

# **ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH NON-LOCAL OPERATORS.**

**Mashudu C. Mathobo**

**Student Number: 2016113298**

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Promoter: Prof. Abdon Atangana

BLOEMFONTEIN

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## **DECLARATION**

I, Mashudu Clifford Mathobo, declare that the thesis hereby submitted by me for Doctor of Philosophy degree at University of the Free State (Institute for Groundwater Studies), is my own independent work and has not previously been submitted by me at another university/ faculty.

I further declare that all sources cited or quoted are indicated and acknowledged by means of a list of references.

I furthermore cede copyright of this thesis in favor of University of the Free State.

**In addition, the following articles will be submitted for review at International Accredited Journals:**

- 1. Mathobo, M. C and Atangana, A., 2020. Analysis of general groundwater flow equation with fractal derivative.**
- 2. Mathobo, M. C and Atangana, A., 2020. Analysis of general groundwater flow equation with fractal-fractional differential operator.**
- 3. Mathobo, M. C and Atangana, A., 2020. Analysis of general groundwater flow equation with Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives.**

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## GREEK NOTATIONS

$\alpha$	alpha
$\beta$	beta
$\tau$	tau
$\partial$	Partial Differentiation
$\delta$	delta
$\varphi$	phi
$\Gamma$	Gamma
$\Delta$	Delta
$\theta$	delta
$\phi$	phi
$\Phi$	Phi
$\sigma$	Sigma
$\Sigma$	Sigma
$\mu$	mu
$\mathcal{L}$	Laplace Transform

## ABBREVIATIONS AND NOTATIONS

$h$	Hydraulic Head
$r$	Radial Distance
$K$	Hydraulic Conductivity
$S$	Storativity
$S_s$	Specific Storage
$S_y$	Specific Yield
$T$	Transmissivity
$t$	Time
$q$	Darcy Flux
$Q$	Discharge
$V$	Volume
$c$	Arbitrary Constant
$r_b$	Ratio of the Borehole
$\Delta r$	Scaling Constant
$f$	Function

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## CHAPTER ONE: INTRODUCTION

### 1.1 BACKGROUND OF THE STUDY

Charles Vernon Theis (1900-1987) was a hydrologist who developed groundwater flow equations for transient flow of water. Theis mathematical models was based on physical resemblance between heat flow in solids and groundwater flow in porous media. (Theis, 1935) equation is one of the fundamental solutions for the deterministic mathematical models of groundwater flow (Xiang, 2014). However, Theis equation was derived based on assumptions that are not entirely true in reality. Therefore, this result in limitations in the equation. Based on the assumptions (aquifer is homogeneous, has infinite aerial extent, isotropic and of uniform thickness, etc.) Theis simplified his equation and decided to ignore high order terms in the derivation of the groundwater flow equation. In real world scenarios, such is not the case as it has been proven timeously that aquifers tend to be heterogeneous, with finite aerial extent due to impermeable boundaries and are pumped at distinct discharge rates. It is in this regard that many authors in the preceding years have tried coming up with the exact groundwater flow equations.

Recently, (Atangana and Mathobo, 2017) derived the exact groundwater flow equation in a confined aquifer. In their research, they argued that the Theis groundwater flow model is an approximation of the formulation of the model as he removed some components of the equation to have a simple model. In their equation, Atangana and Mathobo included all high order terms that was removed by Theis and also to take into account the assumptions that was used during the derivation of the groundwater flow by Theis. Thereafter, it was proved that the new groundwater flow equation in confined aquifers has a unique solution. The derivation of the exact solution was derived and shown using the Boltzmann transform. Numerical simulations of the suggested equation for different values of scale factor were presented and compared with Theis model. The numerical simulations were obtained using some theoretical values of aquifers parameters. (Atangana and Mathobo, 2017) simulations showed the importance of scaling factor which was removed from the Theis groundwater flow equation.

The simulations also show that the change in drawdown depend on the scaling factor. This physical parameter helps to observe the gradual change in drawdown without overestimating or underestimating the change. The smaller the size of the cell, the better the change in drawdown but if the cell is too small, the drawdown can be underestimated. If the cell is too large, the drawdown can be overestimated. However, the research has limitations. The exact solution is limited in terms of derivatives. It does not take into account heterogeneity, random walk and memory effect etc. Therefore, it can only explain Markovian processes but not Non-Markovian.

This dissertation aims at using differentiation operators which are non-local, meaning that they are able to introduce the mathematical formulation of memory and random walk. A differential operator (which is generally discontinuous, unbounded and non-linear on its domain) is an operator defined by some differential expression, and acting on a space (usually vector valued) functions (or sections of a differentiable vector bundle) on differentiable manifolds or else on a space dual to a space of this type (Encyclopedia of mathematics, 2011). A differential operator is an operator defined as a function of the differential operator. A differential operator is used to consider differentiation as an abstract operation, accepting a function and returning another. The most commonly used differential operator is the action of taking the derivative itself (Wikipedia, 2012). Common notation for this operator include:  $D \equiv \frac{d}{dx}$  and if generalize  $D^N \equiv \frac{d^N}{dx^N}$ , where  $D$  is an operator and must therefore be followed by some expression on which it operates. This operator includes the Atangana-Baleanu fractional derivative, Caputo fractional derivative and Caputo-Fabrizio fractional derivative. A fractal operator is a local operator and is able to describe self-similarity in a geological formation (Ramotsho and Atangana, 2017).

Joseph Liouville was the first researcher to carry out a rigorous investigation of fractional derivatives from 1832-1837. Bernhard Riemann further developed on what had been done by Liouville and this led to the popularly known Riemann-Liouville fractional integration operator. In previous years, the Riemann-Liouville operator had been mostly used when fractional integration is performed. According to (Munkhammar, 2004) Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an n-fold integral:

$$\int_g^x dx_1 \int_g^{x_1} dx_2 \dots \int_g^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_g^x \frac{f(t)}{(x-t)^{1-n}} dt \quad (1.1)$$

and since  $(n-1)! = \Gamma(n)$ , Riemann observed that the RHS of (1) still has meaning even when  $(n)$  takes non-integer values. Therefore, it made sense to define fractional integration as follows:

If  $f(x) \in C([g, h])$  and  $g < x < h$  then,

$$I_g^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_g^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad (1.2)$$

where  $\alpha \in ]-\infty, \infty[$ , is called the Riemann-Liouville fractional of order  $\alpha$ . In the same manner for  $\alpha \in ]0, 1[$  we let:

$$D_g^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_g^x \frac{f(t)}{(x-t)^\alpha} dt \quad (1.3)$$

which is called the Riemann-Liouville fractional derivative of order  $\alpha$ .

(Caputo and Fabrizio, 2015) came up with a new fractional order derivative with singular kernel. The new fractional derivative has a smooth kernel which takes on two different representations for the temporal and spatial variable. According to (Caputo and Fabrizio, 2015) the first definition works on the time variables; thus, it is suitable to use the Laplace transform. The second definition is related to the spatial variables by a non-local fractional derivative, for which it is more convenient to work with the Fourier transform. (Atangana and Alqahtani, 2016) further emphasized that the Caputo-Fabrizio is able to describe substance heterogeneities and configurations with different scales. The derivative can also be applied to investigate macroscopic behaviors of certain materials, connected with non-local communications between atoms, which are recognized to be important on the properties of material. (Atangana and Alqahtani, 2016) presented the definition of Caputo fractional derivative as follows:

The Caputo fractional order old editor of a function  $f$  is given as:

$${}_0^c D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} (f(t)) dt, n-1 < \alpha \leq n \quad (1.4)$$

Let  $f \in H(m, n), n > m, \alpha \in [0, 1]$ .

Then the new Caputo fractional derivative is defined as:

$${}_0^c D_t^\alpha (f(t)) = \frac{M(\alpha)}{1-\alpha} \int_\alpha^t f'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \quad (1.5)$$

where  $M(\alpha)$  is a normalization function such that  $M(0) = M(1) = 1$ . However, if the function does not belong to  $H^1(a, b)$  then, the derivative can be redefined as:

$${}_0^c D_t^\alpha (f(t)) = \frac{\alpha M(\alpha)}{1-\alpha} \int_\alpha^t (f(t) - f(x)) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \quad (1.6)$$

(Atangana and Baleanu, 2016) recently introduced a new fractional derivative. The Atangana-Baleanu derivative was based on the generalized Mittag-leffler function, since the function is more suitable in expressing nature than power function (Alqahtani, 2016). The derivative is much better and satisfies all issues that were previously pointed out on the Caputo-Fabrizio fractional derivative and all other fractional order derivatives.

Atangana and Baleanu (2016) define this derivative as follows: Let  $f \in H'(a, b), b > a, \alpha \in [0, 1]$  then:

$${}^{ABC} D_b^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx \quad (1.7)$$

$B(\alpha)$  has the same properties as in Caputo-Fabrizio case. (Atangana and Baleanu, 2016) further stressed that this definition is important for discussing real-world problems and has great advantage when applying the Laplace transform to solve physical problems within initial condition.

If alpha becomes zero, the original function can be recovered only at the origin and function vanishes. We then get the following definition: Let  $f \in H^1(a, b)$ ,  $b > a$ ,  $\alpha \in [0,1]$ , then the definition of the new fractional derivative is given as:

$${}^{ABR}D_t^\alpha(f(t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_b^t f(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx \quad (1.8)$$

With,

$$B(\alpha) = 1 - \alpha + \frac{\alpha(\alpha)}{\Gamma(\alpha)}$$

It's also important to note that the above equations have a non-local kernel.

## 1.2 PROBLEM STATEMENT

(Mathobo and Atangana, 2017) developed or rather modified the general groundwater flow equation for a confined aquifer. They analysed the exact solution for groundwater flow equation in a confined aquifer. The equation included all high order terms that was removed by Theis and took into account the assumptions that was used during the derivation of groundwater flow by Theis. Following this, a new numerical scheme for singular partial differential equations was also developed. (Mathobo and Atangana, 2017) further presented numerical simulations of the suggested equation using different values of scale factor and compared simulations with Theis model. However, there are limitations which are associated with this research. For instance, the exact solution is limited in terms of derivatives. It does not take into account heterogeneity, random walk, diffusion and memory effect. As a result, it can only explain Markovian processes but not non-Markovian. This study aims at addressing such problems by introducing the concept of differential operators, and therefore are able to introduce mathematical formulation of memory and random walk. This research will also apply Atangana-Baleanu operator which is a derivative with non-local and non-singular kernel. This operator overcomes all limitations imposed by the use of fractional derivatives. Differential operators will help to introduce many structural aspects (i.e. heterogeneity, advection, memory etc.) to our numerical simulations.

### **1.3 AIMS AND OBJECTIVES**

The main aim of this research is to analyse the general groundwater flow equation with Non-Local operators.

***Objectives include:***

1.3.1 Derivation of exact general groundwater flow equation within a confined aquifer.

1.3.2 Analyze the general groundwater flow equation using a fractal operator.

1.3.3 Analyze the general groundwater flow equation using differential operators (Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Atangana-Baleanu fractional derivative, Caputo fractal-fractional and Atangana-Baleanu fractal-fractional derivative).

1.3.4 Develop new numerical simulations for both fractal, fractional and fractal-fractional differential operators.

### **1.4 RESEARCH OUTLINE**

The dissertation has six chapters: chapter one provides a background of the study, the main aim and objectives with which this research aims to achieve, the overall framework of this research and lastly, show the derivation of general groundwater flow equation for a confined aquifer. Chapter two presents analysis of general groundwater flow equation with fractal derivative. Stability analysis using the fractal solution was presented. Chapter three presents analysis of general groundwater flow equation with Riemann Liouville fractional derivative. The operator was defined, its properties shown but analyses was done using the Caputo fractional derivative since the two methods are more or less similar. Stability analysis of Caputo solution was shown. Numerical simulations are created, displayed and discussed in chapter seven. Chapter four presents analysis of general groundwater flow equation with Caputo-Fabrizio fractional derivative. A new method called the Newton Polynomial which was introduced by (Atangana and Araz, 2019) was used in the analysis. Stability analysis was presented and numerical simulations were shown in chapter seven. Chapter five presents analysis of general groundwater flow equation with Atangana-Baleanu fractional derivative.

Similar to previous chapters, a solution was derived and stability analysis presented. Chapter six presents analysis of general groundwater flow equation with fractal-fractional derivative. Analysis was conducted using the Caputo fractal-fractional derivative and Atangana-Baleanu fractal-fractional derivative. Newton Polynomial was used in both instances for discretization. Numerical simulations were developed, discussed and shown in chapter seven. Chapter seven shows the numerical simulations for different fractional orders and also presents the discussion based on the simulations. Conclusions were made, relevant authors were acknowledged by means of a reference list and lastly, a short summary of the research was given in a form of abstract.

## 1.5 RESEARCH FRAMEWORK

The following framework was followed to successfully complete this research:



**Figure 1:** Research framework followed in this study.

## 1.6 DERIVATION OF GENERAL GROUNDWATER FLOW EQUATION FOR CONFINED AQUIFER.

The section aims at showing the exact groundwater flow equation for confined aquifer as derived by (Mathobo and Atangana, 2017). Almost all mathematical models are developed from general equations. In groundwater flow, the physical principles of Darcy's law and Mass balance are applied. Combination of this principles results in the formulation of general groundwater flow equation. These equations are known as partial differential equations (PDE). Every groundwater flow is initially formulated from Darcy's law. This principle was developed in 1856 by an engineer by the name of Henry Darcy.

According to (Zhang, 2016) darcy's law is an equation that defines the ability of a fluid to flow through a porous media such as a rock. It relies on the fact that the amount of flow between two points are directly related to the pressure between the points, the distance between the points, and the interconnectivity of flow pathways in the rock between the points.

In 1935, Theis derived the basic equation of unsteady flow towards the well from the analogy between the flow of groundwater and the conduction of heat. The equation was defined as follows:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial h}{\partial r} = \frac{S}{T} \cdot \frac{\partial h}{\partial t} \quad (1.9)$$

where  $h$  is the head,  $r$  is radial distance from the well,  $s$  is storage coefficient,  $T$  is transmissivity and  $t$  is the time since the beginning of pumping.

However, (Mathobo and Atangana, 2017) argued that Theis equation has limitations and does not give a true representation of what actually happens in reality. Theis groundwater flow equation is based on certain assumptions such as that the aquifer is homogeneous, has infinite aerial extent, discharged at a constant rate and of uniform thickness etc. (Mathobo and Atangana, 2017) further stated that Theis groundwater flow model is an approximation of the real formulation of the model as he removed some components of the equation to have a simple model.

This led to the development of exact groundwater flow equation for confined aquifer and included all high order terms which was initially removed by Theis and also taking into account the assumptions that was used during derivation of groundwater flow by Theis.

Newly developed exact groundwater flow equation for confined aquifer is as follows (Mathobo and Atangana, 2017):

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{\partial h}{r \partial r} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (1.10)$$

The above equation has a new physical parameter. Based on the numerical simulations created and compared with Theis model, it was observed that the new parameter plays an important role. The new physical parameter is called scaling factor. (Mathobo and Atangana, 2017) concluded this is also the parameter which was removed by Theis so as to get a simple model. Based on the numerical simulations and contour plots, Mathobo and Atangana also observed that change in drawdown depends on the physical parameter. The smaller the size of the cell, we better observe the change in drawdown but if cell size is too small, drawdown can be underestimated. Nonetheless, if the cell size is too large, the change in drawdown tends to be overestimated. It is observed that Mathobo and Atangana model differs with Theis model by introduction of a physical parameter called the scaling factor.

Mathobo and Atangana groundwater flow equation also has limitations which are associated with it. The newly developed equation is limited in terms of its derivatives and as such cannot take into account heterogeneity, random walk and memory effect. This research aims at extending the model suggested by Mathobo and Atangana by replacing conventional differential operator with non-local operator. We are going to analyze the new equation using Atangana-Baleanu operator, which is non-local and has non-singular kernel. Other differential operators to be used include the Riemann-Liouville, Caputo-Fabrizio, Caputo and Atangana-Baleanu fractal-fractional differential operators. These operators will help extend our model by introducing structural aspects (heterogeneity, random walk, memory, advection, dispersivity etc.) into our model.

The equation will also be analyzed using a fractal operator. Based on the above, numerical simulations will be developed and compared with the general groundwater flow model.

## CHAPTER TWO: ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH FRACTAL DERIVATIVE.

This section aims at analyzing the general groundwater flow equation with a fractal derivative. We shall also discuss the various properties of fractals. If it happens that the fractal derivative may not be solved analytically, we shall adopt a numerical approximation approach, specifically the Forward, Backward and Crank-Nicolson scheme. Subsequently, we shall prove the stability of the solution using Crank-Nicolson method. Stability analysis will additionally be conducted using Atangana-Batogna numerical scheme which will be compared to that of Crank-Nicolson.

### 2.1 FRACTAL DERIVATIVES

According to (Liu and He, 2015) fractal derivative is a natural extension of Leibniz's derivative for discontinuous fractal media. (Yang, 2012) argued that fractal derivative can be categorized as a special local fractional derivative. This type of derivative can be applied to solve problems associated with a discontinuous media and fractal derivatives can be easily solved. Fractal derivative has also been employed as an alternative modelling approach to the classical Fick's second law, where it was used to derive linear anomalous transport-diffusion equation underlying diffusion process (Chen *et al.*, 2010). Fractal derivative can also model heat transfer and water permeation in multi-scale fabric and wool fibers (Fan *et al.*, 2012a; Fan *et al.*, 2012b; Fan *et al.*, 2013a; Fan *et al.*, 2013b; Chen *et al.*, 2013; He, 2013a).

Various authors have defined fractal derivatives.

**Definition 1.** (Chen *et al.*, 2010; Chen, 2006) defined fractal derivative as follows:

$$\frac{du(z)}{dz^\alpha} = \lim_{s \rightarrow x} \frac{u(z) - u(s)}{z^\alpha - s^\alpha} \quad (2.1)$$

where  $\alpha$  is the order of the fractal derivative. According to (Liu and He, 2015) the above definition lacks physical meaning even though it is simple to understand.

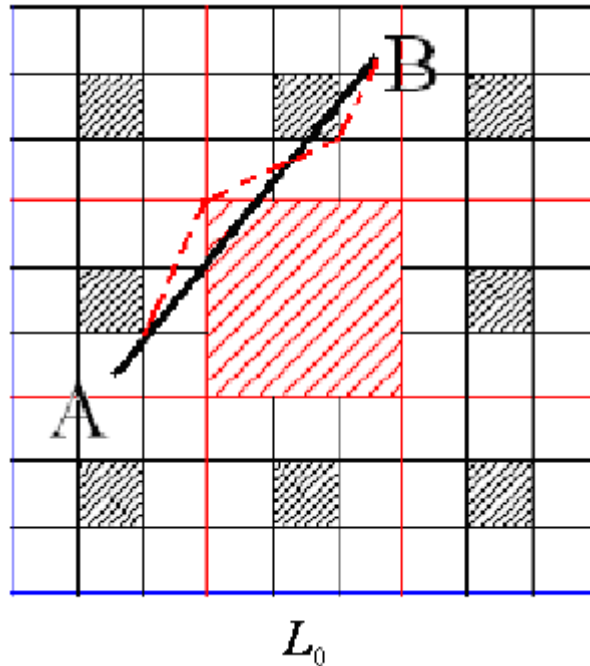
**Definition 2.** Based on figure 2, (Liu and He, 2015) assumed that the smallest measure is  $L_0$  (with any discontinuity less than  $L_0$  being ignored). The distance between point A and point B can then be expressed using fractal geometry. (He, 2011a) indicated that fractal derivative will have the form:

$$\frac{D_u}{Dz^\alpha} = \lim_{\Delta z \rightarrow L_0} \frac{u(A) - u(B)}{d_s} = \lim_{\Delta z \rightarrow L_0} \frac{u(A) - u(B)}{KL_0^\alpha} \quad (2.2)$$

where  $\alpha$  is fractal dimension, the distance ( $d_s$ ) between two points in a discontinuous space can be expressed as:

$$d_s = KL_0^\alpha \quad (2.3)$$

where  $K$  is a function of fractal dimensions,  $K = K(\alpha)$ , and it follows that  $K(1) = 1$ .



**Figure 2:** The distance between point A and B in a discontinuous space or space time (He et al.,2012; He, 2009).

**Definition 3.** (Liu and He, 2015) further stated that fractal derivative can alternatively be defined as:

$$\frac{Du}{Dz^\alpha} = \Gamma(1 + \alpha) = \lim_{\Delta z = z_A \rightarrow L_0} \frac{u(A) - u(B)}{(z_A - z_B)^\alpha} \quad (2.4)$$

where  $\alpha \rightarrow 1$  and  $L_0 \rightarrow 0$ , equation (2.3) becomes ordinary differentiation.

## 2.2 APPLICATIONS OF FRACTAL DERIVATIVES

The section shows how fractal derivatives have been applied by different authors to overcome discontinuous problems which had been a challenge using classical methods.

Fractal derivatives were successfully applied in anomalous diffusion as follows (Chen *et al.*, 2010): on their paper, anomalous diffusion modelling by fractal and fractional derivatives. They came up with a fundamental solution of fractal derivative equation for anomalous diffusion, which characterizes a clear power law. The new anomalous model was compared to corresponding fractional derivative model in terms of computational efficiency, diffusion velocity and heavy tail property.

$$\frac{du(z, t)}{dt^\alpha} = D \frac{\partial}{\partial z^\beta} \left( \frac{\partial u(z, t)}{\partial z^\beta} \right), -\infty < z < +\infty, \quad (2.5)$$

$$u(z, 0) = \delta(z)$$

where  $0 < \alpha < 2$ ,  $0 < \beta < 1$ , and  $\sigma(z)$  is the Dirac Delta Function.

In order to obtain the fundamental solution, we apply the transformation of variables,

$$t' = t^\alpha, z' = z^\beta.$$

Then the equation (2.5) becomes the normal diffusion form equation, the solution of equation (2,5) has the stretched Gaussian form:

$$u(z, t) = \frac{1}{2\sqrt{\pi t^\alpha}} e^{-\frac{z^{2\beta}}{4t^\alpha}} \quad (2.6)$$

The mean squared displacement of above fractal derivative diffusion equation has the asymptote:

$$\langle z^2(t) \rangle \propto t^{\frac{(3\alpha-\alpha\beta)}{2\beta}} \quad (2.7)$$

These allowed them to create both fractal and fractional models, thereby enabling them to compare the two models.

(He, 2011) applied fractal derivative in heat conduction. By considering Fourier's law of heat conduction which is as follows;

$$\frac{\partial q_h}{\partial t} = c \frac{dT}{dn} \quad (2.8)$$

where  $T$  is thermal potential,  $qh$  is heat flow. In discontinuous media, Fourier's law can be simply modified as:

$$\frac{\partial q_h}{\partial t} = -c \frac{DT}{Dn^\alpha} \quad (2.9)$$

where  $\frac{DT}{Dn^\alpha}$  is a fractal derivative defined in equation (2.2).

In another instance, (He, 2011) examined a relaxation equation with fractal derivative;

$$\frac{Du}{Dz^\alpha} + \sigma u = 0, 0 < D < 1 \quad u(0) = 1 \quad (2.10)$$

The above equation can be equivalently re-written in the form:

$$\frac{du}{ds} + \sigma u = 0 \quad (2.11)$$

Its solution is thus obtained as:

$$u = e^{-\sigma s} \quad (2.12)$$

Equation (2.3) can be approximately expressed in the form,

$$ds = KL_0^{\alpha-1} dz \quad (2.13)$$

or

$$s = KL_0^{\alpha-1} z \quad (2.14)$$

Therefore, the solution of equation (2.10) can be obtained:

$$u = e^{-\sigma KL_0^{q-1} z} \quad (2.15)$$

The solution depends on geometrical parameter ( $K$ ), measure scale ( $L_0$ ) and fractal dimension.

Based on the above examples, it can be observed that fractal derivatives may be used with ease for any discontinuous problems.

### **2.3 PROPERTIES OF FRACTALS**

Initially, fractals were only used in mathematics to describe self-similarity in structures. Due to continuous research and new developments in mathematics, fractals are being used in various areas of science as they aid in describing structures in nature that conventional mathematical tools cannot solve. Fractals are easy and a convenient way of solving and creating things which may be difficult or impossible.

(Chen and Huang, 2017) defined fractal as a rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally self-similar and independent of scale. Characteristics of fractals are as follows:

- I. Parts exhibit the same form or structure as the whole, except that they are at a different scale and maybe slightly deformed;
- II. Its form is extremely irregular or fragmented, and remains so, whatever the scale or examination;
- III. Fractals have distinct elements whose scales are very varied and cover a large range;
- IV. Formation by iteration;
- V. Fractional dimension

## 2.4 ANALYSIS OF GENERAL GROUNDWATER FLOW WITH FRACTAL DERIVATIVE.

As discussed in the introduction of these chapter, this particular section aims at analyzing the general groundwater flow equation using a fractal derivative. We are going to derive and propose a new derivative operator that is capable of describing the groundwater flow in fractures and heterogeneity of the system. This could not be accounted for by the general groundwater flow equation. If the differential operator cannot be solved analytically, we are going to use the Forward, Backward and Crank-Nicolson scheme to develop a numerical approximation of the operator and show the stability analysis.

The fractal differentiable operator is given by,

$$\frac{du(z)}{dz^\alpha} = \lim_{s \rightarrow x} \frac{u(z) - u(s)}{z^\alpha - s^\alpha} \quad (2.16)$$

$$\frac{d}{dt}(f) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (2.17)$$

By expressing the fractal differentiable operator with (2.17) we get,

$$\frac{d}{dt^\beta} f(x) = \lim_{x \rightarrow s} \frac{f(x) - f(s)}{x^\alpha - s^\alpha} = \lim_{x \rightarrow s} \frac{f(x) - f(s)}{x - s} \left[ \frac{x - s}{x^\alpha - s^\alpha} \right] \quad (2.18)$$

Therefore,

$$\lim_{x \rightarrow s} \frac{f(x) - f(s)}{x - s} \cdot \left( \frac{1}{\frac{x^\alpha - s^\alpha}{x - s}} \right) \quad (2.19)$$

$$= f'(s) \frac{1}{\alpha s^{\alpha-1}} \quad (2.20)$$

and,

$$= \frac{f'(s)}{\alpha} s^{1-\alpha} \quad (2.21)$$

In order to further extent, the work done by Mathobo and Atangana in 2017, we replace the general groundwater flow derivative with fractal derivative. The initial phase of our analysis starts from the general groundwater flow equation,

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{\partial h}{r \partial r} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (2.22)$$

The above equation is local and does not account for groundwater flow within fractures in a geological environment, heterogeneity of the aquifer, random walk, diffusion and memory effect. We aim to apply a differential operator that will be able to account for groundwater flow in fractures and also the heterogeneity of the system. This differential operator is known as the fractal differential operator which will be integrated into the general groundwater flow equation.

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial h}{\partial r^\alpha} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (2.23)$$

Due to the physical problem under investigation, the advection in groundwater flow exists, therefore the function may act as a differentiable, thus the general groundwater flow equation can be re-formulated as,

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial h}{\partial r} \cdot \frac{1}{\alpha r^{\alpha-1}} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (2.24)$$

By cancelling out the like terms, we get;

$$\frac{S}{T} \frac{\partial h}{\partial t} = \frac{1}{\alpha r^{\alpha-1}} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (2.25)$$

By discretization of the first and second order derivatives, we get;

$$\frac{d}{dt} f(t_n) = \frac{f^{n+1} - f(t_n)}{\Delta t} \quad (2.26)$$

For 2<sup>nd</sup> order,

$$\frac{\partial^2}{\partial x^2} f(x_i, t_n) = \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\Delta x)^2} \quad (2.27)$$

$$\frac{\partial^2}{\partial x^2} f(x_i, t_{n+1}) = \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2} \quad (2.28)$$

We then discretize the forward, backward and Crank-Nicolson scheme into our equation which was re-formulated as follows,

$$\frac{S}{T} \frac{\partial h}{\partial t} = \frac{1}{\alpha r^\alpha} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right].$$

To represent our numerical approximation using forward Euler, we get;

$$\frac{\partial}{\partial x} f(x_i, t_n) = \frac{f_{i+1}^n - f_i^n}{\Delta x} \quad (2.29)$$

Backward Euler gives us,

$$\frac{\partial}{\partial x} f(x_i, t_{n+1}) = \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x} \quad (2.30)$$

Substituting equation (2.29) and (2.30) into equation (2.25) the following is obtained,

$$\frac{S}{T} \frac{h_i^{n+1} - h_i^n}{\Delta t} = \frac{1}{\alpha r_i^\alpha} \frac{h_{i+1}^n - h_i^n}{\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \quad (2.31)$$

$$\frac{S}{T} \frac{h_i^{n+1} - h_i^n}{\Delta t} = \frac{1}{\alpha r_i^\alpha} \frac{h_{i+1}^{n+1} - h_i^{n+1}}{\Delta r} + \frac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{(\Delta r)^2} \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \quad (2.32)$$

$$\frac{S}{T} \frac{h_i^{n+1} - h_i^n}{\Delta t} = \frac{1}{\alpha r_i^\alpha} \frac{h_{i+1}^n - h_i^n}{\Delta r} \quad (2.33)$$

$$+ \left\{ \frac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{2(\Delta r)^2} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \right\}$$

By putting like terms together, we get:

$$\begin{aligned} \frac{S}{T\Delta t} + \frac{1}{(r)^2} \left[ \left( 1 + \frac{r_{i+1} - r_i}{r_i} \right) \right] h_i^{n+1} &= \left( \frac{S}{T\Delta t} - \frac{1}{\alpha r_i^\alpha \Delta r} - \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \right) h_i^n \\ &+ \frac{1}{2(\Delta r)^2} \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] h_{i+1}^{n+1} \\ &+ h_{i+1}^n \left( \frac{1}{\alpha r_i^\alpha \Delta r} + \frac{1}{2(\Delta r)^2} \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \right) \\ &+ h_{i-1}^{n+1} \left( \frac{1}{2(\Delta r)^2} \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \right) \\ &+ h_{i-1}^n \left( \left[ 1 + \frac{r_{i+1} - r_i}{r_i} \right] \frac{1}{2(\Delta r)^2} \right) \end{aligned} \quad (2.34)$$

By simplifying the above in terms of alpha, we get:

$$a_1 h_i^{n+1} = a_2 h_i^n + a_3 h_{i+1}^{n+1} + a_4 h_{i+1}^n + a_3 h_{i-1}^{n+1} + a_3 h_{i-1}^n \quad (2.35)$$

Above is the solution derived by analyzing the general groundwater flow equation using fractal derivative. We then replace the above with epsilon variant to get the following,

$$\begin{aligned} a_1 e^{a(t+\Delta t)} e^{ik_m x} &= a_2 e^{at} e^{ik_m x} \\ + a_3 e^{a(t+\Delta t)} e^{ik_m(x+\Delta x)} &+ a_4 e^{at} e^{ik_m(x+\Delta x)} \\ + a_3 e^{a(t+\Delta t)} e^{ik_m(x-\Delta x)} &+ a_3 e^{at} e^{ik_m(x-\Delta x)} \end{aligned} \quad (2.36)$$

by simplification, we get:

$$\begin{aligned} a_1 e^{a\Delta t} &= a_2 + a_3 e^{a\Delta t} e^{ik_m \Delta x} + a_4 e^{ik_m \Delta x} \\ &+ a_3 e^{a\Delta t} e^{-ik_m \Delta x} + a_3 e^{-ik_m \Delta x} \end{aligned} \quad (2.37)$$

by putting like terms together, we get:

$$e^{a\Delta t} (a_1 - a_3 e^{ik_m \Delta x} - a_3 e^{-ik_m \Delta x}) = a_2 + a_4 e^{ik_m \Delta x} + a_3 e^{-ik_m \Delta x} \quad (2.38)$$

Thus,

$$e^{a\Delta t} = \frac{a_2 + a_4 e^{ik_m \Delta x} + a_3 e^{-ik_m \Delta x}}{a_1 - a_3 e^{ik_m \Delta x} - a_3 e^{-k_m \Delta x}} \quad (2.39)$$

We now have to prove that,

$$|e^{a\Delta t}| < 1$$

Therefore,

$$e^{a\Delta t} = \frac{a_2 + a_4 e^{ik_m \Delta x} + a_3 e^{-ik_m \Delta x}}{a_1 - a_3 (e^{ik_m \Delta x} + e^{-ik_m \Delta x})} \quad (2.40)$$

$$e^{a\Delta t} = \frac{a_2 + a_4 (\cos(k_m \Delta x) + i \sin(k_m \Delta x)) + a_3 (\cos(k_m \Delta x) - i \sin(k_m \Delta x))}{a_1 - 2a_3 \cos(k_m \Delta x)} \quad (2.41)$$

by simplification,

$$e^{a\Delta t} = \frac{a_2 + (a_4 + a_3) \cos(k_m \Delta x) + i(a_4 - a_3) \sin(k_m \Delta x)}{a_1 - 2a_3 \cos(k_m \Delta x)} \quad (2.42)$$

$$|e^{a\Delta t}| < 1 \Rightarrow \frac{|a_2 + (a_4 + a_3) \cos(k_m \Delta x) + i(a_4 - a_3) \sin(k_m \Delta x)|}{|a_1 - 2a_3 \cos(k_m \Delta x)|} < 1 \quad (2.43)$$

Based on the above, equation (2.43) can be rewritten as:

$$|e^{a\Delta t}|^2 < 1 \Rightarrow \frac{|a_2 + (a_4 + a_3) \cos(k_m \Delta x) + i(a_4 - a_3) \sin(k_m \Delta x)|^2}{|a_1 - 2a_3 \cos(k_m \Delta x)|^2} < 1 \quad (2.44)$$

$$\Rightarrow \frac{(a_2 + (a_4 + a_3) \cos(k_m \Delta x))^2 + (\sin(k_m \Delta x)(a_4 - a_3))^2}{(a_1 - 2a_3 \cos(k_m \Delta x))^2} < 1 \quad (2.45)$$

For conditions of stability, the following must be satisfied;

$$\frac{a_2^2 + 2a_2(a_4 + a_3) \cos(k_m \Delta x) + (a_4 + a_3)^2 \cos^2(k_m \Delta x) + (a_4 - a_3)^2 \sin^2(k_m \Delta x)}{a_1^2 - 4a_1 a_3 \cos(k_m \Delta x) + 4a_3^2 \cos^2(k_m \Delta x)} < 1 \quad (2.46)$$

By simplification and putting like terms together, we get;

$$a_1^2 - a_1^2 + [2a_2(a_4 + a_3 + 4a_1a_3)]\cos(k_m\Delta x) + [(a_4 + a_3)^2 - 4a_3^2]\cos^2(k_m\Delta x) + (a_4 - a_3)^2\sin^2(k_m\Delta x) < 0 \quad (2.47)$$

The above is conditionally stable only when the above conditions are satisfied.

We are now going to apply Atangana-Batogna numerical scheme to the general groundwater flow equation and then present the stability analysis using the Iterative method. Firstly, we will present how the Atangana Batogna method was developed. The method derives a two-step Adam-Bashforth numerical scheme in Laplace space and the solution is taken back into real space via Inverse Laplace transform form (Atangana and Batogna, 2018).

By considering the following partial differential equation,

$$\frac{\partial u(x, t)}{\partial t} = Lu(x, t) + Nu(x, t) \quad (2.48)$$

Where  $L$  is linear operator and  $N$  is a non-linear operator.

By applying Laplace transform on both sides of equation (2.48) we get,

$$\mathcal{L}\left(\frac{\partial u(x, t)}{\partial t}\right) = \mathcal{L}(Lu(x, t) + Nu(x, t)) \quad (2.49)$$

$$\frac{\partial u(p, t)}{\partial t} = \mathcal{L}(Lu(x, t) + Nu(x, t)) \quad (2.50)$$

$$v(t) = u(p, t) \quad (2.51)$$

$$\frac{d}{dt}(v(t)) = F(t, v(t)) \quad (2.52)$$

Where  $v(t) = u(p, t)$  and  $F(t, v(t)) = \mathcal{L}(Lu(x, t) + Nu(x, t))$ .

By applying principles of calculus, we get;

$$v(t) - v(0) = \int_0^t F(v, \tau) d\tau \quad (2.53)$$

and

$$v(t) = v(0) + \int_0^t F(v, \tau) d\tau \quad (2.54)$$

When  $t = t_{n+1}$

$$v(t_{n+1}) = v(0) + \int_0^{t_{n+1}} F(v, \tau) d\tau \quad (2.55)$$

When  $t = t_n$

$$v(t_n) = v(0) + \int_0^{t_n} F(v, \tau) d\tau \quad (2.56)$$

It follows that,

$$v(t_{n+1}) - v(t_n) = \int_0^{t_{n+1}} F(v, \tau) d\tau - \int_0^{t_n} F(v, \tau) d\tau \quad (2.57)$$

$$v(t_{n+1}) - v(t_n) = \int_0^{t_{n+1}} F(v, \tau) d\tau + \int_{t_n}^0 F(v, \tau) d\tau \quad (2.58)$$

$$v_{n+1} - v_n = \int_{t_n}^{t_{n+1}} F(v, \tau) d\tau \quad (2.59)$$

By making use of the Lagrange polynomial, we approximate  $F(t, v(t))$

$$p(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F(v, t_n) + \frac{t - t_n}{t_{n-1} - t_n} F(v, t_{n-1}) \quad (2.60)$$

Therefore,

$$v_{n+1} - v_n = \int_{t_n}^{t_{n+1}} \frac{t - t_{n-1}}{t_n - t_{n-1}} F(v, t_n) + \frac{t - t_n}{t_{n-1} - t_n} F(v, t_{n-1}) dt \quad (2.61)$$

$$= \int_{t_n}^{t_{n+1}} \frac{t - t_{n-1}}{\Delta t} F(v, t_n) - \frac{t - t_n}{\Delta t} F(v, t_{n-1}) dt \quad (2.62)$$

$$\frac{1}{\Delta t} F(v, t_n) \int_{t_n}^{t_{n+1}} (t - t_{n-1}) dt - F\left(\frac{v, t_{n-1}}{\Delta t}\right) \int_{t_n}^{t_{n+1}} (t - t_n) dt \quad (2.63)$$

$$\frac{1}{\Delta t} F(v, t_n) \left[ \frac{t^2}{2} - tt_{n-1} \right]_{t_n}^{t_{n+1}} - \frac{F(v, t_{n-1})}{\Delta t} \left[ \frac{t^2}{2} - tt_n \right]_{t_n}^{t_{n+1}} \quad (2.64)$$

Replacing in terms of  $v(t) = u(p, t)$ , we get;

$$u(p, t_{n+1}) = u(p, t_n) + \frac{3}{2} \Delta t F(u(p, t_n), t_n) - \frac{\Delta t}{2} F(u(p, t_{n-1}), t_{n-1}) \quad (2.65)$$

The above has been discretized using the Adam Bash forth method in terms of time. We applied the Inverse Laplace transform in order to return into real space, we get

$$u(x, t_{n+1}) = u(x, t_n) + \frac{3}{2} \Delta t F(u(x, t_n), t_n) - \frac{\Delta t}{2} F(u(x, t_{n-1}), t_{n-1}) \quad (2.66)$$

$$u_i^{n+1} = u_i^n + \frac{3}{2} \Delta t F_i^n - \frac{\Delta t}{2} F_i^{n-1} \quad (2.67)$$

We going to apply the above directly into the general groundwater flow equation and present stability analysis.

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{\partial h}{r \partial r} + \frac{\partial^2 h}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right]$$

The general groundwater flow equation is rearranged as follows;

$$\frac{\partial h}{\partial t} = \frac{S}{T} \left[ \frac{\partial h}{r \partial r} + \frac{\partial^2 h}{\partial r^2} \left( 1 + \frac{\Delta r}{r} \right) \right] \quad (2.68)$$

Therefore,

$$h(r, t) - h(r, 0) = \int_0^t F(h, \tau) d\tau \quad (2.69)$$

$$F(h, t) = \frac{T}{S} \left[ \frac{\partial h}{r \partial r} + \frac{\partial^2 h}{\partial r^2} \left( 1 + \frac{\Delta r}{r} \right) \right] \quad (2.70)$$

We then replace  $u$  in terms of  $h$ ,

$$h_i^{n+1} = h_i^n + \frac{3}{2} \Delta t F_i^n - \frac{\Delta t}{2} F_i^{n-1} \quad (2.71)$$

By discretization, we get

$$F_i^n = \left[ \frac{1}{r_i} \frac{h_i^n - h_{i-1}^n}{\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S} \quad (2.72)$$

$$F_i^{n-1} = \frac{T}{S} \left[ \frac{1}{r_i} \frac{h_i^{n-1} - h_{i-1}^{n-1}}{\Delta r} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \quad (2.73)$$

We substitute equation (2.73) and (2.72) into equation (2.71),

$$\begin{aligned}
h_i^{n+1} = h_i^n + \frac{3}{2} \Delta t \left[ \frac{1}{r_i} \frac{h_i^n - h_{i-1}^n}{\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S} \\
- \frac{\Delta t}{2} \left[ \frac{1}{r_i} \frac{h_i^{n-1} - h_{i-1}^{n-1}}{\Delta r} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S}
\end{aligned} \tag{2.74}$$

We are now going to present stability analysis of the above solution using a method used for solving classical and the non-local derivatives.

The round-off error is defined as follows:

$$\epsilon_i^n = L_i^n - h_i^n \tag{2.75}$$

With  $u_i^n$ , as the solution of the discretized equation (2.74). The difference equation error is linear, thus,

$$\epsilon_i^n = \exp[at] \exp[jk_m x] \tag{2.76}$$

By replacing the above into equation (2.74), we get;

$$\begin{aligned}
& \exp[a(t + \Delta t)] \exp[jk_m x] = \exp[at] \exp[jk_m x] \\
& + \frac{3}{2} \Delta t \left[ \frac{1}{r_i} \frac{\exp[at] \exp[jk_m x] - \exp[at] \exp[jk_m(x - \Delta x)]}{\Delta r} \right] \\
& + \frac{\exp[at] \exp[jk_m x + \Delta x] - 2\exp[at] \exp[jk_m x] + \exp[at] \exp[jk_m(x - \Delta x)]}{(\Delta r)^2} \\
& \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \frac{T}{S} \\
& - \frac{\Delta t}{2} \left[ \frac{1}{r_i} \frac{\exp[a(t - \Delta t)] \exp[jk_m x] - \exp[a(t - \Delta t)] \exp[jk_m x - \Delta x]}{\Delta r} \right]
\end{aligned} \tag{2.77}$$

$$\frac{+exp[a(t - \Delta t)]exp[jk_m(x + \Delta x)] - 2exp[a(t - \Delta t)]exp[jk_mx] + exp[a(t - \Delta t)]exp[jk_mx - \Delta x]}{(\Delta r)^2}$$

$$\left(1 + \frac{r_i - r_{i-1}}{r_i}\right) \frac{T}{S}$$

Simplification yields,

$$exp[a\Delta t] = 1 + \frac{3}{2} \frac{\Delta t}{r_i \Delta t} - \frac{3}{2} exp[-jk_m \Delta x] + exp[jk_m \Delta x]$$

$$- \frac{2T}{(\Delta r)^2 S} \left(1 + \frac{\Delta r}{r_i}\right) - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) exp[-jk_m \Delta x] - \frac{\Delta t}{(\Delta r)^2} exp[-a\Delta t]$$

$$+ \frac{\Delta t}{2(\Delta r)r_i} exp[-a\Delta t] exp[-jk_m \Delta x] + \frac{exp[-a\Delta t] exp[jk_m \Delta x]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \frac{T}{S} \quad (2.78)$$

$$- \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) exp[-a\Delta t] + \frac{exp[-a\Delta t] exp[-jk_m \Delta x]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right)$$

We have,

$$exp[a\Delta t] = \frac{\epsilon_i^{n+1}}{\epsilon_i^n} \quad (2.79)$$

The numerical scheme will be stable if,

$$\|exp[a\Delta t]\| = \left\| \frac{\epsilon_i^{n+1}}{\epsilon_i^n} \right\| < 1 \quad (2.80)$$

Therefore equation (2.78) can be transformed to become,

$$\begin{aligned}
\exp[a\Delta t] &= 1 + \frac{3}{2} \frac{\Delta t}{r_i \Delta t} - \frac{3}{2} [\cos(k_m \Delta x) - i \sin(k_m \Delta x)] \\
&+ \cos(k_m \Delta x) + i \sin(k_m \Delta x) - \frac{2T}{(\Delta r)^2 S} \left(1 + \frac{\Delta r}{r_i}\right) - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \\
&[\cos(k_m \Delta x) - i \sin(k_m \Delta x)] - \frac{\Delta t}{(\Delta r)^2} \exp[-a\Delta t] \\
&+ \frac{\Delta t}{2(\Delta r)r_i} \exp[-a\Delta t] [\cos(k_m \Delta x) - i \sin(k_m \Delta x)] \\
&+ \frac{\exp[-a\Delta t] [\cos(k_m \Delta x) + i \sin(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \frac{T}{S} \\
&\quad - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \exp[-a\Delta t] \\
&+ \frac{\exp[-a\Delta t] [\cos(k_m \Delta x) - i \sin(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right)
\end{aligned} \tag{2.81}$$

By simplification,

$$\begin{aligned}
\exp[a\Delta t] &= 1 + \frac{3}{2} \frac{\Delta t}{r_i \Delta t} - \frac{3}{2} \cos(k_m \Delta x) + \cos(k_m \Delta x) - \frac{2T}{(\Delta r)^2 S} \left(1 + \frac{\Delta r}{r_i}\right) \\
&\quad - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) [\cos(k_m \Delta x)] - \frac{\Delta t}{(\Delta r)^2} \exp[-a\Delta t] \\
&+ \frac{\Delta t}{2(\Delta r)r_i} \exp[-a\Delta t] [\cos(k_m \Delta x)] + \frac{\exp[-a\Delta t] [\cos(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \frac{T}{S} \\
&\quad - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \exp[-a\Delta t] + \frac{\exp[-a\Delta t] [\cos(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right)
\end{aligned} \tag{2.82}$$

$$+i \left\{ \begin{array}{l} -\frac{3}{2}[\sin(k_m \Delta x)] + \sin(k_m \Delta x) - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \\ -\frac{3}{2}\sin(k_m \Delta x) + \sin(k_m \Delta x) - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) [-\sin(k_m \Delta x)] \\ + \frac{\Delta t}{2(\Delta r)r_i} \exp[-a\Delta t] \left[ -\sin(k_m \Delta x) + \frac{\exp[-a\Delta t][\sin(k_m \Delta x)]}{(\Delta r)^2} \right] \left(1 + \frac{\Delta r}{r}\right) \frac{T}{S} \\ + \frac{\exp[-a\Delta t][-\sin(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right\}$$

The above can be further simplified as

$$\begin{aligned} \exp[a\Delta t] &= 1 + \frac{3}{2} \frac{\Delta t}{r_i \Delta t} - \frac{3}{2} \cos(k_m \Delta x) + \cos(k_m \Delta x) - \frac{2T}{(\Delta r)^2 S} \left(1 + \frac{\Delta r}{r_i}\right) \\ &\quad - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) [\cos(k_m \Delta x)] - \frac{\Delta t}{(\Delta r)^2} \exp[-a\Delta t] \\ &\quad + \frac{\Delta t}{2(\Delta r)r_i} \exp[-a\Delta t] [\cos(k_m \Delta x)] + \frac{\exp[-a\Delta t][\cos(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \frac{T}{S} \\ &\quad - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \exp[-a\Delta t] + \frac{\exp[-a\Delta t][\cos(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{aligned} \quad (2.83)$$

$$+i \left\{ \begin{array}{l} -\sin(k_m \Delta x) - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) - \frac{2T}{S(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) [-\sin(k_m \Delta x)] \\ + \frac{\Delta t}{2(\Delta r)r_i} \exp[-a\Delta t] \left[ -\sin(k_m \Delta x) + \frac{\exp[-a\Delta t][\sin(k_m \Delta x)]}{(\Delta r)^2} \right] \left(1 + \frac{\Delta r}{r}\right) \frac{T}{S} \\ + \frac{\exp[-a\Delta t][-\sin(k_m \Delta x)]}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right\}$$

Based on the above, the equation cannot be further simplified or analyzed and hence does not give us conditions for stability. It is in this regard that we propose an alternative method.

We are now going to use the Iterative method to determine the conditions for stability.

Analysis commences from equation (2.74) as follows,

$$h_i^{n+1} = h_i^n + \frac{3}{2}\Delta t \left[ \frac{1}{r_i} \frac{h_i^n - h_{i-1}^n}{\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S}$$

$$- \frac{\Delta t}{2} \left[ \frac{1}{r_i} \frac{h_i^{n-1} - h_{i-1}^{n-1}}{\Delta r} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S}$$

By applying the iterative method, we get;

$$\delta_{n+1}e^{ik_mx} = \delta_n e^{ik_mx} + \frac{3}{2}\Delta t \left[ \frac{1}{r_i} \frac{\delta_n e^{ik_mx} - \delta_n e^{ik_m(x-\Delta x)}}{\Delta r} \right.$$

$$\left. + \frac{\delta_n e^{ik_m(x+\Delta x)} - 2\delta_n e^{ik_mx} + \delta_n e^{ik_m(x-x)}}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S} \quad (2.84)$$

$$- \frac{\Delta t}{2} \left[ \frac{1}{r_i} \frac{\delta_{n-1} e^{ik_mx} - \delta_{n-1} e^{ik_m(x-\Delta x)}}{\Delta r} \right.$$

$$\left. + \frac{\delta_{n+1} e^{ik_m(x-\Delta x)} - 2\delta_{n-1} e^{ik_mx} + \delta_{n-1} e^{ik_m(x-\Delta x)}}{(\Delta r)^2} \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \right] \frac{T}{S}$$

Therefore,

$$\text{if } a = \frac{3}{2}\Delta t + \frac{1}{r_i} - \frac{1}{\Delta r}, b = \left( 1 + \frac{r_i - r_{i-1}}{r_i} \right) \frac{T}{S} \frac{1}{(\Delta r)^2}, C = \frac{\Delta t}{2} + \frac{1}{r_i} \frac{1}{\Delta r}$$

Simplification using the above yields,

$$\delta_{n+1}e^{ik_mx} = \delta_n e^{ik_mx} + \delta_n a e^{ik_mx} - \delta_n a e^{ik_m(x-\Delta x)} + \delta_n b e^{ik_m(x+\Delta x)} - 2\delta_n b e^{ik_mx}$$

$$+ \delta_n b e^{ik_m(x-\Delta x)} - \delta_{n-1} c e^{ik_mx} - \delta_{n-1} c e^{ik_m(x-\Delta x)} + \delta_{n-1} b e^{ik_m(x-\Delta x)} \quad (2.85)$$

$$- 2\delta_{n-1} b e^{ik_mx} + 2\delta_{n-1} b e^{ik_m(x-\Delta x)}$$

By removing  $e^{ik_mx}$  we get,

$$\begin{aligned}
\delta_{n+1} = & \delta_n + \delta_n a - \delta_n a e^{ik_m x} \cdot e^{-ik_m \Delta x} + \delta_n b e^{ik_m x} \cdot e^{ik_m \Delta x} - 2\delta_n b \\
& + \delta_n b e^{ik_m x} \cdot e^{ik_m \Delta x} - \delta_{n-1} c - \delta_{n-1} c e^{ik_m x} \cdot e^{-ik_m \Delta x} \\
& + \delta_{n+1} b e^{ik_m x} \cdot e^{-ik_m \Delta x} - 2\delta_{n-1} b + \delta_{n-1} b e^{ik_m x} \cdot e^{-ik_m \Delta x}
\end{aligned} \tag{2.86}$$

Further simplification yields,

$$\begin{aligned}
\delta_{n+1} = & \delta_n + \delta_n a - \delta_n a e^{-ik_m \Delta x} + \delta_{n-1} b e^{ik_m \Delta x} - 2\delta_{n-1} b + \delta_n b e^{ik_m \Delta x} - \delta_{n-1} c \\
& - \delta_{n-1} c e^{-ik_m \Delta x} + \delta_{n+1} b e^{-ik_m \Delta x} - 2\delta_{n-1} b + \delta_{n-1} b e^{-ik_m \Delta x}
\end{aligned} \tag{2.87}$$

By factorization, we get

$$\begin{aligned}
\delta_{n+1} = & \delta_n (1 + a - a e^{-ik_m \Delta x}) \\
& + \delta_{n-1} (b e^{ik_m \Delta x} - 4b + 3b e^{-ik_m \Delta x} - c)
\end{aligned} \tag{2.88}$$

Equation (2.88) can be written in terms of cos and sin,

$$\begin{aligned}
\delta_{n+1} = & \delta_n (1 + a - a(\cos(k_m \Delta x) - i \sin(k_m \Delta x))) \\
& + \delta_{n-1} (b(\cos(k_m \Delta x) + i \sin(k_m \Delta x)) - 4b + 3b(\cos(k_m \Delta x) - i \sin(k_m \Delta x)) - c)
\end{aligned} \tag{2.89}$$

By induction when  $n = 0$

$$\delta_1 = \delta_0 (1 + a - a \cos(k_m \Delta x) - a i \sin(k_m \Delta x)) \tag{2.90}$$

$$\left| \frac{\delta_1}{\delta_0} \right| < 1 = |1 - a - a \cos(k_m \Delta x) - a i \sin(k_m \Delta x)| \tag{2.91}$$

$$|a + ib| = \sqrt{a^2 + b^2} \tag{2.92}$$

$$= \sqrt{1 - a - a \cos(k_m \Delta x)^2 + (a \sin(k_m \Delta x))^2} < 1 \tag{2.93}$$

$$(1 - a - a\cos(k_m\Delta x))^2 + (a\sin(k_m\Delta x))^2 \quad (2.94)$$

$$(1 - a)^2 - 2(1 - a)a\cos(k_m\Delta x) + (a\cos(k_m\Delta x))^2 + (a\sin(k_m\Delta x))^2 < 1 \quad (2.95)$$

$$(1 - a)^2 - 2(1 - a)a\cos(k_m\Delta x) + a^2(\cos^2(k_m\Delta x) + \sin^2(k_m\Delta x)) < 1 \quad (2.96)$$

$$(1 - a)^2 - 2(1 - a)a\cos(k_m\Delta x) + a < 1 \quad (2.97)$$

$$(1 - a)^2 - 2(1 - a)\cos(k_m\Delta x) < 1 - a \quad (2.98)$$

If  $1 - a < 1$

$$(1 - a) - 2\cos(k_m\Delta x) < 1 \quad (2.99)$$

$$-a - 2\cos(k_m\Delta x) < 0 \quad (2.100)$$

$$a + 2\cos(k_m\Delta x) > 0 \quad (2.101)$$

$$\cos(k_m\Delta x) > \frac{-a}{2} \quad (2.102)$$

This is one of the conditions for stability.

$$\delta_{n+1} = \delta_n(1 + a - a(\cos(k_m\Delta x) - i\sin(k_m\Delta x)))$$

$$+ \delta_{n-1}(b(\cos(k_m\Delta x) + i\sin(k_m\Delta x)) - 4b + 3b(\cos(k_m\Delta x) - i\sin(k_m\Delta x))) - c)$$

for simplification, we let

$$\mathbf{z}_1 = (1 + a - a(\cos(k_m\Delta x) - i\sin(k_m\Delta x))) \quad (2.103)$$

$$\mathbf{z}_2 = (b(\cos(k_m\Delta x) + i\sin(k_m\Delta x)) - 4b + 3b(\cos(k_m\Delta x) - i\sin(k_m\Delta x))) - c) \quad (2.104)$$

$$\delta_{n+1} = \delta_n \mathbf{z}_1 + \delta_{n-1} \mathbf{z}_2 \quad (2.105)$$

$\forall n \geq 0$  we assume that  $\|\delta_n\| \leq \|\delta_0\|$

We then want to find the conditions for which  $\|\delta_{n+1}\| < \|\delta_0\|$

By definition we have,

$$\delta_{n+1} = \delta_n \mathbf{z}_1 + \delta_{n-1} \mathbf{z}_2$$

$$\|\delta_{n+1}\| = \|\delta_n \mathbf{z}_1 + \delta_{n-1} \mathbf{z}_2\| \quad (2.106)$$

$$\leq \|\delta_n \mathbf{z}_1\| + \|\delta_{n-1} \mathbf{z}_2\| \quad (2.107)$$

$$\leq \|\delta_n\| \|\mathbf{z}_1\| + \|\delta_{n-1}\| \|\mathbf{z}_2\| \quad (2.108)$$

By induction, we have  $\|\delta_n\| < \|\delta_0\|$  and  $\|\delta_{n-1}\| < \|\delta_0\|$ .

Therefore,

$$\|\delta_{n+1}\| < \|\delta_0\| \|\mathbf{z}_1\| + \|\delta_0\| \|\mathbf{z}_2\| \quad (2.109)$$

$$< \|\delta_0\| (\|\mathbf{z}_1\| + \|\mathbf{z}_2\|) \quad (2.110)$$

$$\|\delta_{n+1}\| < \|\delta_0\| \text{ if } \|\mathbf{z}_1\| + \|\mathbf{z}_2\| < 1$$

$$\mathbf{z}_1 = (1 + a - a(\cos(k_m \Delta x) - i \sin(k_m \Delta x)))$$

$$\mathbf{z}_2 = (b(\cos(k_m \Delta x) + i \sin(k_m \Delta x)) - 4b + 3b(\cos(k_m \Delta x) - i \sin(k_m \Delta x)) - c)$$

Therefore,

$$\mathbf{z}_1 = (1 + a - a \cos(k_m \Delta x)) - i a \sin(k_m \Delta x) \quad (2.111)$$

$$\mathbf{z}_2 = (4b \cos(k_m \Delta x) - 4b - c) - i 2b \sin(k_m \Delta x) \quad (2.112)$$

$$\|\mathbf{z}_1\| = \sqrt{(1 + a - a\cos(k_m\Delta x))^2 + (a\sin(k_m\Delta x))^2} \quad (2.113)$$

$$\|\mathbf{z}_2\| = \sqrt{(4bc\cos(k_m\Delta x) - 4b - c)^2 + (2b\sin(k_m\Delta x))^2} \quad (2.114)$$

$$\|\mathbf{z}_1\| + \|\mathbf{z}_2\| < 1 \Rightarrow$$

$$\begin{aligned} & \sqrt{(1 + a - a\cos(k_m\Delta x))^2 + (a\sin(k_m\Delta x))^2} \\ & + \sqrt{(4bc\cos(k_m\Delta x) - 4b - c)^2 + (2b\sin(k_m\Delta x))^2} < 1 \end{aligned} \quad (2.115)$$

Simplification yields,

$$\begin{aligned} & \sqrt{(1 + a)^2 - 2(1 + a)a\cos(k_m\Delta x)} \\ & + \sqrt{(4bc\cos(k_m\Delta x) - 4b - c)^2 + (2b\sin(k_m\Delta x))^2} < 1 \end{aligned} \quad (2.116)$$

The above satisfy condition for stability.

We are now going to determine self-similarity with respect to time.

$$\frac{T}{S} \frac{\partial h(r, t)}{\partial t} = \frac{1}{\alpha^{\alpha-1}} \frac{\partial h(r, t)}{\partial r} + \left(1 + \frac{\Delta r}{r}\right) \frac{\partial^2 h}{\partial r^2} \quad (2.117)$$

$$\frac{T}{S} \frac{\partial h(r, t)}{\partial t^\beta} = \frac{1}{\alpha r^{\alpha-1}} \frac{\partial h(r, t)}{\partial r} + \left(1 + \frac{\Delta r}{r}\right) \frac{\partial^2 h}{\partial r^2} \quad (2.118)$$

Since  $h(r, t)$  is differentiable,

$$\frac{T}{S} \frac{\partial h(r, t)}{\partial t} \frac{1}{\beta t^{\beta-1}} = \frac{1}{\alpha r^{\alpha-1}} \frac{\partial h(r, t)}{\partial r} + \left(1 + \frac{\Delta r}{r}\right) \frac{\partial^2 h}{\partial r^2} \quad (2.119)$$

$$\frac{T}{S} \frac{\partial h(r, t)}{\partial t} = \frac{\beta t^{\beta-1}}{\alpha r^{\alpha-1}} \frac{\partial h(r, t)}{\alpha r} + \beta t^{\beta-1} \left(1 + \frac{\Delta r}{r}\right) \frac{\partial^2 h}{\partial r^2} \quad (2.120)$$

$$\frac{T}{S} \frac{\partial h(r, t)}{\partial t} = V_{\beta}^{\alpha}(r, t) \frac{\partial h(r, t)}{\alpha r} + D^{\beta}(r, t) \frac{\partial^2 h}{\partial r^2} \quad (2.121)$$

We now discretize the above equation,

$$\begin{aligned} \frac{T}{S} \frac{h_i^{n+1} - h_i^n}{\Delta t} &= V_{\beta}^{\alpha}(r_i, t_n) \frac{h_{i+1}^n - h_i^n}{\Delta r} \\ &+ D^{\beta}(r_i, t_n) \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \left[1 + \frac{r_{i+1} - r_i}{r_i}\right] \end{aligned} \quad (2.122)$$

$$\begin{aligned} \frac{T}{S} \frac{h_i^{n+1} - h_i^n}{\Delta t} &= V_{\beta}^{\alpha}(r_i, t_n) \frac{h_{i+1}^{n+1} - h_i^{n+1}}{\Delta r} \\ &+ D^{\beta}(r_i, t_n) \frac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{(\Delta r)^2} \left[1 + \frac{r_{i+1} - r_i}{r_i}\right] \end{aligned} \quad (2.123)$$

$$\begin{aligned} \frac{T}{S} \frac{h_i^{n+1} - h_i^n}{\Delta t} &= V_{\beta}^{\alpha}(r_i, t_n) \frac{h_{i+1}^n - h_i^n}{\Delta r} \\ &+ \left[ \frac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{2(\Delta r)^2} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{2(\Delta r)^2} \right] \left[1 + \frac{r_{i+1} - r_i}{r_i}\right] \end{aligned} \quad (2.124)$$

Simplification yields,

$$\begin{aligned} h_i^{n+1} \left[ \frac{T}{S\Delta t} + \frac{1}{(\Delta r)^2} \left[1 + \frac{r_{i+1} - r_i}{r_i}\right] \right] &= h_i^n \left[ \frac{1}{(\Delta r)^2} \left(1 + \frac{r_{i+1} - r_i}{r_i}\right) \right] \\ &+ h_{i+1}^n \left[ V_{\beta}^{\alpha} \frac{(r_i t_n)}{\Delta r} + \left( \frac{1 + \frac{r_{i+1} - r_i}{r_i}}{2(\Delta r)^2} \right) \right] + h_{i-1}^n \left[ V_{\beta}^{\alpha} \frac{(r_i t_n)}{\Delta r} + \left( \frac{1 + \frac{r_{i+1} - r_i}{r_i}}{2(\Delta r)^2} \right) \right] \end{aligned}$$

$$+ h_{i-1}^{n+1} \left( \frac{1 + \frac{r_{i+1} - r_i}{r_i}}{2(\Delta r)^2} \right) \quad (2.125)$$

Simplification in terms of alpha yields,

$$a_1 h_i^{n+1} = a_2 h_i^n + a_3 h_{i+1}^n + a_3 h_{i-1}^n + a_4 h_{i-1}^{n+1} \quad (2.126)$$

The above solution is derived by analyzing self-similarity with respect to time.

Hence,

$$h_i^{n+1} = \delta_{n+1} e^{ik_m x}$$

$$h_i^n = \delta_n e^{ik_m x}$$

$$h_{i+1}^n = \delta_n e^{ik_m(x+\Delta x)}$$

$$h_{i-1}^n = \delta_n e^{-ik_m(x-\Delta x)}$$

$$h_{i-1}^{n+1} = \delta_{n+1} e^{ik_m(x-\Delta x)}$$

By replacing the above to our solution, we get

$$\begin{aligned} a_1 \delta_{n+1} e^{ik_m \Delta x} &= a_2 \delta_n e^{ik_m \Delta x} + a_3 \delta_n e^{ik_m(x+\Delta x)} \\ &+ a_3 \delta_n e^{-ik_m(x-\Delta x)} + a_4 \delta_{n+1} e^{ik_m(x-\Delta x)} \end{aligned} \quad (2.127)$$

Simplification yields,

$$a_1 \delta_{n+1} = a_2 \delta_n + a_3 \delta_n e^{ik_m \Delta x} + a_3 \delta_n e^{-ik_m \Delta x} + a_4 \delta_{n+1} e^{-ik_m \Delta x} \quad (2.128)$$

Putting like terms together and factorization, we get

$$\delta_{n+1} (a_1 - a_4 e^{ik_m \Delta x}) = \delta_n (a_2 + a_3 e^{ik_m \Delta x} + a_3 e^{-ik_m \Delta x}) \quad (2.129)$$

Therefore,

$$\frac{\delta_{n+1}}{\delta_n} = \frac{a_2 + a_3(e^{ik_m\Delta x} - e^{-ik_m\Delta x})}{a_1 - a_4e^{ik_m\Delta x}} \quad (2.130)$$

We now have to prove that,

$$\left| \frac{\delta_{n+1}}{\delta_n} \right| < 1$$

Therefore,

$$\left| \frac{\delta_{n+1}}{\delta_n} \right| < 1 = \frac{|a_2 + 2a_3i\sin(k_m\Delta x)|}{|a_1 - a_4\cos(k_m\Delta x) + i\sin(k_m\Delta x)|} < 1 \quad (2.131)$$

Since  $|a + ib| = \sqrt{a^2 + b^2}$  we get,

$$\sqrt{\frac{a_2^2 + (2a_3\sin(k_m\Delta x))^2}{(a_1 - a_4\cos(k_m\Delta x))^2 + \sin^2(k_m\Delta x)}} < 1 \quad (2.132)$$

Removal of square root gives us,

$$\frac{a_2^2 + (2a_3\sin(k_m\Delta x))^2}{(a_1 - a_4\cos(k_m\Delta x))^2 + \sin^2(k_m\Delta x)} < 1 \quad (2.133)$$

For conditions of stability, the above conditions must be satisfied.

# **CHAPTER THREE: ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE.**

The section aims to discuss Riemann-Liouville fractional derivative. We aim to show analysis of general groundwater flow equation using Caputo fractional derivative since there is a connection to Riemann Liouville. We shall give a general background theory on the concept of fractional derivative, review different definitions from the various authors, explore its properties, provide examples of where the derivative was applied and reveal its main advantages and limitations. We will also show the stability analysis.

## **3.1 RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE**

In previous years, Riemann-Liouville fractional derivative has been the most used generalization of the derivative. The derivative was developed based on the Cauchy's formula for calculating iterated integrals. In 17<sup>th</sup> century, a mathematician by the name G.W Leibniz was the first to investigate fractional derivatives. Later on, between 1832-1837, Liouville carried out an in-depth research and produced a number of papers on this topic. He was the first mathematician to define the first outcast of an operator of fractional integration (Munkhammar, 2004). This raised much interest from other researcher's and in the following years, Riemann developed an integral based Riemann-Liouville fractional integral operator which has since played an important role in fractional calculus. Other mathematicians and scientists kept on developing and improving on what had been already done and this has led to groundbreaking novelty ever since.

Riemann-Liouville fractional derivative is a local operator with singular memory kernel. This means that this operator cannot incorporate advection, diffusion, random walk etc. (Atangana and Koca, 2017) recently raised an argument that fractional derivatives such as Riemann-Liouville have been misused in the past pointing specifically to papers that were written by Jumarie in 2006 and Podlubny in 2002. They further stated that this derivative had being used in modelling most real-world problems but without clear explanation.

Based on a paper by Munkhammar (2004); the Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an  $n$ -fold integral (Samko, *et al.*, 1993).

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{(x-t)^{1-n}} dt \quad (3.1)$$

Since  $(n-1)! = \Gamma(n)$ , Riemann observed that the right-hand side of the above equation would still make sense even when  $n$  takes non-integer values.

**Definition 1.** If  $f(x) \in C([a, b])$  and  $a < x < b$  then:

$$I_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt \quad (3.2)$$

where  $\alpha \in ]-\infty, \infty[$ , is called the Riemann-Liouville fractional integral or order  $\alpha$ . In the same way for  $\alpha \in ]0, 1[$

$$D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt \quad (3.3)$$

which is called the Riemann-Liouville derivative of order  $\alpha$ .

All of the above are known as Riemann-Liouville fractional integral operators. The case where  $\alpha = \frac{1}{2}$  is known as semi-derivative. According to Munkhammar (2004), the connection between the Riemann-Liouville fractional integral and derivative may be traced back to the solvability of Abel's integral equation for any  $\alpha \in ]0, 1[$ ,

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\Phi(t)}{(x-t)^{1-\alpha}} dt, x > 0 \quad (3.4)$$

**Definition 2.** According to Munkhammar (2004), if  $\alpha > 0$  is not an integer then we define:

$$D_a^\alpha f = \frac{d^{[\alpha]}}{dx^\alpha} D_a^{\{\alpha\}} f = \frac{d^{[\alpha] + 1}}{dx^{[\alpha] + 1}} I_a^{1-[\alpha]} f \quad (3.5)$$

Therefore,

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t) dt}{(x - t)^{\alpha - n + 1}} \quad (3.6)$$

for any  $f \in C^{[\alpha]+1}([a, b])$  if  $n = [\alpha] + 1$ . When  $\alpha < 0$ , we get:

$$D_a^\alpha f = I_a^{-\alpha} f \quad (3.7)$$

which according to Munkhammar (2004) may be used as a definition.

When  $\alpha < 0$  the fractional derivative  $D_a^\alpha f(x)$  exists for all  $f \in C([a, b])$  and all  $x \in [a, b]$ . It was observed that for  $\alpha > 0$ , the fractional derivative  $D_a^\alpha f(x)$  certainly exists for all  $f \in C^{[\alpha]+1}([a, b])$  and all  $x \in ]a, b$ , this may not be necessarily the case for  $x = a$ .

**Definition 3.** For  $\alpha > 0$  let  $I_a^\alpha([a, b])$  denote the space of functions which can be represented by a Riemann-Liouville integral of order  $\alpha$  of some  $C([a, b])$ -function (Munkhammar, 2004).

The above definition led to the development of the following theorem:

Let  $f \in C([a, b])$  and  $\alpha > 0$ . In order that  $f(x) \in I_a^\alpha([a, b])$ , it is important:

$$I_a^{n-\alpha} f \in C^n([a, b]) \quad (3.8)$$

where  $n = [\alpha] + 1$ , so that

$$\left( \frac{d^k}{dx^k} I_a^{n-\alpha} f(x) \right) \Big|_{x=a} = 0, \quad k = 0, 1, 2, \dots, n - 1. \quad (3.9)$$

(Munkhammar, 2004) further provided proof for the above theorem and showed its uniqueness.

**Definition 4.** (Kimeu, 2009) for his MSc thesis stated that Riemann-Liouville fractional derivative may be defined using the definition of the fractional integral. Assuming that  $v = n - u$ , where  $0 < v < 1$  and  $n$  is the smallest integer greater than  $u$ . Then, the fractional derivative of  $f(x)$  of order  $u$  is,

$$D^u f(x) = D^n [D^{-v} f(x)] \quad (3.10)$$

Kimeu (2009) further gave examples of fractional derivatives and used the above equation to get the semi-derivative.

There are many other definitions by different authors but for the sole purpose of this research, the above definitions were stated and used.

### **3.2 APPLICATION OF THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE**

(Tateishi *et al.*, 2017) applied Riemann-Liouville fractional operator to generalize the usual diffusion equation. The main aim of their study was to determine how the three fractional operators (Riemann-Liouville, Atangana-Baleanu and Caputo-Fabrizio) under study alter fractional diffusion equation. They compared the classical Riemann-Liouville operator with the recently developed fractional operators which are characterized by a non-singular kernel based on their main goal. Based on their study, they proved without reasonable doubt that the recently developed fractional operators are an improvement to what had been previously developed by Riemann and Liouville and that these operators afford the opportunity to simply and effectively deal with challenges that could not be solved using classical fractional derivative methods.

(Tateishi *et al.*, 2017) used the concept of random walk which is characterized by Gaussian, Markovian and ergodic properties. This concept is able to give various characteristics for anomalous diffusion. They obtained that for the fractional diffusion equation,

$$\frac{\partial}{\partial t} p(x, t) = {}_0D_t^{1-\alpha} (\mathcal{L}\{p(x, t)\}) \quad (3.11)$$

The nonusual relaxation may be linked with the concept of continuous time random walk where waiting time distribution is a power law.

This property can be observed with Riemann-Liouville fractional operator. When considering different forms for the kernel  $\mathcal{K}(t)$  they noticed singularity at the origin  $t = 0$  for the Riemann-Liouville operator. The use of  $\mathcal{K}(t) = \delta t$  lead to the development

$$\omega(t) = \frac{1}{\tau_c} t^{\alpha-1} E_{\alpha,\alpha} \left( -\frac{1}{\tau_c} t^\alpha \right) \quad (3.12)$$

for Riemann-Liouville fractional operator, where  $E_{\alpha,\alpha}(\dots)$  is understood to be generalized Mittag-Leffler function whose asymptotic behavior is a power law,  $\omega(t) \sim \frac{1}{t^{1+\alpha}}$  for  $t \rightarrow \infty$ .

They also noticed that the Atangana-Baleanu and Riemann-Liouville operators produced the same power law decay for  $\omega(t)$  at long times. When deriving formal solution using Riemann-Liouville operator, the mean square displacement recovers the usual diffusion.

Numerical simulations for probability distribution showed a tent-shaped distribution with tails longer than the Gaussian distribution of usual diffusion (Tateishi. 2017).

### 3.3 PROPERTIES OF RIEMANN-LIOUVILLE DERIVATIVE

The following are Riemann-Liouville Fractional derivative properties as proposed on the work which was previously done by various researchers (Dao and Guo,2009; Li and Deng, 2007; Podlubny, 1999).

#### Property 1.

1. For  $\alpha \in (0,1)$ ,  $x(t) \in L^1_{loc}(R^+)$ ,

$${}^{RL}D_t^\alpha x(t) = {}^C D_t^\alpha x(t) - \delta(t) [{}^{RL}D_t^{\alpha-1} x(t)]_{t=0} \quad (3.13)$$

$$= \frac{d}{dt} \int_0^t Y_{1-\alpha}(t-\tau) x(\tau) d\tau \quad (3.14)$$

2. For  $n - 1 < \alpha < n \in N^+$ ,  $x(t) \in L^1_{loc}(R^+)$ ,

$${}^{RL}D_t^\alpha x(t) = {}^G D_t^\alpha x(t) - \delta(t)[{}^{RL}D_t^{\alpha-1}x(t)]_{t=0} \quad (3.15)$$

$$- \sum_{k=1}^{n-1} Y_{-k} [{}^{RL}D_t^{\alpha-k-1}]_{t=0}$$

$$= \frac{d^n}{dt^n} \int_0^t Y_{n-\alpha}(t-\tau)x(\tau)d\tau \quad (3.16)$$

3. Assume  $\alpha > 0$ , then  $\left(\frac{d^n}{dt^n}\right) {}^{RL}D_t^\alpha x(t) = {}^{RL}D_t^\alpha x(t)$  if  $n - \alpha > 0$ , and  $\left(\frac{d^n}{dt^n}\right) {}^{RL}D_t^{-\alpha}x(t) = D_t^{n-\alpha}x(t)$  if  $n - \alpha < 0$ , hold for any  $n \in N^+$ .

4.  ${}^{RL}D_t^\alpha \cdot D_t^{-\alpha}x(t)$  for all  $\alpha > 0$ . More generally  ${}^{RL}D_t^\alpha \cdot D_t^{-\beta}x(t) = {}^{RL}D_t^{\alpha-\beta}x(t)$  for all  $\beta > 0$ . If  $\alpha < \beta$ , then  ${}^{RL}D_t^{\alpha-\beta}x(t) = D_t^{\alpha-\beta}x(t)$ .

5.  $D_t^{-\alpha} \cdot {}^{RL}D_t^\alpha x(t) = x(t) - \sum_{k=1}^n (t^{\alpha-k} \frac{[{}^{RL}D_t^{\alpha-k}x(t)]_{t=0}}{\Gamma(\alpha-k+1)})$ , where  $n - 1 < \alpha < n \in Z^+$ .

6.  $D_t^{-n} \cdot {}^{RL}D_t^n x(t) = x(t) - \sum_{k=0}^{n-1} \left(\frac{t^k}{k!}\right) x^{(k)}(0)$ .

7.  ${}^{RL}D_t^\alpha c = \left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right) c$ , where  $\alpha > 0$  and  $c$  is an arbitrary constant.

8.  $\mathcal{L}[{}^{RL}D_t^\alpha x(t)] = s^\alpha X(s) - \sum_{k=0}^{n-1} s^k [{}^{RL}D_t^{\alpha-k-1}x(t)]_{t=0}$ , in which  $\mathcal{L}$  is Laplace transform, and  $\mathcal{L}[x(t)] = X(s)$ .

### Property 2.

1. Composition with integral operator: for  $\alpha > 0, \beta > 0$ , then  ${}^{RL}D_t^{\alpha-\beta} = {}^{RL}D_t^\alpha \cdot D_t^{-\beta} \neq D_t^{-\beta} \cdot {}^{RL}D_t^\alpha$ .

2. Composition with integer derivative operator: for  $\alpha \in (n - 1, n), n \in Z^+$ , then  $\frac{d^m}{dt^m} \cdot {}^{RL}D_t^\alpha = {}^{RL}D_t^{\alpha+m} \cdot \left(\frac{d^m}{dt^m}\right)$ .

**Property 3.**

1. If  $x(t) \in C^1[0, T]$ ,  $\alpha_i \in (0, 1)$  ( $i = 1, 2$ ) (then the trivial case  $\alpha_i = 0$  or  $1$  is simple and removed here), and  $\alpha_1 + \alpha_2 \in (0, 1]$ , then  ${}^{RL}_0D_t^{\alpha_1} \cdot {}^{RL}_0D_t^{\alpha_2} x(t) = {}^{RL}_0D_t^{\alpha_1 + \alpha_2} x(t)$ .

**Property 4.**

1. Let  $\mathcal{A} = \{x(t) \in R, t \geq 0, x(t) \text{ is analytical for any } t \geq 0\}$ . If  $\alpha \in (0, 1)$ , then Riemann-Liouville derivative operator  ${}^{RL}_0D_t^\alpha$  defined in  $\mathcal{A}$  can be expressed as:

$${}^{RL}_0D_t^\alpha = \sum_{k=0}^{\infty} \frac{[d^k/dt^k]_{t=0}}{\Gamma(k - \alpha + 1)} t^{k-\alpha} \quad (3.17)$$

If  $n - 1 < \alpha < n \in Z^+$ , then  ${}^{RL}_0D_t^\alpha$  defined in  $\mathcal{A}$  may also have the following form:

$${}^{RL}_0D_t^\alpha = \sum_{k=0}^{\infty} \frac{[d^k/dt^k]_{t=0}}{\Gamma(k - \alpha + 1)} t^{k-\alpha} \quad (3.18)$$

### 3.4 ANALYSIS OF GENERAL GROUNDWATER FLOW WITH CAPUTO FRACTIONAL DERIVATIVE

We are going to analyze the general groundwater flow equation for confined aquifer using Caputo fractional derivative. Discretization will be done using the new novel method called the Newton polynomial. This method was developed by (Atangana and Araz, 2019). This method has attracted worldwide attraction. Stability analysis will also be derived. Numerical models will be created and compared to models created using numerical approximation of general groundwater flow equation.

Using Caputo fractional derivative,

$${}^C_0D_t^\alpha y(t) = f(t, y(t)) \quad (3.19)$$

A numerical scheme is developed to solve the above equation.

The Caputo equation was transformed into,

$$y(t_{n+1}) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y(\tau))(t_{n+1} - \tau)^{\alpha-1} d\tau \quad (3.20)$$

at point  $t_{n+1} = (n + 1) \Delta t$ , we get

$$y(t_{n+1}) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y(\tau))(t_{n+1} - \tau)^{\alpha-1} d\tau \quad (3.21)$$

We also have

$$y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \int_0^{t_{n+1}} f(\tau, y(\tau))(t_{n+1} - \tau)^{\alpha-1} d\tau \quad (3.21)$$

We replace the Newton polynomial into equation (3.21) to get,

$$y^{n+1} = y^0 + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \int_{t_j}^{t_{j+1}} \left\{ \begin{array}{l} f(t_{l-2}, y^{l-2}) \\ + \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{l-2}) \\ \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ \times (\tau - t_{l-2})(\tau - t_{l-1}) \\ (t_{n+1} - \tau)^{\alpha-1} d\tau \end{array} \right\} \quad (3.22)$$

We then get

$$y^{n+1} = y^0 + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \left\{ \begin{array}{l} \int_{t_j}^{t_{l+1}} f(t_{l-2}, y^{l-2}) (t_{n+1} - \tau)^{\alpha-1} \\ + \int_{t_j}^{t_{j+1}} \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{l-2}) \\ + \int_{t_l}^{t_{l+1}} \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ \times \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \end{array} \right\} \quad (3.23)$$

For the above integrals, we can get

$$\int_l^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^\alpha}{\alpha} [(n-l+1)^\alpha - (n-l)^\alpha] \quad (3.24)$$

$$\int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau \quad (3.25)$$

$$= \frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)} [(n-l+1)^\alpha (n-l+3+2\alpha) - (n-l)^\alpha (n-l+3+3\alpha)]$$

$$\int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} \quad (3.26)$$

$$\left[ \begin{array}{l} (n-l+1)^\alpha [2(n-l)^2 + (3\alpha+10)(n-l)] \\ - (n-l)^\alpha [2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{array} \right]$$

By putting into above equality, the following scheme was obtained,

$$y^{n+1} = y^0 + \frac{(\Delta t)^\alpha}{\alpha(\alpha+1)} \sum_{j=2}^n f(t_{l-2}, y^{l-2}) (n-l+1)^\alpha - (n-l)^\alpha$$

$$\begin{aligned}
& + \frac{(\Delta t)^\alpha}{\alpha(\alpha + 2)} \sum_{l=2}^n f[(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})] \\
& \times [(n-l+1)^\alpha(n-l+3+2\alpha) - (n-l+1)^\alpha(n-l+3+3\alpha)] \tag{3.27} \\
& + \frac{(\Delta t)^2}{2\Gamma(\alpha + 3)} \sum_{l=2}^n f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2}) \\
& \left[ \begin{array}{l} (n-l+1)^\alpha[2(n-l)^2 + (3\alpha + 10)(n-l)] \\ -(n-l)^\alpha[2(n-l)^2 + (5\alpha + 10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{array} \right]
\end{aligned}$$

## **CHAPTER FOUR: ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE.**

The section aims to show analysis of general groundwater flow equation using Caputo-Fabrizio fractional derivative. We will also discuss properties of Caputo-Fabrizio derivative, applications of the derivative, advantages and its limitations, and also discuss the benefits of using this type of derivative. A graph illustrating this type of derivative will also be shown. Stability analysis of the solution will be shown using the Von Neumann method while also proving that the solution exists and is unique.

### **4.1 CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE AND PROPERTIES**

(Caputo and Fabrizio, 2015) derived a new fractional derivative that has a smooth (exponential) kernel which takes into account different representations for the temporal and spatial variable. According to (Caputo and Fabrizio, 2015) the first definition works on time variables and this allows the use of Laplace transform. They further stated that the second definition is associated to spatial variables, by a non-local fractional derivative, and therefore found it is more appropriate to use Fourier transform. One fundamental benefit of using this approach is its ability to describe material heterogeneities and fluctuations of different scales. (Atangana and Alqahtani, 2016) acknowledged that the newly proposed derivative has several interesting properties that are useful for modelling in many branches of sciences. That is something classical methods cannot do.

According to (Atangana and Koca, 2017) previously used fractional derivatives such as Riemann-Liouville and Caputo derivatives have been misused. They further stated that these derivatives were used to model real world problems with no clear explanations. Many thought that Caputo derivative can be used to solve all problems in real world. The concept of power law that was used to justify the use of fractional derivative in solving real world problem cannot always be observed in nature (Atangana and Koca, 2017). In order to overcome these derivatives shortcomings, Caputo-Fabrizio fractional derivative was developed with exponential kernel. However, (Atangana and Koca, 2017) argued that the kernel used for Caputo-Fabrizio derivative was not non-linear and the anti-derivative associate was only the average of the function and it's integral.

**Definition 1.** (Caputo and Fabrizio, 2015) started by showing Caputo fractional time derivative ( $UFD_t$ ) of order  $\alpha$ , given by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} dt \quad (4.1)$$

with  $\alpha \in [0,1]$  and  $a \in [-\infty, t)$ ,  $f \in H'(a, b)$ ,  $b > a$ . By changing the kernel  $(t-\tau)^{-\alpha}$  with the function  $\exp\left(\frac{-\alpha}{1-\alpha}t\right)$  and  $\frac{1}{\Gamma(1-\alpha)}$  with  $\frac{M(\alpha)}{1-\alpha}$ , they obtained the following new definition of fractional time derivative  $NFD_t$

$$\mathcal{D}_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) \exp\left[-\alpha \frac{(t-\tau)}{1-\alpha}\right] d\tau \quad (4.2)$$

where  $M(\alpha)$  is a normalization function such that  $M(0) = M(1) = 1$ . Based on definition(3.2), the  $NFD_t$  is zero when  $f(t)$  is constant, as in the  $UFD_t$ , but, contrary to the  $UFD_t$ , the kernel does not have singularity for  $t = \tau$ . According to (Caputo and Fabrizio, 2015), this new derivative may be applied to functions that do not belong to  $H'(a, b)$ . They further stated that the definition (3.2) can be formulated also for  $f \in L^1(-\infty, b)$  and for any  $\alpha \in [0,1]$  as:

$$\mathcal{D}^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_{-\infty}^t (f(t) - f(\tau)) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \quad (4.3)$$

They observed that by putting,

$$\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty], \alpha = \frac{1}{1+\sigma} \in [0,1]$$

Definition (2.2) assumes the form

$$\tilde{\mathcal{D}}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t \dot{f}(\tau) \exp\left[-\frac{t-\tau}{\sigma}\right] dt \quad (4.4)$$

where  $\sigma \in [0, \infty]$  and  $N(\sigma)$  is the corresponding normalization term of  $M(\alpha)$ , such that  $N(0) = N(\infty) = 1$ . This is due to that:

$$\lim_{\alpha \rightarrow 1} \frac{1}{\sigma} \exp \left[ -\frac{t - \tau}{\sigma} \right] = \delta(t - \tau) \quad (4.5)$$

and for  $\alpha \rightarrow 1$ , we have  $\sigma \rightarrow 0$ .

$$\lim_{\alpha \rightarrow 1} \mathcal{D}_t^\alpha f(t) = \lim_{\alpha \rightarrow 1} \frac{M(\alpha)}{1 - \alpha} \int_a^t f(\tau) \exp \left[ \frac{\alpha(t - \tau)}{1 - \alpha} \right] d\tau \quad (4.6)$$

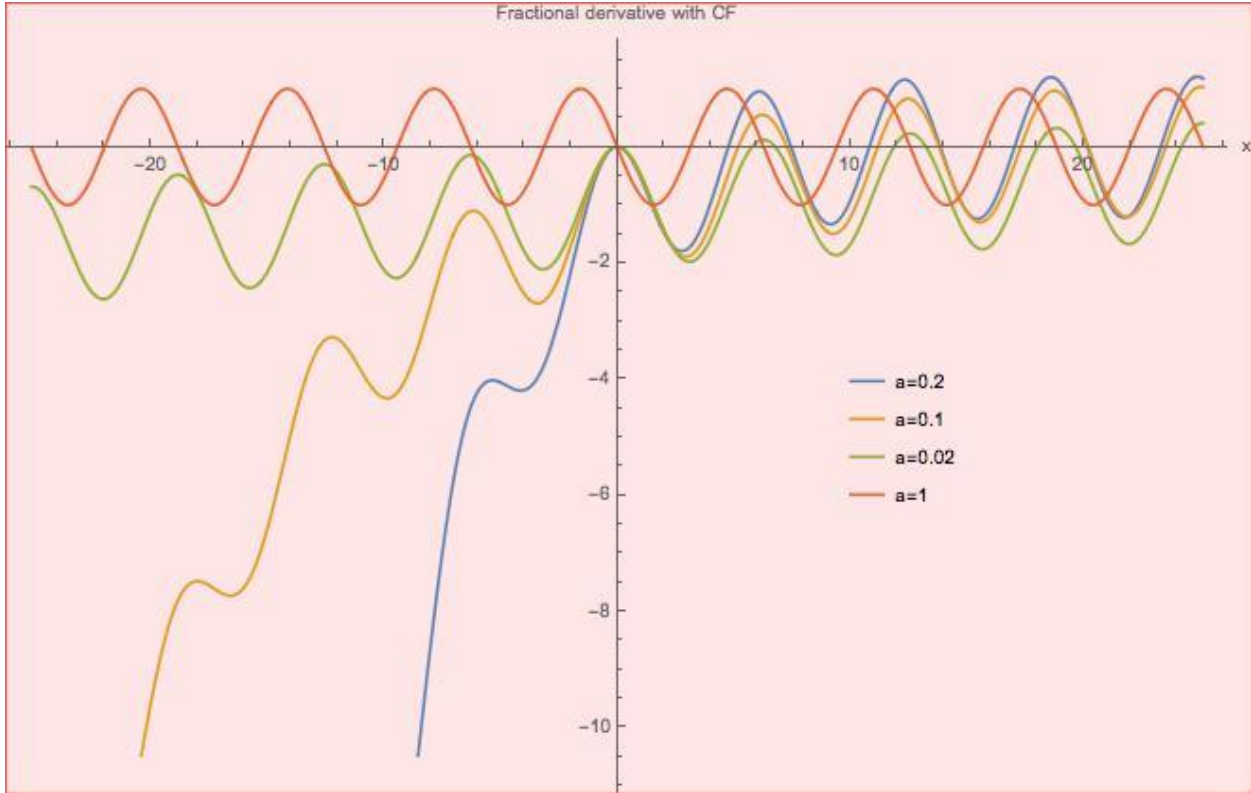
$$\lim_{\sigma \rightarrow 0} \frac{N(\sigma)}{\sigma} \int_a^t \dot{f}(\tau) \exp \left[ -\frac{t - \tau}{\sigma} \right] dt = \dot{f}(t) \quad (4.7)$$

when  $\alpha \rightarrow 0$ , then  $\sigma \rightarrow +\infty$ . Therefore,

$$\lim_{\alpha \rightarrow 0} \mathcal{D}^{(\alpha)} f(t) = \lim_{\alpha \rightarrow 0} \frac{M(\alpha)}{1 - \alpha} \int_a^t \dot{f}(\tau) \exp \left[ -\frac{\alpha(t - \tau)}{1 - \alpha} \right] d\tau \quad (4.8)$$

$$\lim_{\sigma \rightarrow +\infty} \frac{N(\sigma)}{\sigma} \int_a^t \dot{f}(\tau) \exp \left[ -\frac{t - \tau}{\sigma} \right] d\tau = f(t) - f(a) \quad (4.9)$$

The graph below (figure 3) shows the difference between the Caputo-Fabrizio and normal derivative for a simple function.



**Figure 3:** Fractional Derivative with Caputo-Fabrizio Fractional Derivative.

(Losada and Nieto, 2015) studied the properties of Caputo-Fabrizio fractional derivative. According to (Losada and Nieto 2015) the fractional integral of Caputo-Fabrizio must be defined as follows,

Let  $0 < \alpha < 1$ . The fractional integral of order  $\alpha$  of a function  $f$  is defined by,

$${}^{CF}I^\alpha f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u-(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s) ds, \quad t \geq 0 \quad (4.10)$$

Enforcing

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2\alpha)M(\alpha)} \quad (4.11)$$

This led to the following definition;

Let  $0 < \alpha < 1$ . The fractional Caputo-Fabrizio derivative of order  $\alpha$  of a function  $f$  is given by (Losada and Nieto, 2015),

$${}^{\text{CF}}_0D_*^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s) ds, \quad t \geq 0 \quad (4.12)$$

They further studied fractional differential equations and non-linear fractional differential equations using the Caputo fractional derivative.

## 4.2 APPLICATIONS OF THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

Caputo-Fabrizio fractional derivative has been applied in many branches of science. This derivative has been used in real life problems as it is much more suitable than the classical derivative. (Al-Salti *et al.*, 2016) noted that this new derivative has advantageous properties such as that its able to show material heterogeneities and configurations with different scales. That is something that cannot be achieved with previously used popular derivatives. Following are cases where the new Caputo-Fabrizio derivative was applied.

(Atangana and Alqahtani, 2015) applied the newly developed derivative to improve the mathematical model describing the flow of water within a leaky aquifer. They further created numerical simulations that displayed the efficiency of the method. (Goufo, 2016) in his research applied the newly developed Caputo-Fabrizio fractional derivative to explore the possibility of extending the analysis of Korteweg-de-Vries-Burgers equation with two perturbation levels.

By use of Caputo-Fabrizio derivative, Goufo (2016) wanted to get an understanding on the unusual irregularities observed in wave motions and liquids movements. Previously, the Korteweg-de-Vries-Burgers equation was applied to describe and analyze physical contexts related to liquids and wave dynamics, including investigation of propagation of waves in an elastic tube with viscous fluid etc. (Alqahtani and Atangana, 2016) examined the Vallis model for El Nino using fractional derivatives, namely Caputo derivative and Caputo-Fabrizio fractional derivative. Using fractional derivatives, they demonstrated the existence and uniqueness of the solution for the model. These were compared with solutions obtained by local derivatives. Numerical simulations of both local derivatives and fractional derivatives were created and compared. Based on the simulations, (Alqahtani and Atangana 2016) observed that the Caputo-Fabrizio model shows and gives more information when compared to the local derivative model. A chaotic behavior was also observed on the Caputo-Fabrizio model.

(Tateishi *et al.*, 2017) carried out a research on the role of fractional time-derivative operators on anomalous diffusion and their results showed that fractional operators may be a simple and efficient way for incorporating different structural aspects into the system. While investigating general solutions and processes related to the diffusion equation when considering various choices for the kernel  $\mathcal{K}(t)$ , they observed that with Caputo-Fabrizio operator, its connection with continuous time random walk is complex, therefore not compatible with its standard interpretation. The diffusion equation associated with this operator is connected to a diffusive process with stochastic resetting, where the waiting time distribution is exponential (Tateishi *et al.*, 2017; Shkilev, 2017; Mendez and Campos, 2016).

Based on the equation,

$$\frac{\partial}{\partial t} p(x, t) = {}_0D_t^{1-\alpha}(\mathcal{L}\{p(x, t)\}) \quad (4.13)$$

(Tateishi *et al.*, 2017) derived solutions of the above equation using fractional operators.

The solutions were acquired by applying inverse of Fourier and Laplace transforms. For Caputo-Fabrizio operator, they observed that the distribution is more or less the same as a Gaussian for small times, while showing a tent shape for longer times. For  $t \rightarrow \infty$ , a stationery behavior was observed and this was interpreted to correspond with confined diffusion.

This type of behavior is also apparent for the mean square displacement which showed linear behavior in time for small times and saturates for long times. (Evans and Majumdar, 2011) noted that Caputo-Fabrizio fractional operator may be related to a diffusion with stochastic resetting. This was also proved on the work done by Tateishi, Ribeiro and Lenzi in 2015 and Hristov in 2016. (Tateishi, *et al.*, 2017) established an acceptable continuous time random walk formulation by taking into account a density of particle  $J(x, t)$  whose dynamics is governed by,

$$\begin{aligned} J(x, t) = & \delta(t)\delta(x) + r\delta(x) \int_0^t dt' \omega(t')J(x, t - t') \\ & + (1 - r) \int_0^t dt' \Psi(x', t')J(x - x', t - t'). \end{aligned} \quad (4.14)$$

where  $r$  is a resetting rate,  $\Psi(x, t)$  is joint distribution of jump length and waiting time,  $\lambda x = \int_0^\infty dt \Psi(x, t)$  is the jump length distribution, and  $\omega(t) = \int_{-\infty}^\infty dx \Psi(x, t)$  is waiting time distribution. They also concluded that the diffusion equation with Caputo-Fabrizio operator leads to the same waiting time distribution of the usual diffusion which is an exponential.

### **4.3 ANALYSIS OF GENERAL GROUNDWATER FLOW WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE.**

In this section, we are going to analyze the general groundwater flow equation for confined aquifer using Caputo-Fabrizio fractional derivative. Discretization will be done using a new novel method called the Newton Polynomial. Stability analysis will also be conducted on the discretized formula.

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial h}{\partial r} + \frac{\partial h^2}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (4.14)$$

$${}^c D_0^{\alpha} f(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s) ds, t \geq 0 \quad (4.15)$$

In order to analyze the equation, we then discretize using the Newton polynomial as follows;

$$\frac{dy(t)}{dt} = f(t, y(t)) \quad (4.16)$$

$$y(t) - y(0) = \int_0^t f(t, y(\tau)) d\tau \quad (4.17)$$

at  $t = t_{n+1}$

$$y(t_{n+1}) - y(0) = \int_{t_n}^{t_{n+1}} f(\tau, y(\tau)) d\tau \quad (4.18)$$

at  $t = t_n$

$$y(t_n) - y(0) = \int_0^{t_n} f(\tau, y(\tau)) d\tau \quad (4.19)$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(\tau, y(\tau)) d\tau \quad (4.20)$$

$$P_n(\tau) = F(t_{n-2}, y(t_{n-2})) \cdot \frac{F(t_{n-1}, y(t_{n-1})) - F(t_{n-2}, y(t_{n-2}))(\tau - t_{n-2})}{\Delta t} + \frac{F(t_n, y(t_n)) - 2F(t_{n-1}, y(t_{n-1})) + F(t_{n-2}, y(t_{n-2}))(\tau - t_{n-1})(\tau - t_{n-2})}{2(\Delta t)^2} \quad (4.21)$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} P_n(\tau) d\tau \quad (4.22)$$

$$y(t_{n+1}) - y(t_n) = \frac{5}{12} F((t_{n-2}))\Delta t - \frac{4}{3} \Delta t \left[ F(t_{n-1}, y(t_{n-1}))\Delta t + \frac{23}{12} F(t_n, y(t_n)) \right] \quad (4.23)$$

thus

$$y(t_{n+1}) - y(t_n) = \frac{1-\alpha}{M(\alpha)} [F(t_n, y(t_n)) - F(t_{n-1}, y(t_{n-1}))] \quad (4.24)$$

$$+ \frac{\alpha}{M(\alpha)} \left[ \frac{5}{12} F((t_{n-2}))\Delta t - \frac{4}{3} \Delta t \left[ F(t_{n-1}, y(t_{n-1}))\Delta t + \frac{23}{12} F(t_n, y(t_n)) \right] \right]$$

If we want to apply it in our equation, then;

$${}^{cf}D_t^\alpha h(r, t) = \frac{T}{S} \left[ \frac{1}{r} \frac{\partial h(r, t)}{\partial r} + \frac{\partial^2 h(r, t)}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \right] \quad (4.25)$$

$${}^{cf}D_t^\alpha h(r, t) = F(t, r, h(r, t)) \quad (4.26)$$

Applying the Caputo-Fabrizio integral, we have

$$h(r, t) - h(r, 0) = \frac{1-\alpha}{M(\alpha)} F(t, r, h(r, t)) + \frac{\alpha}{M(\alpha)} \int_0^t F(r, \tau, h(r, \tau)) d\tau \quad (4.27)$$

at  $t = t_{n+1}$  and  $t = t_n$ , following the derivation presented earlier, we have

$$h(r, t_{n+1}) - h(r, t_n) = \frac{1-\alpha}{M(\alpha)} [F(r, t_n, h(r, t_n)) - F(r, t_{n-1}, h(r, t_{n-1}))]$$

$$+ \frac{\alpha}{M(\alpha)} \left[ \frac{5}{12} \Delta t F(r, t_{n-2}, h(r, t_{n-2})) - \frac{4}{3} F(r, t_{n-1}, h(r, t_{n-1})) \right. \quad (4.28)$$

$$\left. + \frac{23}{12} F(r, t_n, h(r, t_n)) \right]$$

Now at  $r = r_i$ , we have

$$\begin{aligned}
h(r_i, t_{n+1}) - h(r_i, t_n) &= \frac{1 - \alpha}{M(\alpha)} [F(r_i, t_n, h(r_i, t_n)) - F(r_i, t_{n-1}, h(r_i, t_{n-1}))] \\
&+ \frac{\alpha}{M(\alpha)} \left[ \frac{5}{12} \Delta t F(r_i, t_{n-2}, h(r_i, t_{n-2})) - \frac{4}{3} F(r_i, t_{n-1}, h(r_i, t_{n-1})) \right. \\
&\quad \left. + \frac{23}{12} F(r_i, t_n, h(r_i, t_n)) \right]
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
F(r_i, t_n, h(r_i, t_n)) &= \frac{T}{S} \left[ \frac{1}{r_i} \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \right] \\
&+ \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2}
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
F(r_i, t_{n-2}, h(r_i, t_{n-2})) &= \frac{T}{S} \left[ \frac{1}{r_i} \frac{h(r_{i+1}, t_{n-2}) - h(r_{i-1}, t_{n-2})}{2\Delta r} \right] \\
&+ \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_{n-2}) - 2h(r_i, t_{n-2}) + h(r_{i-1}, t_{n-2})}{\Delta r^2}
\end{aligned} \tag{4.31}$$

We now replace into our equation,

$$\begin{aligned}
h(r_i, t_{n+1}) - h(r_i, t_n) &= \frac{1 - \alpha T}{M(\alpha) S} \left[ \frac{1}{r_i} \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \right] \\
&+ \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2} \\
&- \left( \frac{T}{S} \left[ \frac{1}{r_i} \frac{h(r_{i+1}, t_{n-1}) - h(r_{i-1}, t_{n-1})}{2\Delta r} \right] \right. \\
&\quad \left. + \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_{n-1}) - 2h(r_i, t_{n-1}) + h(r_{i-1}, t_{n-1})}{\Delta r^2} \right) \\
&+ \frac{\alpha}{M(\alpha)} \left[ \frac{5}{12} \Delta t \frac{T}{S} \left( \frac{1}{r_i} \frac{h(r_{i+1}, t_{n-2}) - h(r_{i-1}, t_{n-2})}{2\Delta r} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_{n-2}) - 2h(r_i, t_{n-2}) + h(r_{i-1}, t_{n-2})}{\Delta r^2} \\
& \quad - \frac{4}{3} \left( \frac{T}{S} \left[ \frac{1}{r_i} \frac{h(r_{i+1}, t_{n-1}) - h(r_{i-1}, t_{n-1})}{2\Delta r} \right] \right) \\
& + \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_{n-1}) - 2h(r_i, t_{n-1}) + h(r_{i-1}, t_{n-1})}{\Delta r^2} \\
& \quad + \frac{23T}{12S} \left[ \frac{1}{r_i} \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \right] \\
& + \left[ 1 + \frac{\Delta r}{r_i} \right] \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2}
\end{aligned} \tag{4.32}$$

by factorization, we now put like terms together;

$$\begin{aligned}
h(r_i, t_{n+1}) & = h(r_i, t_n) \left( 1 - \frac{2h}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{1 - \alpha T}{M(\alpha) S} - \frac{2h}{\Delta r^2} \frac{\Delta r}{r_i} \right) \\
& + h(r_i + 1, t_n) \left( \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{1 - \alpha T}{M(\alpha) S} + \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{23T}{12S} \right) \\
& - h(r_{i-1}, t_n) \left( \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{1 - \alpha}{M(\alpha)} - \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{23T}{12S} \right) \\
& - h(r_{i+1}, t_{n-1}) \left( \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \frac{T}{S} + \frac{1}{\Delta r^2} \frac{\Delta r}{r_i} \frac{\alpha}{M(\alpha)} - \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \frac{\Delta r}{r_i} \frac{4T}{3S} \right) \\
& - h(r_{i-1}, t_{n-1}) \left( \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{T}{S} - \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{4T}{3S} \right) \\
& - 2h(r_i, t_{n-1}) \left( \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{T}{S} - \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r} \right) \frac{4T}{3S} \right) \\
& + h(r_{i+1}, t_{n-2}) \left( \frac{1}{2\Delta r} \left( \frac{\alpha}{M(\alpha)} \right) \right)
\end{aligned} \tag{4.33}$$

$$- h(r_{i-1}, t_{n-2}) \left( \frac{1}{2\Delta r} + \frac{1}{\Delta r^2} \left( \frac{\Delta r}{r_i} \right) \frac{\alpha}{M(\alpha)} \right)$$

Simplification in terms of alpha yields,

$$\begin{aligned} \alpha_1 h(r_i, t_{n+1}) &= \alpha_2 h(r_i, t_n) + \alpha_3 h(r_{i+1}, t_n) - \alpha_3 h(r_{i-1}, t_n) \\ &- \alpha_4 h(r_{i+1}, t_{n-1}) - \alpha_5 h(r_{i-1}, t_{n-1}) - \alpha_5 2h(r_i, t_{n-1}) \\ &+ \alpha_6 h(r_{i+1}, t_{n-2}) - \alpha_7 h(r_{i-1}, t_{n-2}) \end{aligned} \quad (4.34)$$

Hence,

$$h_i^n = \delta_n e^{ik_m x} \quad (4.35)$$

$$h_{i-1}^{n-2} = \delta_{n-2} e^{ik_m(x-\Delta r)} \quad (4.36)$$

$$h_i^{n-2} = \delta_{n-2} e^{ik_m x} \quad (4.37)$$

$$h_{i+1}^n = \delta_n e^{ik_m(x+\Delta r)} \quad (4.38)$$

$$h_{i-1}^n = \delta_n e^{-ik_m x} \quad (4.39)$$

By replacing the above to our solution, we get

$$\begin{aligned} \alpha_1 \delta_{n+1} e^{ik_m x} &= \alpha_2 \delta_n e^{ik_m x} + \alpha_3 \delta_n e^{ik_m(x+\Delta r)} - \alpha_3 \delta_n e^{-ik_m(x-\Delta r)} \\ &- \alpha_4 \delta_{n-1} e^{ik_m(x+\Delta r)} - \alpha_5 \delta_{n-1} e^{-ik_m(x-\Delta r)} - \alpha_5 2\delta_{n-1} e^{ik_m x} \\ &+ \alpha_6 \delta_{n-2} e^{ik_m(x+\Delta r)} - \alpha_7 \delta_{n-2} e^{-ik_m(x-\Delta r)} \end{aligned} \quad (4.40)$$

Simplification yields,

$$\alpha_1 \delta_{n+1} = \alpha_2 \delta_n + \alpha_3 \delta_n e^{ik_m \Delta r} - \alpha_3 \delta_n e^{-ik_m \Delta r}$$

$$\begin{aligned}
& -\alpha_4\delta_{n-1}e^{ik_m\Delta r} - \alpha_5\delta_{n-1}e^{-ik_m\Delta r} - \alpha_5 2\delta_{n-1} \\
& + \alpha_6\delta_{n-2}e^{ik_m\Delta r} - \alpha_7\delta_{n-2}e^{-ik_m\Delta r}
\end{aligned} \tag{4.41}$$

Putting like terms together and factorization, we get;

$$\begin{aligned}
\delta_{n+1}(\alpha_1) &= \delta_n(\alpha_2 + \alpha_3e^{ik_m\Delta r} - \alpha_3e^{-ik_m\Delta r}) \\
& - \delta_{n-1}(\alpha_4e^{ik_m\Delta r} - \alpha_5e^{-ik_m\Delta r} - 2\alpha_5) \\
& + \delta_{n-2}(\alpha_6e^{ik_m\Delta r} - \alpha_7e^{-ik_m\Delta r})
\end{aligned} \tag{4.42}$$

Therefore,

$$\begin{aligned}
\alpha_1\delta_{n+1} &= \delta_n(\alpha_2 + \alpha_3(2isink_m\Delta r)) \\
& - \delta_{n-1}(\alpha_4(\cosk_m\Delta r + isink_m\Delta r) - \alpha_5(\cosk_m\Delta r + isink_m\Delta r) - 2\alpha_5) \\
& + \delta_{n-2}[\alpha_6(\cosk_m\Delta r + isink_m\Delta r) - \alpha_7(\cos(k_m\Delta r) - isin(k_m\Delta r))]
\end{aligned} \tag{4.43}$$

Since,

$$e^{ik_m\Delta r} = \cos(k_m\Delta r) + \sin(k_m\Delta r)$$

$$e^{-ik_m\Delta r} = \cos(k_m\Delta r) - \sin(k_m\Delta r)$$

$$\begin{aligned}
\alpha_1\delta_{n+1} &= \delta_n(\alpha_2 + 2i\alpha_3\sin(k_m\Delta r)) \\
& - \delta_{n-1}[(\alpha_4 - \alpha_5)\cosk_m\Delta r - 2\alpha_5 + i(\alpha_4 - \alpha_5)\sin(k_m\Delta r)] \\
& + \delta_{n-2}[(\alpha_6 - \alpha_7)\cos(k_m\Delta r) + i(\alpha_6 - \alpha_7)\sin(k_m\Delta r)]
\end{aligned} \tag{4.44}$$

When  $n = 0$ , we have

$$\alpha_1\delta_1 = \delta_0(\alpha_2 + 2i\alpha_3)\sin(k_m\Delta r) \tag{4.45}$$

Hence,

$$\left| \frac{\delta_1}{\delta_0} \right| = \left| \frac{\alpha_2 + 2i\alpha_3 \sin(k_m \Delta r)}{\alpha_1} \right| \quad (4.46)$$

$$= \frac{\sqrt{\alpha_2^2 + (2\alpha_3 \sin k_m \Delta r)^2}}{|\alpha_1|} \quad (4.47)$$

$$\left| \frac{\delta_1}{\delta_0} \right| < 1 \Rightarrow \sqrt{\alpha_2^2 + (2\alpha_3 \sin k_m \Delta r)^2} < |\alpha_1| \quad (4.48)$$

We assume that  $\forall_n \geq 1$

If  $\left| \frac{\delta_n}{\delta_0} \right| < 1$  then, we want to prove that  $\left| \frac{\delta_{n+1}}{\delta_0} \right| < 1$

However,

$$\begin{aligned} |\delta_{n+1}| &\leq |\delta_n| |\alpha_2 + 2i\alpha_3 \sin(k_m \Delta r)| \\ &+ |\delta_{n-1}| |(\alpha_4 - \alpha_5) \cos k_m \Delta r - 2\alpha_5 + i(\alpha_4 + \alpha_5) \sin(k_m \Delta r)| \\ &+ |\delta_{n-2}| |(\alpha_6 - \alpha_7) \cos(k_m \Delta r) + i(\alpha_6 - \alpha_7) \sin(k_m \Delta r)| \end{aligned} \quad (4.49)$$

However, according to the inductive formula,

$$|\delta_n| < |\delta_0|, |\delta_{n-1}| < |\delta_0| \text{ and } |\delta_{n-2}| < |\delta_0|$$

Therefore,

$$|\delta_{n+1}| < |\delta_0| \left[ \frac{\sqrt{\alpha_2^2 + (2\alpha_3 \sin k_m \Delta r)^2}}{|\alpha_1|} + \sqrt{((\alpha_4 - \alpha_5) \cos(k_m \Delta r) - 2\alpha_5)^2 + (\alpha_4 + \alpha_5)^2 \sin^2(k_m \Delta r)} + \sqrt{(\alpha_6 - \alpha_7) \cos(k_m \Delta r)^2 + ((\alpha_6 - \alpha_7) \sin(k_m \Delta r))^2} \right] \quad (4.50)$$

$$\frac{|\delta_{n+1}|}{|\delta_0|} < 1 \quad (4.51)$$

The scheme is stable if

$$\text{Max} \left\{ \frac{\sqrt{\alpha^2 + (2\alpha_3 \sin k_m \Delta r)^2}}{|\alpha_1|}; \left( \begin{array}{l} \sqrt{\alpha^2 + (2\alpha_3 \sin k_m \Delta r)^2} \\ + \sqrt{((\alpha_4 - \alpha_5) \cos(k_m \Delta r) - 2\alpha_5)^2 + (\alpha_4 + \alpha_5)^2 \sin^2(k_m \Delta r)} \\ + \sqrt{(\alpha_6 - \alpha_7) \cos(k_m \Delta r)^2 + ((\alpha_6 - \alpha_7) \sin(k_m \Delta r))^2} \end{array} \right) \right\} \quad (4.52)$$

The above completes our stability.

## CHAPTER FIVE: ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH ATANGANA-BALEANU FRACTIONAL DERIVATIVE.

The section discusses the newly developed Atangana-Baleanu fractional derivative. Applying Atangana-Baleanu fractional operator, we will analyze general groundwater flow equation for confined aquifer. Properties of this new operator, where it has been applied, comparison with other fractional derivatives as well as its main advantages will be discussed under this section.

### 5.1 ATANGANA-BALEANU FRACTIONAL DERIVATIVE.

(Atangana and Baleanu, 2016) proposed a fractional derivative that is based upon the generalized Mittag-Leffler function, since the Mittag-Leffler function is more suitable in expressing nature than power function. Atangana-Baleanu fractional derivative is an improvement to what had been previously developed by various authors. Recently, (Caputo and Fabrizio, 2015) had developed a fractional derivative which is a great improvement to the classical fractional derivatives. It has many added advantages when compared to Riemann-Liouville and Caputo fractional derivatives that had been earlier developed. There are certain limitations that has been noted by different authors. For instance, Alqahtani (2016) observed that the kernel utilized was not non-local and the anti-derivative associated is an average of the function and its integral. Therefore, it was concluded that the operator was a filter than a fractional derivative (Alqahtani, 2016). (Atangana and Baleanu, 2016) suggested a new fractional derivative with no singular and non-local kernel. The new fractional derivative overcomes all challenges that could not be solved using previously developed fractional derivatives. In their work published in 2016, they defined the new fractional derivative as follows;

**Definition 1.** Let  $f \in H^1(a, b)$ ,  $b > a$ ,  $\alpha \in [0, 1]$  then, the new fractional derivative was defined as;

$${}^{ABC}D_t^\alpha(f(t)) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx. \quad (5.1)$$

Furthermore, Atangana and Baleanu (2016) stressed that definition 1 will aid in addressing real world problems and has great advantage when using Laplace transform to solve physical problems that has initial conditions. They also observed that when alpha ( $\alpha$ ) is zero (0), the original function may not be recovered except when at the origin the function disappears. The second definition aims at overcoming such a problem.

**Definition 2.** Let  $f \in H'(a, b)$ ,  $b > a$ ,  $\alpha \in [0, 1]$  then, the new fractional derivative was defined as,

$${}^{ABR}D_t^\alpha(f(t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_b^t f(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx. \quad (5.2)$$

The novelty about their work is that the above equations have a non-local kernel.

**Definition 3.** The fractional integral associate to the new Atangana-Baleanu fractional derivative was defined as,

$${}^{AB}D_t^\alpha\{f(t)\} = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(y)(t-y)^{\alpha-1} dy. \quad (5.3)$$

Initial function is recovered when  $\alpha$  becomes zero but when  $\alpha$  is 1, ordinary integral is obtained.

## 5.2 PROPERTIES OF THE ATANGANA-BALEANU FRACTIONAL DERIVATIVE.

Here, properties of the newly developed Atangana-Baleanu fractional derivative are discussed as written on their work which was published in thermal science journal in 2016.

Initially, equation (6.1) and equation (6.2) were analyzed using the Laplace transform. Atangana and Baleanu (2017) found that,

$$\mathcal{L}\{{}^{ABR}D_t^\alpha(f(t))\}(p) = \frac{B(\alpha)}{1-\alpha} \frac{p^\alpha \mathcal{L}\{f(t)\}(p)}{p^\alpha + \frac{\alpha}{1-\alpha}} \quad (5.4)$$

and

$$\mathcal{L}\{{}^{ABC}_0D_t^\alpha(f(t))\}(p) = \frac{B(\alpha)}{1-\alpha} \frac{p^\alpha \mathcal{L}\{f(t)\}(p) - p^{\alpha-1}f(0)}{p^\alpha + \frac{\alpha}{1-\alpha}} \quad (5.5)$$

The authors then developed the following theorem;

**Theorem 1.** Let  $f \in H'(a, b), b > a, \alpha \in [0,1]$  then,

$${}^{ABC}_0D_t^\alpha(f(t)) = {}^{ABR}_0D_t^\alpha(f(t)) + H(t) \quad (5.6)$$

**Proof:** Definition 6.6 and the Laplace transform was applied on both sides to obtain the following;

$$\mathcal{L}\{{}^{ABC}_0D_t^\alpha(f(t))\}(p) = \frac{B(\alpha)}{1-\alpha} \frac{p^\alpha \mathcal{L}\{f(t)\}(p)}{p^\alpha + \frac{\alpha}{1-\alpha}} - \frac{p^{\alpha-1}f(0)}{p^\alpha + \frac{\alpha}{1-\alpha}} \frac{B(\alpha)}{1-\alpha} \quad (5.7)$$

Based on the equation 6.4, we get:

$$\mathcal{L}\{{}^{ABC}_0D_t^\alpha(f(t))\}(p) = \mathcal{L}\{{}^{ABR}_0D_t^\alpha(f(t))\}(p) - \frac{p^{\alpha-1}f(0)}{p^\alpha + \frac{\alpha}{1-\alpha}} \frac{B(\alpha)}{1-\alpha} \quad (5.8)$$

Atangana and Baleanu applied the inverse Laplace on both sides of equation (6.8):

$${}^{ABC}_0D_t^\alpha(f(t)) = {}^{ABR}_0D_t^\alpha(f(t)) - \frac{B(\alpha)}{1-\alpha} f(0) E_\alpha\left(-\frac{\alpha}{1-\alpha} t^\alpha\right) \quad (5.9)$$

The above completes the proof.

**Theorem 2.** Let  $f$  be a continuous function on a closed interval  $[a, b]$ . The following inequality is obtained on  $[a, b]$ ,

$$\|{}^{ABR}_0D_t^\alpha(f(t))\| < \frac{B(\alpha)}{1-\alpha} k, \|h(t)\| = \max_{a \leq t \leq b} |h(t)| \quad (5.10)$$

**Proof:**

$$\begin{aligned} \|{}^{ABR}_0D_t^\alpha(f(t))\| &= \left\| \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t f(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx \right\| \\ &< \frac{B(\alpha)}{1-\alpha} \left\| \frac{d}{dt} \int_0^t f(x) dx \right\| = \frac{B(\alpha)}{1-\alpha} \|f(x)\|. \end{aligned} \quad (5.11)$$

Taking  $k$  to be  $\|f(x)\|$  the proof is completed.

Theorem 3. According to (Atangana and Baleanu, 2016) their derivative in Riemann and Caputo sense possess the Lipschitz condition, meaning that for a given couple function  $f$  and  $h$ , subsequent inequalities may be established:

$$\|{}^{ABR}_0D_t^\alpha(f(t)) - {}^{ABR}_0D_t^\alpha(h(t))\| \leq H\|f(t) - h(t)\| \quad (5.12)$$

and

$$\|{}^{ABC}_0D_t^\alpha(f(t)) - {}^{ABC}_0D_t^\alpha(h(t))\| \leq H\|f(t) - h(t)\| \quad (5.13)$$

**Proof:**

$$\begin{aligned} &\|{}^{ABR}_0D_t^\alpha(f(t)) - {}^{ABR}_0D_t^\alpha(h(t))\| = \\ &\left\| \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t f(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx - \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t h(x) E_\alpha \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx \right\|. \end{aligned} \quad (5.14)$$

They used the Lipschitz condition of the first order derivative to find a small positive constant such that:

$$\begin{aligned} \|{}^{ABR}_0D_t^\alpha(f(t)) - {}^{ABR}_0D_t^\alpha(h(t))\| &< \frac{B(\alpha)\theta_1}{1-\alpha} E_\alpha \left[ -\alpha \frac{t^\alpha}{1-\alpha} \right] \|f(x) - h(x)\| t \\ &= H\|f(x) - h(x)\|, \end{aligned} \quad (5.15)$$

which produced the requested result.

Let  $f$  be an  $n$  – times differentiable with natural number and  $f^{(k)}(0) = 0, k = 1, 2, 3 \dots n$ , through inspection they obtained,

$${}^{ABC}D_t^\alpha \left( \frac{d^n f(t)}{dt^n} \right) = \frac{d^n}{dt^n} \left( {}^{ABR}D_t^\alpha (f(t)) \right) \quad (5.16)$$

By applying inverse Laplace transform and using convolution theorem, Atangana and Baleanu (2016) proved that the time fractional ordinary differential equation:

$${}^{ABC}D_t^\alpha (f(t)) = u(t) \quad (5.17)$$

has unique solution which is obtained as:

$$f(t) = \frac{1 - \alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t u(y)(t - y)^{\alpha-1} dy \quad (5.18)$$

### 5.3 APPLICATION OF ATANGANA-BALEANU DERIVATIVE.

In this section, we are going to discuss application of Atangana-Baleanu fractional derivative on the general diffusion equation and its comparison to other fractional derivatives based on the work which was conducted by Tateishi, Ribeiro and Lenzi.

(Tateishi *et al.*, 2017) investigated the role of fractional time-derivative operators on anomalous diffusion which was published on *Frontiers in Physics* journal. In their work, they compared the classical Riemann-Liouville fractional operator with the newly developed fractional derivatives with non-singular memory kernels, namely Caputo-Fabrizio and Atangana-Baleanu fractional operators. The three fractional time operators were used to generalize the usual diffusion equation. Initially, they investigated a one-dimensional diffusive process which is best described by the general fractional diffusion equation. Three different forms for the kernel  $\mathcal{K}(t)$  (singular and non-singular) were considered for the different fractional operators. The Atangana-Baleanu operator corresponded to the following kernel,

$$\mathcal{K}(t) = bE_{\alpha}\left(-\frac{\alpha t^{\alpha}}{1-\alpha}\right) \quad (5.19)$$

where  $E_{\alpha}(\dots)$  is the Mittag-Leffler function.

The authors observed that both Caputo-Fabrizio and Atangana-Baleanu fractional operators are non-singular operators. Secondly, they investigated general solutions and processes related with diffusion equation when considering different choices (singular and non-singular) for the kernel. (Tateishi *et al.*, 2017) noted that Atangana-Baleanu operator yields a cross over between a stretched exponential and power-law distribution. Based on their simulations and findings, it was observed that for long times, the operators of Riemann-Liouville and Atangana-Baleanu yield the same power-law decay for  $\omega(t)$ . Furthermore, they noted that the Atangana-Baleanu yields a non-divergent  $\omega(t)$ , a feature that is not observed for the singular kernel of Riemann-Liouville. While finding the formal solutions for the fractional diffusion equation using the Atangana-Baleanu operator, the profile of  $p(x, t)$  resembles a Gaussian for small times, while displaying a tent-shape for long times. Moreover, the distribution does not have a stationary solution for Atangana-Baleanu operator. There is also a cross over between usual and sub-diffusion which can also be observed in several biological systems.

In their summary of the changes caused by Atangana-Baleanu operator, (Tateishi *et al.*, 2017) observed that for the waiting time distribution, they observed a cross over from stretched exponential to power-law, meanwhile for the mean square displacement, they observed a crossover from the usual to sub-diffusion. Lastly, for the probability distribution, they noted a crossover from Gaussian to non-Gaussian.

#### **5.4 ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH ATANGANA-BALEANU FRACTIONAL DERIVATIVE**

The section aims at analyzing general groundwater flow equation for confined aquifer using Atangana-Baleanu fractional operator. A numerical scheme will be developed and then applied on our general equation. Stability analysis will be conducted and numerical simulations developed.

$$\frac{S}{T} \cdot \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial h}{\partial r} + \frac{\partial h^2}{\partial r^2} \left[ 1 + \frac{\Delta r}{r} \right] \quad (5.20)$$

$${}^{ABC}D_t^\alpha y(t) = f(t, y(t)) \quad (5.21)$$

Atangana-Baleanu integral was applied to the above equation (5.21) into the following,

$$y(t) - y(0) = \frac{1 - \alpha}{AB(\alpha)} f(t, y(t)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t f(\tau, y(\tau))(t - \tau)^{\alpha-1} d\tau \quad (5.22)$$

at  $t_{n+1} = (n + 1)\Delta t$ , we get

$$\begin{aligned} y(t_{n+1}) - y(0) &= \frac{1 - \alpha}{AB(\alpha)} f(t_n, y(t_n)) \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y(\tau))(t_{n+1} - \tau)^{\alpha-1} d\tau \end{aligned} \quad (5.23)$$

It can also be expressed as,

$$y(t_{n+1}) = y(0) + \frac{1 - \alpha}{AB(\alpha)} f(t_n, y(t_n)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} f(\tau, y(\tau)) d\tau \quad (5.24)$$

The approximation of the function  $f(t, y(t))$ , we use the Newton polynomial which is given by,

$$\begin{aligned} P_n(\tau) &= f(t_{n-2}, y(t_{n-2})) + \frac{f(t_{n-1}, y(t_{n-1})) - f(t_{n-2}, y(t_{n-2}))}{\Delta t} (\tau - t_{n-2}) \\ &+ \frac{f(t_n, y(t_n)) - 2f(t_{n-1}, y(t_{n-1})) + f(t_{n-2}, y(t_{n-2}))}{2(\Delta t)^2} \times (\tau - t_{n-2})(\tau - t_{n-2}) \end{aligned} \quad (5.25)$$

Taking the above polynomial to equation (5.24), we get

$$\begin{aligned}
y^{n+1} &= y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} \left\{ \begin{aligned} &f(t_{l-2}, y^{l-2}) + \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{n-2}) \\ &+ \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ &\quad \times (\tau - t_{l-2})(\tau - t_{l-1}) \\ &\quad (t_{n+1} - \tau)^{\alpha-1} d\tau \end{aligned} \right\} \quad (5.26)
\end{aligned}$$

The above can also be rearranged into,

$$\begin{aligned}
y^{n+1} &= y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \left\{ \begin{aligned} &\int_{t_l}^{t_{l+1}} f(t_{l-2}, y^{l-2}) (t_{n+1} - \tau)^{\alpha-1} d\tau \\ &+ \frac{\int_{t_l}^{t_{l+1}} f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \\ &+ \int_{t_l}^{t_{l+1}} \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ &\quad \times (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \end{aligned} \right\} \quad (5.27)
\end{aligned}$$

Thus,

$$\begin{aligned}
y^{n+1} &= y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n f(t_{l-2}, y^{l-2}) \int_{t_l}^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} \\
&\quad \times \int_{t_l}^{t_{l+1}} (\tau - t_{l-2}) (t_{n+1} - \tau)^{\alpha-1} d\tau \quad (5.28)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\
& \quad \times \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau
\end{aligned}$$

Calculating the above integrals gives us,

$$\int_{t_l}^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^\alpha}{\alpha} [(n-l+1)^\alpha - (n-l)^\alpha] \quad (5.29)$$

$$\int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)} \left[ \begin{aligned} & (n-l+1)^\alpha(n-l+3+2\alpha) \\ & - (n-l)^\alpha(n-l+3+3\alpha) \end{aligned} \right] \quad (5.30)$$

$$\begin{aligned}
& \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} \\
& \times \left[ \begin{aligned} & (n-l+1)^\alpha[2(n-l)^2 + (3\alpha+10)(n-l)] \\ & - (n-l)^\alpha[2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{aligned} \right] \quad (5.31)
\end{aligned}$$

By putting all equalities into above scheme, the following scheme can be derived,

$$\begin{aligned}
& y^{n+1} = y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
& + \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+1)} \sum_{l=2}^n f(t_{l-2}, y^{l-2}) [(n-l+1)^\alpha - (n-l)^\alpha] \\
& + \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{l=2}^n [f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})] \\
& [(n-l+1)^\alpha(n-l+3+2\alpha) - (n-l)^\alpha(n-l+3+3\alpha)]
\end{aligned} \quad (5.32)$$

$$\frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+3)} \sum_{l=2}^n [(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})]$$

$$\times \left[ \begin{array}{l} (n-l+1)^\alpha [2(n-l)^2] + (3\alpha+10)(n-l) - (n-l)^\alpha \\ [2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{array} \right]$$

Here,

$$y^{n+1} = h(r, t_{n+1}) \quad (5.33)$$

$$f(t_n, y(t_n)) = f(r, t_n, h(r, t_n)) \quad (5.34)$$

$$f(t_{l-2}, y^{l-2}) = f(r, t_{l-2}, h(r, t_{l-2})) \quad (5.35)$$

We now discretize in space,

$$x_i = \Delta x i$$

$$f(r_i, t_n, h(r_i, t_n)) = \frac{T}{S} \left[ \begin{array}{l} \frac{1}{r_i} \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \\ + \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right] \quad (5.36)$$

$$f(r_i, t_{l-2}, h(r_i, t_{l-2})) = \frac{T}{S} \left[ \begin{array}{l} \frac{1}{r_i} \frac{h(r_{i+1}, t_{l-2}) - h(r_{i-1}, t_{l-2})}{2\Delta r} \\ + \frac{h(r_{i+1}, t_{l-2}) - 2h(r_i, t_{l-2}) + h(r_{i-1}, t_{l-2})}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right] \quad (5.37)$$

$$f(r_i, t_{l-1}, h(r_i, t_{l-1})) = \frac{T}{S} \left[ \begin{array}{l} \frac{1}{r_i} \frac{h(r_{i+1}, t_{l-1}) - h(r_{i-1}, t_{l-1})}{2\Delta r} \\ + \frac{h(r_{i+1}, t_{l-1}) - 2h(r_i, t_{l-1}) + h(r_{i-1}, t_{l-1})}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right] \quad (5.38)$$

## **CHAPTER SIX: ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH FRACTAL-FRACTIONAL DERIVATIVE.**

In this section, we are going to discuss fractal-fractional derivative. Applying the fractal-fractional operator, we will analyze the general groundwater flow equation. The fractal-fractional operators to be used are Caputo fractal-fractional derivative and Atangana-Baleanu fractal-fractional derivative. Their properties and application will be discussed. To fully understand fractal-fractional derivatives, it is important to define both derivatives separately. This will allow us to fully understand their properties and applications thus far.

### **6.1 FRACTAL-FRACTIONAL DERIVATIVE.**

The fractal derivative is a Non-Newtonian type of derivative in which the variable such as  $t$  has been scaled according to  $t^\alpha$ . This derivative was defined in fractal geometry. Porous media, aquifer, turbulence and other media exhibit fractal properties. Physical concepts such as distance and velocity in fractal media are required to be redefined; the scales for space and time should be transformed according to  $(x^\beta, t^\alpha)$  (Wikipedia, 2012). The elementary physical concepts such as velocity in a fractal spacetime  $(x^\beta, t^\alpha)$  can be redefined by:

$$V' = \frac{dx'}{dt'} = \frac{dx^\beta}{dt^\alpha}, \quad \alpha, \beta > 0 \quad (6.1)$$

Where  $S^{\alpha, \beta}$  represents the fractal spacetime with scaling indices  $\alpha$  and  $\beta$  (Wikipedia, 2012).

The concept of the fractal derivative of a function  $u(t)$ , with respect to a fractal measure  $t$  was developed as follows;

$$\frac{\partial f(t)}{\partial (t)^\alpha} = \lim_{t_1 \rightarrow t} \frac{f(t_1) - f(t)}{t_1^\alpha - t^\alpha}, \quad \alpha > 0 \quad (6.2)$$

A more generalized definition was given by,

$$\frac{\partial^\beta f(t)}{\alpha t^\alpha} = \lim_{t_1 \rightarrow t} \frac{f^\beta(t_1) - f^\beta(t)}{t_1^\alpha - t^\alpha}, \quad \alpha > 0, \beta > 0 \quad (6.3)$$

According to (Matlob and Jamali, 2017) fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Fractional calculus is a field of mathematics that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer values (Dalir and Bashour, 2010). Different scholars defined fractional derivatives as follows;

1. L. Euler (1730):

Euler generalized the formula,

$$\frac{d^n x^m}{dx^n} m(m-1) \dots (m-n+1) x^{m-n} \quad (6.4)$$

By using of the following property of gamma function,

$$\Gamma(m+1) = m(m-1) \dots (m-n+1) x^{m-n} \quad (6.5)$$

To obtain,

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \quad (6.6)$$

Gamma function is defined as follows,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re}(z) > 0 \quad (6.7)$$

2. J. B. J Fourier (1820-1822):

By means of integral representation, he wrote

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) dz \int_{-\infty}^{\infty} \cos\left(px - pz + n\frac{\pi}{2}\right) dp \quad (6.8)$$

3. N. H Abel (1823-1826):

Abel considered the integration representation  $\int_0^{\infty} \frac{s'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x)$  for arbitrary  $\alpha$  and then wrote,

$$s(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}} \quad (6.9)$$

4. L. Liouville (1832-1855):

I. In his first definition, according to exponential representation of a function  $f(x) = \sum_{n=0}^{\infty} c_n e^{\alpha_n x}$ , he generalized the formula  $\frac{d^m e^{\alpha x}}{dx^m} = a^m e^{\alpha x}$  as,

$$\frac{d^v f(x)}{dx^v} = \sum_{n=0}^{\infty} c_n a_n^v e^{\alpha_n x} \quad (6.10)$$

II. His second type of definition was fractional integral

$$\int^{\mu} \Phi(x) dx^{\mu} = \frac{1}{(-1)^{\mu}\Gamma(\mu)} \int_0^{\infty} \Phi(x+\alpha)\alpha^{\mu-1} d\alpha \quad (6.11)$$

$$\int^{\mu} \Phi(x) dx^{\mu} = \frac{1}{\Gamma(\mu)} \int_0^{\infty} \Phi(x-\alpha)\alpha^{\mu-1} d\alpha \quad (6.12)$$

By substituting of  $r = x + \alpha$  and  $r = x - \alpha$  in the above, the following was obtained,

$$\int^{\mu} \Phi(x) dx^{\mu} = \frac{1}{(-1)^{\mu} \Gamma(\mu)} \int_0^{\infty} \Phi(r - \alpha)^{\mu-1} \Phi(r) dr \quad (6.13)$$

$$\int^{\mu} \Phi(x) dx^{\mu} = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{\infty} \Phi(x - r)^{\mu-1} \Phi(r) dr \quad (6.14)$$

III. The third definition includes fractional derivative,

$$\frac{d^{\mu} f(x)}{dx^{\mu}} = \frac{1}{h^{\mu}} \left( f(x) \frac{\mu}{1} f(x+h) + \frac{\mu(\mu-1)}{1.2} f(x+2h) - \dots \right) \quad (6.15)$$

$$\frac{d^{\mu} f(x)}{dx^{\mu}} \frac{1}{h^{\mu}} \left( f(x) \frac{\mu}{1} f(x-h) + \frac{\mu(\mu-1)}{1.2} f(x-2h) - \dots \right) \quad (6.16)$$

5. G. F. B Riemann (1847-1876)

His definition of fractal integral is,

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t) dt + \psi(t) \quad (6.17)$$

6. N.YA. Sonin (1869), A.V Letnikov (1872), H, Laurent (1884), N. Nekrasove (1888), K. Nishimoto (1987):

Using the Cauchy Integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_c \frac{f(t)}{(t-z)^{n+1}} dt \quad (6.18)$$

7. Riemann Liouville

The popular definition of fractional calculus shows joining of two previous definitions,

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(r)dr}{(t-r)^{\alpha-n+1}}, \quad (n-1 \leq \alpha < n) \quad (6.19)$$

8. Grunwald-Letnikov

Below is another joined definition which may be useful at times,

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^j \binom{\alpha}{j} f(t-jh) \quad (6.20)$$

9. M. Caputo (1967)

This is the second most popular definition,

$${}_a^c D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^n(r)dr}{(t-r)^{\alpha+1-n}}, \quad (n-1 \leq \alpha < n) \quad (6.21)$$

10. K. S. Miller, B. Ross (1993)

Differential operator  $D$  was used as

$$D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \quad (6.22)$$

Which  $D^{\alpha_i}$  is Riemann-Liouville or Caputo definitions.

## 6.2 PROPERTIES OF THE FRACTAL-FRACTIONAL DERIVATIVE.

Properties of fractals include;

1. Expansion Coefficients

Similar to the Taylor series expansion, the coefficients  $l_m$  can be expressed in terms of the fractal derivatives of order  $m$  of  $f$ :

$$l_m = \frac{1}{m!} \left( \frac{d}{dx^\alpha} \right)^m f(x = x_0) \quad (6.23)$$

Proof idea: assuming  $\left( \frac{d}{dx^\alpha} \right)^m f(x = x_0)$  exists,  $l_m$  can be written as  $l_m = a_m \cdot \left( \frac{d}{dx^\alpha} \right)^m f(x = x_0)$

We can now use  $f(x) = (x^\alpha - x_0^\alpha)^n \Rightarrow \left( \frac{d}{dx^\alpha} \right)^k f(x = x_0) = n! \delta_n^k$  and  $l_n = 1 \Rightarrow a_n = \frac{1}{n!}$

## 2. Connection with derivative

If for a given function  $f$  both the derivative  $df$  and the fractal derivative  $D_\alpha f$  exists, we can find an analog to the chain rule;

$$\frac{df}{dx^\alpha} = \frac{df}{dx} \frac{dx}{dx^\alpha} = \frac{1}{\alpha} x^{1-\alpha} \frac{df}{dx} \quad (6.24)$$

The last step was motivated by the implicit function theorem which, under appropriate conditions, gives us  $\frac{dx}{dx^\alpha} = \frac{dx^\alpha}{dx^{1-\alpha}}$ .

Similarly, for the more general definition

$$\frac{d^\beta f}{d^\alpha x} = \frac{d(f^\beta)}{d^\alpha x} = \frac{1}{\alpha} x^{1-\alpha} \beta f^{\beta-1}(x) f'(x) \quad (6.25)$$

## 6.3 APPLICATION OF FRACTAL-FRACTIONAL DERIVATIVE.

(Atangana and Araz, 2019) developed a new numerical method for ordinary and partial differential equations, including those with non-integers called Newton Polynomial. In their work, Atangana and Araz mentioned that as much as Adam Bash forth method have been recognized to be a very efficient numerical method to solve linear and nonlinear orders, the method was developed using the Lagrange interpolation, which is less accurate as compared to their newly developed Newton Polynomial.

The new method was further used with Caputo-Fabrizio fractal-fractional derivative, Atangana-Baleanu fractal-fractional derivative and with Caputo fractal-fractional derivative. Numerical simulations using various examples were shown.

#### 6.4 ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH ATANGANA-BALEANU FRACTAL-FRACTIONAL DERIVATIVE.

General groundwater flow equation for confined aquifer will be analyzed using the Atangana-Baleanu Fractal-Fractional derivative. Discretization will be conducted using the Newton Polynomial. Stability analyses will be shown. Numerical simulations will be developed using the new scheme.

By considering the following Cauchy problem,

$${}^{FFM}D_t^{\alpha,\beta} y(t) = f(t, y(t)) \quad (6.26)$$

By integrating the Atangana-Baleanu fractal-fractional derivative, we get

$$\begin{aligned} y(t) - y(0) &= \frac{1 - \alpha}{AB(\alpha)} \beta t^{\beta-1} f(t, y(t)) \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \int_0^t \tau^{\beta-1} f(\tau, y(\tau)) (t - \tau)^{\alpha-1} d\tau \end{aligned} \quad (6.27)$$

by taking  $F(t, y(t)) = \beta t^{\beta-1} f(t, y(t))$ , the above equation can be reformulated as

$$\begin{aligned} y(t) - y(0) &= \frac{1 - \alpha}{AB(\alpha)} f(t, y(t)) \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \int_0^t f(\tau, y(\tau)) (t - \tau)^{\alpha-1} d\tau \end{aligned} \quad (6.28)$$

at the point  $t_{n+1} = (n + 1) \Delta t$ , we have

$$\begin{aligned}
y(t_{n+1}) - y(0) &= \frac{1 - \alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y(\tau))(t_{n+1} - \tau)^{\alpha-1} d\tau
\end{aligned} \tag{6.29}$$

We also get,

$$\begin{aligned}
y(t_{n+1}) &= y(0) + \frac{1 - \alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} f(\tau, y(\tau))(t_{n+1} - \tau)^{\alpha-1} d\tau
\end{aligned} \tag{6.30}$$

Newton polynomial was used to approximate the function  $f(\tau, y(\tau))$  as follows,

$$\begin{aligned}
p_n(\tau) &= f(t_{n-2}, y(t_{n-2})) + \frac{f(t_{n-1}, y(t_{n-1})) - f(t_{n-2}, y(t_{n-2}))}{\Delta t} (\tau - t_{n-2}) \\
&+ \frac{f(t_n, y(t_n)) - 2f(t_{n-1}, y(t_{n-1})) + f(t_{n-2}, y(t_{n-2}))}{2(\Delta t)^2} \\
&\quad \times (\tau - t_{n-2})(\tau - t_{n-1})
\end{aligned} \tag{6.31}$$

By putting polynomial into equation (6.6), we derive it as

$$\begin{aligned}
y^{n+1} &= y^0 + \frac{1 - \alpha}{AB(\alpha)} f(t_n, y(t_n)) \\
&+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} \left\{ \begin{aligned} &+ \frac{f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{n-2}) \\ &+ \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ &\quad \times (\tau - t_{l-2})(\tau - t_{l-1}) \end{aligned} \right\}
\end{aligned} \tag{6.32}$$

$$(t_{n+1} - \tau)^{\alpha-1} d\tau$$

If we order the above equality, we obtain

$$y^{n+1} = y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} \left\{ \begin{array}{l} \frac{f(t_{l-2}, y^{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau}{\Delta t} \\ + \int_{t_l}^{t_{l+1}} \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} \\ \times (\tau - t_{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau \\ + \int_{t_l}^{t_{l+1}} \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ \times (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \end{array} \right\} \quad (6.33)$$

and write the following,

$$y^{n+1} = y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n f(t_{l-2}, y^{l-2}) \int_{t_l}^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} \times \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{l=2}^n \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \times \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \quad (6.34)$$

The above integrals were calculated as,

$$\int_{t_l}^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^\alpha}{\alpha} [(n-l+1)^\alpha - (n-l)^\alpha] \quad (6.35)$$

$$\int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)} \left[ (n-l+1)^\alpha(n-l+3+2\alpha) - (n-l)^\alpha(n-l+3+3\alpha) \right] \quad (6.36)$$

$$\int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} \quad (6.37)$$

$$\times \begin{bmatrix} (n-l+1)^\alpha [2(n-l)^2 + (3\alpha+10)(n-l) + 2\alpha^2 + 9\alpha + 12] \\ -(n-l)^\alpha [2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{bmatrix}$$

by replacing into the above scheme, we get

$$\begin{aligned} y^{n+1} &= y^0 + \frac{1-\alpha}{AB(\alpha)} f(t_n, y(t_n)) \\ &+ \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+1)} \sum_{l=2}^n f(t_{l-2}, y^{l-2}) (n-l+1)^\alpha - (n-l)^\alpha \\ &+ \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{l=2}^n [f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})] \\ &\times [(n-l+1)^\alpha(n-l+3+2\alpha) - (n-l)^\alpha(n-l+3+3\alpha)] \quad (6.38) \\ &+ \frac{\alpha(\Delta t)^\alpha}{2AB(\alpha)\Gamma(\alpha+3)} \sum_{l=2}^n [f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})] \\ &\times \begin{bmatrix} (n-l+1)^\alpha [2(n-l)^2 + (3\alpha+10)(n-l) + 2\alpha^2 + 9\alpha + 12] \\ -(n-l)^\alpha [2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{bmatrix} \end{aligned}$$

The following approximation is thus obtained,

$$\begin{aligned}
y^{n+1} &= y^0 + \frac{1-\alpha}{AB(\alpha)} \beta t_n^{\beta-1} f(t_n, y(t_n)) \\
&+ \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+1)} \sum_{l=2}^n t_{j-2}^{\beta-1} f(t_{l-2}, y^{l-2}) [(n-l+1)^\alpha - (n-l)^\alpha] \\
&+ \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{l=2}^n t_{l-2}^{\beta-1} (t_{l-1}, y^{l-1}) - t_{l-2}^{\beta-1} (t_{l-2}, y^{l-2}) \\
&\times [(n-l+1)^\alpha (n-l+3+2\alpha) - (n-l)^\alpha (n-l+3+3\alpha)] \\
&+ \frac{\alpha\beta(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+3)} \sum_{l=2}^n t_l^{\beta-1} f(t_n, y(t_n)) - 2t_{l-2}^{\beta-1} f(t_{l-1}, y^{l-1}) + t_{l-2}^{\beta-1} (t_{l-2}, y^{l-2}) \\
&\left[ \begin{aligned} &[(n-l+1)^\alpha [2(n-l)^2 + (3\alpha+10)(n-l) + 2\alpha^2 + 9\alpha + 12]] \\ &[-(n-l)^\alpha [2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12]] \end{aligned} \right]
\end{aligned} \tag{6.39}$$

We now discretize in space,

$$\begin{aligned}
x_i &= \Delta xi \\
f(r_i, t_n, h(r_i, t_n)) &= \frac{T}{S} \left[ \begin{aligned} &\frac{1}{r_i} \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \\ &+ \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n))}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{aligned} \right] \\
f(r_i, t_{l-2}, h(r_i, t_{l-2})) &= \frac{T}{S} \left[ \begin{aligned} &\frac{1}{r_i} \frac{h(r_{i+1}, t_{l-2}) - h(r_{i-1}, t_{l-2})}{2\Delta r} \\ &+ \frac{h(r_{i+1}, t_{l-2}) - 2h(r_i, t_{l-2}) + h(r_{i-1}, t_{l-2}))}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{aligned} \right] \\
f(r_i, t_{l-1}, h(r_i, t_{l-1})) &= \frac{T}{S} \left[ \begin{aligned} &\frac{1}{r_i} \frac{h(r_{i+1}, t_{l-1}) - h(r_{i-1}, t_{l-1})}{2\Delta r} \\ &+ \frac{h(r_{i+1}, t_{l-1}) - 2h(r_i, t_{l-1}) + h(r_{i-1}, t_{l-1}))}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{aligned} \right]
\end{aligned}$$

## 6.5 ANALYSIS OF GENERAL GROUNDWATER FLOW EQUATION WITH CAPUTO FRACTAL-FRACTIONAL DERIVATIVE.

We are now going to analyze general groundwater flow equation for confined aquifer using the Caputo fractal-fractional derivative. Similar to previous sections, we will apply the Newton polynomial to discretize our formula. This method has been developed by Atangana and Araz (2019) as an addition to Adam Bash forth that has been used by many scholars in their respective researches.

Analyses begins by handling the following Cauchy problem,

$${}^{FFP}D_t^{\alpha,\beta} y(t) = f(t, y(t)) \quad (6.40)$$

Integrating equation (6.13), we get

$$y(t) - y(0) = \frac{\alpha\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} f(\tau, y(\tau)) (t - \tau)^{\alpha-1} d\tau \quad (6.41)$$

By taking  $F(t, y(t)) = \beta t^{\beta-1} f(t, y(t))$ . Thus, we get

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau, y(\tau)) (t - \tau)^{\alpha-1} d\tau \quad (6.42)$$

where the function  $F$  is non-linear. A numerical scheme to solve the above was provided. At the point  $t_{n+1} = (n + 1) \Delta t$ , we have

$$y(t_{n+1}) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y(\tau)) (t_{n+1} - \tau)^{\alpha-1} d\tau \quad (6.43)$$

Hence,

$$y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} f(\tau, y(\tau)) (t_{n+1} - \tau)^{\alpha-1} d\tau \quad (6.44)$$

The Newton polynomial was then substituted to the Caputo Fractal-fractional derivative, we get

$$y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \int_{t_l}^{t_{l+1}} \left\{ \begin{array}{l} f(t_{l-2}, y^{l-2}) \\ + \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{l-2}) \\ + \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ \times (\tau - t_{l-2})(\tau - t_{l-1}) \\ (t_{n+1} - \tau)^{\alpha-1} d\tau \end{array} \right\} \quad (6.45)$$

The above equation can also be written as,

$$y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \left\{ \begin{array}{l} \int_{t_l}^{t_{l+1}} f(t_{l-2}, y^{l-2}) \\ + \int_{t_l}^{t_{l+1}} \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} (\tau - t_{l-2}) (t_{n+1} - \tau)^{\alpha-1} d\tau \\ + \int_{t_l}^{t_{l+1}} \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\ \times (\tau - t_{l-2})(\tau - t_{l-1}) (t_{n+1} - \tau)^{\alpha-1} d\tau \end{array} \right\} \quad (6.46)$$

Thus, the following equality is obtained

$$y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n f(t_{l-2}, y^{l-2}) \int_{t_l}^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau \\ + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \frac{f(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})}{\Delta t} \int_{t_l}^{t_{l+1}} (\tau - t_{l-2}) (t_{n+1} - \tau)^{\alpha-1} d\tau$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \sum_{l=2}^n \frac{f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})}{2(\Delta t)^2} \\
& \quad \times \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau
\end{aligned} \tag{6.47}$$

The above integrals are then calculated as follows;

$$\begin{aligned}
& \int_{t_l}^{t_{l+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^\alpha}{\alpha} [(n-l+1)^\alpha - (n-l)^\alpha] \\
& + \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)} \left[ \begin{array}{l} (n-l+1)^\alpha(n-l+3+2\alpha) \\ -(n-l)^\alpha(n-l+3+3\alpha) \end{array} \right] \\
& + \int_{t_l}^{t_{l+1}} (\tau - t_{l-2})(\tau - t_{l-1})(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+2)} \\
& \left[ (n-l+1)^\alpha \left[ \begin{array}{l} 2(n-l)^2 + (3\alpha+10)(n-l) + 2\alpha^2 + 9\alpha + 12 \\ -(n-l)^\alpha [2(n-l)^2 + (5\alpha+10)(n-l)^\alpha + 6\alpha^2 + 18\alpha + 12] \end{array} \right] \right]
\end{aligned} \tag{6.48}$$

When replacing them into above equality, we have the following,

$$\begin{aligned}
y(t_{n+1}) & = y(0) + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)} \sum_{l=2}^n f(t_{l-2}, y^{l-2}) [(n-l+1)^\alpha - (n-l)^\alpha] \\
& + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{l=2}^n [(t_{l-1}, y^{l-1}) - f(t_{l-2}, y^{l-2})] \\
& \quad \times \left[ \begin{array}{l} (n-l+1)^\alpha(n-l+3+2\alpha) \\ -(n-l)^\alpha(n-l)^\alpha(n-l+3+3\alpha) \end{array} \right] \\
& + \frac{(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{l=2}^n [f(t_l, y^l) - 2f(t_{l-1}, y^{l-1}) + f(t_{l-2}, y^{l-2})]
\end{aligned} \tag{6.49}$$

$$\begin{bmatrix} (n-l+1)^\alpha [2(n-l)^2 + (3\alpha+10)(n-l) + 2\alpha^2 + 9\alpha + 12] \\ -(n-l)^\alpha [2(n-l)^2 + (5\alpha+10)(n-l) + 6\alpha^2 + 18\alpha + 12] \end{bmatrix}$$

The above completes the analysis.

We now discretize in space,

$$x_i = \Delta x_i$$

$$f(r_i, t_n, h(r_i, t_n)) = \frac{T}{S} \left[ \begin{array}{c} \frac{1}{r_i} \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \\ + \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right]$$

$$f(r_i, t_{l-2}, h(r_i, t_{l-2})) = \frac{T}{S} \left[ \begin{array}{c} \frac{1}{r_i} \frac{h(r_{i+1}, t_{l-2}) - h(r_{i-1}, t_{l-2})}{2\Delta r} \\ + \frac{h(r_{i+1}, t_{l-2}) - 2h(r_i, t_{l-2}) + h(r_{i-1}, t_{l-2})}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right]$$

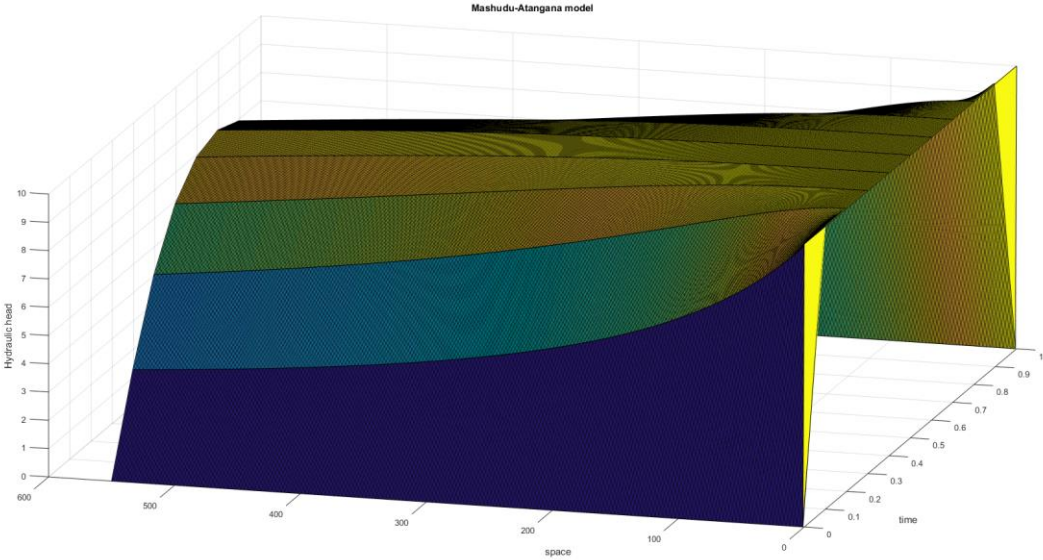
$$f(r_i, t_{l-1}, h(r_i, t_{l-1})) = \frac{T}{S} \left[ \begin{array}{c} \frac{1}{r_i} \frac{h(r_{i+1}, t_{l-1}) - h(r_{i-1}, t_{l-1})}{2\Delta r} \\ + \frac{h(r_{i+1}, t_{l-1}) - 2h(r_i, t_{l-1}) + h(r_{i-1}, t_{l-1})}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \end{array} \right]$$

## CHAPTER SEVEN: NUMERICAL SIMULATIONS AND DISCUSSION.

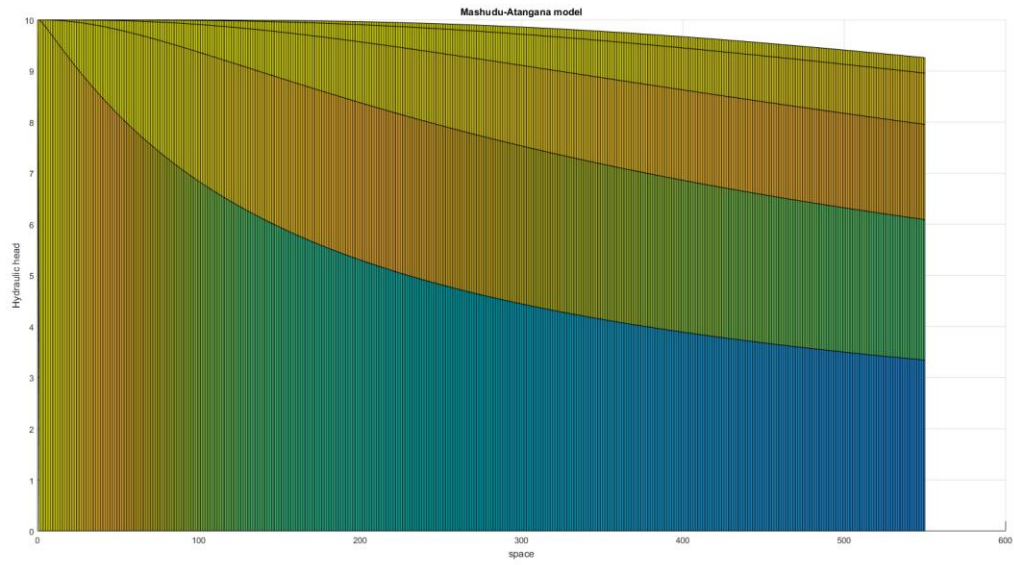
Classical differential and integral operators have been used and applied in many fields of science, technology and engineering. While many new technological achievements were accomplished using these differential and integral operators, while many discoveries were done from these. One will recognize that, many problems in nature were unable to be depicted due to limitations in concept of rate of change. In general, we shall recognize that these operators are the foundation of classical mechanics. They can only be used to replicate processes following the Markovian process. In general, Markovian processes are random processes for which the future depends upon the most recent values, in this case the initial condition and the generators even play a very important role to determine the future events. Such differential and integral operators do not include into mathematical formula, the effect of heterogeneities, memory, elasticity and many other physical problems with complex behaviors. They cannot therefore be used for complex systems, for instance, the flow of groundwater within a confined geological formation. While the Theis groundwater flow model has been used in many situations, one has to first acknowledge that such model was derived from the heat flow. The heat flow within a homogeneous media, therefore the memory and elastic properties are not requested to be included. More importantly, the Theis model is nothing more than the approximation of the heat flow model as all terms were not included in the mathematical equation. In 2017, Mashudu and Atangana introduced, a more general groundwater flow model which considers the forgotten terms by Theis; however, their model uses the concept of classical differential operators and therefore could not help capture complexities of the media through which the sub-surface water flows. The model was extended to the concept of non-local operators including Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Atangana-Baleanu fractional derivative and fractal-fractional differential operators. New numerical methods were used to solve the new models.

In this section then, numerical simulations are depicted in the following figures below. The figures suggest three different types of flow according to the fractional orders. For fractional order from 0.99 to 0.51, one can see the depiction of slow flow, which means the geological formation, does not really connect, helping water to pass through easily.

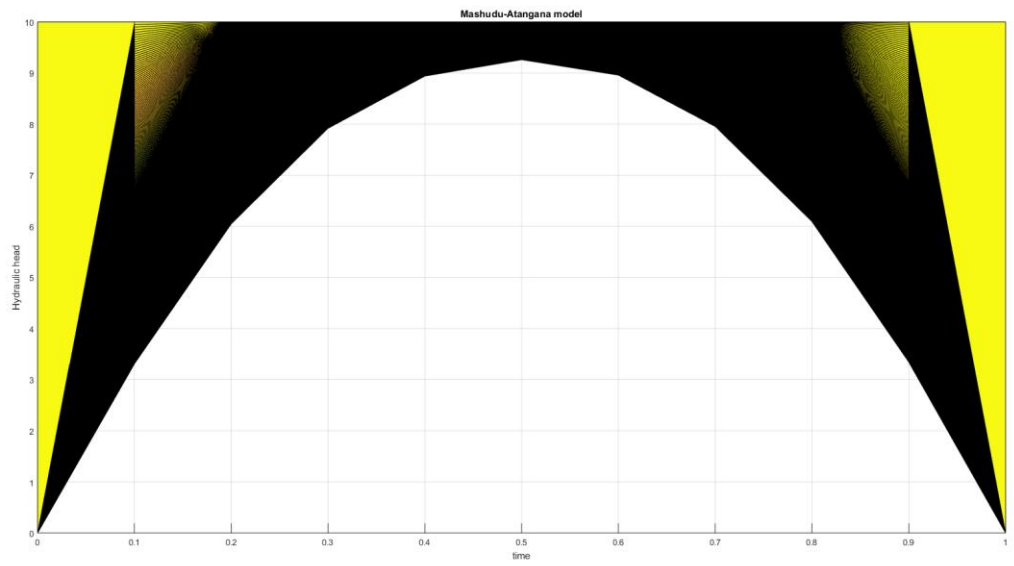
This situation can be observed in real world situations, and sometime researchers change storativity or transmissivity to match the mathematical formula to the observed fact, such results could be misleading, however, here the fractional order help the mathematical formula to capture such physical behavior. When the fractional order is below 0.5, we can observe a fast flow and finally when the fractional order is 1, we recover normal flow where the geological formation is assumed to be homogenous. Below, figure with T = time, S = space and H = hydraulic head.



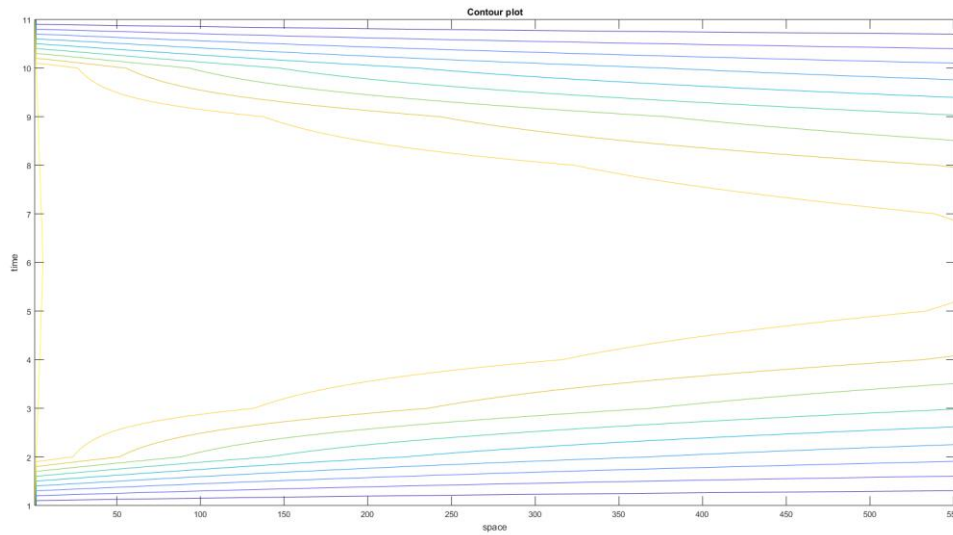
**Figure 4:** Numerical Simulation of Caputo with theoretical value of 1 showing H, S, and T.



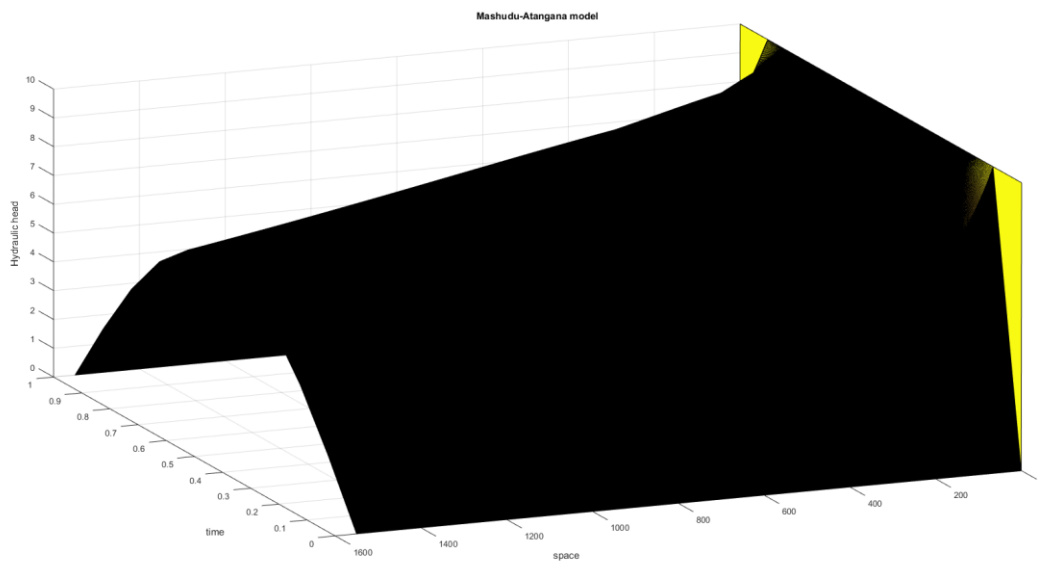
**Figure 5:** Numerical Simulation of Caputo with theoretical value of 1 showing H and S.



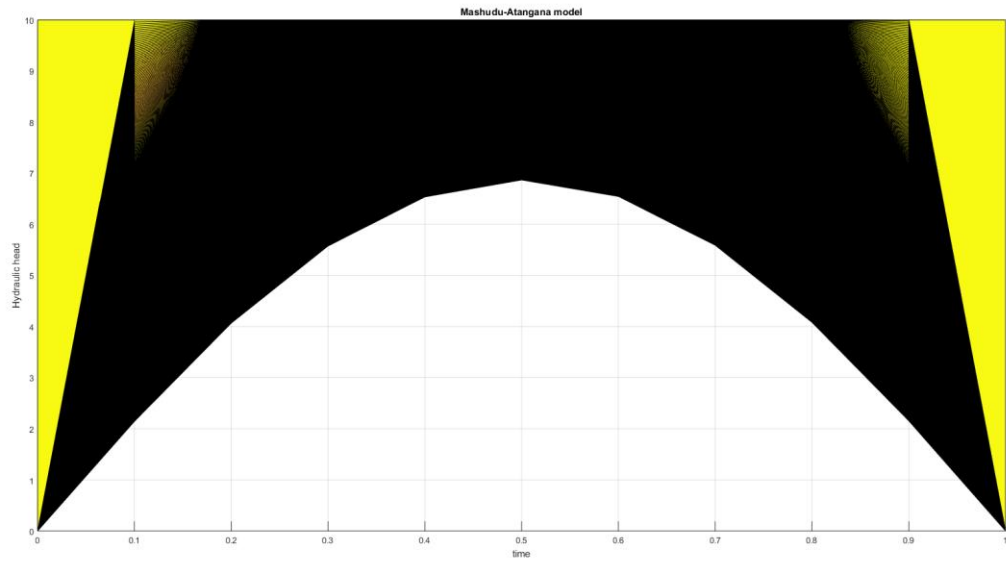
**Figure 6:** Numerical Simulation of Caputo with theoretical value of 1 showing H and T.



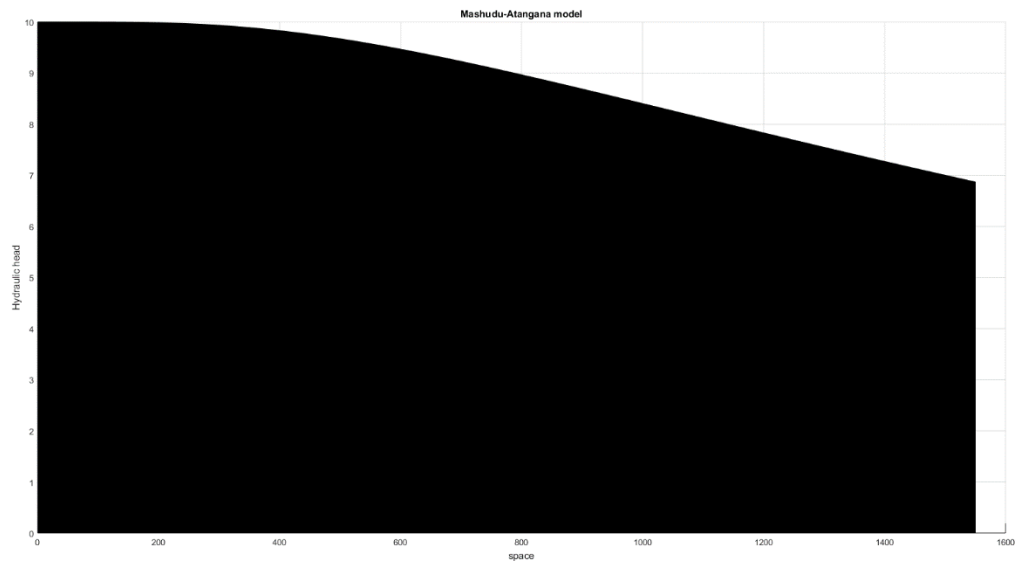
**Figure 7:** Contour Plot of Caputo with theoretical value of 1.



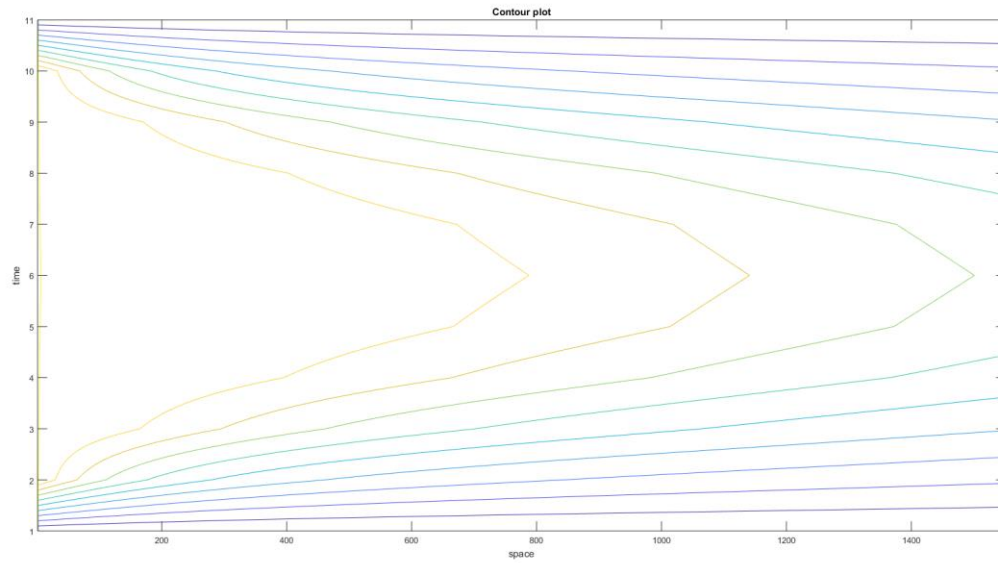
**Figure 8:** Numerical Simulation of Caputo with theoretical value of 0.8 showing H, S and T.



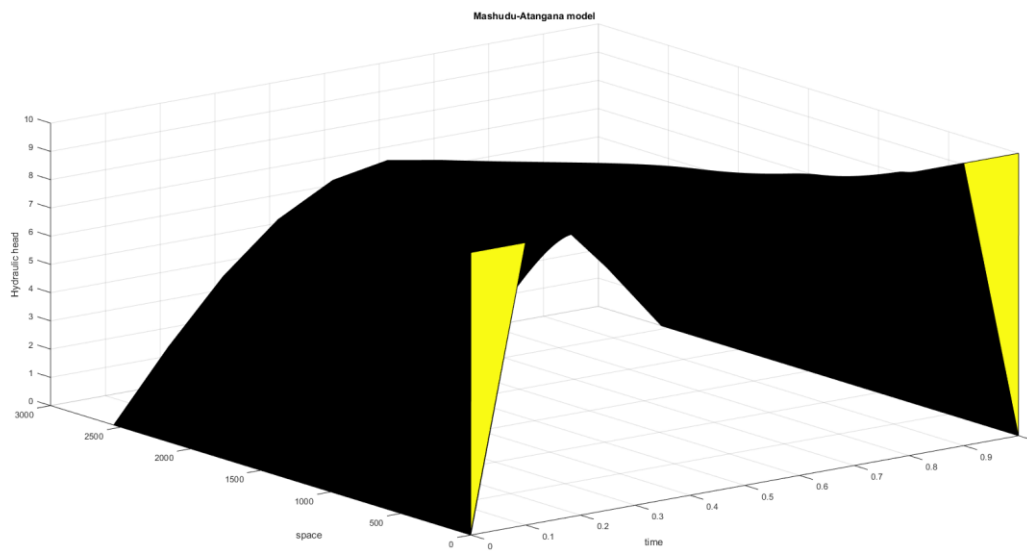
**Figure 9:** Numerical Simulation of Caputo with theoretical value of 0.8 showing H and T.



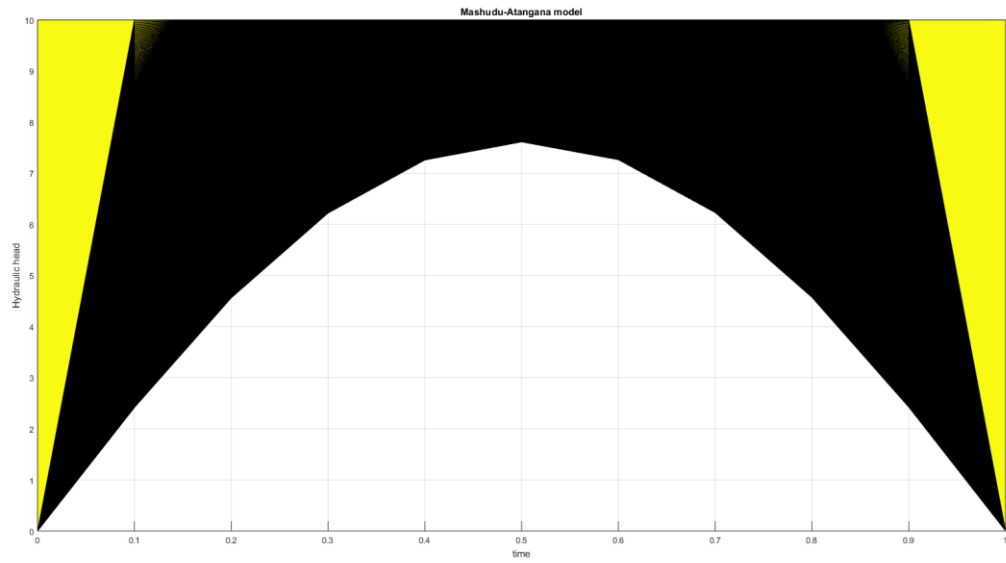
**Figure 10:** Numerical Simulation of Caputo with theoretical value of 0.8 showing H and S.



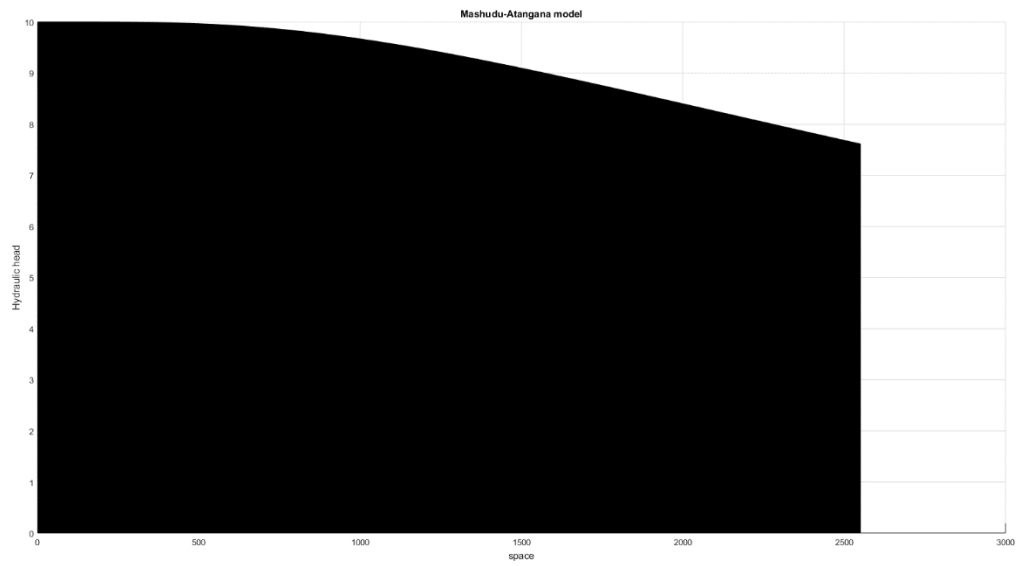
**Figure 11:** Contour Plot of Caputo with theoretical value of 0.8.



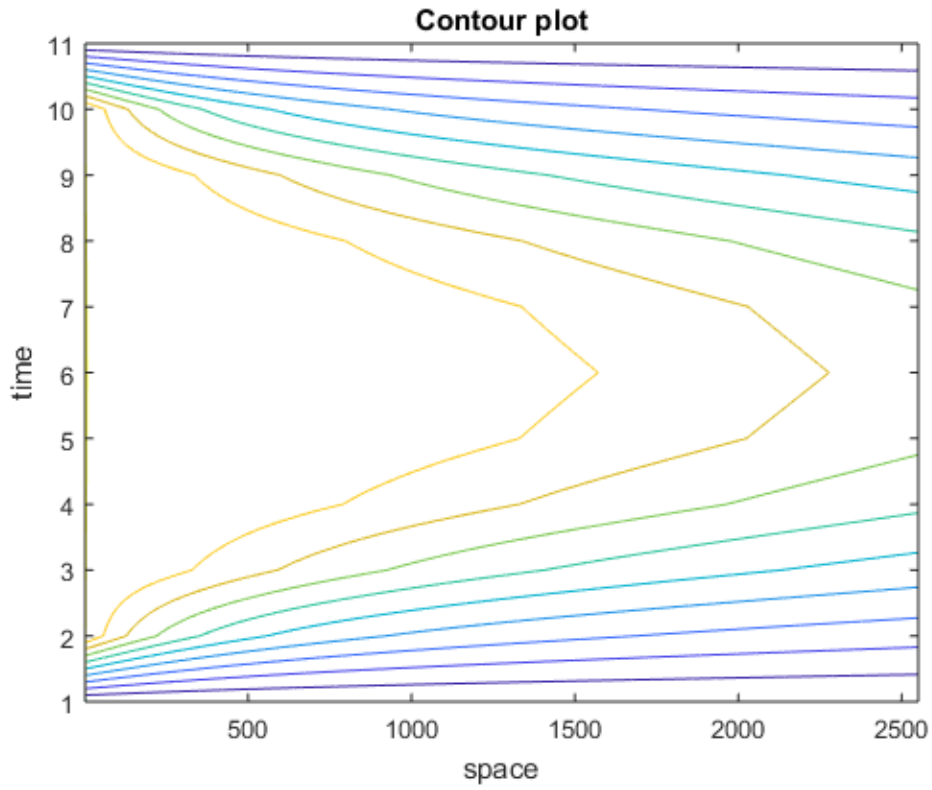
**Figure 12:** Numerical Simulation of Caputo with theoretical value of 0.4 showing H, S and T.



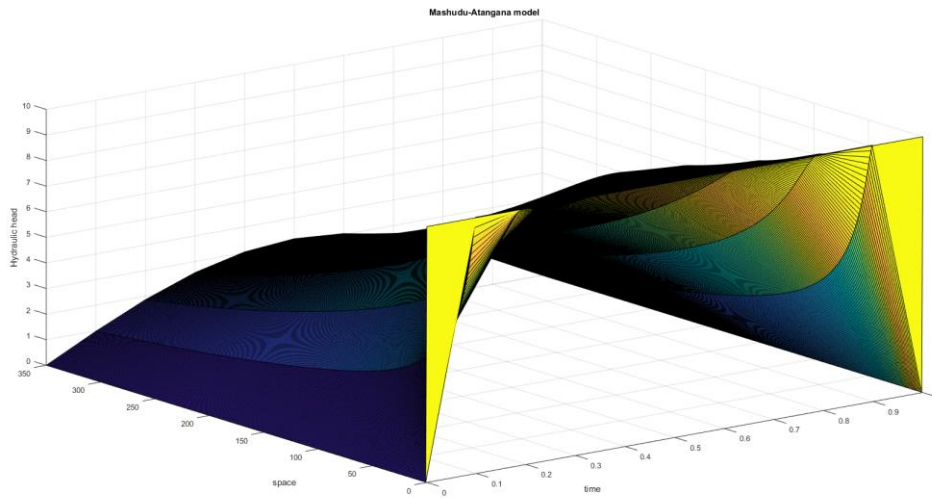
**Figure 13:** Numerical Simulation of Caputo with theoretical value of 0.4 showing H and T.



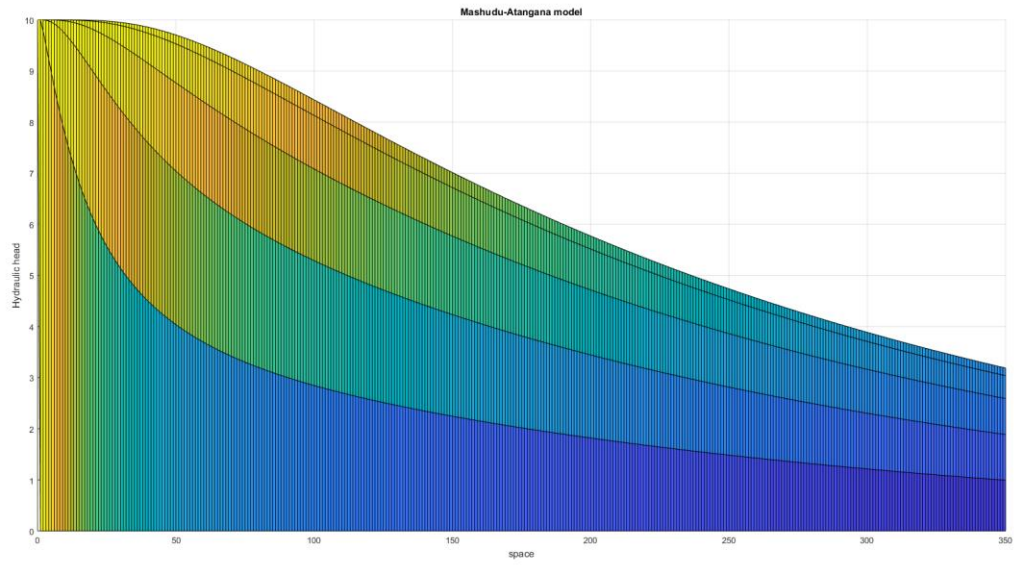
**Figure 14:** Numerical Simulation of Caputo with theoretical value of 0.4 showing H and S.



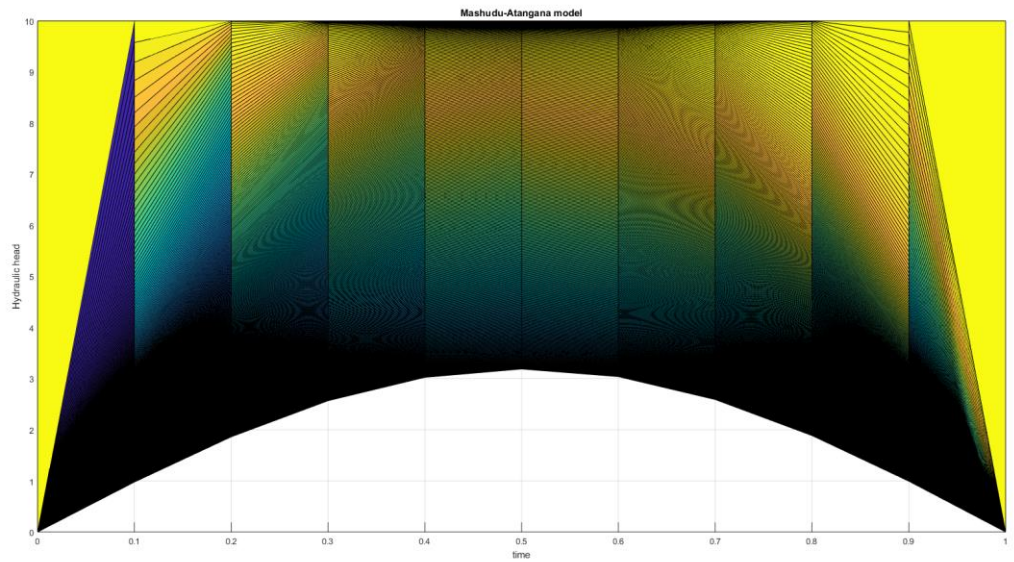
**Figure 15:** Contour Plot of Caputo with theoretical value of 0.4.



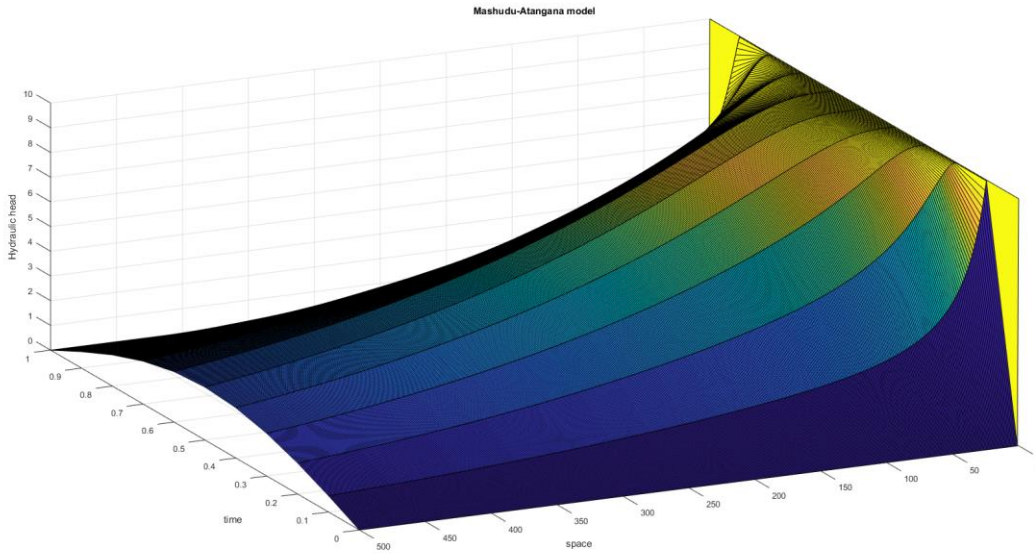
**Figure 16:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 1 showing H, S and T.



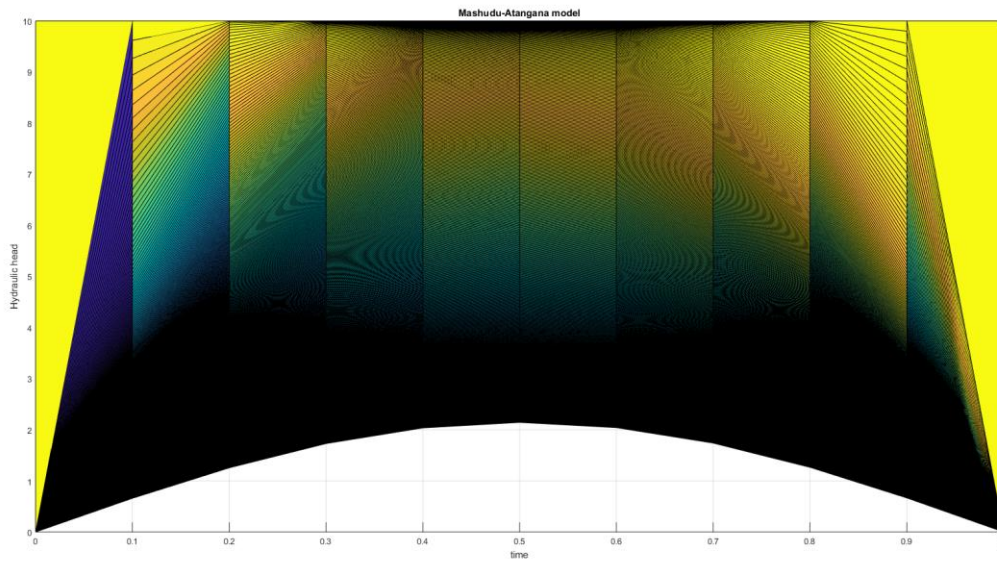
**Figure 17:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 1 showing H and S.



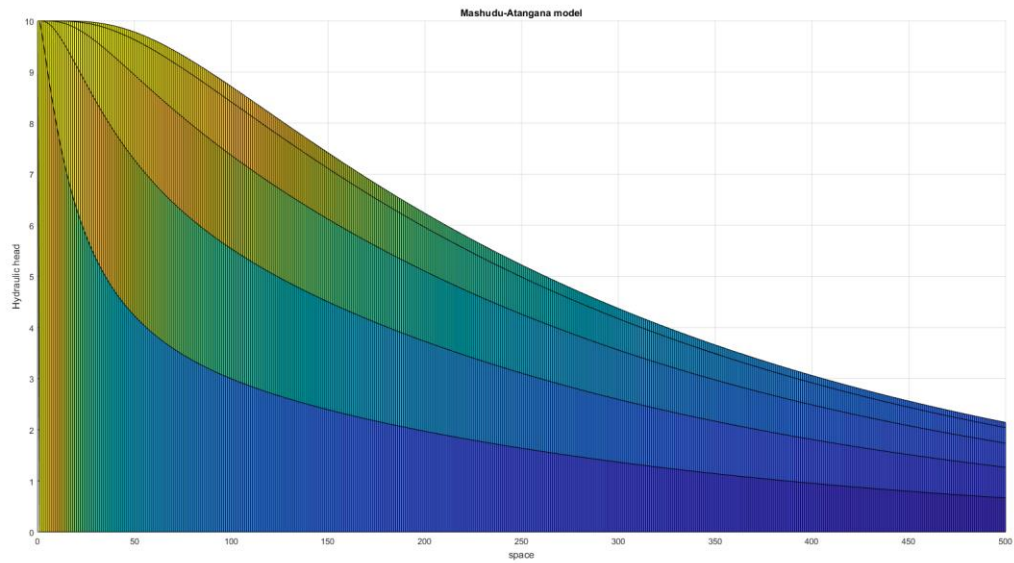
**Figure 18:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 1 showing H and T.



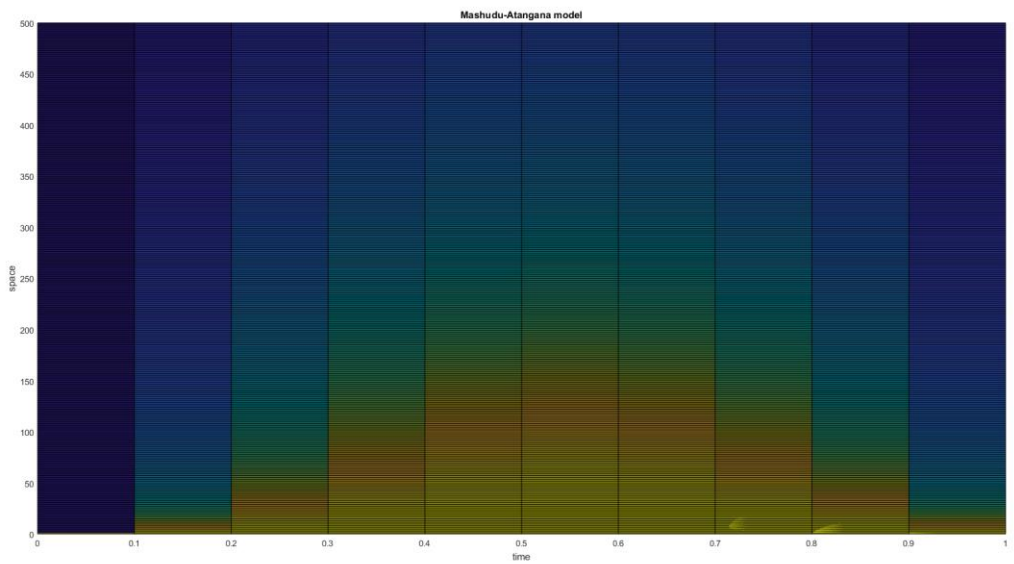
**Figure 19:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.9 showing H, S and T.



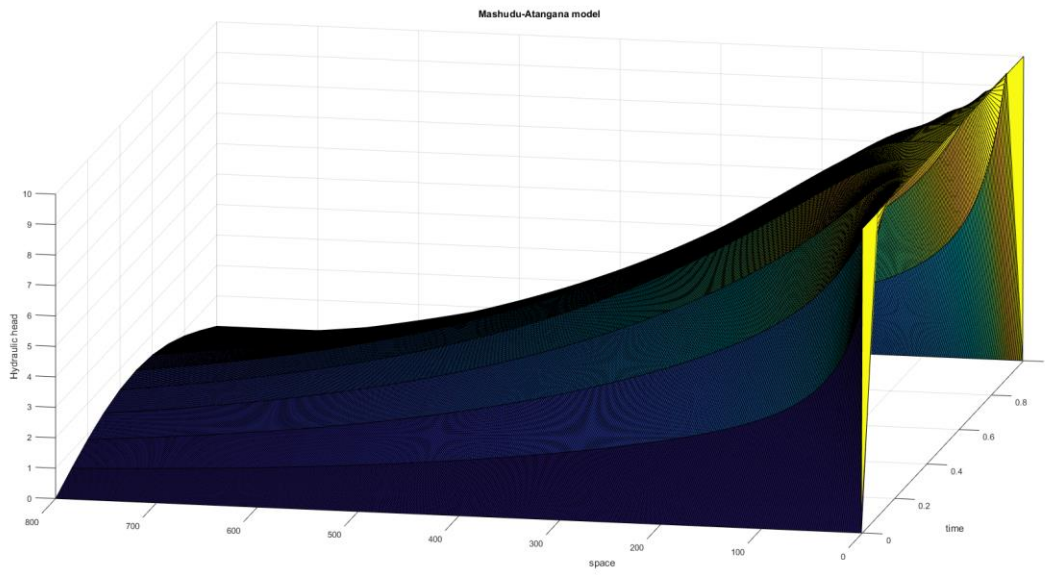
**Figure 20:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.9 showing H and T.



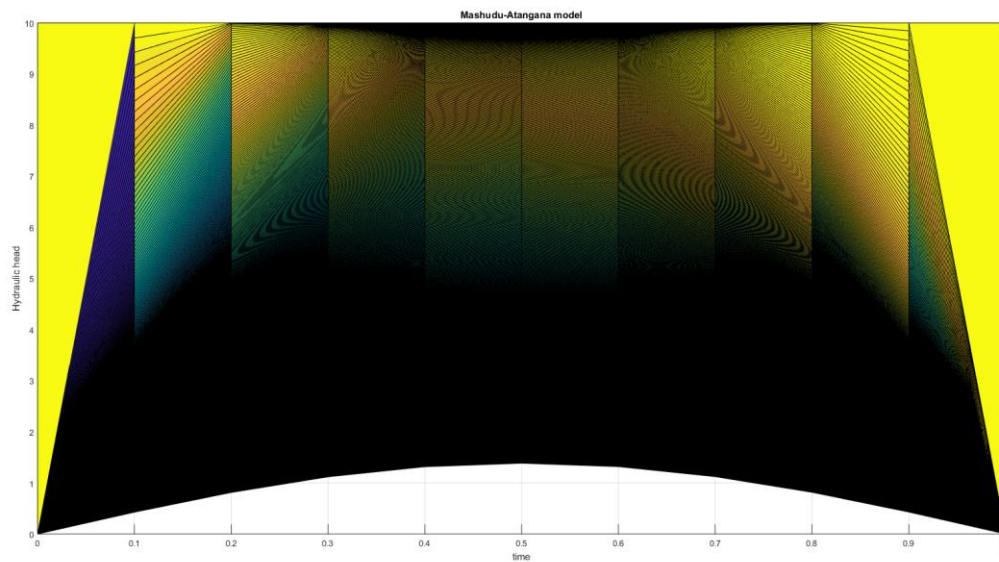
**Figure 21:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.9 showing H and S.



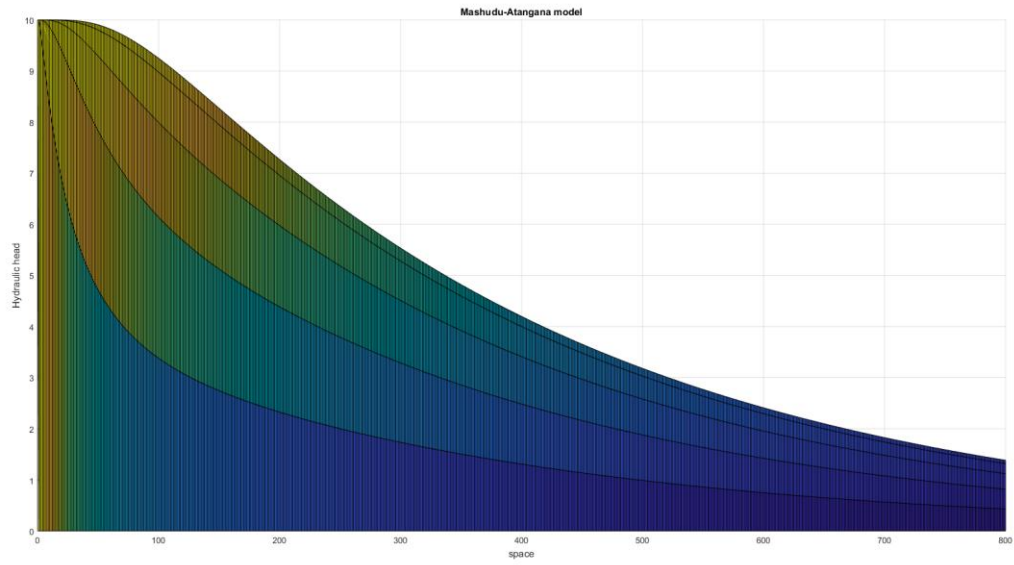
**Figure 22:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.9 showing S and T.



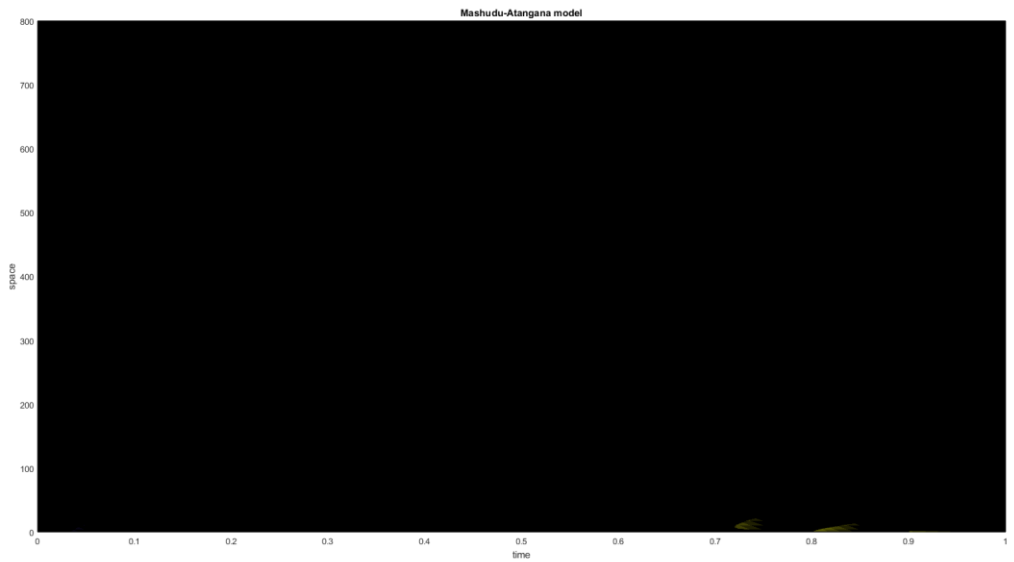
**Figure 23:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.7 showing H, S and T.



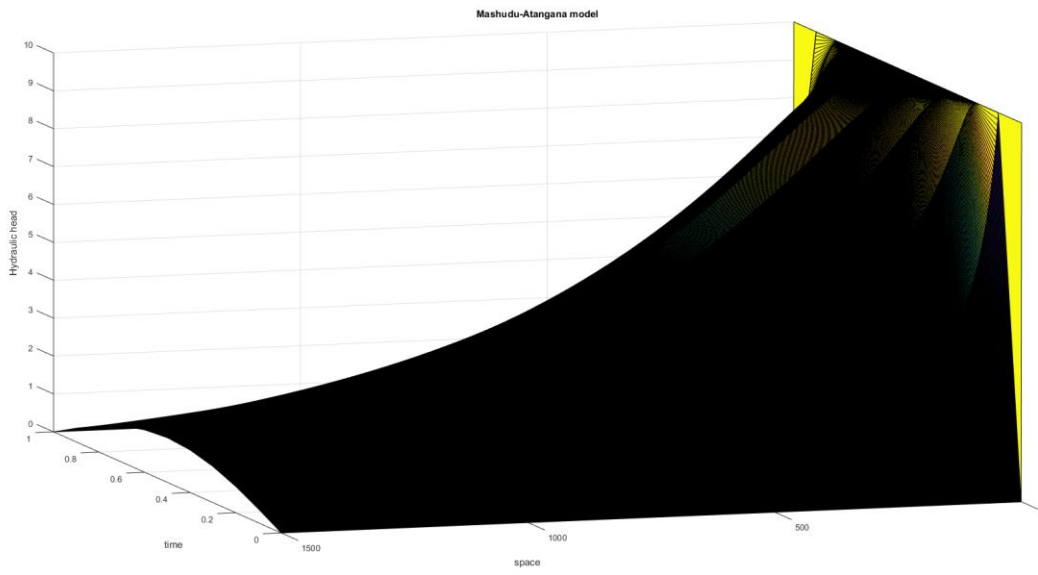
**Figure 24:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.7 showing H and T.



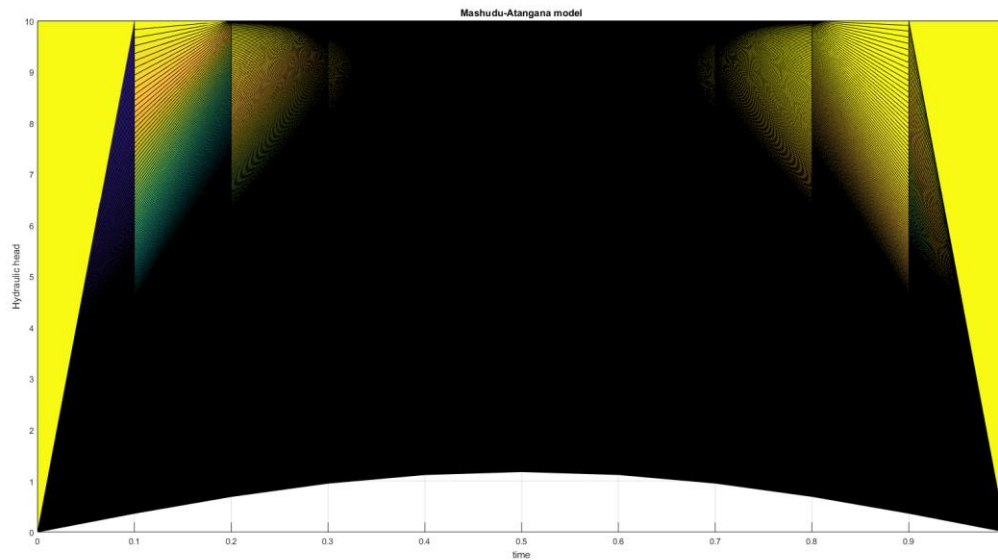
**Figure 25:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.7 showing H and S.



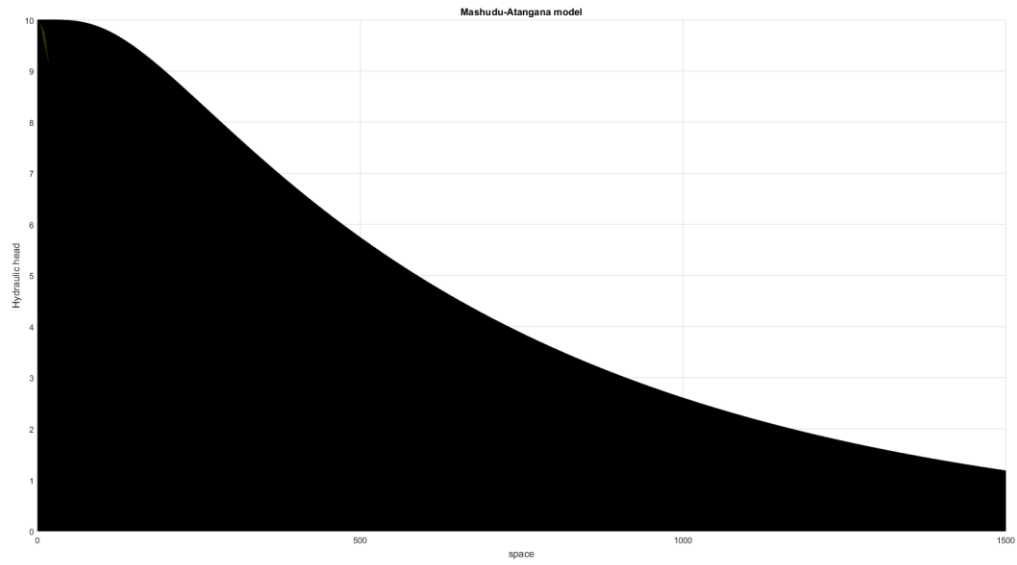
**Figure 26:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.7 showing S and T.



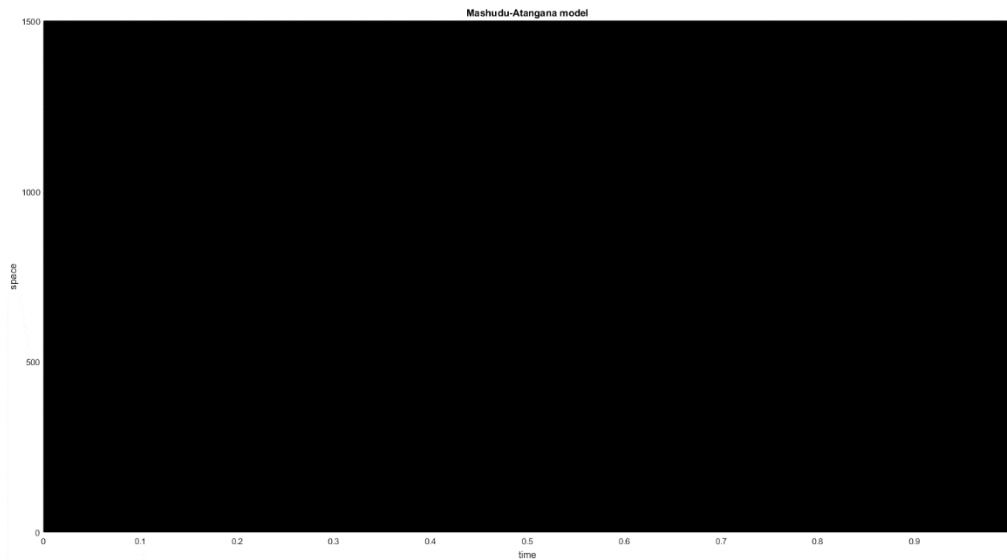
**Figure 27:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.4 showing H, S and T.



**Figure 28:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.4 showing H and T.



**Figure 29:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.4 showing H and S.



**Figure 30:** Numerical Simulation of Caputo-Fabrizio with theoretical value of 0.4 showing S and T.

## CONCLUSION

The main aim of the research was to analyze general groundwater flow equation for confined aquifer with non-local operators which includes Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Atangana-Baleanu fractional derivative and fractal-fractional differential operators. In addressing this aim, a newly developed general groundwater flow equation for confined aquifer which was developed by Mashudu and Atangana (2017) was adopted. The equation was then analyzed using the above-mentioned derivatives. New numerical schemes were developed using the classical Adam Bashforth method and the newly developed Newton polynomial, which was developed by Atangana and Araz (2019). Atangana and Araz mentioned that as much as the Adam Bash forth method has been recognized to be a very efficient numerical method to solve linear and nonlinear orders, the method was developed using the Lagrange interpolation, which is less accurate as compared to their newly developed Newton Polynomial.

Numerical simulations were developed for Caputo fractional derivative and Caputo-Fabrizio fractional derivative. Based on the simulations, it can be observed that for fractional order from 0.99 to 0.51, there is slow flow meaning the geological formation is not really connected and as such, does not helping water to pass through easily. When fractional order is below 0.5, we observe a fast flow and when the fractional order is 1, normal flow is observed and thus the geology is assumed to be homogenous. In conclusion, this research helped to capture complexities of the geological formation through which the sub-surface water flows.

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## ABSTRACT

The main aim of this research was to analyze the general groundwater flow equation with non-local operators. Non-local operators are able to introduce the mathematical formulation of memory and random walk. Recently, Mashudu and Mathobo (2017) introduced a more general groundwater flow model which considers the forgotten terms by Theis; however, their model uses the concept of classical differential operators and therefore could not help capture complexities of the media through which the sub-surface water flows. It is in this regard that their model was extended to the concept of non-local operators. The general groundwater flow equation for a confined aquifer was analyzed using the Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Atangana-Baleanu fractional derivative and fractal-fractional differential operators. New numerical schemes were developed using the classical Adam Bash fort method and the newly developed Newton Polynomial method which was developed by Atangana and Araz in 2019. Stability analysis of the numerical schemes were presented. Numerical simulations were developed and showed evidence of three different types of flow according to fractional orders. Slow flow was observed in fractional order from 0.99 to 0.51, depicting that the geological media is not well connected and thus not helping water to pass easily. When the fractional order is below 0.5, fast flow was observed and when the fractional order is 1, normal flow was recovered and the geology was assumed to be homogenous.

**Key words:** Caputo, Caputo-Fabrizio, Atangana-Baleanu, Fractal, Fractional, Fractal-Fractional differential operators, Groundwater, Confined aquifer, Memory, Numerical Scheme, Newton Polynomial.