta_0){\psi_b}^{\lambda}\ \dfrac{\lambda}{\psi_i^{\lambda+1}}} \left(K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda} \right)}{\Delta z} \frac{\big(\psi_{i+1}^{j-1} - \psi_i^{j-1}\big)}{\Delta z} \right. \\
&\left. + K_s \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda} \times \frac{\big(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\big)}{(\Delta z)^2} - K_s \frac{\left(\left(\dfrac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda} \right)}{\Delta z} \right)
\end{aligned}
\tag{6.5.4.2}
$$

The final solution for the unsaturated groundwater flow equation is given as:

$$
\begin{aligned}
&\psi_i^{n+1}-\psi_i^0\\
&=\frac{(1-\alpha)}{B(\alpha)}\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}}\Bigg(K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\right)}{\Delta z}\frac{(\psi_{i+1}^n-\psi_i^n)}{\Delta z}+K_s\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\\
&\times\frac{(\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n)}{(\Delta z)^2}-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\right)}{\Delta z}\Bigg)\\
&+\frac{(\Delta t)^\alpha}{\Gamma(2+\alpha)}\sum_{j=0}^{n}\Bigg[\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}}\Bigg(K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}\right)}{\Delta z}\frac{(\psi_{i+1}^j-\psi_i^j)}{\Delta z}\\
&+K_s\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}\times\frac{(\psi_{i+1}^j-2\psi_i^j+\psi_{i-1}^j)}{(\Delta z)^2}\\
&-K_s\frac{\left(\left(\frac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}\right)}{\Delta z}\Bigg)\big((n-j+1)^\alpha(n-j+2+\alpha)\\
&-(n-j)^\alpha(n-j+2+2\alpha)\big)\\
&-\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}}\Bigg(K_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{j-1}-\psi_i^{j-1}}{\Delta z}+K_s\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\\
&\times\frac{\psi_{i+1}^{j-1}-2\psi_i^{j-1}+\psi_{i-1}^{j-1}}{(\Delta z)^2}-K_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z}\Bigg)\big((n-j+1)^{\alpha+1}\\
&-(n-j)^\alpha(n-j+1+\alpha)\big)\Bigg]
\end{aligned}
\tag{6.5.5}
$$

For simplicity of the above equation let:

$$\mathscr{p} = -(\theta_s - \theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}$$

$$\mathscr{a} = \frac{(1-\alpha)}{B(\alpha)}$$

$$\mathscr{b} = \frac{K_s}{\Delta z}$$

$$c = \frac{1}{\Delta z}$$

$$\mathscr{d} = K_s$$

$$e = \frac{1}{(\Delta z)^2}$$

$$\mathscr{f} = \frac{(\Delta t)^{\alpha}}{\Gamma\,(2-\alpha)}$$

$$R_{n.j}^{1,\alpha} = \big((n-j+1)^{\alpha}(n-j+2+\alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\big)$$

$$R_{n.j}^{2,\alpha} = \big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)$$

$$
\begin{aligned}
\psi_i^{n+1}-\psi_i^{0}={}& a\Bigg(\frac{1}{p}\Bigg(b\left(\left(\frac{\psi_b}{\psi_{i+1}^{n}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}\right)c\left(\psi_{i+1}^{n}-\psi_i^{n}\right)+d\left(\frac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}\\
&\times e\left(\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}\right)-b\left(\left(\frac{\psi_b}{\psi_{i+1}^{n}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}\right)\Bigg)\Bigg)\\
&+a\sum_{j=0}^{n}\Bigg[\frac{1}{p}\Bigg(b\left(\left(\frac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\right)c\left(\psi_{i+1}^{j}-\psi_i^{j}\right)+d\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\\
&\times e\left(\psi_{i+1}^{j}-2\psi_i^{j}+\psi_{i-1}^{j}\right)-b\left(\left(\frac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\right)\Bigg)R_{n.j}^{1,\alpha}\\
&-\frac{1}{p}\Bigg(b\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)c\left(\psi_{i+1}^{j-1}-\psi_i^{j-1}\right)+d\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\\
&\times e\left(\psi_{i+1}^{j-1}-2\psi_i^{j-1}+\psi_{i-1}^{j-1}\right)\\
&-b\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)\Bigg)R_{n.j}^{2,\alpha}\Bigg]
\end{aligned}
\tag{6.5.6}
$$

6.4. NUMERICAL SOLUTION OF THE SATURATED-UNSATURATED GROUNDWATER FLOW EQUATION USING THE GHANBARI & ATANGANA NUMERICAL SCHEME

Numerical methods such as Adams-Bashforth, two-step Laplace transform, and Atangana-Toufik including others have been applied successfully when solving complexities associated with non-linear differential and integral equations. While these methods have been intensively applied to the case of power-law differential and integral operators, one needs to mention that many of them have not really been applied when dealing with differential and integral equations generated by the generalised Mittag-Leffler function (Ghanbari & Atangana, 2019). In this section, we are going to apply the newly introduced numerical scheme by Ghanbari and Atangana to our saturated-unsaturated equation. This newly established numerical scheme is based on the product-integral

(PI) rule to solve a functional initial-value problem. Let us consider an AB fractional derivative equation as:

$$ {}^{ABC}_{t_0}D_t^{\alpha} = F\big(t, \psi(t)\big) \tag{6.6} $$

where $F\big(t, \psi(t)\big)$ are continuous functions and the initial condition $\psi(t_0) = \psi_0$. Applying the integral operator on both sides of equation (6.6) above we provide the following Volterra integral equation:

$$ \psi(t) - \psi(t_0) = \frac{(1-\alpha)}{B(\alpha)} F\big(t, \psi(t)\big) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{t_0}^{t} F\big(\tau, \psi(\tau)\big)\,(t-\tau)^{\alpha-1} d\tau \tag{6.6.1} $$

Taking $t = t_n = t_0 + n\hbar$ in the above equation, where $\hbar = \frac{t_f - t_0}{N}$ is a constant step-size, we achieve:

$$ \begin{aligned} &\psi(t_n) - \psi(t_0) \\ &\qquad = \frac{(1-\alpha)}{\mathrm{B}(\alpha)} F\big(t_n, \psi(t_n)\big) \\ &\qquad + \frac{\alpha}{\mathrm{B}(\alpha) \times \Gamma(\alpha)} \sum_{i=0}^{n-1} \int_{t_j}^{t_{j+1}} F\big(\tau, \psi(\tau)\big)(t_n - \tau)^{\alpha-1} d\tau \qquad 1 \le n \le N \end{aligned} \tag{6.6.2} $$

Now, we can approximate the function $F\big(\tau, \psi(\tau)\big)$by the first-order Lagrange interpolation:

$$ F\big(\tau, \psi(\tau)\big) \approx F\big(t_{j+1}, \psi_{j+1}\big) + \frac{\tau - t_{j+1}}{\hbar}\Big(F\big(t_{j+1}, \psi_{j+1}\big) - F\big(t_j, \psi_j\big)\Big) \;\; \tau \in \big[t_j, t_{j+1}\big] \tag{6.6.3} $$

where the notation $\psi_j = \psi\big(t_j\big)$ is used. Substituting equation 6.6.3 into 6.6.2 along with doing some algebraic manipulations, the following implicit AB-Caputo PI (ABC-PI) rule is achieved:

$$ \psi^n - \psi^0 = \frac{(1-\alpha)}{\mathrm{B}(\alpha)} F(t_n, \psi_n) + \frac{\alpha\hbar^{\alpha}}{\mathrm{B}(\alpha)} \left(\gamma_n F(t_n, \psi_n) + \sum_{J=0}^{N} \beta_{n-j} F\big(t_J, \psi_J\big)\right) \tag{6.6.4} $$

where:

$$ \gamma_n = \frac{(n-1)^{\alpha+1} - n^{\alpha}(n-\alpha-1)}{\Gamma(\alpha+1)} $$

$$B_J\begin{cases}\frac{1}{\Gamma(\alpha+2)}, j=0\\ \frac{(j-1)^{\alpha+1}-2j^{\alpha+1}+(j+1)^{\alpha+1}}{\Gamma(\alpha+2)}, j=1,2,3\ldots\ldots..n-1\end{cases}$$

Introducing the above equation 6.6.4 into our 1-d saturated-unsaturated groundwater flow equation, we get the following equation for the saturated zone:

$$\begin{aligned}\psi^n-\psi^0 &\\ &=\frac{(1-\alpha)}{\mathrm{B}(\alpha)}\frac{K_s}{S_s}\frac{\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n}{(\Delta z)^2}\\ &+\frac{\alpha\hbar^\alpha}{\mathrm{B}(\alpha)}\left(\gamma_n\frac{K_s}{S_s}\frac{\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n}{(\Delta z)^2}+\sum_{J=0}^{N}\beta_{n-j}\frac{K_s}{S_s}\frac{\psi_{i+1}^j-2\psi_i^j+\psi_{i-1}^j}{(\Delta z)^2}\right)\end{aligned}\qquad(6.6.5)$$

and the following for the unsaturated zone:

$$
\begin{aligned}
&\psi^{n}-\psi^{0}\\
&=\frac{(1-\alpha)}{\mathrm{B}(\alpha)}\left(\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\dfrac{\lambda}{\psi_i^{\lambda+1}}}\left(K_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n}-\psi_i^{n}}{\Delta z}+K_s\left(\frac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}\right.\right.\\
&\left.\left.\times\frac{\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}}{(\Delta z)^{2}}-K_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}}{\Delta z}\right)\right)\\
&+\frac{\alpha\hbar^{\alpha}}{\mathrm{B}(\alpha)}\left(\gamma_n\left(\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\dfrac{\lambda}{\psi_i^{\lambda+1}}}\left(K_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n}-\psi_i^{n}}{\Delta z}+K_s\left(\frac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}\right.\right.\right.\\
&\left.\left.\left.\times\frac{\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}}{(\Delta z)^{2}}-K_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{n}}\right)^{2+3\lambda}}{\Delta z}\right)\right)\right)\\
&+\sum_{J=0}^{N}\beta_{n-j}\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\dfrac{\lambda}{\psi_i^{\lambda+1}}}\left(K_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{j}-\psi_i^{j}}{\Delta z}+K_s\left(\frac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}\right.\\
&\times\frac{\psi_{i+1}^{j}-2\psi_i^{j}+\psi_{i-1}^{j}}{(\Delta z)^{2}}\\
&\left.-K_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{j}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{j}}\right)^{2+3\lambda}}{\Delta z}\right)\Bigg) \qquad (6.6.6)
\end{aligned}
$$

To simplify the above equation, let’s use the following parameters:

$$
\wp=-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}
$$

$$𝒶 = \frac{(1-\alpha)}{\mathrm{B}(\alpha)}$$

$$𝒷 = \frac{K_s}{\Delta z}$$

$$c = \frac{1}{\Delta z}$$

$$𝒹 = K_s$$

$$e = \frac{1}{(\Delta z)^2}$$

$$𝒻 = \frac{\alpha \hbar^{\alpha}}{\mathrm{B}(\alpha)}$$

$$
\begin{aligned}
(\,\psi^{n}-\psi^{0}) = \mathscr{a}\,\frac{1}{\mathscr{p}}\Bigg(\mathscr{b}\,\bigg(\Big(\frac{\psi_b}{\psi_{i+1}^{n}}\Big)^{2+3\lambda} - \Big(\frac{\psi_b}{\psi_{i}^{n}}\Big)^{2+3\lambda}\bigg) c(\psi_{i+1}^{n}-\psi_{i}^{n}) + \mathscr{d}\,\Big(\frac{\psi_b}{\psi_{i}^{n}}\Big)^{2+3\lambda} \\
\times e(\psi_{i+1}^{n}-2\psi_{i}^{n}+\psi_{i-1}^{n}) - \mathscr{b}\,\bigg(\Big(\frac{\psi_b}{\psi_{i+1}^{n}}\Big)^{2+3\lambda} - \Big(\frac{\psi_b}{\psi_{i}^{n}}\Big)^{2+3\lambda}\bigg)\Bigg) \\
+ \mathscr{f}\,\Bigg(\gamma_n \frac{1}{\mathscr{p}}\Bigg(\mathscr{b}\,\bigg(\Big(\frac{\psi_b}{\psi_{i+1}^{n}}\Big)^{2+3\lambda} - \Big(\frac{\psi_b}{\psi_{i}^{n}}\Big)^{2+3\lambda}\bigg) c(\psi_{i+1}^{n}-\psi_{i}^{n}) + \mathscr{d}\,\Big(\frac{\psi_b}{\psi_{i}^{n}}\Big)^{2+3\lambda} \\
\times e(\psi_{i+1}^{n}-2\psi_{i}^{n}+\psi_{i-1}^{n}) - \mathscr{b}\,\bigg(\Big(\frac{\psi_b}{\psi_{i+1}^{n}}\Big)^{2+3\lambda} - \Big(\frac{\psi_b}{\psi_{i}^{n}}\Big)^{2+3\lambda}\bigg)\Bigg) \\
+ \sum_{J=0}^{N} \beta_{n-j}\,\frac{1}{\mathscr{p}}\Bigg(\mathscr{b}\,\bigg(\Big(\frac{\psi_b}{\psi_{i+1}^{j}}\Big)^{2+3\lambda} - \Big(\frac{\psi_b}{\psi_{i}^{j}}\Big)^{2+3\lambda}\bigg) c\big(\psi_{i+1}^{j}-\psi_{i}^{j}\big) + \mathscr{d}\,\Big(\frac{\psi_b}{\psi_{i}^{j}}\Big)^{2+3\lambda} \\
\times e\big(\psi_{i+1}^{j}-2\psi_{i}^{j}+\psi_{i-1}^{j}\big) \\
- \mathscr{b}\,\bigg(\Big(\frac{\psi_b}{\psi_{i+1}^{j}}\Big)^{2+3\lambda} - \Big(\frac{\psi_b}{\psi_{i}^{j}}\Big)^{2+3\lambda}\bigg)\Bigg)\ \Bigg)
\end{aligned}
\tag{6.6.7}
$$

CHAPTER 7

APPLICATION OF FRACTIONAL-STOCHASTIC APPROACH TO 1-D SATURATED-UNSATURATED ZONE MODEL

7.1. INTRODUCTION

Recently, a new concept that brings us much closer to the reality of complex systems was suggested by Atangana and Bonyah. This new approach is a combination of fractional differentiation and stochastic modelling. These two important approaches have been used independently in previous years and has attained great success. The differential operators have been used to capture the non-Markovian process; meanwhile, the stochastic approach has been used to capture heterogeneity in a closed system for a Markovian process (Atangana & Bonyah, 2019). The concept of differentiation is used to construct mathematical formulas called differential equations with constant coefficients, wherein the differential operators can be local, non-local with the singular kernel, local with non-singular, and non-local with the non-singular kernel (Atangana & Araz, 2019; Atangana & Bonyah, 2019). However, these approaches have failed to capture some statistical settings of nature for larger space and time. Contrarily, the stochastic approach has been used in many situations to capture statistical settings; nonetheless, this approach is unable to capture long-range behaviour and also fading memory. Combining these two approaches can help us capture physical problems appearing in nature, with fading memory, long-range and statistical settings for larger space and time (Atangana & Bonyah, 2019). In this chapter, we aim to apply this approach to the 1-d saturated-unsaturated subsurface flow equation. We will make use of numerical solutions that were derived in previous chapters using the Caputo fractional derivative, Caputo-Fabrizio fractional derivative, and Atangana-Baleanu fractional derivative. The constant coefficients of the equations will be converted into distributions; this will enable us to capture processes with statistical settings.

7.2. APPLICATION OF THE STOCHASTIC APPROACH

Under consideration is the 1-d saturated-unsaturated groundwater flow equation. This equation has two constant coefficients; the hydraulic conductivity K_s and the specific storage S_S of the porous media. To convert these constant coefficients into distributions, we need to find the mean and the variance of the two constants.

7.2.1 The mean and variance of the hydraulic conductivity

The mean for the hydraulic conductivity K_s is given by:

$$\bar{K}_s = \frac{\sum_{j=0}^{n} K_{sj}}{n} \tag{7.1}$$

The variance is given by:

$$\sigma^2 = \frac{\sum_{j=0}^{n}\left(K_{sj} - \bar{K}_s\right)^2}{(n-1)} \tag{7.2}$$

Now we make use of the log-normal distribution to convert the constant-coefficient K_s into distribution, we convert the parameter $\widehat{K_s}$ to:

$$\widehat{K_s} = \bar{K}_{sj} + \gamma logN\left(\bar{K}_{sj}, \sigma^2\right) \tag{7.3}$$

In the case of equation (7.1-7.3), n is the number of data points, K_{sj} is each of the values of the data, σ^2 is the variance, and $\overline{K_s}$ is the mean of K_{sj}, γ is the stochastic constant, and $logN$ is the log-normal distribution given as:

$$fx(\sigma) = \frac{1}{x}\frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{\left(lnx - \bar{a}_j\right)}{2\sqrt{2}}\right] \tag{7.4}$$

7.2.2 The mean and variance of the specific storage

The mean for the specific storage S_s is given by:

$$\bar{S}_s = \frac{\sum_{j=0}^{n} S_{sj}}{n} \tag{7.5}$$

The variance is given by:

$$\sigma^2 = \frac{\sum_{j=0}^{n}\left(S_{sj} - \bar{S}_s\right)^2}{(n-1)} \tag{7.6}$$

where, $n =$ to the number of data points, $S_{sj} =$ each of the values of the data, and $\bar{S}_s =$ the mean of S_{sj}.

Now we make use of the log-normal distribution to convert the constant-coefficient S_s into distribution, we convert the parameter $\widehat{S_s}$ to:

$$\widehat{S_s} = \bar{S}_{sj} + \gamma logN\left(\bar{S}_{sj}, \sigma^2\right) \tag{7.7}$$

In the case of equation (8.5-8.7), n is the number of data points, S_{sj} is each of the values of the data, σ^2 is the variance, and $\overline{S_s}$ is the mean of S_{sj}, γ is the stochastic constant, and $logN$ is the log-normal distribution given as:

$$fx(\sigma) = \frac{1}{x}\frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{\left(lnx - \bar{a}_j\right)}{2\sqrt{2}}\right] \tag{7.8}$$

7.2.3 Stochastic 1-d saturated-unsaturated groundwater flow equation

Now introducing the stochastic approach to the 1-d saturated zone equation (3.8) we obtain the following:

$$\frac{\partial\psi}{\partial t} = \frac{\widehat{K}_s}{\hat{S}_s}\frac{\partial^2\psi}{\partial z^2} \tag{7.9}$$

The stochastic unsaturated zone equation is given by:

$$\begin{aligned}
\frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta t} = \frac{1}{-(\theta_s - \theta_0)\psi_b{}^{\lambda}\ \dfrac{\lambda}{\psi_i^{\lambda+1}}} \Bigg(& \widehat{K}_s \frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta z} \\
& + \widehat{K}_s \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{(\Delta z)^2} \\
& - \widehat{K}_s \frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z} \Bigg)
\end{aligned} \tag{7.10}$$

7.3. APPLICATION OF FRACTIONAL STOCHASTIC APPROACH

In previous chapters, new equations for modelling the subsurface flow of water in the saturated-unsaturated zone were derived using the Caputo, Caputo-Fabrizio, and the Atangana-Baleanu fractional derivatives. Above, the constant coefficients of the equation under consideration were converted into distributions. Now we combine the two approaches to convert the equations into a stochastic differential equation then perform a stability check for each equation where applicable.

7.3.1 Stochastic differential equation using the Caputo fractional derivative

In this section, numerical solutions for the 1-d saturated-unsaturated groundwater flow equation are obtained using three different approximation methods of ψ and replacing the time derivative of equation (7.9) with the Caputo fractional derivative. The equation below gives the stochastic differential equation in Caputo sense:

$$ {}_{0}^{C}D^{\alpha}\psi(z,t) = \frac{\widehat{K}_s}{\hat{S}_s}\frac{\partial^2\psi}{\partial z^2} \tag{7.11}$$

And

$$\begin{aligned} {}_{0}^{C}D^{\alpha}\psi(z,t) \; = \; & \frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\dfrac{\lambda}{\psi_i^{\lambda+1}}}\left(\widehat{K}_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n+1}-\psi_i^{n+1}}{\Delta z} \right. \\ & + \widehat{K}_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{(\Delta z)^2} \\ & \left. - \widehat{K}_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\dfrac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z} \right) \end{aligned} \tag{7.12}$$

7.3.1.1 Explicit forward Euler method

Applying the explicit forward Euler method on the right-hand side of equation (7.11) we obtain the following stochastic differential equation for the saturated zone:

$$\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\sum_{j=0}^{n}\left(\psi_i^{j+1}-\psi_i^{j}\right)\left((n-j)^{\alpha}-(n-j+1)^{\alpha}\right)=\frac{\widehat{K}_s}{\hat{S}_s}\left[\frac{\psi_{i+1}^{n}-2\psi_i^{n}+\psi_{i-1}^{n}}{(\Delta z)^2}\right] \tag{7.13}$$

Before we can perform stability analysis for the above numerical solution, we need to simplify the equation. To do that we make use the following parameters:

$$a = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)},$$

$$b = \frac{\widehat{K}_s}{\widehat{S}_s(\Delta z)^2}$$

$$R_{n,j}^{\alpha} = ((n-j)^{\alpha} - (n-j+1)^{\alpha})$$

Now we can rewrite the equation with parameter:

$$\sum_{j=0}^{n} \left(\psi_i^{j+1} - \psi_i^{j}\right) a R_{n,j}^{\alpha} = b(\psi_{i+1}^{n} - 2\,\psi_i^{n} + \psi_{i-1}^{n}) \qquad (7.13.1)$$

The above equation is simplified into:

$$(\psi_i^{n+1} - \psi_i^{n}) a R_{n,n}^{\alpha} + \sum_{j=0}^{n-1} \left(\psi_i^{j+1} - \psi_i^{j}\right) a R_{n,j}^{\alpha} = b({\psi_{i+1}}^{n} - 2{\psi_i}^{n} + {\psi_{i-1}}^{n}) \qquad (7.13.2)$$

Now we have this equation for a numerical scheme:

$$\psi_i^{n+1} a R_{n,n}^{\alpha} = \psi_i^{n}\left(a R_{n,n}^{\alpha} - 2b\right) + b({\psi_{i+1}}^{n} + {\psi_{i-1}}^{n}) - \sum_{j=0}^{n-1} \left(\psi_i^{j+1} - \psi_i^{j}\right) a R_{n,j}^{\alpha} \qquad (7.13.3)$$

For stability analysis, we assume that:

$$\psi_i^{n} = \sigma^{n} e^{i k_m z} \qquad (7.13.4)$$

Substituting the above into equation (7.13.4), we obtain the following:

$$\begin{aligned} &\sigma^{n+1} e^{i k_m z} a R_{n,n}^{\alpha} \\ &\qquad = \sigma^{n} e^{i k_m z}\left(a R_{n,n}^{\alpha} - 2b\right) + b e^{i k_m z}\left(\sigma^{n} e^{i k_m \Delta z} + \sigma^{n} e^{i k_m \Delta z}\right) \\ &\qquad - \sum_{j=0}^{n-1} \left(\sigma^{j+1} - \sigma^{j}\right) e^{i k_m z} a R_{n,j}^{\alpha} \end{aligned} \qquad (7.13.5)$$

If we simplify the above equation, we obtain the following:

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} - 2b + b\left(e^{ik_m\Delta z} + e^{-ik_m\Delta z}\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (7.13.6)$$

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} + 2b\left(\frac{e^{ik_m\Delta z} + e^{-ik_m\Delta z}}{2} - 1\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (7.13.7)$$

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} + 2b(\cos(k_m\Delta z) - 1)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (7.13.8)$$

Using the double angle for cos, we can simplify the equation into:

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} + 2b\left(1 - 2\sin^2\left(\frac{k_m\Delta z}{2}\right) - 1\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (7.13.9)$$

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} - 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (7.13.10)$$

If $n = 0$ the above equation becomes:

$$a\sigma^{0} R_{0,n}^{\alpha} = \sigma^{0}\left[aR_{0,n}^{\alpha} - 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)\right] \qquad (7.13.11)$$

$$\left|\frac{\sigma^1}{\sigma^0}\right| < 1 \rightarrow \left|\frac{\sigma^1}{\sigma^0}\right| = \left|\frac{aR_{0,n}^{\alpha} - 4b\sin^2\left(\frac{k_m\Delta z}{2}\right)}{aR_{0,n}^{\alpha}}\right| < 1 \qquad (7.13.12)$$

$$aR_{0,n}^{\alpha} - 4b\sin^2\left(\frac{k_m\Delta z}{2}\right) < aR_{0,n}^{\alpha} \qquad (7.13.13)$$

$$0 < 4b\sin^2\left(\frac{k_m\Delta z}{2}\right) \qquad (7.13.14)$$

$$0 < 4\frac{\widehat{K}_s}{\hat{S}_s(\Delta z)^2}\sin^2\left(\frac{k_m\Delta z}{2}\right) \qquad (7.13.15)$$

We assume that $\forall\, n > 0$:

$$\left|\frac{\sigma^{n}}{\sigma^{0}}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^{0}}\right| < 1$$

$$\sigma^{n+1} aR_{n,n}^{\alpha} = \sigma^{n}\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right] - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \qquad (7.13.16)$$

$$\sigma^{n+1} = \frac{\sigma^{n}\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right]}{aR_{n,n}^{\alpha}} - \frac{\sum_{j=0}^{n-1} \sigma^{n-j} aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}} \qquad (7.13.17)$$

$$|\sigma^{n+1}| \leq |\sigma^{n}| \left|\frac{\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right]}{aR_{n,n}^{\alpha}}\right| - \frac{\sum_{j=0}^{n-1} |\sigma^{n-j}| aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}} \qquad (7.13.18)$$

Using the recursive hypothesis:

$$|\sigma^{n+1}| < |\sigma^{0}| \left[\frac{aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right)}{aR_{n,n}^{\alpha}}\right] + \frac{\sum_{j=0}^{n-1} |\sigma^{0}| aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}} \qquad (7.13.19)$$

$$|\sigma^{n+1}| < |\sigma^{0}| \left[\frac{\left[aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right]}{aR_{n,n}^{\alpha}} + \frac{\sum_{j=0}^{n-1} aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}}\right] \qquad (7.13.20)$$

$$\frac{|\sigma^{n+1}|}{|\sigma^{0}|} < 1$$

$$\left|\frac{aR_{n,n}^{\alpha} - 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right) + \sum_{j=0}^{n-1} aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha}}\right| < 1 \qquad (7.13.21)$$

$$aR_{n,n}^{\alpha} - 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) + \sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < aR_{n,n}^{\alpha} \tag{7.13.22}$$

The stability condition is given as:

$$\sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{7.13.23}$$

7.3.1.2 Implicit backward Euler method

Applying the implicit backward Euler method on the right-hand side of equation (7.11), we obtain the following stochastic differential equation for the saturated zone:

$$\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)} \sum_{j=0}^{n} (\psi_i^{n+1} - \psi_i^{n})((n-j)^{\alpha} - (n-j+1)^{\alpha}) = \frac{\widehat{K}_s}{\hat{S}_s}\left[\frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{(\Delta z)^2}\right] \tag{7.14}$$

Before we can perform stability analysis for the above implicit backward Euler method, we need to simplify the equation. To do that, we use the following parameters:

$$a = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)},$$

$$b = \frac{\widehat{K}_s}{\hat{S}_s (\Delta z)^2}$$

$$R_{n,j}^{\alpha} = ((n-j)^{\alpha} - (n-j+1)^{\alpha})$$

Now we can rewrite the equation with parameter:

$$\sum_{j=0}^{n} (\psi_i^{n+1} - \psi_i^{n}) aR_{n,j}^{\alpha} = b(\,\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) \tag{7.14.1}$$

$$(\psi_i^{n+1} - \psi_i^{n}) aR_{n,n}^{\alpha} + \sum_{j=0}^{n-1} \left(\psi_i^{j+1} - \psi_i^{j}\right) aR_{n,j}^{\alpha} = b(\,\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) \tag{7.14.2}$$

Now we have the above equation as a numerical scheme:

$$(\psi_i^{n+1}-\psi_i^n)aR_{n,n}^{\alpha}=b(\,\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1})-\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^{j}\right)aR_{n,j}^{\alpha} \qquad (7.14.3)$$

Which can be further simplified:

$$-\psi_i^n aR_{n,n}^{\alpha}=-\psi_i^{n+1}\left(aR_{n,j}^{\alpha}+2b\right)+b(\,\psi_{i+1}^{n+1}+\psi_{i-1}^{n+1})-\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^{j}\right)aR_{n,j}^{\alpha} \qquad (7.14.4)$$

For stability analysis, we assume that:

$$\psi_i^n=\sigma^n e^{ik_m z} \qquad (7.14.5)$$

Substituting the above into equation (8.14.4), we obtain the following:

$$\begin{aligned}-\sigma^n e^{ik_m z}aR_{n,n}^{\alpha}&\\ &=-\sigma^{n+1}e^{ik_m z}\left(aR_{n,n}^{\alpha}+2b\right)\\ &+b\left(\sigma^{n+1}e^{ik_m(z+\Delta z)}+\sigma^{n+1}e^{ik_m(z-\Delta z)}\right)\\ &-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)e^{ik_m z}aR_{n,j}^{\alpha} \qquad (7.14.6)\end{aligned}$$

$$\begin{aligned}-\sigma^n e^{ik_m z}aR_{n,n}^{\alpha}&\\ &=-\sigma^{n+1}e^{ik_m z}\left(aR_{n,n}^{\alpha}+2b\right)+b\sigma^{n+1}e^{ik_m z}\left(e^{ik_m\Delta z}+e^{-ik_m\Delta z}\right)\\ &-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)e^{ik_m z}aR_{n,j}^{\alpha} \qquad (7.14.7)\end{aligned}$$

$$\begin{aligned}-\sigma^n e^{ik_m z}aR_{n,n}^{\alpha}&\\ &=-\sigma^{n+1}e^{ik_m z}\left(aR_{n,n}^{\alpha}+2b-b\left(e^{ik_m\Delta z}+e^{-ik_m\Delta z}\right)\right)\\ &-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)e^{ik_m z}aR_{n,j}^{\alpha} \qquad (7.14.8)\end{aligned}$$

$$\begin{aligned}-\sigma^{n} e^{ikmz} aR_{n,n}^{\alpha} \\ &= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - 2b\left(\frac{e^{ik_m \Delta z} + e^{-ik_m \Delta z}}{2}\right)\right) \\ &- \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}\end{aligned} \tag{7.14.9}$$

$$\begin{aligned}-\sigma^{n} e^{ik_m z} aR_{n,n}^{\alpha} \\ &= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - 2b(\cos(k_m \Delta z)\right) \\ &- \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}\end{aligned} \tag{7.14.10}$$

$$\begin{aligned}-\sigma^{n} e^{ik_m z} aR_{n,n}^{\alpha} \\ &= -\sigma^{n+1} e^{ik_m z}\left(aR_{n,n}^{\alpha} + 2b - 2b\left(1 - 2\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right)\right) \\ &- \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) e^{ik_m z} aR_{n,j}^{\alpha}\end{aligned} \tag{7.14.11}$$

$$-\sigma^{n} aR_{n,n}^{\alpha} = -\sigma^{n+1}\left(aR_{n,n}^{\alpha} + 4\mathrm{b}\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right) - \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right) aR_{n,j}^{\alpha} \tag{7.14.12}$$

If $n = 0$ then the above equation becomes:

$$\sigma^{0}\left(aR_{0,n}^{\alpha} + 4\mathrm{b}\sin^{2}\left(\frac{k_m \Delta z}{2}\right)\right) = \sigma^{0} aR_{0,n}^{\alpha} \tag{7.14.13}$$

$$\left|\frac{\sigma^{1}}{\sigma^{0}}\right| < 1 \rightarrow \left|\frac{\sigma^{1}}{\sigma^{0}}\right| = \left|\frac{aR_{0,n}^{\alpha}}{aR_{0,n}^{\alpha} + 4b\sin^{2}\left(\frac{k_m \Delta z}{2}\right)}\right| < 1 \tag{7.14.14}$$

$$aR^{\alpha}_{0,n} < aR^{\alpha}_{0,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{7.14.15}$$

$$0 < 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{7.14.16}$$

We assume that $\forall\, n > 0$:

$$\left|\frac{\sigma^n}{\sigma^0}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^0}\right| < 1$$

$$\sigma^{n+1}\left(aR^{\alpha}_{n,n} + 4\mathrm{b}\sin^2\left(\frac{k_m \Delta z}{2}\right)\right) + \sum_{j=0}^{n-1}\left(\sigma^{j+1} - \sigma^{j}\right)aR^{\alpha}_{n,j} = \sigma^n aR^{\alpha}_{n,n} \tag{7.14.17}$$

$$\sigma^{n+1} = \frac{\sigma^n aR^{\alpha}_{n,n}}{\left[aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)\right]} - \frac{\sum_{j=0}^{n-1} \sigma^{n-j} aR^{\alpha}_{n,j}}{aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)} \tag{7.14.18}$$

$$|\sigma^{n+1}| \leq |\sigma^n| \left|\frac{aR^{\alpha}_{n,n}}{\left[aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)\right]}\right| - \frac{\sum_{j=0}^{n-1} \left|\sigma^{n-j}\right| aR^{\alpha}_{n,j}}{aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)} \tag{7.14.19}$$

Using the recursive hypothesis:

$$|\sigma^{n+1}| < |\sigma^0| \left|\frac{aR^{\alpha}_{n,n}}{aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)}\right| + \frac{\sum_{j=0}^{n-1} |\sigma^0| aR^{\alpha}_{n,j}}{aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)} \tag{7.14.20}$$

$$|\sigma^{n+1}| < |\sigma^0| \left[\frac{aR^{\alpha}_{n,n}}{aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)} + \frac{\sum_{j=0}^{n-1} aR^{\alpha}_{n,j}}{aR^{\alpha}_{n,n} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)}\right] \tag{7.14.21}$$

$$\frac{|\sigma^{n+1}|}{|\sigma^0|} < 1$$

$$\left|\frac{aR_{n,n}^{\alpha} + \sum_{j=0}^{n-1} aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right)}\right| < 1 \tag{7.14.22}$$

$$aR_{n,n}^{\alpha} + \sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < aR_{n,n}^{\alpha} + 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{7.14.23}$$

The stability condition is given as:

$$\sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < 4b\sin^2\left(\frac{k_m \Delta z}{2}\right) \tag{7.14.24}$$

8.3.1.3 Implicit Crank-Nicolson method

We obtain other numerical solution by applying the implicit Crank- Nicolson discretization on the right-hand side of equation (7.11):

$$\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\left(\psi_i^{n+1} - \psi_i^{n}\right)((n-j)^{\alpha} - (n-j+1)^{\alpha})$$
$$= \frac{\widehat{K}_s}{\widehat{S}_s}\left[\frac{\psi_{i+1}^{n+1} - 2\,\psi_i^{n+1} + \,\psi_{i-1}^{n+1}}{2(\Delta z)^2} + \frac{\psi_{i+1}^{n} - 2\,\psi_i^{n} + \,\psi_{i-1}^{n}}{2(\Delta z)^2}\right] \tag{7.15}$$

Before we can perform stability analysis for the above equation, we need to simplify the above equation by using the following parameters:

$$a = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)},$$

$$c = \frac{\widehat{K}_s}{\widehat{S}_s 2(\Delta z)^2}$$

$$R_{n,j}^{\alpha} = ((n-j)^{\alpha} - (n-j+1)^{\alpha})$$

Now we can rewrite the equation as:

$$aR_{n,j}^{\alpha} = c[(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) + (\psi_{i+1}^{n} - 2\psi_i^{n} + \,\psi_{i-1}^{n})] \tag{7.15.1}$$

$$\sum_{j=0}^{n}(\psi_i^{n+1} - \psi_i^{n})aR_{n,j}^{\alpha} = c\left[\left(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}\right) + \left(\psi_{i+1}^{n} - 2\psi_i^{n} + \,\psi_{i-1}^{n}\right)\right] \tag{7.15.2}$$

$$\left(\psi_i^{n+1}-\psi_i^n\right)aR_{n,n}^{\alpha}+\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^j\right)aR_{n,j}^{\alpha}$$
$$=c\left[\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right)+\left(\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n\right)\right] \tag{7.15.3}$$

$$\psi_i^{n+1}aR_{n,n}^{\alpha}=c\left(\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}\right)+c\left(\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n\right)+\psi_i^n aR_{n,n}^{\alpha}$$
$$-\sum_{j=0}^{n-1}\left(\psi_i^{j+1}-\psi_i^j\right)aR_{n,j}^{\alpha} \tag{7.15.4}$$

For stability analysis, we assume that:

$$\psi_i^n=\sigma^n e^{ik_m z} \tag{7.15.5}$$

$$\sigma^{n+1}e^{ik_m z}aR_{n,n}^{\alpha}$$
$$=c\left(\sigma^{n+1}e^{ik_m(z+\Delta z)}-2\sigma^{n+1}e^{ik_m z}+\sigma^{n+1}e^{ik_m(z-\Delta z)}\right)$$
$$+c\left(\sigma^n e^{ik_m(z+\Delta z)}-2\sigma^n e^{ik_m z}+\sigma^n e^{ik_m(z-\Delta z)}\right)+\sigma^n e^{ik_m z}aR_{n,n}^{\alpha}$$
$$-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^j\right)e^{ik_m z}aR_{n,j}^{\alpha} \tag{7.15.6}$$

$$\sigma^{n+1}e^{ik_m z}aR_{n,n}^{\alpha}-c\sigma^{n+1}e^{ik_m z}\left(e^{ik_m\Delta z}-2+e^{-ik_m\Delta z}\right)$$
$$=c\sigma^n e^{ik_m z}\left(e^{ik_m\Delta z}+\sigma^n e^{-ik_m\Delta z}\right)+\sigma^n e^{ik_m z}\left(aR_{n,n}^{\alpha}-2c\right)$$
$$-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^j\right)e^{ik_m z}aR_{n,j}^{\alpha} \tag{7.15.7}$$

$$\sigma^{n+1}aR_{n,n}^{\alpha}-c\sigma^{n+1}\left(e^{ik_m\Delta z}-2+e^{-ik_m\Delta z}\right)$$
$$=c\sigma^n\left(e^{ik_m\Delta z}+e^{-ik_m\Delta z}\right)+\sigma^n\left(aR_{n,n}^{\alpha}-2c\right)$$
$$-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^j\right)aR_{n,j}^{\alpha} \tag{7.15.8}$$

$$\sigma^{n+1}\left(aR^{\alpha}_{n,n}-2c\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right)$$
$$=\sigma^{n}\left(aR^{\alpha}_{n,n}+2c\left(\frac{e^{ik_m\Delta z}+e^{-ik_m\Delta z}}{2}-1\right)\right) \tag{7.15.9}$$
$$-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)aR^{\alpha}_{n,j}$$

$$\sigma^{n+1}\left(aR^{\alpha}_{n,n}-2c(cos(k_m\Delta z)-1)\right)$$
$$=\sigma^{n}\left(aR^{\alpha}_{n,n}+2c(cos(k_m\Delta z)-1)\right)-\sum_{j=0}^{n-1}\left(\sigma^{j+1}-\sigma^{j}\right)aR^{\alpha}_{n,j} \tag{7.15.10}$$

If $n=0$ the above equation becomes:

$$\sigma^{1}\left(aR^{\alpha}_{n,n}-2c(cos(k_m\Delta z)-1)\right)=\sigma^{0}\left(aR^{\alpha}_{n,n}+2c(cos(k_m\Delta z)-1)\right) \tag{7.15.11}$$

$$\left|\frac{\sigma^1}{\sigma^0}\right|<1\rightarrow\left|\frac{\sigma^1}{\sigma^0}\right|=\left|\frac{aR^{\alpha}_{n,n}+2c(cos(k_m\Delta z)-1)}{aR^{\alpha}_{n,n}-2c(cos(k_m\Delta z)-1)}\right|<1 \tag{7.15.12}$$

$$aR^{\alpha}_{n,n}+2c(cos(ik_m\Delta z)-1)<aR^{\alpha}_{n,n}-2c(cos(k_m\Delta z)-1) \tag{7.15.13}$$

$$0<-4c(cos(k_m\Delta z)-1) \tag{7.15.14}$$

$$4ccos(k_m\Delta z)<4c \tag{7.15.15}$$

$$cos(k_m\Delta z)<1 \tag{7.15.16}$$

We can write 1 as $cos2\pi$ such that:

$$cos(k_m\Delta z)<cos2\pi \tag{7.15.17}$$

$$ik_m\Delta z<2\pi \tag{7.15.18}$$

Therefore:

$$k_m<\frac{2\pi}{\Delta z} \tag{7.15.19}$$

We assume that $\forall\, n>0$:

$$\left|\frac{\sigma^{n}}{\sigma^{0}}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^{0}}\right| < 1$$

$$\begin{aligned}&\sigma^{n+1}\left(aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)\right) \\ &\qquad = \sigma^{n}\left(aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)\right) - \sum_{j=0}^{n-1}\sigma^{n-j}aR_{n,j}^{\alpha}\end{aligned} \tag{7.15.20}$$

$$\sigma^{n+1} = \sigma^{n}\left[\frac{aR_{n,n}^{\alpha} + 2c(cos(ik_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right] - \frac{\sum_{j=0}^{n-1}\sigma^{n-j}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} \tag{7.15.21}$$

$$|\sigma^{n+1}| \leq |\sigma^{n}|\left|\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right| + \frac{\sum_{j=0}^{n-1}|\sigma^{n-j}|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} \tag{7.15.22}$$

Using the recursive hypothesis:

$$|\sigma^{n+1}| < |\sigma^{0}|\left|\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right| + \frac{\sum_{j=0}^{n-1}|\sigma^{0}|aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} \tag{7.15.23}$$

$$|\sigma^{n+1}| < |\sigma^{0}|\left[\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1)}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)} + \frac{\sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right] \tag{7.15.24}$$

$$\left|\frac{aR_{n,n}^{\alpha} + 2c(cos(k_m\Delta z) - 1) + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha}}{aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1)}\right| < 1 \tag{7.15.25}$$

$$aR_{n,n}^{\alpha} + 2c(cos(km\Delta z) - 1) + \sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < aR_{n,n}^{\alpha} - 2c(cos(k_m\Delta z) - 1) \tag{7.15.26}$$

$$\sum_{j=0}^{n-1}aR_{n,j}^{\alpha} < -4c(cos(k_m\Delta z) - 1) \tag{7.15.27}$$

The stability condition is given as:

$$4ccos(k_m \Delta z) + \sum_{j=0}^{n-1} aR_{n,j}^{\alpha} < 4c \tag{7.15.28}$$

The above stability condition can be simplified into:

$$4ccos(k_m \Delta z) < 4c - \sum_{j=0}^{n-1} aR_{n,j}^{\alpha} \tag{7.15.29}$$

$$k_m \Delta z < cos^{-1}\left(1 - \frac{\sum_{j=0}^{n-1} aR_{n,j}^{\alpha}}{4c}\right) \tag{7.15.30}$$

8.3.1.4 New model of the unsaturated zone in Caputo sense

Now we consider equation (7.12) to provide the fractional stochastic numerical solution for the unsaturated zone in Caputo sense:

$$\begin{aligned}
&\frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)}\left(\psi_i^{n+1} - \psi_i^{n}\right)\left((n-j)^{\alpha} - (n-j+1)^{\alpha}\right) \\
&= \frac{1}{-(\theta_s - \theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}}\left(K_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^{n+1} - \psi_i^{n+1}}{\Delta z} \right. \\
&+ K_s \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}}{(\Delta z)^2} \\
&\left. - K_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z} \right)
\end{aligned} \tag{7.16}$$

To make further simplification on the above equation, let us say:

$$\wp = \left(-(\theta_s - \theta_0){\psi_b}^{\lambda}\,\frac{\lambda}{\psi_i^{\lambda+1}}\right)$$

$$\mathscr{a} = \frac{(\Delta t)^{-\alpha}}{\Gamma\,(2-\alpha)} \quad ,$$

$$\mathscr{b} = \frac{K_s}{\Delta z} \quad ,$$

$$c = \frac{1}{\Delta z} \quad ,$$

$$\mathscr{d} = \frac{1}{(\Delta z)^2} ,$$

$$R_{n,j}^{\alpha} = ((n-j)^{\alpha} - (n-j+1)^{\alpha})$$

Now we rewrite the equation as:

$$\begin{aligned}
&\sum_{j=0}^{n} (\psi_i^{n+1} - \psi_i^{n})\mathscr{a} R_{n,j}^{\alpha} \\
&\qquad = \frac{1}{\mathscr{p}}\left(\mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \right) c(\,\psi_{i+1}^{n+1} - \psi_i^{n+1}) \right. \\
&\qquad + K_s \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \times \mathscr{d}(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) \\
&\qquad \left. - \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda} \right) \right)
\end{aligned} \tag{7.16.1}$$

The above numerical scheme is highly non-linear; therefore, it is not possible to perform a stability analysis.

7.3.2 Stochastic differential equation using the Caputo-Fabrizio fractional derivative

In this section, new numerical solutions for the 1-d saturated-unsaturated groundwater flow equation using the Caputo Fabrizio fractional derivative and the stochastic approach will be produced. Firstly, let us recall some interesting properties of the Caputo-Fabrizio operator. The C-F operator can capture fading memory, and it provides a waiting distribution-based exponential decay law.

7.3.2.1 New model of the saturated zone using in Caputo-Fabrizio sense

Introducing the Caputo-Fabrizio fractional derivative to equation (7.9), we obtain the following stochastic differential equation for the saturated zone:

$$ {}_{0}^{CF}D^{\alpha}\psi(z,t) = \frac{\widehat{K}_s}{\hat{S}_s}\frac{\partial^2\psi}{\partial z^2} \tag{7.17}$$

where:

$$ {}_{0}^{CF}D^{\alpha}\psi(z,t) = f\big(z,t,\psi(z,\alpha)\big) \tag{7.17.1}$$

We apply the Caputo-Fabrizio integral to the above equation:

$$\psi(z,t) = \psi(z,0) + \frac{1-\alpha}{M(\alpha)}f\big(z,t,\psi(z,t)\big) + \frac{\alpha}{M(\alpha)}\int_0^t f\big(z,\tau\ \ \psi(z,\tau)\big)d\tau \tag{7.17.2}$$

When $t = t_{n+1}$Now we have the following equation:

$$\psi(z_i,t_{n+1}) = \psi(z_i,0) + \frac{1-\alpha}{M(\alpha)}f\big(z_i,t_n,\psi(z_i,t_n)\big) + \frac{\alpha}{M(\alpha)}\int_0^{t_{n+1}} f\big(z_i,\tau\ \ \psi(z_i,\tau)\big)d\tau \tag{7.17.3}$$

When $t = t_n$:

$$\psi(z_i,t_n) = \psi(z_i,0) + \frac{1-\alpha}{M(\alpha)}f\big(z_i,t_{n-1},\psi(z_i,t_{n-1})\big) + \frac{\alpha}{M(\alpha)}\int_0^{t_n} f\big(z_i,\tau\ \ \psi(z_i,\tau)\big)\,d\tau \tag{7.17.4}$$

Subtracting the above solution for t_n from t_{n+1} , we obtain the following:

$$\begin{aligned}\psi_i^{n+1} - \psi_i^n &= \frac{1-\alpha}{M(\alpha)}[f(z_i,t_n,\psi_i^n) - f(z_i,t_{n-1},\psi_i^{n-1})] \\ &\quad + \frac{\alpha}{M(\alpha)}\int_{t_n}^{t_{n+1}} f\big(z_i,\tau,\psi(x_i,\tau)\big)d\tau\end{aligned} \tag{7.17.5}$$

$$\begin{aligned}\psi_i^{n+1} &= \psi_i^n + \frac{1-\alpha}{M(\alpha)}[f(z_i,t_n,\psi_i^n) - f(z_i,t_{n-1},\psi_i^{n-1})] \\ &\quad + \frac{\alpha}{M(\alpha)}\left[\frac{3}{2}\Delta t f(z_i,t_n,\psi_i^n) - \frac{\Delta t}{2}f(z_i,t_{n-1},\psi_i^{n-1})\right]\end{aligned} \tag{7.17.6}$$

where:

$$F\left(z_i, t_j, \psi_i^n\right) = \frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} \tag{7.17.7}$$

$$F\left(z_i, t_{j-1}, \psi_i^{n-1}\right) = \frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2} \tag{7.17.8}$$

We substitute the above equation (7.17.7) and (7.17.8) into equation (7.17.6) to obtain the final numerical solution for the saturated groundwater flow equation:

$$\begin{aligned} \psi_i^{n+1} = \psi_i^n + \frac{1-\alpha}{M(\alpha)} & \left[\frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} - \frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2}\right] \\ & + \frac{\alpha}{M(\alpha)}\left[\frac{3\Delta t}{2}\left(\frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2}\right)\right. \\ & \left. - \frac{\Delta t}{2}\left(\frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2}\right)\right] \end{aligned} \tag{7.17.9}$$

To check the stability of the above numerical solution, we assume a Fourier expansion in the space of:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{7.17.10}$$

We also assume the following parameters:

$$a = \frac{1-\alpha}{M(\alpha)}$$

$$b = \frac{\alpha}{M(\alpha)}$$

$$c = \frac{\widehat{K}_s}{\hat{S}_s(\Delta z)^2}$$

$$d = \frac{\Delta t}{2}$$

Then our equation becomes:

$$\begin{aligned}\sigma^{n+1}e^{ik_m z} = \sigma^n e^{ik_m z} &\\ + a\big[c\big(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\big) &\\ - c\big(\sigma^{n-1}e^{ik_m(z+\Delta z)} - 2\sigma^{n-1}e^{ik_m z} + \sigma^{n-1}e^{ik_m(z-\Delta z)}\big)\big] &\\ + b\big[3dc\big(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\big) &\\ - dc\big(\sigma^{n-1}e^{ik_m(z+\Delta z)} - 2\sigma^{n-1}e^{ik_m z} + \sigma^{n-1}e^{ik_m(z-\Delta z)}\big)\big] &\end{aligned} \tag{7.17.11}$$

$$\begin{aligned}\sigma^{n+1}e^{ik_m z} = \sigma^n e^{ik_m z} &\\ + a\big[\sigma^n e^{ik_m z}c\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big) &\\ - \sigma^{n-1}e^{ik_m z}c\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big] &\\ + b\big[3\sigma^n e^{ik_m z}dc\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big) &\\ - \sigma^{n-1}e^{ik_m z}dc\big(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\big)\big] &\end{aligned} \tag{7.17.12}$$

$$\begin{aligned}\sigma^{n+1} = \sigma^n + a[\sigma^n c(2cos(k_m\Delta z) - 2) - \sigma^{n-1}c(2cos(k_m\Delta z) - 2)] &\\ + b[3\sigma^n dc(2cos(k_m\Delta z) - 2) - \sigma^{n-1}dc(2cos(k_m\Delta z) - 2)] &\end{aligned} \tag{7.17.13}$$

$$\begin{aligned}\frac{\sigma^{n+1}}{\sigma^n} = a\left[2c(cos(k_m\Delta z) - 1) - 2c\frac{\sigma^{n-1}}{\sigma_n}(cos(k_m\Delta z) - 1)\right] &\\ + b\left[3dc(2cos(k_m\Delta z) - 2) - dc\frac{\sigma^{n-1}}{\sigma^n}(2cos(k_m\Delta z) - 2)\right] &\end{aligned} \tag{7.17.14}$$

$$\frac{\sigma^{n+1}}{\sigma^n} = 2ac(cos(k_m\Delta z) - 1)\left[1 - \frac{\sigma^{n-1}}{\sigma^n}\right] + bdc(2cos(k_m\Delta z) - 2)\left[3 - \frac{\sigma^{n-1}}{\sigma^n}\right] \tag{7.17.15}$$

For $n = 0$, the equation becomes:

$$\left|\frac{\sigma^n}{\sigma^0}\right| = |2ac(cos(k_m\Delta z) - 1) + 3bdc(2cos(k_m\Delta z) - 2)| \tag{7.17.16}$$

The stability is given by:

$$|2ac(cos(k_m\Delta z) - 1) + 3bdc(2cos(k_m\Delta z) - 2)| < 1 \tag{7.17.17}$$

Now we assume that $\forall\, n > 0$:

$$\left|\frac{\sigma^n}{\sigma^0}\right| < 1$$

Using the above assumption, we will proof that:

$$\left|\frac{\sigma^{n+1}}{\sigma^{0}}\right| < 1$$

The equation then becomes:

$$\begin{aligned}\sigma^{n+1} = \sigma^{n} + a[\sigma^{n}2c(cos(k_m\Delta z) - 1) - \sigma^{n-1}2c(cos(k_m\Delta z) - 1)] \\ + b[\sigma^{n}3dc(2cos(k_m\Delta z) - 2) - \sigma^{n-1}dc(2cos(k_m\Delta z) - 2)]\end{aligned} \tag{7.17.18}$$

Which implies:

$$\begin{aligned}|\sigma^{n+1}| = |\sigma^{n}|\big(1 + 2ac(cos(k_m\Delta z) - 1)\big) - |\sigma^{n-1}|2ac(cos(k_m\Delta z) - 1) \\ + |\sigma^{n}|3bdc(2cos(k_m\Delta z) - 2) - |\sigma^{n-1}|dbc(2cos(k_m\Delta z) - 2)\end{aligned} \tag{7.17.19}$$

Using recursive hypothesis, the equation becomes:

$$\begin{aligned}|\sigma^{n+1}| < |\sigma^{0}|\big(1 + 2ac(cos(k_m\Delta z) - 1)\big) - |\sigma^{0}|2ac(cos(k_m\Delta z) - 1) \\ + |\sigma^{0}|3bdc(2cos(k_m\Delta z) - 2) - |\sigma^{0}|dbc(2cos(k_m\Delta z) - 2)\end{aligned} \tag{7.17.20}$$

$$\begin{aligned}|\sigma^{n+1}| < |\sigma^{0}|\big[\big(1 + 2ac(cos(k_m\Delta z) - 1)\big) - 2ac(cos(k_m\Delta z) - 1) \\ + 3bdc(2cos(k_m\Delta z) - 2) - dbc(2cos(k_m\Delta z) - 2)\big]\end{aligned} \tag{7.17.21}$$

$$|\sigma^{n+1}| < |\sigma_0|[1 + 2bdc(2cos(k_m\Delta z) - 2)] \tag{7.17.22}$$

$$1 + 4bdc(cos(k_m\Delta z) - 1) < 1 \tag{7.17.23}$$

$$4bdccos(k_m\Delta z) - 4bdc \ < 0 \tag{7.17.24}$$

The stability condition is given by:

$$4bdccos(k_m\Delta z) < 4bdc \tag{7.17.25}$$

$$cos(k_m\Delta z) < 1 \tag{7.17.26}$$

8.3.2.2 New model for the unsaturated zone in Caputo-Fabrizio sense

Now, we replace the time derivative on the unsaturated zone equation (7.9) with the Caputo-Fabrizio fractional derivative, such that:

$$
{}^{CF}_{\ 0}D^{\alpha}\psi(z,t) = \frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}}\left(\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n+1}-\psi_i^{n+1}}{\Delta z}\right.
$$

$$
+\widehat{K}_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{(\Delta z)^2}
$$

$$
\left.-\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\right) \tag{7.18}
$$

where

$$
{}^{CF}_{\ 0}D^{\alpha}\psi(z,t) = f\big(z,t,\psi(z,\alpha)\big) \tag{7.18.1}
$$

We apply the Caputo-Fabrizio integral to the above equation:

$$
\psi(z,t) = \psi(z,0) + \frac{1-\alpha}{M(\alpha)}f\big(z,t,\psi(z,t)\big) + \frac{\alpha}{M(\alpha)}\int_0^t f\big(z,\tau\ \ \psi(z,\tau)\big)d\tau \tag{7.18.2}
$$

When $t = t_{n+1}$Now we have the following equation:

$$
\psi(z_i,t_{n+1}) = \psi(z_i,0) + \frac{1-\alpha}{M(\alpha)}f\left(z_i,t_j,\psi\big(z_i,t_j\big)\right) + \frac{\alpha}{M(\alpha)}\int_0^{t_{n+1}} f\big(z_i,\tau\ \ \psi(z_i,\tau)\big)d\tau \tag{7.18.3}
$$

When $t = t_n$:

$$
\psi(z_i,t_n) = \psi(z_i,0) + \frac{1-\alpha}{M(\alpha)}f\big(z_i,t_{n-1},\psi(z_i,t_{n-1})\big) + \frac{\alpha}{M(\alpha)}\int_0^{t_n} f\big(z_i,\tau\ \ \psi(z_i,\tau)\big)\,d\tau \tag{7.18.4}
$$

Subtracting the above solution for t_n from t_{n+1} , we obtain the following:

$$
\psi_i^{n+1}-\psi_i^{n} = \frac{1-\alpha}{M(\alpha)}\left[f(z_i,t_n,\psi_i^n)-f(z_i,t_{n-1},\psi_i^{n-1})\right] + \frac{\alpha}{M(\alpha)}\int_{t_n}^{t_{n+1}} f\big(z_i,\tau,\psi(x_i,\tau)\big)d\tau \tag{7.18.5}
$$

$$
\begin{aligned}
\psi_i^{n+1} = \psi_i^n &+ \frac{1-\alpha}{M(\alpha)}\left[f(z_i,t_n,\psi_i^n) - f(z_i,t_{n-1},\psi_i^{n-1})\right] \\
&+ \frac{\alpha}{M(\alpha)}\left[\frac{3}{2}\Delta t f(z_i,t_n,\psi_i^n) - \frac{\Delta t}{2} f(z_i,t_{n-1},\psi_i^{n-1})\right]
\end{aligned}
\tag{7.18.6}
$$

where:

$$
\begin{aligned}
F(z_i,t_n,\psi_i^n) = \frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\dfrac{\lambda}{\psi^{\lambda+1}}}\Bigg(&\widehat{K}_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^n-\psi_i^n}{\Delta z} \\
&+ \widehat{K}_s\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} \\
&- \widehat{K}_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z}\Bigg)
\end{aligned}
\tag{7.18.7}
$$

$$
\begin{aligned}
&F(z_i,t_{n-1},\psi_i^{n-1}) \\
&= \frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\,\dfrac{\lambda}{\psi_i^{\lambda+1}}}\Bigg(\widehat{K}_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n-1}-\psi_i^{n-1}}{\Delta z} \\
&+ \widehat{K}_s\left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2} - \widehat{K}_s\frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}}{\Delta z}\Bigg)
\end{aligned}
\tag{7.18.8}
$$

We substitute the above equation (7.18.7) and (7.18.8) into equation (7.18.6) to obtain the final numerical solution for the saturated groundwater flow equation:

$$\psi_i^{n+1} = \psi_i^n + \frac{1-\alpha}{M(\alpha)} \Bigg[\frac{1}{-(\theta_s - \theta_0)\psi_b{}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}} \Bigg(\hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^n - \psi_i^n}{\Delta z}$$

$$+ \hat{K}_s \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} - \hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z} \Bigg)$$

$$- \frac{1}{-(\theta_s - \theta_0)\psi_b{}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}} \Bigg(\hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^{n-1} - \psi_i^{n-1}}{\Delta z}$$

$$+ \hat{K}_s \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2} - \hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z} \Bigg) \Bigg]$$

$$+ \frac{\alpha}{M(\alpha)} \Bigg[\frac{3\Delta t}{2} \frac{1}{-(\theta_s - \theta_0)\psi_b{}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}} \Bigg(\hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^n - \psi_i^n}{\Delta z}$$

$$+ \hat{K}_s \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} - \hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z} \Bigg)$$

$$- \frac{\Delta t}{2} \frac{1}{-(\theta_s - \theta_0)\psi_b{}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}} \Bigg(\hat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^{n-1} - \psi_i^{n-1}}{\Delta z}$$

$$+ \hat{K}_s \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}}{(\Delta z)^2}$$

$$- \widehat{K}_s \frac{\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}}{\Delta z} \Bigg)\Bigg] \tag{7.18.9}$$

Now, let:

$$\mathscr{p} = \left(-(\theta_s - \theta_0){\psi_b}^{\lambda} \ \frac{\lambda}{\psi_i^{\lambda+1}}\right)$$

$$\mathscr{a} = \frac{1-\alpha}{M(\alpha)}$$

$$\mathscr{b} = \frac{\alpha}{M(\alpha)}$$

$$c = \frac{\widehat{K}_s}{\Delta z}$$

$$\mathscr{d} = \frac{1}{\Delta z}$$

$$e = \frac{1}{(\Delta z)^2}$$

$$\mathscr{f} = \frac{3\Delta t}{2}$$

$$\mathscr{g} = \frac{\Delta t}{2}$$

Using the above substitutions, equation (7.18.9) becomes:

$$
\begin{aligned}
\psi_i^{n+1} = \psi_i^n + \mathscr{a}\Bigg[&\frac{1}{\mathscr{p}}\Bigg(c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\Bigg)\mathscr{d}(\psi_{i+1}^n - \psi_i^n) + \widehat{K}_s\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \\
&\times e(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) - c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\Bigg)\Bigg) \\
&-\frac{1}{\mathscr{p}}\Bigg(c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}\Bigg)\mathscr{d}(\psi_{i+1}^{n-1} - \psi_i^{n-1}) + \widehat{K}_s\left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda} \\
&\times e(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}) - c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\Bigg)\Bigg)\Bigg] \\
&+ b\Bigg[\Bigg(\mathscr{f}\frac{1}{\mathscr{p}}\Bigg(c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\Bigg)\mathscr{d}(\psi_{i+1}^n - \psi_i^n) + \widehat{K}_s\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \\
&\times e(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) - c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\Bigg)\Bigg)\Bigg) \\
&- \mathscr{g}\frac{1}{\mathscr{p}}\Bigg(c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}\Bigg)\mathscr{d}(\psi_{i+1}^{n-1} - \psi_i^{n-1}) + \widehat{K}_s\left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda} \\
&\times e(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1}) \\
&- c\Bigg(\left(\frac{\psi_b}{\psi_{i+1}^{n-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{n-1}}\right)^{2+3\lambda}\Bigg)\Bigg)\Bigg]
\end{aligned}
\tag{7.18.10}
$$

7.3.3 Stochastic differential equation using the Atangana-Baleanu fractional derivative

Now we make use of the recently introduced Atangana-Baleanu fractional derivative combining it with the stochastic approach so to produce a numerical solution with a statistical setting. Let us recall on interesting properties of this operator. The A-B fractional derivative is more natural, and it is the first of its kind to show the cross-over performance amid the stretched exponential law for earlier time and power-law for a latter time; patterns like this can be found in nature. Therefore, applying this operator to this equation while help in capturing a more complex environment.

7.3.3.1 New model of the saturated zone in A-B sense

Using the Atangana-Baleanu fractional derivative on the saturated zone equation (7.9), we obtain the following fractional stochastic equation:

$$ {}_{0}^{AB}D^{\alpha}\psi(z,t) = \frac{\widehat{K}_s}{\hat{S}_s}\frac{\partial^2\psi}{\partial z^2} \tag{7.19}$$

where

$$ {}_{0}^{AB}D^{\alpha}\psi(z,t) = f\big(z,t,\psi(z,\alpha)\big) \tag{7.19.1}$$

Then the AB integral is written as:

$$\psi(z_i,t) - \psi(z_i,0) = \frac{(1-\alpha)}{B(\alpha)} f\big(z,t,\psi(z,t)\big) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f\big(z_i,\tau,\psi(z_i,\tau)\big)\,(t-\tau)^{\alpha-1}d\tau \tag{7.19.2}$$

At point $t_{n+1}, n = 0, 1, 2, 3, 4 \ldots\ldots\ldots\ldots$

$$\psi(z_i,t_{n+1}) - \psi(z_i,0) = \frac{(1-\alpha)}{\mathrm{B}(\alpha)} f\big(z,t,\psi(z,t)\big) + \frac{\alpha}{\mathrm{B}(\alpha)\times\Gamma(\alpha)}\int_0^{t_{n+1}} f\big(z_i,\tau,\psi(z_i,\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau \tag{7.19.3}$$

$$\begin{aligned}\psi(z_i,t_{n+1}) - \psi(z_i,0) &= \frac{(1-\alpha)}{\mathrm{B}(\alpha)} f\big(z,t,\psi(z,t)\big) \\ &+ \frac{\alpha(\Delta t)^{\alpha}}{\mathrm{B}(\alpha)\times\Gamma(2+\alpha)}\sum_{j=0}^{n}\Big[f\big(z_i,t_j,\psi_i^j\big)\big((n-j+1)^{\alpha}(n-j+2+\alpha) \\ &- (n-j)^{\alpha}(n-j+2+2\alpha)\big) \\ &- f\big(z_i,t_{j-1},\psi_i^{j-1}\big)\big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)\Big]\end{aligned} \tag{7.19.4}$$

where:

$$F\left(z_i, t_j, \psi_i^j\right) = \frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{(\Delta z)^2} \tag{7.19.4.1}$$

$$F\left(z_i, t_{j-1}, \psi_i^{j-1}\right) = \frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}}{(\Delta z)^2} \tag{7.19.4.2}$$

The final numerical solution for the saturated groundwater flow equation is given as:

$$\begin{aligned}
&\psi(z_i, t_{n+1}) - \psi(z_i, 0) \\
&\quad = \frac{(1-\alpha)}{\mathrm{B}(\alpha)} \left(\frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{(\Delta z)^2} \right) \\
&\quad + \frac{\alpha(\Delta t)^\alpha}{\mathrm{B}(\alpha)\Gamma(2+\alpha)} \sum_{j=0}^{n} \left[\frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{(\Delta z)^2} \left((n-j+1)^\alpha (n-j+2+\alpha) \right. \right. \\
&\quad \left. - (n-j)^\alpha (n-j+2+2\alpha) \right) \\
&\quad - \frac{\widehat{K}_s}{\hat{S}_s} \frac{\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}}{(\Delta z)^2} \left((n-j+1)^{\alpha+1} \right. \\
&\quad \left. \left. - (n-j)^\alpha (n-j+1+\alpha) \right) \right]
\end{aligned} \tag{7.19.5}$$

We also assume the following parameters:

$$a = \frac{(1-\alpha)}{\mathrm{B}(\alpha)}$$

$$b = \frac{\widehat{K}_s}{\hat{S}_s (\Delta z)^2}$$

$$c = \frac{\alpha(\Delta t)^\alpha}{\mathrm{B}(\alpha)\Gamma(2+\alpha)}$$

$$R_{n.j}^\alpha = \left((n-j+1)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha) \right)$$

$$\delta_{n,j}^\alpha = \left((n-j+1)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right)$$

Then our equation is written as:

$$\begin{aligned}\psi_i^{n+1} - \psi_i^n &= a[b(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)] \\ &\quad + c\sum_{j=0}^{n}\left[b\left(\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j\right)R_{n,j}^\alpha - b\left(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\right)\delta_{n,j}^\alpha\right]\end{aligned} \tag{7.19.6}$$

$$\begin{aligned}\psi_i^{n+1} - \psi_i^n &= a[b(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)] \\ &\quad + c\left[b(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n)R_{n,n}^\alpha - b(\psi_{i+1}^{n-1} - 2\psi_i^{n-1} + \psi_{i-1}^{n-1})\delta_{n,n}^\alpha\right] \\ &\quad + c\sum_{j=0}^{n-1}\left[b\left(\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j\right)R_{n,j}^\alpha - b\left(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\right)\delta_{n,j}^\alpha\right]\end{aligned} \tag{7.19.7}$$

To check the stability of the above numerical solution, we assume a Fourier expansion in the space of:

$$\psi_i^n = \sigma^n e^{ik_m z} \tag{7.19.8}$$

Using the above equation, equation 7.19.7 becomes:

$$\begin{aligned}&\sigma^{n+1}e^{ik_m z} - \sigma^n e^{ik_m z} \\ &\quad = a\left[b\left(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\right)\right] \\ &\quad + c\left[b\left(\sigma^n e^{ik_m(z+\Delta z)} - 2\sigma^n e^{ik_m z} + \sigma^n e^{ik_m(z-\Delta z)}\right)R_{n,n}^\alpha\right. \\ &\quad \left. - b\left(\sigma^{n-1} e^{ik_m(z+\Delta z)} - 2\sigma^{n-1} e^{ik_m z} + \sigma^{n-1} e^{ik_m(z-\Delta z)}\right)\delta_{n,n}^\alpha\right] \\ &\quad + c\sum_{j=0}^{n-1}\left[b\left(\sigma^j e^{ik_m(z+\Delta z)} - 2\sigma^j e^{ik_m z} + \sigma^j e^{ik_m(z-\Delta z)}\right)R_{n,j}^\alpha\right. \\ &\quad \left. - b\left(\sigma^{j-1} e^{ik_m(z+\Delta z)} - 2\sigma^{j-1} e^{ik_m z} + \sigma^{j-1} e^{ik_m(z-\Delta z)}\right)\delta_{n,j}^\alpha\right]\end{aligned} \tag{7.19.9}$$

$$\begin{aligned}&e^{ik_m z}(\sigma^{n+1} - \sigma^n) \\ &\quad = a\left[b\sigma^n e^{ik_m z}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right] \\ &\quad + c\left[R_{n,n}^\alpha b\sigma^n e^{ik_m z}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right. \\ &\quad \left. - \delta_{n,n}^\alpha b\sigma^{n-1} e^{ik_m z}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right] \\ &\quad + c\sum_{j=0}^{n-1}\left[R_{n,j}^\alpha b\sigma^j e^{ik_m z}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right. \\ &\quad \left. - \delta_{n,j}^\alpha b\sigma^{j-1} e^{ik_m z}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right]\end{aligned} \tag{7.19.10}$$

$$\begin{aligned}\sigma^{n+1} - \sigma^{n} = {} & a\left[b\sigma^{n}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right] \\ & + c\left[R_{n,n}^{\alpha} b\sigma^{n}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right. \\ & \left. - \delta_{n,n}^{\alpha} b\sigma^{n-1}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right] \\ & + c\sum_{j=0}^{n-1}\left[R_{n,j}^{\alpha} b\sigma^{j}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right. \\ & \left. - \delta_{n,j}^{\alpha} b\sigma^{j-1}\left(e^{ik_m\Delta z} - 2 + e^{-ik_m\Delta z}\right)\right]\end{aligned} \tag{7.19.11}$$

$$\begin{aligned}\sigma^{n+1} - \sigma^{n} = {} & 2ab\sigma^{n}\left(\frac{e^{ik_m\Delta z} + e^{-ik_m\Delta z}}{2} - 1\right) \\ & + c\left[2R_{n,n}^{\alpha} b\sigma^{n}\left(\frac{e^{ik_m\Delta z} + e^{-ik_m\Delta z}}{2} - 1\right)\right. \\ & \left. - 2\delta_{n,n}^{\alpha} b\sigma^{n-1}\left(\frac{e^{ik_m\Delta z} + e^{-ik_m\Delta z}}{2} - 1\right)\right] \\ & + c\sum_{j=0}^{n-1}\left[2R_{n,j}^{\alpha} b\sigma^{j}\left(\frac{e^{ik_m\Delta z} + e^{-ik_m\Delta z}}{2} - 1\right)\right. \\ & \left. - 2\delta_{n,j}^{\alpha} b\sigma^{j-1}\left(\frac{e^{ik_m\Delta z} + e^{-ik_m\Delta z}}{2} - 1\right)\right]\end{aligned} \tag{7.19.12}$$

Considering the definition for a hyperbolic cosine, we can rewrite the above equation (7.19.12) as:

$$\begin{aligned}\sigma^{n+1} - \sigma^{n} = {} & 2ab\sigma^{n}(cos(k_m\Delta z) - 1) \\ & + c\left[2R_{n,n}^{\alpha} b\sigma^{n}(cos(k_m\Delta z) - 1)\right. \\ & \left. - 2\delta_{n,n}^{\alpha} b\sigma^{n-1}(cos(k_m\Delta z) - 1)\right] \\ & + c\sum_{j=0}^{n-1}\left[2R_{n,j}^{\alpha} b\sigma^{j}(cos(k_m\Delta z) - 1)\right. \\ & \left. - 2\delta_{n,j}^{\alpha} b\sigma^{j-1}(cos(k_m\Delta z) - 1)\right]\end{aligned} \tag{7.19.13}$$

If n=0 the equation above becomes:

$$\sigma_1 - \sigma^0 = 2ab\sigma^0(cos(k_m\Delta z) - 1) + c\left[2R_{n,n}^{\alpha} b\sigma^0(cos(k_m\Delta z) - 1)\right] \tag{7.19.14}$$

$$\sigma_1 = \sigma^0\left[1 + 2ab(cos(k_m\Delta z) - 1) + R_{n,n}^{\alpha} cb(cos(k_m\Delta z) - 1)\right] \tag{7.19.15}$$

$$\left|\frac{\sigma^1}{\sigma^0}\right| < 1$$

$$\left|\frac{1}{1 + 2ab(cos(k_m\Delta z) - 1) + R_{n,n}^{\alpha} cb(cos(k_m\Delta z) - 1)}\right| < 1 \tag{7.19.16}$$

The denominator of the above equation will always be greater than the numerator; therefore, the A-B fractional derivative for the 1-d saturated groundwater flow equation is unconditionally stable.

Now we assume $\forall n > 0$ to prove that:

$$\left|\frac{\sigma^n}{\sigma^0}\right| < 1$$

$$\begin{aligned}\sigma^{n+1} = \sigma^n\big[&1 + 2ab(cos(k_m\Delta z) - 1) + R_{n,n}^{\alpha}cb(cos(k_m\Delta z) - 1)\big]\\ &- 2\delta_{n,n}^{\alpha}cb\sigma^{n-1}(cos(k_m\Delta z) - 1)\\ &+ c\sum_{j=0}^{n-1}\big[2R_{n,j}^{\alpha}b\sigma^j(cos(k_m\Delta z) - 1)\\ &- 2\delta_{n,j}^{\alpha}b\sigma^{j-1}(cos(k_m\Delta z) - 1)\big]\end{aligned} \tag{7.19.17}$$

We let:

$$A = \big[1 + 2ab(cos(k_m\Delta z) - 1) + R_{n,n}^{\alpha}cb(cos(k_m\Delta z) - 1)\big]$$

$$B = 2\delta_{n,n}^{\alpha}cb(cos(k_m\Delta z) - 1)$$

$$A_1 = 2R_{n,j}^{\alpha}b(cos(k_m\Delta z) - 1)$$

$$A_2 = 2\delta_{n,j}^{\alpha}b(cos(k_m\Delta z) - 1)$$

Then equation (8.23) becomes:

$$\sigma^{n+1} = \sigma^n A - \sigma^{n-1}B + c\sum_{j=0}^{n-1}\big[\sigma^j A_1 - \sigma^{j-1}A_2\big] \tag{7.19.18}$$

$$|\sigma^{n+1}| \le |\sigma^n||A| - |\sigma^{n-1}||B| + c\sum_{j=0}^{n-1}\big[|\sigma^j||A_1| - |\sigma^{j-1}||A_2|\big] \tag{7.19.19}$$

$$|\sigma^{n+1}| \le |\sigma^0|\left(|A| - |B| + c\sum_{j=0}^{n-1}(|A_1| - |A_2|)\right) \tag{7.19.20}$$

$$\frac{|\sigma^{n+1}|}{|\sigma^0|} \le |A| - |B| + c\sum_{j=0}^{n-1}(|A_1| - |A_2|) \tag{7.19.21}$$

The stability condition is therefore given by:

$$\left|\frac{1}{|A|-|B|+c\sum_{j=0}^{n-1}(|A_1|-|A_2|)}\right| \leq 1 \tag{7.19.22}$$

7.3.3.1 New model of the unsaturated zone in A-B sense

Now, using the AB fractional derivative and stochastic the new model for the unsaturated zone is given by the following equation:

$$\begin{aligned} &{}^{ABC}_{\ \ 0}D^{\alpha}\psi(z,t) \\ &= \frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}}\left(\hat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{n+1}-\psi_i^{n+1}}{\Delta z}\right. \\ &\left.+\hat{K}_s\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^{n+1}-2\psi_i^{n+1}+\psi_{i-1}^{n+1}}{(\Delta z)^2}-\hat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{n+1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{n+1}}\right)^{2+3\lambda}}{\Delta z}\right)\end{aligned} \tag{7.20}$$

where

$${}^{ABC}_{\ \ 0}D^{\alpha}\psi(z,t)=F\big(z,t,\psi(z,\alpha)\big) \tag{7.20.1}$$

Then the AB integral is given as:

$$\begin{aligned}&\psi(z_i,t)-\psi(z_i,0)\\ &\qquad=\frac{(1-\alpha)}{B(\alpha)}f\big(z,t,\psi(z,t)\big)\\ &\qquad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f\big(z_i,\tau,\psi(z_i,\tau)\big)(t-\tau)^{\alpha-1}d\tau\ \ (8.20.2)\end{aligned} \tag{7.20.2}$$

At point $t_{n+1}, n=0,1,2,3,4\ldots\ldots\ldots\ldots$

$$\begin{aligned}
&\psi(z_i, t_{n+1}) - \psi(z_i, 0) \\
&\qquad = \frac{(1-\alpha)}{B(\alpha)} f\big(z, t, \psi(z,t)\big) \\
&\qquad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} F\big(z_i, \tau, \psi(z_i, \tau)\big)(t_{n+1} - \tau)^{\alpha-1} d\tau
\end{aligned} \tag{7.20.3}$$

$$\begin{aligned}
&\psi(z_i, t_{n+1}) - \psi(z_i, 0) \\
&\qquad = \frac{(1-\alpha)}{B(\alpha)} f\big(z, t, \psi(z,t)\big) \\
&\qquad + \frac{(\Delta t)^{\alpha}}{B(\alpha)\Gamma(2+\alpha)} \sum_{j=0}^{n} \Big[F\big(z_i, t_j, \psi_i^j\big)\big((n-j+1)^{\alpha}(n-j+2+\alpha) \\
&\qquad - (n-j)^{\alpha}(n-j+2+2\alpha)\big) \\
&\qquad - F\big(z_i, t_{j-1}, \psi_i^{j-1}\big)\big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)\Big]
\end{aligned} \tag{7.20.4}$$

where:

$$\begin{aligned}
F\big(z_i, t_j, \psi_i^j\big) = \frac{1}{-(\theta_s - \theta_0){\psi_b}^{\lambda}\ \dfrac{\lambda}{\psi_i^{\lambda+1}}} \Bigg(& \widehat{K}_s \frac{\left(\dfrac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^j - \psi_i^j}{\Delta z} \\
& + \widehat{K}_s \left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{(\Delta z)^2} \\
& - \widehat{K}_s \frac{\left(\dfrac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}}{\Delta z} \Bigg)
\end{aligned} \tag{7.20.4.1}$$

$$F\left(z_i, t_{j-1}, \psi_i^{j-1}\right) = \frac{1}{-(\theta_s - \theta_0){\psi_b}^{\lambda}\ \dfrac{\lambda}{\psi_i^{\lambda+1}}} \left(\widehat{K}_s \frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z} \frac{\psi_{i+1}^{j-1} - \psi_i^{j-1}}{\Delta z} + \widehat{K}_s \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda} \times \frac{\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}}{(\Delta z)^2} - \widehat{K}_s \frac{\left(\dfrac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\dfrac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z} \right) \quad (7.20.4.2)$$

The final numerical solution for the unsaturated groundwater flow equation is given as:

$$(\psi_i^{n+1}-\psi_i^0) = \frac{(1-\alpha)}{B(\alpha)}\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\frac{\lambda}{\psi_i^{\lambda+1}}}\left(\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^n-\psi_i^n}{\Delta z}\right.$$

$$\left.+\widehat{K}_s\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n}{(\Delta z)^2}-\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}}{\Delta z}\right)$$

$$+\frac{(\Delta t)^{\alpha}}{\Gamma(2+\alpha)}\sum_{j=0}^{n}\left[\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}}\left(\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^j-\psi_i^j}{\Delta z}\right.\right.$$

$$+\widehat{K}_s\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^j-2\psi_i^j+\psi_{i-1}^j}{(\Delta z)^2}$$

$$\left.-\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}}{\Delta z}\right)\big((n-j+1)^{\alpha}(n-j+2+\alpha)$$

$$-(n-j)^{\alpha}(n-j+2+2\alpha)\big)$$

$$-\frac{1}{-(\theta_s-\theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}}}\left(\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z}\frac{\psi_{i+1}^{j-1}-\psi_i^{j-1}}{\Delta z}\right.$$

$$+\widehat{K}_s\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\times\frac{\psi_{i+1}^{j-1}-2\psi_i^{j-1}+\psi_{i-1}^{j-1}}{(\Delta z)^2}$$

$$\left.-\widehat{K}_s\frac{\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda}-\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}}{\Delta z}\right)\big((n-j+1)^{\alpha+1}$$

$$\left. - (n-j)^{\alpha}(n-j+1+\alpha)\big) \right] \tag{7.20.5}$$

For simplicity of the above equation let:

$$\mathscr{p} = \left(-(\theta_s - \theta_0){\psi_b}^{\lambda}\ \frac{\lambda}{\psi_i^{\lambda+1}} \right)$$

$$\mathscr{a} = \frac{(1-\alpha)}{B(\alpha)}$$

$$\mathscr{b} = \frac{\widehat{K}_s}{\Delta z}$$

$$c = \frac{1}{(\Delta z)^2}$$

$$\mathscr{d} = \frac{1}{\Delta z}$$

$$e = \frac{(\Delta t)^{\alpha}}{\Gamma\,(2-\alpha)}$$

$$R_{n.j}^{1,\alpha} = \big((n-j+1)^{\alpha}(n-j+2+\alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\big)$$

$$R_{n,j}^{2,\alpha} = \big((n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\big)$$

$$\begin{aligned}
\psi_i^{n+1} - \psi_i^0 = \mathscr{a}\frac{1}{\mathscr{p}}\Bigg(& \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\right)\mathscr{d}(\psi_{i+1}^n - \psi_i^n) + K_s\left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda} \\
& \times c(\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n) - \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^n}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^n}\right)^{2+3\lambda}\right)\Bigg) \\
& + e\sum_{j=0}^{n}\Bigg[\frac{1}{\mathscr{p}}\Bigg(\mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}\right)\mathscr{d}\left(\psi_{i+1}^j - \psi_i^j\right) + K_s\left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda} \\
& \times c\left(\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j\right) - \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^j}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^j}\right)^{2+3\lambda}\right)\Bigg)R_{n.j}^{1,\alpha} \\
& - \frac{1}{\mathscr{p}}\Bigg(\mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)\mathscr{d}\left(\psi_{i+1}^{j-1} - \psi_i^{j-1}\right) + K_s\left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda} \\
& \times c\left(\psi_{i+1}^{j-1} - 2\psi_i^{j-1} + \psi_{i-1}^{j-1}\right) \\
& - \mathscr{b}\left(\left(\frac{\psi_b}{\psi_{i+1}^{j-1}}\right)^{2+3\lambda} - \left(\frac{\psi_b}{\psi_i^{j-1}}\right)^{2+3\lambda}\right)\Bigg)R_{n,j}^{2,\alpha}\Bigg] \qquad (7.20.6)
\end{aligned}$$

CHAPTER 8
SUMUDU, LAPLACE TRANSFORMS AND THEIR APPLICATION TO GROUNDWATER

8.1. INTRODUCTION

Historically, the genesis of the integral transforms can be traced back to the 1970s from the work of P.S Laplace (Maitama & Zhao, 2010). The Laplace transform is a potent tool for explaining ordinary and partial differential equations; it transforms the ordinary differential equation into a simple algebraic expression. This can simply be transformed back into the original solution (Schiff, 1999). A fascinating mathematical modelling technique called the transfer function uses the Laplace transform to make sense of complex processes. A transfer function is a mathematical relationship existing between the numerical input and the subsequent output in a dynamic system. For a linear, time-invariant system, the transfer function is defined as a ratio of the Laplace transform of the output variable $Y(s) = \mathcal{L}\{y(t)\}$, to the Laplace transform of the input variable $F(s) = \mathcal{L}\{f(t)\}$ with all zero initial conditions (Zwart, 2004). Together with the Bode plots, the transfer function can show us the frequency response of a linear time-invariant system. This system could be any system that experiences a change in behavior due to a change in frequency, an example of such a system is the flow of groundwater. The theory surrounding the transfer functions of linear time-invariant systems has been available for many years and was predominantly used in connection with electrical and mechanical systems described in continuous time, until recently (Pollock, 2001; Cook, 1979). There exists another integral transform called the Sumudu transform. This transform was established by G.K. Watugala in 1993 to give solutions to control engineering problems and differential equations. It is theoretical a dual of the Laplace transform (Panchal *et al.*,2016; Atangana& Kilicman, 2013). This new transform challenges the Laplace transform in solving problems. The main advantage of the Sumudu transform is that it can provide solutions to problems resorting to a new frequency domain, due to its ability to preserve the same scales and physical unit properties (Tchvenche & Mbare, 2009). Other advantages of the Sumudu transform over the Laplace transform includes the resemblance between the function $f(x,t)$ in the (x,t) domain and the subsequent function $\mathcal{S}\{f(x,t)\}$ in the (u,v) domain, the equality of $f(x,t)$ and $\mathcal{S}\{f(x,t)\}$ for constant functions and the limit of $f(t)$ as t approach zero is equal to the limit of $F(u)$ as u approach zero (Jarad & Tas, 2012). For these set of functions, the Sumudu transform is defined as (Kilicman & Eltayeb, 2012):

$$\mathcal{A} = \left\{ f(t) \colon \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{t/\tau_j}, \ \ if\ t \in (-1)^j \times [0, \infty) \right\}$$

By:

$$F(u) = \mathcal{S}[f(t)] = \int_0^\infty e^{-t} f(ut) dt, \qquad u \in (-\tau_1, \tau_2) \tag{8.1}$$

If $F(u)$ is the Sumudu transform of $f(t)$, then the Sumudu transform of the n^{th} derivative is as follows:

$$\mathcal{S}\left[\frac{d^n f(t)}{dt^n}\right] = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k \left.\frac{d^n f(t)}{dt^n}\right|_{t=0}\right] \tag{8.1.1}$$

And the Sumudu of the first derivative is given by:

$$\mathcal{S}\left[\frac{df(t)}{dt}\right] = \frac{1}{u}\left[F(u) - f(0) - u \left.\frac{df(t)}{dt}\right|_{t=0}\right] \tag{8.1.2}$$

`And the Sumudu of the second derivative is given by:

$$\mathcal{S}\left[\frac{d^2 f(t)}{dt^2}\right] = \frac{1}{u^2}\left[F(u) - f(0) - u \left.\frac{df(t)}{dt}\right|_{t=0}\right] \tag{8.1.3}$$

In this section, we illustrate the application of the Laplace transform and the Sumudu transform in groundwater. We apply these transforms to the 1-d saturated-unsaturated classical equation and on the equation with Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives. The aim of this analysis is to find out if we could get better results when using the Sumudu transform than the Laplace transform. Note that the application will be on the saturated part of the equation since the unsaturated part of the equation is highly nonlinear. We view our partial differential equation as a transfer function from time to space. One of the importance of a transfer function is that once you have a transfer function of a system you can estimate the output of that system for any given input. Solutions for each case are detailed below. However, we shall note the theory under analysis in this section is not yet fully developed. This will open doors for new investigation within the field of partial differential equations and their applications in real world problems.

8.2. APPLICATION OF THE LAPLACE TRANSFORM TO THE SATURATED GROUNDWATER EQUATION

Let us recall the 1-d saturated groundwater flow equation (3.8) presented in chapter 3. Applying the Laplace transform in time, we find:

$$\mathcal{L}\left(\frac{\partial \psi}{\partial t}\right) = \mathcal{L}\left(\frac{K_s}{S_s}\frac{\partial^2 \psi}{\partial z^2}\right) \tag{8.2}$$

$$s\tilde{\psi} - \psi(0) = \mathcal{L}\left(\frac{K_s}{S_s}\frac{\partial^2 \psi}{\partial z^2}\right) \tag{8.2.1}$$

$$s\tilde{\psi} - \psi(0) = \frac{K_s}{S_s}\frac{\partial^2 \tilde{\psi}}{\partial z^2} \tag{8.2.2}$$

$$\frac{s\tilde{\psi} - \psi(0)}{\frac{K_s}{S_s}\frac{\partial^2 \tilde{\psi}}{\partial z^2}} = \frac{1}{s} = R(s) \tag{8.2.3}$$

Now we find the Laplace transform in space:

$$\mathcal{L}\left(\frac{\partial \psi}{\partial t}\right) = \frac{K_s}{S_s}\left(s^2\psi(z,s) - s\psi(z,0) + \psi'(z,0)\right) \tag{8.2.4}$$

$$\frac{\frac{\partial \tilde{\psi}}{\partial t}\frac{S_s}{K_s} + s\tilde{\psi}(z,s) + \psi'(z,0)}{\psi(z,s)} = s^2 \tag{8.2.5}$$

$$\frac{\psi(z,s)}{\frac{\partial \tilde{\psi}}{\partial t}\frac{S_s}{K_s} + s\tilde{\psi}(z,0) + \psi'(z,0)} = \frac{1}{s^2} = R(s) \tag{8.2.6}$$

For Caputo case we have:

$${}_0^C D_t^\alpha \psi(z,t) = \frac{K_s}{S_s}\frac{\partial^2 \psi(z,t)}{\partial z^2} \tag{8.2.7}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\partial \psi(z,t)}{\partial t}(t-\tau)^{-\alpha} d\tau = \frac{K_s}{S_s}\frac{\partial^2 \psi(z,t)}{\partial z^2} \tag{8.2.8}$$

Applying the Laplace transform on both sides with respect to t we have:

$$s^{\alpha}\tilde{\psi}(z,s) - s^{\alpha-1}\psi(z,s) = \frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2} \tag{8.2.9}$$

$$\frac{s^{\alpha}\tilde{\psi}(z,s) - s^{\alpha-1}\psi(z,s)}{\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2}} = \frac{1}{s^{\alpha}} = R^{\alpha}(s) \tag{8.2.10}$$

For the Laplace transform with respect to space, we get the same solution like the one above.

Now, with Caputo-Fabrizio, we have:

$${}_{0}^{CF}D_t^{\alpha}\psi(z,t) = \frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.2.11}$$

$$\frac{M(\alpha)}{1-\alpha}\int_0^t \frac{d\psi(z,\tau)}{d\tau} exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau = \frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.2.12}$$

$$\frac{M(\alpha)}{1-\alpha}\frac{d\psi(z,t)}{dt} * exp\left[-\frac{\alpha}{1-\alpha}t\right] = \frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.2.13}$$

Applying the Laplace transform on either side of equation (8.2.13) with respect to time we get:

$$\frac{M(\alpha)}{1-\alpha}\left(s\tilde{\psi}(s) - \psi(0)\right)\frac{1}{s+\frac{\alpha}{1-\alpha}} = \frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2} \tag{8.2.14}$$

$$s\tilde{\psi}(s) - \psi(0) = \left(s+\frac{\alpha}{1-\alpha}\right)\frac{1-\alpha}{M(\alpha)}\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2} \tag{8.2.15}$$

$$\tilde{\psi}(s) = \frac{\psi(0)}{s} + \frac{\left(s+\frac{\alpha}{1-\alpha}\right)}{s}\frac{1-\alpha}{sM(\alpha)}\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2} \tag{8.2.16}$$

We find the same solution for the Laplace transform in space.

Now, with the Atangana-Baleanu derivative, we have:

$${}_{0}^{ABC}D_t^{\alpha}\psi(z,t) = \frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.2.17}$$

$$\frac{AB(\alpha)}{1-\alpha}\int_0^t \frac{d\psi(z,\tau)}{d\tau} E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right] d\tau = \frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.2.18}$$

Applying the Laplace transform, we obtain:

$$\frac{AB(\alpha)}{1-\alpha}\left(s\tilde{\psi}(z,s)-\psi(z,0)\right)\frac{s^{\alpha-1}}{s^\alpha+\frac{\alpha}{1-\alpha}} = \frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2} \tag{8.2.19}$$

$$\frac{s\tilde{\psi}(z,s)-\psi(z,0)}{\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,s)}{\partial z^2}} = \frac{1-\alpha}{AB(\alpha)}\frac{s^\alpha+\frac{\alpha}{1-\alpha}}{s^{\alpha-1}} = R(s) \tag{8.2.20}$$

We obtain the same solution for the Laplace transform in space.

8.3. APPLICATION OF THE SUMUDU TRANSFORM TO THE SATURATED GROUNDWATER EQUATION

Let us recall the 1-d saturated groundwater flow equation (3.8) presented in chapter 3. Applying the Sumudu transform in time, we find:

$$\mathcal{S}\left(\frac{\partial\psi}{\partial t}\right) = \mathcal{S}\left(\frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2}\right) \tag{8.3}$$

$$\frac{\mathcal{S}(\psi)-\psi(z,0)}{p} = \frac{K_s}{S_s}\mathcal{S}\left(\frac{\partial^2\psi}{\partial z^2}\right) \tag{8.3.1}$$

$$\frac{\mathcal{S}(\psi)-\psi(z,0)}{\frac{K_s}{S_s}\mathcal{S}\left(\frac{\partial^2\psi}{\partial z^2}\right)} = p = R(p) \tag{8.3.2}$$

Now we find the Sumudu transform in space:

$$\mathcal{S}_z\left(\frac{\partial\psi}{\partial t}\right) = \mathcal{S}_z\left(\frac{K_s}{S_s}\frac{\partial^2\psi}{\partial z^2}\right) \tag{8.3.3}$$

$$\frac{\partial\,\tilde{\psi}}{\partial t} = \frac{K_s}{S_s}\,\mathcal{S}_z\left(\frac{\partial^2\psi}{\partial z^2}\right) \tag{8.3.4}$$

$$\frac{\partial\tilde{\psi}}{\partial t} = \frac{K_s}{S_s}\frac{\tilde{\psi}(z,p)-\psi(z,0)}{p^2} - \frac{\psi'(z,0)}{p} \tag{8.3.5}$$

$$\frac{K_s}{S_s}\frac{\tilde{\psi}(z,p)-\psi(z,0)}{\frac{\partial\tilde{\psi}}{\partial t}}-\psi'(z,0)p=p^2=R(p) \tag{8.3.6}$$

For Caputo case we have:

$${}_{0}^{C}D_t^{\alpha}\psi(z,t)=\frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.3.7}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\partial\psi(z,t)}{\partial t}(t-\tau)^{-\alpha}d\tau=\frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.3.8}$$

Applying the Sumudu transform on both sides with respect to t we have:

$$\frac{\mathcal{S}\psi(z,p)-\psi(z,0)}{p^{\alpha}}=\frac{K_s}{S_s}\mathcal{S}\left(\frac{\partial^2\psi(z,p)}{\partial z^2}\right) \tag{8.3.9}$$

$$\frac{\tilde{\psi}(z,p)-\psi(z,0)}{\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,p)}{\partial z^2}}=p^{\alpha}=R(p) \tag{8.3.10}$$

For Sumudu transform with respect to space, we get the same solution as the one above.

Now, with Caputo-Fabrizio we have:

$${}_{0}^{CF}D_t^{\alpha}\psi(z,t)=\frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.3.11}$$

$$\frac{M(\alpha)}{1-\alpha}\int_0^t\frac{d\psi(z,\tau)}{d\tau}exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau=\frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.3.12}$$

Applying the Sumudu transform on both sides of equation (8.3.12) with respect to time we get:

$$\frac{M(\alpha)}{1-\alpha}\mathcal{S}\left(\int_0^t\frac{d\psi(z,\tau)}{d\tau}exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau\right)=\frac{K_s}{S_s}\mathcal{S}\left(\frac{\partial^2\psi(z,p)}{\partial z^2}\right) \tag{8.3.13}$$

$$\left(\mathcal{S}(\psi)-\psi(z,0)\right)\frac{M(\alpha)}{\alpha p+1-\alpha}=\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,p)}{\partial z^2} \tag{8.3.14}$$

$$\frac{\left(\tilde{\psi}(z,p)-\psi(z,0)\right)M(\alpha)}{\alpha p+1-\alpha}=\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,p)}{\partial z^2} \tag{8.3.15}$$

$$\frac{\tilde{\psi}(z,p)-\psi(z,0)}{\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,p)}{\partial z^2}}=\frac{\alpha p+1-\alpha}{M(\alpha)}=R(p) \tag{8.3.16}$$

We find the same solution for the Sumudu transform in space. Now, with the Atangana-Baleanu derivative, we have:

$${}^{ABC}_{\ \ 0}D_t^{\alpha}\psi(z,t)=\frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.3.17}$$

$$\frac{AB(\alpha)}{1-\alpha}\int_0^t\frac{d\psi(z,\tau)}{d\tau}E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right]d\tau=\frac{K_s}{S_s}\frac{\partial^2\psi(z,t)}{\partial z^2} \tag{8.3.18}$$

Applying the Sumudu transform with respect to time, we obtain:

$$\frac{AB(\alpha)}{1-\alpha}\mathcal{S}\left(\int_0^t\frac{d\psi(z,\tau)}{d\tau}E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right]d\tau\right)=\frac{K_s}{S_s}\mathcal{S}\left(\frac{\partial^2\psi(z,t)}{\partial z^2}\right) \tag{8.3.19}$$

$$\frac{\left(\tilde{\psi}(z,p)-\psi(z,0)\right)AB(\alpha)}{1-\alpha+\alpha p^\alpha}=\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,p)}{\partial z^2} \tag{8.3.20}$$

$$\frac{\tilde{\psi}(z,p)-\psi(z,0)}{\frac{K_s}{S_s}\frac{\partial^2\tilde{\psi}(z,p)}{\partial z^2}}=\frac{1-\alpha+\alpha p^\alpha}{AB(\alpha)}=R(p) \tag{8.3.21}$$

We obtain the same solution for Sumudu transform in space.

8.4. BODE PLOTS OF THE LAPLACE AND SUMUDU TRANSFORM

A Bode plot is a graph that shows the system's frequency response. It was initially considered by Hendrick Wade Bode in the 1930s. It combines two logarithmic plots, one expressing the magnitude and the other, the phase shift of a system with respect to a given input frequency (York, 2009). The x axis displays frequency, whereas the y axis displays magnitude and phase angle (Summer, 2004). The Bode plot is used to test and analyse filters of a system. Two types of filters are recognised, a high pass filter allows passage of signals with frequencies higher than the cut-off frequency and restricts signals with frequencies lower than the cut-off frequency. The second filter is a low pass filter, it permits signals with a frequency less than a selected cut-off frequency to pass and restricts signals with frequencies greater than the cut-off frequency. In the next few pages, we give the subsequent Bode plots for the Laplace and Sumudu transform with respect to time and space as well with Caputo, Caputo-Fabrizio and Atangana-Baleanu derivatives. Analysis and comparison between the Bode plots is also presented. The following plots were produced using the MATLAB software

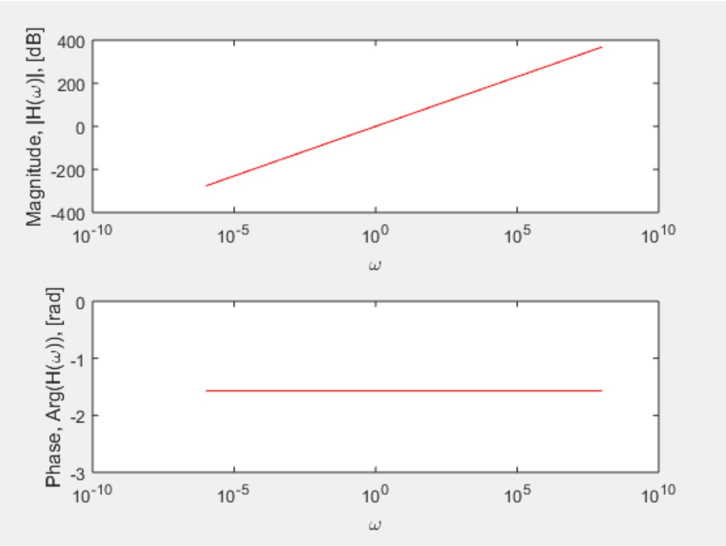

Figure 6: Transfer function of the model with the classical equation using the Laplace transform in time indicates that the system behaves like a high pass filter

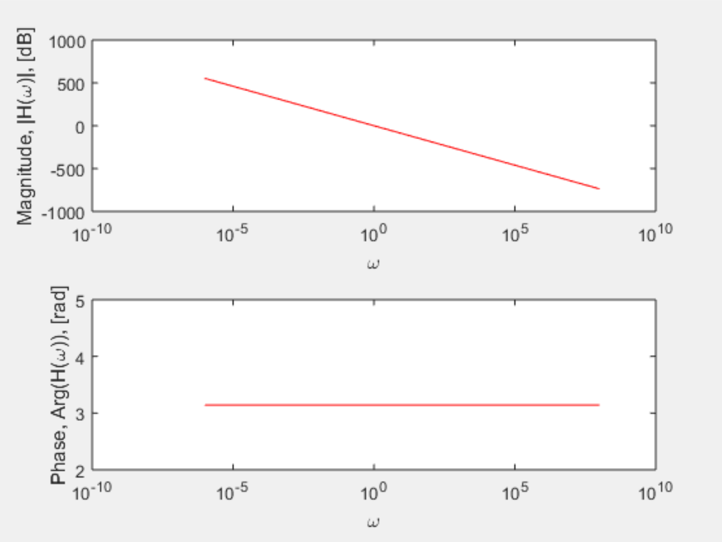

Figure 7: Transfer function of the model with the classical equation using the Laplace transform in space indicates that the system behaves like a low pass filter

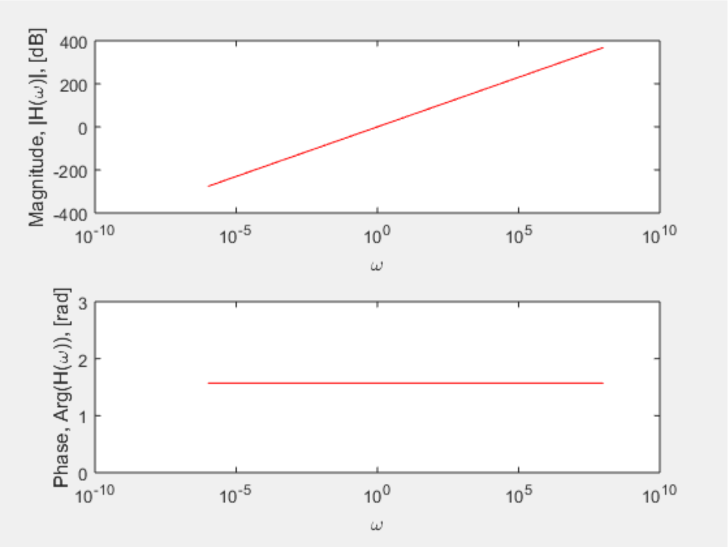

Figure 10: Transfer function of the model with the classical equation using the Sumudu transform in space indicates that the system behaves like a high pass filter

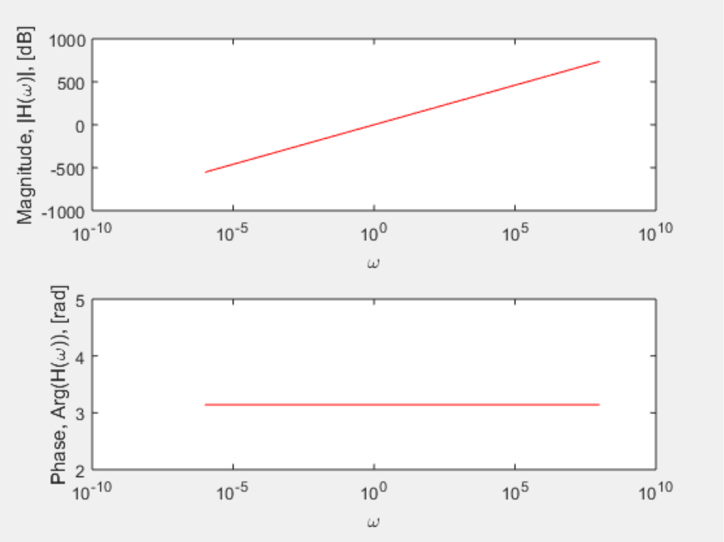

Figure 11: Transfer function of the model with the classical equation using the Sumudu transform in space indicates that the system behaves like a low pass filter

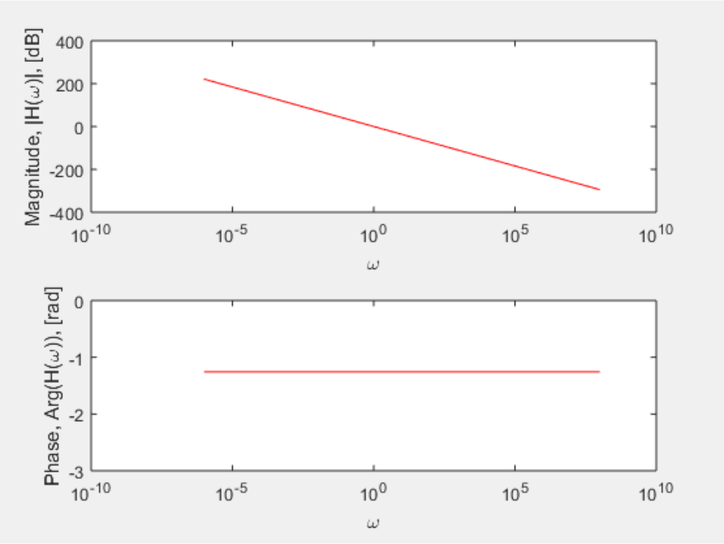

Figure 8: Transfer function of the model with the Caputo derivative using the Laplace transform shows that the system behaves like a low pass filter.

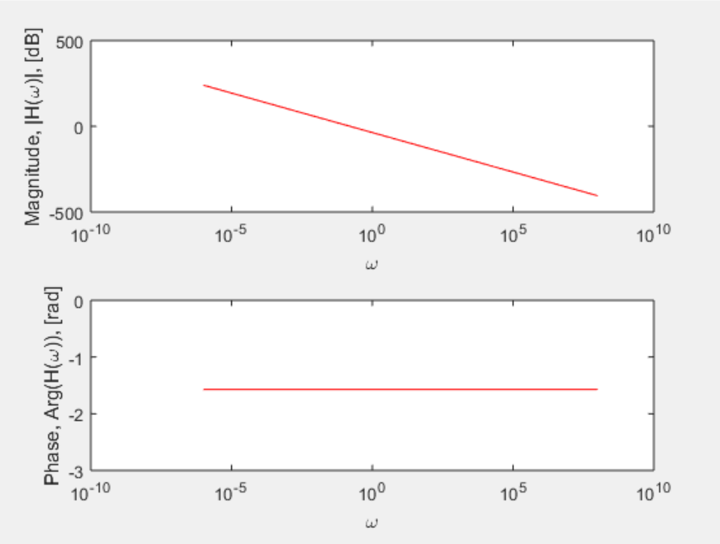

Figure 9: Transfer function of the model with Caputo-Fabrizio derivative using the Laplace transform shows that the system behaves like a low pass filter.

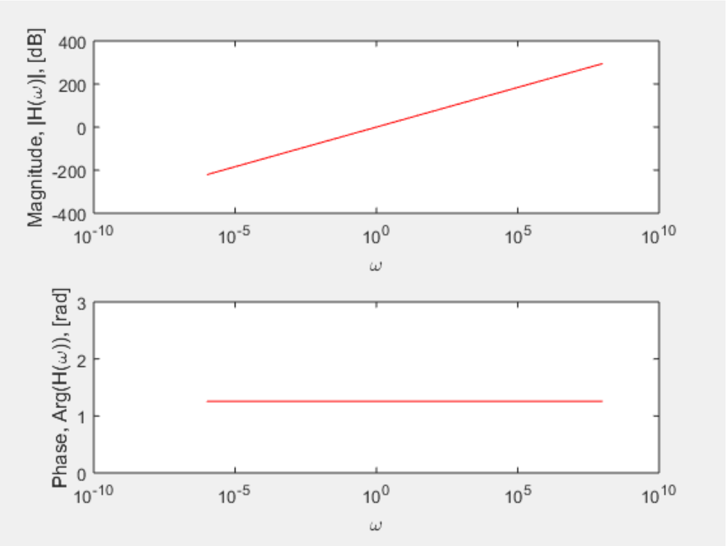

Figure 12: Transfer function of the model with Caputo derivative using the Sumudu transform with the indicates that the system behaves like a high pass filter.

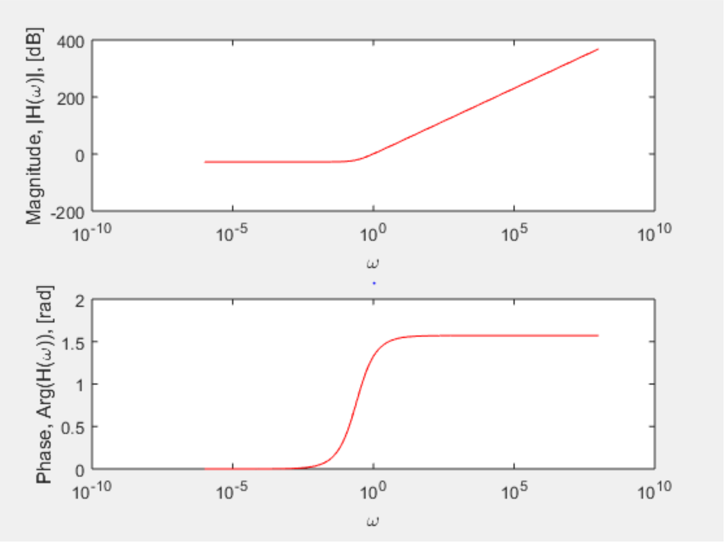

Figure 13: Transfer function of the model with the Sumudu transform using the Caputo-Fabrizio derivative shows that the system behaves like a high pass filter.

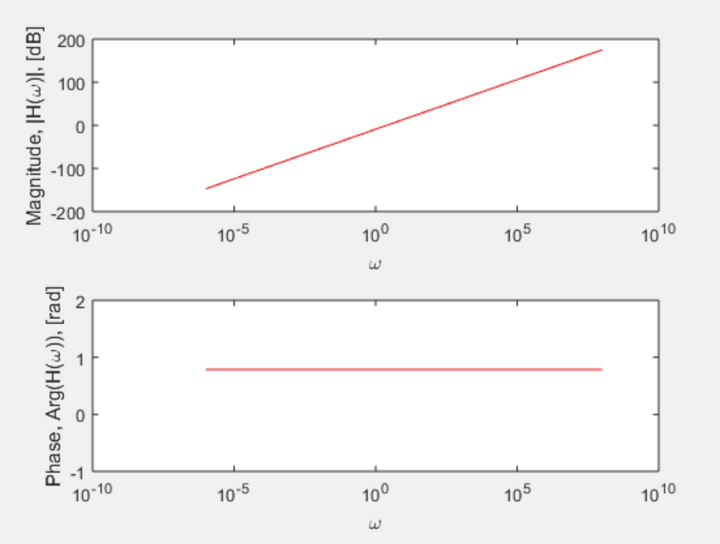

Figure 14: Transfer function of the model with Atangana-Baleanu derivative using the Laplace transform indicates that the system behaves like a low pass filter.

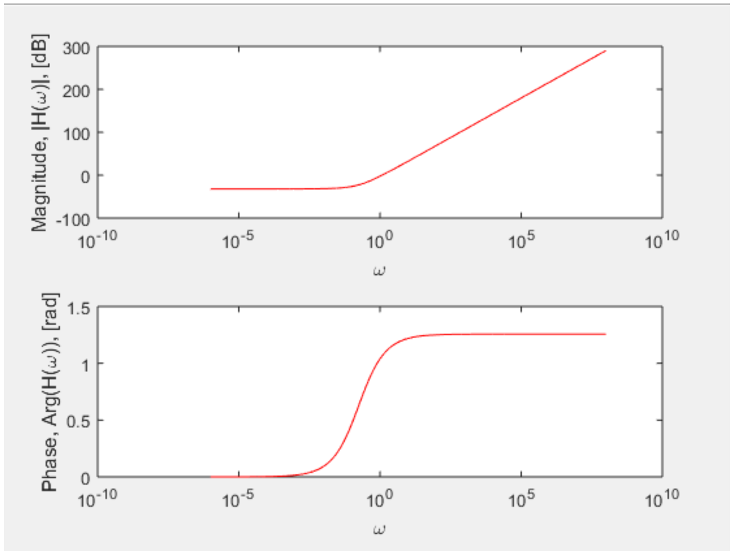

Figure 15: Transfer function of the model with Atangana-Baleanu derivative using the Sumudu transform indicates that the system behaves like a high pass filter.

From the Bode plots presented above, it is evident that the plots do not represent similar behaviours for the system. Most plots for the Laplace transform indicate that the system is a low pass filter as we observe the most gain at low frequencies and less gain at higher frequencies. All the Sumudu transform Bode plots and Laplace transform with the Atangana-Baleanu derivative and with respect to time indicates that the system is a high pass filter as it provides less gain at low frequencies and the most gain at high frequencies. Now, the bigger question here is, which transform between the two represents the correct behavior of the system or which transform could give better results? However, from analysing the two transform, one could suggest that the Sumudu give better results due to the advantages it has over the Laplace transform. Based on everything presented in this chapter, it is suggested that future research should try and interpret groundwater flow and other real-world problems as a transfer function using both the Laplace and Sumudu transform.

CHAPTER 9

NUMERICAL SIMULATIONS, DISCUSSIONS, RESULTS, AND CONCLUSION

9.1. INTRODUCTION

In this section, we introduce an example illustrating the performance of the suggested methods. First, we demonstrate that of the already existing classical groundwater flow equation. Thereafter, we also demonstrate the use and effectiveness of the new equation with the fractal-fractional approach using the power-law kernel. The resulting figures are then compared to see the effect each method introduces to the equation. The MATLAB software was used to plot the figures presented below.

9.2. NUMERICAL SIMULATIONS

For simulations, we only demonstrated flow in the saturated zone. The numerical simulations presented below will enable us to follow the evolution of groundwater flow in the saturated zone. Let us consider the equation with the following boundary conditions for the saturated zone:

$$\psi(0,t) = 0,\ and\ \psi(1,t) = 0 \qquad t > 0$$

And the following initial condition:

$$\psi(z,0) = \sin(n\pi z) \qquad z \in\ [0,L]$$

For computation work, we use the following theoretical parameters $K_s = 2.5cm/hr,\ S_s = 0.023cm, \Delta t = 0.001, h\ (space\ step) = 0.1$

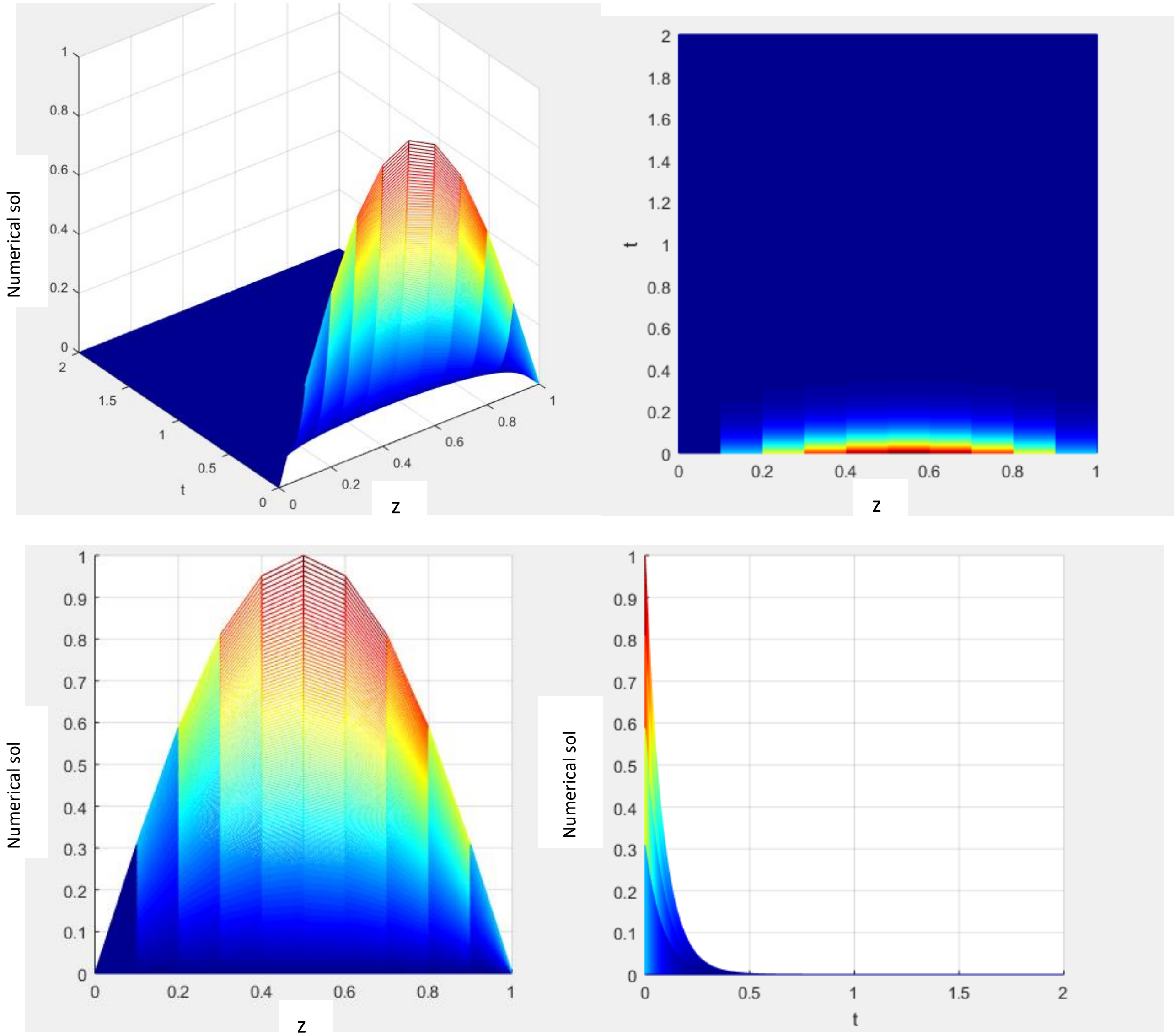

Figure 16: Numerical simulations for the classical saturated groundwater flow equation

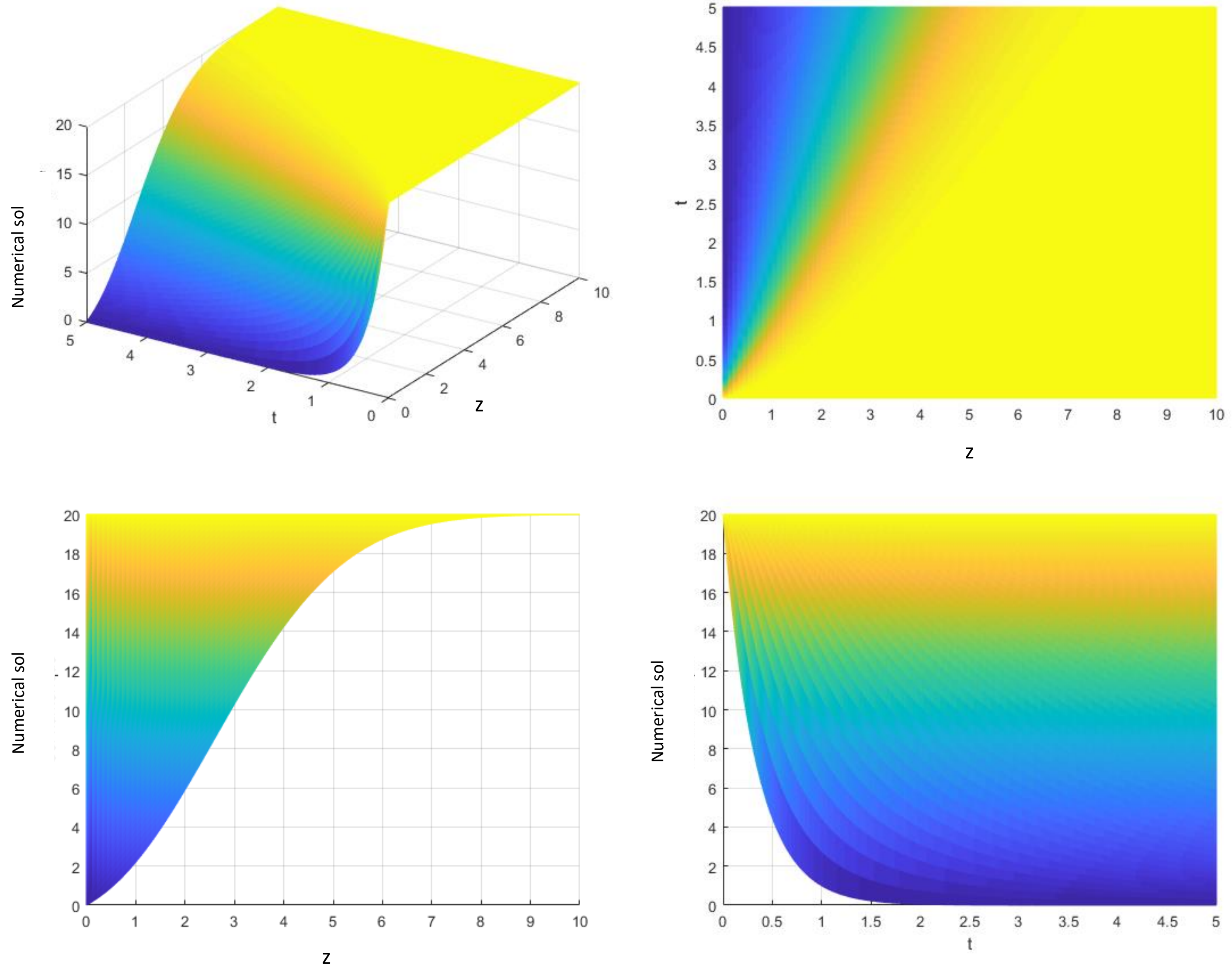

Figure 17: Numerical solution of the saturated groundwater flow equation using fractal and power law kernel with $\beta = 0.9$ and $\alpha = 1$

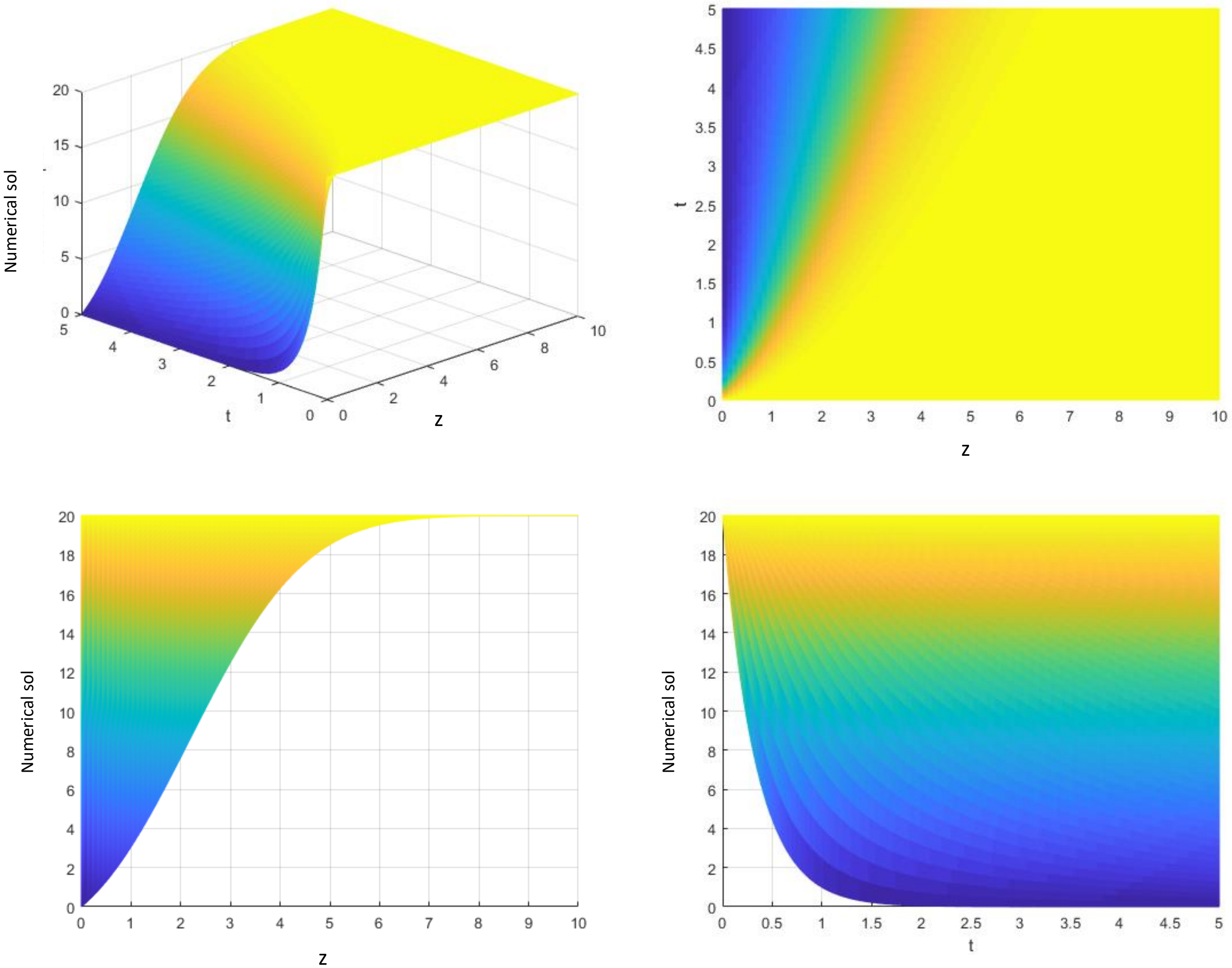

Figure 18: Numerical solution of the saturated groundwater flow equation using fractal and the power law kernel with $\beta = 1$ and $\alpha = 0.9$

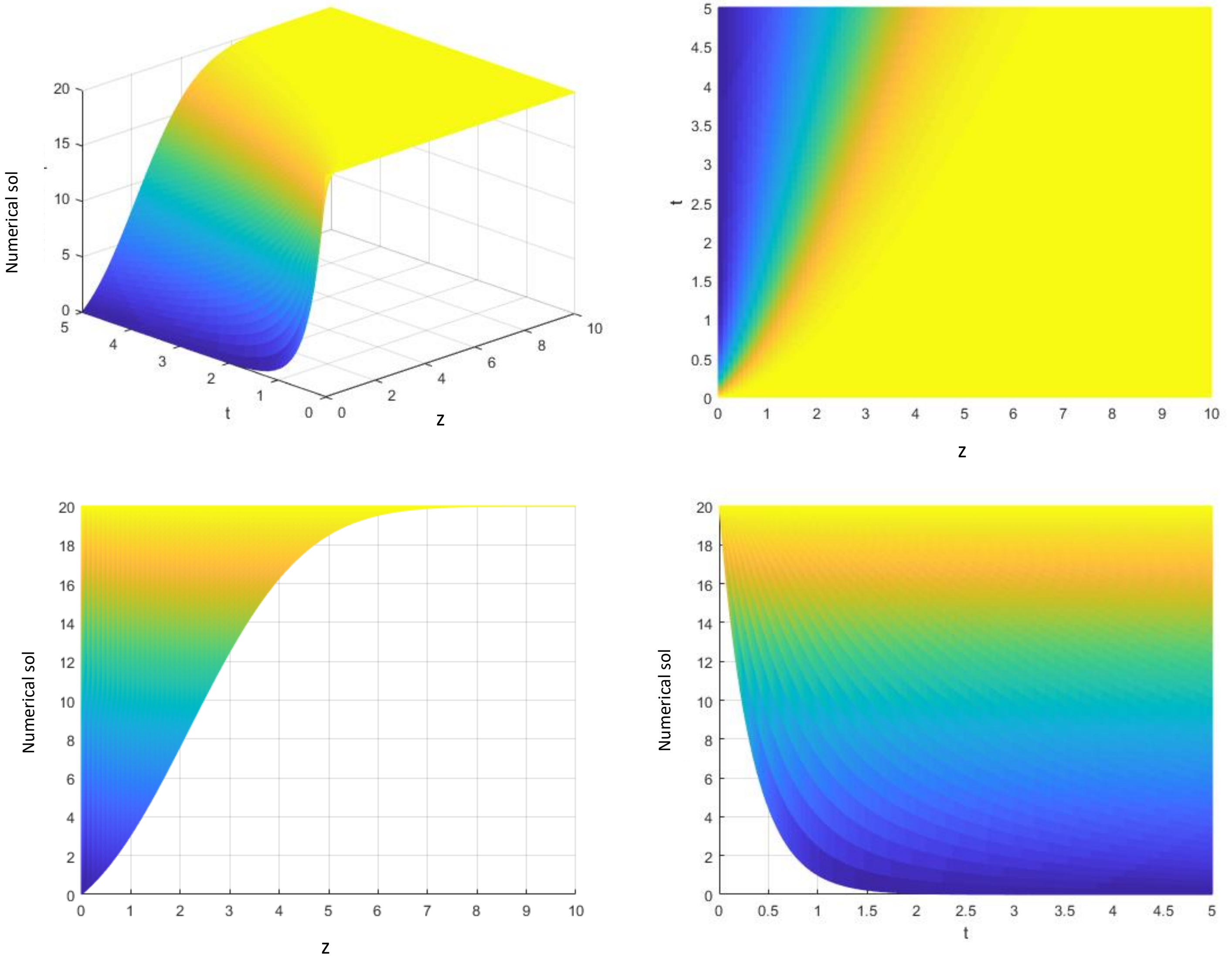

Figure 19: Numerical solution of the saturated groundwater flow equation using fractal and power law kernel with $\beta = 0.95$ and $\alpha = 1$

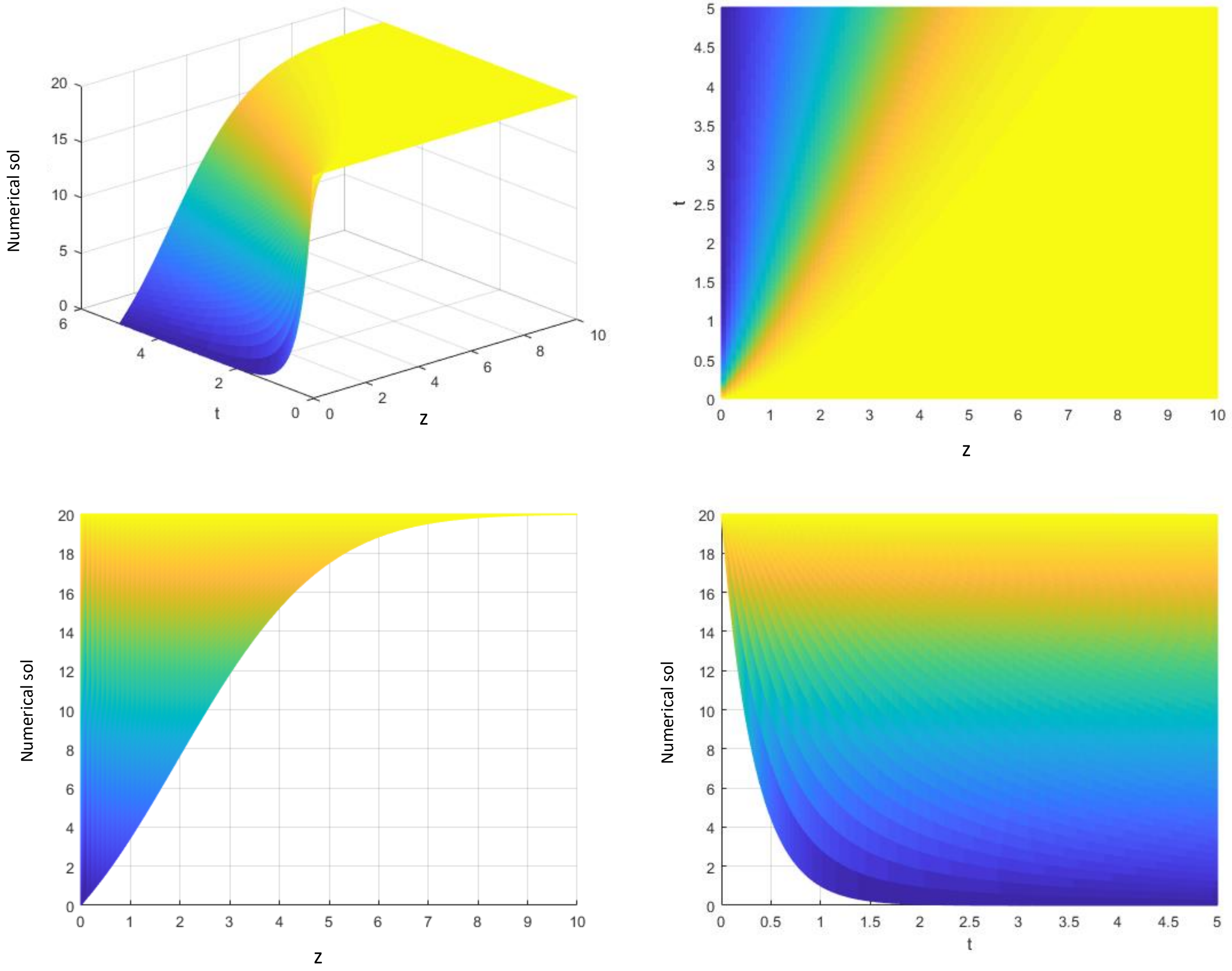

Figure 20: Numerical solution of the saturated groundwater flow equation using fractal and power law kernel with $\beta = 0.9$ and $\alpha = 0.9$

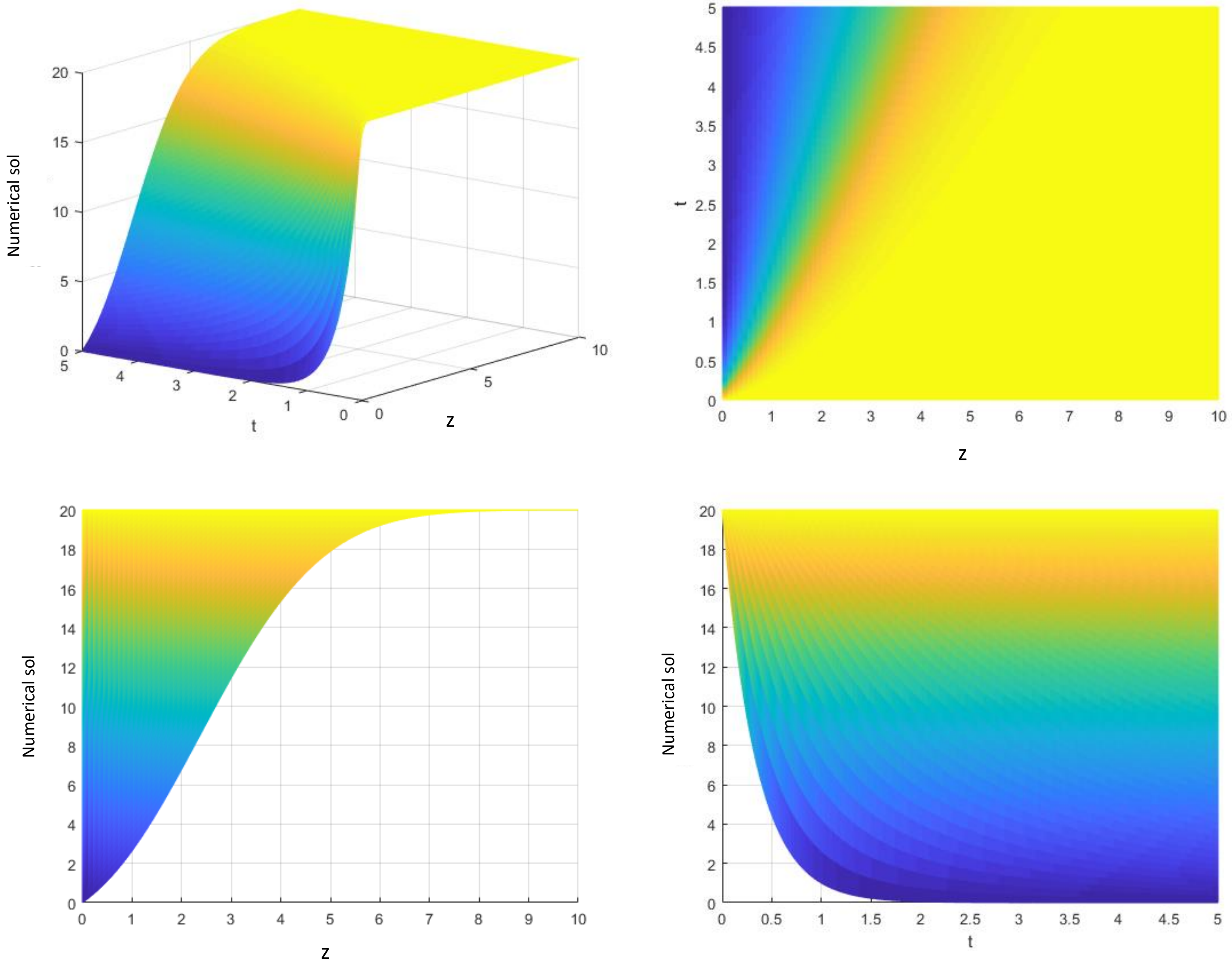

Figure 21: Numerical solution of the saturated groundwater flow equation using fractal and power law kernel with $\beta = 0.95$ and $\alpha = 0.95$

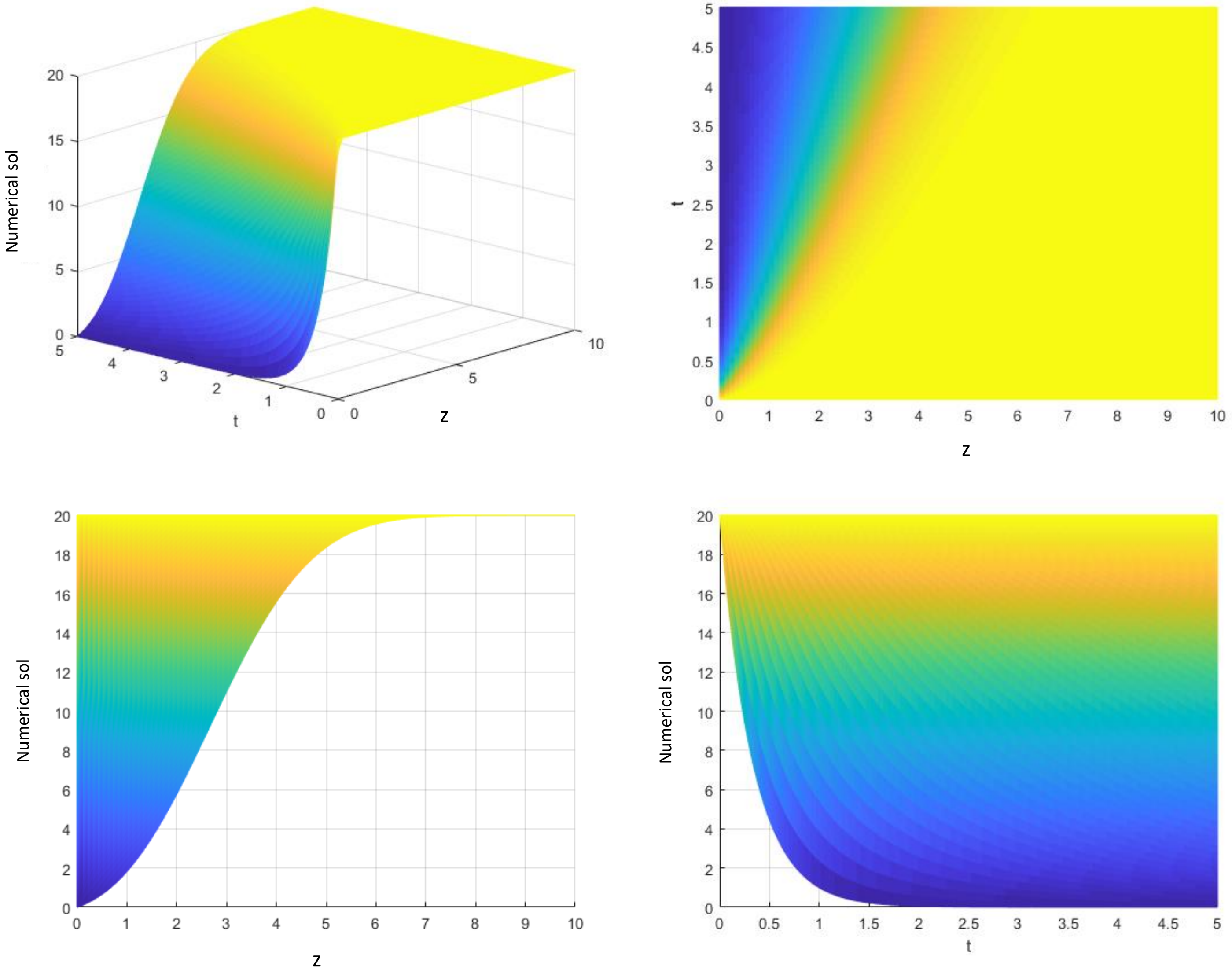

Figure 22: Numerical solution of the saturated groundwater flow equation using fractal and power law kernel with $\beta = 1$ and $\alpha = 1$

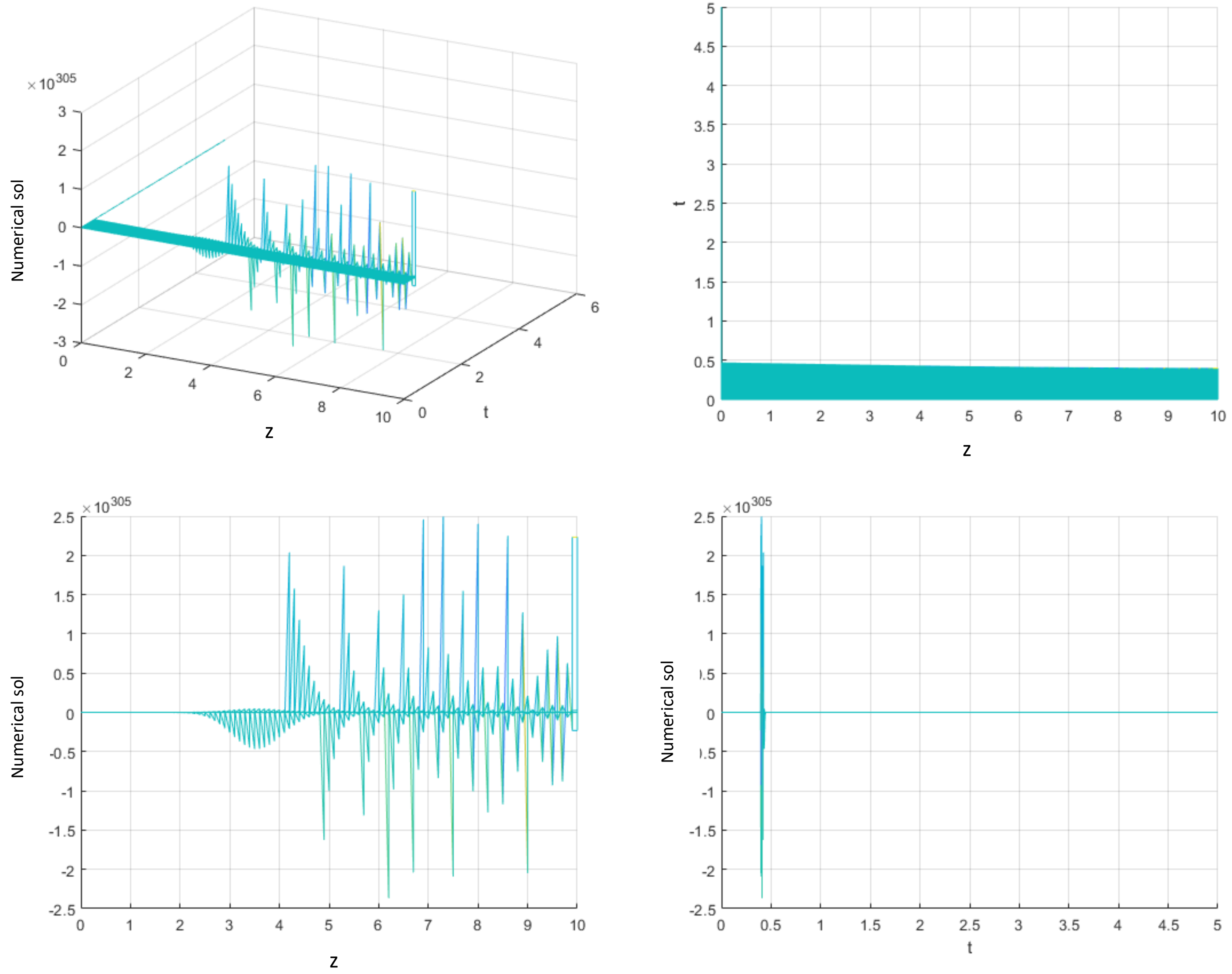

Figure 23: Numerical solution of the saturated groundwater flow equation using fractal and power law kernel with $\beta = 0.45$ and $\alpha = 1$

9.3. DISCUSSIONS AND RESULTS

While classical differentiation and integration have been used for many decades to depict real world problems observed by humankind in their different daily activities. The concept was used under behaviours that are natural and give a possible prediction of their future behaviours. Within the framework of classical mechanics, such a concept was used to depict very accurately classical problems where complexities were not included. For instance, the decay of a lifeless body could be predicted using classical differential operators. The velocity of a given object in a free media could be depicted very accurately using classical differentiation and integration. The cruise control concept was developed based on such a concept, as this does not really consider the involvement of other cars in a high way. The prediction of plane movement with no turbulence flow could be replicated and predicted using classical differential operators. In fact, diffusion, flow of water and spread of infectious diseases could accurately be depicted using this concept as the media through which the dynamical system takes place is free of heterogeneity and complexities. Nevertheless, when dealing with complex real-world problems, classical differential and integral operators are not suitable candidate to be used as they cannot really replicate the non-Markovian processes. The flow of sub-surface water within a saturated and unsaturated media has attracted the attention of some researchers, while some mathematical models were developed to capture such physical problems, one has to point out that heterogeneity of the geological formation were not really taken into account as many models use classical operators. In order to include into mathematical formulation some complexity of nature for instance, a flow taking place in fracture, a new concept was introduced called fractional calculus. One of the great properties of fractional derivative with power law kernel is the ability to replicate the long-tailed, which can be assimilated with the flow within a fracture. On the other fractional derivative with exponential are suitable mathematical operators to depict processes with fading memory. This situation is similar to the flow taking place in elastic media where the geological formation with slow transmissivity and high storativity. Finally, the differential operators with Mittag-Leffler function can replicate crossover behavior, for instance, the operator is able to replicate a passage from random walk to power law. This property can be used to depict the flow from the matrices soil to fracture. Nevertheless, it was recently revealed that even these new operators were unable to depict some complicate flow. For instance, flow within a geological formation where the system of network has self-similar properties. Self-similar problems with power law, fading memory and crossover behavior have been recognized as complex real-world problems that could not be represented with classical and fractional differential operators. Thus, fractal-fractional differential operators were used in this thesis. Numerical simulation of the saturated model with classical is depicted in figure 16 and models with fractal-fractional derivative are depicted in figures 17, 18, 19, 20, 21,22 and 23 for different values of fractional orders and fractal dimensions. Numerical simulation suggests that a change in fractional order provide a different type of flow within a saturated media. Such a fractional order, therefore represent a given geological properties that was included into the mathematical formulation. On the other hand, a change in fractal dimension shows a case of a preferential path that can be understood as fracture presence. And finally changing fractional order and fractal one obtains a more complex scenario. However, figure 23 shows very strange behaviors that cannot be found in nature, this figure is included in this work to show the important role of the stability analysis presented earlier. The figure is obtained under a violation of the stability condition. Therefore, in general, it is important to meet the stability condition before giving any interpretation of the obtained numerical results. Fractal-fractional differential and integral operators appear to be more efficient mathematical tools able to reconstruct observed real world problems.

CONCLUSION

The study of groundwater within the saturated-unsaturated zone has attracted the attention of few scholars in the last decades, perhaps due to its complexity when evaluating the passage from unsaturated to the saturated zone. Most attention was not devoted to model such a dynamical process using mathematical equations, in particular, to provide solutions to the already existing mathematical models. While much work has not been done in this direction, one will inform that the process is one of the important ones as the groundwater recharge depends upon it. It has been discussed that the vertical groundwater flow through the saturated-unsaturated media cannot be accurately described by the existing classical 1-d saturated-unsaturated groundwater flow equation because it is based on the concept of rate of change. This concept doesn't incorporate heterogeneity and therefore inaccurately represents the observed reality. In order to extend the existing model to capture complex real-world problems such as flow in heterogeneous media, we introduced new reliable and effective models. The new models include the equation with power law kernel, exponential decay law, Mittag-Leffler function, fractal-fractional and stochastic-fractional approach. A new model using the transfer function of a Sumudu and Laplace transform is also suggested. Each of these methods introduces a special property that helps to capture complexities that can be found in nature. We illustrated the transfer function in time and space using the Laplace and Sumudu transform by means of bode plots. Viewing the system of groundwater as a transfer function could be extremely helpful as it would analyse the evolution of flow from the given input variable to the output variable at different time and space. The classical model and fractal-fractional model with the power-law kernel are also illustrated using the figures presented above, and from the figures obtained it is clear that changing fractional order and fractal one obtains a more complex scenario as compared to the figures obtained for the classical model. It is clear that the model with fractal-fractional is an effective mathematical tool and can reconstruct observed real world problems.

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