The teaching of Euclidean geometry: A Universal Design for Learning Approach

by

Matheko Thabo Thamae

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Supervisor: Dr M.M. Moleko

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DECLARATION

I declare that the dissertation, *THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH,* hereby handed in for the qualification of Magister Artium at the University of the Free State is my sovereign work, and I have not previously submitted the same work for a qualification at/in another university/faculty.

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ABSTRACT

Numerous studies have shown the effectiveness of Universal Design for Learning (UDL) in supporting learners with extensive needs, disabilities and planning inclusive mathematics curricula. However, little has been documented about the effects of implementing UDL in the teaching Euclidean geometry. Hence the focus of this study was to explore the implementation of UDL to enhance the teaching of Euclidean geometry. Learners' performance in Euclidean geometry had been a concern to all stakeholders in the education system locally and internationally. Several studies affirmed that the root cause of the poor performance in Euclidean geometry were the ineffective teaching strategies which resulted in teachers not meeting the needs of the learners in their classrooms. UDL is a curriculum framework designed to address diverse learners' needs and create a conducive and enabling learning environment for all learners.

The study was qualitative in nature, underpinned by social constructivism and adopted a participatory action research (PAR) as a research design. Data were collected from a team of five mathematics teachers with experience of more than ten years of teaching from Grades 8-10, the Mathematics HOD, the subject advisor, Mathematics coordinator for the senior phase, the Curriculum Education Specialist, and the UDL coach. The coach raised awareness about the diversity in classrooms and how the principles of UDL could be implemented to teach Euclidean geometry effectively. The lesson observations and focus group discussions were used as the primary data collection instruments. Data were generated through active engagement and discussion among the co-researchers using free attitude interview technique (FAI). The findings revealed that implementing UDL principles in teaching Euclidean geometry could make geometry content accessible and perceptible to all learners. This study could add to the body of knowledge as there was little documented about how UDL could be implemented to create inclusive and flexible teaching strategies for teaching Euclidean geometry and making the content accessible to the society of diverse learners.

Keywords: Approach, Euclidean geometry, diverse learners, Universal Design, Universal Design for Learning, and universal teaching.

LIST OF ABBREVIATIONS

AMESA	Association for Mathematics Education of South Africa
CAPS	Curriculum and Assessment Policy Statement
CES	Chief Education Specialist
CPD	Continuing Professional Development
DBE	Department of Basic Education
DoE	Department of Education
ET	English Translations
FAI	Free Attitude Interview
FET	Further Education and Training
HOD	Head of Department
ICT	Information and Communications Technology
INSET	Education and Training In-Service
MMAE	Multiple Means of Action and Expressions
MME	Multiple Means of Engagement
MMR	Multiple Means of Representations
NSC	National Senior Certificate
PAR	Participatory Action Research
PCK	Pedagogical Content Knowledge
PDW	Professional Development Workshops
PLCs	Professional Learning Communities
SA	South Africa
SC	Social Constructivism
SGBs	School Governing Bodies

SMTs	School management Teams
SPTD	Senior Primary Teacher Diploma
SWOT	Strengths, Weaknesses, Opportunities and Threats
UD	Universal Design
UDL	Universal Design for Learning
USA	United States of America
ZPD	Zone of Proximal Development

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CHAPTER 1
INTRODUCTION

1.1 INTRODUCTION AND BACKGROUND

Euclidean geometry, the study of planes and solid figures on the basis of axioms and theorems employed by the Greek mathematician Euclid (c.300 BCE), is one of the critical topics in the mathematics curriculum, with a certain history attached to it in the South African education system. It was included Paper 3 as an optional mathematics assessment standard for examination for Grades 10, 11, and 12 from 2006 to 2011 (Assessment Instruction 26 of 2009). It was re-introduced in Grade 10 as Paper 2 in 2012 (Dlamini, 2012). Since the inception of the Curriculum Assessment Policy Statement (CAPS) in 2012 (Maharajh et al., 2016), this topic has been made a compulsory examinable section of the Mathematics second paper in Grades 10, 11, and 12. Prior to 2012, most schools opted not to do Mathematics Paper 3 to prevent learners' results in first and second papers from being negatively affected by Paper 3 results, especially Euclidean geometry, since it was regarded as challenging with the likelihood that learners would fail. However, such a decision to opt out of Paper 3 (Euclidean geometry) negatively impacted the economic development of countries that made this decision, including South Africa, since Euclidean geometry is the bedrock of engineering, architecture and technological development (Makhubele, 2014).

Euclidean geometry is one component of geometry, dealing with axioms and proofs of theorems and their converses through deductive thinking (Mamali, 2015). It is of great practical value as geometrical skills are critical in construction work, architectural design, engineering, mechanical drawing and even in deciding the shapes of houses (Alex & Mammen, 2014). Euclidean geometry is, however, challenging to teach and learn because teachers often lack content knowledge and the knowledge of inclusive strategies to teach it (Ubah & Bansilal, 2019). According to Mthembu (2007), the leading cause of poor performance in Euclidean geometry is how teachers present it, especially in Grades 11 and 12. Brodie and Borch (2004) point out that the ‘chalk and talk’ approach, which promotes rote learning, is predominantly used to teach Euclidean geometry in South African classrooms.

Consistent with Brodie's findings, Boggan et al. (2010) also confirm that many teachers still prefer the traditional teaching approach for geometry and other mathematics topics and are reluctant to use manipulatives to enhance metacognitive skills and strategies to enable high thinking skills. However, the traditional teaching methods are not productive since they do not encourage deep learning (Moleko & Mosimege, 2020). They also cause mathematics anxiety, stir resentment towards mathematical concepts and promote rote learning (Howie, 2003). These forms of teaching (traditional teaching methods) further lead to the poor development of learners' reasoning strategies rather than problem-solving and critical thinking skills (Snyder et al. (2008). According to Guo et al. (2019), learners generally become bored and disinterested in teaching and learning because teachers use pedagogies that promote rote memorisation. Teachers are ignorant about learners' need for equal recognition and attention when addressing their preferences and interests (Ashraf et al., 2021). Research shows that most teachers did not study Euclidean geometry at any level of their training (Machisi, 2021). They lack inclusive instructional strategies, which would promote learners' equal access to instruction that would address their differing learning style needs (Nyahunda et al., 2020).

The complexity of teaching Euclidean geometry is exacerbated by its textual nature. Moleko (2018) avows that the problems presented in the form of text are often difficult for most teachers to teach and for learners to learn. This is supported by Kutama (2002), who explains that Euclidean geometry comprises the types of problems mainly presented in the form of text, requiring teachers to develop learners' spatial knowledge and reasoning skills. However, these problems (geometry problems) are complex to teach and engage learners since teachers lack both the content and instructional knowledge. I, the researcher, am a Mathematics teacher with the experience of more than ten years. I conducted training workshops for in-service teachers. Through such interactions, I realised that the contextual nature of Euclidean geometry was not the only factor contributing to learners' poor performance. Teachers' lack of knowledge of flexible and inclusive teaching strategies to cater for diverse learners is another contributing factor. When I consulted the literature, Possi & Milinga (2018), Buli-Holmberg & Jeyaprathaban (2016), to mention a few, they concur with my findings. Chidziva (2021) states that unless teachers create an engaging environment wherein learners can talk, write, draw, and become practically involved, learners will continue to experience challenges in learning Euclidean geometry. Hence, the study's interest was on implementing UDL as a fresh approach that caters to all learners regardless of the characteristics they bring to the classroom.

Previous research has shown that poor performance in Euclidean geometry is a problem not only in South Africa, but in developing and advanced countries such as the United Kingdom (UK) (Mamali, 2015). Research indicate that mathematics teachers leave the institutions of higher education not fully equipped to deal with diversity in their classrooms (Panthi & Belbase, 2017). The study conducted on Euclidean geometry in secondary schools in Rivers State, Nigeria, revealed that the foundation of most mathematics teachers in geometry was poor (Adolphus, 2011) with teachers, lacking conceptual understanding of the components of Euclidean geometry. A study in South Africa (SA) indicates that the mathematics teachers use strategies that facilitate procedural understanding of Euclidean geometry and do not use instructional materials to enhance the learning of the concepts thereof (Howie, 2003). Although there have been attempts to address the challenges of teaching geometry, teachers struggle to teach this topic due to poor pedagogical choices influenced by economic status such as lack of teaching aids and technological tools (Panthi & Belbase, 2017). According to Shulman (1986), Pedagogical Content Knowledge (PCK) and understanding of teachers is defined as involving the relationship between knowledge of teaching materials, how to transfer the subject matter, and the knowledge of students in mathematics on algebraic functions that the subject matter may be understood by students.

The fact that teachers lack knowledge of the inclusive and flexible teaching strategies to make geometry content accessible to all learners calls for exploring other teaching alternatives. Against this backdrop, this study seeks to explore the implementation of the universal design for learning (UDL) (a framework that guides inclusive and flexible teaching) to enhance the teaching of Euclidean geometry.

1.2 PROBLEM STATEMENT

The application of Universal Design for Learning (UDL) has generally found immense recognition in addressing the issues of learners with different learning styles and preferences, particularly applied in mathematics instruction to assist learners with extensive support needs (Yavuzarslan & Arslan, 2020), used to plan inclusive mathematics curricula (Lambert et al., 2021) and to support learners with disabilities, to mention a few (Ross, 2019). Despite its successes, little has been reported on how UDL can be implemented to enhance the teaching of Euclidean geometry, which is the focus of the study.

1.3 PURPOSE AND OBJECTIVES OF THE STUDY

This study aims to explore the implementation of the Universal Design for Learning (UDL) to enhance the teaching of Euclidean geometry. In line with the purpose of the study, the objectives formulated are as follows:

1. to identify the challenges pertaining to the teaching and learning of Euclidean geometry;
2. to highlight ways in which UDL can be used in the teaching of Euclidean geometry;
3. To determine mitigating factors/circumstances against threats that may hinder the optimal benefit of using UDL in the teaching of Euclidean geometry.

1.4 RESEARCH QUESTIONS

The main research question of this study is: *How can the Universal design for learning be implemented to enhance the teaching of Euclidean geometry?*

The main research question necessitated the formulation of secondary research questions:

1. What are the challenges pertaining to the teaching and learning of Euclidean geometry?
2. How can the Universal Design for Learning (UDL) be applied in the teaching of Euclidean geometry?
3. What are the mitigating factors that can be used to circumvent the threats that may hinder the optimal benefit of the use of UDL in the teaching of Euclidean geometry?

1.5 THEORETICAL FRAMEWORK OF THE STUDY

Social constructivism was adopted as the theoretical framework this study. According to Geels (2020), social constructivism is considered an eminent approach in social sciences. It promotes the notion of knowledge construction through collaborative working and sharing of ideas (Rannikmae et al., 2020). For instance, Vygotskian ideas suggest that learners can create new knowledge through social interaction (Adam, 2017). Verwey (2010) states that social constructivists view learning as a social process in which new knowledge is built based on the current knowledge and experiences. Social constructivist theory is considered a suitable theoretical framework for this study because it provides the teachers with the platform to share their experiences and best practices in teaching Euclidean geometry. Through shared debates, teachers would be empowered with knowledge that will assist them in productively teaching Euclidean geometry. The shared debates, conducted through social interactions, would enable teachers to construct new knowledge (Tlali, 2013) of alternate teaching approaches and strategies and realise that the unproductive practices they were using could be replaced (Moleko, 2014). Social constructivism was thus deemed pertinent for this study because it allowed the teachers to be exposed to diverse perspectives and teamwork (Shangase, 2013) to enhance the teaching of Euclidean geometry, thus deepening their knowledge.

1.6 RESEARCH METHODOLOGY

Participatory action research (PAR) was followed as an approach to generate data in this study. PAR is a qualitative research method that emphasises the collaboration of the researcher with the co-researchers (MacDonald, 2012). In the context of PAR, the participants are viewed as co-researchers because they operate at the same power level as the researcher and are also involved in all the stages of the research project (Motsoeneng & Mahlomaholo, 2015). Therefore, participants in this study are referred to as co-researchers. For a profound understanding of how to address the challenge at hand, the researcher and co-researchers worked together and operated on the same power basis. Contrary to the last few decades, where the research process assumed participants to be the 'research subject'; where their ideas and feelings were not of significance, PAR advocates the consideration of participants as influential and valuable individuals for the research process in educational settings (Sokhanvar & Salehi, 2018). PAR is deemed apposite for this study because of its potential to create a conducive environment for both the researcher and co-researchers to collectively share ideas pertinent to addressing the identified problem. It resonates with the paradigm of this study since they both advocate knowledge construction through collaborative working.

According to Tetui et al. (2017), the PAR approach is a cyclical process that consists of the following stages/phases (see Table 1.1):

Table 1.1: Stages of PAR

Stage Number	Stage Name	Description
1.	Problem identification	- this is the stage in the study wherein the problem will be collectively identified.
2.	Planning	- this is the step where the best strategies to address the problem were identified and prioritised to promote a positive change. The planning would include a UDL training workshop by the UDL coach with more focus on planning UDL customised lessons.
3.	Implementation	- this is the phase where strategies and plans are executed. This includes the implementation of UDL principles.
4.	Action observation	- the researcher will sit in class to observe and audio-video tape the lessons. An observation tool will be used to collect data from this stage.
5.	Reflection	- After the lesson's presentation, the researcher and the co-researchers will meet to reflect upon the representations and shared experiences, identifying areas of weakness and strengths. The team will also re-plan to address the identified areas of weakness - [session will be audio-recorded].

(Source Tetui *et al.,* 2017)

The above stages were used to guide data generation in this study.

1.7 RESEARCH INSTRUMENTS

Data were generated through class observation, focus group discussions and reflection sessions. According to Johnson and Christensen (2012), observation is one of the most effective tools for collecting data. A researcher can see and hear what is happening at the site without interacting with the co-researchers. On the other hand, focus group discussion is a technique that offers the researcher and co-researchers an opportunity to deeply explore the issue under discussion. The researcher acts as the facilitator to guide the group's meeting (Nyumba et al., 2018). The focus group discussion was appropriate for the study because it allowed for sharing and comparing of understandings and ideas and yielded more insights about implementing UDL in the teaching of Euclidean geometry. Therefore, the focus groups were essential in assisting the researcher is focusing on the issues during the discussions. The sessions also allowed the researcher to conduct group follow-up discussions and provide more clarity to the co-researchers. In addition, the reflection sessions allowed the co-researchers to reflect on their teaching practices and experiences.

1.8 DATA COLLECTION PROCEDURES

A series of class observations and focus group discussions were conducted. During these meetings, the discussions were centred on the challenges of teaching Euclidean geometry, identifying suitable solutions to the challenges, and mitigating threats that may hamper the implementation of the identified solutions. The conversations/discussions were audio and video recorded. The unproductive practices observed during the lesson presentations were also discussed during the reflection sessions. A standardised lesson observation tool was used to assess geometry according to UDL principles (Moleko, 2018). The Free Attitude Interview (FAI) technique allowed for the initiation of conversations/discussions. According to Meulenberg-Buskens (2011), when FAI is used, the co-researchers get to communicate as in normal day-to-day conversations. This technique elicited as much information as possible from the co-researchers. The technique encourages open-ended questions, which enabled the co-researchers to say more than they would in surveys with close-ended questions.

1.9 DATA ANALYSIS

Thematic analysis was used to analyse the collected data. Nowell, Norris, White and Moules (2017) note that thematic analysis, which consists of six stages, is fundamental for examining different co-researchers' points of view. The thematic analysis allows flexibility in interpreting collected data and highlights similarities and differences in generating unanticipated insights in response to the research question (Nowell et al., 2017). This technique, described in Chapter 3, assisted in terms of identifying and organising the emerging themes.

1.10 TRUSTWORTHINESS

The trustworthiness of a study concerns the level of trust and confidence in the data, interpretation and methods used to ensure the quality of the study (Polit & Beck, 2012). In contrast to the positivistic approach where the method is structured and detailed, the PAR approach allows the flexibility of the researcher and co-researchers because it accommodates a wide range of individual contributions throughout all phases of the research process. Triangulation, which refers to multiple data generation, was used as a strategy to test the 'validity of data' through the convergence of information from various sources (MacDonald, 2012). Moreover, it was used to clarify meaning, verify the repeatability of observations, and interpret the generated data.

Audio and videotape recordings, transcriptions, and the documentation of minutes by the researcher and co-researchers influenced the credibility, transferability, dependability, and confirmability of the findings and their interpretations. Member checking was done with the co-researchers. According to Birt et al. (2016), member checking is a tool or a technique that enhances the credibility and trustworthiness of results. This technique, which makes it possible for researcher and the co-researchers to agree on what exactly was discussed or pointed out without data being misinterpreted. It enabled the researcher to capture and interpret the data correctly and in context. Member checking further assisted in addressing the researcher's biases towards the data. Before analysing data, the researcher familiarised herself with the depth and breadth of data to be reported (Braun & Clarke, 2006), which enabled the researcher to write credible data and package it systematically.

1.11 SELECTION OF THE CO-RESEARCHERS

In participatory action research (PAR), similar to the other modes of research, selecting co-researchers is seen as a vital step that requires a thoughtful approach as it allows the co-researchers to deliberate on their lived experiences (Alase, 2017). The study followed purposive sampling, which involved selecting co-researchers knowledgeable about the concept under study (Creswell et al., 2011). The study included ten co-researchers. Some were from one senior secondary school in the

Motheo district in the Free State, while others were departmental officials. The co-researchers had teaching experience ranging between ten and fifteen years. There were five mathematics teachers, one subject advisor for Mathematics, one head of the department of mathematics (HOD), 1 Chief Education Specialist (CES), one Free State mathematics coordinator, and 1 UDL coach. The purposive sampling technique was deemed pertinent for the study. All participants provided crucial information that any person not in mathematics education could not give.

1.12 VALUE OF THE STUDY

The results of this study hopefully provide teachers of Mathematics, subject advisors and curriculum developers with a deeper understanding of how to teach Euclidean geometry through UDL. Teachers would consider diversity in mathematics classrooms and take learners through every level of Van Hiele's theory of geometry thinking (Bonyah & Larbi, 2021).

The results could inspire teachers and learners because teaching Euclidean geometry through UDL creates a conducive learning environment for learners with diverse learning styles. This study could possibly add to the body of knowledge since there is limited literature on how UDL could be used in teaching Euclidean geometry. The study thus contributes to expanding the existing knowledge of teaching and learning theories about geometry.

1.13 ETHICAL CONSIDERATIONS

The researcher applied for ethical clearance from the University of the Free State to ensure that the study adheres to the ethical principles of research. The study was ethically cleared and assigned the following reference: UFS-HSD2020/1868 (see Appendix A1). Permission to conduct the study was also sought from the Department of Education in the Free State province and the school principal, where data was collected and permission was granted (see Appendices A2 and A3). Parents were requested to sign the assent forms allowing learners below 18 years of age to participate in the research (see Appendix A4). The co-researchers (teachers, HOD, subject advisor, mathematics district coordinator, CES and UDL coach) were requested to sign the consent forms (see Appendices A5; A6; A7; A8; A9, and A10). Plowright (2013) states that confidentiality is essential to consider once the data have been collected, stored, and analysed. In line with the above principle, the researcher used pseudonyms to protect the confidentiality and dignity of the co-researchers. Data will be stored in a safe filing cabinet for three years after the completion of the study, after which it will be destroyed.

1.14 OUTLINE OF THE DISSERTATION

The study will consist of five (5) chapters. *Chapter 1* introduced the research and outlined the background of the study: the problem statement, the purpose of the study; research questions; research design and methodology; the significance of the study; ethical considerations, and the layout of the chapters. *Chapter 2* covers a review of the literature on the challenges encountered in teaching Euclidean geometry and how UDL can address these challenges. *Chapter 3* outlines the research design and methodology used. *Chapter 4* presents the analysis of the data and the presentation and discussion of findings on the implementation of UDL. *Chapter 5* summarises the findings, draws conclusions, presents the limitations of the study, opportunities for further research, implications for practice, and offers recommendations.

CHAPTER 2
LITERATURE REVIEW

2.1 INTRODUCTION

The study aimed to explore implementing the universal design for learning (UDL) teaching approach to enhance the teaching and learning of Euclidean geometry. In line with this purpose, this chapter first discusses social constructivist theory as a paradigm that guides the study. This is followed by the literature review section, which reviews the literature on challenges pertaining the teaching and learning of Euclidean geometry, the solutions implemented and suggested to address the identified challenges, threats that may impede the successful implementation of the strategies and ways to mitigate against risks that may hinder optimal benefits of the use of the identified strategies for teaching and learning Euclidean geometry. The chapter also provides the definitions of the operational concepts, namely, approach, diverse learners, Euclidean geometry, Universal Design, universal teaching and Universal Design for Learning, since they serve as pillars on which the study is anchored.

2.2 THEORETICAL FRAMEWORK

According to Swanson and Chermack (2013), a theoretical framework is a structure that supports the theory of a research study. Alaidaros et al. (2020) state that a theoretical framework gives researchers a direction to interpret, develop tools for analysing, designing and monitoring the progress, and explain, evaluate and generalise from their findings. The theoretical framework helps a reader understand why the researcher conducts the study on a particular topic. It also gives the researcher a different perspective about the study and relevant ways to explain how and why things happen the way they do (Lynam et al., 2007). According to Kivunja (2018), the theoretical framework serves as a lens that magnifies the contents of the data. It divulges interconnections that make sense of the research questions and address the problem the researcher wants to investigate.

Moleko (2018) indicates that the role of the theoretical framework is to demonstrate the relationship between new ideas and existing knowledge. Adam (2017) concurs with Moleko that creating knowledge is based on understanding the existing knowledge. Moleko (2018) further states that the study's objectives should influence the choice of the theoretical framework. For the current study, the purpose is to explore the implementation of UDL to enhance the teaching of Euclidean geometry. With this in mind, social constructivism was adopted as a suitable theoretical framework that underpins the study. According to Bozkurt (2017), Social Constructivism (SC) promotes active interaction in the teaching and learning process. Rankhumise and Imenda (2014) noted that social constructivists believe that knowledge is socially constructed and that knowledge is often gained as people interact with one another. Social constructivists advocate social interaction for learning to be socially constructed.

The application of social constructivism in this study would enable the researcher and the co-researchers to interact and engage in exploring the implementation of UDL to enhance the teaching and learning of Euclidean geometry. This would result in the construction of new knowledge on how UDL could be implemented to cater to diverse learners and make content accessible. According to Kivunja (2018), in a constructivist classroom, a facilitator must create a highly dynamic teaching and learning environment to allow learners to participate as partners in knowledge construction. In this study, the researcher would play a similar role to ensure that the co-researchers participate fully in the research project in a safe environment wherein their voices are heard and respected (Dold & Chapman, 2012).

2.2.1 Historical Background of Social Constructivist Theory

The social constructivism theory of learning was coined by post-revolutionary Soviet Psychologist Lev Vygotsky, who believed that knowledge is mutually constructed through social interaction (Nassaji & Tian 2018). Vygotsky was a cognitivist; however, he had a different viewpoint from other cognitivists, such as Piaget and Perry, who believed that learning could be separated from social context (Bozkurt, 2017). According to Adam (2017), Piaget and Perry considered knowledge construction as a distinctive process. In contrast, Vygotsky believed that people create meaning through interacting with each other and the objects in the environment under the guidance of a more skilled peer or an adult.

Bozkurt (2017) points out that Vygotsky developed the Zone of Proximal Development (ZPD) concept to explain this social and participatory learning with the more informed peer. ZPD refers to the range of abilities one can carry out under the guidance of an expert but cannot execute on their own (Chang, 2021). By the same token, Knestrick (2012) considers ZPD as the difference between what a child can do independently and what they can do through scaffolding. This confirms that social interaction facilitates meaningful learning to a greater extent than what one can learn individually without interacting with others (Kim, 2001). According to Bozkurt (2017), social constructivism theory indicates that knowledge is built and constructed actively; therefore, social interaction plays a vital role in the learning of Mathematics.

Social constructivist theory is adopted in the teaching and learning of Mathematics as implied as follows in the Curriculum and Assessment Policy Statement (CAPS):

> This can be done through observations, discussions, practical demonstrations, learner-teacher conferences, informal classroom interactions … (DBE, 2012: p. 51).

Additionally, Department of Basic Education (DBE) emphasises the importance of applying active and critical approach to learning, rather than memorisation and uncritical learning of given truths (Booysen, 2018). Social constructivism theory fits precisely into the context of this study because it advocates for collaboration, knowledge sharing amongst the co-researchers, and mutual interaction to construct new knowledge. Holmes (2020) states that it is crucial for the researcher to declare his/her position in research so that they may integrate a reflexive perspective into their research. Therefore, my stance as a researcher in this study is that of a social constructivist. Similar to other social constructivists such as Akpan et al. (2020), I believe that when teachers collaborate and share knowledge with each other, they can construct new knowledge. I also believe that knowledge sharing and shared debates would provide teachers with opportunities to learn and be empowered. I think that teachers are life-long learners and that platforms such as focus group discussions and reflection sessions wherein the teaching experiences and best practices are shared, provide them with opportunities to learn and socially construct new knowledge.

2.2.2 Objectives of Social Constructivism

Thomas et al. (2014) state that social constructivism focuses on revealing ways in which individuals and groups participate in knowledge construction. Thomas et al., (2014) further affirm that social constructivist theory informs the researcher on how to construct a conducive environment for building new knowledge. According to Adam (2017), the theory establishes the influence of socio-cultural background on cognitive development. It highlights the crucial role played by semiotic mediation in knowledge construction. As stated by Bozkurt (2017), Vygotsky defines semiotic mediation as an investigation of how knowledge is constructed using language, various systems of counting, mnemonic techniques, algebraic symbol systems, diagrams, and mechanical drawings, to mention a few.

In this study, social constructivism enables the researcher to understand how knowledge is constructed using language, various systems of counting, mnemonic techniques, algebraic symbol systems, diagrams, etc.

2.2.3 Nature of Reality

Epistemologically, social constructivists maintain that knowledge is constructed through social interaction (Bozkurt, 2017). Social constructivists also believe in multiple truths since people differ according to their experiences and backgrounds (Moleko, 2014). This, therefore, means that there can never be one truth. The truth is, consequently, multi-layered. In this study, the co-researchers engaged in social interactions and discussions to explore the implementation of UDL to enhance the teaching of Euclidean geometry. Therefore, the co-researchers (teachers) shared different perspectives and experiences. Thus, there can never be only one perspective considered in the process but multiple perspectives. Tsimane and Downing (2020) point out that one way to improve teaching and learning is to allow researchers and co-researchers to work as a team to share diverse ideas and experiences in education. For diverse perspectives to be shared, researchers need to create a conducive environment where everyone is free to participate without being judged (Hlomuka, 2014).

2.2.4 Role of the Researcher as informed by Social Constructivist Theory

The role of the researcher is to gather the co-researchers together and create a platform for them to engage in discussions (Tsotetsi, 2013). The discussions empower the co-researchers with the knowledge to employ the UDL principles in teaching Euclidean geometry. Additionally, the researcher should explicitly explain the aim and purpose of the study and assist in clarifying the roles of the co-researchers (Hlomuka, 2014). Furthermore, the researcher's role is to interpret the co-researchers' ideas and opinions to make sense of them (Moleko, 2014).

According to Moleko (2014), engaging the co-researchers enables them to manage the present situation and develop ownership of the research project's outcomes. Within the context of social constructivist theory, the researcher acts as a facilitator of the discussion. The researcher works collaboratively with co-researchers because the social constructivist approach is participatory and advocates for teamwork (Shangase, 2013). For research to succeed, Tsimane (2019) affirms that a researcher should be trustworthy, honest, friendly, patient, transparent and a team player. Shangase (2013) further points out that researchers must be compassionate, patient, and transparent as they interact with the co-researchers to allow reflexivity and humbleness among themselves.

2.3 DEFINITION OF OPERATIONAL TERMS

The following sections provide the comprehensive definitions of the operational concepts that underpin this study. It is important to define these concepts according to the context of this study for the readers to be enlightened and have a deeper understanding of their meaning. They are Euclidean geometry, approach, diverse learners, universal teaching, Universal Design (UD), and Universal Design for Learning (UDL).

2.3.1 Euclidean Geometry

According to Merriam Webster dictionary, Euclidean geometry is a Mathematical system contributed by Alexandrian Greek Mathematician by the name of Euclid. In his textbook *Euclid's Elements*, he wrote about a small set of instinctively interesting axioms and deduced many other theorems. Euclid was the first to show the applicability of theorems in deductive and logical systems. He wrote about the geometric properties of two and three-dimensional figures. Güven and Kosa (2008) define Euclidean geometry as a study of shapes and space, which can be conceptually understood if one has well-developed spatial skills. On the other hand, Mamali (2015) states that Euclidean geometry is a field of mathematics dealing with axioms and proofs of theorems through deductive thinking. Similarly, according to Artmann (2012), Euclidean geometry studies planes and solid figures based on the axioms and theorems, as engaged by the Greek Mathematician Euclid. Euclidean geometry is commonly taught in secondary schools and in the South African curriculum, it has a weighting of about 33,3% in Mathematics Paper 2 (DBE, 2012, p.12).

Table 2.1: The weighting of content areas

Paper 2: Grades 11 and 12: theorems and or trigonometric proofs: maximum 12 marks			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Statistics	15 ± 3	20 ± 3	20 ± 3
Analytical Geometry	15 ± 3	30 ± 3	40 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3
Euclidean Geometry and Measurement	**30 ± 3**	**50 ± 3**	**50 ± 3**
Total	**100**	**150**	**150**

(Source: DBE, 2012, p.12)

Brannan et al (2011) state that Euclidean geometry includes the theory of points, lines, angles and circles on a flat plane. According to Jones (2000), the aims of teaching Euclidean geometry are to:

- develop learners' spatial awareness and ability to visualise.
- encourage learners to use conjecture, deductive thinking and proofs.
- enable the development of learners' conceptual understanding of geometrical properties and theorems to solve problems in a real-world context.

Even though various scholars define Euclidean geometry diversely, in the context of my study, I adopt definitions provided by Brannan et al. (2012) and Mamali (2015), which state that Euclidean geometry is a field of mathematics that includes the theory of points, lines, angles and circles on a flat plane as well as dealing with axioms and proofs of theorems through deductive thinking. Learners can provide solid support to a conclusion and establish geometric truth after reasoning from one or more statements to reach a logical conclusion. In the teaching of Euclidean geometry, teachers must ensure that learners are engaged in using descriptions, demonstrations and rational justifications for strategic learning and the construction of proofs of theorems. According to Machisi (2021), Van Hiele's theory-based instruction has a positive impact on students' attitudes and confidence towards Euclidean geometry.

2.3.1.1 Van Hiele's theory of geometric thinking

Bishop (2020) declares that two mathematics teachers, Dina, and Pierre Van Hiele from the Netherlands, observed that their students had difficulties learning geometry. In the 1950s, they developed Van Hiele's theory of geometric thinking to prove that learners' structure for reasoning is crucial in teaching and learning geometry (Prayito et al., 2019). Van Hiele’s theory of geometric thinking describes how learners learn geometry, comprising five levels, namely; (1) visualisation/recognition, (2) analysis, (3) abstraction, (4) formal deduction and (5) rigor (Vojkuvkova, 2012). These are the levels that a learner must hierarchically pass through to progress from recognising figures to writing formal geometric proofs. Vojkuvkova (2012) further mentions that Van Hiele believed that instruction informs cognitive progress in geometry learning.

2.3.2 Approach

Merriam Webster’s dictionary defines approach as taking preliminary steps toward a particular purpose. Therefore, the teaching approach is a method of teaching ‘something’ using learning techniques. In line with this, the current study proposed using UDL as an approach to enhance the teaching of Euclidean geometry. This teaching (the UDL approach) entailed multiple and flexible ways in which:

- Content is presented using various formats.
- Teaching is designed in a manner that seeks to remove barriers to learning.
- Learners are afforded opportunities to demonstrate their knowledge (or what they know or what they have learned) in diverse ways.
- Different (varied) strategies are used to keep learners motivated, engaged with learning and focused on the mathematical tasks.

An approach in the context of this study refers to teaching that is tailor-made to cater to diverse learner populations (that is, inclusive, equitable, and accessible teaching.

2.3.3 Diverse Learners

Diver learners refer to learners that are racially, ethnically, critically and linguistically different. In the context of this study, they are the learners who have different learning styles and preferences. This calls for teachers to create a classroom environment cognisant to learners’ cultural background so every learner feels safe and welcome.

2.3.4 Universal Teaching

The term ‘universal’, as defined by Merriam-Webster dictionary, refers to ‘something that applies throughout the universe to many things and is accessible to all people.’ It is adjustable to many sizes, uses, and devices. In line with this definition, therefore, the term ‘universal teaching’ refers to the teaching suitable to meet all learners' needs in a classroom. It is a form of teaching that encompasses varied strategies to cater to all learners' needs. It is the type of teaching which provides flexibility in the ways learners access learning material, engage with it and demonstrate what they know. In the context of this study, the term universal teaching means teaching that is planned in a manner that caters to a diverse learner population or teaching that considers learner variations.

2.3.5 Universal design (UD)

Jones (2014) describes UD as an environment or situation design that can be accessible to everyone regardless of age, size, ability or disability. According to Burgstahler (2009), the architect Ronald Mace invented this concept known as UD when describing barrier-free products and physical environments that integrate people with disability into the mainstream. UD in the education system is a supporting structure in designing various educational products, including a curriculum that is accessible, perceptible, simple, and intuitive to all learners (Burgstahler, 2009). In this study, UD is about designing a learning environment that caters to all learners regardless of the characteristics (visual, tactile and auditory) they bring into the mathematics classrooms.

2.3.6 Universal Design for Learning (UDL)

Dalton (2017) refers to UDL as a framework for curriculum design that is informed by the values of UD. Barteaux (2014) explains that UDL is intended to create a conducive and enabling learning environment for all learners. Therefore, if learners' variability is recognised, curriculum design and instruction should thus address learners' diverse needs. Dalton (2017) states that UD aims to create a barrier-free physical environment, whereas UDL is designed to eliminate barriers from the learning environment. In addition to access to the content and information, learners need to have engagement and connection to what they are learning (Chan et al., 2014)

LaRocco and Willken (2013) report that UDL is a scientifically sound teaching framework that provides flexible guidance on how information is presented to learners with diverse learning styles and preferences, ways to demonstrate what they know and various ways in which they can be engaged in the process of learning. UDL reduces barriers to teaching and learning and provides apposite support to learners with different learning styles (Moleko, 2018) and assists in ensuring the accessibility of learning to all learners (Boothe & Lohmann, 2020). In addition, Stolz (2020) defines UDL as a framework that congregates flexible curriculum and pedagogy that respond to the diversity of learners. UDL has been adopted as a framework for designing and delivering barrier-free strategies for teaching and learning (Capp, 2020).

Rose and Strangman (2007) indicate that research on cognitive science gives a hint of three brain networks in cognition and learning. The first brain network is for the recognition of patterns, while the second network is for planning and generating patterns. The third network is for determining which patterns are essential to learning. According to Dalton et al. (2012), these brain networks are referred to as recognition (recognition of information to be learned), strategic (application of strategies to process the information) and affective network (engagement in the learning task). Additionally, Rose and Strangman (2020) point out that the UDL framework is developed on these three brain networks, as a guide to creating a flexible curriculum that embraces learner variability. UDL comprises three core principles: namely, Multiple Means of Representation (MMR), Multiple Means of Action and Expression (MMAE) and Multiple Means of Engagement (MME)

2.3.6.1 Multiple Means of Representation (MMR)

MMR provides multiple and flexible ways of presenting content and information to create a barrier-free learning environment for all learners. The presentation may be given using visual, auditory or tactile material (Dalton et al., 2012). MMR provides learners with options for perception, comprehension, language and mathematical expressions and symbols (Meyer et al., 2014).

2.3.6.2 Multiple Means of Action and Expressions (MMAE)

MMAE is characterised by learners displaying more than one way to interact with different materials as they demonstrate what they have learned (Dalton 2017). There is no strategy that all learners most favour; hence, it is important to afford opportunities for action and expressions to learn in a class of diverse learners. MMAE provides learners with options to use multiple formats of planning, organising, and initiating purposeful activities and how they demonstrate the mastery of the acquired knowledge (CAST, 2018).

2.3.6.3 Multiple Means of Engagement (MME)

This principle enhances learning by providing multiple flexible ways for engagement in learning (Dalton, 2017). It is about different ways in which learners can be motivated. It advocates for teachers to tap into learners' meta-cognition development, which could reveal their differences in neurology, culture, personal relevance and background. There is no one means of engagement that all learners favour in all contexts. MME provides learners with options for promoting expectations and beliefs that optimise motivation, facilitating personal skills and strategies, as well as developing self-assessment and reflection (Novak & Rodriguez, 2018).

2.3.7 Relationship between UDL Principles and Brain Networks

According to Balt et al. (2021), UDL is a framework based on cognitive neuroscience that concentrates on engaging multiple brain networks. UDL provides a comprehensive guide in terms of how learning takes place. Its three principles are classified in line with the three brain networks, thus providing a clear exposition of how learning occurs. Studies of the brain have long established that the three main networks are active during learning (Connell et al., 2012). The networks include affective, recognition, and strategic. Affective networks are responsible for the 'why' of learning. They regulate the emotional involvement with learning, such as our motivation and our ability to focus and remain engaged with tasks. Recognition networks regulate the 'what' of learning and are responsible for receiving information and concept formation. Strategic networks govern the 'how' of learning. They are responsible for planning, executing, and monitoring our actions. In line with this, Neuroscience has shown and confirmed that learner variability is the rule rather than the exception.

Figure 2.1 illustrates classification of UDL principles in line with the brain networks, thus explaining how learning takes place.

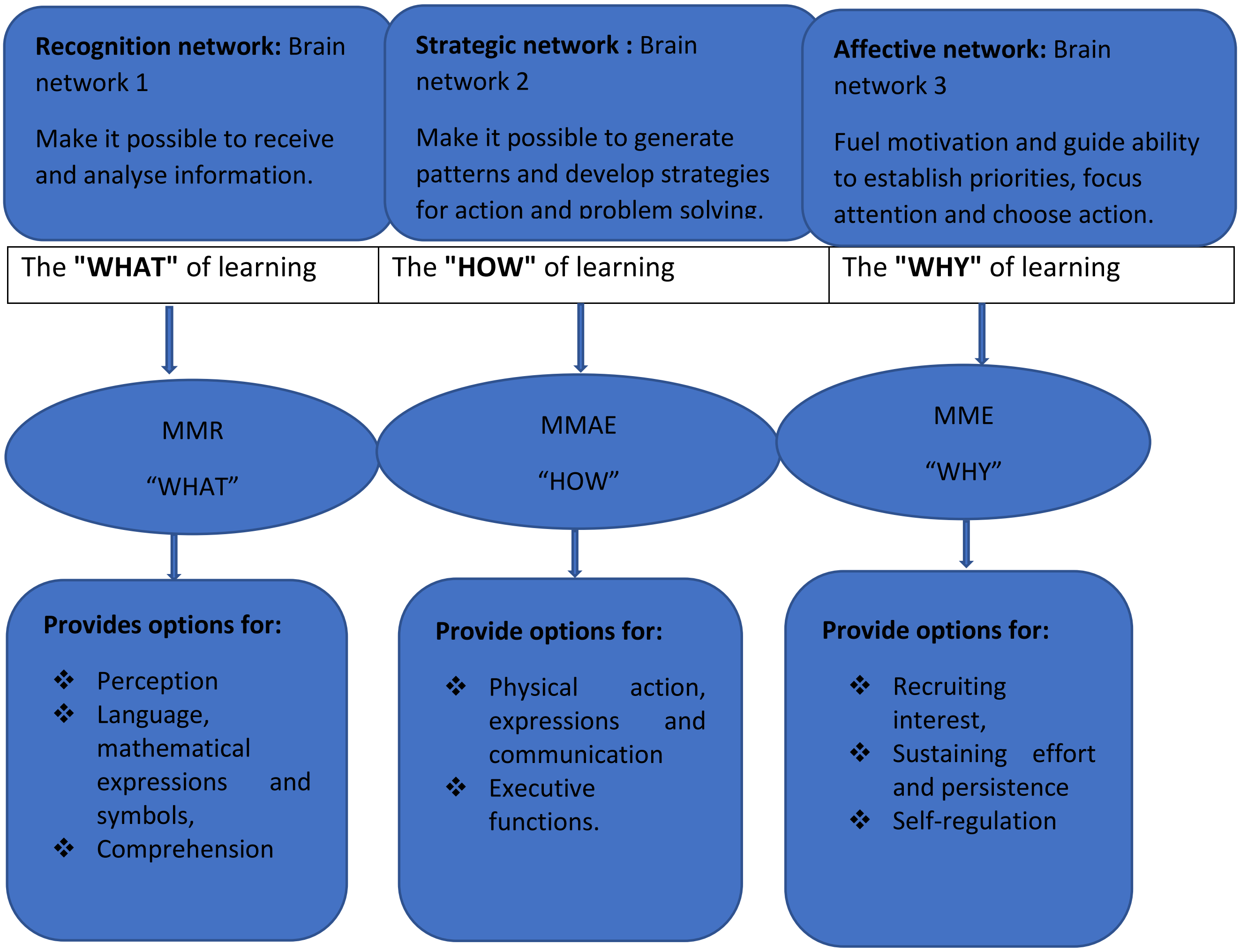


Figure 2.1: Classification of UDL principles

(Source: adapted from Balta, Supple & O'Keeffe, 2020)

MMR supports recognition of brain networks. These brain networks make it possible for learners to receive, analyse information and recognise the object of learning (mathematical concept); hence it addresses the 'what of learning' (Ross, 2019). In a classroom setting, teachers apply the MMR principle by customising information display (that is, representing content in multiple ways and thus meeting the needs of the various learners (Parrish, 2019).

Strategic brain networks. MMAE supports them. They facilitate the demonstration of the acquired knowledge using multiple and flexible physical actions and expression methods. They address the 'how of learning'. In a classroom situation, teachers may allow learners to interact with one another and the material to demonstrate what they learned in diverse ways (Meyer et al., 2014)

MME supports affective brain networks. According to García-Campos et al. (2020), affective brain networks trigger enthusiasm and interest in learning. They harness the power of emotions and motivation in learning. Affective brain networks are responsible for the 'why of learning'. They control learners' emotional involvement with learning and motivate them to focus and remain engaged with tasks.

2.3.8 Definition of UDL in the Context of the Study

Various scholars have provided definitions of the UDL concept (Barteaux, 2014; Dalton, 2017; Stolz, 2020). They all consider UDL to consist of proactive strategies to create a barrier-free learning environment. Barteaux's (2014) definition, which states that UDL makes information and learning activities accessible to a diverse classroom, is the most appropriate one for the study's purpose.

2.4 REVIEW OF THE RELATED LITERATURE

This section discusses literature based on the teaching and learning of Euclidean geometry. The subsequent sections are organised in line with the objectives and research questions of the study as highlighted in (1.3). The section, therefore, covers the following:

- challenges on the teaching and learning of Euclidean geometry,
- the solutions implemented to address the challenges and
- ways to mitigate against risks that may hinder optimal benefits of the use of the identified strategies for teaching and learning

2.4.1 Challenges

The subsequent sections highlight some of the challenges pertaining to the teaching and learning of Euclidean geometry.

2.4.1.1 Lack of knowledge of inclusive teaching strategies for teaching Euclidean geometry

According to Ubah and Bansilal (2019), teaching Euclidean geometry is difficult and complex because teachers lack the knowledge of content and inclusive teaching strategies. Some Mathematics teachers lack knowledge of Euclidean geometry and how to teach proof and reasoning because they did not study the topic at any level of their studies (Machisi, 2020), especially those who did matric prior to the introduction of the Curriculum Assessment Policy Statement (CAPS) in 2012 (DBE, 2012). Machisi further articulates that teachers often use teacher-centred approaches where they copy theorems and proofs onto the chalkboard and then teach these theorems using a traditional lecture method. Learners thus copy and memorise the notes written on the board so they can reproduce them in-class tests and examinations. Learners are therefore not given the opportunity to discover the concepts independently, which impedes their ability to develop conceptual understanding. Mthembu (2007) refers to this traditional method as an 'explain-memorise' teaching approach. This method promotes memorisation and procedural understanding. According to Boggan, Harper and Whitmire (2010), some teachers seem reluctant to use resources such as manipulatives, making it difficult for learners' metacognitive skills to be enhanced and to develop high thinking skills.

Ubah and Bansilal (2019) confirm that poor performance in Euclidean geometry is due to non-inclusive teaching strategies. It should be noted that there are learners with different learning styles and preferences in every classroom. Some are visual learners, while some are tactile (touching and doing) or auditory (through listening). This means that learners' learning styles should be taken into consideration by the teacher in the choice of approaches and strategies. Thus, the teachers' failure to vary and use inclusive teaching strategies makes it difficult for meaningful learning to take place.

Euclidean geometry is a practical topic that requires teachers to convey principles and explain concepts verbally and make demonstrations and drawings, which are embedded in levels 1 and 2 (recognition and analysis) of Van Hiele's theory of geometrical thinking. A study conducted by Ngirishi and Bansilal (2019) revealed that most learners are operating at visual (level 1) and analysis (level 2) of Van Hiele's theory of geometric thinking. This is problematic because grade 11 learners must think critically and use deductive reasoning to prove theorems as suggested by the Department of Basic Education (Machisi, 2021). This means that teachers must plan their forms of instruction aligned to learners' levels of geometric thinking and be mindful of the different ways learners assimilate information in the process. Mthembu (2007) observed that teachers consider teaching Mathematics as manipulation of numbers. They overlook the importance of spatial representation and language, which are crucial in developing and communicating mathematical ideas, particularly in the teaching of Euclidean geometry. Mthembu (2007) further testifies that teachers feel that the investigative approach of teaching, which is in level 3(abstraction) of Van Hiele's theory of geometric thinking, is time-consuming and delays the completion of the syllabus. This explains why the teachers practice the drilling method in teaching Mathematics, especially Euclidean geometry.

Studies conducted by Abdullah and Zakaria (2013) and Siyepu and Mtonjeni (2014) show that Euclidean geometry is taught through a traditional teacher-centred approach. According to Govender and Govender (2019), some teachers are expected to teach Euclidean geometry, yet they never had any contact with the topic. Machisi (2021) points out that in many geometry classrooms, teachers write theorems and proofs on the chalkboard and give a lecture, and then learners are asked to copy them onto their books. The learners are bound to memorise and reproduce the theorems and proofs during the class tests and examinations. Teachers' shortcomings in using inclusive teaching strategies to meet the needs of diverse learners in the mathematics classroom emanate from their Lack of knowledge of the geometric concepts.

2.4.1.2 Teachers' inability to create an engaging environment for meaningful learning to take place

The engaging learning environment is the one that encourages learners to embrace their uniqueness and motivates them to strive for excellence (Boligger & Martin, 2018). The learning environment is engaging if it allows multiple flexible ways of instruction to ensure that learners can work cooperatively. Burgstahler (2009) ascertains that in cooperative learning, learners can be divided into small groups to discover and help each other understand new concepts. However, in my observation, many classrooms' seating arrangements do not encourage collaborative work since teachers are often at the front. At the same time, learners are seated facing the same direction, implying that the teacher is the only source of knowledge. This sitting arrangement does not promote an engaging environment. Burgstahler (2009) further indicates that to create an engaging learning environment, physical space in the classroom should be such that it allows interaction of learners and maximises meaningful learning for all.

According to Makina (2010), meaningful learning, as opposed to rote learning, is characterised by the learner's ability to gather information efficiently, sort it creatively, and come to a reliable and trustworthy conclusion. It is the ability to relate new events to already existing concepts. Bolliger and Martin (2018) point out that meaningful learning is influenced by three forms of interaction in the learning environment: learner-content, learner-instructor, and learner-learner interaction. Kutama (2002) notes that unless teachers create an engaging learning environment wherein learners can read and write with understanding, talk, draw, and show with their hands, learners will continue to experience challenges in the learning of Euclidean geometry; thus, they will fail to develop connections and conceptual understanding. Conceptual understanding is characterised by learners' ability to transfer acquired knowledge into a new situation and use it for different purposes (Alex & Mammen 2018). Teachers overlook the importance of creating a positive, engaging classroom environment for all learners. According to Sepeng and Webb (2012), this leads to productive discussions that deepen the learners' understanding of the content. Teachers who do not create an engaging environment deny learners an opportunity to interact and engage with each other and the teaching material to explore and practice high-level critical thinking skills. This makes it difficult for meaningful learning to take place and for learners to develop conceptual understanding.

2.4.1.3 Lack of visualisation skills

Visualisation is a cognitive process of forming images or constructing mental representations (Klerkx et al., 2014). It enhances critical thinking, which is the most crucial skill in learning mathematics (Makina 2010). Although visualisation plays a vital role in teaching and learning, Makina (2010) indicates that teachers often present geometry lessons without understanding how learners think during problem-solving. They do not allow learners to imagine and think about the concepts that they are learning but instead, they focus on teaching them rules without engaging them in reflective and independent thinking. Strakova & Cimermanova (2018) affirm that learners' ability to think critically does not develop naturally; teachers must facilitate it. According to Mudaly and Dowlath (2016), learners who cannot visualise information have difficulty sketching their diagrams from given information. This becomes a barrier to learning Euclidean geometry because this concept includes types of problems mainly in the form of text (Kutama, 2002). These problems require learners to read and create mental pictures concurrently of what they are reading, which becomes a challenge since many teachers do not incorporate strategies that help develop the learners' visualisation skills in their teaching (Moleko, 2021).

Moreover, according to Djumanova (2021), teachers seem to be ignorant about the impact of learners' prior knowledge; consequently, they tend to build new knowledge on an unstable shaky foundation. Moleko (2014) affirms that insufficient prior knowledge negatively impacts learning. For example, learners cannot visualize and prove theorems if they neither know the properties of angles, lines, and shapes nor visualise these properties.

Thornton (2001) states that in any classroom, there are learners who learn efficiently from either concrete, pictorial imagery (pictures in mind), pattern imagery, memory images of formulae, or kinaesthetic imagery (involving muscular activity) or dynamic (moving) imagery. Mathematics teachers are not mindful of this range of visualisation skills generated by individuals in their classrooms. Studies have been conducted on the role played by visualisation in many disciplines, including health and psychology (Makina, 2010). These studies affirm that visualisation plays an important role in making large-scale decisions in life. Making decisions based on visualisation involves cognitive processes, including thinking, knowing, remembering, judging, and problem-solving. Bradford (2004) witnesses that learners are variant in learning and have diverse preferences. Bradford further declares that about 65% of the population are visual learners. They learn best with pictures, images, and mind maps, to mention a few. About 30% of the population consists of auditory learners who learn best through hearing. Lastly, kinaesthetic learners form 5% of the population. Their best way of learning is to involve them in physical activity (Bradford 2016). Based on the above, teachers must not ignore the impact of visual representations in teaching and learning mathematics. Buentello-Montoya, Lomelí-Plascencia & Medina-Herrera (2021) ascertain that visualisation is an essential cornerstone of teaching for understanding, and critical thinking is the determining factor in understanding quality. As a matter of fact, poor visualisation skills negatively impact teaching and learning of Euclidean geometry, which predominantly affects. For instance, traditional tools such as textbooks and chalkboard methods do not support spatial visualisation skills (Alqahtani et al., 2017). According to Algahtani et al. (2017) instructors indicate that students who come to tertiary institutions are challenged to process visual objects and mental images of the mathematical models. The reason could be that learners' visualisation skill was never promoted and stimulated at the high school level.

2.4.1.4 Lack of knowledge of geometry terminology and symbolic representation

Burgstahler (2009) states that learners are distinct in character because they come from various ethnic and racial backgrounds; hence they have diverse learning styles and ways in which they assimilate information. Unfortunately, for many of the learners in South Africa, English is not their first language (Mahlambi & Mawela, 2021). Kodisang (2015) describes language as a vehicle for communication and in the mathematics classroom, using appropriate terminology and symbols is crucial to enhancing meaningful understanding of concepts in the discussion. Being contextual, as stated by Makhubele at el. (2015), Euclidean geometry challenges learners because teachers fail to give proper attention to the vocabulary used and ways to unpack statements and diagrams in the question. They provide learners with information and steps on how to answer questions without analysing and understanding the given information and symbols in the question. According to Sinclair, Bartolini Bussi, de Villiers, Jones, Kortenkamp, Leung & Owens (2016), the threats of learning geometry, even from the primary level, consist of understanding of visuospatial reasoning, the use of diagrams, understanding the use of technologies, teaching and learning of geometric terminology and moving beyond traditional teaching strategies. Problems written in the form of text are complex for most teachers to teach Moleko (2018), and as Euclidean geometry includes problems mainly in the form of text and symbols, teachers need to develop learners' spatial knowledge and reasoning skills (Kutama, 2002). Learners' challenge lies in the analysis and correct interpretation of geometric statements; for example, if a learner does not know what a bisector is and what it does, it becomes challenging to understand theorems such as *"a line drawn from the centre of a circle, perpendicular to the chord, bisects the chord."* Terms including bisector, perpendicular, parallel, segment, cyclic quadrilateral, chord, and others make learning theorems understandable and perceivable. Appropriate terminology in any field is critical for acquiring a sound understanding of the content, hence it is imperative that learners develop this mathematics vocabulary even if it is in English.

Makhubele (2014) avows that the performance in Mathematics depends on language for meaningful teaching and learning and this is true with the teaching and learning of Euclidean geometry as teachers in South Africa are challenged to teach Euclidean geometry because of language-related technicalities (Mamali, 2015).

2.4.2 Solutions to the Identified Challenges

The following sections highlight the solutions to the identified challenges.

2.4.2.1 Inclusive teaching strategy for teaching Euclidean Geometry

Zambo and Zambo (2008) consider Professional Development Workshops (PDW) as a tool to influence teachers' content knowledge and skills required to teach Mathematics. PDW creates a platform for teachers to learn new ideas and ways to help learners (Zambo & Zambo, 2008). This is in line with the UDL framework, which advocates fostering collaboration and community amongst the teachers (CAST, 2018) to build their capacity to teach mathematical concepts. CAPS (DBE, 2012) 1.3c points out that one of the general aims of the South African curriculum is to encourage an active and critical approach to learning rather than rote learning of given truths.

Furthermore, CAPS (DBE, 2012), 1.3e states that inclusivity should be central in planning and teaching; therefore, teachers should have a sound understanding of embracing diversity in classrooms. An inclusive teaching strategy requires teachers to use multiple flexible forms of instruction to cater to learners' uniqueness in learning styles and preferences. Teachers must create a learning environment that, according to the UDL framework, should afford learners options to demonstrate what they have learned and promote their expectations and beliefs that optimise motivation (CAST, 2018). This kind of learning environment is essential to promote active learning, foster critical thinking and motivate learners to own the process of learning solving activities (Chan et al., 2014). According to Mthembu (2007) learners who work in pairs or groups perform better than when they work individually.

For this reason, teachers should structure their instruction on geometry so that learners could interact with one another. It means that learner involvement is critical in the teaching and learning of Euclidean geometry. In summary, teachers must know their learners and the way they learn. Burgstahler (2009) established that teachers should use equitable and flexible teaching strategies that promote accessibility and maximise learning. When teachers know that learners present with different learning styles, they can design inclusive lessons to make content accessible to all learners and effectively communicate necessary information. For example, suppose a teacher wants the learners to show the validity of the theorem that says *" tangents drawn from the same point outside of the circle are equal in length."* The teacher must allow flexibility in approaching this question. Some learners may decide to accurately draw a diagram of a circle and two tangents from the same point and then measure the lengths of the tangents to check if indeed they are equal. Alternatively, others may prefer to calculate the lengths of the tangents if given the radius or diameter of the circle and the distance from the centre of the circle to the point of intersection of the two tangents.

2.4.2.2 Creating an engaging environment for meaningful learning

An engaging learning environment is an interactive environment that can be developed by integrating technology into the teaching and learning mathematical concepts. Knowledge of learners and their learning styles is indispensable when creating an engaging environment because one must appreciate learners' diversity in the way they absorb and assimilate information. Lebid and Shevchenko (2020) affirm that technology improves learners' memory skills, visual-spatial skills, and multitasking abilities. Bolliger and Martin (2018) point out that in a classroom situation, collaboration involves a working relationship between a teacher and the learners and amongst learners themselves. Bolliger and Martin further state that engagement strategies allow learners to participate in collaborative group work where they can facilitate presentations and discussions and be involved in hands-on activities.

As identified by Einfeld (2014), three forms of interaction in promoting engagement in teaching and learning are fundamental. They are

Learner-content: Learners must engage with content, such as being involved in hands-on activities. However, not all learners would learn from hands-on activities. Bradford (2016) confirms that learners assimilate information in diverse ways. Bradford further indicates that visual learners rely on pictorial representations. They need to see pictures, diagrams, and graphs to visualise given information. A saying that a picture is worth a thousand words is valid for visual learners. For auditory learners, the best way to stimulate communication and learning is through discussions, oral presentations, repetitions, and group chats, to mention a few. They prefer to discuss what they hear. Lastly, Bradford (2016) further affirms that kinaesthetic learners need to get up and be involved in the action for the information to sink into their memory. For instance, learners may create geometry vocabulary cards for conceptual understanding of geometry terminology, or they may prefer to sing and rap properties of shapes. They may use geoboards to explore the properties of shapes, angles, lines and their relationships. According to Mukunthan (2013), Geogebra was invented in the 1950s by an Egyptian Mathematician called Caled Gattegno. It is an excellent tool for exploring basic geometric concepts such as properties of 2-dimensional shapes, area, perimeter and geometry of straight lines.

Learner-instructor means learners must work in collaboration with the teacher. For instance, Machisi (2020) points out that in the teaching of Euclidean geometry, a teacher may start by giving learners a brief history of its origin, reasons for offering it at the secondary school level, the role it plays in a real-life context for careers that are key in the economic development of any country. They will be motivated to study the concept and take ownership of their learning.

Learner-learner means learners are encouraged to work as a team. According to Moleko (2014), learners' engagement creates a platform for them to work together in actively constructing knowledge that enhances meaningful learning. They do most of the discussion among themselves while the teacher contributes minimally. For example, some learners understand the theorem about the sum of opposite angles of a cyclic quadrilateral equal 180°, if they could draw the cyclic quadrilateral and measure the opposite angles.

Khan et al. (2017) encourage teachers to target active learner involvement and user-friendly material. Moreover, teachers may engage learners in a small group discussion or assign different roles to learners in their discussion. For instance, a learner may be assigned to demonstrate that angles in the same segment are equal. This affords them avenues for content mastery, delivery and knowledge of communication.

2.4.2.3 Strategies to enhance Learner Visualisation Skills

Bradford (2016) affirms that generally, a population consists of groups of people with invariant learning styles. This is also true in a classroom situation. About 65% are visual learners, about 30% are auditory and 5% are kinaesthetic learners. The skill of visualisation is pivotal in teaching and learning mathematics and other areas such as maps and drawings (Klerkx et al., 2014). It is dependent on the connection between prior knowledge and new knowledge and if prior knowledge is not sufficient, the whole learning process would be negatively affected (Moleko, 2014).

Similarly, Masilo (2018) confirms that considering learners' prior knowledge enables them to connect concrete and abstract levels of Van Hiele's theory of geometric thinking. For instance, Masilo is very vocal about probing to stimulate critical thinking. Using manipulatives promotes perception, encourages recognition, and ensures conceptualisation and visualisation of analysis and conjecturing (Masilo, 2018). This supports learners' ability to create pictures mentally based on the words they hear and the texts they read to critically analyse them (Palavan, 2020). According to Shatri and Buza (2017), visualisation positively influences development and increases learners' critical thinking, including analysing, interpreting, presenting and evaluating. A teacher may give learners a geometrical puzzle which can be solved simply by looking, thinking, and imagining; for example, learners may be provided with a set of 2-dimensional polygons and asked to identify, without calculating, the ones that are congruent or similar.

2.4.2.4 Enhancing knowledge of Euclidean Geometry vocabulary

According to Alex and Mammen, (2018) knowledge and correct terminology are central in any field of study. The use of geometric terms such as point, line angle, parallel, perpendicular, plane, square, triangle and rectangle, to mention a few, enable one to communicate their ideas either in writing or speaking in precise forms in a diverse society. Teachers must use the correct terminology when teaching Euclidean geometry to avoid misconceptions and confusion. Peng and Lin (2019) state that the study's findings conducted in the United States of America (USA) at the University of Texas in Austin, revealed that Mathematics vocabulary made a noticeable contribution to word problems, particularly in measurements and geometry. According to Cain, A. (2014), knowledge of the terminology of a subject creates a positive classroom environment, improves attention, supports emotional regulation and reduces the anxiety of learners. The correct use of Mathematics vocabulary is critical for improved mathematics performance, and for this reason, Moleko and Mosimege (2020) encourage the explicit teaching of mathematics vocabulary to maximise learning. In my experience as a Mathematics teacher, I realised that for a meaningful understanding of any topic in Mathematics, knowledge of terminology and jargon is very important. The mathematics curriculum is spiral. This means that the curriculum allows a logical progression from simple to complex ideas. Most topics build on each other with increasing complexity. I believe that if teachers in the lower grades could actively involve learners by allowing them to draw lines, angles and shapes, explore their properties and make conjectures, then that would affect the use of a many geometrical terms, which are vital in the teaching and learning of Euclidean geometry. In addition to the activities, a glossary of geometrical terms and their symbol representation could stimulate a meaningful understanding of geometry. That would enhance their knowledge of geometrical vocabulary and symbols.

2.5 ANTICIPATED THREATS TO THE TEACHING AND LEARNING EUCLIDEAN GEOMETRY

The following sections highlight the conditions that hamper effective teaching of Euclidean geometry.

2.5.1 Teacher Training on how to Teach Euclidean Geometry

Professional Development Workshops (PDWs) focus on empowering teachers with content knowledge (Euclidean geometry) to afford them the opportunity to learn new ideas about the aspects of geometry (Zambo & Zambo, 2008; Machisi, 2021). Although the PDWs assist in enhancing teachers' content knowledge, they often do not equip teachers with knowledge of inclusive pedagogies to cater to diverse learner populations since most teachers and training facilitators lack knowledge of these pedagogies. For instance, when Euclidean geometry was brought back to mainstream mathematics curriculum, the Department of Basic Education (DBE) conducted training workshops throughout all provinces in the country (Machisi, 2021). According to Dube (2016), some of the CAPS training facilitators seemed to be week and not able to equip teachers with necessary skills to teach Euclidean geometry effectively. In addition, they did not have sound knowledge of inclusive teaching strategies. It is noteworthy that these workshops violated one of the specific aims explicitly stated in the CAPS 1.3e p. 5) which states that:

> *Inclusivity should become a central part of the organisation, planning, and teaching at each school. This can only happen if all teachers have*

a sound understanding of how to recognise and address barriers to learning, and how to plan for diversity.

The teacher development training workshops which do not provide teachers with hands-on activities where they could engage in exercises that involve Euclidean geometry, would make it difficult for the teachers' knowledge of Euclidean geometry to be advanced. Teachers' negative attitude and dislike for Euclidean geometry may also make it difficult for the PDW to be effective and for their knowledge to be advanced. Furthermore, their sense of disliking the topic (Euclidean geometry) may cause them to absent themselves from the training workshops thus rendering the workshops ineffective.

To address the above-mentioned issues, teacher development workshops should be organised in such a manner that they give ample opportunities for teachers to become empowered with in-depth knowledge and skills on how to teach Euclidean geometry. According to Tsotetsi and Mahlomaholo (2013), there is a need to design a strategy for effective implementation. This study aims at designing a strategy to effectively implement Continuing Professional Development (CPD) programmes for teachers. The teachers should also be actively engaged in practical hands-on sessions wherein they can engage in solving Euclidean geometry problems. The Euclidean geometry activities that are empowering and enabling will assist in eradicating the teachers' negative attitude, recruit their interest and develop their love for the topic. Constant monitoring must also be conducted in order to ensure that the teachers are supported even beyond the workshop sessions. A continued support of this nature will be motivational to the teachers and instil them with the necessary confidence to teach Euclidean geometry.

2.5.2 Lack of Knowledge of Learners and Geometric Thinking

Teachers' competency with Euclidean geometry content is pointless if they do not know their learners and understand their thinking processes because their choice of the learning material would not appropriately address individuals' diverse learning needs. The issue of teachers who do not know their learners is crucial because even if learners have a mastery of the mathematical concepts (Euclidean geometry). they will continue to display poor performance when teachers fail to apply the strategies that cater for their needs. One of the important aspects of Pedagogical Content

Knowledge (PCK) is knowledge of the learners (Bam & Mavhunga, 2015; Rollnick & Davidowitz, 2015). This knowledge is important for it enables the teacher to understand how the learners develop, this allows the teacher to know how their learners work and how they can relate to them in the process. Knowledge of the learners also makes it possible for teachers to understand how their learners think and how they can reach them intellectually. However, if teachers lack this knowledge, it will be challenging to understand the learners intellectually and cater to their varied needs. Equally important to note is that even if teachers' Euclidean content knowledge is advanced, they lack knowledge of levels of geometric thinking that the learners need to pass through to learn geometry, learners will continue to struggle to understand this topic resulting in poor performance.

To address the issues mentioned above, teachers would have to be introduced to workshops wherein they receive empowerment and enlightenment on aspects of knowing their learners. The teachers should also be empowered with knowledge on how to differentiate the curriculum as it would enable them to develop strategies that meet individual learners' needs in their classes. Similarly, the workshops on geometric thinking by Van Hiele where learners' levels of geometric thinking are studied (Chidziva, 2021), would help the teachers scaffold learning appropriately in a manner that allows the learners to develop an understanding of Euclidean geometry and its mastery.

2.5.3 Time Factor and Shortage of Funds

Teachers need considerable time to know their learners and find ways to facilitate their critical thinking, which would inform teachers' decisions on the choices of teaching strategies. Unfortunately, most teachers complain about the time since the curriculum is overloaded. As a result, teachers fail to invest much time in devising productive teaching and learning strategies that accommodate a diverse learner population. The expectations on teachers to cover the topics, as outlined in the pacesetter, discourages the teachers from making efforts in terms of devising strategies that would meet the needs of the learners.

Furthermore, financial status within school systems might impede the implementation of flexible teaching strategies. For instance, Quintile 1 and 2 schools, where most learners come from a disadvantaged background, may not have sufficient funds to buy the teaching aids. Therefore, lack of teaching resources may hinder the teaching and learning processes.

One way to overcome the shortage of learning material due to the unavailability of funds is that schools in one cluster may share the resources among themselves. Kabi (2016) states that neighbouring schools should share resources to have a sustainable learning environment. Furthermore, teachers may be trained to develop their manipulatives from readily available material. For instance, they may use egg cartons/trays, nails, and rubber bands to create geo-boards. Additionally, teachers' training institutions should work closely with institutions of higher education, particularly with people who have published studies on how to teach Euclidean geometry. These people may share findings of their studies on the best practices of teaching Euclidean geometry, particularly in different contexts such as urban, rural and farm schools, and the importance of knowing learners' levels of geometric thinking to inform the choice of effective teaching strategies of Euclidean geometry. According to Chidziva (2021), teachers must study learners' levels of geometric thinking and in addition, teachers need time to develop knowledge of curriculum saliency, which is one of the essential components of teacher knowledge (Rolnick & Davidowitz, 2011). This vital knowledge will enable the teachers to ‘omit certain aspects of the topic’ without necessarily disadvantaging the learners and denying them access to other aspects of the topic. This knowledge will also enable the teachers to sequence the topics so that learning can occur meaningfully.

2.6 CONCLUSION

This chapter discussed social constructivism as a theoretical framework underpinning the study. The origins, historical background, and suitability of the theory were thus discussed. The key operational terms such as Euclidean geometry, universal teaching, Universal Design for Learning, approach, and Van Hiele's theory of geometric thinking were also defined since they are the pillars on which the study is built. The chapter further reviewed the literature on the challenges pertaining to the teaching and learning of Euclidean geometry. The chapter also outlines some of the UDL principles vital in the teaching of Euclidean geometry. Furthermore, the solutions identified to address the challenges were noted while indicating the role of UDL in the process. The chapter also highlighted the threats that may impede the successful implementation of the solutions.

The next chapter will cover the study's research methodology and design.

CHAPTER 3
RESEARCH DESIGN AND METHODOLOGY

3.1 INTRODUCTION

The study aimed to explore implementing the Universal Design for Learning (UDL) approach to enhance the teaching of Euclidean geometry. This chapter explains Participatory Action Research (PAR) methodology as the research design to generate empirical data to respond to the research question: *How can the Universal Design for Learning be implemented to enhance the teaching of Euclidean geometry?* Therefore, methods and techniques for generating data and analysis are discussed in this chapter

The chapter covers the definition of PAR by various authors, its evolution and historical background and its objectives. The five phases of PAR are also elucidated in this chapter, detailing what transpired in each stage. Additionally, the chapter gives the rationale behind the suitability of PAR in the study. This chapter also covers the SWOT analysis detailing the team's Strengths, Weaknesses, Opportunities, and Threats (SWOT). The chapter further specifies the roles of the researcher and the co-researchers according to their profiling, their selection, data collection instruments and procedures, quality assurance, trustworthiness, triangulation, and data analysis techniques followed.

3.2 PARTICIPATORY ACTION RESEARCH

Mubuuke and Leibowitz (2013) consider PAR a form of research involving co-researchers in every research study phase. PAR is efficient because it allows the co-researchers to share their voices and play a meaningful role in the study. PAR has a longstanding history of researching marginalised communities (Malebese, 2016). Through PAR, the researcher collaborates with the co-researchers to understand a problematic situation and change it for the better. MacDonald (2012) defines PAR as a qualitative research methodology that is democratic, equitable and liberating because the feelings and views of co-researchers are revealed without manipulation from the researcher.

Contrary to the last few decades, where the research process assumed co-researchers to be the 'research subject', where their ideas and feelings were not of significance, PAR advocates the consideration of co-researchers as influential and valuable individuals for the research process in educational settings (Sokhanvar & Salehi, 2018). Moleko (2014) attests that PAR elevates marginalised people to the position of the co-researcher, not the researched. That is, the researcher and co-researchers function from the same power basis.

Tetui *et al.* (2018) confirm that the PAR approach treats the communities of inquiry as generators of knowledge. The power relations are redistributed when deciding what is researched and the benefits of the research across all stakeholders. In such collaborations, communities play a role in decision-making. Hence the PAR approach is appropriate for resolving social issues. Additionally, Moleko (2014) authenticates that the PAR approach creates a platform for people habitually excluded from decision-making to voice their viewpoints towards achieving the desired change in their situation. It accommodates those whose perceptions are never acknowledged.

Miranda, Fine, Torre and Cabana (2018) noted that PAR is not just a methodological approach but also an ethical stance, an honest way to build a relationship with others. Jacobs (2016) asserts that PAR is a tool for addressing challenges encountered by the marginalised community. The beauty of PAR is that topics for investigation emerge directly from co-researchers' experiences, aspirations and fears. They are allowed to identify the solutions to their problems (Mahlomaholo, 2013) and this makes them develop a sense of ownership

3.2.1 Origins and Historical Background of PAR

Wallerstein, (2020) notes that participatory research emerged from the emancipatory philosophy of Paulo Freire (1970), which is about the movement challenging social inequalities. According to Duke (2020), PAR is known to have originated with a social psychologist, Kurt Lewin, whose research began in the late 1930s, and Paulo Freire, who was born in 1921 in Brazil. MacDonald (2015) points out that Lewin believed that people perform outstandingly well if allowed to participate in decision-making. Lewin thus focused on studying the social system to demonstrate the importance of including society when solving social problems. Therefore, Lewin's form of action research involved investigation of discrimination challenges and segregation, assisting people in initiating change and studying the effect of the change (Tsimane & Downing, 2020).

MacDonald (2015) pointed out that Freire believed that conceptualising, applying, analysing, synthesising and evaluating information is crucial for social change. Freire was more concerned about empowering the marginalised community regarding literacy and land reform analysis issues. He was an adult educator and author of the book related to pedagogy, challenging traditional education grounded on power. He openly challenged social relationships and traditional education. In summary, Lewin's and Freire's focus was on the impact of democratic participation in factories and community settings. Therefore, PAR is about bringing people together to participate collaboratively in solving practical problems using data. Gill and Jackson (2000) indicate that the origin of PAR emphasises a strong relationship between theory and practice. On the other hand, Wallerstein (2020) considers PAR as the integration of ‘Action Research’ which Kurt Lewin (1999) defined as the action to engage multiple stakeholders in research, and ‘participatory research’.

Reason and Bradbury (2001) note that PAR cannot be viewed as a research methodology only but also as an agent of positive change in the philosophy of life, which involves development of the critical consciousness of the researcher and the co-researchers, improvement of the lives of the co-researchers and modifications in societal structures and associations. Therefore, PAR is a potential approach to generate multiple solutions for teaching Euclidean geometry to diverse learners.

3.2.2 Objectives of PAR

The goal of PAR is to solve real-world problems by creating a positive change and encouraging learning among the community closest to the change (Tlali, 2013). PAR involves all stakeholders in all stages of the research, encompassing defining the problem, inventing questions, collecting and interpreting data and preparing recommendations (McGarvey, 2007). According to Baloyi-Mothibeli, (2018) PAR creates a platform for the researcher and co-researchers to collaboratively work together to understand an issue of concern and change it for the better. It promotes learning among the marginalised community to revive hope that their situation will improve (Kabi, 2013).

Following Engagement (2021), PAR is meant to contest or mitigate social problems by involving the culprits of the situation in producing new knowledge. Therefore, another objective of the PAR process is to erode class disparity between the researcher and the researched. Engagement (2021) declares that separating the researcher and the researched often results in disassociated outcomes from the society where the data was collected. A wrong problem might have been investigated, or the findings could have been misled and manipulated. Moreover, McGarvey (2007) states that the PAR process aims to produce objective data which the affected community can confidently use to address their challenges and reach a sustainable solution.

3.2.3 Suitability of PAR for this Study

PAR was deemed apposite to be used as a research design for this study. It is informed by the point of view articulated by Kurshumlia and Vula (2020). They aver that participatory research is a pedagogical perception that enables co-researchers to take charge of their actions to transform their teaching practices. According to Leavy (2017), PAR allows the co-researchers to search for writings and learn and practise new methods to alter their teaching practices. Furthermore, PAR complements the social constructivism theory of learning, which is a lens that underpins the study in that they both advocate the notion of working collaboratively to construct knowledge. They further support creating a conducive environment for researchers and co-researchers to share ideas pertinent to addressing the identified problem collectively. Besides, they advocate empowerment, emancipation and liberation through social interaction of the people as they address the challenge in their situation (Armstrong, 2019). The belief is that the co-researchers can provide rich information through PAR. It is probable that through social interaction and sharing of experiences and ideas, the implementation of the principles of UDL to enhance the teaching of Euclidean geometry could be realised.

PAR is a relevant research design for this study because it justifies Briffett Aktaş, (2021) declaration that the PAR approach creates a platform for the co-researchers to receive recognition and start to consider themselves not as victims but as active community members. Educational researchers commonly use this research design to critically analyse educational issues to transform and improve the situation (Dania &

Griffin, 2020). Teachers and learners may regard themselves as direct victims of the poor performance of Euclidean geometry. However, in the PAR project, they become a prominent and active team component in solving problems and creating new knowledge. An individual cannot bring about change, but a group of people with diverse knowledge, experiences and skills can. It is attested by Leavy (2017) when he pointed out that in PAR, co-researchers work in partnership from diagnosing the problem to the final stage. They collectively evaluate and reflect on the results. As advocated by the social constructivism theory of learning and the PAR approach, the co-researchers will arrive at solutions to their problems through their interaction.

PAR is preferred over other research designs because of its nature. It focuses on planning, doing, observing and reflecting. According to Mahlomaholo (2013), it promotes equity, inspires hope and gives freedom to the marginalised communities usually excluded in the research process. PAR complements social constructivism (the lens underpinning the study) in that it is rooted in the fact that knowledge is socially constructed (Engagement, 2021). PAR and social constructivism form a significant component of participation in learning and creating new knowledge through social interaction. It is worth noting that they all promote the observation, collaboration, community involvement, democracy, exploration and learning in their broadest sense. They collectively pursue the removal of barriers that can impede people's ability to get optimal results and successful experiences. Removing learning barriers must be the focus when teaching Euclidean geometry to learners with diverse learning styles (Armstrong, 2019).

Although PAR is selected to generate empirical data in this study, it is worth noting that its implementation may come with possible challenges. The challenges may include lack of participation, lack of self-confidence, scepticism among the co-researchers, power hierarchies and status issues, to mention but a few (Moleko, 2018).

3.2.4 Participatory Action Research Phases

According to Tsotetsi (2012), the exceptional strength of PAR lies in its sequential stages, collective participation and action, education and indigenous knowledge. These qualities allow the integration of mindful and cautious thinking and reflection on the matter of discussion. The stages of PAR that make it possible for change to be created are outlined below. Figure 3.1 shows the stages of PAR followed in this study.

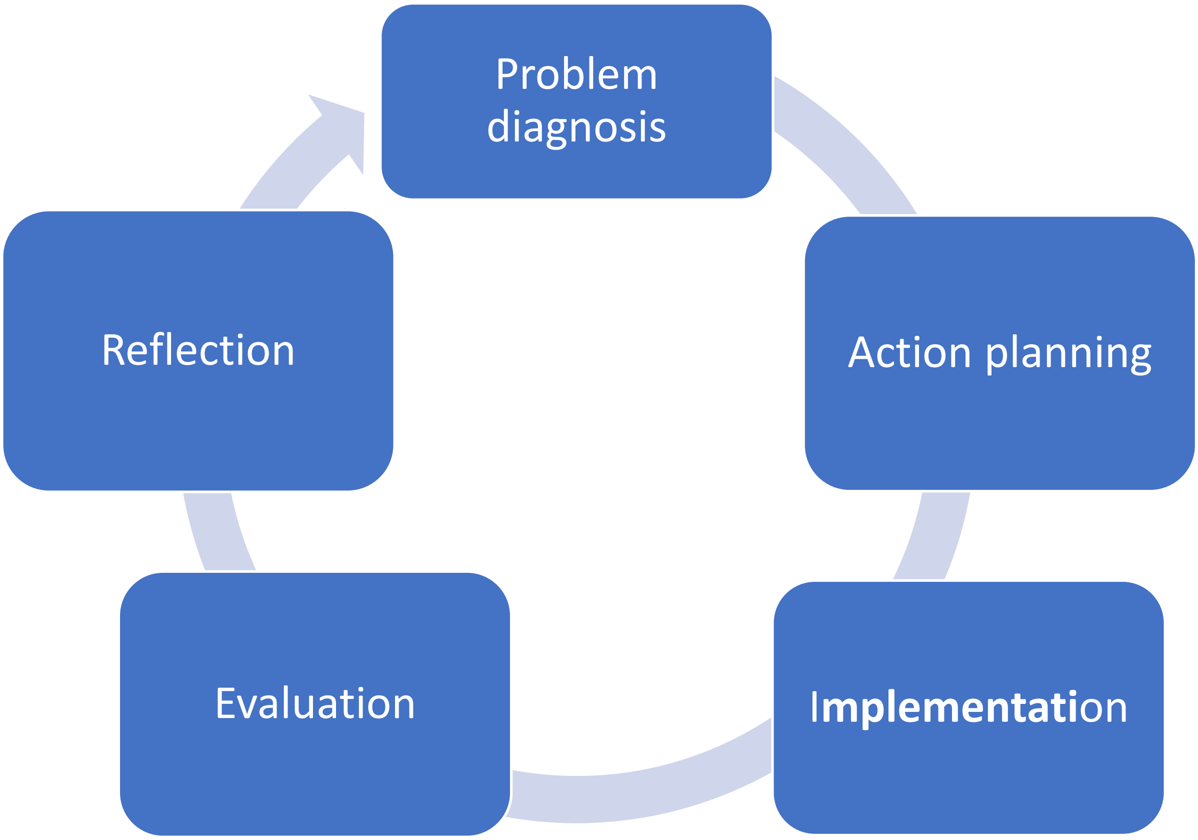


Figure 3.1: Stages of PAR

(Source: Tetui et al., 2017)

Figure 3.1 illustrates the cyclic nature of PAR while Table 3.1 offers a further elaboration of the phases and how they were followed in the study.

Table 3.1: The phase of PAR and how they were followed in this study

Stage	Stage Description	Activities in the study
Problem diagnosis	According to Wolk (2001), the researcher and co-researchers in this phase might have to consider the following questions:	• This is the stage where the researcher and co-researchers collectively identified a problem

Stage	Stage Description	Activities in the study
	• How do they know that it is a problem for their society? • How do they feel about the situation? • Why does the problem exist? • How does it affect their community? • What resources do they already have and other additional information they might need?	which was informed by examiners' diagnostic reports. • The DBE conducts workshops to empower teachers, but there is no noticeable improvement in learners' performance.
Action planning	This phase is about: Finding the root cause of the problem and suggesting strategies to remedy the situation. • What will be done to get the solution? • Who will be involved? • Where and when the research will occur? • What skills we need? • How to use the information acquired?	• All stakeholders in the education system highlighted that the root cause might be that • Teachers lack knowledge of Euclidean geometry, which leads to a lack of confidence, • Learners may have a negative attitude towards the topic because of the teaching methods. Teachers do not meet their learning needs. • Inclusivity nowadays is a matter of concern; it is discussed in all life spheres, including teaching and learning. • UDL coach was invited to join the team of knowledgeable people to raise awareness about diversity in our classrooms and how to remove barriers to learning.
Implementation	The phase is about: • implementing the identified strategies.	The co-researchers, who included teachers, subject advisor, Mathematics HOD, Mathematics coordinator, CES, and UDL coach,

Stage	Stage Description	Activities in the study
		who is also a mathematics specialist, shared their experiences and challenges in teaching Euclidean geometry. The coach conducted a session on how UDL could be incorporated into teaching Euclidean geometry to address learning barriers in mathematics classrooms. Later on, class observations, audio, and video recordings were made.
Evaluation	The phase involved: • discussion of the observations made during the implementation stage. • The researcher and co-researchers engage in focus group discussions to encourage positive performance and strengthen weak articulations to inform decision-making processes.	Focus group discussions were held, possible threats were identified, and ways to mitigate them.
Reflection	The stage is all about: • Reflecting upon the whole process from phase one. • Checking whether the goal is achieved. • Reflecting on the upshots, • The team looked back over the previous stages.	We reflected on the entire process and re-planned for a better version of the process.

3.3 SELECTION OF CO-RESEARCHERS

According to Vella et al. (2021), people recruited to participate in the research project must consent. They must be equitably recruited to provide rich data for the study. However, it is also significant to include vulnerable persons, mainly if the project's outcomes could be of value in their society. The selection of co-researchers in this study was made through the purposive sampling technique. Purposive sampling involves the deliberate choice of co-researchers knowledgeable about the concept under the study (Creswell & Clark, 2011). The purposive sampling technique of the co-researchers met the needs of the study in that every one of them provided crucial information which could not be given by any other person who is not in the field of mathematics education.

The co-researchers in this study were well informed about learners' poor performance in Euclidean geometry, which could have resulted from an ineffective teaching approach. According to Colucci (2007), individual self-disclosure and willingness to participate are instrumental in generating rich data for the research. The co-researchers were also conscious that there was a need to conduct the study on effective teaching of Euclidean geometry. Moleko (2014) attests that it could be difficult for the co-researchers to participate effectively in the research project without a clear understanding of the study. Wright (2020) witnesses that research tended to pay less attention to teachers and the challenges they experience in the mathematics classroom. They were often treated as passive recipients of the research findings and expected to implement them without participating. In this study, teachers' contribution is the priority and inescapable. According to Sokhanvar and Salehi (2018), PAR is welcomed in the education system and used in students' learning development and teachers' successful instruction. In this study, PAR was applied in developing teachers' effective teaching strategies of Euclidean geometry. PAR motivates teachers to feel free to participate in the research project alongside other stakeholders.

3.3.1 Co-researchers and their Roles

In this study, the team consisted of the following people: five teachers, three female (Ms. Mofokeng, Ms. Ndweni, and Ms. Tau), and two male (Mr. Mosola and Mr. Mokwena); (all the names used were pseudonyms), the Universal Design for Learning (UDL) coach (who was only responsible for giving training on UDL); the Curriculum Education Specialist (CES); the Subject advisor, the Mathematics coordinator; and the

Head of the Department of Mathematics (HoD). The entire team had experience in teaching Euclidean geometry and knowledge of the issues pertaining to the teaching and learning of geometry.

3.3.1.1 Universal Design for Learning coach

The UDL coach had been a Mathematics lecturer for more than six years. Her role was to train and introduce the concept of Universal Design for Learning (UDL) to the co-researchers (teachers), thus empowering them to use the principles in the mathematics classroom. The UDL coach explained the principles and how they can be implemented in the classroom. (See the training programme in Appendix B)

3.3.1.2 Curriculum Education Specialist

Generally, the CES is an expert on curriculum development (in this case, mathematics curriculum). He facilitates and trains teachers to improve their performance on high standards of teaching and learning and supports teachers in promoting key activities for implementing the curriculum. In addition to the general responsibilities, CES is also a chief marker for the NSC examinations. He has conducted in-service teacher training workshops on Euclidean geometry countrywide since 2012, at the inception of Euclidean geometry in Mathematics Paper 2. In this study, the role of the CES is to share with the co-researchers the common errors and misconceptions that learners have displayed when attempting questions on Euclidean geometry, which could be due to unproductive geometry teaching strategies.

3.3.1.3 Mathematics coordinator in the senior phase

The mathematics coordinator was one of the co-researchers in this study. He has been with the Department of Education for more than ten years. His role was to share teaching strategies of mathematics teachers and the effective teaching of Grade 9 geometry, including the geometry of straight lines and geometry of 2-dimensional shapes, which inform the learning of Euclidean geometry.

3.3.1.4 Subject Advisor

Mathematics advisors interact with many mathematics teachers in different schools. Including him in the study was crucial because of his experience of ten years in the office as a subject advisor. He shared teaching strategies used by other teachers when teaching Euclidean geometry. He also shared some of the areas where teachers struggle when teaching geometry.

3.3.1.5 HOD of Mathematics

The HoD's role is to run the department of Mathematics. He was included in the research because he has been teaching mathematics for the past 12 years and has been managing and leading his school's mathematics department for six years. He thus could suggest the best practices for quality teaching and learning of mathematics.

3.3.1.6 Mathematics teachers

Five mathematics teachers were selected based on their teaching experience of more than ten years, teaching Grades 9 to 12. Their role was to shed light on their challenges in teaching Euclidean geometry.

3.3.2 SWOT analysis

According to Nasreen and Afzal (2020), a SWOT analysis is a tool to divulge an organisation's strengths and weaknesses; to detect the opportunities for enhancement and the threats that need to be addressed for optimal output. Analysis of strengths, weaknesses, opportunities and threats (SWOT) has become a central tool for companies for strategic planning (Benzaghta et al., 2012). SWOT analysis is commonly used to analyse an organisation's internal and external situations. Strengths and weaknesses form the company's internal situation because they enable it to achieve the set goal and raise awareness about internal elements that may inhibit its success. Opportunities and threats are external conditions that empower the company to reach its goals. On the other hand, threats to the company are external aspects that are a potential barrier to accomplishing the goal (Benzaghta et al., 2012). Nasreen and Afzal (2020) confirm that a SWOT analysis was initially used for strategic planning by business organisations, but now it is commonly used by educational institutions.

In the context of my study, a SWOT analysis was performed to explore the strengths, weaknesses, and opportunities that the team had and develop ways to mitigate the identified threats. The analysis would inform the choice of activities to be executed in the study and align them with the role assigned to each co-researcher.

Table 3.2: SWOT analysis

SWOT	Explanation
Strength	• The team consisted of experienced people teaching Euclidean geometry and marking Mathematics Paper 2 for the NSC examinations. • They were dedicated and willing to engage in discussions to enhance the teaching of Euclidean geometry. They were optimistic about interacting and sharing their expertise to benefit the entire group. • The teachers and the officials from the DBE were acquainted with the challenges encountered in teaching Euclidean geometry.
Weakness	• A team that comprises different people with different levels of expertise and experiences puts specific individuals in a position wherein they do not feel comfortable participating, especially in group activities, as some may feel shy and overpowered by others.
Opportunities	• Having the UDL coach as a team member was an excellent opportunity to learn and expand classroom diversity knowledge. Consequently, this would inform the choice of teaching strategies. • The school gave us a classroom to conduct our face-to-face session. In addition, the team members had laptops, making it possible for us to conduct our meetings virtually.
Threats	• The time factor was the main threat because the co-researchers had different responsibilities in their places of work. Finding a meeting time that suited everyone during regular school hours was challenging. Additionally, the COVID-19 pandemic was another limiting factor as we had to observe the protocols and rules.

3.3.3 Mitigation of the threats and the weaknesses

McCarron et al. (2021) set out that it is crucial to establish ground rules for well-defined teamwork procedures. Based on the statement above, ground rules for effective meetings were set during the planning session. The team agreed that they should treat one another with respect for everyone to participate freely with confidence. It was clear that the co-researchers should avoid belittling each other and that everyone's contribution was valuable. Additionally, the researcher and co-researchers agreed on a convenient time for in-person and virtual meetings. They had different work schedules according to their varied workplaces, making it difficult to have free slots during the same period. Some focus group sessions were conducted on social platforms like Microsoft teams and zoom when the researcher and co-researchers could not gather in person.

Furthermore, we shared our thoughts through emails and WhatsApp messages. Some meetings were held on Saturday when all the co-researchers were available, adhering to the COVID-19 protocols, such as maintaining social distance, sanitizing the hands and the equipment used and wearing masks.

3.4 DATA GENERATION

It is pertinent for the researcher to select an appropriate research instrument to collect data so that analysis could result in reliable and credible answers to the research question (Adosi, 2020). The tools may include tests, questionnaires, interview schedules and focus group discussions. I used class observations and focus group discussions to collect the data in this study. According to Smit and Onwuegbuzie (2018), observation in qualitative research is a tool that systematically collects data using one's senses of sight and hearing. The observation approach is perfect for this study because one can easily pick it up if there is inclusivity or not. Smit and Onwuegbuzie (2018) further point out that researchers should be open to what they see and hear during observational procedures. Focus group discussions are ideal because they give an in-depth understanding of social issues. Pearson and Vossler (2016) affirm that focus groups form a reliable tool to explore co-researchers' experiences, perceptions, attitudes, opinions, beliefs, and ideas on a discussion topic of the study. This qualitative data collection approach creates a platform for world views to uncover society's perception and worth.

The researcher and co-researchers collected data through meetings and dialogues. Data were generated through active engagement and discussion among the co-researchers (Moleko, 2014). As previously indicated, the group consisted of teachers, CES, HOD, UDL coach, subject advisor, and mathematics coordinator. They are people with great expertise and knowledge of Euclidean geometry. I presented PAR and its phases to the group as an approach to collecting data for this study. PAR considers every person as a valuable human being, so ground rules were set. One of the rules was that we should all show humility and patience and create a non-threatening environment (see Appendix C).

3.4.1 Procedure

The team conducted a series of eight focus group discussions on teaching Euclidean geometry by highlighting the challenges and identifying suitable solutions to the challenges. The conversations were audio and video recorded. We discussed unproductive teaching practices observed during the lesson presentations during these meetings. We used a standardised observation tool designed as an instrument to assess the teaching of geometry in line with UDL principles (see Appendix D1). We used the Free Attitude Interview (FAI) technique by Meulenberg-Buskens (2011) to initiate conversations. The technique (FAI) requires one comprehensive question to be asked to initiate the discussions. According to Meulenberg-Buskens (2011), when FAI is used, the co-researchers get to communicate as in typical day-to-day conversations. This technique elicited as much information as possible from the co-researchers. The technique uses open-ended questions, which enabled the co-researchers to say more than they would in surveys with close-ended questions. In this study, the main question asked to initiate the discussion was, "How can the Universal design for learning (UDL) be implemented to enhance the teaching of Euclidean geometry?"

To ensure that the co-researchers did not deviate from the focus of the discussion, the researcher reminded them of the question asked and rephrased it where necessary to ensure that everybody understood it and interpreted it correctly (Mahlomaholo, 2009; Tsotetsi, 2013).

3.4.2 Quality Assurance

The trustworthiness of a study concerns the level of trust and confidence in the data, interpretation and methods used to ensure the quality of the study (Polit & Beck, 2012). In contrast to the positivistic approach, where the process is structured and detailed, the PAR approach allows the flexibility of the researcher and co-researchers because it accommodates a wide range of individual contributions throughout all phases of the research process. Triangulation, which refers to multiple data generation, was used as a strategy to test the validity of data through the convergence of information from various sources (MacDonald, 2012). Moreover, it was used to clarify meaning, verify observations' repeatability and interpret the generated data.

Audio and videotape recordings, transcriptions, and the documentation of minutes by the researcher and co-researchers increased the chances of the findings and the interpretations being credible, transferable, dependable and confirmable. These additional techniques discouraged biases, as one member could dominate and influence the team's discussion. Following the member checking process, I returned the analysed data to the co-researchers to access and validate them. According to Birt, Scott, Cavers, Campbell and Walter (2016), member checking reduces the chances of researcher's biases. Dube and Tsotetsi (2020), concurrent with Birt et al., note that member checking verifies the findings' trustworthiness, validity, and transferability. The researcher familiarised herself with the depth and breadth of data to be reported (Braun & Clarke, 2006).

3.4.3 Instrumentation

Adosi (2020) believes that selecting an appropriate instrument is vital to capture data analysed to give credible answers to the research question. The data was generated through observation, focus group discussions, and reflection sessions. According to Johnson and Christensen (2012), observation is one of the practical tools for collecting data. A researcher could see and hear what was happening at the site without interacting with the co-researchers. Focus group discussion is a technique that offers the researcher and co-researchers an opportunity to explore the issue under discussion deeply. The researcher acts as the facilitator to guide the group's discussion (Nyumba et al., 2018). The focus group discussion was appropriate for the study because it allowed sharing and comparing understandings and ideas, which yielded more insights about implementing UDL to teach Euclidean geometry effectively.

3.5 DATA ANALYSIS

Thematic analysis was used to analyse the collected data. Nowell, Norris, White, and Moules (2017) note that thematic analysis is fundamental for examining different co-researchers' points of view. The thematic analysis allows flexibility in interpreting collected data and highlights similarities and differences in generating unimagined insights in response to the research question (Nowell et al., 2017). This technique thus assisted in terms of identifying and organising the emerging themes.

Table 3.3: Six stages of the thematic analysis followed in this study

Stages		
1	**Familiarisation**	In this stage, we familiarised ourselves with the data by reading and re-reading the transcripts and listening to audio recordings. Before analysing the items, it is vital to get a thorough overview of all the collected data.
2	**Coding**	We generated codes by identifying patterns of meaning in the data, categorised and labelled relevant features concerning the research question. We sorted the data to make it easy to identify and describe each category. The codes made it easy to gain a condensed overview of the recurring responses.
3	**Searching for themes**	We identified patterns among the codes we created, combined several codes to a single theme, and disregarded those not relevant enough to the research question.
4	**Reviewing themes**	We went back to the data to check if it correlated with the themes created. We checked if anything was missing and whether the themes were really present in the data.
5	**Defining and naming themes**	We formulated names for the themes and figured out how they could help us understand the data.
6	**Writing the report**	We wrote the report on the analysis of the data**.**

According to Gauthier and Wallace (2022), researchers must consider contextual ethics and transparency when engaging with data. They encourage using thematic analysis to manage ethics and transparency throughout the analysis process. The researcher and the team must closely study the data and classify repeated ideas and patterns of the data. In this study, data analysis followed six stages of thematic analysis by Braun and Clarke (2006). In the first stage, the researcher and co-researcher read and re-read the data to familiarise themselves with it before analysing it. Secondly, they created shorthand labels or codes for sentences and phrases to gain a condensed overview of the main points and meaning that recur throughout the data. Thirdly, we generated themes from the codes we had created by combining several codes into a single theme. Fourthly, we returned to the data set and compared the themes against it to check if we had missed anything. Fifthly, we defined and named the themes to create a final list of themes. Lastly, we wrote our analysis of the data.

3.6 VALUE OF THE STUDY

This study could provide mathematics teachers, subject advisors and mathematics coordinators with a deeper understanding of inclusive and flexible teaching strategies for teaching Euclidean geometry and making the content accessible. The study could empower the teachers with valuable lessons and skills on how to use UDL in the teaching of Euclidean geometry, and this could enhance their confidence in teaching the topic

This study will add to the body of knowledge since there is limited literature on how UDL could be applied in the teaching of Euclidean geometry. The study will thus contribute to expanding knowledge of theories about teaching Euclidean geometry.

3.7 ETHICAL CONSIDERATIONS

The researcher applied for permission to conduct the research (Cohen et al., 2017). The researcher was granted permission to conduct the study from the University of the Free State, and the study was ethically cleared. The study was conducted in one senior secondary school in the Motheo district of education. Parents signed the assent forms as their children participate in the study during class observation (see Appendix A4). Additionally, the co-researchers signed the consent forms to indicate their voluntary participation in the study (see Appendices A5; A6; A7; A8; A9 and A10) to audio and video-record the sessions (see Appendix D2). The co-researchers were also assured that their identities would be kept anonymous and confidential. Therefore, pseudonyms were used as names of the co-researchers to hide their identities. The generated data was stored in an encrypted file with a password. The laptop where the data was stored also required a password to be opened. The data will be kept for a period of six months after the completion of the study, and thereafter, it will be destroyed.

3.8 CONCLUSION

This chapter focused on implementing PAR as a methodology to generate data. It further indicates how the social constructivism theory and PAR intertwined in advocating social interaction. The chapter outlines the activities carried out during the different stages of PAR. Additionally, the profiling of the co-researchers was discussed. The chapter further deliberated on data collection procedures and instruments. The quality assurance issues and steps taken to analyse the data were also covered in this chapter.

Chapter 4will focus on data analysis and interpretation of the generated data. It will divulge a change that the study contributed to in people's lives.

CHAPTER 4

PRESENTATION, ANALYSIS, AND INTERPRETATION OF DATA

4.1 INTRODUCTION

This chapter presents the data, which is primarily qualitative, generated from class observations and focus groups discussion. It provides challenges pertaining to the teaching and learning of Euclidean geometry and the possible solutions thereof. Additions, it highlights solutions to the potentials threat that may inhibit the implementation of the identified strategies.

Please note: As indicated in the previous chapter (Chapter 3), the co-researchers used two languages, Sesotho and English, during the focus group discussions and observation lessons. Therefore, ET is used in this chapter as an abbreviation for English Translations.

4.2 CHALLENGES IN THE TEACHING OF AND LEARNING OF GEOMETRY

The subsequent sections highlight the challenges of teaching and learning geometry as highlighted by the participants. The data in the subsequent sections were gathered during lesson observations and the focus group discussions.

4.2.1 Knowledge of Flexible and Inclusive Teaching Strategies

According to Dube (2016), teachers must develop multiple flexible and inclusive teaching strategies to cater to learners' diverse needs and address learning barriers in mathematics classrooms. In line with this, Dalton et al. (2012), in their research on the neurological basis of learning styles, point out that teachers ought to apply flexible and inclusive teaching strategies that involve amongst others, the multiple means of representation through a variety of media to gratify a range of learning needs. However, the teaching of Euclidean geometry remains a challenge to teachers because of their lack of knowledge of flexible and inclusive teaching strategies. The 'chalk and talk' approach is dominant, while the textbook is the only resource used. The predominant use of these traditional teaching strategies signifies those learners do not get to experience various forms of teaching that could support their learning. These methods (that is, traditional teaching methods) thus deprive learners of opportunities to engage in meaningful learning and do not afford them a chance to own their learning.

Consequently, learners need to memorise what they are taught and reproduce it in tests and examinations. During our focus group discussion, we learned that teachers predominantly use teacher-centred approaches, which promote memorisation and procedural understanding. This was evidenced by the comments made by the teachers during the problem identification stage as follows:

> ***Ms. Tau:*** *To be honest, geometry is my weak point, and I think I am not the only one who struggles with it. I find it challenging for me to teach it. I rely mainly on the textbook. I often focus only on the examples made in the textbook to teach it".*
>
> ***Mr. Mosola:*** *"…Lenna ke etsa jwalo hore ke intshe ditabeng. Ke tla etsang kannete? E thata geometry!" [**ET**: I also do that to stay out of trouble. What can I do really? Geometry is complicated!]*
>
> ***Ms. Mofokeng:*** *"…Many colleagues, including myself, struggle with Euclidean geometry, and you find most of the time teaching what is inside the textbook and redraw the same shapes provided as examples in the textbook…".*
>
> **Mr. Mokwena:** "…*the fact that most of us do not master Euclidean geometry makes it difficult for us to try different strategies and even be creative in teaching geometric concepts*".

From the comments made above, it was evident that teachers lacked knowledge of the teaching strategies to teach Euclidean geometry. Ms. Tau perceived herself as weak when it came to the teaching of Euclidean geometry. As a result, the only resource she used to teach Euclidean geometry was the textbook, and she focused mainly on the examples. She mentioned that she could not use any other resource or think of another teaching strategy because she lacked content knowledge. Therefore, she had to rely entirely on the examples in the textbook. Mr. Mosola shared the same sentiment about the difficulty and complexity of Euclidean geometry, so he taught the topic just for compliance. The teachers' sole use of textbooks for teaching Euclidean geometry violates the UDL framework, which advocates the need to build flexibility in teaching by using flexible teaching strategies that cater to diverse learner populations. The use of textbooks indicates that learners are only introduced to one format of Euclidean geometry content representation, which according to the UDL framework, is unaccepted. Learners assimilate content differently; therefore, one form of representation cannot be used as a 'blanket approach'.

Additionally, Ms. Mofokeng indicated that many teachers, including herself, struggle with the Euclidean geometry content. She thus attributed their choice of using a textbook as the predominant resource to a lack of Euclidean geometry content knowledge. As a result, these teachers (herself included) resort to the practice of writing examples (drawn from the textbook) and solutions onto the chalkboard and making the learners copy them into their books. The learners, therefore, have no option other than to memorise all the work. Mr. Mokwena confirmed that it was difficult for many teachers to think of different ways to teach Euclidean geometry because they had not mastered the topic. Lack of content knowledge made it difficult for teachers to devise the appropriate strategies for teaching Euclidean geometry. Ubah and Bansilal (2019) also associate the complexity of Euclidean geometry teaching with teachers' lack of content knowledge and inclusive teaching strategies (See 2.4.1.1). As a result, teachers are in a predicament in teaching Euclidean geometry. They are thus compelled to embark on the 'telling' method approach, which promotes memorisation and procedural understanding.

What transpired in our focus group discussion tallies with Machisi's viewpoint; that is, due to a lack of knowledge of Euclidean geometry and pedagogy, teachers often opt for the 'telling' method and rely entirely on copying from textbooks to the chalkboard (Machisi, 2020) (See 2.4.1.1). However, Tsimane and Downing (2020) and Moleko (2018) discourage this form of teaching since it does not provide meaningful learning and does not cater to the needs of the various learners within the classrooms. Furthermore, the teacher-centred approach is disadvantageous since it does not provide the learners with options for comprehension.

4.2.2 Knowledge of Euclidean Geometry Vocabulary and Expressions

Mathematics is "a language that uses symbols and notations to describe numerical, geometric, and graphical representation" (DBE, 2012, p. 8). Simpson and Cole (2015) define Mathematics vocabulary as a significant component of Mathematics language used to describe specific Mathematical concepts and procedures. Kleemans et al. (2018) believe that academic language is closely associated with academic performance, as the proper use of Mathematics terminology helps learners better comprehend Mathematical concepts. Moleko (2018) holds the same sentiment that knowledge of the mathematical vocabulary is fundamental to facilitating comprehension of the mathematical concepts. Judging by the comments made during the focus group discussions, although knowledge of the mathematical vocabulary is essential to master and understand the mathematical concepts, many teachers seem to lack this knowledge. This is reflected in the teachers' reflection below:

> ***Ms. Mofokeng:*** *I attended a workshop, and one of the topics of discussion was the geometry of straight lines. The facilitator gave us a task to explicitly explain what we meant by the expression "angles on a straight line add up to* 180°*. We were all puzzled because we thought the statement was straightforward and could not be clarified further. I thought I understood very well what the statement meant. Then the facilitator drew two angles on a straight line and asked if they added up to 180°* (See Figure 4.1).

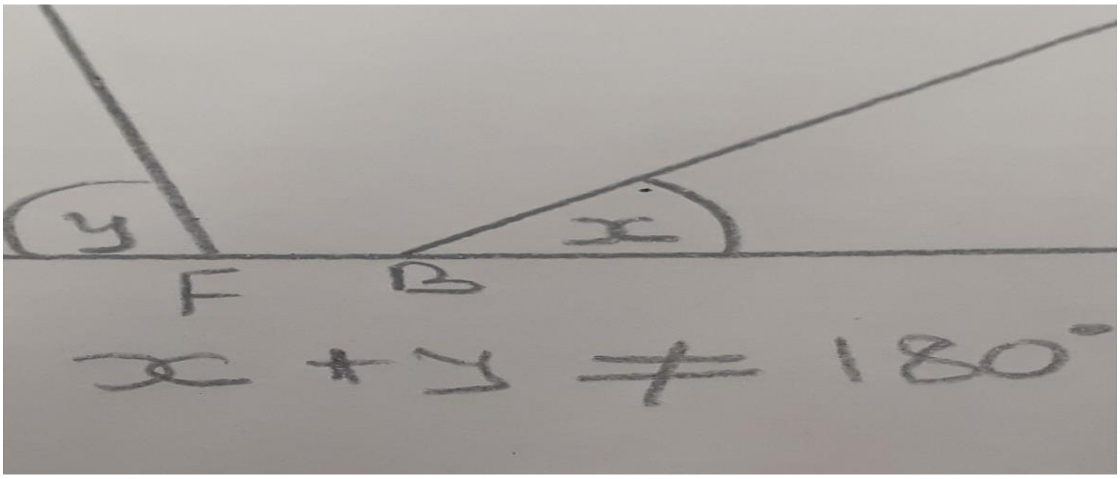


Figure 4.1: Example of two angles on a straight line

> ***Mr. Mosola****: "Is it not true that angles on a straight line add up to 180°? Nna ha ke utlwisisi kannete. (Honestly, I do not understand). I have been using this statement many times when I am teaching geometry. Vertex yona ke eng? (ET: what is a vertex?). I never emphasised that angles on a straight line add up to* 180° *if they have a common vertex. Honestly, I did not mean what is illustrated in the diagram above.*

> ***Ms. Tau:*** *While discussing the theorem of a circle that says, "angles subtended by the same chord or arc are equal." My learners could not explain the meaning of 'angles subtended by the same arc or chord. I never paid attention to explaining the statement. I only demonstrated to them through sketches. Now it is evident that I cannot describe many Mathematical terms such as a circle, proportionality, and subtends, to mention a*

few. Yes, I can demonstrate what a circle is, for example, but I never bothered to describe it verbally. I can show what is meant by angles subtended by the same arc or chord but cannot explain the word "subtend." I think we are ignorant about the mathematical vocabulary and register knowledge.

Subject-advisor: *Ms. Tau, you are not alone. During my school visits, I joined a grade 8 teacher in class. He was going to present a lesson on a circle and its parts. He started by drawing a circle and showing the radius, diameter, sector, and major and minor segments. He was pointing at and giving the names of parts of the circle. The Teacher admitted that he did not know how to define a circle.*

In the discussion above, Ms. Mofokeng and Mr. Mosola attest that they have been teaching the learners the expression, "the sum of angles on a straight line equals 180°" without explicitly explaining the 'necessary condition' and qualifying it. In this instance, the angle must *'share the common vertex'*. If this condition is not highlighted, the statement can be misleading as some learners may think that any set of angles on the straight line sums up to180°. Ms. Mofokeng believed that it was clear that the two or more angles on a straight line always add up to 180°, without emphasising that they should have a common vertex or common arm. She did not know this as it was also news to the other teachers who attended the workshop. Mr. Mosola also confirmed that he never paid attention to specifying the required condition. He further indicated that he has used the statement many times when teaching geometry without highlighting this condition (that is, angles on a straight line add up to 180° if they are adjacent to one another - if they have a common vertex). Based on Ms. Mofokeng's and Mr. Mosola's discussion, Ms. Tau alluded that generally, teachers are ignorant about the importance of knowledge of mathematics vocabulary hence the comment, *"I think we are ignorant about mathematical vocabulary and register knowledge."* She also highlighted her experience regarding the terminology that is used in geometry. She indicated that she never bothered to explain terms such as subtend, tangent and proportionality, to mention a few. She further stated that she could illustrate these terms using the drawings but could not verbally explain them. In confirming that teachers were ignorant and lacked knowledge of Mathematics vocabulary and register, the subject-advisor pointed out that the challenge was prevalent among the teachers as he had also attested to the struggles of the other teachers in that regard citing the Grade 8 teacher who could not verbally define a circle and its related parts. He could point at the circle and its different parts but could not explain/describe them verbally. UDL requires the vocabulary to be clarified and explicitly taught to reinforce learners' understanding. Mthembu (2007) asserts that teachers underestimate the effect of spatial representation and language in spreading and transmitting mathematical ideas, particularly in teaching Euclidean geometry (See 2.4.1.1). However, in this case, teachers seem to lack knowledge of the Euclidean geometry vocabulary and expressions, creating knowledge gaps for learners.

Considering the statement uttered by the subject advisor, "Ms. Tau, you are not alone," it proves that many teachers have a challenge in explaining the geometrical terminology. Most of them teach Euclidean geometry without understanding the geometric language. Deducing from the teachers' engagement in our focus group discussion, it is evident that teachers do not make enough effort to learn more about the geometrical terminology. Vocabulary is the pillar of knowledge. Our focus group discussion was concurrent with Alex and Mammen (2018), who denoted that basic knowledge and terminology are fundamental in any field of study (see 2.4.2.4).

Makhubele (2014) affirms that Euclidean geometry challenges teachers because they do not give proper attention to the vocabulary and ways to unpack statements and diagrams in the question (see 2.4.1.4). Euclidean geometry questions are primarily in the form of text and sketches. Therefore, teachers must have a better understanding of mathematical terms and the ability to explain them, enabling them to enhance learners' spatial knowledge and reasoning skills. If teachers cannot elucidate terms such as bisector, perpendicular, parallel, segment, tangent, cyclic, and tangent, to name but a few, it would be difficult for them to bring the goal of UDL to fruition in terms of making the concepts (Euclidean geometry concepts in this case) simple, perceptible and intuitive to all learners.

4.2.3 Resources and Competency for Teaching Euclidean Geometry

Learning resources include anything that could create a conducive classroom environment for effective teaching and learning (Mkhasibe et al., 2020). According to Hylen (2021), learning resources are crucial intellectual property in education. Resources may be textbooks, computers, humans (teachers), finances, infrastructure and time needed to accomplish the educational system's goals. Makofane (2019) and Maile (2019) state that a resource shortage contributes to poor mathematics performance. They further allude to a high need for mathematics and science teachers in South Africa. This situation compels teachers to teach Mathematics, yet it is not in their specialties. Our focus group discussion revealed that even though qualified teachers were deemed necessary, the DBE was not holding frequent in-service workshop for teachers.

> ***Ms. Tau:*** *Teaching Euclidean geometry is a nightmare because I am not qualified to teach Grades 10 to 12. I have Senior Primary Teacher Diploma (SPTD). There was a shortage of Mathematics teachers in my school. The lady who was teaching Grades 10 to 12 was promoted to the position of curriculum advisor. I think the Department of Education does not pay enough attention to the scarcity of human resources. I struggle to teach Mathematics in Grades 10 to 12, and it is even more challenging to teach Euclidean geometry because learners have a negative attitude toward geometry from the lower grades.*

> ***Mr. Mokwena: …*** *due to family matters, I had to relocate to another school in my village. The school did not have a computer laboratory. I was used to using technology in teaching mathematics. It was a challenge for me to meet the needs of diverse learners to understand the properties of geometric shapes and how to prove theorems…. I think the DoE does not distribute manipulatives and electronic devices equally because some schools do not have even one computer.*

> ***Mr. Mosola:*** *In my school, we have a shortage of Mathematics textbooks. Sometimes learners must share textbooks, for instance, one textbook for three learners. I consider textbooks as essential tools for teaching and learning. It becomes a challenge for me to explain Euclidean geometry, which is mainly about lines, shapes, axioms, and theorems, without referring the learners to the concepts in the textbook. It is only fair to engage learners fully, but it becomes difficult without having enough textbooks. Sometimes I must sketch diagrams from the textbook. It is time-consuming, and unfortunately, I am poor at making drawings.*

> ***Ms. Mofokeng:*** *I think my situation is the worst. As a Funza Lushaka bursary holder, I specialised in Mathematics though it was not my favourite subject in Matric. In fact, geometry was a big challenge for me. I am now duty-bound to teach this concept that was a monster to me. Ho boima (ET: it is not easy). I wish I could access resources such as IBP videos, Heymath, and Master Maths to learn from specialists' presentations.*

From the extracts above, it is obvious that teachers are aware of essential resources in teaching and learning Euclidean geometry. Ms. Tau has a Senior Primary Teacher Diploma (SPTD) and feels that it has not equipped her to teach Grades 10 to 12.

Therefore, teaching mathematics is huge challenge and she has developed Mathematics anxiety, which is often transferred to the learners. She struggles to teach Euclidean geometry due to a lack of content knowledge and confidence. Ubah and Bansilal (2019) confirm that Euclidean geometry is challenging to teach because teachers have content gap (See 2.4.1.1). A human resource (that is, a qualified teacher) is vital for teaching and learning. What seemed to complicate Ms. Tau's situation was that as much as she had limited content knowledge and struggled to teach geometry, learners had developed negative attitudes toward geometry stemming from lower grades.

Judging from Mr. Mokwena's contribution, computers were deemed the most crucial resource for teaching and learning geometry at his previous school. When comparing his experience teaching Euclidean geometry using computers, he pointed out that his teaching was more effective than when using textbooks as the only source of information. He suggested that technology would make teaching and learning more attractive and exciting to the present generation because incorporates visual representations, audio-visual material and GeoGebra software to teach the properties of shapes and their relationships, all of which facilitate the teaching of geometry. Teachers may use to teach the properties of shapes and their relationships. Mr. Mokwena's view was that teaching should be done in a manner that learners find appealing; however, his move to another school (an under-resourced school) made it challenging to present geometry effectively. The Shortage of resources such as computers made it difficult to deliver the exciting geometry lessons used previously. It thus impeded his teaching as he could no longer use multiple modes of presentation, which UDL advocates, to demonstrate the geometrical figures in a manner that makes the concepts perceptible (see 2.5.3).

According to Mr. Mosola, a textbook is another essential resource for teaching and learning, creating a framework for class activities. Mr. Mosola regarded the textbook as a vital resource allowing teachers to refer to the concepts, they are teaching combined with the drawing of figures on the board for explanation. However, if learners do not have textbooks or must share textbooks, it makes it more challenging for teachers. Shortage of textbooks contributes to learners' Mathematics anxiety. It hinders learners' problem-solving skills development as textbooks have many exercises that inspire learners to solve problems during their free time. Ms. Mofokeng is in a predicament with the teaching of Mathematics, which was not her favourite subject whilst at school. She studied teaching and specialised in Mathematics, not because she was passionate about teaching the subject but because of the availability of the Funza Lushaka bursary. Even though she specialised in Mathematics, teaching geometry is a significant challenge to her. She says "ho boima" (ET: not easy). Deducing from her statement, it was evident that being a qualified Mathematics teacher is a vital resource. In addition, one must be passionate and optimistic about teaching Euclidean geometry effectively.

4.2.4 Factors that affect Teaching and Learning of Euclidean Geometry

> Engagement involves positive student behaviors, such as attendance, paying attention, and participation in class, as well as the psychological experience of identification with school and feeling that one is cared for, respected, and part of the school environment (Anderson et al., 2004, p.97)

Learners who are fully engaged in teaching and learning feel motivated and put more effort into increasing their performance and developing knowledge and skills in their areas of interest (Douglas et al., 2020). For example, if they are engaged, they would be motivated to take ownership of their learning. They would draw their mnemonic and mind-maps to organise newly-acquired information and connect to the subject or topic. They would be highly dedicated to their studies and likely to succeed. According to Yengin et al. (2010), engaging learners in teaching and learning allows them to own their learning because it puts them at the centre. Although learner engagement significantly impacts learning by improving learners' behavioural, cognitive, and emotional well-being, it remains a challenge for teachers to use this good practice. Moleko (2021) confirms that teaching Mathematics is becoming more challenging as teachers must work on learners' lack of motivation and negative attitudes towards the subject. Similarly, Williams and Choudhury (2016) divulge that pedagogy transmission (teacher-centred approach) fast-track learners' decision to opt-out of studying mathematics at higher levels of education.

The discussion in our focus group unfolded as follows:

> ***Ms. Tau:*** *Teaching Euclidean geometry has been a challenge because most learners in Grade 10 have negative thoughts and feelings toward geometry. They show a sense of*

isolation, a lack of interest and purpose in learning geometry, and no emotional connection to the topic. I tend to think that the negative attitude in Grade 10 is due to disengagement in the previous grades.

Mr. Mokwena: *Mam Tau, this is precisely my observations in teaching geometry. Sometimes I tend to think that we are to blame because we seldom give sufficient information about Euclidean geometry, what it entails, its applicability in real-life situations, why it is named after Euclid, and how it is different from Analytical geometry. I don't know, but I think learners would be motivated and interested in learning Euclidean geometry if they could be provided with information about the topic.*

Mr. Mosola: *What makes my life difficult when teaching Euclidean geometry is my learners' level of commitment. If I give them homework or classwork, they don't complete the work or claim they forgot books at home. I created a WhatsApp group as a platform for them to share knowledge and ask questions when necessary. I hoped the platform would enhance their curiosity and interest in learning geometry. I thought those who feel isolated would use the media to connect to others and the topic. To my disappointment, their level of engagement remained low.*

HOD: *Colleagues, I agree with you 100%. Learner disengagement is the primary source of their negative attitude towards Euclidean geometry. But now, a million-dollar question is how do we rectify that?*

Ms. Tau's response indicated that learners do not only develop a negative attitude towards geometry at Grade 10-level; it seems this has developed in the lower grades. Ms. Tau tends to think that the negative attitude displayed by grade 10 learners might be due to disengagement in lower grades resulting from the teaching methods where teachers did not actively engage learners in the process of learning. Sometimes negative past experiences with geometry can foster low self-esteem and create a learning barrier.

Mr. Mokwena's acknowledged that teaching Euclidean geometry is challenging and suggested that teachers offer information about the topic, the importance of understanding the topic, show its applicability in real-life situations, and state what it entails in order to motivate them (which is what UDL requires to fuel motivation for learning). In that way, learners would have a clear purpose to learn the concept and take ownership of their learning. In many cases, teachers are restricted in the way they teach Euclidean geometry due to a number of factors such as lack of subject and pedagogical content knowledge and a lack of resources. Teacher-centred approach does not allow learners to connect the concepts they are learning in class with real-life situations and violates the UDL principle of multiple means of engagement, which requires the lessons to be tailor-made to exemplify authentic real-life situations.

Mr. Mosola's main challenge is that learners do not complete their tasks which is vital for them to confirm and extend their learning. Learners offer a range of excuses such as exercise books or textbooks left at home. Even though he created a learners' WhatsApp group hoping they would participate, interact and collaborate with those feeling isolated and disconnected from the topic, affording them avenues for content mastery, they displayed a lack of interest, which makes it difficult to engage in meaningful learning. The HOD admitted that learners' lack of interest in geometry and disengagement made it challenging to teach Euclidean geometry. The practice of disengagement violates MMAE, which provides options for expression and communication through multiple tools for constructing and composing knowledge.

4.2.5 Visualisation Skills

Visualisation, a process of forming mental images and pictures and using them effectively for mathematical discovery and understanding, is key to developing critical thinking skills (Makina, 2010). As visualisation is fundamental in teaching and learning because it enables learners to create mental images and pictures that make it easy to solve problems (Moleko, 2021), it is essential in the teaching and learning of Euclidean geometry. Although visualisation plays a vital role in learning, teachers seldom pay attention to the way learners think during problem-solving. They teach them what to think, not how to think.

The focus group discussion revealed that learners struggle to solve riders in geometry because they cannot visualise the given information. Thus, their deductive reasoning skill is not at the expected level. The participants reflected as follows:

> ***Ms. Tau:*** *My Grade 11 learners struggle to apply theorems and giving appropriate reasons when solving riders. I taught them how to scrutinise the given statements in conjunction with diagrams before attempting to answer questions. Unfortunately, they still could not draw inferences from the specified data.*

> ***Mr. Mosola:*** *I have the same problem with my Grade 11 learners. Kannete ha ke sa tseba na ke etse joang (ET: honestly, I don't know what to do). I have taught them the*

theorems, their converses, and axioms. I have made diagrammatical representations of the theorems. I was convinced that they understood the theorems because they could state them correctly in a written form and verbally.

Ms. Tau and Mr. Mosola's discussion illustrates that as learners did not fully understand the theorems, they could not apply them to construct proofs and give valid reasons. Even when they had taught them how to scrutinise given statements in conjunction with diagrams, they struggled to solve riders. The use of diagrams in some cases, could increase learners' understanding of geometry, however, teachers need to understand how learners think during problem-solving to help them think about the given problems effectively. From the teachers' statements, it can de deduced that Ms. Tau and Mr. Mosola focus on transmitting knowledge of theorems and pay little or no attention to how learners visualise (make mental pictures) the given information. There was no significant effort to ensure that they develop the learners' visualisation skills, teaching theorems from the book without and unpacking their related diagrams to reinforce understanding and stimulate visualisation. Teachers, it seems, do not create a platform for learners to think critically about the geometry concepts, limiting their deductive reasoning skills, which are crucial in promoting visualisation.

The discussion continues.

Ms. Mofokeng: *My Grade 11 learners are doing well in writing the theorems and giving them verbally but struggle to construct proofs because they cannot form mental pictures from given information.*

Ms. Mofokeng's Grade 11 learners also struggling to solve riders, shying away from questions requiring proofs. Ms. Mofokeng states that learners memorise the theorems, but this do help them understand as they have not been trained to build pictures and images in their minds. Learners who lack visualisation skills find it difficult to solve geometry problems. Even though Mr. Mokwena let the learners draw diagrams representing the theorems, they still struggled to unpack scenarios that involved two or more theorems in one diagram or to show two or more theorems in one diagram, He attributed this struggle to learners' inability to visualise and form the mental pictures of the theorems in their heads to understand the relationships between the theorems. As a result, they could not express these theorems diagrammatically on paper.

4.3 SUMMARY

The above sections highlighted the challenges of teaching and learning Euclidean geometry. The empirical data reveals that the complexity of teaching geometry results from teachers' lack of knowledge of flexible and inclusive teaching strategies, lack of understanding of the mathematical vocabulary and expressions, shortage of teaching and learning resources, learners' disengagement and negative attitude, lack of motivation and teachers' inability to promote visualisation techniques which could be compounded by teachers lack of subject and pedagogical content knowledge.

4.4 COMPONENTS OF THE SOLUTIONS

This section outlines the components of the proposed strategies responding to the challenges discussed in section 4.1. The section also highlights some UDL strategies used to address challenges outlined in 4.1.

4.4.1 Varied Strategies to teach Euclidean Geometry

Ngirish and Bansilal (2019) note that learners apart from learners coming from different backgrounds, they also have different learning styles. For teachers to maximise learning, they need to acknowledge that learners are diverse and could be visual, auditory, or kinaesthetic learners. Thus, teachers need to devise strategies that suit the all learner needs. In our focus group discussion, teachers indicated that they were aware of the importance of using multiple ways of teaching Euclidean geometry to cater to diverse groups of learners by developing responsive lessons. The discussion unfolded as follows:

> ***Ms. Mofokeng*** *"…to sketch shapes on the board is the right thing to do because it caters to visual learners; however, we should supplement that with manipulatives such as a geo-board* [see Figure 4.2] *to accommodate tactile learners. Using the geo-board enhances a deep understanding of the properties of the geometry of straight lines and 2-Dimensional shapes*.

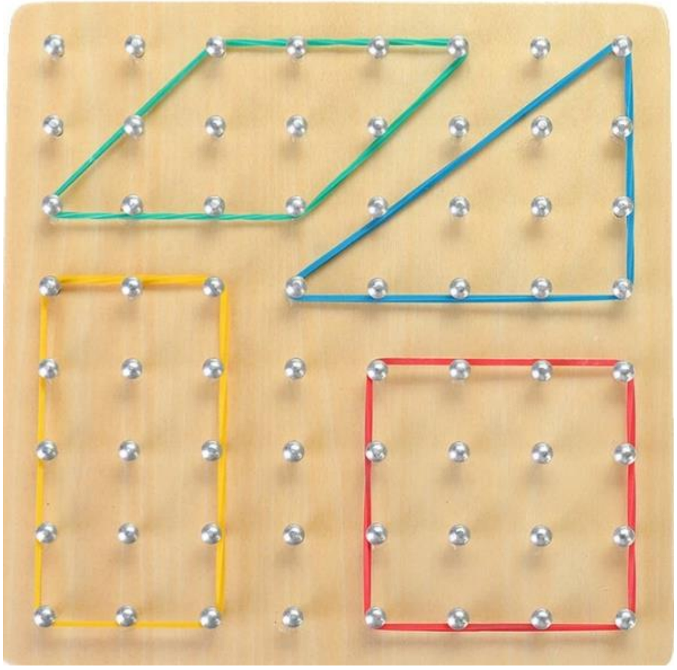


Figure 4.2: Geo-board

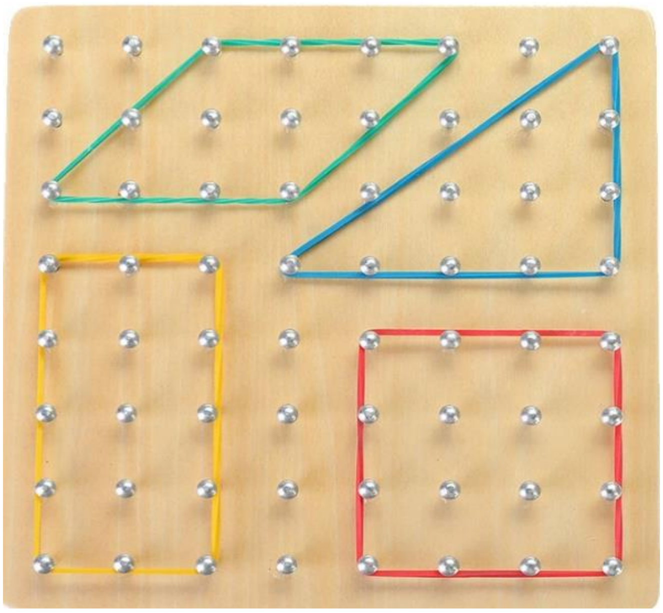
Ms. Mofokeng suggested using a geo-board to maximise Euclidean geometry learning as it explores basic concepts in plane geometry and promotes understanding of the properties of straight lines and 2-dimensional shapes to make the concept accessible to all learners. The statement, *"…to sketch shapes on the board is the right thing to do because it caters to visual learners; however, we should supplement that with the use of manipulatives such as a geo-board, to accommodate tactile learners"*, means that teachers should not rely only on the chalkboard to teach Euclidean geometry, but need to use other tools, such as geo-boards, which make it possible to engage all the learners in the learning process. The geo-board makes the teaching and learning of Euclidean geometry flexible since it can engage the learners in the learning process. Learners can use the geo-board to discover more about straight lines, angles, and shapes or to form parallel lines with more than one transversal in different orientations as examples, making it possible for learners to discover the concepts on their own Through the use of the geo-board, learners can quickly, on their own. Using the geo-board serves as an excellent strategy to accommodate different learning styles and reinforce learner understanding of the properties of the geometry of straight lines and 2-dimensional shapes, which constitute prerequisite knowledge for Euclidean geometry.

The MMR principle in UDL, as the teaching framework, advocates the notion of representing the concepts in various formats since learners differ in the way they assimilate and understand the content (Ross 2019). According to Mukunthan (2013), teachers must design and use appropriate teaching-learning materials in Mathematics. The use of the geo-board in this instance, thus serves as another way in which the same Euclidean geometry concepts highlighted in the textbook could be represented, ensuring that the teaching of Euclidean geometry concepts simple, intuitive and perceptible. Therefore, using multiple representations serves as one way to provide options for comprehension (CAST, 2011).

Using the geo-board and manipulatives as other representations seemed to be useful in accommodating and engaging all the learners. This was evidenced by how the learners participated during the practical lesson, as shown in the extract below:

Lesson description: Lesson on the different geometric shapes and their properties. The teacher wanted to assess the learners' understanding of the different shapes and their related properties.

Mr. Mosola: I want you to pay attention to the different shapes I will create using this geo-board and the rubber bands. Please write down the name of each shape and also provide its characteristics. There we go! *(Designing the different shapes using a geo-board)*



Mr. Mosola: I will give you 5 minutes to mention the shapes and give at least one property of each shape, rea utlwana? (ET: Are we clear?)

Learners: *(all responded)* Yes, Teacher!

Mr. Mosola: Can someone give answers regarding the first shape

Learner 1: the figure is a rhombus.

Mr. Mosola: What do you notice about the opposite sides?

Learner 2: They are equal and parallel.

Mr. Mosola: What do you notice opposite interior angles?

Learner 3: They are equal in size.

Mr. Mosola: Now, look at the following shape (the blue one). Which shape is this?

Learners: (all responded) It is the right-angle triangle.

Mr. Mosola: Yes, you are correct...

The extract dialogue above indicates the role played by the geo-board in engaging all the learners in the learning process and assisting the teacher in providing visual representations of the geometric shapes. Thus, the geo-board served as an effective instrument since it provided precise shapes with accurate measurements, making it easy for learners to identify the shapes and assign their properties. Changing the shapes easily demonstrates the different orientations of the shapes, a helpful exercise as learners tend to be confused when the shapes are represented using different orientations. The geo-board thus provides flexibility for such exercises to be performed successfully in class. According to Vojkuvkova (2012), this exercise is important because Van Hiele's theory of geometric thinking (level 2) requires learners to know the shapes by their look and their properties.

The idea of introducing learners to the different orientations of the shapes was also highlighted as significant by Ms. Mofokeng during the focus group discussion.

> *Ms. Mofokeng: I think the emphasis on the properties of parallel lines and shapes should be a focal point in teaching Euclidean geometry. Learners would not struggle with shapes in different orientations if they had a conceptual understanding of the properties of lines and shapes. I used a geoboard with rubber bands of different colours to demonstrate various orientations of parallel lines with more than one transversal. In*

addition to that, on the classroom walls, I put a shape collage showing shapes in different orientations.

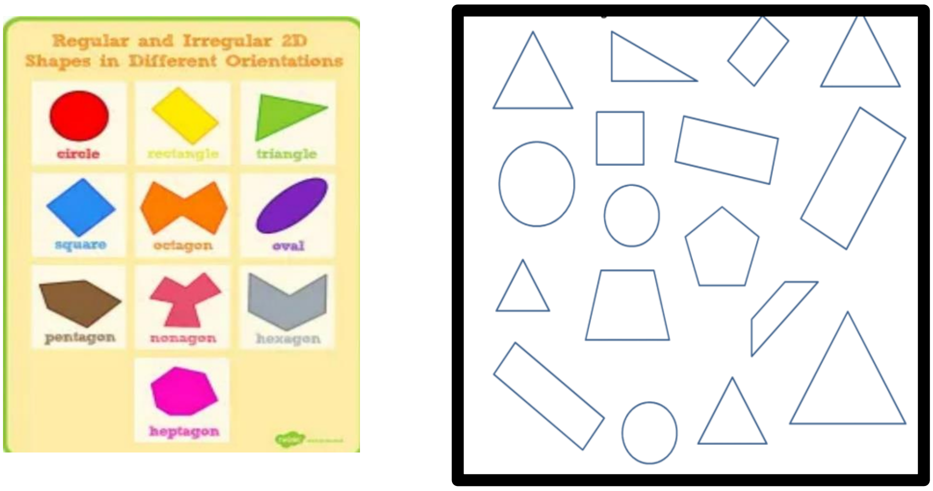


Figure 4.3: Shapes at different orientations

It seems that learners are used to working on one pair of horizontal parallel lines with one transversal but have not been introduced to multiple pairs of horizontal lines with more than one transversal. Furthermore, the learners were not introduced to the different orientations of lines and shapes. The use of geoboards with coloured rubber bands and posters of collages of different shapes would enhance visual perception because it would enable learners to see different orientations. According to Dalton (2017), teaching encompassing various representations provides learners with accessible learning opportunities responsive to their diverse learning styles.

Another vital comment regarding the notion of devising the appropriate strategies for teaching Euclidean geometry was made during the focus group discussion. The statement was made as follows:

> ***Mr. Mokwena:*** *The teaching of Euclidean geometry used to frustrate me, and I have realised it was because I did not know my learners and how they comprehend new knowledge. I never allowed them the opportunity to explore concepts in the lesson.*

To devise effective and appropriate strategies to teach learners Euclidean geometry, Mr. Mokwena indicated that it is vital to know his learners and how they comprehend new knowledge. Knowing learners enables the teacher to plan and organise lessons and devise effective and appropriate strategies to teach Euclidean geometry. Learners develop understanding of the concepts if their teachers know how they learn, thus devising varied strategies that cater to their needs allowing them to interact through a learner-centred approach.

4.4.2 The Explicit Teaching of Mathematics Vocabulary and Expressions

Mathematics is a language that has its vocabulary and register (Wilkinson, 2018). To understand any language, one must be well versed in the terminology. A study conducted in the United States of America (USA) at the University of Texas in Austin revealed that knowledge of Mathematics vocabulary made a noticeable contribution to word problems, particularly in measurements and geometry (Peng & Lin, 2019). According to Moleko (2018), if one understands the vocabulary, one can easily translate word problems into mathematical structures using symbols and diagrams (Moleko, 2018).

In our focus group discussion, the co-researchers shared valuable strategies to address the Euclidean geometry vocabulary and expressions. Their comments unfolded as follows:

> ***Ms. Mofokeng:*** *I think we must avoid using shortened mathematical statements at the introductory stage of any topic. For instance, "**adjacent angles** on a straight line add up to 180°" instead of "angles on a straight line add up to 180°."*
>
> ***Subject advisor****: I agree with you, Ms. Mofokeng. We must use complete statements at the introductory stage of teaching geometry to enhance conceptual understanding. The necessary condition for angles on a straight line to add up to 180° is that they must have a common vertex* (see Figure 4.4)*. Shortened statements may be used when the concept is well understood and the learners are at the advanced stage.*

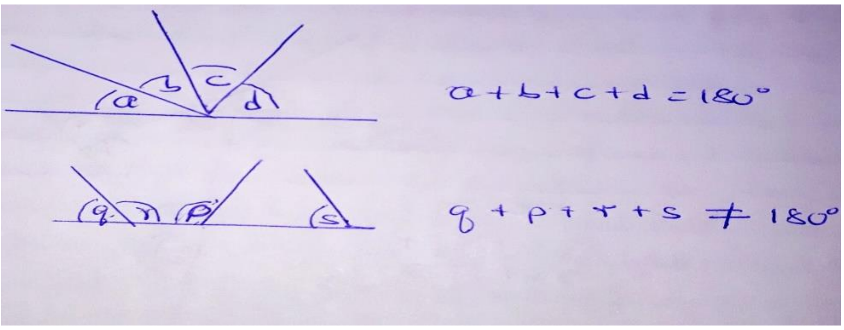


Figure 4.4: A condition for angles on a straight line to add up to 180◦ (sharing of a common vertex

CES: *Colleagues, talking of shortened statements at the introductory stage of a topic reminds me of the concept of the Sum of angles of a triangle. Specifying whether the Sum of 180° is for interior or exterior angles is crucial. This is what I mean* (referring to Figure 4.4):

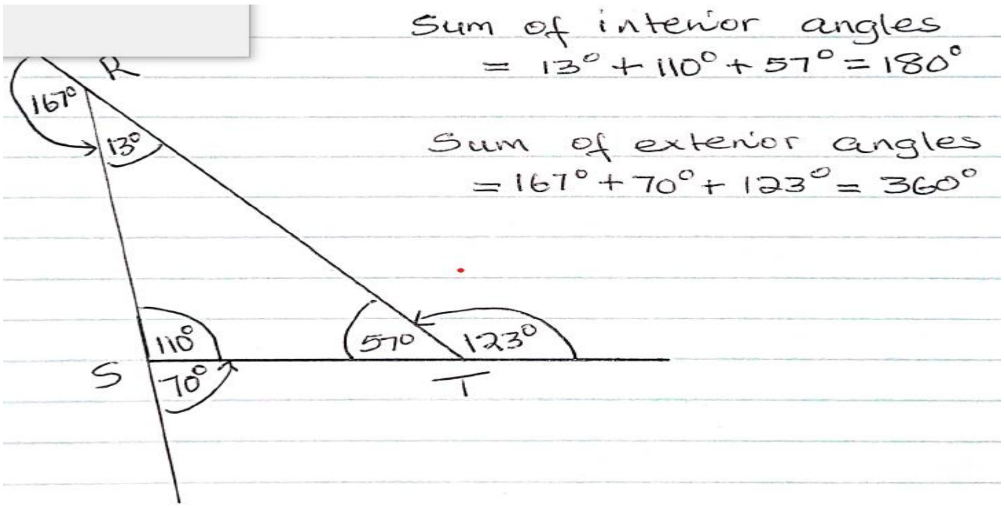


Figure 4.5: Interior and exterior angles of triangles

Emphasis on the difference between interior and exterior angles is important in this instance, as it would ensure that learners later understand the geometry of a circle with the following theorem: "The measure of an exterior angle at a vertex of a cyclic quadrilateral is equal to the opposite interior angle" (see Figure 4.5).

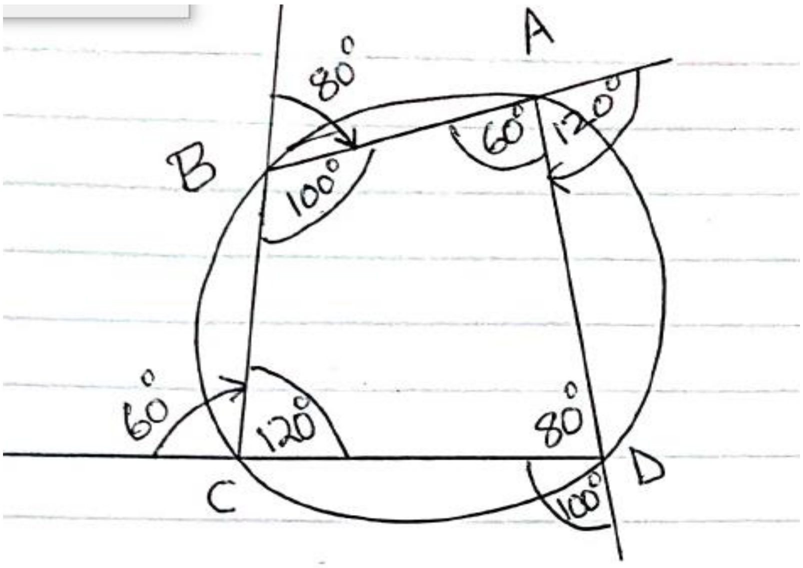


Figure 4.6: Interior and exterior angles of a cyclic quadrilateral

The above responses confirm that mathematical vocabulary and expressions are central in the teaching and learning of mathematics. According to Mishra et al. (2019), knowledge of the terminology creates a positive classroom environment, improves attention, supports emotional instruction and reduces learners' anxiety.

Ms. Mofokeng suggested that using shortened mathematical statements right at the beginning of the geometry topic, was not the right approach. The subject advisor concurred and confirmed that it would be acceptable to use the abridged mathematical statements when learners have entirely learned the concepts and fully understood them; hence the statement "*Shortened statements may be used when the concept is well understood, and the learners are at the advance stage*".

Ms. Mofokeng raised the issue regarding the expression; *"angles on a straight line add up to,"* which can be understood differently by the learners. Based on the statement, one learner might conclude that two or more angles on a straight line add up to 180° (See Figure 4.7).

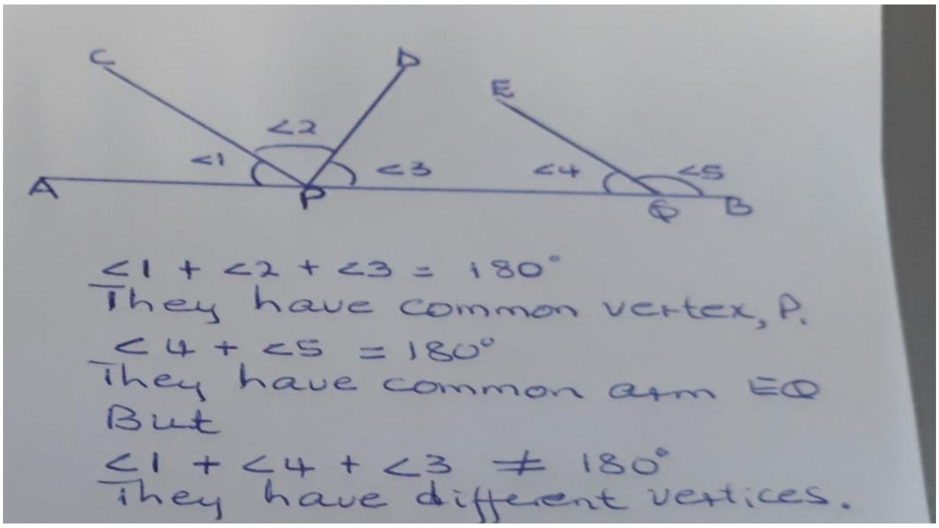


Figure 4.7: The sum of angles on a straight line

To address any misconceptions, it is vital to emphasise at the introductory phase of the geometry lesson that angles on a straight line add up to 180° … **if and only if they have a common vertex or arm**. The CES cautioned participants about the expression *"the sum of angles of a triangle equals 180*°", encouraging teachers to be specific about whether it is the Sum of interior or exterior angles of a polygon. Learners need to know that any polygon has the same number of interior and exterior angles and that the Sum of interior and exterior angles at one vertex equals 180°. This initial understanding builds a solid foundation for a better conception of the circle theorem about cyclic quadrilateral. The emphasis on clarifying the Euclidean geometry vocabulary and expressions is in alignment with the UDL's MMR principle, which emphasises the need to reinforce understanding of the mathematical language, symbols and expressions for meaningful learning to take place (CAST, 2011).

During class observation, Ms. Tau introduced the topic of "congruency of triangles" and displayed her deep understanding of UDL principles. She indicated that she learned that MMR provides options for language, mathematical expressions, and symbols to clarify vocabulary and symbols and that these can be illustrated through multiple media. She thus engaged the learners in hands-on activities to enhance conceptual understanding and promote self-discovery skills. She introduced congruent triangles by taking the learners through an activity (see Activity A). She took advantage of the exercise by teaching the learners mathematical vocabulary and symbols that UDL deems important to pay attention to promote effective learning.

Please note: *Rule for congruency of triangles is as follows : Mathematically we say that if two figures are of the same shape and size. they are 'CONGRUENT'.

Activity A:

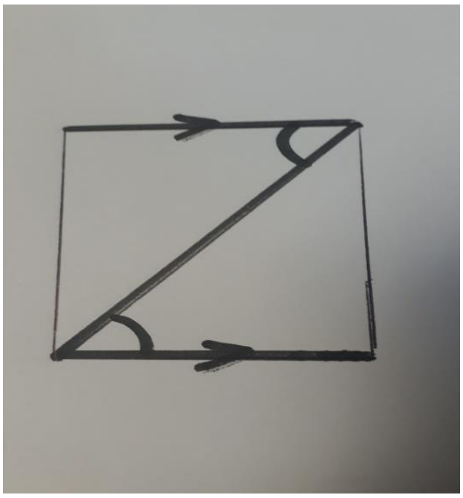
Instructions were given to the learner by the Teacher (Miss. Tau):

- ***Take an A-4 blank page and label the vertices ABCD;***
- ***Join two opposite vertices with a straight line, a DIAGONAL***

- ***Cut the page along the diagonal to get two triangles***
- ***Check if the two triangles map onto each other;***
- ***Measure three angles of each of the triangles;***

- ***What do you notice about them?***
- ***Are triangles the same in shape and size?***

I am sure you noticed that when drawing a diagonal line. the letter formed is a zed or ‘Z’. This letter forms the two angles we call the "alternating angles". (Referring to the angles in the drawing). These angles alternate; can you see them? Look at how they are situated; that's why they are called alternating angles.



Ms. Tau's presentation promotes knowledge of geometry terminology and registers through self-discovery. Even though the learners were told that the line joining the two opposite corners is called diagonal, they were first asked to draw it, which enabled them to define it later. A diagonal line joins two opposite corners passing through the centre of a polygon. The two triangles formed by the diagonal line mapped exactly onto each other. By engaging the learners in hands-on activities, Ms. Tau addressed the ‘what of learning’, thus making it possible for learners to recognise and analyse the information (See 2.3.7). The recognition brain networks are responsible for MMR and make it possible for learners to receive, analyse data and recognise the object of learning. In a classroom setting, teachers customise information display by representing content in multiple ways to cater to the various learners' needs (see 2.3.7).

The use of simple resources such as paper, a ruler, and a pair of scissors made the concept understandable. The step-by-step guidance on forming the two triangles was a good strategy for "guiding information processing and manipulation" (CAST, 2011). After the learners had found out that the two triangles were of the same size and shape, the term ‘congruent’ was then used and explained to the learners. The next step was to introduce the concept of alternating angles while engaging the learners in this practical exercise. Miss Tau indicated that the diagonal line drawn to join the two vertices forms a zed or ‘Z’ shape, forming a pair of angles called the ‘alternating angles’.

Engaging learners in hands-on activities enables them to discover the concepts on their own, making it possible for them to learn vocabulary both theoretically and practically. Teachers can bring in an element of flexibility in the teaching and learning of the geometrical terms (e.g., congruent, diagonal, polygon and vertices) by using simple resources such as paper and engaging learners in a practical activity. This experimental activity made the concepts more understandable.

4.4.3 Appropriate Resources for Teaching and Learning Euclidean Geometry

According to Magano (2014), the education system would function well if the planners are made aware of the shortage of resources. Human resources (supply and demand of teachers), enough textbooks, and technological appliances are essential resources in teaching. Mokotjo and Mokhele (2021) consider technology a fundamental part of teaching and learning for the present generation such as GeoGebra an essential software that could enhance and uplift the performance of teachers and learners. Therefore, one can conclude that technology integration is a means to improve mathematics education. In our focus group discussion, the participants concurred with the above statement.

> ***Mr. Mokwena:*** *In my previous school, where there were computers, laptops, and data projectors, I used GeoGebra to illustrate the properties of various polygons. My learners used it to investigate and form conjectures about the properties of different polygons, draw triangles, and measure angles. I think GeoGebra and other manipulatives should be used as a scaffold for teaching Mathematics from the primary to the university level. Again, they should be fairly distributed among schools. And the neighbouring schools should work together and borrow each other’s e teaching resources. Those that are more resourceful should assist the ones with no resources.*

Ms. Mofokeng: *I agree with you, Mr. Mokwena. Such electronic devices would make teaching easy because learners could also watch Mathematics videos available to their schools. Another factor that needs to be considered is that teachers must be trained to equip them with the knowledge to use GeoGebra and other electronic devices in teaching and learning Euclidean geometry.*

Ms.Tau: *As a remedy to the shortage of qualified Mathematics teachers in our cluster, the DBE, School Governing Bodies (SGBs), and School Management Teams (SMTs) decided to hire veteran expert teachers to teach Euclidean geometry to learners from combined schools in our cluster during winter holidays. I regarded the suitability of human resources in the education system as a crucial matter. I also learned a lot from their presentations.*

From the above extracts, it is evident that Mr. Mokwena and Ms. Mofokeng believes that using resources such as electronic devices and manipulatives contributes to teachers' improvement in Euclidean geometry. participants suggested that Geogebra and other manipulatives could be used from the lower grades to develop the solid foundation necessary for an understanding of Euclidean geometry. Learners would be able to investigate, analyse, evaluate, and subsequently acquire meaningful learning of Euclidean geometry. The use of resources such as GeoGebra is encouraged by the UDL framework, which advocates of the use of multiple. It was felt that the Department should ensure that schools are equipped with resources such as manipulatives and electronic devices which could be shared among the schools in the cluster particularly is they are under-resourced. This system is supported by Kabi (2013), who ascertains that to achieve sustainable learning, neighbouring schools could work together in terms of borrowing teaching and learning resources.

However, capacitating teachers to use electronic devices and manipulatives is crucial. Ms. Mofokeng suggested that training teachers on using electronic devices and manipulatives should be highly prioritised, otherwise, it would be a futile attempt if they were not adequately prepared. This incorporates workshop equipping teachers with the knowledge and skill of integrating ICT into the teaching and learning of Euclidean geometry which would also focus on improving teachers' content knowledge and using appropriate resources for teaching Euclidean geometry.

A further point raised was the use of quality human resources such as calling on expert veteran Mathematics teachers to hold workshops to assist teachers acquire and develop content and pedagogical knowledge on Euclidean geometry.

4.4.4 Promoting Learner Engagement in Euclidean Geometry

According to Abla and Fraumeni (2019), learner engagement, which is a degree of frequency and quality of their participation during teaching and learning, is interrelated with academic success. Learner engagement depends entirely on the learning environment, how teachers appreciate learners' diversity and how they assimilate information, which is the reason from knowledge of learners' different learning styles to engage them in meaningful learning. Kivunja (2018) stated that it is critical for teachers to create a constructivist classroom environment, allowing all learners to participate as partners in knowledge construction. Imenda (2014) attests that social constructivism believes learners gain knowledge as they interact. Therefore, teachers must use multiple flexible ways of instruction when teaching Euclidean geometry. Moreover, they must make it a point that learners get the opportunity to work cooperatively to allow cross-pollination of ideas. The teaching of Euclidean geometry would be easily facilitated because learners at different levels of understanding could support others to improve their overall performance.

During focus group discussion, the co-researchers reflected on how to engage learners to increase their attention and focus on acquiring meaningful learning of Euclidean geometry. They commented as follows

> ***Mr. Mokwena:*** *I observed the same behaviour in my Grade 11 class, willingness to learn. However, what I did differently was that at the introductory stage of the lesson, I listed key concepts in understanding the topic and communicated the learning outcomes of the topic. Additionally, I referred to the benefits of mastering Euclidean geometry and its relevance to most careers. Subsequently, in response, they exerted more effort to acquire more knowledge.*

> ***Mr. Mosola:*** *I have realised that my learners' low commitment resulted from not knowing them and their learning styles. Improving their engagement by providing various options for perception decreased the rate of dropouts and positively impacted their behaviour. My lesson planning was responsive to their varying methods of answering questions. Some of the learners easily grasped information in the statement if it was illustrated with a diagram*

HOD: *Thank you, colleagues. In summary, you are saying that to make Euclidean geometry teaching easy, we should engage learners to investigate theorems and find information on their own. We need to give background and relevance of the topic in a real-life context. Most importantly, we need to know our learners' diverse learning styles.*

The above extracts confirm that learner engagement is essential to enhance the teaching and learning of Euclidean geometry. According to Mr. Mokwena, one of the ways to nurture learner achievement is by giving learners reasons for studying the topic. Learners become stimulated to learn if they know how the topic is relevant to fields of life. This is in line with the UDL principle (multiple means of engagement), which requires teaching to be tailor-made to show learners the relevance of the concepts they are learning in class to the outside world. If they become aware of the relevancy of the content, they will strive for better performance by exerting more effort. Moleko (2021) confirms that for successful learner engagement, it is crucial to let learners know the expectations of the topic to maximise academic success.

For effective teaching to take place, teachers should use a learner-centred approach where learners interact, ask questions, and explore the topic. Teachers should also tap into learners' background knowledge (prior knowledge) to build new skills which is in line with the UDL principle (multiple means of representation). Engaging learners can be a valuable strategy for achieving success since many ideas could emerge from robust engagements and discussions in class, thus leading to a situation where learners are open to new ideas, share ideas and come up with solutions to problems.

The HOD summed up the discussion stating that teachers must know their learners and how they learn to plan their lessons to accommodate different learning styles as this would enable them to plan their lessons accordingly and devise the appropriate engagement strategies to promote meaningful learning. Teachers must engage learners giving them options to demonstrate their learning processes, allowing them to investigate and make conjectures from their findings, as emphasised by the UDL principle of multiple means of actions and expression. When teaching Euclidean geometry, teachers should give a brief background about the topic and its relevance in a real-life context, which would provide learners with a solid reason for learning the topic. Showing the relevance and applicability of Euclidean geometry in real life would deepen the learners' understanding of the topic and help develop their conceptual understanding. This aligns with MME, which provides options for recruiting interest to optimise Euclidean geometry's relevance, value, and authenticity.

1. ***Draw a circle of radius 3cm, or 4cm or 5cm;***
2. ***Mark 4 points anywhere on the circumference;***
3. ***Label the them A, B, C, and D in an anti-clockwise direction;***
4. ***Join the points such that they form a cyclic quadrilateral ABCD;***
5. ***Measure the sizes of the interior angles of figure ABCD;***
6. ***What do you notice about opposite interior angles of the figure ABCD?***
7. ***Produce one of the arms/lines of*** $B\hat{A}D$ ***to point E to form exterior angle*** $E\hat{A}D$***;***
8. ***What is the size of*** $E\hat{A}D$***?***
9. ***Name the interior angle opposite to*** $E\hat{A}D$***?***
10. ***What can you say about the size of exterior angle*** $E\hat{A}D$ ***and its opposite interior angle?***
11. ***Repeat steps 7 to 10 on the three remaining vertices of figure ABCD;***
12. ***What do you notice about the size of exterior angle and its opposite interior in a cyclic quadrilateral?***

The above discussion was reinforced by what I observed in Ms. Tau's Grade 11 class. To introduce the theorem stating, "Exterior angle of a cyclic quadrilateral is equal to the opposite interior angle," she instructed the learners to sit in groups of three. In each group, learners A, B, and C drew circles of radii 5cm, 4cm, and 3cm, respectively. She engaged them in an activity to draw meaningful conclusions from their findings. The instructions were as follows:

The above activities confirm that learner engagement is essential to enhance the teaching and learning of Euclidean geometry. For example, the learners used protractors to measure the interior angles of a cyclic quadrilateral and found the Sum of the opposite angles in three different activities. They established that for any inscribed quadrilateral, opposite angles add up to 180°. In that way, they acquired a conceptual understanding of the theorem that says; "Sum of opposite angles of a cyclic quadrilateral is 180°. They measured exterior angles of the cyclic quadrilateral and established that the measure of an exterior angle of a cyclic quadrilateral is equal to its opposite interior angle. They worked cooperatively to develop a positive attitude and changed their attitude and behaviour towards learning Euclidean geometry because they discovered ‘things’ independently with hands-on activities to investigate other circle theorems.

Ms. Tau's approach resonates with MME because it creates positive motivational cycles. Learners become more willing to investigate information due to intrinsic and extrinsic motivation resulting from being actively engaged. A learner-centred approach leads to learners' deep engagement in the content, promoting profound learning that depicts academic success. In a learner-centred and constructivist approach, teachers act as facilitators for cooperative and active learning. Bolliger and Martin (2018) affirm that teacher-to-learner and learner-to-learner interaction could promote conceptual understanding of Euclidean geometry and encourages engagement in hands-on activities (see 2.4.1.2).

4.4.5 Enhancing Visualisation Skills for Euclidean Geometry

Moleko (2021) points out that visualisation is an integral part of teaching and learning word problems; hence, it is crucial in teaching Euclidean geometry because of its textual nature. For correct interpretation of geometric statements and/or scenarios, learners must be able to create meaningful mental pictures and images. Similarly, Patahuddin et al. (2020) articulate that visualisation skills play a critical role in teaching mathematics, which also relates to the teaching of Euclidean geometry. Visualisation is considered a tool that actuates learners' imagination and develops conceptual understanding, which may be displayed through drawings (see 2.4.1.3). As the discussion about the effects of visualisation unfolded in our focus group, the researcher realised that the participants were also aware of the critical role of visualisation skills in teaching Euclidean geometry. The discussion unfolded as follows:

> ***Subject advisor:*** *During my school visits, I was overwhelmed by the Grade 11 teacher while presenting circle theorems. He would read a sentence in the given statement in conjunction with the diagram and dissect the information using different coloured pencils to demonstrate and explain it piece by piece. For example, he coloured parallel lines to be distinctively visible. Learners easily recognise pairs of corresponding, co-interior, and or alternate angles.*
>
> ***Ms. Tau:*** *… In addition to teaching them how to scrutinise given information in conjunction with the diagram, I engaged them in drawing inferences from given data*

by visualising abstract concepts. For instance, if it is stated that two radii of a circle meet both ends of a chord on the circumference, then the learners should infer the given data and visualise equal angles opposite equal sides, the radii.

Mr. Mosola*: ne ke tlo cho joalo Ms. Tau (ET: I was about to say exactly that). "I have noticed that questions are not only based on the given information and the diagram in Euclidean geometry. One is expected to make inferences and visualise abstract aspects. Hence as a teacher, I should focus more on enhancing visualisation skills by asking probing questions so that learners can find out more than what is given in the statement and the diagram.*

Ms. Mofokeng: *I think as a teacher, I have to use techniques that promote self-discovery skills among the learners. That would assist them in conceptualising and simplifying abstract ideas in Euclidean geometry. I think including technology, such as Geogebra, in teaching mathematics in South Africa could yield a positive attitude towards teaching and learning Euclidean geometry. Learners' ability to form mental pictures would be strengthened.*

Based on the discussions above, it is evident that the subject advisor observed that highlighting some aspects of given data using different coloured pencils when explaining given information piece by piece is practical and enhances the perception of the data. This is supported by MMR (UDL principle), which advocates for improving visual perception to reinforce understanding of the mathematical concepts (CAST, 2011). Therefore, highlighting the parallel lines allowed the learners to see the different pairs of angles formed (for example, corresponding, co-interior, and *alternate angles*). Reading the geometric sentences or statements while referring to the diagram(s) and dissecting the information using different coloured pencils to demonstrate and explain the data 'piece by piece' was an excellent strategy to develop learner visualisation skills. Thus, scaffolding conceptual understanding of Euclidean geometry by demonstrations and explanations is an efficient way to enhance the visualisation of abstract aspects of the topic.

Ms. Tau believed it was important for the learners to master ways to scrutinise the given information in conjunction with diagrams. Scrutiny of the provided information should be coupled with an emphasis on making inferences and drawing conclusions based on the observations and the learners' mental pictures. Drawing inferences is essential in teaching and learning geometry because it enhances visualisation. It helps learners understand the information even if it is not directly stated but implied. Ms. Tau referred to the example: *In a circle of radius 30 mm, two radii, OA and OB, and the chord AB form an isosceles triangle AOB.* See Figure 4.8

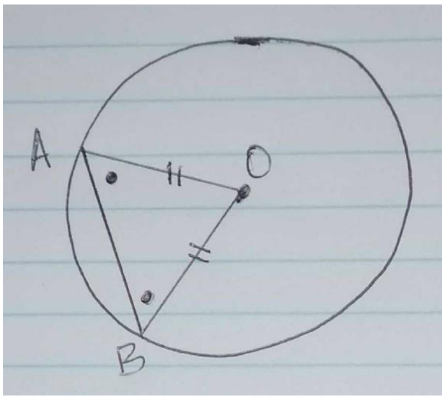


Figure 4.8: Radii and the chord of a circle

A learner is given information about two radii, OA and OB and a chord AB. Based on this information, a learner could form a mental picture of an isosceles triangle and deduce that angles opposite the radii are equal and could further recognise that any angle on the circumference subtended by the same chord AB will be half of the angle AOB at the centre. Therefore, one way to enhance learners' visualisation skills could be by asking probing questions. In that way, learners can make proper deductions, make sound conclusions, and correctly respond to questions by integrating the information on the paper-based on what it implies and what they have visualised. Probing was, therefore, a good strategy for developing the learners' ability to create mental pictures and find out what is implied by the given information.

Visualisation techniques are dependent on inference and comprehension skills. Through visualisation, learners could fill in what is not included in the statement or work out what the information was implying yet not explicitly stipulated. Nuuyoma and Makhene (2020) state that learners' ability to create mental pictures is based on critical analysis of the words they hear and the texts they read (see 2.4.1.3).

Ms. Mofokeng believes that teachers should promote self-discovery skills amongst learners. She also highlighted the importance of including technology (e.g., Geogebra) in teaching and learning Euclidean geometry as applications such as Geogebra would assist learners in conceptualising and simplifying abstract aspects, which could enhance visualisation and, subsequently, their academic performance. This concurs with Mokotjo and Mokhele (2021), who attest that using Geogebra could enhance deep learning because it would help learners visualise the related concepts.

4.5 FACTORS THAT IMPEDE THE EFFECTIVE IMPLEMENTATION OF UDL STRATEGIES IN TEACHING EUCLIDEAN GEOMETRY

This section highlights some anticipated threats to the successful implementation of solutions discussed previously.

4.5.1 Workshops

A workshop should serve as a platform for teachers' empowerment and be equipped with flexible ways to teach mathematics. It should be where teachers' interactions with the subject advisors and their peer teachers result in improved content knowledge. The participants unanimously agreed that the In-Service Education and Training (INSET) workshops are a brilliant move for equipping teachers with content knowledge. It is the platform where a lack of content knowledge is habitually addressed. Even for teachers who are not qualified to teach Grade 12, the workshops have a positive impact by improving content knowledge. However, it was indicated that the facilitators focus more on improving teachers' content knowledge rather than the pedagogical knowledge, which is equally essential in facilitating meaningful learning. This is evidenced by the comments made below by Ms. Ndweni and Mr. Mokwena, respectively.

> ***Ms. Ndweni****: I have gained a lot from this workshop and am happy that I clarified the concept of the triangle proportionality theorem and its converse. Feela ho thata ho e ruta (ET: but it is challenging to teach it).*

> ***Mr. Mokwena:*** *That's precisely my challenge. I have understood the concept, but I struggle to make my learners understand it. This section of geometry is always poorly performed by my learners. Workshop facilitators should also teach us how to teach this theorem.*

Ms. Ndweni reported that she did not understand the triangle proportionality theorem and its converse before attending the workshop but had gained knowledge and understanding of the concept that had been her challenge for many years. However, the workshop did not make teaching the theorem easy, hence the statement, *"feela ho thata ho e ruta (ET: but it is difficult to teach it)."* The researcher deduced from Miss Ndweni's remark that the workshop provided teachers with opportunities to improve their content knowledge. However, the workshop did not provide sufficient pedagogical expertise for teachers to teach Euclidean geometry effectively. The workshop that focuses only on content knowledge poses a threat since it does not address the 'how to teach' part, which is also vial for the learners' overall performance. It seemed that the facilitators of the workshops also lacked the skill to empower the teachers with pedagogical knowledge on how to teach Euclidean geometry; hence their focus was on content knowledge and not on sharing various strategies on how to teach Euclidean geometry. Workshops that do not focus on empowering teachers with various teaching strategies violates the UDL framework, which advocates the notion of promoting varied teaching strategies to make teaching flexible, inclusive and accessible (Moleko & Mosimege, 2020).

Mr. Mokwena was also able to develop understanding of the concept during the workshop but was not able to effectively teach the topic to the learners make his learners understand it; hence his learners continued to perform poorly in geometry. His remark, "*I have understood the concept, but I struggle to make my learners understand it,"* confirms that the workshop was only one-dimensional in developing the teachers, thus focusing only on content knowledge. Although content knowledge is important for teaching, it cannot make much of a positive impact if teachers do not have pedagogical knowledge, which is a vehicle through which content should be conveyed. Mr. Mokwena thus believed that it would positively impact if the facilitators could also give special attention to the teaching strategies. The workshop, which lacks an empowering element of pedagogical knowledge, thus makes it difficult for meaningful learning to take place and for learners' performance in Euclidean geometry to be improved.

Generally, workshops are meant to support and develop teachers holistically, that is, to upgrade and augment their content and pedagogy knowledge. However, deducing from the teachers' comments above, little attention seems to be paid to developing pedagogical knowledge to empower the teachers with strategies to teach Euclidean geometry. This is a threat in that it becomes futile to refine teachers' content knowledge without giving special attention to improving their pedagogical knowledge (how they teach), which is equally crucial for the overall learner performance. Zambo (2008) affirms that the Professional Development Workshops (PDWs) focus on empowering teachers with content knowledge to afford them opportunities to learn new ideas about the aspects of geometry (see 2.4.2.1). However, the workshop that overlooks the elements of pedagogy violates the UDL teaching framework, which advocates the use of varied and inclusive teaching strategies to accommodate diverse learner populations. Drawing from Ms. Ndweni's and Mr. Mokwena's responses, it was clear that the workshop did not improve their pedagogical knowledge since the focus was only placed on content knowledge. Consequently, their instructional strategies were not responsive to the learners' needs, which explains why their learners' performance was poor in the section on triangle proportionality theorem and its converse.

To address the identified threat, the CES and HOD commented, respectively, as follows:

> ***CES:*** *Colleagues, please remember to teach from known to unknown, simple to complex. Ms. Ndweni, it would not be challenging to introduce the concept of triangle proportionality theorem and its converse if you could review your learners' knowledge of similarity and congruency that they learned in Grade 10.*
>
> ***HoD:*** *I support Mr. Mokwena's suggestion. Facilitators should train us on how to teach this theorem effectively. In addition, I think it is high time we implement what we have learned from workshops. We can conduct micro-teaching whereby one teacher will plan and teach a lesson while others observe and give constructive feedback on the presentation. We can do this in our schools.*

The CES reminded the participants that new knowledge is built on existing knowledge and teachers should teach from known to unknown and simple to complex. For teachers to discover what their learners know, they should make efforts to know them in terms of how they learn. The CES further pointed out that it would not be challenging to teach the theorem and its converse if learners' knowledge of Grade 10 geometry is reviewed such as similarity, and congruency of triangles. Masilo (2018) supports this practice and confirms that activating learners' prior knowledge when introducing a new concept is crucial. That would enable them to understand and thus learn the new concepts. UDL espouses the practice of activating the learners' prior knowledge to allow learners to build on new knowledge and to make connections between the mathematical concepts in creating new knowledge. Prior knowledge is also vital to fulfilling Van Hiele's theory of geometric thinking, which seeks for learners to be moved from one level to the next without compromising the other (see 2.4.2.3).

The HOD concurred with Mr. Mokwena that the workshop facilitators should equip teachers with teaching strategies for learners with diverse learning abilities, thus espousing the teaching goal of UDL (Dalton, 2017). She proposed engagement in micro-teaching where one teacher could do a presentation while others observe to give constructive feedback on display. This could be another way to sharpen their instructional methods. The more they practise micro-teaching, as a platform to discuss multiple ways of teaching theorems and their converses to learners with diverse learning styles, the better their teaching will be. This social constructivist way of working, which is one of the tenets of the lens couching this study (social constructivism) (see 2.2.1), will therefore afford the teachers opportunities to share ideas, enlighten and empower one another with knowledge of the various strategies to use when teaching the theorem.

4.5.2 Lack of Expertise in Teaching Euclidean Geometry

One of the distinguishing characteristics of a good teacher is their ability to teach effectively, using different methods to cater to all learners (Ida, 2017). Teachers' lack of expertise threatens the efficiency of their teaching approach. For instance, if they lack conceptual understanding of geometry basics, it would be challenging to go deeper into the concept and teach it efficiently. Even if the content gap is attended to during the workshop, it may not be enough for a teacher who is under-qualified to teach at the FET level or a teacher who did not study geometry adequately in their initial teacher training, as indicated by Ms. Tau and Ms. Mofokeng.

> ***Ms. Tau:*** *Teaching Euclidean geometry is a nightmare because I am not qualified to teach Grade 12. I have Senior Primary Teacher Diploma (SPTD). Being not qualified makes me lose interest in teaching the topic."*
>
> ***Ms. Mofokeng:*** *I think my situation is worse than yours, Ms. Tau. I have a Bachelor of Education degree, but unfortunately, my high school teacher had a challenge with Euclidean geometry. He used to copy theorems and diagrams from the textbook to the chalkboard. I did not learn anything, and this has affected me badly.*

Both participants acknowledge that teaching Euclidean geometry was challenging as they were not qualified to at that level, they there had a content gap This is the reason for In-Service Education Training (INSET) workshops but as reported, only content knowledge is addressed which means that Ms Tau was not confident in teaching the topic. According to Machisi (2021), some teachers lack knowledge of Euclidean geometry and effective ways of teaching it because they did not study it at any level of their studies (see 2.4.2.1). Ms. Mofokeng reported that even though she had specialised in Mathematics, she lacked a conceptual understanding of Euclidean geometry. at the high school level, she had been taught by an incompetent teacher who copied theorems and diagrams from the textbook to the chalkboard. Ms. Mofokeng thus found it difficult to learn Euclidean geometry under such conditions, which badly affected her as she could not execute the topic later when she became a teacher.

INSET workshops become helpful for under-qualified teachers to improve their content knowledge. However, in this case, the threat is that even though the workshop attempted to address the content knowledge, its development takes time and the low self-confidence the teachers continue to display makes it even more difficult for them to master and teach the geometry concepts. The major challenge is in the realisation that teachers who are struggling with geometry concepts also lack the pedagogical knowledge to execute the geometric ideas. As a result, these teachers resort to teacher-centred approaches (telling methods and copying the theorems as they are), which help them teach the topic for compliance. Lack of pedagogical knowledge thus makes it difficult for teachers to teach geometry in a manner that motivates learners to learn. The teachers' self-knowledge about their lack of pedagogical knowledge causes them to be discouraged and lose interest in teaching the topic. This is confirmed by the extracts below:

> ***Mr. Mosola:*** *Kannete nna ke na le mathata (ET: Indeed, I have problems). I am struggling with geometry concepts and lack pedagogical knowledge. I think it will take time to be confident to teach Euclidean geometry. One INSET workshop per year is not enough for me to master this topic*

> ***Ms. Mofokeng:*** *I know I am struggling with geometry, which makes me feel discouraged and less interested in teaching the topic."*

Subject advisor: *Absolutely, Ms. Mofokeng! Teachers' lack of confidence and interest in teaching the topic affects learners' emotions, leading to poor performance.*

From the above extracts, one can deduce that the issue of teachers' content gap is attempted to be addressed during the workshop workshops. However, teachers do not gain the confidence to teach the topic. They experience mathematics anxiety which unfortunately is inherited by the learners in the process. According to Mr. Mosola, the time allocated for workshops is inadequate for them to become experts in teaching Euclidean geometry. The subject advisor confirmed what Ms. Mofokeng had uttered. He further pointed out that it is true that teachers' lack of confidence and loss of interest in the topic negatively impact the learners' performance.

Generally, the workshops focus more on the content and minimally pay attention to the pedagogy. The CES, as the official from the Department of Education (DoE), also shared the same sentiment and further suggested what needs to be done to improve the situation going forward. He commented as follows:

"Colleagues, you have raised an important point which I agree with you a hundred percent! However, as the DoE, I think we should restructure our planning for INSET workshops. Our programs should focus more on imparting skills that address diverse learning needs. It becomes a futile exercise to focus on addressing the content gap but ignore varied ways of delivering the content".

Mr. Mokwena*: To add to what you have just said meneer, I think the DoE, working closely with higher education institutions, should urgently develop a policy on effective teaching strategy that caters to diverse learning styles.*

Subject advisor*: Besides restructuring our planning on what should be covered in the workshop, we should also strive for at least three seminars per year. I agree with Mr. Mosola that more time is needed to train one for effective teaching of Euclidean geometry. Hence one workshop per year is not enough.*

HoD: *It is crucial to prioritise teacher development on pedagogy; otherwise, learners' emotions will be harshly affected if their teacher is discouraged and loses interest in teaching the topic. Every learner is critical, and we must try to meet their learning needs.*

The CES pointed out that the DoE must restructure the program for INSET workshops. It should pay special attention to teaching strategies that address diverse learning needs. The CES considered it a waste of time to conduct workshops that focus only on content. Mr. Mokwena, in accord with the CES, suggested that the DoE should work in partnership with higher education institutions to effectively develop a policy for teaching learners with diverse learning needs.

According to the subject advisor, in addition to restructuring the INSET workshop programs, the frequency of the workshops should also be a focal point of the DoE. He suggested a minimum of three workshops per year because one may not be enough to build a competent teacher of Euclidean geometry. The HoD believes that if teachers lack the expertise to teach Euclidean geometry, their negative attitude may affect learners' emotions. This infringes the UDL principle (MME) since such a lack of expertise may deprive learners of the opportunity to become purposeful and motivated (CAST, 2011). It would therefore be difficult to achieve this goal if learners are emotionally not stable due to the teachers' incompetence. It is therefore important to ensure that every learner's learning style is met because all learners are important.

The participants seemed to unanimously agree on the need to improve the standard of the workshops by giving special attention to pedagogy and multiple ways of teaching Euclidean geometry and investing sufficient time to reinforce understanding of Euclidean geometry necessary to raise their confidence levels to introduce the topic.

4.5.3 Resources

According to participants, limited resources are another factor that could threaten the effective teaching of Euclidean geometry. Limited time and a high teacher-learner ratio may hamper the implementation of some of the strategies that could enhance the teaching and learning of Euclidean geometry accessible. The participants' discussion unfolded as follows:

> ***Mr. Mosola:*** *I agree that we should deliver content to become accessible to learners from different backgrounds and with diverse learning needs. My main challenge is planning for different learning styles requires more time. Euclidean geometry is jam-packed with content. How do I ensure that my lesson plan caters to audio, visual, and tactile learners?*

> ***Mr. Mokwena:*** *Additionally, the teacher-learner ratio is far more than what is suggested by the education policy, which is 1: 40. This makes it difficult for us to teach this topic.*

Mr. Mosola supports the idea of making the learning of Euclidean geometry accessible to all learners. However, the challenge is creating lesson plans that caters to all learning styles since the time is limited. The time factor hampers the implementation of inclusive teaching strategies, mainly because Euclidean geometry is jam-packed with content. The planning of the lessons did not follow UDL principles; thus, there were no efforts in terms of using multiple representations, using varied strategies to assess learners and to engage them in meaningful learning processes (Scott et al., 2003).

Participants raised the issue of high *"teacher-learner ratio … far more than what is suggested by the policy of education, that is 1: 40,"* which made it difficult for the teachers to teach this topic and is thus a challenge in catering to all learning styles. Working with too many learners in a limited time is difficult since they have different learning needs (Burgstahler, 2009). This, therefore, means that over and above-lacking geometry pedagogical knowledge, the teachers, are faced with teaching large classes. Bozzi et al., (2021) assert that traditional lecturing is predominantly the only viable teaching approach for large classes, yet this approach can impede learning.

As part of the solution to the identified threats, the following strategies were suggested:

> ***Subject advisor:*** *The use of technology and manipulatives could make it easy to teach the topic. Although many people think that manipulatives waste time, I beg to differ. I think they save time and make learning practicable and engaging.*
>
> ***CES:*** *In addition to using technology and manipulatives, you may consider adopting the flipped classroom model. Flipped classroom model empowers learners to take control of their learning because they may have access to information videos or audio-recorded material in advance in preparation for teacher-guided discussions. The model is responsive to a high teacher-learner ratio.*
>
> ***Subject advisor:*** *Colleagues, you may also incorporate elements of the practical-based learning model where learners can explore, determine, and investigate independently.*

Based on the extracts above, participants offered a range of solutions such as the used of manipulatives and technology which makes teaching Euclidean geometry effective, more engaging and practical as multiple means of representation ensures that content is represented and taught differently.

UDL supports the flipped classroom model because it allows learners to take responsibility for their learning and provides options for illustrations through numerous media (CAST, 2011). The model is responsive to a high teacher-learner ratio because learners may be given information earlier to work through on their own and then later engage in teacher-guided discussions.

After reflecting on the threats discussed above, the CES promised that even though the time factor and the teacher-learner ratio may impact negatively on effective teaching of Euclidean geometry, support is given by Department officials in the form of frequent interactions with teachers after the training workshops. The CES did point out that the Department should work hand in hand with the institutions of higher education to develop a policy on what inclusive teaching strategies entail, how to equip pre-service teachers with knowledge of the strategy, and how to implement it. All this was deduced from CES's input where he said,

> ***CES:*** *The DoE officials have to frequently visit the schools to monitor and evaluate the impact of the workshops. Moreover, higher education institutions should work collaboratively with the DoE to develop a policy on the inclusive teaching strategy and how to train teachers on implementing it.*

In summary, the participants mutually maintained that if teachers could be trained on inclusive teaching strategies, it could be possible to teach Euclidean geometry effectively amidst high numbers of learners and limited time. However, they pointed out that INSET workshops focus more on content knowledge and minimally pay attention to inclusive teaching strategies. Additionally, teachers lack the expertise in teaching Euclidean geometry as they only have a procedural understanding of the topic. This could be because they were under-qualified to teach at FET level or did not study the topic at any level of pre-service training.

4.6 CONCLUSION

The chapter presented the findings and analysis of the data. It highlighted the challenges pertaining to the teaching and learning Euclidean geometry, outlined solutions to address the identified challenges, and highlighted the threats to implementing the UDL strategies. The next chapter discusses the findings, limitations, recommendations and conclusions of the study.

CHAPTER 5
CONCLUSIONS

5.1 INTRODUCTION

The study aimed to explore how the universal design for learning (UDL) can be implemented to enhance the teaching and learning of Euclidean geometry. In pursuance of the outlined aim, the chapter restates the primary purpose and objectives of the study. Thereafter it confers the findings based on the challenges pertaining to the teaching of Euclidean geometry. The chapter further highlights the solutions and necessary conditions for successfully implementing UDL to address the identified challenges. It also provides recommendations for future research and highlights the limitations of the study, implications for practice, and conclusion.

5.2 THE OBJECTIVES OF THE STUDY

Learners' poor performance in Mathematics has been a cause of concern to all stakeholders. According to Chihambakwe (2017), the NSC diagnostic report revealed that Euclidean geometry was the worst-performed topic (See 1.1). Although it is a fact that learners are different and have diverse learning styles that should be catered for, teachers seem to lack knowledge of the teaching strategies for teaching Euclidean geometry. As they have not been adequately trained to teach Euclidean geometry. Thus they lack knowledge of the teaching strategies to teach the topic effectively. The teachers thus fail to apply the teaching strategies that are inclusive, flexible, accessible, and responsive to the needs of the learners. Based on this, the study sought to explore how the universal design for learning can be applied to enhance the teaching of Euclidean geometry. The study was guided by the following research question: *How can the Universal Design for Learning be implemented to strengthen the teaching of Euclidean geometry*?

To achieve the purpose of the study, three objectives were formulated, namely;

1. to identify the challenges encountered in the teaching and learning of Euclidean geometry;

2. to determine how UDL principles can be used for effective teaching of Euclidean geometry;
3. to determine the mitigating factors against threats that may hinder the optimal benefit of using UDL principles in the teaching of Euclidean geometry.

5.3 SUMMARY OF THE STUDY

The learning theory adopted as the lens for this study was social constructivism theory. This theory was selected because it advocates similar principles to the research methodology of the study, namely, the collaboration between the researcher and the participants (co-researchers in the context of this study), knowledge sharing by the 'mass' and not an individual (Moleko, 2014), acknowledging multiple perspectives in addressing the issue (Tlali, 2013), and co-construction of new knowledge (Tsotetsi, 2013). The study revealed interconnections of some progressive and productive approaches that address the teacher-centred approach, which is the leading cause of poor performance in Euclidean geometry. Kivunja (2018) affirms that in a constructivist classroom, the facilitator creates a dynamic teaching and learning environment to ensure that all learners contribute toward constructing and transmitting new knowledge. Therefore, it is significant to create an active learning environment and use an effective teaching strategy because learners' cultural backgrounds influence the learning process (Weldeana, 2016).

Social constructivist theory interweaves with UDL principles in that they both acknowledge learner diversity, hence the need for multiple ways of teaching, striving to use inclusive teaching practices that cater to diverse learner populations, and using the varied learner engagement strategies since no single means of engagement may be optimal for all the learners (Grier-Reed & Williams-Wengerd 2018; Moleko, 2021). Nkoane (2012) believes that the researcher and the co-researchers should be encouraged to work on the core source of the unwanted circumstances mutually. Based on Nkoane's stance, the social constructivism approach opened the platform for discussing the root cause of poor performance in Euclidean geometry and how to alleviate the challenge. PAR was therefore considered a suitable research methodology for this study because it created a collegial environment for the researcher and the co-researchers to collectively discuss the challenges regarding the teaching and learning of Euclidean geometry and share ideas on how these challenges can be addressed using UDL principles. PAR gave the people who experienced the problem the opportunity to identify challenges and suggest solutions for the challenges they have identified. The PAR approach is a cyclical process that comprises the following phases: problem identification, planning, implementation, observation and reflection, stages that afforded the co-researchers the opportunity to engage in discussions around the challenges in the teaching and learning of Euclidean geometry and to also engage in the discussions leading to the determination of the UDL-based strategies to address the identified challenges.

Data were generated from a team that consisted of mathematics teachers, mathematics HoD, subject advisor, and CES. SWOT analysis was executed to divulge the strengths and weaknesses of the team. Together, the participants spotted the opportunities for improving the situation and ways to mitigate the threats that could impede the implementation of UDL principles in teaching Euclidean geometry. Since the study is about implementing UDL principles to enhance the teaching of Euclidean geometry, the team shared information about the content knowledge gap of Euclidean geometry and how to teach the topic to learners that lacked prerequisite knowledge and had a negative attitude towards the topic. UDL as a teaching strategy attempted to reduce barriers to learning since it enabled teachers to plan for all learners. Teachers need to ensure that all learners access and engage actively in learning Euclidean geometry. UDL principles were ultimately found to help learners become motivated and purposeful through the implementation of the MME principle, knowledgeable and resourceful through the implementation of the MMR principle, goal-oriented and strategic about learning Euclidean geometry through the implementation of the MMAE principle.

5.4 FINDINGS ON THE CHALLENGES PERTAINING TO TEACHING OF EUCLIDEAN GEOMETRY

Several challenges were identified regarding the teaching of Euclidean geometry, as discussed in the previous chapters, which subsequently necessitated implementing the UDL-based strategies. The following sections provide a summary of these challenges.

5.4.1 Teachers' Lack of Knowledge of Inclusive Teaching Strategies

The study revealed that teachers did not use effective teaching strategies mainly because they had not been trained on how to teach geometry (see 4.2.1). Over and above, they were never trained to cater to learners with diverse learning styles (Machisi 2021). As a result of lacking both the content (Euclidean geometry) and pedagogical knowledge, the teachers thus resorted to using traditional methods (teacher-centred approaches) of teaching geometry because they were not conversant with the flexible, inclusive, and accessible strategies that accommodate all learners and maximise learning (see 2.4.1.1). This, therefore, means that the learners were not exposed to various teaching styles that could meet their learning needs. Learners were also subjected to rote learning and expected to memorise the Euclidean geometry concepts through traditional teaching methods (see 4.2.1). The findings also show that teachers relied on textbooks as the only teaching material/resource and hardly considered the different characteristics that learners bring into their classrooms (see 4.2.1), hence the need for varied forms of instruction and learning materials for more accessible teaching and learning. Teachers were also found ignorant about the effects of creating a positive classroom environment for all learners.

Due to a lack of in-depth knowledge of Euclidean geometry and how to teach it, many teachers paid little or no attention to the effect of spatial representation and language, which are vital in communicating geometric concepts. According to Moleko and Mosimege (2020), teachers still display incompetency in facilitating learners' ability to read with understanding and interpreting geometry problems. The study's findings revealed that teachers focus on the examples given in the textbooks and lack the creativity to think of multiple effective ways of instruction because of a lack of knowledge of inclusive teaching strategies (see 4.2.1). As articulated by Machisi (2020) (See 2.4.1.1), teachers opt for the 'telling' method and copying from textbooks to the chalkboard due to a lack of knowledge of strategy responsive to diverse learning styles. Moleko (2018) and Tsimane and Downing (2020) do not support a teacher-centred approach since it promotes rote learning and memorisation of concepts and does not cater to diverse learning styles within classrooms.

5.4.2 Shortage of Resources

Both theoretical and empirical data revealed that there was a shortage of resources that were key for the effective teaching of Euclidean geometry (see 2.5.3 and 4.2.3). As alluded by Hylen (2021) in 4.2.3, learning resources are the primary academic property in education. The shortage thereof makes it difficult for learning to take place and for the goal to be achieved. The scarcity of resources included qualified teachers, technological appliances (computers, GeoGebra software), textbooks and teachers passionate about teaching the topic. All these led to the teaching done for compliance and did not cater to learners' different learning styles (see 4.2.1).

The participants realised that the shortage of human resources contributes to the poor teaching of Euclidean geometry. Due to a major shortage of qualified mathematics teachers, in-service teachers are obliged to teach the subject even though it is not their specialisation, as confirmed by Makofane and Maile (2019) (see 4.2.3). The shortage of resources thereof infringes the principle of UDL, MMR, which provides options for mathematical expressions, vocabulary, and symbols. According to the literature (see 2.4.1.4) and what became apparent from our discussion (see 4.2.2), it is challenging to teach Euclidean geometry when one lacks knowledge of basic concepts of geometry and not aware or knowledgeable of teaching methods that are responsive to different learning abilities. Govender and Govender (2019) also affirmed that some teachers lacked knowledge of Euclidean geometry because they did not study it at any level. Hence, they struggle to teach the concept (2.4.1.1). Indeed, there is some evidence that for teachers to enable epistemological access in their classrooms, they must have a sound understanding of the subject matter and be able to relate it to other disciplines of knowledge. However, this seemed to be a challenge, according to the findings of this study (see 4.2.3).

5.4.3 Teachers' Inability to develop Engaging Learning Environments

Although social constructivists consider learner interaction with one another as a way to create new knowledge, the findings of this study and the literature revealed that teachers treat learners as passive recipients of knowledge (see 2.4.1.2 and 4.2.4). Teachers present their lessons authoritatively without engaging learners so that they can make sense of the concept taught to them. Inability to develop engaging environments was the cause of learners' negative attitude towards Euclidean geometry. Findings also revealed that the inability to teach geometry in a manner that shows its applicability to real-life situations, also contributed to learners' development of negative attitudes (see 4.2.4).

Findings further revealed that sometimes learners come to school without completing their homework or leave their exercise books and textbooks at home because of a lack of purpose and interest in learning the topic (see 4.2.4). As documented in Chapter 2 (see 2.4.1.2), learners' seating arrangement could be another factor that hinders learner engagement. If learners are seated facing the same direction, and their teacher is standing in front, it might imply that the teacher is the only source of knowledge. That means the created environment would not afford the learners a platform to interact with each other. Additionally, it does not allow the use of multiple flexible ways of instruction for different learning abilities.

Several studies (Boligger & Martin, 2018; Burgstahler, 2009; Kutama, 2002; Moleko, 2021) noted that teaching mathematics becomes difficult if learners are not motivated and engaged. Learner engagement is one way to address negative attitudes towards learning in general.

5.4.4 Teachers' Inability to develop Learners' Visualisation Skills

Even though visualisation was deemed essential in learning Euclidean geometry, the findings revealed that teachers had difficulty developing this skill in learners (see 4.4.4). This leads to learners not being able to create mental images and pictures of what they are expected to learn, and this subsequently makes it difficult to solve Euclidean geometry problems. Alex and Mammen (2017) consider visualisation skills as a catalyst in teaching and learning because they improve abilities, boost confidence and enhance motivation. Findings indicate that even though learners seemed to understand the theorems, their converses and axioms, they struggled to solve riders because they could not visualise the given information. For instance, they could not construct proofs and solve related problems because they could not visualise relationships amongst the theorems.

5.4.5 Workshops do not address Teachers' Lack of Expertise

Findings of the study revealed that the INSET workshops which the Department organised only addressed the teachers' content knowledge gap while paying minimal attention to pedagogical knowledge when teaching learners that have diverse learning styles (see 4.3.2). The programme of the workshops does not seem to include teaching methods. It was further revealed that the Department conducted workshops that did not develop the teachers holistically (see 4.5.1). Furthermore, it was revealed that the time invested in the training workshops was not sufficient. Thus, the workshops did not equip the teachers with the relevant knowledge and skills to become experts in teaching Euclidean geometry (see 4.5.2). This study showed that training workshops that overlooked pedagogical knowledge desecrated MME, the UDL principle, which facilitates coping skills and strategies (see 4.5.1).

5.4.6 Lack of Knowledge of Euclidean Geometry Vocabulary and Expressions

The study's findings revealed that in certain instances, the teachers displayed a lack of knowledge of Euclidean geometry vocabulary and statements (see 4.2.2). If one does not know geometry vocabulary or misuses it, the concepts will not make sense to learners. The study revealed that the shortened geometric statements might lead to a misunderstanding of concepts if they are not well explained, especially at the beginning of a lesson (see 4.4.2). The findings further revealed that some Euclidean geometry expressions could be misinterpreted due to the conditions which are not stipulated within the questions. Inadequate knowledge of geometry basics leads to the transgression of the UDL principle of MMR, which offers a variety of ways of customising the display of information. For example, the statement stating that angles on a straight line are supplementary may be misinterpreted because it is silent about the fact that the angles should have a common arm (see 4.4.2). Angles on a straight line are supplementary if they have a common arm or vertex. As outlined by Kodisang (2015), language is a vehicle for learning and developing conceptual understanding (see 2.4.1.4) because it influences our thinking and how we perceive and interpret the world around us. The study's findings further proved that a failure to pay attention to vocabulary leads to learners' inability to unpack statements and diagrams in Euclidean geometry questions (see 2.4.1.4).

5.5 STRATEGY FOR EFFECTIVE IMPLEMENTATION OF UDL PRINCIPLES IN TEACHING EUCLIDEAN GEOMETRY

This section presents a strategy for the effective implementation of UDL, which comprises events and activities that unfolded when designing the framework that would be sustainable for teaching Euclidean geometry. The key aspects of the effective implementation of UDL are discussed in the subsequent sections. The sections thus indicate how UDL could inform the productive teaching of Euclidean geometry.

5.5.1 Knowledge of Inclusive Teaching Strategies

The composition of the team of co-researchers in this study was exposed to the principles of UDL. The team, therefore, understood the importance of using flexible, inclusive, and accessible teaching strategies to provide meaningful learning. The SWOT analysis which was conducted indicates that the strengths of the team outweighed the weaknesses. PAR as a research methodology for the study was deemed ideal because it fostered a conducive environment where everyone was free to share their experiences and challenges in teaching Euclidean geometry. The team was very active in assisting one another to make sense of the discussions leading to the effective implementation of the UDL principles. The UDL mini-workshop conducted by UDL coach for the participants raised awareness about the need for multiple effective teaching strategies to cater to diverse learning styles in any classroom. It has been crystal clear that for effective implementation of the UDL principles in teaching Euclidean geometry, teachers' content knowledge, knowledge of mathematical vocabulary and geometry expressions, pedagogical knowledge, knowledge of learners, and learners' learning styles should be the focal point for effective teaching and learning of Euclidean geometry.

5.5.2 Availability of Teaching and Learning Resources

It is pertinent that all stakeholders in the education system ensure that schools have appropriate resources to address the diverse learning styles in classrooms (see 4.4.3). The current study indicates that it becomes challenging to teach Euclidean geometry to learners in under-resourced schools, especially when one first started by teaching in well-resourced schools and later has to adapt to teaching in an under-resourced school (see 4.2.3). This finding indicates that teachers need to know the different contexts as this will assist them in teaching effectively regardless of the present contextual factors. This finding also suggests the need for more time to be invested in training to ensure the capacity development of the teachers in this regard. It is evidenced by the current study (see 4.4.3) and the literature (see 2.5.3) that the availability of the appropriate resources enhances teaching and learning. One of the UDL principles, MMR, requires the information to be presented and represented in various formats to make it possible for learners to recognise and analyse the concepts, thus catering to the needs of the diverse learner population. This, therefore, means that the materials/resources for teaching Euclidean geometry need to be represented in different formats to cater to all learners. In addition, teachers (as human resources) must have relevant qualifications to teach Mathematics to interact with all learners confidently. These teachers must also receive training on how to teach Euclidean geometry from time to time to advance their content and pedagogical knowledge.

Additionally, learners must have textbooks, but these textbooks should not be used solely as some learners may not derive benefit from them. This means that efforts need to be made to use various resources to accommodate all learners not only textbooks. However, each learners needs a textbook to continue to work on activities as homework. However, these textbooks must be used with other resources to supplement them, which is the notion that UDL supports since learners learn and assimilate content differently. Varying the resources thus makes it easy for the teacher to accommodate all learners (Moleko, 2021). Furthermore, literature (see 2.5.3) and the empirical data for this study (see 4.4.3) emphasise the need for schools to avail the appropriate resources, including manipulatives and technological appliances such as geo-Gebra to make the teaching of Euclidean geometry flexible and accessible to all learners (Moleko, & Mosimege, 2021).

5.5.3 Sustainability of the Outcomes of INSET Workshops

The training workshops should target holistic development of teachers and not just focus on developing the teachers' content knowledge. The study's findings support the notion of workshops, which must address both teachers' content knowledge gap and pay more attention to teachers' pedagogical knowledge gap (see 2.6.1). For the sustainability of outcomes of the workshops and teacher development, the facilitators should create a follow-up tool kit to evaluate the professional development process. The officials from the Department must support teachers in implementing what they have learned from the workshops.

According to the UDL framework, this is important to support planning and strategy development and enhance capacity for monitoring progress (CAST 2011). However, this calls for the transformation of the mindset of the stakeholders (teachers and Department officials) because teachers would need more time to plan and prepare lessons that cater to learners with diverse learning styles. The officials would have to monitor and evaluate the training programmes by increasing the frequency of school visits to ensure the implementation of UDL principles beyond the training workshops. Micro-teaching practice at the school level could be another activity where teachers can practice effective teaching strategies (that ism UDL strategies) for learners with diverse learning styles. The more they practise micro-teaching, the better their teaching practice because the teachers would receive constructive feedback about what has worked, what did not and what improvements could be made to improve their teaching (see 4.5.1).

5.5.4 Creating an Engaging Learning Environment

Teachers and other stakeholders in the education system must acknowledge learners' diversity by availing different forms of motivation at their disposal. Some of the learners may be stimulated and excited by the inclusion of technological appliances, drawings on the board, or when given hands-on activities to explore more on the concept (see 2.6.1 and 4.4.2). For example, learners may draw a cyclic quadrilateral and measure the sizes of angles to confirm that the interior angle of a cyclic quadrilateral equals its opposite exterior angle (see 4.4.4). This approach enhances conceptual understanding and discourages memorisation. It is an approach espoused by UDL's MME principle, which advocates using varied teaching strategies to engage and motivate learners (Scott et al., 2003). Creating an engaging environment for learners is helpful because it allows learners to engage in meaningful learning, subsequently improving their learning skills and deepening their knowledge (Moleko, 2021). According to UDL, MME principle, creating an engaging learning environment facilitates learners' coping skills and strategies (CAST, 2011). This is important to empower the learners with the coping skills and strategies, particularly for a challenging topic such as geometry which involves multiple theorems and requires logic and the ability to make sense of the geometric statements. Therefore, teachers must use flexible, inclusive teaching strategies to make learning accessible and intuitive to all learners in a classroom (Burgstahler, 2009).

5.5.5 Commitment

Based on the study's findings, it is evident that for effective implementation of the UDL principles, teachers must be committed to transforming their mindset regarding the learner-centred approach. This means that teachers must be willing to try other teaching strategies. Teachers need to realise that to provide meaningful learning of Euclidean geometry, they need to alter their teaching strategies and be committed to the course. To succeed in doing this, teachers need to know their kind of learners. They need to understand how learners learn and what learning means to them. Moreover, teachers must train their learners to think critically when solving geometry problems. Although it might be challenging to pay attention to every learning style in the classroom or focus on every learner, some teaching strategies are effective in teaching geometry, especially in large classes. In such instances, a flipped classroom and problem-based learning models seemed to be the appropriate models to address teaching and learning challenges often presented by the high teacher-learner ratio situations (see 4.5.3). These models enabled teachers to actively engage the learners in the learning process and made them own their learning as they interact with one another in building new knowledge.

5.6 IMPLEMENTATION OF UDL IN TEACHING EUCLIDEAN GEOMETRY

In this section, threats that may hamper the successful implementation of UDL in the teaching of Euclidean geometry as well as strategies that can address the threats are discussed.

5.6.1 Threats that may hamper the Successful implementation of UDL

The current study's findings revealed that the training workshops create a platform for teachers to learn content, practise new techniques, and share their experiences with others to become better teachers. However, it becomes a futile exercise if teachers are not willing to go the extra mile to devise flexible, inclusive and accessible strategies. The teachers' resistance to strive to plan their lessons using the UDL principles may pose a threat to the teaching of Euclidean geometry. Moreover, even if teachers have been trained to apply inclusive teaching strategies, a threat in this instance would be lack of resources to support the implementation of UDL. The study revealed that the workshops focus mainly on bridging the content gap but not on the pedagogy to teach mathematical topics, including Euclidean geometry (see 4.5.1).

Furthermore, the training is often conducted without follow-up training to further empower the teachers with knowledge and skills to teach Euclidean geometry (see 4.5.2). This kind of training would therefore pose a serious threat and not assist in improving the teaching of Euclidean geometry

Although the teachers may be willing to implement UDL strategies to teach Euclidean geometry, their lack of confidence (since they are not knowledgeable about geometry concepts), as well as the learners' negative attitude towards Euclidean geometry, may demotivate and discourage teachers from attempting to apply the UDL strategies (see 4.5.2). This may result in the teaching of Euclidean geometry not improving and not being meaningful. The perception that UDL takes too much time to implement may also discourage teachers from using the strategy (see 4.5.3), thus resulting in unproductive teaching of Euclidean geometry.

5.6.2 Strategies that can Address the Threats

It is of utmost importance for the Department to implement UDL principles. Subject advisors and mathematics HODs have to follow up on the workshops by embarking on class visits to ensure a conducive environment for teachers to apply what they have learned from the workshops. Additionally, as UDL involves the use of manipulatives, SMTs ought to make it a priority that schools have mathematics laboratories. This would motivate teachers to apply their knowledge of UDL principles. Teachers and learners would gain confidence and be actively involved in teaching and learning Euclidean geometry. For schools that have no resources, the teachers may have to be trained on how to design materials and resources for themselves using the resources they have at their disposal. Continuous developmental workshops and Professional Learning Communities (PLCs) may assist in terms of boosting the teachers' confidence to teach Euclidean geometry. These platforms will further provide the teachers with ideas on how best to teach Euclidean geometry.

5.7 LIMITATIONS OF THE STUDY

The study was conducted in one of the schools in the Motheo district of the Free State province. Therefore, the results of the study cannot be generalised. However, the schools that are in the same context could benefit from this study. The findings are influenced by teachers' level of knowledge, type of learners and availability of resources in that school. This means that different perspectives would be established if the same study was conducted in a different context. The challenges, solutions and ways to overcome the threats that may impede the implementation of the UDL principle in teaching Euclidean geometry in this study would be completely different if the study were conducted in another school. The data were collected when the Covid-19 pandemic was prevalent; therefore, it was not easy to conduct numerous observations. The many restrictions, including social distancing, made it challenging to observe the lessons. The school's principal, where the study was conducted, was also reluctant to allow the researcher to sit in class to observe the lessons. The findings of this study on observations were therefore affected by this; hence fewer sessions on UDL implementation are reported.

5.8 IMPLICATIONS FOR PRACTICE

Learners' poor performance in Euclidean geometry has been a matter of concern to all stakeholders in the education system. The study's findings show that one of the contributing factors could be the ineffective teaching strategies teachers use. It is a fact that every classroom has learners from different backgrounds who have diverse learning styles. The subsequent sections highlight the implications for practice when teaching Euclidean geometry using the three principles of UDL.

Multiple Means of Representation (MMR) is one of the UDL principles that make it simple for learners to recognise and analyse information. MMR promotes using different representations to cater to diverse learning styles in classrooms. When teaching Euclidean geometry in line with this principle, one would have to plan a lesson such that it provides options for comprehension, language, mathematical expressions, and symbols and perceptions.

To provide options for comprehension, teachers must first check learners' prior knowledge, which will inform their choice of representations of the content. The teacher would have to design material that reinforces the visualisation and manipulation of concepts. Additionally, teachers should encourage learners to highlight critical features of the geometry concepts that they are learning.

To provide options for language, mathematical expressions and symbols, teachers ought to plan their lessons such that they clarify geometry vocabulary and symbols. The lesson should allow learners to switch between verbal, diagrammatical, symbolic and concrete representations of the geometric concepts. For instance, in parallel lines, learners should be able to demonstrate what is meant by parallel lines using various representation formats. They can verbally describe parallel lines as lines that are always the same distance apart and never intersect; represent them symbolically as $//$ (e.g. $PQ//RS$) represent the lines diagrammatically as shown in Figure 5.1 and represent them concretely using geo-board as shown in Figure 5.2

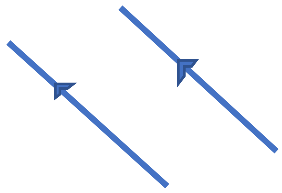


Figure 5.1: Diagrammatical representation of parallel lines

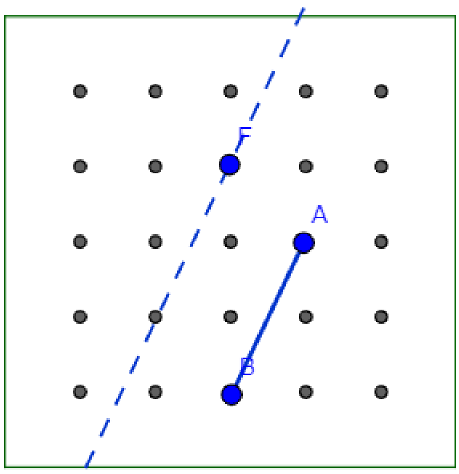


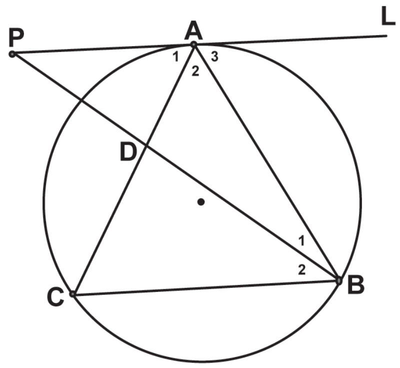
Figure 5.2: Parallel lines using a geo-board

Being well versed with the different representations will make it easy for learners to recognise and analyse the given information (This would help them to analyse geometric statements correctly). For example. Consider the example below:

Given: $AB = AC. AP//BC,$ and $\hat{A}_2 = \hat{B}_2$

Prove that:

1. PAL is a tangent to circle *ABC*;



2. *AB* is a tangent to circle *ADP*.

To answer this question, learners should recognise and highlight critical features such as parallel lines, alternate angles, and co-interior or corresponding angles. Though the parallel lines are not shown in the sketch, the symbolic expression $AP//BC$ tells how the two lines are related, and the learner can recognise this and thus show them with arrowed lines.

To provide perception options, teachers should plan their lessons so that the concept becomes perceptible to all learners. It should offer alternatives to auditory and visual learners. For example, they can use tangible objects such as geo-board to teach 2-dimensional geometry, the pre-requisite knowledge for Euclidean geometry. They can also use technology such as Geo-Gebra as it provides equitable teaching, allowing learners to access content easily.

Multiple Means of Action and Expression (MMAE): This is the principle of UDL that provides options for executive functions, expressions and communication and physical action.

When MMAE is used in teaching Euclidean geometry to provide options for executive functions, teachers ought to plan their lessons to provide prompts and scaffolds to achieve the set objective regarding the Euclidean geometry. In addition, teachers should create an opportunity and platform for the learners to use various ways to demonstrate what they have learned. For instance, one learner may prefer to demonstrate their understanding of theorems by verbally stating them, contrary to the one who would opt for a diagrammatic representation of the theorems.

To provide options for expressions and communication, the planning of the lesson should allow the use multiple media for communication and modelling a geometric problem; for instance, the ability to draw a picture relating to a problem. Moreover, teachers should incorporate opportunities for learners to give feedback on the lesson, which would subsequently help them identify learners' misconceptions.

To provide options for physical action, teachers should vary the response methods and optimise access to tools and assistive technologies. Teachers need to allow learners to solve problems and make mistakes, such as practising freely in handling geometric shapes and making visually appealing notes on new concepts. Teachers need to teach learners geometry in an attractive way by using technology to create a conducive environment in the class and make geometry enjoyable for everyone.

Multiple Means of Engagement (MME): This is the principle that provides options for self-regulation, sustaining effort and persistence and for recruiting interest.

Using MME in teaching Euclidean geometry motivates learners to participate in the teaching and learning process. Motivation, be it intrinsic or extrinsic, is the catalyst of learning.

To provide options for self-regulation, teachers ought to plan their lessons to optimise motivation, improve abilities and boost confidence. Learners ought to be motivated to a point where they can set realistically achievable personal goals. For instance, teachers may create geometry activities that will give learners hands-on experience and engage them in learning.

To provide options for sustaining effort and persistence, teachers should inform learners about the lesson's goal for them to become optimistic about achieving the set goal. When teaching Euclidean geometry, teachers should share the aims of studying the topic, maybe as they appear in the CAPS document. For example:

> *"Solve problems and prove riders using the properties of parallel lines, triangles, and quadrilaterals" (*DBE, 2012, p. 28)

To provide options for recruiting interest, teachers ought to plan their lessons to include examples showing relevance of Euclidean geometry in real-life situations. Learners become more connected to learning understandings that speak to their lives outside the classroom. Therefore, teachers must ensure that Euclidean geometry lessons provide socially relevant tasks appropriate for diverse ability levels.

5.9 RECOMMENDATIONS

5.9.1 Recommendations for Practice

The following are offered as recommendations for practice:

To address workshops that minimally empower teachers on effective teaching strategies, the Department should redesign and restructure workshop programmes to focus on content knowledge and ensure that pedagogy becomes the main focal point for meaningful learning.

The Department should collaborate with teachers' training institutions to design a curriculum that would enhance teachers' knowledge of content that they are expected to teach, knowledge of learners, and ability to create a conducive learning environment for all learners as guided by the UDL teaching framework.

Raising awareness about diversity has become a matter of concern in many spheres of life, including education. However, little support is given to teachers on how to recognise and appreciate diverse learning styles in their classrooms. The recommendation from the current study could be that teacher training institutions consider the use of the UDL principles as part of teacher training. The in-service teacher training should incorporate the empowerment of teachers with the content knowledge and skills to implement the UDL principles to cater to diverse learners through PLCs, micro-teaching, and lesson study sessions in schools.

The institutions of higher education should consider including the concept of universal design for learning in their modules to fully prepare the pre-service teachers to cater for diverse learners' needs.

As solutions to the high teacher-learner ratio, the subject advisor and CES should encouraged teachers to incorporate the flipped classroom elements and practical-based learning models in their teaching. These models would be responsive to the lack of resources and high teacher-learner ratio.

Teachers need to be exposed to the idea of attending the Association for Mathematics Education of South Africa (AMESA) conference where they can learn more about concepts such as UDL. This platform also provides teachers with opportunities to share their best practices through presentation sessions such as *"how I teach".*

Therefore, teachers exposed to the UDL concept can share some insights into how they teach using UDL principles.

5.9.2 Recommendations for Future Research

The study was conducted in one school in a rural area in Free State, where the learners had almost the same cultural and socio-economic background. Nonetheless, it has constituted new knowledge and awareness of the unrelenting challenge of teaching Euclidean geometry in South African schools. I suggest the same research problem be addressed in a different setting: e.g. learner population from diverse cultures and locations; different sample sizes; the use of different analytic tools etc. Additionally, I propose further research to address the unanswered question of how teachers prepare inclusive and flexible lessons for all learners using MMR, MME, and MMAE.

5.10 A FINAL WORD

This chapter highlighted the objectives of the study as stipulated in Chapter 1. It also covers a summary of the study. It discusses the literature and empirical data findings as they appear in chapters 2 and 4, respectively. The chapter also highlights the strategies to address the threats that may impede the effective implementation of UDL. It also highlights the limitations of the study, implications for practice, and recommendations for future research. This research which followed a participatory action research approach, meant that the teachers were the co-researchers, that they identified the problem and were able through class observation, discussion and peer learning to find solutions that would improve their practice. This shows the resilience coupled with the professionalism of the South African teacher, who has the best interests of the students at heart.

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APPENDICES

Appendix A1: Ethical clearance

UNIVERSITY OF THE FREE STATE
UNIVERSITEIT VAN DIE VRYSTAAT
YUNIVESITHI YA FREISTATA
UFS UV

GENERAL/HUMAN RESEARCH ETHICS COMMITTEE (GHREC)

05-Jul-2021

Dear Mrs Matheko Thamae

Application Approved

Research Project Title:
The teaching of Euclidean geometry: A Universal Design for Learning Approach.

Ethical Clearance number:
UFS-HSD2020/1868

We are pleased to inform you that your application for ethical clearance has been approved. Your ethical clearance is valid for twelve (12) months from the date of issue. We request that any changes that may take place during the course of your study/research project be submitted to the ethics office to ensure ethical transparency. furthermore, you are requested to submit the final report of your study/research project to the ethics office. Should you require more time to complete this research, please apply for an extension. Thank you for submitting your proposal for ethical clearance; we wish you the best of luck and success with your research.

Yours sincerely

Dr Adri Du Plessis
Chairperson: General/Human Research Ethics Committee

205 Nelson Mandela Drive
Park West
Bloemfontein 9301
South Africa

P.O. Box 339
Bloemfontein 9300
Tel: +27 (0)51 401 9337
duplessisA@ufs.ac.za
www.ufs.ac.za

Appendix A2: Approval to conduct research

Enquiries: MZ Thango
Ref: Notification of research: MT Thamae
Tel. 082 537 2654
Email: MZ.Thango@fseducation.gov.za

education
Department of Education
FREE STATE PROVINCE

District Director
Motheo District

Dear Mr. Moloi

NOTIFICATION TO CONDUCT RESEARCH PROJECT IN YOUR DISTRICT BY MT THAMAE

The above mentioned candidate was granted permission to conduct research in your district as follows:

Topic: The teaching of Euclidean geometry: A Universal design for learning approach.

1. **List of schools involved:** Qibing Senior Secondary School.
2. **Target Population:** Five Mathematics Grade 11 Learners, Four Mathematics Teachers and One Mathematics Subject Advisor at the selected Secondary school.
3. **Period of research**: From the first week of February 2021 until 30 September 2021. Please note the department does not allow any research to be conducted during the fourth term (quarter) of the academic year nor during normal school hours. The researcher is expected to request permission from the school principals to conduct research at schools.
4. **Research benefits:** The research may inspire subject advisors and teachers of mathematics to use productive methods of teaching because through the use of universal design for learning, they would meet diverse needs of the learners. Learners would have conceptual understanding of mathematics and achieve good results. They would qualify to pursue their studies in the fields of engineering, architecture and constructions to mention a few. This would impact positively in the economic growth of province of the Free State.
5. Strategic Planning, Policy and Research Directorate will make the necessary arrangements for the researchers to present the findings and recommendations to the relevant officials in the district.

Yours sincerely

04/12/2020

Mr. M.M. Sithole
DDG: Corporate Services

DATE:

RESEARCH NOTIFICATION. M.T. THAMAE. 01 DECEMBER 2020. MOTHEO DISTRICT

Strategic Planning, Research & Policy Directorate Private Bag X20565, Bloemfontein, 9300 - Thuto House, Room 101, 1st Floor, St Andrew Street, Bloemfontein

www.fsdoe.fs.gov.za

Appendix A3: Request to conduct research in school

8th June,2021

Dear Sir/Madam

Re: Application to conduct research in one Senior Secondary school

I, Matheko Thabo Thamae, hereby request permission to conduct research study in one Senior Secondary school in the Motheo Education District. I am a Masters student, with student number 2016431559, at the University of the Free State. The research will be in a ‘Participatory Action Research’ form in this school with grade 11 learners and will last for 1 month. This will take place once every week. My focus will be on how Universal Design for Learning (UDL) approach can be used in the teaching of Euclidean geometry.

Yours sincerely,

M. T. Thamae (Mrs)

Researcher: Principal:

…

Appendix A4: Information sheet for assent form (parent/child)

Dear Participant,

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH".** I request your assistance in conducting this study to share your knowledge and expertise in teaching mathematics.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry. UDL principles create barrier-free learning environment for all learners regardless of their background.

Participation is not compulsory and if you decide to withdraw from the project, that will not be held against you. Confidentiality, anonymity, and legality issues about this project will be discussed with you, teacher and principal, as it is imperative that you fully understand the nature and purpose of this study.

This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like your child to participate in this research, please give your consent by signing below:

Name: --

Surname: --

Signature: ---

Date: ---

Contact details: --

Email address: ---

Appendix A5: Information sheet and consent form (teacher)

Dear participant,

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH"**. I request your assistance in conducting this study to share your knowledge and expertise in teaching mathematics.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry. UDL principles create barrier-free learning environment for all learners regardless of their background.

Participation is not compulsory and if you decide to withdraw from the project, that will not be held against you. Confidentiality, anonymity and legality issues about this project will be discussed with you, as it is imperative that you fully understand the nature and purpose of this study. This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like to participate in this research, please give your consent by signing below:

Name: ---

Surname: --

Signature: ---

Date: --

Contact details: --

Email address: --

Appendix A6: Information sheet and consent form (HoD)

Dear participant,

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH".** I request your assistance in conducting this study to share your knowledge and expertise in teaching mathematics.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry. UDL principles create barrier-free learning environment for all learners regardless of their background.

Participation is not compulsory and if you decide to withdraw from the project, that will not be held against you. Confidentiality, anonymity, and legality issues about this project will be discussed with you, as it is imperative that you fully understand the nature and purpose of this study.

This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like to participate in this research, please give your consent by signing below:

Name: ---

Surname: ---

Signature: ---

Date: --

Contact details: --

Email address: ---

Appendix A7: Information sheet and consent form (subject advisor)

Dear participant,

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH".** I request your assistance in conducting this study to share your knowledge and expertise in teaching mathematics.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry. UDL principles create barrier-free learning environment for all learners regardless of their background.

Participation is not compulsory and if you decide to withdraw from the project, that will not be held against you. Confidentiality, anonymity, and legality issues about this project will be discussed with you, as it is imperative that you fully understand the nature and purpose of this study. This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like to participate in this research, please give your consent by signing below:

Name: ---

Surname: ---

Signature: --

Date: ---

Contact details: --

Email address: ---

Appendix A8: Information sheet and consent for (Mathematics coordinator)

Dear Participant

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH".** I request your assistance in conducting this study to share your knowledge and expertise in teaching mathematics.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry. UDL principles create barrier-free learning environment for all learners regardless of their background.

Participation is not compulsory and if you decide to withdraw from the project, that will not be held against you. Confidentiality, anonymity and legality issues about this project will be discussed with you, as it is imperative that you fully understand the nature and purpose of this study. This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like to participate in this research, please give your consent by signing below:

Name: ---

Surname: ---

Signature: ---

Date: --

Contact details: --

Email address: ---

Appendix A9: Information sheet and consent form (Curriculum Education Specialist)

Dear Participant

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH".** I request your assistance in conducting this study to share your knowledge and expertise in teaching mathematics.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry. UDL principles create barrier-free learning environment for all learners regardless of their background.

Participation is not compulsory and if you decide to withdraw from the project, that will not be held against you. Confidentiality, anonymity and legality issues about this project will be discussed with you, as it is imperative that you fully understand the nature and purpose of this study. This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like to participate in this research, please give your consent by signing below:

Name: ---

Surname: ---

Signature: ---

Date: --

Contact details: ---

Email address: ---

Appendix A10: Information sheet and consent form (UDL coach)

Dear Participant

I am currently doing research with the University of the Free State on the study titled **"THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH".** I request your assistance in conducting this study to share your knowledge by giving sessions on UDL to the participants.

The study aims at exploring the implementation of UDL for effective teaching of Euclidean geometry.

Confidentiality, anonymity and legality issues about this project will be discussed with you, as it is imperative that you fully understand the nature and purpose of this study.

This study will comply with rules and regulations of conducting research.

For further clarification, you may contact me on 051 505 1252 or at the following e-mail address: Thamaemt@ufs.ac.za

Thank you.

Matheko.

If you would like to participate in this research, please give your consent by signing below:

Name: ---

Surname: ---

Signature: --

Date: ---

Contact details: ---

Email address: --

Appendix B: The training programme

Item	Responsible person	Duration
Welcome and introduction of the participants	Researcher	10 minutes
Purpose of the meeting	Researcher	20 minutes
Title of the study	Researcher	10 minutes
Discussion of the title	Everyone	30 minutes
Break	Break	Break
UDL training	UDL coach	2 hrs

Appendix C: Ground rules

The participants were expected to:

- Show up on time and come prepared for meetings;
- Stay mentally and physically prepared;
- Treat everyone with respect and avoid belittling one another;
- Make sure that everyone gets a chance to contribute;
- Critique ideas, not people.

Appendix D1: Observation tool

Areas to observe	(What will be observed) Observation will be carried out to check whether the teachers:	• Comments
• **Multiple Means of Representation (MMR)** • **Provide options for:**		
comprehension	+Activate or supply background knowledge +Highlight patterns, critical features, big ideas, and relationships + Guide information processing, visualization, and manipulation + Maximize transfer and generalization	•
Mathematical language, mathematical expressions and mathematical symbol	+ Clarify vocabulary and symbols + Clarify syntax and structure + Support decoding of text, mathematical notation, and symbols + Promote understanding across languages + Illustrate through multiple media	•
perception	+ Offer ways of customizing the display of information + Offer alternatives for auditory information	•

	+ Offer alternatives for visual information	
• **Multiple Means of Action and Expression (MMAE)** • **Provide options for:**		
executive functions	+ Guide appropriate goal-setting + Support planning and strategy development + Enhance capacity for monitoring progress	•
expression and communication	+ Use multiple media for communication + Use multiple tools for construction and composition + Build fluencies with graduated levels of support for practice and performance	•
physical action	+ Vary the methods for response and navigation + Optimize access to tools and assistive technologies	•

Appendix D2: Transcripts

To be honest, geometry is my weak point, and I think I am not the only one who struggles with it. I find it challenging for me to teach it. I rely mainly on the textbook. I often focus only on the examples made in the textbook to teach it".

*"…Lenna ke etsa jwalo hore ke intshe ditabeng. Ke tla etsang kannete? E thata geometry!" [**ET**: I also do that to stay out of trouble. What can I do really? Geometry is complicated!]*

"...Many colleagues, including myself, struggle with Euclidean geometry, and you find most of the time teaching what is inside the textbook and redraw the same shapes provided as examples in the textbook…".

"*…the fact that most of us do not master Euclidean geometry makes it difficult for us to try different strategies and even be creative in teaching geometric concepts*".

I attended a workshop, and one of the topics of discussion was the geometry of straight lines. The facilitator gave us a task to explicitly explain what we meant by the expression "angles on a straight line add up to $180°$. *We were all puzzled because we thought the statement was straightforward and could not be clarified further. I thought I understood very well what the statement meant. Then the facilitator drew two angles on a straight line and asked if they added up* $tttt$ $180°$ (see Figure 4.1).

"Is it not true that angles on a straight line add up to 180°? Nna ha ke utlwisisi kannete. (Honestly, I do not understand). I have been using this statement many times when I am teaching geometry. Vertex yona ke eng? (ET: what is a vertex?). I never emphasised that angles on a straight line add up to $180°$ *if they have a common vertex. Honestly, I did not mean what is illustrated in the diagram above.*

While discussing the theorem of a circle that says, "angles subtended by the same chord or arc are equal." My learners could not explain the meaning of 'angles subtended by the same arc or chord. I never paid attention to explaining the statement. I only demonstrated to them through sketches. Now it is evident that I cannot describe many Mathematical terms such as a circle, proportionality, and subtends, to mention a few. Yes, I can demonstrate what a circle is, for example, but I never bothered to describe it verbally. I can show what is meant by angles subtended by the same arc or chord but cannot explain the word "subtend." I think we are ignorant about the mathematical vocabulary and register knowledge.

Ms. Tau, you are not alone. During my school visits, I joined a grade 8 teacher in class. He was going to present a lesson on a circle and its parts. He started by drawing a circle and showing the radius, diameter, sector, and major and minor segments. He was pointing at and giving the names of parts of the circle. The Teacher admitted that he did not know how to define a circle.

Teaching Euclidean geometry is a nightmare because I am not qualified to teach Grades 10 to 12. I have Senior Primary Teacher Diploma (SPTD). There was a shortage of Mathematics teachers in my school. The lady who was teaching Grades 10 to 12 was promoted to the position of curriculum advisor. I think the Department of Education does not pay enough attention to the scarcity of human resources. I struggle to teach Mathematics in Grades 10 to 12, and it is even more challenging to teach Euclidean geometry because learners have a negative attitude toward geometry from the lower grades

> *… due to family matters, I had to relocate to another school in my village. The school did not have a computer laboratory. I was used to using technology in teaching mathematics. It was a challenge for me to meet the needs of diverse learners to understand the properties of geometric shapes and how to prove theorems…. I think the DoE does not distribute manipulatives and electronic devices equally because some schools do not have even one computer.*
>
> *In my school, we have a shortage of Mathematics textbooks. Sometimes learners must share textbooks, for instance, one textbook for three learners. I consider textbooks as essential tools for teaching and learning. It becomes a challenge for me to explain Euclidean geometry, which is mainly about lines, shapes, axioms, and theorems, without referring the learners to the concepts in the textbook. It is only fair to engage learners fully, but it becomes difficult without having enough textbooks. Sometimes I must sketch diagrams from the textbook. It is time-consuming, and unfortunately, I am poor at making drawings.*
>
> *I think my situation is the worst. As a Funza Lushaka bursary holder, I specialised in Mathematics though it was not my favourite subject in Matric. In fact, geometry was a big challenge for me. I am now duty-bound to teach this concept that was a monster to me. Ho boima (ET: it is not easy). I wish I could access resources such as IBP videos, Heymath, and Master Maths to learn from specialists' presentations*
>
> *Teaching Euclidean geometry has been a challenge because most learners in Grade 10 have negative thoughts and feelings toward geometry. They show a sense of isolation, a lack of interest and purpose in learning geometry, and no emotional connection to the topic. I tend to think that the negative attitude in Grade 10 is due to disengagement in the previous grades.*
>
> *Mam Tau, this is precisely my observations in teaching geometry. Sometimes I tend to think that we are to blame because we seldom give sufficient information about Euclidean geometry, what it entails, its applicability in real-life situations, why it is named after Euclid, and how it is different from Analytical geometry. I don't know, but I think learners would be motivated and interested in learning Euclidean geometry if they could be provided with information about the topic.*

What makes my life difficult when teaching Euclidean geometry is my learners' level of commitment. If I give them homework or classwork, they don't complete the work or claim they forgot books at home. I created a WhatsApp group as a platform for them to share knowledge and ask questions when necessary. I hoped the platform would enhance their curiosity and interest in learning geometry. I thought those who feel isolated would use the media to connect to others and the topic. To my disappointment, their level of engagement remained low.

Colleagues, I agree with you 100%. Learner disengagement is the primary source of their negative attitude towards Euclidean geometry. But now, a million-dollar question is how do we rectify that?

My Grade 11 learners struggle to apply theorems and giving appropriate reasons when solving riders. I taught them how to scrutinise the given statements in conjunction with diagrams before attempting to answer questions. Unfortunately, they still could not draw inferences from the specified data.

I have the same problem with my Grade 11 learners. Kannete ha ke sa tseba na ke etse joang (ET: honestly, I don't know what to do). I have taught them the theorems, their converses, and axioms. I have made diagrammatical representations of the theorems. I was convinced that they understood the theorems because they could state them correctly in a written form and verbally.

My Grade 11 learners are doing well in writing the theorems and giving them verbally but struggle to construct proofs because they cannot form mental pictures from given information.

"…to sketch shapes on the board is the right thing to do because it caters to visual learners; however, we should supplement that with manipulatives such as a geo- board (see Figure 4.2) to accommodate tactile learners. Using the geo-board enhances a deep understanding of the properties of the geometry of straight lines and 2-Dimensional shapes".

I think the emphasis on the properties of parallel lines and shapes should be a focal point in teaching Euclidean geometry. Learners would not struggle with shapes in different orientations if they had a conceptual understanding of the properties of lines and shapes. I used a geoboard with rubber bands of different colours to demonstrate various orientations of parallel lines with more than one transversal. In addition to that, on the classroom walls, I put a shape collage showing shapes in different orientations.

The teaching of Euclidean geometry used to frustrate me, and I have realised it was because I did not know my learners and how they comprehend new knowledge. I never allowed them the opportunity to explore concepts in the lesson.

*I think we must avoid using shortened mathematical statements at the introductory stage of any topic. For instance, "**adjacent angles** on a straight line add up to 180°" instead of "angles on a straight line add up to 180°."*

I agree with you, Ms. Mofokeng. We must use complete statements at the introductory stage of teaching geometry to enhance conceptual understanding. The necessary condition for angles on a straight line to add up to 180◦ is that they must have a common vertex (see Figure 4.4*). Shortened statements may be used when the concept is well understood and the learners are at the advanced stage.*

Colleagues, talking of shortened statements at the introductory stage of a topic reminds me of the concept of the Sum of angles of a triangle. Specifying whether the Sum of 180° is for interior or exterior angles is crucial. This is what I mean (referring to Figure 4.5):

In my previous school, where there were computers, laptops, and data projectors, I used GeoGebra to illustrate the properties of various polygons. My learners used it to investigate and form conjectures about the properties of different polygons, draw triangles, and measure angles. I think GeoGebra and other manipulatives should be used as a scaffold for teaching Mathematics from the primary to the university level. Again, they should be fairly distributed among schools. And the neighbouring schools should work together and borrow each other's e teaching resources. Those that are more resourceful should assist the ones with no resources.

I agree with you, Mr. Mokwena. Such electronic devices would make teaching easy because learners could also watch Mathematics videos available to their schools. Another factor that needs to be considered is that teachers must be trained to equip them with the knowledge to use GeoGebra and other electronic devices in teaching and learning Euclidean geometry.

As a remedy to the shortage of qualified Mathematics teachers in our cluster, the DBE, School Governing Bodies (SGBs), and School Management Teams (SMTs) decided to hire veteran expert teachers to teach Euclidean geometry to learners from combined schools in our cluster during winter holidays. I regarded the suitability of human resources in the education system as a crucial matter. I also learned a lot from their presentations.

I observed the same behaviour in my Grade 11 class, willingness to learn. However, what I did differently was that at the introductory stage of the lesson, I listed key concepts in understanding the topic and communicated *the learning outcomes of the topic. Additionally, I referred to the benefits of mastering Euclidean geometry and its relevance to most careers. Subsequently, in response, they exerted more effort to acquire more knowledge.*

I have realised that my learners' low commitment resulted from not knowing them and their learning styles. Improving their engagement by providing various options for perception decreased the rate of dropouts and positively impacted their behaviour. My lesson planning was responsive to their varying methods of answering questions. Some of the learners easily grasped information in the statement if it was illustrated with a diagram

Thank you, colleagues. In summary, you are saying that to make Euclidean geometry teaching easy, we should engage learners to investigate theorems and find information on their own. We need to give background and relevance of the topic in a real-life context. Most importantly, we need to know our learners' diverse learning styles.

During my school visits, I was overwhelmed by the Grade 11 teacher while presenting circle theorems. He would read a sentence in the given statement in conjunction with the diagram and dissect the information using different coloured pencils to demonstrate and explain it piece by piece. For example, he coloured parallel lines to be distinctively visible. Learners easily recognise pairs of corresponding, co- interior, and or alternate angles.

... In addition to teaching them how to scrutinise given information in conjunction with the diagram, I engaged them in drawing inferences from given data by visualising abstract concepts. For instance, if it is stated that two radii of a circle meet both ends of a chord on the circumference, then the learners should infer the given data and visualise equal angles opposite equal sides, the radii.

ne ke tlo cho joalo Ms. Tau (ET: I was about to say exactly that). "I have noticed that questions are not only based on the given information and the diagram in Euclidean geometry. One is expected to make inferences and visualise abstract aspects. Hence as a teacher, I should focus more on enhancing visualisation skills by asking probing questions so that learners can find out more than what is given in the statement and the diagram.

I think as a teacher, I have to use techniques that promote self- discovery skills among the learners. That would assist them in conceptualising and simplifying abstract ideas in Euclidean geometry. I think including technology, such as Geogebra, in teaching mathematics in South Africa could yield a positive attitude towards teaching and learning Euclidean geometry. Learners' ability to form mental pictures would be strengthened.

I have gained a lot from this workshop and am happy that I clarified the concept of the triangle proportionality theorem and its converse. Feela ho thata ho e ruta (ET: but it is challenging to teach it).

That's precisely my challenge. I have understood the concept, but I struggle to make my learners understand it. This section of geometry is always poorly performed by my learners. Workshop facilitators should also teach us how to teach this theorem.

Colleagues, please remember to teach from known to unknown, simple to complex. Ms. Ndweni, it would not be challenging to introduce the concept of triangle proportionality theorem and its converse if you could review your learners' knowledge of similarity and congruency that they learned in Grade 10.

I support Mr. Mokwena's suggestion. Facilitators should train us on how to teach this theorem effectively. In addition, I think it is high time we implement what we have learned from workshops. We can conduct micro-teaching whereby one teacher will plan and teach a lesson while others observe and give constructive feedback on the presentation. We can do this in our schools.

Teaching Euclidean geometry is a nightmare because I am not qualified to teach Grade 12. I have Senior Primary Teacher Diploma (SPTD). Being not qualified makes me lose interest in teaching the topic."

I think my situation is worse than yours, Ms. Tau. I have a Bachelor of Education degree, but unfortunately, my high school teacher had a challenge with Euclidean geometry. He used to copy theorems and diagrams from the textbook to the chalkboard. I did not learn anything, and this has affected me badly.

"Colleagues, you have raised an important point which I agree with you a hundred percent! However, as the DoE, I think we should restructure our planning for INSET workshops. Our programs should focus more on imparting skills that address diverse learning needs. It becomes a futile exercise to focus on addressing the content gap but ignore varied ways of delivering the content".

To add to what you have just said meneer, I think the DoE, working closely with higher education institutions, should urgently develop a policy on effective teaching strategy that caters to diverse learning styles.

Besides restructuring our planning on what should be covered in the workshop, we should also strive for at least three seminars per year. I agree with Mr. Mosola that more time is needed to train one for effective teaching of Euclidean geometry. Hence one workshop per year is not enough.

It is crucial to prioritise teacher development on pedagogy; otherwise, learners' emotions will be harshly affected if their teacher is discouraged and loses interest in teaching the topic. Every learner is critical, and we must try to meet their learning needs.

I agree that we should deliver content to become accessible to learners from different backgrounds and with diverse learning needs. My main challenge is planning for different learning styles requires more time. Euclidean geometry is jam- packed with content. How do I ensure that my lesson plan caters to audio, visual, and tactile learners?

Additionally, the teacher-learner ratio is far more than what is suggested by the education policy, which is 1: 40. This makes it difficult for us to teach this topic.

The use of technology and manipulatives could make it easy to teach the topic. Although many people think that manipulatives waste time, I beg to differ. I think they save time and make learning practicable and engaging.

In addition to using technology and manipulatives, you may consider adopting the flipped classroom model. Flipped classroom model empowers learners to take control of their learning because they may have access to information videos or audio-recorded material in advance in preparation for teacher-guided discussions. The model is responsive to a high teacher-learner ratio.

Colleagues, you may also incorporate elements of the practical-based learning model where learners can explore, determine, and investigate independently.

The DoE officials have to frequently visit the schools to monitor and evaluate the impact of the workshops. Moreover, higher education institutions should work collaboratively with the DoE to develop a policy on the inclusive teaching strategy and how to train teachers on implementing it.

Appendix E: Proof of editing

To whom it may concern

This letter serves to confirm that editing and proofreading was done for:

MATHEKO THABO THAMAE

Magister Educationis (Mathematics Education)

Faculty of Education

University of the Free State

THE TEACHING OF EUCLIDEAN GEOMETRY: A UNIVERSAL DESIGN FOR LEARNING APPROACH

Cilla Dowse
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Cilla Dowse PhD in Assessment and Quality Assurance in Education and Training: University of Pretoria 2014 Basic Editing and Proofreading: McGillivray Linnegar Associates 2008 Programme on Editing Principles and Practices: University of Pretoria 2009 Editing and Proofreading for Academic Purposes: McGillivray Linnegar Associates 2021 Professional Editors' Guild Associate Member, DOW003	Rosedale Farm P.O. Box 48 Van Reenen Free State cilla.dowse@gmail.com Cell: 084 900 7837

Appendix F: Similarity Index

Matheko Thabo Thamae

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