MIXTURE FAILURE RATE MODELING WITH APPLICATIONS

By

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Declaration

I declare that this thesis on “mixture failure rate modeling with some applications” hereby submitted by Taoana Thomas Kotelo for the degree Doctor of Philosophy in the subject of Mathematical Statistics at the University of the Free State is my own work, except of course where indicated otherwise. I have not previously submitted it, either in part or entirety, at any University and/or Faculty degree. I have adhered to the best academic writing styles within the specific area of specialization and my own work and what is from other sources could be clearly discerned. I further acknowledge, improperly referenced material may lead to plagiarism.

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Summary

This thesis is mostly on mixture failure rate modeling with some applications. The topic is very important in the modern statistical analysis of real world populations, as mixtures is the tool for modeling heterogeneous populations. Neglecting heterogeneity can result in serious errors in analyzing the corresponding statistical data. Many populations are heterogeneous in nature and the homogeneous modeling can be considered as some approximation. It is well known that the failure (mortality) rate in heterogeneous populations tends (as time increases) to that of the strongest subpopulation. However, this basic result had to be considered in a much more generality dealing with the shape of the failure rate and the corresponding properties for other reliability indices as well. This is done in the dissertation, which is (we believe), its main theoretical contribution which can have practical implications as well.

We focus on describing aging characteristics for heterogeneous populations. A meaningful case of a population which consists of two subpopulations, which we believe was not sufficiently studied in the literature, is considered. It is shown that the mixture failure rate can decrease or be a bathtub (BT) shaped: initially decreasing to some minimum point and eventually increasing as $t \to \infty$ or show the reversed pattern (UBT). Otherwise, the IFR property is preserved.

The mean residual life's (MRL) 'shape properties' are analyzed and some relations with the failure rate are highlighted. We show that this function for some specific cases with, e.g., IFR or UBT shaped failure rates is decreasing for certain values of parameters, whereas it is UBT for other values.

Some findings on the bending properties of the mixture failure rates are presented. It follows from conditioning on survival in the past interval of time that the mixture failure rate is majorized by the unconditional one. These results are extended to other main reliability indices.

The mixture failure rate before and after a shock for ordered heterogeneous populations are compared. It turns out that the failure rate after the shock is smaller than the one without a

shock, which means that shocks under some assumptions can improve the probabilities of survival for items in a heterogeneous population.

We show that the population failure/mortality rate decreases with age and, even tend to reach a plateau for some specific cases of mortality (hazard) rate process induced by the non-homogeneous Poisson process of shocks. Our model can be used to model and analyze the damage accumulated by organisms experiencing external shocks. In this case, the cumulated damage is reflected by jumps in the failure rate.

The focus in the literature has been mostly on the study of expectations for mixtures, however, the obtained results show that the variability characteristics in heterogeneous populations may change dynamically.

Key Words:

Increasing (decreasing) failure rate,

Mixtures of distributions,

Mixture failure rate,

Stochastic ordering

Mortality (failure) rate process, and

Shocks.

Dedication

To: ‘Maletšaba, Retšepile, Letšaba and ‘Malekeba Kotelo and the rest other family members

Contents

CHAPTER 1: Introduction

1.1. Motivational Aspects

This thesis is mostly on mixture failure rate modeling with some applications. The topic is very important in the modern statistical analysis of real world populations, as mixtures is the tool for modeling heterogeneous populations. Neglecting heterogeneity can result in serious errors in analyzing the corresponding statistical data. Many populations are heterogeneous in nature and the homogeneous modeling can be considered as some approximation. Quite a number of examples and applications of theoretical modeling in this thesis are from the fields of reliability and demography. It is well known that the failure (mortality) rate in heterogeneous populations tends (as time increases) to that of the strongest subpopulation. However, this basic result had to be considered in a much more generality dealing with the shape of the failure rate and the corresponding properties for other reliability indices as well. This is done in the dissertation, which is (we believe), its main theoretical contribution which can have practical implications as well.

It should be noted that we do not provide specific engineering applications for the obtained results (only sometimes mention them), but rather emphasize 'general, natural applicability' of the obtained and discussed results. For instance, the apparent decrease in the observed failure rate was first acknowledged for the heterogeneous set of aircraft engines with each subpopulation described by the constant failure rate [66]. However, all obtained and discussed results can be directly applied to various engineering settings with homogeneous subpopulations. The same refers to the demographic context as well. Therefore, we believe, that our text is indeed of a prospective applied nature, which is reflected in the title.

In the subsequent section, we discuss some general reliability notions relevant to our study and present a brief introductory literature survey, whereas the more detailed analysis of specific references will be conducted throughout the text at appropriate places while discussing relevant issues. It should be noted that the literature on mixture failure rate modeling is quite abundant, however, there are still a lot of topics and problems to be considered. We hope that our work fills this gap, at least, to some extent. In the rest of this chapter, we pay also a considerable attention to describing various (aging) classes of distributions, as it will be important and useful while discussing the statistical modeling for mixtures of distributions.

As usual, we have adopted in this thesis, the convention of using increasing (decreasing) for non-decreasing (non-increasing). The terms failure (hazard) rate and mortality rate will be used in this dissertation.

1.2. Some general reliability notions: Definitions and basic concepts

We consider nonnegative random variables, usually called *lifetimes* (i.e., $T \geq 0$). Realizations of these random variables may generally be manifested by some 'end event'. The time to failure of the man-made devices, the wear accumulated by a degrading system up to some predetermined threshold or death of an organism are all relevant examples of lifetimes.

Our main focus in this study will be mostly on four main reliability indices: the failure rate (FR), the mean residual lifetime (MRL), the reversed failure rate (RFR) and the mean waiting (inactivity) time (MIT). At appropriate places we will consider the corresponding mixture models for these indices.

Denote the cumulative distribution function (Cdf) that describes a lifetime T by, $F(t) = P(T \leq t)$ and its probability density function (pdf) by $f(t)$. Then the corresponding failure rate, $\lambda(t)$, which will be one of the prime objects in heterogeneous settings as well (to be defined further), in this homogeneous setting is defined as

$$\lambda(t) = \frac{f(t)}{\overline{F}(t)}, \tag{1.1}$$

where, $\overline{F}(t) = 1 - F(t) = S(t)$ is the corresponding survival function (otherwise also known as the reliability function), i.e., $S(t) = P(T > t)$. From this, the famous exponential representation (also sometimes referred to as the product integral formula, [185]) can be obtained, i.e.

$$S(t) = \exp(-u), \tag{1.2}$$

where $u = \int_0^t \lambda(v)\, dv$ is the cumulative failure rate. Result (1.2) already provides the simplest characterization of $S(t)$ via the failure rate.

Reliability characteristics uniquely describe a lifetime, T, whereas, e.g., the shape of the failure rate provides powerful tools for describing the aging properties of the corresponding lifetime distributions. Therefore, hereafter, we present and describe the main ageing classes of distributions.

1.2.1. IFR (DFR) and IFRA (DFRA) Classes

"The increasing failure rate (IFR) is an indication of deterioration or ageing of some kind, for a lifetime and this is an important property in various applications", [66]. On the other hand, if the failure rate is decreasing (DFR), the object's lifetime is improving. The increasing (decreasing) failure rate average, IFRA (DFRA) are the corresponding simplest generalizations of these classes.

The lifetime distribution, Cdf, $F(t)$ is IFR (DFR) if, the conditional survival function, (i.e. conditioned on survival to t) is decreasing (increasing) in $t \geq 0$ for each $x \geq 0$. Alternatively $F(t)$ is defined in reference [218] to be IFR (DFR) if $-\ln S(t)$ is concave (convex). These are equivalent to the failure rate $\lambda(t)$ being increasing (decreasing) in $t \geq 0$.

The Cdf $F(t)$ is said to be IFRA (DFRA) if $-(1/t)\ln S(t)$ is increasing (decreasing) in $t \geq 0$. The forgoing means that: $S(\alpha t) \geq (\leq) S^{\alpha}(t)$ for $0 < \alpha < 1$, $t \geq 0$ and/or $-\ln S(t)$ is a star-shaped function: i.e. $-\ln S(\alpha t) \geq (\leq) -\alpha \ln S(t)$, $0 < \alpha < 1$, $t \geq 0$.

Thus, IFRA (DFRA) classes, already, relax some stringent requirements of the corresponding IFR (DFR) classes. See, e.g., also references [66], [77], [86], [179], [218], [226], [238] and [242] just to mention a few, for some other relevant discussions on these classes and other related classes.

1.2.2. IMRL (DMRL) classes based on mean residual life

The mean residual life (also known in demography as life expectancy at age t), also, plays a pivotal role for studying the aging characteristics of lifetime distributions. Let T_x be the remaining lifetime of an item of age t, $(T_x = (T - t \mid T > t))$. Then, the lifetime distribution of T_x is, therefore, uniquely characterized by the conditional survival function,

$$S_x\,(t) = P\,(T_x > t) = \exp\,(-\vartheta), \tag{1.3}$$

where $S_x(t) = S(t \mid x)$ and $\vartheta = \int_t^{t+x} \lambda(v)\,dv$. The MRL function is, therefore, defined as the expectation of the random variable, T_x via (1.3), as follows [66]:

$$m(t) = E[T_x] = \frac{\int_t^{\infty} S(v)\,dv}{S(t)} = \int_0^{\infty} \exp(-\vartheta)\,dx. \tag{1.4}$$

See, e.g., reference [161], for the corresponding necessary and sufficient conditions for the relationship (1.4) to be the mean residual life function of a non-negative random variable. Intuitively, the IFR property that characterizes deterioration (ageing) imply a decreasing MRL of an item (object). “This is not true in general, as the MRL function may be monotone whereas the corresponding failure rate is not”, [129]. Hereafter, we consider the corresponding aging classes for the MRL and also some relevant generalizations.

The lifetime distribution $F(t)$ describes the increasing (decreasing) mean residual lifetime IMRL (DMRL) if its mean residual life function,

$$m(t)=\int_0^\infty S_x(t)\,dt=\frac{\int_t^\infty S(v)\,dv}{S(t)}\ ,\quad t\ge 0 \text{ is increasing (decreasing) in } t.$$

The DMRL class, already, defines some kind of deterioration or ageing of an item as time increases, whereas the IMRL defines the corresponding improvement as age. Other classes that arise from comparing the original distribution with the distribution of the remaining lifetime include: the new better (worse) than used NBU (NWU) and the new better (worse) than used in expectation NBUE (NWUE).

The lifetime distribution $F(t)$ is said to be NBU (NWU), if $S_x(t)\le(\ge)S(t)$, i.e. $S(t+x)\le(\ge)S(t)\,S(x)$ for $t,x\ge 0$. Equivalently, $\ln S(t+x)\le(\ge)\ln S(t)+\ln S(x)$ having the same support, $t,x\ge 0$. Utilizing relations (1.2) and (1.3), then it means, $F(t)$ is NBU if,

$$u=\int_0^t\lambda(v)\,dv\le\int_t^{t+x}\lambda(v)\,dv=\vartheta\quad\text{for } t,x\ge 0.$$

Whereas, it is NWU if the inequality above is reversed. This means that, “an item of any age has a stochastically smaller remaining lifetime than does a new item”, [66].

The lifetime distribution $F(t)$ is NBUE if,

$$\int_0^\infty S_x(t)\,dx\le m(t)\ \text{for } t\ge 0\ \text{ or }\ \int_x^\infty S(t)\,dt\le m(t)\,S(x).$$

In fact, when $F(t)$ is IFRA, $-(1/t)\ln S(t)$ is increasing in $t\ge 0$. It follows, immediately, from the definition of NBU that, $-(1/t+x)\ln S(t+x)\ge-(1/t)\ln S(t)$ for $t,x\ge 0$, and $-(1/t+x)\ln S(t+x)\ge-(1/x)\ln S(x)$ for all t and x greater than (or equal) zero. This means that $S(t+x)\le(\ge)S(t)\,S(x)$ with the same support. Hence, if $F(t)$ is IFRA it is also NBU. As pointed out, in reference [86 p. 115], $F(t)$ is DMRL implies NBUE, whereas IMRL implies NWUE.

The aging classes presented here are extensively studied in literature and a comprehensive treatment can be found, e.g., in reference, [86]. On the other hand, some further discussions of other properties of the NBU, NWU, NBUE, and the NWUE classes could be found in reference [234], some moment bounds (in particular, lower and upper bounds) for the IFR (DFR), IFRA (DFRA) classes as well as for the NBU (NWU) classes are also established in [201]. We also refer to references [63], [86], [157], [158], [179], [206], [213], [214], [218] and [230] to mention a few on other aspects of the classes considered here and other more general ageing classes.

1.2.3. DRFR and IMIT Classes

"The concept of the reversed failure rate (RFR) was introduced by Von Mises in 1936", [66]. It was mainly regarded in the literature as dual to the hazard rate, see, e.g., references [198] and [233] for the detailed discussions. This notion is more, intuitive for random variables with support in $0 \le a < b \le \infty$. Assuming this finite interval of support, in reference [159], some cases were considered when the RFR is constant or increasing. However, it was later showed in reference [129], that RFR cannot be constant or an increasing function. In this light, we focus on describing classes with decreasing RFR (DRFR).

"The reversed failure rate, denoted, $r(t)$ is defined as the ratio of the density (pdf), $f(t)$ and $F(t)$", [233],

$$r(t) = \frac{f(t)}{F(t)}. \tag{1.5}$$

Hence, $F(t)$ is DRFR if, (1.5) is a decreasing function for $t \ge 0$, which means that the distribution function is log-concave.

Another useful notion, which is of interest, is the mean inactivity time. For a positive random variable, T, let T_ω (where, $T_\omega(t) = [T - t \mid T < t]$), define the time elapsed since the last failure. This random variable is known in literature as the inactivity (waiting) time (or in some references the reversed residual life, see, e.g., [123]). We define the expectation of this random variable, the mean inactivity (waiting) time (MIT) as,

$$\varpi(t) = E[T_{\omega}] = \frac{\int_0^t F(v)\,dv}{F(t)} \quad . \tag{1.6}$$

The properties of (1.6) have been studied by [135], whereas some further characterizations via the mean inactivity time could be found in references [40], [90], and [123]. In particular, we refer to references [102], [105], and [127] to name a few on some further results on this class. As a result, various classes of lifetime distributions with increasing mean inactivity time (IMIT) are defined in the literature. See for instance, references [62] and [90]:

The lifetime distribution $F(t)$ is said to be IMIT if $\varpi(t)$ is an increasing function for $t \geq 0$.

Alternatively, as pointed out in reference, [62], $F(t)$ is said to be IMIT iff $\int_0^t F(v)\,dv$ is log-concave for $t \geq 0$, or equivalently,

$$\int_0^{t+x} F(v)\,dv \Big/ \int_0^t F(v)\,dv \text{ is decreasing in } t \geq 0 \text{ for all } x \geq 0.$$

This result, already, implies that, if T is DRFR, then it is also IMIT.

The corresponding aging properties of the MIT function could, already, be obtained from the corresponding properties of the RFR. The following relation exists between the RFR and the MIT, [127]:

$$r(t) = \frac{1 - \varpi'(t)}{\omega(t)}. \tag{1.7}$$

In fact, using the above relations, some results were obtained that are useful for describing different maintenance policies in reliability in reference, [125]. We also recall and discuss some of these properties in section 2.6.2. Whereas these properties and other aspects of the MIT modeling have been studied in the literature, there are no results based on relative stochastic comparisons of the mean inactivity time that have been reported so far, at least, to the best of our knowledge. We discuss the corresponding relative mean inactivity order in section 3.6.1.

1.2.4. Non-Monotonic Ageing Classes

The shapes of the failure rate of mechanical and electronic items, often exhibits a non-monotonic aging behavior: e.g., bathtub-shaped failure rates (BT), initially decreasing to a certain minimum point then increasing as time increases or show the reversed pattern, i.e., the upside-down bathtub (UBT) shape. There are also other shapes. See, for example, references [73], [77], [86], [157] and [202] to name a few. We focus on the first two in this work. We also discuss further some simple models for the failure (mortality) rate with unimodal change point under some shock settings in section 5.4.

1.2.4.1. Bathtub (BT) and upside-down (UBT) shape failure rate classes

Formal definitions of the above non-monotonic ageing classes are contained in Glazer's theorem (which is also proved in [66 p.32]). The authors of reference [138] extended these definitions to a situation, in which the failure rate exhibits several change points. Accordingly with the forgoing authors, the following establishes some sufficient conditions for the monotonic or the BT (UBT) shapes of the failure rate using the function, $g(t)$, which is defined as:

$$g(t) = -\frac{f^{'}(t)}{f(t)} \quad . \tag{1.8}$$

As pointed out in reference [1], "$\lambda(t)$ and $g(t)$ are asymptotically equivalent when, $\lim_{t\to\infty} f(t) = 0$, e.g. $\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} f(t)/S(t) = \lim_{t\to\infty} - f(t)/f(t)$". See e.g., also reference [86]. Therefore, the behavior of the failure rate, $\lambda(t)$ can easily be analyzed via the monotonicity properties of $g(t)$.

Let "the density $f(t)$ be strictly positive and differentiable on, $[0, \infty)$, such that $\lim_{t\to\infty} f(t) = 0$", then, [1 p.18] and [77 pp134-135]:

i) If $g(t)$ is increasing, then the failure rate, $\lambda(t)$ is also increasing.

ii) If $g(t)$ is decreasing, then the failure rate, $\lambda(t)$ is also decreasing.

iii) If there exists a point, t_1 for which $g(t)$ is decreasing in $t \le t_1$ and increasing in, $t \ge t_1$, then there exists $t_2 (0 \le t_2 \le t_1)$ such that $\lambda(t)$ is decreasing in $t \le t_2$ and increasing in $t \ge t_2$.

iv) If there exists a point, t_1 for which $g(t)$ is increasing in $t \le t_1$ and decreasing, $t \ge t_1$, then there exists $t_2 (0 \le t_2 \le t_1)$such that $\lambda(t)$ is increasing in $t \le t_2$ and decreasing in $t \ge t_2$.

The first two conditions naturally define monotonic failure rates, e.g., the IFR (DFR) classes. The last two characterize the non-monotonic failure rates, e.g., the BT (UBT) classes. On the other hand, if $t_1 = t_2$ then we can define an interval $t_1 \le t \le t_2$, where $g(t)$ is constant, which ultimately translates to a constant $\lambda(t)$. This description, which defines the traditional BT (UBT) that includes also the constant failure rate is used in reference [182]. For $t_1 = t_2$, the corresponding aging classes are defined with a single change point. We adopt this latter definition, in the rest of this dissertation. See, e.g., also references [66], [72], [73], [110], [120], [138], [162] and [176] just to mention a few for some further discussions on the above ageing classes.

1.2.4.2. Mean residual life (MRL) classes with bathtub (BT) or (UBT) shapes

It is well-known that, for a monotonically increasing (decreasing) failure rate, e.g., IFR (DFR), the corresponding MRL function is decreasing (DMRL) (increasing (IMRL)), whereas the reverse has been shown not to be true in general by the authors of reference, [129]. We define, "the increasing, then decreasing mean residual life (IDMRL) and decreasing, then increasing mean residual life (DIMRL) classes" in the following, [66]:

The lifetime distribution, $F(t)$ is said to be IDMRL, if there is a, $t_0 \ge 0$, for which the MRL is initially increasing on $[0, t_0)$ and then decreasing on, $[t_0, \infty)$. If MRL is initially decreasing on $[0, t_0)$ and increasing on $[t_0, \infty)$, we have the corresponding, DIMRL class . Therefore, $F(t)$ is IDMRL when, $m(t) \in UBT$ whereas it is DIMRL if, $m(t) \in BT$.

The considered here ageing classes will be useful to further explore other lifetime distributions and/or mixtures of distributions exhibiting the bathtub or UBT shapes and for discussing our results in the subsequent chapters. See, e.g., also references [27], [111], [176], [181], [182], [213], [217], and [248] on some other relevant discussions.

Remark 1.1

1. We, also, generalize the properties of the failure rate to the discrete case in chapter 4. There are "some important differences between the failure rates in the discrete setting as compared to the failure rate in the continuous case", [4]. We investigate the impacts of these differences in describing the corresponding aging characteristics. For example, the shapes of the failure rate for some specific distributions in the class of discrete Weibull distributions will be analyzed.
2. Other specific situations to be considered as well:

a) The case when items (objects) for study have some random (unknown) ages (section 2.7). Determining the impacts of this random delay on reliability characteristics of the baseline lifetime distribution of an object is of interest in this case. A specific model of mixing, where the unknown initial age, is the mixing parameter will be studied.

b) The corresponding aging characteristics will be also discussed under some shock settings in chapter 5.

1.3. Mixtures

Most of the thesis is devoted to modeling heterogeneity via considering the corresponding mixtures of distributions. Homogeneous populations present the simplest models for analyzing the shapes of the main reliability characteristics in this thesis. "It is well known that in this case, the failure rate, $\lambda(t)$ characterizes a lifetime random variable (i.e. $T \geq 0$) for items (objects) operating in a fixed (or specified) environmental conditions", [39]. However, many populations are heterogeneous in nature. This heterogeneity may "arise in situations in which data is pooled from two parent distributions to enlarge the sample size or when physical mixing identical items, albeit from different manufacturers", see e.g., references, [66], [72] and [81]. The shapes of reliability characteristics may change quite significantly under these settings. In fact, it is a common knowledge that mixtures of decreasing failure rate distributions (i.e., DFR) are always DFR. However, "the pattern of population aging could change considerably from IFR aging to DFR aging", [66] even for "mixtures of distributions with strictly increasing failure rates", [177]. It was, also, shown that the population failure rate tends to bend down with time when compared with the corresponding unconditional characteristic. This observed deceleration, [195] already has the meaningful interpretation: "*the weakest populations are dying out first as time increases*" principle in heterogeneous populations. As a result, the failure (mortality) rate in heterogeneous populations tends (as time increases) to that of the strongest subpopulation. We are intrigued by this finding, which can be considered as counter-intuitive. This principle, can also be extended to explain the recently observed mortality rate plateau in human populations [157] for the oldest-old populations in developed countries as a result of improved health care quality. However, this result had to be dealt with in more generality and with respect to other indices as well. This topic has a wide applicability in a number of areas dealing with lifetime modeling and analysis. Therefore, this thesis is rolled over five forthcoming chapters covering different aspects of the problem.

1.4. Brief overview

Chapter 2

Lifetimes for heterogeneous populations are often induced by changing environment conditions and/or other random effects. We focus, on describing the corresponding aging characteristics for heterogeneous populations. Henceforth, we firstly consider the random failure rate. This notion, which is a generalization of (1.1), will be particularly important for the corresponding analysis of the mixture failure rate and also for formulating our results in the subsequent sections. In section 2.2., some aspects of general mixture models, which will be useful in the rest of this thesis are discussed. In particular, some simple frailty (mixture) models are studied in section 2.3., and some initial results are discussed. Mixtures of distributions often present the simplest corresponding modeling and analysis. To illustrate some applications of the models of these sections, we consider some specific cases of a mixed population, which consists of two subpopulations. The shapes of the corresponding mixture failure rates are discussed. Another specific case, which also explicitly illustrate some further applications of the models of this section to a case, where the mixing parameter is the initial (usual unknown) random age is analyzed in section 2.7.

The mean residual life (MRL), also, plays a pivotal role for studying aging characteristics of lifetime distributions. We present, some useful general results on the properties of the MRL to be used in obtaining the corresponding shape properties for mixtures in section 2.5.1. At the same time, in this section, the corresponding shapes properties are also analyzed for some specific cases and some relations with the failure rate are highlighted. We also revisited a specific case of the proportional MRL model and briefly discussed some results relating to this model.

A literature survey on the reversed failure rate (RFR) and the mean waiting (inactivity) time is presented in sections 1.2.3. We discuss the corresponding general properties and consider the shapes of 'the reversed failure rate' in section 2.6.1. To illustrate the applications of these models, we analyze two specific cases, e.g., when the failure rate is increasing or is of the UBT-type shape. The specific frailty mixture model for the reversed failure rate, e.g., the proportional reversed failure rate (PRFR) is also considered in this section. We briefly discuss some results relating to this model. The properties of the MIT could easily be analyzed via the properties of the RFR. This is done in the last section 2.6.

Chapter 3

In this chapter, we firstly deal with some essential aspects of stochastic orderings, particularly, in section 3.2. Mixture failure rates are important in studying heterogeneous populations in different environments. We discuss some results on the corresponding aging properties of the mixture failure rates when compared with a specific form of our model (2.1). These results are extended to other main reliability indices: the mean residual life in section 3.4.2, the reversed failure rate in section 3.4.3 and the mean waiting (inactivity) time in section 3.4.4. We, also, analyze the failure (mortality) rate for heterogeneous populations in section 3.5, e.g. when the subpopulations are ordered (in some stochastic sense).

We, also discuss some results with respect to vitality modeling in section 3.6. In the final section 3.7, we consider the notion of relative aging and discuss results on relative aging of the main reliability indices. In particular, we propose ordering of lifetimes in terms of monotocity properties of the ratio of the mean waiting (inactivity) times.

Chapter 4

In this chapter, the properties of the failure rate are generalized to the discrete case. We highlight and briefly discuss some differences in the failure rate in discrete and continuous settings. The corresponding shapes of the failure rate are investigated for some discrete Weibull distributions. For the type II discrete Weibull distribution the classical failure rate increases (IFR), whereas the alternative failure rate is of the UBT type. This obvious difference should be taken into account in practical applications. It means that the alternative failure rate may be an appropriate choice in the modeling and analysis of various aging characteristics as compared to the usual ("classical") failure rate. It is, also, interesting to explore further this behavior for other discrete lifetime distributions.

The shapes of the corresponding failure rate of a mixture of two distributions are studied in section 4.2. We show that the mixture failure rate bends down when compared with the expectation of the conditional failures rates. Specifically, some selected discrete lifetime distributions are studied. We show, e.g., that, under the defined settings, the corresponding failure rate of the mixture of the discrete geometric distribution and the Type I discrete Weibull distribution is decreasing for some values of parameters. For the mixture of geometric distribution and the discrete modified Weibull distribution the corresponding mixture failure rate is UBT. This property is also reflected for some values of the parameters when the latter distribution is mixed with the discrete gamma distribution whereas it shows the reversed pattern (BT) for other values. This means that the proportion of surviving items (objects) in the mixed population is increasing, e.g., the population lifetime is improving somehow as the "weakest subpopulations are dying out first".

Finally, some results on the *general properties* of discrete mixture failure rates are briefly discussed and some simple models of heterogeneity are presented. In the final section of this chapter, we also define the MRL in the discrete setting and highlight some useful relations with the corresponding failure rate.

Chapter 5

We consider stochastically ordered heterogeneous populations. The shapes of mixture failure rate for this population under some shock settings are analyzed for two specific cases. When the frailty W is a continuous random variable, we show that the failure rate for an object that experienced a shock, is less than the one which has not a shock. Therefore, shocks under some assumptions can improve the probabilities of survival for a heterogeneous population. These results are also extended to the case when frailty is a discrete random variable. Shocks as an alternative kind of burn-in is theoretically justified in these cases.

In section 5.3., a specific increasing mortality rate process induced by the non-homogeneous Poisson process of shocks is considered. The shape of the observed (marginal) failure rate is analyzed in this case. In particular, we show for some specific cases: the overall failure/mortality rate decreases with age and in some instances reaches a plateau. This result is obtained, already, shows an improvement of our population with time. Our model can, therefore, be used to model and analyze the damage accumulated by organisms experiencing external shocks. In this case, the cumulated damage is reflected by jumps on the failure rate.

An overview of results on mortality rate processes with a single change point is presented and discussed in section 5.4. Variability characteristics in heterogeneous populations are also discussed in section 5.5. We focus on the variance of the conditional random variable, $W \mid t \equiv (W \mid T > t)$, for a subpopulation of items that survived the operational interval, $[0,t)$. Two specific cases are considered: the case, when the random variable (frailty), W is discrete and when it is continuous. Another, useful measure, which we considered in this section, is the coefficient of variation of the random variable, $W \mid t$.

Chapter 6

In this chapter, we summarize conclusions and recommendations of the previous chapters

CHAPTER 2: Main model settings and some initial results

Heterogeneity in real-world populations (of items) is often induced by changing environment conditions and/or other random effects. We focus, on describing the corresponding aging properties of items from heterogeneous populations. Henceforth, we firstly consider the random failure rate. This notion, which is a generalization of (1.1), will be particularly useful in the analysis of the mixture failure rate and also for formulating our results in the subsequent sections.

2.1. Random failure rate

Let W be a positive random variable, which represents the unobserved heterogeneity. The lifetime random variable T of an item from a heterogeneous population may, therefore, be characterized via the random failure rate by the following specific but meaningful model:

$$\lambda_t = \lambda\left(t \mid W\right), \tag{2.1}$$

which is defined for each realization $W = w$. This means that, the failure rate is indexed by the random variable, W. We will consider specific cases for (2.1) later. Thus, the expectation of this random failure rate is given by

$$\lambda\left(t\right) = E\left[\lambda\left(t \mid W\right) \mid T > t\right]. \tag{2.2}$$

It is evident from (2.2) that the observed failure rate (1.1) is simply the expectation (with respect to *W*) of the random failure rate (2.1) conditioned on survival in $[0,t)$.

Our main focus is in the analysis of the model (2.1). This model is important for our further analysis of the shapes of mixture failure rates under different settings. It should be noted that "monotocity properties of the failure rate, $\lambda(t)$ change when compared with monotonicity properties of the family of conditional failure rates, $\lambda\left(t \mid W = w\right)$", [66].

2.2. Continuous mixtures

Let "T be a continuous lifetime random variable with the failure rate (2.1) defined for each realization, $W = w$" see, e.g. reference, [66]. The corresponding survival function is given by:

$$S(t \mid w) = \exp(-u), \tag{2.3}$$

where

$$u = \int_0^t \lambda(v \mid w)\, dv, \tag{2.4}$$

is the cumulative failure rate 'indexed by each realization' of the random variable W. As pointed out by the forgoing authors, " this setting can be interpreted in terms of mixtures". The random variable W in this case plays a role of a mixing parameter. Hence, the marginal distribution survival, $S(t)$ function is obtained respectively by taking the corresponding expectation with respect to W,

$$S(t) = \Pr(T > t) = E\left[\exp\left\{-\int_0^t \lambda(v \mid W)\, dv\right\}\right]. \tag{2.5}$$

From (2.5), the observed failure rate is not equivalent to the random failure rate, i.e. $\lambda(t) \neq \lambda(t \mid W)$, where, $\lambda(t)$ denotes the observed failure rate (1.1) and $\lambda(t \mid W)$ is a random failure rate. In fact, using Jensen's inequality and Fubuni's theorem when the condition $E[W] < \infty$ is assumed to hold and $S(t \mid w)$ is a strictly convex function, the following important result can be obtained,

$$S(t) = \Pr(T > t) > \left[\exp\left\{-\int_0^t E[\lambda(v \mid W)\, dv]\right\}\right]. \tag{2.6}$$

Under this setting, the following relation exists

$$\lambda(t) = E[\lambda(t \mid W)] = \lambda_b(t)\, E[W], \tag{2.7}$$

for a specific multiplicative case, $\lambda\left(t\,|\,w\right)=w\,\lambda_b\left(t\right)$, where $\lambda_b\left(t\right)$ is the failure rate of a lifetime distribution in some 'unperturbed' (usually referred to as baseline) environment and W is a positive random variable.

It should be noted that the authors of reference [251] were the first to consider this model for a rather specific case of the gamma-frailty, although the term (frailty) was introduced into demographic literature by [240]. Other types of models considered in the literature are the additive and the accelerated life models whereas the mentioned above is usually referred to as the proportional hazards model. We will consider these types of frailty (mixture) models for some specific cases in section 2.3., and will utilize the results in other subsequent subsections. Before we proceed, let us introduce some important for the presentation to follow notions.

Suppose, as previously, a lifetime T with cumulative distribution (Cdf), $F\left(t\right)$ is indexed by some nonnegative random variable W with support in, $\left[a\,,b\right]$, $a\geq 0; a\leq b\leq\infty$ and having a density $g\left(w\right)$, then

$$F\left(t\,|\,w\right)=\Pr\left(T\leq t\,|\,W=w\right), \tag{2.8}$$

and let $S\left(t\,|\,w\right)=1-S\left(t\right)$ be the corresponding survival function. Therefore, the mixture Cdf can be defined as:

$$F_m\left(t\right)=\int_a^b S\left(t\,|\,w\right)g\left(w\right)dw\,. \tag{2.9}$$

On the other hand, from (1.1), the general mixture (marginal) failure rate is given by,

$$\lambda_m\left(t\right)=\frac{\int_a^b f\left(t\,|\,w\right)g\left(w\right)dw}{\int_a^b S\left(t\,|\,w\right)g\left(w\right)dw}\,, \tag{2.10}$$

where, $f\left(t\,|\,w\right)$ is the conditional pdf of T. In fact, it was also shown by the authors of references, [136] and [143] that the failure rate (2.10) could also be compactly represented by the following conditional form,

$$\lambda_m\left(t\right)=\int_a^b \lambda\left(t\,|\,w\right)g\left(w\,|\,t\right)dw\,. \tag{2.11}$$

Specifically, $g\left(w|t\right)$ in (2.11) represents the pdf of the random variable W, which is conditioned on $T>t$ and given by:

$$g\left(w|t\right)=\frac{S\left(t|w\right)g\left(w\right)}{\int_a^b S\left(t|w\right)g\left(w\right)dw}\ , \tag{2.12}$$

The distribution functions of the unconditional random variables, W and the conditional one, $W\,|\,T>t$; $W\,|\,0=W$, are respectively given by

$$G\left(w\right)=\Pr\left(W\le w\right)=\int_0^w g\left(v\right)dv;\ G\left(w|t\right)=\Pr\left(W\le w|T>t\right)=\frac{\int_0^w S\left(t\,|\,v\right)g\left(v\right)dv}{\int_a^b S\left(t\,|\,w\right)g\left(w\right)dw}\ , \tag{2.13}$$

The conditioning in (2.10) can change monotonicity properties of the mixture failure rate, $\lambda_m\left(t\right)$ as compared with the monotonicity properties of the family of conditional failure rates, $\lambda\left(t\,|\,W=w\right)$. For instance, two cases e.g., when the conditional failure rate is increasing as a power function for each w and when it is an exponentially increasing function, are considered. In both cases W is a Gamma distributed random variable. In particular, it turns out in the first case, that $\lambda_m\left(t\right)$ exhibit an upside-down bathtub shape (UBT). As opposed to the well-known bathtub shaped (BT), which initially decreases and after some time increases, "this function initially increases to a maximum at some point in time and eventually monotonically decreases to zero as $t\to\infty$", [66]. For the latter case, the mixture failure rate, $\lambda_m\left(t\right)$ tends to a constant. We consider further other distributions and/or mixtures of distributions exhibiting these properties later on in this work. See e.g. also a number of other relevant examples in reference [177], albeit concentrating on mixture failure rates that are of BT (bathtub) shape type.

We further, consider other relevant specific cases, which exhibit these important properties later on in this work. In particular, as pointed out, by the authors of reference, [66] these results "provide possible explanations for the mortality rate plateau observed in [156]" for human populations at adult ages. Relations in (2.13) will especially be useful for analysis of bending properties of mixture failure rates, as well as for other related main reliability indices in subsequent sections.

2.3. Additive and Proportional Hazards models

Different functional forms of the conditional failure (hazard) rate can be used to analyze the bending of mortality (failure) rate at advanced ages. In frailty (mixture) models, this phenomenon is modeled via the concept of population heterogeneity. As was already mentioned, the random effects may act multiplicatively or additively on the failure (hazard) rate function. We consider, these simplest models and analyze the shapes of the, $\lambda_m(t)$ for some specific cases. The results obtained at this initial stage are also important for our further analysis in the subsequent sections.

2.3.1. The additive "frailty" model

As pointed out in section 2.2., in particular for models (2.11) and (2.12), the conditional random variable, $W|t\ ; W|0=W$ is characterized via the pdf, $g(w|t)$. Thus, its expectation is:

$$E\,[W|t]=\int_0^t w\ g\ (w|t)\ dw. \tag{2.14}$$

The conditional expectation (2.14) will particularly be useful to investigate the behavior (shape) of the mixture failure rate (2.11). "Let $\lambda(t|w)$ be indexed by the parameter, W in the following additive way", [29]:

$$\lambda\ (t|w)=\lambda_b\ (t)+w, \tag{2.15}$$

where, $\lambda_b(t)$ is the failure rate of some lifetime distribution in some 'unperturbed' (usually referred to as baseline) environment. Then using (2.11) for the model (2.15), it can be shown that:

$$\lambda_m\ (t)=\lambda_b\ (t)+\frac{\int_0^\infty w\ S\ \ (t|w)\ g\ (w)dw}{\int_a^b S(t|w)\ g\ (w)\ dw}=\lambda_b\ (t)+E\ [W|t]\ . \tag{2.16}$$

In fact, it can be easily proved that the derivative of the conditional expectation (2.14) reduces to the following specific form;

$$E^{'}\ [W|t]=-Var\ \ (W|t)<0\ , \tag{2.17}$$

whereas, the right hand side is the conditional variance of the random variable, W (which is also conditioned on the event, $T > t$). Result (2.17) specifies that the derivative of the conditional expectation (2.14) is decreasing as a function of $t \to \infty$. Thus, the shape of the mixture failure rate could be explained by the shapes of the functions in (2.16). Specifically, when $\lambda_b(t)$ is increasing the mixture failure rate could be of the BT type.

2.3.2. The multiplicative "frailty" Model

Consider now, another important mixing model: the case "$\lambda(t \mid w)$ is indexed by the parameter, W, in the multiplicative way" [136]:

$$\lambda(t \mid w) = w\,\lambda_b(t), \tag{2.18}$$

whereas, $\lambda_b(t)$, is again the baseline failure rate as in the additive case (2.15). Similar to (2.11) and (2.14), the corresponding mixture (marginal) failure rate is obtained as

$$\lambda_m(t) = \lambda_b(t)\; E\left[W \mid t\right]. \tag{2.19}$$

It turns out, $\lambda_m(t)$ in this case decreases, only when $Var\,(W \mid t)$ is large, particularly when $\lambda_b(t)$ is also increasing. Differentiating in (2.19) leads to the following useful and important result:

$$\lambda'_m(t) = \lambda'_b(t)\, E\left[W \mid t\right] + \lambda_b(t)\, E'\left[W \mid t\right]. \tag{2.20}$$

It can be easily proved that

$$E'\left[W \mid t\right] = -\,\lambda_b(t)\; Var\,(W \mid t) < 0. \tag{2.21}$$

This means, that "the conditional expectation of W for the multiplicative model (2.18) is a decreasing function of $t \in [0, \infty)$", [137]. Obviously, from (2.19), the mixture failure rate increases in the neighborhood of zero when, $\lambda_b(t)$ is increasing. This result is also obtained for the gamma mixture, where the mixture failure rate, reflects the UBT shape (see Fig. 3). Further, a similar result can be obtained for a mixture of Weibull and gamma distributions with increasing failure rates, [66]. This behavior of the mixture failure rate can already be explained by the effects of the well-known principle: “the weakest populations are dying out first” in heterogeneous populations.

Mixtures may arise naturally from heterogeneous populations. The simplest, however a very meaningful case, is a population, which consists of two subpopulations, is not sufficiently studied in the literature. We firstly consider some simple but pertinent example of continuous mixtures of two distributions and discuss some properties describing the shape of the failure rate under these mixtures. Another specific case, which also explicitly illustrates some further applications of the models of this section to a case, where the mixing parameter is the initial (usual unknown) random age is considered in section 2.7.

2.4. Exponential distributions

Consider a mixture pooled from items having constant failure rates, but produced, e.g., by two different manufacturers. The failure rates of these items may be different, due to different production irregularities at these manufacturing sites.

Suppose the mixing proportion, p of items from manufacturing site 1 and $q = 1 - p$ of items from manufacturing site 2, with the corresponding failure rates, λ_i , $i = 1$, 2 and, $\lambda_1 > \lambda_2$. The time to failure of an item picked up at random from this population, is a random variable with the Cdf, $F_1(t)$, or $F_2(t)$. The survival function in this case is the weighted sum of survival functions for the corresponding subpopulations,

$$S\,(t) = p\ S_1(t) + q\ S_2(t) = p\ \exp\,\{-\lambda_1(t)\} + q\ \exp\{-\lambda_2(t)\}\ , \tag{2.22}$$

where, $q = 1 - p$. The corresponding probability density function (pdf) is

$$f\,(t) = p\,\lambda_1 \exp\,\{-\lambda_1\,(t)\} + q\,\lambda_2 \exp\,\{-\lambda\,(t)\}. \tag{2.23}$$

Consider, for example, the ratio of the mixing proportions to be 0.6: 0.4, then the mixture failure rate, in accordance with the definition of the failure rate (1.1), is obtained as

$$\lambda_m(t) = \frac{0.6\,\lambda_1 \exp\{-\lambda_1(t)\} + 0.4\,\lambda_2 \exp\{-\lambda_2(t)\}}{0.6\exp\{-\lambda_1(t)\} + 0.4\exp\{-\lambda_2(t)\}}\ . \tag{2.24}$$

The corresponding plot of $\lambda_m(t)$ for different values of λ_i , $i = 1$, 2 and $p = 0.6$ is shown on Fig.1 below,

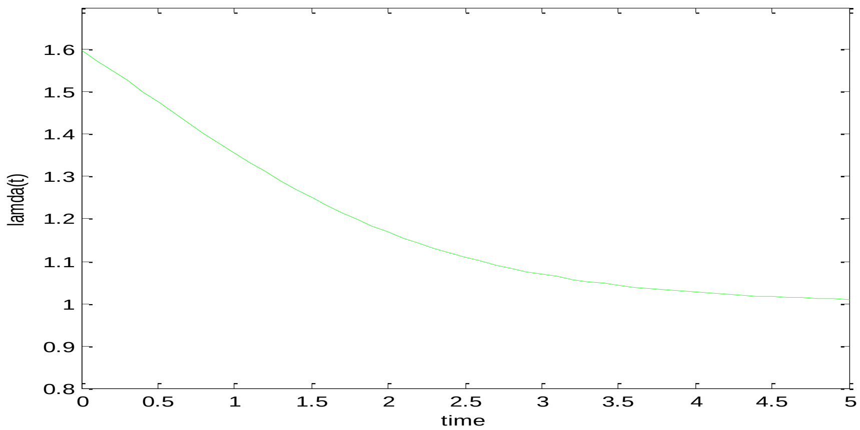

Fig.1: a mixture of exponentials with constant failure rates, $\lambda_1 = 2$, $\lambda_2 = 1$ and $p = 0.6$

From Fig. 1, $\lambda_m(t)$ is decreasing approaching 1 in this case, albeit, constant failure rates are being mixed. This apparent decrease in the observed failure rate was first acknowledged for the heterogeneous set of aircraft engines with each subpopulation described by the constant failure rate, by the authors of reference [243].

Intuitively, the early high failures observed (in Fig.1) may be due to items from the first manufacturing site, as $\lambda_1 > \lambda_2$. As time increases, these items tend to die first, resulting in a mixture failure that is decreasing towards the failure rate of the strongest subpopulation, e.g., from manufacturing site 2 in this case.

2.5. Truncated extreme value distribution (continuous mixture)

Consider the truncated Gumbel distribution, (which is a form of a truncated extreme value distribution) for the operation of mixing as:

$$F(t) = 1 - \exp\{-(w\,v\,(R-1))\}\ ;\ \lambda(t \mid w) = w\lambda_b(t) = v\,w\,R,\ t \geq 0 \text{ and, } v > 0, \tag{2.25}$$

where, $R(t)=\exp(t)$, (and for brevity of notation here and in the rest of the work we omit the argument, i.e., R) and "$\lambda_b(t)$ is some deterministic, increasing (at least for sufficiently large t) continuous function ($\lambda_b(t)\geq 0$, $t\geq 0$)", [137]. Assume further that, $g(w)$, is an exponential pdf with parameter, η. Then,

$$\int_0^{\infty} f\ (t) g(w)\, dw = \int_0^{\infty} w\ v\ R\ \exp\ \{-w\ v\ (R-1)\}\ \eta\ \exp(-w\eta) dw = \frac{v R \eta}{a^2}\ , \tag{2.26}$$

where, $a = v\,[R-1]+\eta$. Therefore,

$$\int_0^{\infty} f\ (t \mid w)\, dw = \eta \int_0^{\infty} \exp\ (-w) dw = \frac{\eta}{a}\ . \tag{2.27}$$

In accordance with (2.11), the mixture failure turns out to be:

$$\lambda_m(t) = 1 + (v-\eta)\,\{v\,[R-1]+\eta\,\}^{-1}, \tag{2.28}$$

which, can be written as,

$$\lambda_m\ (t) = (1+C)^{-1}, \tag{2.29}$$

where, $C = h/b$, $b = vR$, and $h = \eta - v$. It follows, that, $\lambda_m(0) = v/\eta$. When, $h<0$ and $v>\eta$, the mixture failure rate, $\lambda_m(t)$, is monotonically decreasing asymptotically converging (from above) to 1 (the blue curve on Fig. 2), whilst it is monotonically increasing, asymptotically converging (from below) to 1 when $h>0$ and $v<\eta$ (the green curve on Fig. 2) as, $t\rightarrow\infty$. On the other hand, $\lambda_m(t)$ is equal to 1 when, $h=0$ and $v=\eta$, (the red line on Fig. 2). These results show that the mixture failure rate in this case can increase (decrease) or be constant for certain values of parameters.

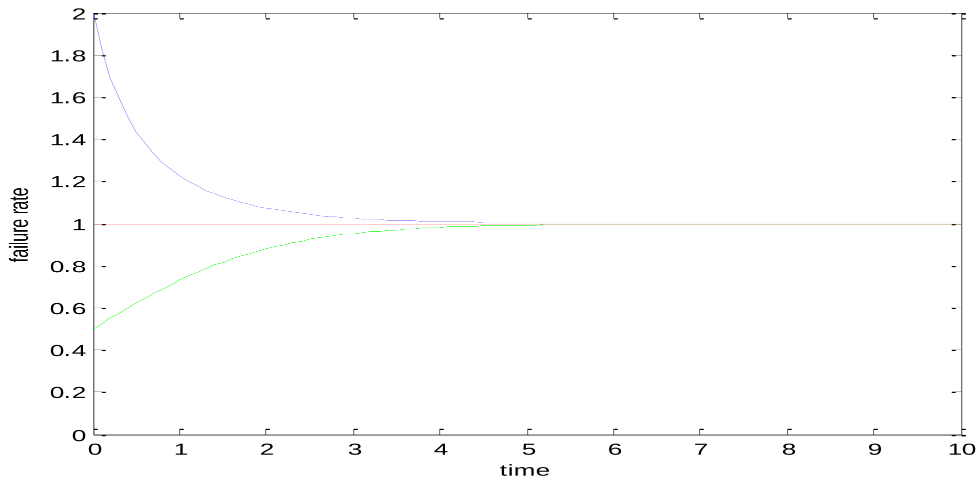

Fig. 2: A mixture failure rate of Gumbel and Exponential distributions for different values of v, η and, $\lambda = 1$.

This result, already, shows that the mixture failure (mortality) rates can also bend down (the upper curve) as time increases (i.e., $t \to \infty$) and even reach a plateau. On the other hand the lower curve provides "a possible explanation for the mortality rate plateau of human populations at great ages observed in [156]", see e.g. also, reference [66].

2.6. The gamma distribution

Consider a Gamma distribution, with the survival function and failure rate given, respectively by:

$$S(t) = \sum_{k=0}^{\alpha-1} \frac{Z(\lambda t)^k}{k!} \;\; ; \; \lambda(t) = \frac{\lambda^k \, t^{k-1}}{\Gamma(k) \sum_{k=0}^{\alpha-1} \frac{(\lambda t)^k}{k!}} \qquad \text{for } t > 0, \qquad (2.30)$$

where, λ is random with distribution: $\mu(w) = \mu \exp(-\mu w)$ and $Z = \exp(-\lambda t)$. The corresponding mixture survival function in this case is

$$\int_0^\infty S(t\mid w)\, g\,(w)\, dw = \int_0^\infty \sum_{k=0}^{\alpha-1} \frac{\mu Z (wt)^k}{k!} \exp(-\mu w) dw$$

$$= \sum_{k=0}^{\alpha-1} \frac{\mu\, t^k}{k!} \int_0^\infty w^k \exp(-w(t+\mu)) dw$$

$$let \quad z = v\,(t+\mu) \Rightarrow v = \frac{z}{(t+\mu)};\; dv = \frac{dz}{(t+\mu)}$$

$$= \sum_{k=0}^{\alpha-1} \frac{\mu\, t^k}{k!\,(t+\mu)^{k+1}} \int_0^\infty z^k \exp\,(-z) dz \qquad \Rightarrow \Gamma(k+1) = k\ \Gamma(k) \quad , \tag{2.31a}$$

$$= \sum_{k=0}^{\alpha-1} \frac{\mu\, t^k}{k\ \Gamma(k)(t+\mu)^{k+1}} k\ \Gamma(k) \qquad \Rightarrow \Gamma(k+1) = k!$$

$$= \sum_{k=0}^{\alpha-1} \frac{\mu\, t^k}{(t+\mu)^{k+1}}$$

as defined in (2.9), whereas the pdf. is determined by the following:

$$\int_0^\infty f\ (t\mid w)\ g\ (w)\, dw = \int_0^\infty \frac{\mu\ Z\ w^k}{(k-1)!} t^{k-1} \exp\,(-\mu\ w)\, dw$$

$$= \frac{\mu\ t^{k-1}}{\Gamma\,(k)} \int_0^\infty w^k \exp\,(-w\ \ (t+\mu))\, dw$$

$$let \qquad z = v\ (t+\mu) \Rightarrow w = \frac{z}{(t+\mu)}; \quad dv = \frac{dz}{(t+\mu)}\ . \tag{2.31b}$$

$$= \frac{\mu\ t^{k-1}}{\Gamma(k).(t+\mu)^{k+1}} \int_0^\infty u^k \exp\ (-z) dz$$

$$= \frac{\mu\ t^{k-1}}{\Gamma(k).(t+\mu)^{k+1}} k\,\Gamma(k) = \frac{\mu\ k\ t^{k-1}}{(t+\mu)^{k+1}}$$

Accordingly with definition (2.11), the mixture failure rate is obtained as:

$$\lambda_m\ (t) = \frac{\mu\ k\ t^{k-1}}{(t+\mu)^{k+1}} \sum_{k=0}^{\alpha-1} \left(\frac{\mu\ t^k}{(t+\mu)^{k+1}} \right). \tag{2.32}$$

From second equation (2.30), $\lambda(t) = \lambda$, when $\alpha = 1$, but when $v = 0$, $\lambda(t) = \infty$. As can be noted if, $0 < \alpha < 1$ then $t \to 0$, but if $\alpha \geq 1$ then $t \to \infty$ and $\lambda(t) = \lambda$ (see the blue dashed curve on Fig. 3). As, $\lambda(0) = 0$ and $\lambda(t)$ is an increasing function asymptotically approaching λ from below when $\alpha > 1$ (see Fig. 3), then $\lambda(t)_{t\to\infty} = \lambda$. But the mixture failure rate reflects an upside-down bathtub shape (UBT): See e.g. the green, red and yellow curves on Fig. 3. Intuitively, the initially increasing failure rate may be due to a large proportion of weak items with large failure rates. As time increases these "die out first", leading to the "bending down" of the mixture failure rate observed in Fig. 3.

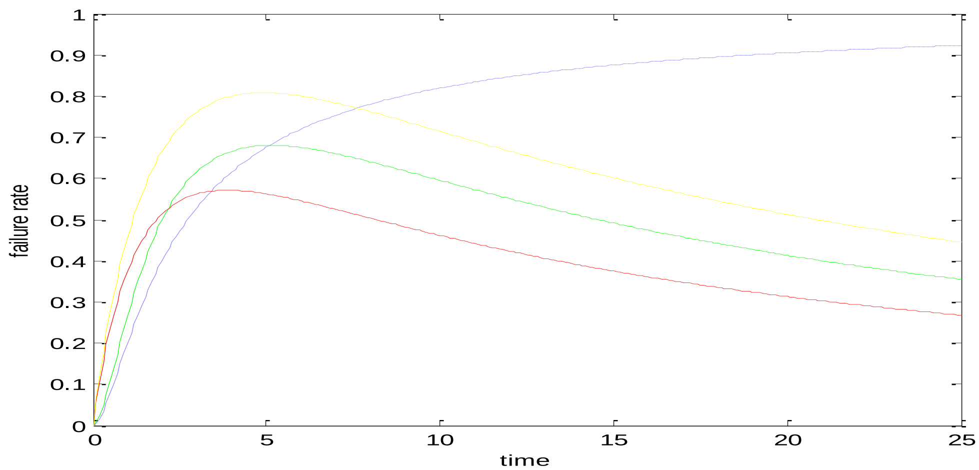

Fig. 3: A plot of mixture gamma failure rates for different values of μ where $k = 2$.

This result was, "experimentally observed for a heterogeneous sample of miniature light bulbs", [94] and for large cohort of medflies, in reference, [195]. It was also proved analytically for some specific cases [94]: the mixture failure rate bents down either in a strong sense or a weak sense when compared with a specific form of our model (2.1): i.e. "the unconditional expectation of the random failure (hazard) rate, $\lambda\left(t \mid w\right)$ with similar to the subpopulation failure rates monotonic properties in the mixtures", see e.g. reference, [58] :

$$\lambda_p\left(t\right) = \int_0^\infty \lambda\left(t \mid w\right) g\left(w\right) dw. \tag{2.33}$$

From (2.11) and (2.12), $\lambda_m\left(0\right) = \lambda_p\left(0\right)$ and if, $\lambda\left(t \mid w\right)$, $w \in \left[0, \infty\right)$ is increasing, then $\lambda_p\left(t\right)$ increases as well. We use a somewhat similar analogy for studying the aging properties of other related main reliability characteristics in the subsequent sections. However, due to the conditioning on (2.10), the mixture failure rate may have a different shape. It may decrease, preserve the IFR property, be BT or UBT (as in Fig. 3), etc.

2.7. Modelling for items (objects) with unknown initial ages

Consider a specific but very important and often encountered in practice model of mixing, where the initial age (usually not known), is the mixing parameter. The failure rate of a distribution describing this random age was considered initially in [128]. The impacts of this random delay (initial age) on reliability characteristics of the baseline lifetime distribution of an object is of interest. "Let W be a random time of delay with support in $[0,\infty)$" , [66] with,

$$S_w(t) = \Pr(T_w > t) = \frac{S(u)}{S(w)} , \tag{2.34}$$

where, T_w, is the remaining lifetime, $u = w + t$ and $W = w$. Then, for this setting, the conditional distribution is

$$F_w(t) = F(t \mid W = w), \tag{2.35}$$

with the corresponding failure rate $\lambda(t \mid w) = \lambda(u)$. Accordingly with (2.11), the mixture failure rate is obtained as,

$$\lambda_m(t) = \int_0^\infty \lambda(u)\, g(w \mid t)\, dw , \tag{2.36}$$

which can be rearranged as,

$$\lambda_m(t) = \lambda_b(t) + \int_0^\infty (\lambda(u) - \lambda_b(t)) g(w \mid t) dw . \tag{2.37}$$

We investigate, the behavior of $\lambda_m(t)$ when the shape of $\lambda_b(t)$ is assumed to be known. However, for this analysis, we assume "a concrete form of the baseline and the mixing distributions and necessarily the distribution function (Cdf), should belong to the IFR class of distributions", see e.g. reference, [66] in this case. Two specific cases are considered in what follows.

2.8. The shape of the mixture failure rate

Consider, "$F(t)$ to be a specific type of Weibull distribution with the linear increasing failure rate: $\lambda_b(t) = v\,t$, $v > 0$" [128], then from (2.15) we have,

$$\lambda(t \mid w) = \lambda\,(u) = \lambda_b\,(t) + v\,w. \tag{2.38}$$

Specifically, from (2.37), the following is obtained,

$$\lambda_m(t) = \lambda_b(t) + v\int_0^\infty w\,g\,(w \mid t)\,dw = vt + v\;E[W \mid t]\,. \tag{2.39}$$

As the baseline failure rate given by $v\,t$ is an increasing function, this means that for, $\lambda_m(t)$ to decrease, the variance of $E[W \mid t]$ should be large. In fact, "$\lambda_m(t)$ seem to decrease in some interval, $[0\,,\,c)$ and increases in $[c\,,\infty)$ where, C can be determined by the equation, $Var(W \mid c) > \lambda\,(c)'$" [81]. Thus in this case, assuming the conditional variance of the random variable $W \mid t$ is decreasing in $t \in [0\,,\,\infty)$, the mixture failure rate is of BT shape type.

The truncated value distribution was considered in section 2.5. Suppose now, the Cdf, $F\,(t)$ is given by

$$F\;(t) = 1 - \exp\;\{-v\;(R-1)\}\,, \tag{2.40}$$

where, $R = \exp\{t\}$ and the corresponding (exponential) failure rate is $\lambda\;(t) = v\,R$, $v > 0$. As opposed to the previous case, the mixture model reduces to the multiplicative model of the form:

$$\lambda\;(t \mid w) = \lambda\;\;(u) = \lambda_b\;\;(t)\;R\,. \tag{2.41}$$

This model can easily be transformed into the following expression,

$$\lambda\;(t \mid w) = \lambda\;(u) = \lambda_b\;(t)\;[R-1] + \lambda_b\;\;(t) = w'\;\lambda_b\;\;(t) + \lambda_b\;\;(t)\,, \tag{2.42}$$

where, $w' = \exp(w) - 1$. Now,

$$\lambda_m(t) = vR + \frac{v-\eta}{v[R-1]+\eta}\ , \tag{2.43}$$

whereas, the difference,

$$\lambda_m(t) - \lambda_b(t) = 1 + \frac{v-\eta}{v\,[R-1]+\eta}, \tag{2.44}$$

enables the analysis of shapes of mixtures failure rates when compared with the corresponding baseline failure rate.

The corresponding plots of $\lambda_b(t)$ and, $\lambda_m(t)$ for different values of, v, η, and $\lambda = 1$ are reflected in Fig. 4. It is apparent from this figure that $\lambda_b(t)$ (see e.g. the blue curve on Fig. 4) is monotonically increasing as a function of, t and the mixture failure rate is also increasing parallel to $\lambda_b(t)$ as $t \to \infty$ (see e.g., for $v = 2$, $\eta = 1$ the yellow curve and $v = 3$, $\eta = 1$ the purple (magenta) curve; $v = 5$, $\eta = 1$ red curve and $v = 1$, $\eta = 2$ green curve). In particular, when v is greater than, η, the curves initially increase slowly, then at some point, t, increase sharply (though still parallel above the baseline failure rate) and deviating further away from $\lambda_b(t)$ as time increases. When v is sufficiently large (i.e. $t \to \infty$), the mixture failure rate exhibits a bathtub (BT) shape (see e.g. the red curve on Fig. 4, where $v = 5$, $\eta = 1$).

On the other hand, when v is less than η, the mixture failure rate increases "parallel" from below approaching the $\lambda_b(t)$ as time increases (see e.g. the green curve on Fig. 3 where $v = 1$, $\eta = 2$). It is worth to note that, a similar behavior may be observed for other larger values of η when v is also fixed and reduces to the shape of $\lambda_b(t)$ when, $v = 1$, $\eta = 1$.

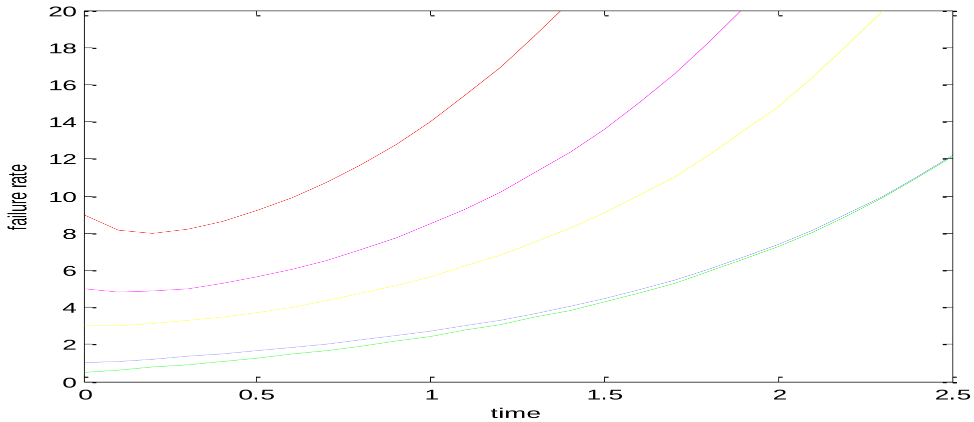

Fig. 4: A plot of truncated extreme value and Exponential mixture failure rate for different values of v , η , and $\lambda = 1$

These results imply that for certain values of parameters for this model, the mixture may either preserve the IFR property or have a bathtub shaped: initially decreasing to some minimum point and eventually increasing as $t \to \infty$ (as in Fig. 3, see the red curve). The IFR preservation property was shown, although under very stringent conditions, by [153] to hold even for mixtures of strongly IFR distributions. See, e.g., [94] for some necessary conditions under which continuous mixtures of increasing failure rates preserve the IFR property. Other related and relevant discussions on this topic could be found in references, [112], [113], [177] and [178] to name but a few.

2.9. Mean residual life (MRL) Model

In what follows, we present some useful general results on the properties of MRL to be used in obtaining the corresponding 'shape properties' for mixtures.

2.9.1. The shape of the MRL

Let T_x as in section 1.2.2., be the remaining lifetime of an item of age t, described by the corresponding conditional survival function (1.3), $S_x(t) = S(t \mid x)$, which we reproduce here for convenience of reference, i.e.,

$$S_x(t) = \Pr(T_x > t) = \exp(-\vartheta), \tag{2.45}$$

where, $\vartheta = \int_t^{t+x} \lambda(v)dv$. The MRL function is defined as the expectation of the random variable, T_x via (2.45) as follows:

$$m(t) = E[T_x] = \frac{\int_t^{\infty} S(v)\, dv}{S\,(t)} = \int_0^{\infty} \exp(-\vartheta)dx\,. \tag{2.46}$$

For $x = 0$, equation (2.46) is equal to the expectation of the random variable, T, e.g., $E[T]$. Some useful properties of model (2.46) are studied in references [66] and [129]. These properties and other results in literature that are useful in the sequel are recalled and discussed in what follows. Among the prominent results are the relations between the failure rates, survival function with MRL, which are respectively obtained by first rearranging (2.46):

$$m(t)S(t) = \int_t^{\infty} S(u)du\,. \tag{2.47}$$

Differentiating with respect to, t, on both sides of (2.47), we obtain the following important relation between the failure rate and the MRL function,

$$\begin{aligned} & m^{'}(t)S(t) + m(t)\,(S(t))^{'} = -S(t) \\ & \frac{f(t)}{S(t)} = \frac{m'(t)+1}{m(t)} \Rightarrow \lambda(t) = \frac{m'(t)+1}{m(t)} \end{aligned}\,. \tag{2.48}$$

See e.g. also references, [66], [86] and [129] to mention a few. Integrating both sides of (2.48) gives a result that immediately leads to the representation of the survival function in terms of the MRL. For instance,

$$\begin{aligned} & \int_0^x \frac{d}{dt}(\ln S\,(t)) = -\int_0^x [m'\,(t)+1]m\,(t)^{-1}\,dt \\ & S\,(t) = \exp\left\{-\int_0^x [m'\,(t)+1]m\,(t)^{-1}dt - C\right\} = \exp\left\{-\int_0^x \frac{d}{dt}(\ln m\,(t)) - C\right\} \end{aligned}\,, \tag{2.49}$$

where, $C = \int_0^x 1/m\,(t)dt$. The famous inversion formula, relating the survival function to the MRL is obtained as,

$$S(t) = m(0)\, m(t)^{-1} \exp\left(-\int_0^t \left(1/m(u)\right) du \right). \tag{2.50}$$

Alternatively MRL can be represented via the survival function (1.2) and the density function in the following way, [229 p.2],

$$m(t) = \frac{\int_t^\infty v\, f(v)\, dv}{S(t)} - t \quad . \tag{2.51}$$

These simple but meaningful results play a pivotal role in the analysis of the shapes of MRL functions. From results (2.48) and (2.50), when the survival and failure rate functions increase (decrease) then MRL is also decreasing (increasing). As a specific case, consider the Lindley distribution with the pdf,

$$f(t \mid \lambda) = D\, Z\, \lambda^2 (1 + t) \quad ,$$

where, $D = 1/(\lambda + 1)$ and similar to section 2.6, $Z = \exp(-\lambda t)$, $t,\, \lambda > 0$. Therefore, the corresponding survival function,

$$S(t \mid \lambda) = Z\, (1 + D \lambda t) \ .$$

This distribution may be considered as a mixture of an exponential distribution with scale parameter λ and a gamma distribution, where λ is the scale parameter and α is the shape parameter (in particular, $\alpha = 2$ in this case). The mixing proportions are, respectively, λD and, D. Hence, in accordance with definition (1.1) we have,

$$\lambda(t) = D Z \lambda^2 (1 + t) \times [\lambda(1 + t)]^{-1} (\lambda + 1) Z \, .$$

Rearranging, we have,

$$\lambda(t) = \frac{\lambda^2}{\lambda(1 + t) + 1} (1 + t),$$

whereas, according to definition (2.46) the corresponding mean residual life is obtained by evaluating the following,

$$m(t) = [\lambda\ Z\ (1 + t)]^{-1} \int_t^\infty \lambda (1 + y) \exp\, (-\lambda\ y) dx \, .$$

As the integration of the integral exists in this case and the integrand is continuous on the interval, $[0,\infty)$, this problem is turned into a limit problem,

$$\int_t^{\infty} \lambda\,(1+x)\exp(-\lambda\,y)\,dy = \frac{\lim}{b\to\infty}\int_t^{\infty}(\lambda\,(y+1)+1)\exp\,(-\lambda\,y\,t\,)dx$$

$$\Rightarrow \frac{\lim}{b\to\infty}\left[\{Z-\exp\,(-\lambda b)\}+\frac{1}{\lambda}\{Z-\exp\,(-\lambda b)\}\right]+Z\left(\frac{1}{\lambda}+y\right)$$

$$= Z\left(\frac{2}{\lambda}+y+1\right)$$

As a result, the mean residual life function for Lindley distribution is:

$$m\,(t)=[\lambda Z\,(1+t)]^{-1}\left[Z\left(\frac{2}{\lambda}+y+1\right)\right]=\frac{\lambda\,(1+t)+2}{\lambda\,\{\lambda\,(1+t)+1\}}.$$

The plots of the failure rate functions and the corresponding shapes of the MRL functions are, respectively, shown for different values of $\lambda>0$ on Fig.5 and Fig.6 below.

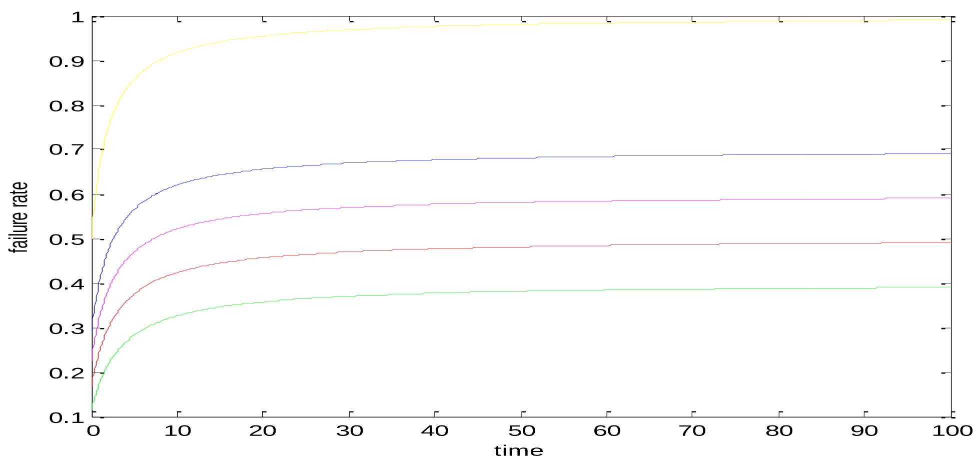

Fig. 5: A plot of failure rate for Lindley distribution for values of, $\lambda > 0$.

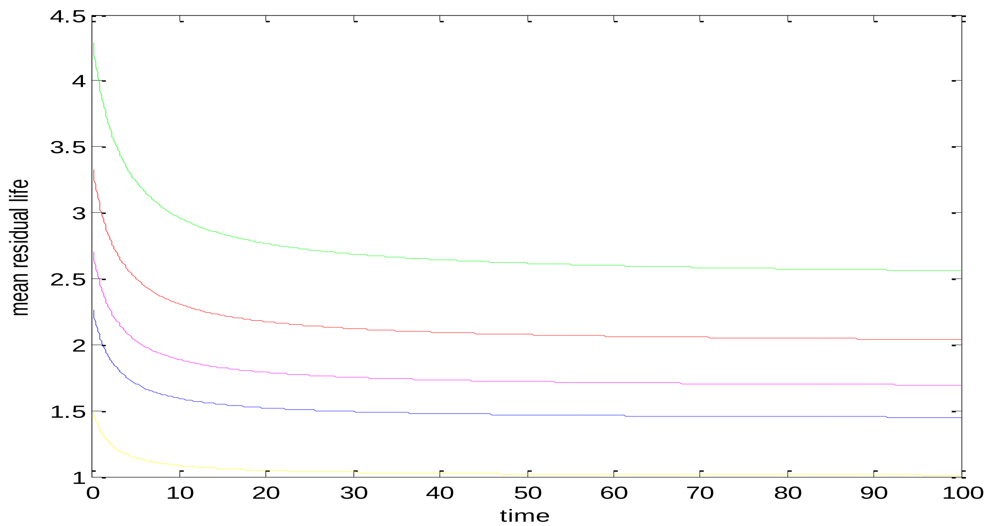

Fig. 6: A plot of mean residual life for Lindley distribution for values of, $\lambda > 0$.

From Fig. 5, the failure rate is increasing and after some interval of time asymptotically approaches the straight line. It is well-known that, "the Gompertz model does not account for mortality rates that increases and after some interval of time reaches a plateau", [58]. The Lindley distribution may be a suitable choice for analysis in this case. The shapes of the corresponding MRL are also decreasing and after some interval of time asymptotically approaches a straight line (see Fig. 6). Thus, the Lindley distribution, as opposed to the Gompertz model can describe the mortality rate plateau, which was also observed in [156] for human populations at great ages.

Relation (2.51) could be used as an alternative to (2.46), when the survival function cannot be obtained in an explicit form and/or requires some tedious numerical integration to obtain it. For instance, consider the two-parameter standard gamma distribution with the pdf,

$$f(t) = \frac{Z \lambda^{\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{for} \quad t \geq 0,$$

where, $Z = \exp(-\lambda t)$ and λ is the scale parameter with α being the shape parameter (all positive) and

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} \exp(-u) du,$$

is, as usual, the gamma function. Although, it is well-known that for the general case, α (particularly, for non-integer, α) the Cdf, of this distribution is not found in the closed form, when, α a positive integer (the Erlangian distribution):

$$F(t) = 1 - \sum_{k=0}^{\alpha-1} \frac{Z\,(\lambda\, t)^k}{k!},$$

which, leads to the survival function, $S(t)$ given by the first equation (2.30) and the corresponding failure rate is given by the second equation (2.30). In this example, the failure rate is an increasing function asymptotically approaching λ from below when, $\alpha > 1$. A similar result is obtained [102 p. 61] using the following identity:

$$(\lambda(t))^{-1} = \int_0^{\infty} \left(1 + \frac{v}{t}\right)^{\alpha-1} \exp(-\lambda v)\, dv \cdot$$

This function is decreasing in, t, for $\alpha \geq 1$. It implies that "the failure rate is also increasing, whereas, it is decreasing for $0 < \alpha \leq 1$", see e.g., reference [66]. The Gamma distribution reduces to a constant when, $\alpha = 1$ (i.e. to an exponential distribution, which exhibits a non-aging property). The corresponding mean residual life, is obtained using relation (2.51) by [229 pp3-4] (see also [66 p. 22] and [86 p. 131]):

$$m(t) = \frac{\lambda^{\alpha-1} t^{\alpha}}{\Gamma(\alpha)\, S(t)} + \alpha\, \lambda^{-1} - t$$

where, $S(t)$ is the survival function for $F(t)$, given by first equation (2.30). Therefore, the MRL for gamma distribution is:

$$m(t) = \frac{\lambda^{\alpha-1} t^{\alpha}}{\Gamma(\alpha) \sum_{k=0}^{\alpha-1} \frac{Z\,(\lambda\, t)^k}{k!}} + \alpha\, \lambda^{-1} - t \cdot$$

The plots of the corresponding MRL for different values of $\alpha \geq 1$ and $\lambda = 1$ are shown in Fig. 7, below.

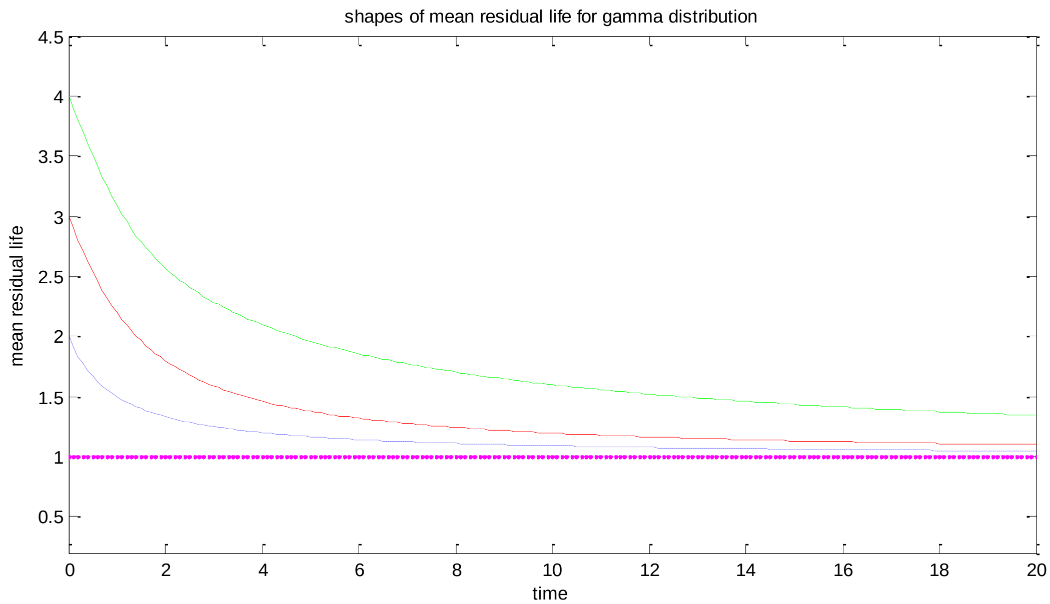

Fig. 7: A plot of MRL for gamma distribution, for different values of, $\alpha_i \geq 1$, $(i = , 2 , 3 , 4)$ and, $\lambda = 1$.

Thus, for an increasing failure rate, the mean residual life decreases (see the red, blue and green curves on Fig. 7). When, $\alpha = 1$, the MRL is constant (horizontal purple line), whereas it is increasing for values of, $0 < \alpha < 1$, see [86]. The MRL is also decreasing for $\alpha > 1$ and increasing when, $\alpha < 1$, for the popular two-parameter standard Weibull distribution and reduces to the constant when $\alpha = 1$. The Gompertz model that is often used to model the increasing mortality rates of biological organisms is also relevant in this case. Its MRL is decreasing as time increase.

The characteristics of the MRL function are, also, often inferred from its failure rate using relation (2.48). As a specific case, consider the simplest example of an exponentially distributed lifetime random variable T with the Cdf,

$$F(t) = 1 - S(t) = 1 - \exp(-\lambda t).$$

In this case, the failure rate is constant, (i.e. $\lambda(t) = \lambda$) and as a result, the corresponding MRL is $m(t) = 1/\lambda$. Denote, $\upsilon(t) = 1/m(t)$, then from (2.48) the following relation is obtained,

$$\lambda(t) = -(\upsilon'(t)/\upsilon(t)) + \upsilon(t).$$

The function $\upsilon(t)$ is, therefore, “asymptotically equivalent to $\lambda(t)$ in the following sense”, [66]:

$$|\lambda(t)-\upsilon(t)| \to 0 \quad \text{as, } t\to\infty, \text{ if and only if } \quad |\upsilon'(t)/\upsilon(t)| = m'(t)/m(t) \to 0 \quad \text{as } t\to\infty.$$

These results, also, hold for power functions of the form: $\upsilon(t)=B\,t^{\alpha-1}$ but does not hold for functions that are sharply increasing, e.g. $\upsilon(t)=\exp(t)$ or, $\upsilon(t)=\exp(t^2)$. The following weaker asymptotic relation holds: $\lambda(t)=\upsilon(t)(1+o(1))$ as $t\to\infty$ ensures that $|m'(t)|\to 0$ for the sharply increasing functions, [66].

Some further results on the limiting and asymptotic properties are studied in [115] and [245] to name a few. We also refer to [66], [71], [86], [129] and [238] for some other detailed discussions on further implications of these relations and results on this topic. On the other hand, sufficient conditions for the monotocity of the failure rate in terms of the monotocity of the MRL function are contained in the theorem 2.5 of [66]. Other alternative representations of MRL in terms of the failure rate are also established in [118]. This also provide some further flexibility for studying the shapes of the MRL function.

2.9.2. Relations between non-monotonic failures rates and non-monotonic MRL

Non-monotonic aging properties of the failure rate and the mean residual life function are often useful for describing the aging properties of many electronic, mechanical or even biological objects.

For a meaningful illustration, consider the inverse Weibull (IW) distribution with the Cdf,

$$F(t)=\exp\{-\beta t^{-\alpha}\},$$

where, $t>0$ and $\alpha>0$ is the shape parameter and $\beta>0$ is the scale parameter. The pdf of the IW distribution is established as,

$$f(t)=\alpha\,\beta(t)^{-(\alpha+1)}\exp\left(-\beta\,t^{-\alpha}\right),$$

where, $t\geq 0$ and $\alpha,\beta>0$. Then, the failure rate in this case is,

$$\lambda(t)=\frac{\alpha\beta(t)^{-(\alpha+1)}\exp\{-\beta t^{-\alpha}\}}{1-\exp\{-\beta t^{-\alpha}\}} \qquad =\frac{\alpha\beta(t)^{-(\alpha+1)}}{\exp\{-\beta t^{-\alpha}\}-1}.$$

Since the shape of the failure rate does not in this case, necessarily, depend on the scale parameter β, we may take $\beta=1$ without loss of generality and the corresponding plots of the failure rate functions for different values of $\alpha>0$ are shown in Fig. 8 below,

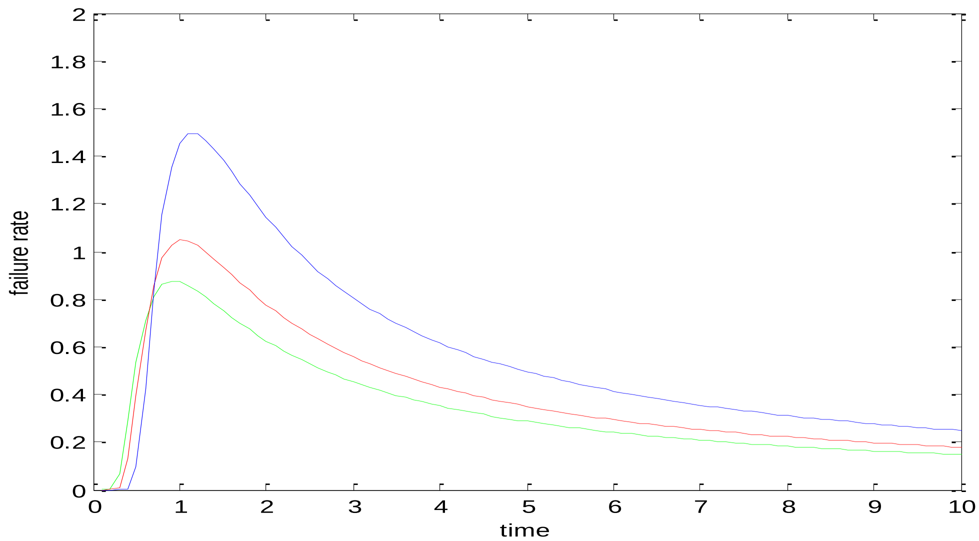

Fig. 8: plots of the failure rate for IW for different values of the shape parameter, $\alpha_i>0$ and scale parameter, $\beta=1$.

The corresponding mean residual life can be obtained by using relation (2.51) and is given as follows:

$$m(t)=(S(t))^{-1}\int_t^{\infty} v\ f(v)dv-t$$

$$=\frac{\alpha\beta}{1-\exp\left(-\beta\, t^{-\alpha}\right)}\int_t^{\infty} v^{-\alpha}\exp\left(-\beta v^{-\alpha}\right)dv-t\,.$$

$$\Rightarrow m(t)=\frac{\beta^{(1/\alpha)}\Gamma\left(1-1/\alpha,\ \beta t^{-\alpha}\right)}{1-\exp\left(-\beta t^{-\alpha}\right)}$$

The plots for different values of the shape parameter, $\alpha\ge1$ and the scale parameter, $\beta\ge1$ shown in Fig. 9 below,

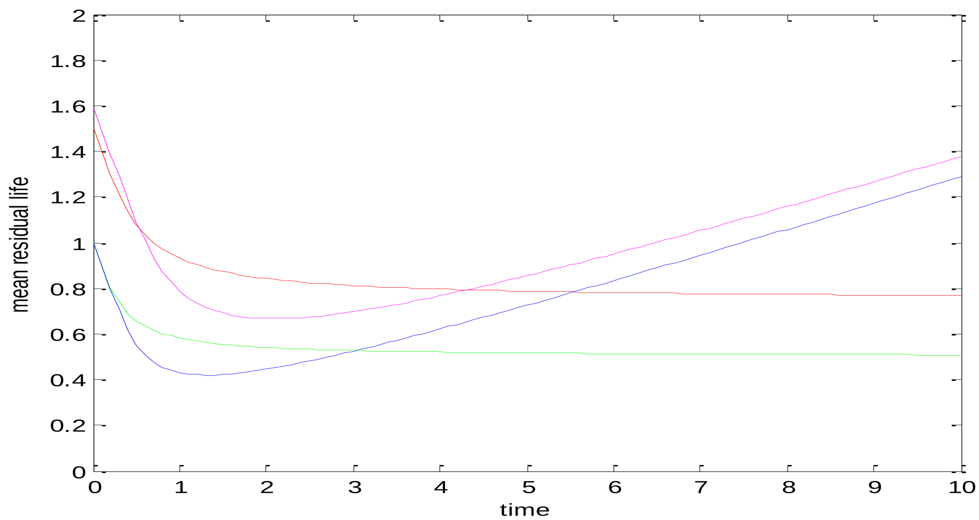

Fig. 9: plots of the mean residual life for IW for different values of the shape parameter, $\alpha \geq 1$ and scale parameter, $\beta \geq 1$.

As can be noted from Fig. 8, for the IW distribution, the failure rate is UBT shaped, whereas the corresponding MRL (see Fig. 9) is decreasing for certain values of parameters and UBT for other values. These results indicate that the inverse Weibull model can be considered in modeling and analysis in many situations. Other results for the case, where the failure rate with the BT shape leads to a decreasing $m(t)$ and the UBT failure rate results in the increasing $m(t)$, are considered in [177]. We also refer to [111] for some generalized Weibull distributions with the BT failure rate that exhibit MRL functions with UBT shapes.

We are, on the other hand, interested in the effects of heterogeneity on the main reliability characteristics. In what follows, we will consider the corresponding mixture operations for the MRL functions and briefly discuss results relating to the proportional mean residual (MRL) model, which will be particularly useful for further discussions of the bending properties of the MRL functions under mixtures in section 3.4.2.

2.9.3. Mixture operations for MRL functions and the Proportional MRL Model

Suppose (as in section 2.2.), $F(t)$ to be indexed by a nonnegative random variable, W. Then in accordance with (2.46) and (2.50), we have

$$m(t \mid W = w) = m(t \mid w) = \frac{\int_t^{\infty} S(v \mid w) dv}{S(t)}, \tag{2.52}$$

with the corresponding mixture mean residual life model defined as

$$m_m(t) = (B(t \mid w))^{-1} \int_t^{\infty} \int_0^{\infty} S(v \mid w) g(w) dw dv = \frac{\int_0^{\infty} S_m(v) dv}{S(t)}. \tag{2.53}$$

where,

$$B(t \mid w) = \int_a^b S(t \mid w) g(w) dw.$$

Relation (2.53) can be written "in terms of the MRL function, $m(t \mid w)$":

$$m_m(t) = (B(t \mid w))^{-1} \int_t^{\infty} \int_0^{\infty} S(v \mid w) g(w) dw dv = \int_0^{\infty} m(t \mid w) g(w \mid t) dw \tag{2.54}$$

where, $g(w \mid t)$ is the probability density function of the random variable W, which is conditioned on the event, $T_x > t$. Usually, the proportional MRL model, as a specific case of (2.54), is considered for the corresponding modeling and analysis. We briefly discuss some results relating to this model. Specifically, we consider, the specific case of the 'proportional MRL' model, to be given by,

$$m(t \mid w) = w^{-1} m(t), \tag{2.55}$$

where, $m(t)$ is the baseline MRL function and $w > 0$. From (2.55),

$$\lambda(t \mid w) = \{m'(t \mid w) + 1\}[m(t \mid w)]^{-1} = m'(t) \lambda_b(t) + w \lambda_b(t), \tag{2.56}$$

where, $m'(t \mid w) > w^{-1} m'(t) > -1$, implying that $m'(t \mid w) > -w$. In this form model (2.56), already contains both models (2.15) and (2.18). Therefore, it follows from (2.16), (2.20) and

(2.11) (and from the fact that now, $w \in [1\ ,\ \infty)$) that the mixture failure rate is given in this case by

$$\lambda_m(t) = \int_1^{\infty} \lambda(t \mid w) g\ (w) dw = m^{'}(t) \lambda_b(t) + \lambda_b(t) E[W \mid t]\ . \tag{2.57}$$

Then using (1.17) and (1.18), we can state: "$E[W \mid t]$ is decreasing in $t \in [0\ ,\ \infty)$", [81]. We, also, refer to reference [129], in which some conditions for preservation of the monotocity properties of, $m_m\ (t)$ are established.

From the above results and utilizing relation (2.48), it may be concluded: if $\lambda_m(t)$ is increasing (decreasing) in $[0, \infty)$ then $m_m(t)$ is decreasing (increasing) in the corresponding interval. Hence, our heterogeneous population is improving as "the weakest populations are dying out first".

2.10. Reversed Failure Rate (RFR) Model

2.10.1. Some general properties of RFR Model

Let T be a lifetime with the Cdf, $F(t)$ and the pdf, $f(t) = F^{'}(t)$. The corresponding reversed failure rate is then defined by the ratio (1.5), which we reproduce here for convenience of reference,

$$r(t) = \frac{f(t)}{F(t)}, \tag{2.58}$$

For a sufficiently small change in time Δt, the model (2.58) i.e., $r(t)\ \Delta t$ may be interpreted as the conditional probability of failure of an item (object), in $[t - \Delta t, t]$, where the failure is assumed to have occurred in the interval $[0, t)$. This additional aspect is important in the descriptions of lifetime distributions. The reversed failure rate also uniquely defines, $F(t)$ via the following analogue of the exponential representation (1.2),

$$F(t) = \exp\ (-\theta)\ , \tag{2.59}$$

where, $\theta = \int_t^{\infty} r\left(v\right) dv$ and

$$f\left(t\right) = r\left(t\right) \exp\left(-\theta\right). \qquad 2.60)$$

As noted by [66] and [129], for proper lifetime distributions and $t > 0: \theta \neq \infty$, whereas

$$\int_0^{\infty} r\left(v\right) dv = \infty \ . \qquad (2.61)$$

This fact implies that $\lim_{t\to 0} r\left(t\right) = \infty$ and the case, $F\left(0\right) = 0$, may also be considered as the corresponding limit.

The above facts will be useful for analysis of the shapes of RFR. The simplest way for investigating the relationship between the reversed failure rate and the failure rate is to consider,

$$r\left(t\right) = \frac{\lambda\left(t\right) S\left(t\right)}{F\left(t\right)} . \qquad (2.62)$$

Using relation (2.62) and considering (2.59) it can be shown that (2.62) reduces to,

$$r\left(t\right) = \frac{\lambda\left(t\right)}{\exp\left(u\right) - 1}, \qquad (2.63)$$

where as in (1.2), $u = \int_0^t \lambda\left(v\right) dv$. From (2.63), if $\lambda\left(t\right)$ is decreasing, then $r\left(t\right)$ is decreasing. The simplest case of an exponential distribution with constant failure rate is considered, by the authors of reference, [66]. This function decreases exponentially as, $t \to \infty$, whereas its behavior as $t \to 0$ is defined by the function t^{-1}. The reversed failure rate is also decreasing for the Lindley distribution, which has an increasing failure rate (see Fig. 10). For example, using relation (2.63), the RFR in this case is obtained as,

$$\begin{aligned} r\left(t\right) &= D\lambda^2\left(1+t\right) / \left(D\left\{\lambda\left(t+1\right)\right\} + Z - 1\right) \\ &= \frac{\lambda^2\left(1+t\right)}{\lambda Z t} \Rightarrow r\left(t\right) = \frac{\lambda\ Z\left(1+t\right)}{t} \end{aligned} \quad ,$$

where, D and Z are as defined earlier.in section 2.5.1. The corresponding plots for different values of $\lambda > 0$ are shown in Fig. 10, below.

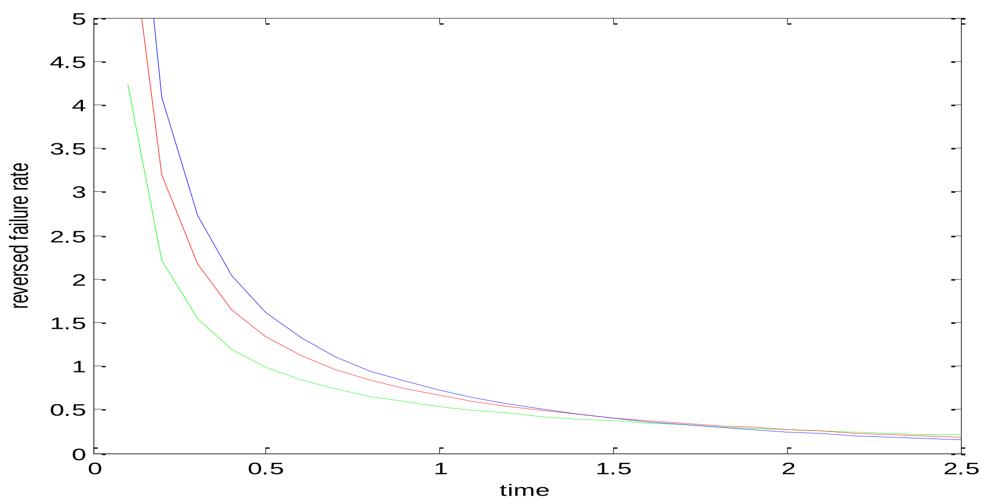

Fig. 10: A plot of reversed failure rate for different values of the shape parameter, $\lambda > 0$.

As another example, consider a lifetime with the distribution function "$F(t) = \exp\left(-c\, t^{-\alpha}\right)$ with $t > 0$ and $c\,,\alpha > 0$". As, $f(t) = F^{'}(t)$, then in accordance with (1.1) the corresponding failure rate is,

$$\lambda(t) = \frac{f(t)}{S(t)} = \frac{c\,\alpha \exp\left(-c\,t^{-\alpha}\right)}{1 - \exp\left(-c\,t^{-\alpha}\right)} \Rightarrow \lambda(t) = \frac{c\,\alpha\; t^{-(\alpha+1)}}{\exp\left(c\,t^{-\alpha}\right) - 1},$$

whereas, utilizing definition (2.58), we have,

$$r(t) = \frac{f(t)}{F(t)} = c\,\alpha\,t^{-(\alpha+1)}.$$

The shapes of the failure rate and reversed failure functions for pairs of $c\,,\alpha > 0$ are shown, respectively, in Fig. 11 and Fig. 12, below.

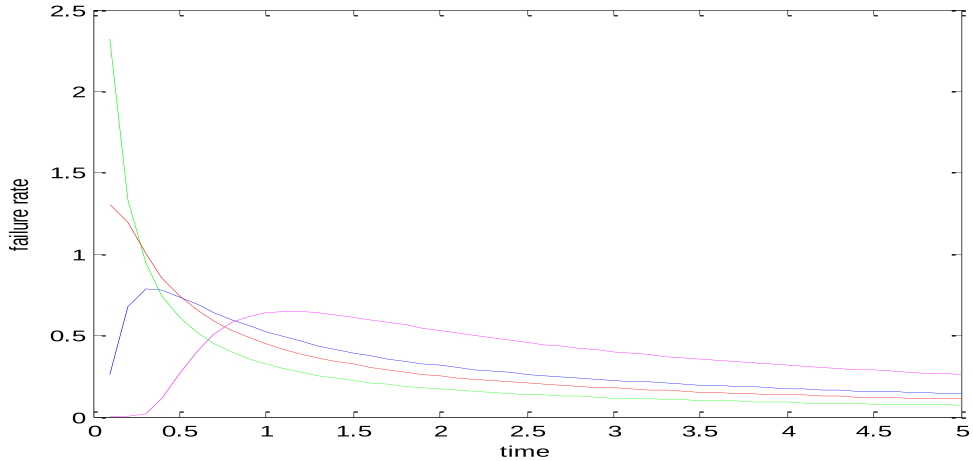

Fig. 11: A plot of failure rate for different pairs of values for $c, \alpha > 0$.

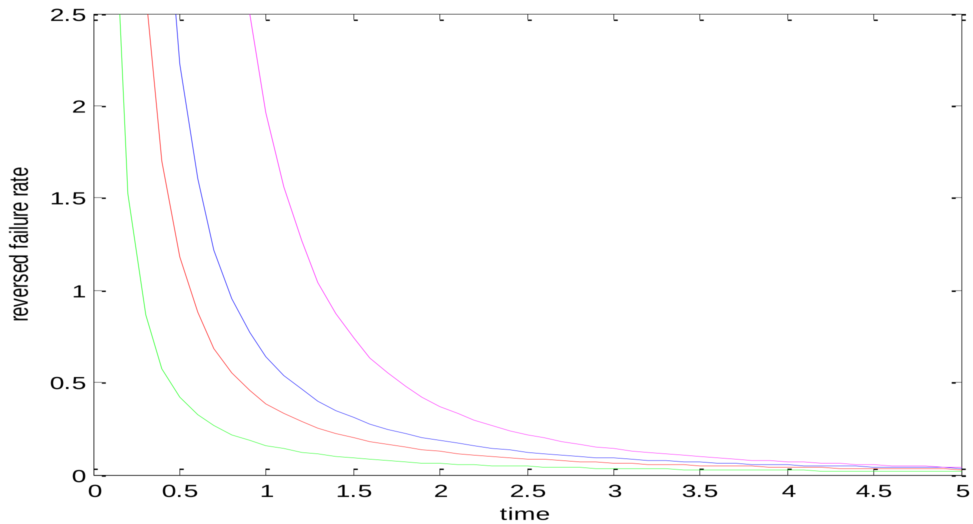

Fig. 12: A plot of reversed failure rate for different pairs of values for $c, \alpha > 0$.

The failure rate is exponentially decreasing for values of c , $\alpha < 0.63$ and is upside-down bathtub (UBT) for the corresponding values of c , $\alpha \geq 0.63$. The corresponding reversed failure rate is decreasing for all pairs of values, c , $\alpha > 0$. Another distribution, which has a decreasing reversed failure rate is the inverse Weibull distribution with the UBT failure rate, (see Fig. 8). For example, from (2.58), the corresponding RFR for the IW distribution is obtained as,

$$r\left(t\right) = \alpha\,\beta\,\left(\beta t\right)^{-(\alpha+1)}.$$

The corresponding plots of RFR for different values of $\alpha > 0$ and $\beta = 1$ are shown in Fig.13 below.

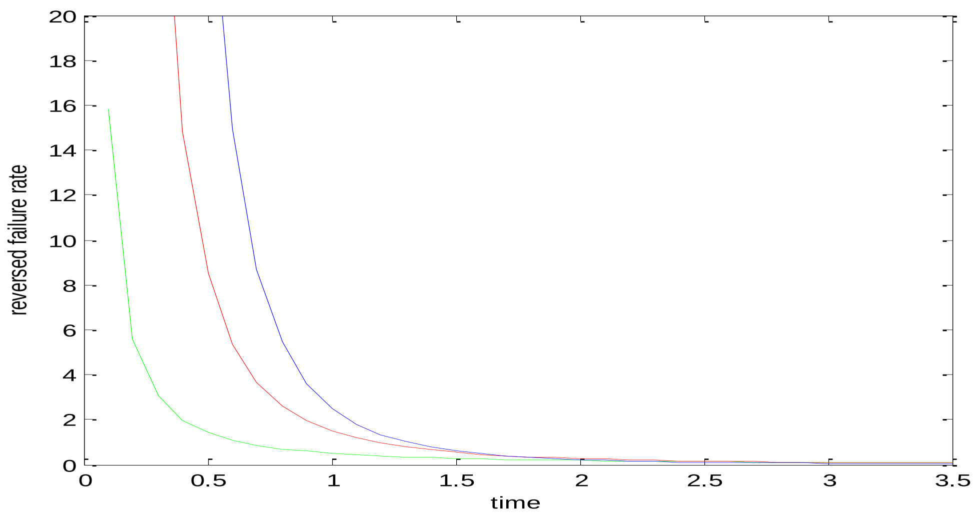

Fig. 13: A plot of reversed failure rate for different pairs of values for $\alpha > 0$ and $\beta = 1$.

2.10.2. Mixture operations for RFR functions and the Proportional RFR model

It is well known that the effects of heterogeneity may change properties of the main reliability indices. Does this mean that the reversed failure rate (which is decreasing even for distributions with the increasing, BT and UBT failure rates) would be increasing in this case? As in "frailty" models, we now investigate the impacts of the unobserved random variable W on the RFR. Let assumptions made in sections 2.1., and 2.2. about the distribution of lifetime random variable T hold, then (as in frailty models), the corresponding distribution indexed by $W = w$ is given by:

$$F\,(t \mid w) = \exp\,(-\vartheta), \tag{2.64}$$

where, now $\vartheta = \int_t^{\infty} r\,(v \mid w) dv$ and $r\,(v \mid w)$ is the conditional reversed failure rate. Therefore, taking the expectation on both sides in (2.64) with respect to W as in (2.3), we have the mixture Cdf, defined similar to (2.9) as,

$$F_m(t) = E\left[F(t \mid W)\right] = \int_a^b F(t \mid w)\; g(w)\; dw,, \tag{2.65}$$

Therefore, the corresponding mixture reversed failure rate is,

$$r_m(t) = \frac{\int_a^b f\left(t \mid w\right) g\left(w\right) dw}{\int_a^b F\left(t \mid w\right) g\left(w\right) dw} = \int_a^b r\left(t \mid w\right) g\left(w\right) dw, \tag{2.66}$$

where, , $w, t \geq 0$ and $g\left(w \mid t\right)$ is the conditional probability density function of W, which is conditioned on the event, $T \leq t$,

$$g\left(w \mid t\right) = g\left(w \mid T \leq t\right) = \frac{g\left(w\right) F\left(t \mid w\right)}{\int_a^b F\left(t \mid w\right) g\left(w\right) dw}. \tag{2.67}$$

As in (2.13), the conditional random variable $W \mid t$, $W \mid 0 \equiv W$, is characterized via the corresponding distribution function,

$$G\left(w \mid t\right) = \Pr\left(W \leq w \mid T \leq t\right) = \frac{\int_a^w F\left(t \mid v\right) g\left(v\right) dv}{\int_a^b F\left(t \mid w\right) g\left(w\right) dw}, \tag{2.68}$$

whereas, the corresponding distribution function of the unconditional random variable, W is its limit as, $t \to \infty$. These general mixture models are useful for the corresponding analysis of the shape of the reversed failure rate under mixtures. Specifically, the proportional reversed failure (hazard) rate (PRFR) model is considered. The PRFR model of the form:

$$F\left(t\right) = \left[F_0\left(t\right)\right]^{\theta}, \tag{2.69}$$

where $F_0\left(t\right)$ is the baseline distribution and $\theta > 0$ is proposed by the authors of reference, [160]. Some results relating to this model were later discussed by among others in [76], [127], [139], [147], and [159]. As pointed out in reference [127], a topic for further investigation is to consider the case when θ in the model (2.69) is random. For the mixture model (2.65), we consider as a specific case,

$$r\left(t \mid w\right) = w\, r_b\left(t\right), \tag{2.70}$$

where, $w, t \geq 0$ and $r_b\ (t)$, is the reversed failure rate of the baseline lifetime T_b and W is the mixing random variable. This model (which is equivalent to (2.69)), already, defines "the reversed failure rate of the subpopulation for each realization $W = w$ with the distribution function" given by (2.64). The unconditional expectation of the random reversed failure rate, $r\ (t \mid w)$ is given by,

$$r_p\ (t) = E\ [r\ (t \mid W)] = \int_0^\infty r\ (t \mid w)\ g\ (w) dw. \tag{2.71}$$

This function captures the monotonic pattern of a family of random variables, $r\ (t \mid w)$, e.g., if $r\ (t \mid w)$ is decreasing, then consequently, $\lambda_p\ (t)$ is decreasing as well". This model, will be particularly useful for discussing the shapes of the mixture reversed failure rate, $r_m (t)$, in section 3.4.3.

2.11. Some General Properties of the Mean Inactivity Model

Let the random variable, $T_\omega (t)$, be the time elapsed since the last failure, with the corresponding survival function given by:

$$S_\omega (t) = \Pr(t - T \mid T \leq t), \tag{2.72}$$

and the pdf,

$$f_\omega (t) = \frac{f\ (t - x)}{F(t)}\ . \tag{2.73}$$

The mean inactivity time is defined as the expectation of the random variable, $T_\omega\ (t)$, defined by (1.6) and reproduced here for convenience,

$$\varpi\ (t) = E[T_\omega] = \frac{\int_0^t F\ (v)\ dv}{(F(t))}\ . \tag{2.74}$$

Some properties of the model (2.74) are considered in [127]. In fact, if (2.73) is differentiable, then a useful relation (1.7) between the reversed failure rate and the mean waiting (inactivity) time is obtained. Thus, with respect to relation (2.60), the following exponential representation can be shown to hold in this case, i.e.,

$$F(t) = \exp\left\{-\int_t^\infty V\,dv\right\}. \tag{2.75}$$

where, $V = \{(1-\varpi'(v))/\varpi(v)\}$. The distribution of the random variable, $T_\omega(t)$, is uniquely defined by the mean waiting (inactivity) time. The corresponding characterization conditions established in [127] are useful for studying the aging behavior of lifetime's distributions by means of the MIT.

Recalled, hereafter,

1. $\varpi(0) = 0$, whereas $\varpi(t) > 0$ for $t > 0$ and $\varpi'(t) < 1$.

2. $\int_t^\infty V\,dv < \infty$ and $\int_0^\infty V\,dv = \infty$

From these results and equation (2.75), it is clear that, we could not have lifetime distributions with decreasing mean waiting time in $[a\ ,\ \infty)$ for all $a \geq 0$. Furthermore, for $\varpi'(t) < 0$, the first part of condition 2 does not necessarily hold. However, if we assume further that $r(t) < \varpi(t)^{-1}$, then $\omega(t)$ is increasing (monotonically) in $[0\ ,\ \infty)$. These results, are also important for our further analysis of the shape of the MIT under the operation of mixing in section 3.5.

2.12. Concluding Remarks

Heterogeneity in populations (of items) is often induced by changing environmental conditions and/or other random effects. We focus, on describing the corresponding aging characteristics for heterogeneous populations.

We have introduced and described the notion of the random failure rate for specific cases. Some aspects of general mixture models, which are useful in the rest of this thesis are considered and briefly discussed.

We focused on some basic frailty (mixture) models and discussed some results that describe the shape of the mixture failure rate. Specifically, a meaningful case of a population which consists of two subpopulations, which we believe was not sufficiently studied in the literature, is considered in detail. The corresponding properties describing the shape of the failure rate under these mixtures are analyzed. It is shown that the mixture failure rate can decrease or be UBT-shaped for some specific cases.

Another important in practice specific case of a random initial age as a mixing variable was also studied in detail. It is shown that this type of mixing can also change the aging properties of an object, e.g., for certain values of parameters, the mixture failure rate may either preserve the IFR property or have a bathtub shape: initially decreasing to some minimum point and eventually increasing as $t \to \infty$.

The mean residual life, also plays a central role in characterization of the lifetime distributions. We present some useful general results on the properties of the MRL function and obtain the corresponding 'shape properties' for the relevant mixtures. The MRL function shape properties are analyzed for some specific cases and some relations with the failure rate are also derived.

We show that for the inverse Weibull baseline distribution with the UBT shaped failure rate, the corresponding MRL is decreasing for certain values of parameters, whereas it is the UBT-shaped for other values. These results already show the flexibility of the inverse Weibull model in describing different aging phenomena.

We further, consider some general properties of the reversed failure rate. The corresponding shapes of the RFR are analyzed. In particular, two specific cases when the failure rate is increasing and when it is UBT-shaped are studied. It turns out that the reversed failure rate is decreasing in both cases. From these results, it is concluded that, there are no lifetime distributions with increasing or constant reversed failure rates.

We also briefly consider a specific proportional reversed failure rate (PRFR) model, which will be important for further analysis of the bending properties of the mixture RFR model.

Some properties of the mean inactivity (waiting) time (MIT) are presented and discussed. Specifically, it is stated that, due to relations between the RFR and the MIT, there are also no lifetime distributions with decreasing mean waiting time.

The obtained in this chapter results on bending down of the mixture failure rates with time are well justified for human populations (Gompertz law of mortality for baseline distributions in mixing models) from developed countries, where there is a gradual availability of validated mortality data on centerians and supercentarians. It is also interesting to investigate this phenomenon utilizing data from less developed countries.

CHAPTER 3: Stochastic ordering for mixtures of random variables

3.1. Brief Overview

We firstly, discuss some essential aspects of stochastic orderings, in section 3.2. A detailed general theory on various stochastic orders and other aspects on this topic, could be found in, reference [80]. However, ordering for mixtures is not sufficiently studied in the literature due to variety of settings and applications.

We start with the most natural and at the same time, non-trivial, in this context ordering of the mixture failure rates. "Usually stochastic ordering of mixture failure rates for stochastically ordered mixing random variables arises when considering heterogeneous populations in different environments" [81]. We are motivated by the results on the aging properties of the failure (hazard) rate of general and some specific mixed models in references [83], [94] and [137]: see, e.g., also reference [54]. The definition of the bending down property of the mixture failure rate is established in references [83] and [94]. The mixture failure rate may either bend down in a weak or strong sense. For instance, if

$$\lambda_m(t) < \lambda_p(t) \qquad \text{for } t > 0, \tag{3.1}$$

where, $\lambda_p(t)$ is given by relation (2.33). In this case, the mixture failure rate is said to bend down in a weak sense. On the other hand, if

$$\lambda_p(t) - \lambda_m(t) \uparrow \qquad \text{for } t \geq 0, \tag{3.2}$$

the mixture failure rate bends down in a strong sense (meaning that this difference increases in time). Along with this idea, we discuss results on the bending properties of the mixture failure rates when compared with a specific form of our model (2.1). These results are also extended to other main reliability indices mentioned earlier.

We, also, analyze the failure (mortality) rate for heterogeneous populations in section 3.5, e.g. "when the subpopulations are ordered (in some stochastic sense)", [41]. As pointed out by the forgoing authors, this setting can be interpreted via the fixed frailty models with one or more frailty parameters. Under this setting, we present some results with respect to vitality modeling in section 3.6. In the final section 3.7, we consider relative aging of the mentioned main reliability indices. Specifically, we propose ordering of lifetimes in terms of monotocity properties of the ratio of the mean waiting (inactivity) times.

3.2. Some essential aspects of stochastic orderings

Let X and Y be non-negative and absolutely continuous random variables with the distribution functions: $F_X(t)$ and $F_Y(t)$, where $\lambda_X(t)$ and $\lambda_Y(t)$ are the corresponding failure rates and $f_X(t)$ and $f_Y(t)$ are the density functions. Assume further that the first moments exist and are finite. Then X is said to be smaller than Y in expectation if $E[X] \le E[Y]$: e.g. from the definition of the expected value of a lifetime random variable,

$$E[X] = \int_0^\infty S(x)\,dx \le \int_0^\infty S(y)\,dy = E[Y]. \tag{3.3}$$

Some other basic stochastic orders widely used in the literature are:

a) X is said to be stochastically smaller than Y, in the sense of the usual stochastic order, denoted, $X \le_{st} Y$, if,

$$S_X(t) \le S_Y(t) \quad \text{for } \forall t \ge 0. \tag{3.4}$$

Hence, the corresponding distribution functions are also ordered (i.e. $F_X(t) \ge F_Y(t)$). In this setting, then $X \le_{st} Y$ "if and only if, $E[\phi(X)] \le E[\phi(Y)]$, for all increasing functions, ϕ", [80]. This, condition, implies that the corresponding random variables, will also be ordered in the sense of increasing convex (concave) order for every increasing, convex (concave) functions ϕ.

b) X is said to be stochastically smaller than Y, in the sense of failure (hazard) rate ordering, denoted, $X \le_{hr} Y$, if, $S_X(t)/S_Y(t)$ is decreasing in t.

Therefore, which follows from the corresponding exponential representations for survival functions,

$$\lambda_X(t) \ge \lambda_Y(t) \ , \quad \text{for} \quad \forall t \in [0\ , \infty]. \tag{3.5}$$

c) X is said to be stochastically smaller than Y, in the sense of reversed failure (hazard) rate ordering, denoted, $X \le_{rhr} Y$, if, $F_X(t)/F_Y(t)$ is decreasing in t.

The reversed failure rate order is denoted as

$$r_X(t) \le r_Y(t), \quad \text{for} \quad \forall t \ , \tag{3.6}$$

where,

$$r_X(t) = \frac{f_X(t)}{F_X(t)} \ ; \ r_Y(t) = \frac{f_Y(t)}{F_Y(t)}.$$

d) X is said to be stochastically smaller than Y, in the mean residual life order, denoted $X \le_{mrl} Y$, if,

$$\frac{S(x)}{\int_t^\infty S(u)du} \ge \frac{S(y)}{\int_t^\infty S(u)du} \quad \text{for all } x\ , y \le t.$$

The mean residual life order is denoted as,

$$m_X(t) \le m_Y(t) \quad \text{for} \quad \forall t \in [0\ , \infty]. \tag{3.7}$$

e) X is said to be stochastically smaller than Y, in the mean waiting (inactivity) order, denoted, $X \le_{mit} Y$, if,

$$\int_0^t F_X(t) / \int_0^t F_Y(t), \text{ is decreasing in } t.$$

Hence, mean inactivity order is denoted as,

$$\varpi_X(t) \ge \varpi_Y(t), \quad \text{for} \quad \forall t. \tag{3.8}$$

f) X is said to be stochastically smaller than Y, in the likelihood ratio order, denoted $X \leq_{lr} Y$ if, $f_X(t)/f_Y(t)$ is decreasing in t or if ,

$$f(x)\, f(y) \geq f(y)\, f(x), \quad \text{for all} \quad x \leq y.$$

Hence, we write,

$$\ell_X(t) \leq_{lr} \ell_Y(t). \tag{3.9}$$

Remark 1.2

- Obviously, from (2.46), the failure rate ordering (3.5) implies MRL ordering (3.7). For the inverse relation to hold, a sufficient condition of increasing $m_Y(t)/m_X(t)$ in t, should be presumed. Some useful monotocity properties of this ratio of the MRL functions are studied in reference [82]. These, will be useful for our analysis of the relative aging behavior of lifetime distributions in section 3.6.

- Note that relation (3.7) and other properties of (2.46) will also be utilized to analyze the bending properties of MRL in section 3.4.2.
- It will be shown that the likelihood ratio order is the most natural order for ordering of mixing distributions.

3.3. Stochastic ordering of mixing distributions

Consider two continuous nonnegative random variables, W_1 and W_2 with the corresponding densities, $g_1(w)$ and $g_2(w)$ with the same support in $[0,\infty)$. Then, "W_1 is stochastically smaller than W_2 in the sense of likelihood ratio order:

$$W_1 \geq_{lr} W_2, \tag{3.10}$$

if, $\dfrac{g_2(w)}{g_1(w)}$ is a decreasing function". Similarly and parallel to this result, we consider, the case, where the frailty parameter, W is modified to $W \mid t \equiv (W \mid T > t)$. This conditional random variable, already, account for the information on subpopulation of objects that survived the operational interval, $[0,t)$. In particular, assuming the family of failure rates $\lambda(t \mid w)$ to be increasing in w, implies,

$$\lambda(t \mid w_1) \le \lambda(t \mid w_2) \quad \text{for } w_1 < w_2 \text{ and } \forall\ w_1,\ w_2 \in [0,\ \infty). \tag{3.11}$$

As a result, we can argue that the family of random variables, $W \mid t$ for all $t \ge 0$ is decreasing in the sense of the likelihood ratio order. In this case, considering the conditional mixing distribution (2.10) in our mixing model (2.11), we can then compare the frailty distribution of surviving subpopulations (i.e. a subpopulation surviving to t_1 and the one surviving to t_2):

$$\frac{g(w \mid t_2)}{g(w \mid t_1)} = \frac{S(w \mid t_2)\int_0^\infty S(t_1 \mid w)\, g(w)\, dw}{S(w \mid t_1)\int_0^\infty S(t_2 \mid w)\, g(w)\, dw}\ . \tag{3.12}$$

If (3.12) is decreasing, then the conditional random variables, $W \mid t$ are ordered in the sense of likelihood ratio order (3.11). As,

$$\frac{S(w \mid t_2)}{S(w \mid t_1)} = \exp\left\{-\int_{t_1}^{t_2} \lambda(v \mid w)\, dv\right\} \tag{3.13}$$

is decreasing in w, for all $t_1 > t_2$, due to ordering (3.11), we can conclude that (3.12) is decreasing and the family of conditional random variables, $W \mid t$ is decreasing for all $t \ge 0$ in the sense of the likelihood ratio order (3.10).

Specifically, we consider different mixing random variables: i.e., frailties W_1 and W_2 with densities, $g_1(w)$ and $g_2(w)$, with the corresponding distribution functions, $G_1(w)$ and $G_2(w)$.
Let

$$g_2(w) = \frac{u(w) g_1(w)}{\int_0^\infty u(w) g_1(w) dw}\ , \tag{3.14}$$

where, $u(w)$ is a decreasing function of w. As W_1 is stochastically larger than W_2 in the sense of (3.5), then $G_1(w) \le G_2(w)$. For instance, using (3.14) we have,

$$G_2(w)=\frac{\int_0^w u(t)\,g_1(t)dt}{\int_0^\infty u(t)\,g_1(t)dt}=\frac{\int_0^w u(t)\,g_1(t)dt}{\int_0^w u(t)\,g_1(t)dt+\int_w^\infty u(t)g_1(t)dt}\quad. \tag{3.15}$$

As $u(w)$ is a decreasing function, it can easily be shown that, $G_1(w)\le G_2(w)$, where $G_1(w)=\int_0^w g_1(t)\,dt$. For example, see e.g. reference [83],

$$G_2(w)=\frac{v(i\,|\,w)\int_0^w u(t)\,g_1(t)dt}{v(i\,|\,w)\int_0^w u(t)\,g_1(t)dt+v(w\,|\,j)\int_w^\infty u(t)\,g_1(t)dt}\ge G_1(w), \tag{3.16}$$

where, $i\ge 0$ and $j\le\infty$. In this case, $v(i\,|\,w)$ and $v(w\,|\,j)$in relation (3.16) represents the respective means of $u(w)$. This result implies that $v(i\,|\,w)\ge v(w\,|\,j)$ and $G_1(w)\le G_2(w)$, $w\in[0\,,\,\infty)$. Obviously, when the two frailty random variables W_1 and W_2 are ordered in the sense of (3.10), then the corresponding random variables $W_1\,|\,t$ and $W_2\,|\,t$, for our subpopulation that survived the operational interval $[0\,,\,t)$ are also accordingly ordered. For instance,

$$G_2(w\,|\,T>t)=\frac{\int_0^w S(t\,|\,v)\,g_2(v)dv}{\int_0^\infty S(t\,|\,v)\,g_2(v)dv}=\frac{\int_0^w S(t\,|\,v)\,u(v)\,g_1(v)dv}{\int_0^\infty S(t\,|\,v)\,u(v)\,g_1(v)dv}\quad. \tag{3.17}$$

On the other hand, by definition $G_1(w\,|\,T>t)$ is given by,

$$G_1(w\,|\,T>t)=\frac{\int_0^w S(t\,|\,v)g_1(v)dv}{\int_0^\infty S(t\,|\,v)g_1(v)dv}\quad. \tag{3.18}$$

As the function, $u(w)$ in (3.14) is decreasing, then from results (3.17) and (3.18), $G_1(w\,|\,T>t)\le G_2(w\,|\,T>t)$. This result, already shows that, $W_1\ge W_2$, and $W_1\,|\,t\ge W_2\,|\,t$. If in addition, ordering (3.5) holds, then “the corresponding (mixture) random variables are also ordered in the sense of the failure (hazard) rate ordering". From the fact that, $G_1(w \mid T > t) \le G_2(w \mid T > t)$, the following result can be shown to hold, see e.g. reference, [83],

$$a - b = \int_0^\infty \lambda(t \mid w)[g_1(t \mid w) - g_2(w/t)dw], \tag{3.19}$$

where,

$a - b = \int_0^\infty -\lambda_w(t \mid w)' [G_1(w \mid t) - G_2(w \mid t)]dw \ge 0,$ for $t > 0$ is the difference between the two mixture failure rates (i.e. $a = \lambda_{m(1)}(t)$ and $b = \lambda_{m(2)}(t)$). This result, already shows the bending down property as signified by the majorization of, $\lambda_m(t)$ by $\lambda_p(t)$ (i.e. from (3.19), $G_1(w \mid T > t) - G_2(w \mid T > t) \ge 0$). In particular, it implies that, "when the mixing distributions are ordered in the sense of (3.10)" see e.g. reference, [81], the mixture failures are consequently ordered in the sense of (3.11). The respective random variables would also be consequently ordered in the sense of the usual stochastic order as the hazard rate order implies the usual stochastic order.

3.4. Stochastic ordering of the main reliability indices

In what follows, we intend to generalize results on the "bending" properties of mixture failure rates to other related main reliability characteristics mentioned earlier. Suppose that $\Psi(w)$ is the family of random variables with W having support in $[0, \infty)$ and let again as before W be the mixing random variable. Denote, for convenience, the respective reliability characteristics in the mixture, $\Psi_m(t)$.

Our idea is to compare, $\Psi_m(t)$ with the unconditional (on survival in $[0,t)$) characteristic, $\Psi_p(t) = E[\Psi(t \mid W)]$, where W is the mixing random variable. The function, $\Psi_p(t)$ is important and it captures the monotonicity properties of the underlying subpopulation lifetimes in the mixture. The following relations (in addition to the considered stochastic orders) are, therefore, important for our analysis of the "bending" properties of the main reliability characteristics under mixtures in the subsequent section.

1. If $\Psi_p(t) > \Psi_m(t)$ and $\Psi_p(t) - \Psi_m(t)$ is decreasing, then $\Psi_m(t)$ is bending down in the weak sense.
2. If $\Psi_p(t) < \Psi_m(t)$ and $\Psi_p(t) - \Psi_m(t)$ increases, then $\Psi_m(t)$ "bends down" in strong sense.
3. If the ratio $\dfrac{\Psi_p(t)}{\Psi_m(t)}$ is increasing in t, then $\Psi_m(t)$, is again bending down in a strong sense.

3.4.1. Overview of important results on stochastic ordering of mixture failure rates

Relations in items 1 and 2 in the preceding section are utilized in establishing the bending down properties of the failure rate under mixtures in references [81], [83] and [94], whereas relations 4 and 5, could already be found in reference, [86].

When the relations in 2 above holds and,

$$\lim_{t\to\infty} \Psi_p(t) - \Psi_m(t) = 0\text{, where, } \Psi_p(0) = \Psi_m(0),$$

it can easily be shown that, the mixture, $\Psi_m(t)$ is also bending up (down) in a weak sense. On the other hand, if

$$\lim_{t\to\infty} \lambda(t \mid w) = \infty, \quad w \in [a, b],$$

then,

$$\Psi_p(t) - \Psi_m(t) \to \infty .$$

It was, also, shown in reference [81] that the mixture failure rate $\Psi_m(t)$ is majorized by $\Psi_p(t)$, in this case. This result is due to the corresponding conditioning in (2.11), which renders "the mixture failure rate to always be smaller than the unconditional one for $t > 0$", [94]. For instance as stated by these authors, if "the conditional and unconditional expectations in (2.11) and (2.33) respectively, are finite for $\forall\, t \in [0, \infty)$, then: the mixture failure rate $\lambda_m(t)$ weakly bends down with time", see e.g. also reference, [83], whereas, $\lambda'(t \mid w)$ also increases in t and $\lambda_p(t) - \lambda_m(t)$ is increasing, then for this case, $\lambda_m(t)$ strongly bends down with time.

Intuitively, from these results, relation in item 3 above is increasing as a function of t and the mixture failure rate also bends down in a strong sense. For instance,

$$\frac{\lambda_p(t) - \lambda_m(t)}{\lambda_m(t)} \text{ is increasing,}$$

as $\lambda_m(t)$ is majorized by $\lambda_p(t)$. Specifically, consider the multiplicative model, (2.19), where $\lambda_b(t)$ is the baseline failure rate. In this case, model (2.11) turns to be given by,

$$\lambda_b(t) E[W \mid t],$$

whereas model (2.33) is defined by

$$\lambda_b(t)\, E[W \mid 0],$$

where, $E[W \mid 0] \equiv E[W]$. As the baseline failure rate $\lambda_b(t)$ is increasing and the conditional expectation $E[W \mid t]$ is decreasing as function of $t \in [0\,,\,\infty)$, we can conclude that,

$$\lambda_m(t) = \lambda_b(t)\, E[W \mid t].$$

is decreasing, only if the variance $Var\,(W \mid t)$ is large. Therefore, we can also conclude that,

$$\lambda_p(t) - \lambda_m(t) = \lambda_b(t) E[W \mid 0] - E[W \mid t],$$

is increasing. From relations in items 2 and 3 above, the mixture failure rate bends down in a strong sense in this case.

Interestingly, when $\lambda'(t \mid W)$ is increasing (decreasing) in W whereas $S(t \mid w)$ is decreasing (increasing) in W and $\Psi(w)$ is IFR for all $W = w$ and, $t > 0$, then the relation in item 3 above increases as well. The mixture failure rate bends down then in a strong sense.

3.4.2. Stochastic ordering of MRL mixtures

In most cases, $\lambda\left(t \mid w\right)$ is expected to increase with time as a result of tear and wear accumulated by an operating item (object). In this sense, aging is taking place and intuitively, $\lambda\left(t \mid w\right)$ ($m\left(t \mid w\right)$) would also be increasing (decreasing). The conditional variance of W would intuitively also be increasing in this case. However, it is proved in [94] that the mixture, $\lambda_m\left(t\right)$ bends down in a weak sense when the failure rates are ordered as in (3.11). On the other hand, it bends down in a strong sense if $\lambda'\left(t \mid w\right)$ increases in t and $\lambda_p\left(t\right)-\lambda_m\left(t\right)$ is increasing as a function, t; that is,

$$\lambda_m\left(t\right) \leq E\left[\lambda\left(t \mid W\right)\right].$$

From the forgoing, the relation in item 3 increases as well, and the mixture failure rate is bending down in a strong sense.

From, the relations between the failure rate and the MRL (see e.g. relations (2.52) and (2.56)); the MRL function under the operation of mixing bends up. Due to ordering (3.5), (which is stronger relative to both the ordering (3.4) and the ordering (3.7)), a decreasing $\lambda\left(t \mid w\right)$ in w for all t, leads to the conclusion that both $S\left(t \mid w\right)$ and $m\left(t \mid w\right)$ are increasing in w for all $t > 0$. As a result,

$$m_m\left(t\right) \geq E\left[m\left(t \mid W\right)\right].$$

Whereas, if $m'\left(t \mid w\right)$ is increasing (decreasing) in W and $m\left(t \mid w\right)$ is decreasing (increasing) in W for all $t > 0$, then, from the relations in item 2, $m_p\left(t\right)-m_m\left(t\right)$ is a decreasing function of, t. The mixture MRL "bends up" in a strong sense in this case.

Additionally, considering the relation in item 3, it is easily concluded from the fact that $m_p\left(t\right)-m_m\left(t\right)$ is increasing, that the shape of the mixture MRL bends up in a strong sense in this case. For example, $m_m\left(t\right)$ is majorized by $m_p\left(t\right)$,

$$\frac{m_p(t)-m_m(t)}{m_m(t)} \text{ is increasing.}$$

From these results, our heterogeneous population is improving, e.g. the proportions of strong subpopulations are increasing as "the weakest populations are dying out first".

3.4.3. Ordering the reversed failure rate mixtures

There are no non-negative random variables for which their lifetime distributions are characterized by a constant or increasing reversed failure rate in a finite interval of support, i.e. $[a,\infty), a \geq 0$. Does, this mean the corresponding shapes of the reversed failure rate do not change under the operation of mixing?

From relation (3.6), which can be shown to be stronger relative to the usual stochastic order (3.4), it is intuitive that $F(t \mid w)$ is increasing (decreasing) in w whereas $r(t \mid w)$ is decreasing (increasing) in w for all $t > 0$. When, $\Psi(w)$ is increasing (decreasing) in w, in the reversed stochastic order (3.6), $r_m(t)$ bends down in a weak sense. It, follows from the above considerations that,

$$r_m(t) \leq E[r(t \mid W)].$$

On the other hand, the relations in item 1 above imply a weak bending down of the reversed failure rate.

Parallel to the multiplicative model (2.19), we consider, the multiplicative reversed failure rate model (2.70), where, $r_b(t)$ is the baseline reversed failure rate in this case. Then, model (2.71) reduces to,

$$r_b(t)\, E[W \mid 0], \text{ where, } E[W \mid 0] \equiv E[W].$$

Then model (2.66) turns to

$$r_b(t)\, E[W \mid t].$$

As the baseline reversed failure rate, $r_b(t)$ and $E[W \mid t]$ are decreasing for $t \geq 0$, then we can conclude that

$$r_m(t) \leq E[r\ (t \mid W)].$$

Thus, the reversed failure rate bends down in a weak sense. Interestingly, when the derivative, $r^{'}(t \mid w)$ is decreasing or alternatively $F(t \mid w)$ is decreasing in t for all $w > 0$, then, $r_p(t) - r_m(t)$ is increasing as a function of $t > 0$ as well. Therefore, from the relations in items 2 and 3 above, the reversed failure rate strongly bends down as time increases.

The results obtained above, show that the mixture reversed failure rate $r_m(t)$ bends down as time increases. However, the implications of locally increasing mixture reversed failure rate observed in reference [54] in lifetime modeling would need some further study.

3.4.4. Ordering the mean waiting time for mixtures

The results on the mixture reversed failure rate $r_m(t)$ show that it bends down as time increases. Does it mean that the corresponding mean inactivity time is also not decreasing for lifetime random variables?

Relations between the RFR and the MIT, specifically relation (1.7) are useful in studying the aging behavior of the MIT. If , $\Psi(w)$ is increasing (decreasing) in w in the mean waiting (inactivity) stochastic order (3.8), then it is also decreasing (increasing) in w in the usual stochastic order (3.4). Therefore,

$$\varpi_m(t) \geq E[\omega(t \mid W)].$$

The mean waiting (inactivity) time is bending up in a weak sense in this case. Whereas, $\varpi_p(t) - \varpi_m(t)$ is increasing, when the derivative $\omega^{'}(t \mid w)$ is increasing in z. Therefore, the corresponding $r(t \mid w)$ is decreasing in w, and by relation (1.7), the mean waiting (inactivity) time bends up in a strong sense in this case. Furthermore, considering the relations in items 2 and 3 above, then,

$$\frac{\varpi_p(t)-\varpi_m(t)}{\varpi_m(t)} \text{ is increasing.}$$

This result is due to the fact that, the numerator is increasing and therefore, the mean waiting (inactivity) time bends up in a strong sense also in this case.

3.5. A case of two frailties

Consider the “Gompertz law of human mortality” defined by the following mortality rate, which is a function of two parameters α, and β,

$$\lambda(t \mid \alpha,\, \beta)=\alpha \, \exp\,(\beta \, t). \tag{3.20}$$

When “α, is randomized and β is fixed, model (3.20) reduces to the ordinary multiplicative frailty model, with asymptotically flat hazard (failure) rate, particularly, when the distribution of frailty is, e.g., Gamma and $t \to \infty$”, [41]. This already reflects an ordering in terms of failure rates. These results may be extended to a more general case with more than one frailty parameter.

Consider the bivariate “frailty” model: $F(t \mid w_1, w_2)$ with the failure rate:

$$\lambda(t, w_1, w_2)=\frac{f\,(t \mid w_1, w_2)}{S\,(t \mid w_1, w_2)},$$

where, w_1 and w_2 are the non-negative random variables (frailties) having the joint pdf, $g(w_1, w_2)$ with support $[0,\infty)$. Two specific cases of this model are considered for a system with two statistically independent components in series. If, the frailties are considered independent, that is:

$$g\,(w_1, w_2)=g\,(w_1)\,g\,(w_2) \text{ and } \quad F\,(t \mid w_1, w_2)=1-S\,(t \mid w_1)\,S(t \mid w_2),$$

then the population failure rate:

$$\lambda(t)=\int_0^\infty\int_0^\infty \lambda\,(t \mid w_1, w_2)\,g\,(w_1, w_2 \mid t)\,dw_1\,dw_2\,,$$

with the corresponding conditional pdf,

$$g(w_1, w_2 \mid t) = \frac{g\,(w_1, w_2)\, S\,(t \mid w_1, w_2)}{\int_0^\infty \int_0^\infty S\,(t \mid w_1, w_2)\, g\,(w_1, w_2)\, dw_1\, dw_2}.$$

Specifically, consider the frailty $W_2 = w_2$ to be fixed, then the failure rate can simply be defined via model (2.11):

$$\lambda\,(t \mid w_2) = \int_0^\infty \lambda\,(t \mid w_1, w_2)\, g\,(w_1, w_2)\, dw_1.$$

In this case, the mortality rate is decreasing, if the failure rates, $\lambda\,(t \mid w_2)$ are ordered in w_2. Specifically, when $\lambda\,(t \mid w_2)$ for each w_2 is decreasing asymptotically as $t \to \infty$, the mixture population failure rate in this case decreases. Thus, considering random $W_2(W_1)$ leads to the strictly decreasing population failure rate. This behavior can be explained as result of the well-known principle, the "weakest subpopulations are dying out first" as time increase.

3.6. Vitality Modeling

3.6.1. Brief Overview

The decline in in vitality of an organism as some aggregate characteristic of health can be effectively modeled and analyzed via some simple vitality models. In first passage and Markov (phase-type) models, the corresponding modeling and analysis usually considers the survival capacity (vitality) as an aggregated characteristic endowed into organisms at birth. The stochastic loss of vitality (i.e. deterioration of "vital" parameters[1]) as age, is usually modeled via some stochastic process (i.e., "the Weiner process with negative drift which, already, describes the (non-monotone) decrease in vitality of organisms" [41]). Usually in these cases, the probability of death (i.e. "the first time passage to the zero boundary", see e.g., reference, [241]), is often modeled via the inverse Gaussian distribution in a number of application areas. For instance, this model which apparently was first considered in [252] is used to explain mortality plateau at advanced ages in [144] and [109], relating natural and xenobiotic stressors to survival of organisms in [145] and extended to a model with vitality-dependent and vitality independent

[1] As they are known in biological literature.

(extrinsic or accidental morality) components by [61]. See, e.g., also a wide range of Markov processes considered in [132] and phase-type distributions in [221]. The review on the development of the inverse Gaussian distribution and its statistical applications is also provided by [241] and later a more general review of some mortality models was given in reference [149]. It is also interesting to determine which "other statistical distributions are characterized by the asymptotically flat hazard rate", [41].

3.6.2. Some results on Vitality Modeling

Vitality loss (i.e. the decline in vitality with fixed initial value) when $z_0 > 0$ can be described via the stochastic process, see e.g., reference, [45]:

$$Z_t = z_0 - U, \tag{3.21}$$

where $U = Vt$. Under, this model, death is assumed to occur when Z_t reaches zero for the corresponding lifetime, T_V with the Cdf:

$$F_V(t) = 1 - F\,(z_0 \mid t),$$

where, V is a non-negative random variable with a corresponding distribution, $F\,(t)$. In this simplest form of model (3.21), the linear process of degradation (i.e. "decline in physiological functions of organisms") can be described for realizations of the random variable, V. In particular, the corresponding subpopulations from the heterogeneous population may be ordered accordingly to their corresponding lifetimes. In fact, it is shown by the authors of references [44] and [45], for a gamma-distributed, V, with the pdf:

$$\frac{1}{\Gamma(\alpha)} \eta\, x^{\eta-1} \exp(\alpha\, x), \;\; \alpha\,, \eta > 0,$$

that the shape of the failure rate of the resulting inverse Gaussian distribution, is bathtub, whereas, for model (3.21), the failure rate tends to zero as $t \to \infty$. Other, more specific models may also be considered for the corresponding analysis. The first and the most popular among these is perhaps, the Wiener process with drift given by:

$$Z_t = z_0 - D_t, \tag{3.22}$$

Where, $D_t = st + W_t$, ($D_t, t \geq 0$), is the Wiener process with s as the drift parameter and W_t , $t \geq 0$, is the standard Wiener process with normally distributed values (for each fixed t) with mean 0 and variance $\sigma^2 t$. See reference, [45]. As pointed out by the forgoing authors, "the probability distribution of this first passage time: i.e. when R_t reaches the boundary v_0 for the first time can be well modeled via the inverse Gaussian distribution" with the pdf,

$$f_V(t) \equiv f_V(t \mid z_0, s, \sigma) = \frac{z_0}{\sigma\sqrt{2\pi}} t^{-3/2} \exp\left\{\frac{(z_0 - st)^2}{2\sigma^2 t}\right\}. \tag{3.23}$$

See e.g. also references, [61], [132], [144], [145] and [241]. In particular, it is argued by [132] that although, there are three parameters in this case (i.e. z_0, s and σ), the probability distribution of the first passage time depends only on, z_0/σ and s/σ. Thus, if we let $\lambda = s^2/\sigma^2$ and $k = s z_0/\sigma^2$, it is easy to see that (3.23) reduces to a simple two-parameter inverse Gaussian distribution with the pdf,

$$f_V(t) \equiv f_V(t \mid z_0, s, \sigma) = \frac{\lambda k}{\sigma\sqrt{2\pi}} (\lambda t)^{-3/2} \exp\left\{\frac{(k - \lambda t)^2}{2\lambda t}\right\}. \tag{3.24}$$

The properties of the model (3.24) were studied for the first time in literature in [45] within the context of randomization of parameters and the corresponding ordering of subpopulations. See, e.g. also, references [4] and [29] for some further results. Specifically, it is shown by the authors of reference [41] that the shape of the failure rate is increasing in a certain interval (i.e. for $t \in (0, t_1)$ particularly when $t_2 \leq t_1 = 2z_0/3\sigma^2$ and decreasing asymptotically to a plateau for $t_1 > t_2$. However, as pointed out by these authors, "only the non-randomized version of this distribution leads to the asymptotically constant failure rate", see e.g. also reference, [44]. As a result, the initial vitality has no impact on the shape of the failure rate at advanced ages. The Gamma distribution, where λ is randomized may also be considered as another relevant example that exhibits asymptotically constant failure rate. Other examples include the Birnbaum-Saunders distribution and the gamma process with monotonic sample paths considered recently together with the inverse Gaussian distribution in references [44] and [45].

These results, show that the shape of the failure (mortality) rate can exhibit a number of possible shapes when the parameters are randomized, e.g., meaning that the corresponding curves are increasing or decreasing asymptotically to a plateau or even tend to zero as $t \to \infty$.

3.7. Relative aging of reliability characteristics

The notion of relative aging in addition to considerations of the usual aging, is important. This concept, actually provides a way to determine which population ages faster between two or more populations in some probabilistic sense. One of the simplest ways to implement this concept would be to compare the "extend of aging" described by the increasing failure rates (IFR) distributions, [44]. We intend to look at this problem in our future work.

In the subsequent section 3.7.1., we firstly present the definitions and some background on the relative aging concepts. Our goal is to utilize the results *and propose* another type of relative aging based on the monotocity properties of ratio of the mean waiting (inactivity) times.

3.7.1. Relative stochastic orders of reliability characteristics

Let X and Y be two absolutely continuous lifetime random variables with the corresponding distribution functions $F_X(t)$ and $F_Y(t)$, failure rates $\lambda_X(t)$ and $\lambda_Y(t)$, mean residual lifetimes $m_X(t)$ and $m_Y(t)$, reversed failure rates $r_X(t)$ and $r_Y(t)$, and mean waiting (inactivity) times $\varpi_X(t)$ and $\varpi_Y(t)$. Assume, further, that the first moments exist and are finite.

The lifetime random variable X is said to be aging faster than the lifetime random variable Y in the following senses:

a) Relative failure (hazard) rate ordering ($X \leq_{rlhr} Y$) if the ratio $\lambda_X(t)/\lambda_Y(t)$ is increasing on $t \in [0, \infty)$.
b) Relative mrl order ($X \leq_{rlmr} Y$) if the ratio $m_X(t)/m_Y(t)$ is increasing in $t \in [0, \infty)$.

c) Relative reversed failure (hazard) rate order ($X \leq_{rlth} Y$) if the ratio $r_X(t)/r_Y(t)$ is increasing on $t \in [0, \infty)$.

The following is the modification for the case of large t :

i) X is said to be ultimately IFR aging faster than Y if $\lambda_X(t)/\lambda_Y(t)$ is increasing for sufficiently large t.

ii) X is said to be ultimately MRL aging faster than Y if $m_X(t)/m_Y(t)$ is increasing for sufficiently large t.

iii) X is said to be ultimately aging faster in RFR than Y if $r_X(t)/r_Y(t)$ is increasing for sufficiently large t.

The foregoing results are, particularly useful for our analysis of the proposed ordering of lifetimes in terms of monotocity properties of the ratio of the mean waiting (inactivity) times. Note that the definition of the mean waiting (inactivity) time, $\varpi(t)$ is given by equation (2.74).

We define the lifetime random variable X as aging faster than the lifetime random variable Y in the relative mean inactivity order ($X \leq_{rlmit} Y$) if the ratio $\varpi_X(t)/\varpi_Y(t)$ is increasing in $t \in [0, \infty)$. The following is intended to establish conditions under which $X \leq_{rlmit} Y$. Similar and parallel to Lemma 1 of reference [82], we can restate in this case that, $X \leq_{rlmit} Y$ if and only if,

$$r_X(t) - m^{-1} \leq r_Y(t) - n^{-1}, \tag{3.28}$$

where, $m = \varpi_x(t)$ and $n = \varpi_y(t)$. Hence, based on the subsequent proof thereof, we can similarly argue that $\varpi_X(t)/\varpi_Y(t)$ increases if,

$$\varpi_X{}'(t)\ \varpi_Y(t) - \varpi_X(t)\ \varpi_Y{}'(t) \geq 0. \tag{3.29}$$

For instance, assuming that $\varpi_X(t)$ and $\varpi_Y(t)$ are differentiable, it can easily be shown that, $(\varpi_X(t)/\varpi_Y(t))' \geq 0$. Since,

$$\varpi_X{}'(t) = 1 - r_X(t)\ \varpi_X(t) \text{ and } \varpi_Y{}'(t) = 1 - r_Y(t)\ \varpi_Y(t),$$

we have,

$$r_X(t) - r_Y(t) - \left(m^{-1} - n^{-1}\right) \geq 0. \tag{3.30}$$

Considering the conditions established in [127 p.2] for $\varpi(t)$ to increase, we can conclude that (3.29) and (3.30) hold and hence the ratio of the mean inactivity times (e.g. $\varpi_X(t)/\varpi_Y(t)$) is increasing.

Alternatively, we can also see from (2.74) and relation (3.8) that when the mentioned conditions hold, then, $\varpi(t)$ is increasing and consequently results (3.29) and (3.30) trivially hold. Hence, the ratio of the mean inactivity times (e.g. $\varpi_X(t)/\varpi_Y(t)$) is increasing.

Moreover, it was also shown by [125] that "$X \leq_{rh} Y$ implies $X \leq_{mit} Y$". Does this result mean that the implication of this chain relation is preserved under this current relative reasoning? Assume, $\varpi_X(t)/\varpi_Y(t)$ to be increasing for $t > 0$, and using definition (2.74), then the foregoing result implies that

$$\frac{\varpi_X(t)}{\varpi_Y(t)} \geq \lim_{t \to 0} \frac{F_Y(t) \int_0^t F_X(u)\, du}{F_X(t) \int_0^t F_Y(u)\, du} = 1. \tag{3.31}$$

Hence, $\varpi_X(t) \geq \varpi_Y(t)$ in this case. This result, already, defines ordering (3.8), which further implies that if X is increasing, the mean inactivity time is IMIT, then Y is also IMIT. Consequently, these results also imply that ordering (3.6), i.e. $r_X(t) \leq r_Y(t)$ also holds in this case. Therefore, from definition c) above and using (3.28), we can conclude that, $X \leq_{rlmit} Y$ is implied by $X \leq_{rlrh} Y$.

3.8. Concluding Remarks

Some general essential aspects of stochastic orderings, which are relevant to our study are presented and discussed in the beginning of this chapter.

We consider ordering of mixing distributions in the sense of the likelihood ratio. Specifically, some relevant and useful results for the case of two frailties are discussed. It turns out that the mixture failure rates are ordered, as functions of time in $[0,\infty)$, when the mixing distributions are ordered in the sense of the likelihood ratio.

Some findings on the bending properties of the mixture failure rates are presented. It follows from conditioning on survival in the past interval of time that the mixture failure rate is majorized by the unconditional one. Hence, the mixture failure rate bends down in a weak sense or a strong sense as time increases.

These results are extended to other main reliability indices. Specifically, we show, that the MRL function under the operation of mixing bends up either in a strong sense or a weak sense as time increases. The reversed failure rate is also bending down either in a weak sense or a strong sense, whereas the corresponding mean inactivity (waiting) time exhibits the reverse behavior.

Some relevant results on failure (mortality) rate when the corresponding parameters are randomized are presented. In this case, it turns out that randomization of parameters may lead to the strictly decreasing population failure rate. Under this setting, we also formulate some relevant findings with respect to the vitality modeling.

Finally, some useful results on relative aging of the mentioned main reliability indices are discussed. Specifically, ordering of lifetimes in terms of monotonicity properties of the ratio of the mean waiting (inactivity) times is proposed and some conditions for a random variable X to be aging faster than the lifetime random variable Y in the relative mean inactivity order are established.

It is also proved that if X is increasing in the sense of the mean inactivity time (IMIT) then Y is also IMIT when the corresponding random variables are ordered accordingly. It is concluded that the relative mean inactivity order is implied by relative reversed failure rate order.

CHAPTER 4: Discrete lifetime modeling

In this chapter, we generalize the properties of the failure rate to the discrete case. “There are some important differences between the failure rates in the discrete setting as compared to the failure rate in the continuous case”, [1]. We investigate the impacts of these differences in describing the corresponding aging characteristics. Hereafter, we analyze the shapes of the failure rate for some specific distributions in the class of discrete Weibull distributions. The shapes of the corresponding failure rate of a mixture of two distributions are studied in section 4.2. On the other hand, some results on the general properties of discrete mixture failure rates are briefly discussed in section 4.3. The forthcoming characterizations of discrete lifetime distributions are studied in the literature, see for instance references, [1], [3], [86], [131], [164] and [184] just to mention a few.

4.1. The failure rate in discrete setting

Let K be a discrete random lifetime with a support in, $N^{+}=\{1, 2, \ldots\}$. Under this setting, the probability of failure occurring at time k is given by,

$$f(k)=\Pr(K=k), \quad k=1,2,\ldots, \tag{4.1}$$

and the corresponding survival function is,

$$S(k)=\Pr(K>k)=\sum_{j=k+1}^{\infty} f(j), \tag{4.2}$$

where, $F(k)=1-S(k)$. The failure rate (called the classical failure rate in reference [35]) is given by,

$$\lambda(k)=\Pr(K=k \mid K>k-1)=\frac{\Pr(K=k)}{\Pr(K>k)}=\frac{f(k)}{S(k-1)}, \tag{4.3}$$

where, $\Pr(K > k) > 0$. The failure rate (4.3) may be interpreted as the conditional probability that an item (object) fails at time k (on condition that it is still operational at time $k-1$), see e.g. reference [86]. The relation (4.3) can also be written in the following form,

$$\lambda(k) = \frac{S(k-1) - S(k)}{S(k-1)}. \tag{4.4}$$

The necessary conditions for a sequence, $\lambda(k),\ k \geq 1$ to define the failure rate (4.4) for some random variable with the support in $N^{+} = \{1, 2, \ldots\}$ are given in references [184] and [236]. These conditions may be restated as follow, [86]:

1. For all $K < m$, then $\lambda(k) < 1$, where the distribution is defined over $\{1, 2, \ldots, m\}$.
2. For $k \in N^{+}$, then $0 \leq \lambda(k) \leq 1$ and $\sum_{j=1}^{\infty} \lambda(j) = \infty$.

The above reliability characteristics, which uniquely describe the distribution of the random variable, K, are related to each other in the following way,

$$f(k) = S(k) - S(k+1), \tag{4.5}$$

and,

$$S(k) = \prod_{j=1}^{k} (1 - \lambda(j)). \tag{4.6}$$

On the other hand,

$$f(k) = S(k)\lambda(k) \Rightarrow \quad f(k) = \lambda(k)\prod_{i=1}^{k-1}(1 - \lambda(i)) \tag{4.7}$$

Remark 1.3

1. For distribution defined over the interval $\{1, 2, \ldots, m\}$, condition 1 above holds and the failure rate is bounded, i.e., $\lambda(k) \leq 1$ for all integer $k > 0$. On the other hand, it is unbounded

in the continuous case. Specifically, this fact implies that the failure rate would not be convex in this case.

2. On the other hand, for a distribution defined over the interval $k \in N^{+}$, the condition 2 above seems to be natural.

It should be noted that, as a consequence of condition 1, some properties of the failure rate in the continuous case do not hold for the discrete one. The failure rates for the series system of independent components are not additive, e.g. "*the failure rate of a system is not equal to the sum of individual failure rates of its components*", [3]. Another implication of condition 1 is that, the failure rate has a meaning of probability in the discrete case, whereas it is not for the continuous case. The probability of failure in the latter is only approximated when the failure rate is multiplied by a sufficiently small unit interval of time. Consequently, the exponential representation does not hold in general settings in the discrete case and the cumulative failure rate is not equivalent to its continuous counterpart, e.g.

$$\Lambda(k) = \sum_{j=1}^{k} \lambda(j) \neq -\log S(k). \tag{4.8}$$

This nonequivalence has been referred in literature as the main reason for the two definitions of IFRA (DFRA) and NBU (NWU) classes in the discrete case as opposed to the corresponding classes in the continuous case. See e.g., references [103 p. 6] and [120 p.11]. Intuitively, this would create some challenges in the choice of appropriate model to be used in the analysis of the corresponding aging classes.

There are approaches, which address the observed underlying differences of the corresponding failure rates in the discrete and continuous cases. For instance, [164] proposed "the ratio of two consecutive probabilities, $f(k+1)/f(k)$ and studied the monotocity properties of a wide class of distributions with increasing failure rates". This approach was also adopted for defining the corresponding IFR (DFR) aging classes in [86]. In particular:

1. The distribution is log-concave if and only if $\dfrac{f(k+1)}{f(k)}$ for $k \geq 1$ is decreasing. As $f(k+2)f(k) < [f(k+1)]^2$ for $k \geq 0$ is log-concave, the failure rate is IFR in this case.

Whereas, it is log-convex if and only if $\frac{f(k+1)}{f(k)}$ for $k \geq 1$ is increasing and $f(k+2)f(k) > [f(k+1)]^2$ for $k \geq 0$. This, means that the corresponding failure rate is DFR.

2. On the other hand, if $\frac{f(k+1)}{f(k)}$ for $k \geq 1$ is constant, i.e. $\left(\frac{f(k+1)}{f(k)} = \frac{f(k+2)}{f(k+1)}\right)$, then $f(k) = C^k f(0)$ for some constant C. For the specific case, the geometric distribution with $f(k) = \theta(1-\theta)^{k-1}$ for $k = 1, 2, \ldots$, has constant failure rate. Whereas, for the uniform distribution with $f(k) = c$ for $k = 0, 1, 2, \ldots, m$ the failure rate is increasing (IFR). The failure rate is increasing (IFR) if also $f(k) = \frac{c^k}{1 + C + C^2 + \ldots + C^m}$ for $j = 0, 1, 2, \ldots, m$.

See e.g. also references, [10], [12] and [34] for other alternative characterizations.

The *alternative failure rate*,

$$\lambda_a(k) = -[\log S(k) - \log S(k-1)] = -\log \frac{S(k)}{S(k-1)} = \log \frac{S(k-1)}{S(k)} \quad \text{for} \quad k = 1, 2, \ldots, \tag{4.9}$$

which is unbounded as in the continuous case, is proposed by the authors of reference [198]. “Despite not having a clear probabilistic meaning, this failure rate is a useful transformation of $\lambda(k)$”, [3]. It reconciles the observed differences in the aging properties in the discrete case. In particular, it is related to the failure rate (4.7) via the following,

$$\lambda_a(k) = -\ln(1 - \lambda(k)) \ , \tag{4.10}$$

where, $\lambda(k) = 1 - \exp(-\lambda_a(k))$. These relations, in addition to the ones in (4.5-4.7), are useful for comparing the aging behavior of items (objects) for any increasing, $\lambda(k)$. From (4.10), “$\lambda(k)$ and $\lambda_a(k)$ exhibit the same monotocity properties, e.g., $\lambda_a(k)$ increases (decreases) if and only if $\lambda(k)$ increases (decreases)”, [131]. We will discuss now the shapes of the corresponding failure rates for some specific distributions in the class of discrete Weibull distributions.

Type I Discrete Weibull Distribution

This distribution was the first discretized analogue of the 2-parameter continuous Weibull distribution (cf. reference [244]). It has the survival function

$$S(k) = v^{k^{\eta}} \ , \ 0 < v < 1 \, , \, \eta > 0 \, . \tag{4.11}$$

Accordingly with (4.4), the corresponding failure rate is,

$$\lambda(k) = \frac{S(k-1) - S(k)}{S(k-1)} = 1 - v^{k^{\eta} - (k-1)^{\eta}} \, . \tag{4.12}$$

On the other hand, in accordance with (4.10), the alternative failure is given by

$$\lambda_a(k) = \log \frac{S(k-1)}{S(k)} = \log \frac{v^{(\eta-1)}}{v^{k^{\eta}}} = \log v \left[(k-1)^{\eta} - k^{\eta}\right]. \tag{4.13}$$

The corresponding failure rate plots for different values of $\eta < 1$ and $\eta > 1$ with $v = 0.5$ are, respectively, shown in Fig.14 and Fig.15.

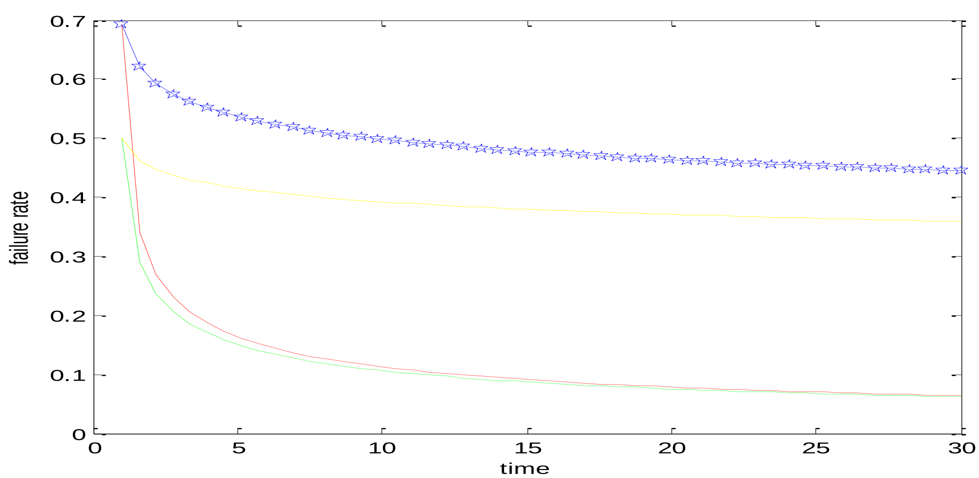

Fig. 14: A plot of $\lambda(k)$ and $\lambda_a(k)$ for various values $\eta < 1$ and $v = 0.5$.

As can be noted, from Fig. 14, the failure rates are both monotonically decreasing as functions of k for the specific case $\eta < 1$. Whereas, from (4.12) and (4.13) it can also be noted that when $\eta = 1$, the type I discrete Weibull distribution reduces to the geometric distribution with non-aging property as the exponential distribution in the continuous case. On the other hand, for $\eta > 1$ the failure rate (4.4) is increasing as a function of k, approaching one from below, whereas the alternative failure rate is continually increasing as time increases (see Fig. 15, below),

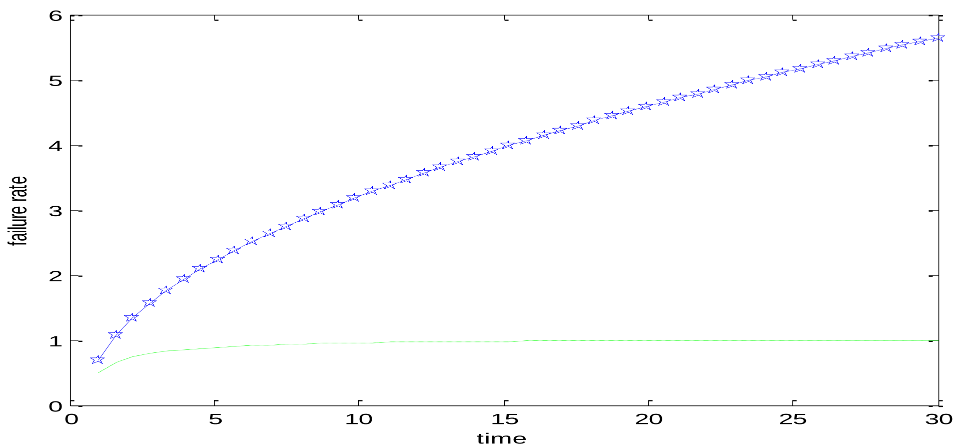

Fig. 15: A plot of $\lambda(k)$ and $\lambda_a(k)$ for different values $\eta > 1.5$ and $v = 0.5$.

Type II Discrete Weibull Distribution

This distribution is introduced into literature by the authors of reference [228] based on the preservation of the power function form of the failure rate. It has failure rate given by,

$$\lambda(k) = h k^{\eta-1} \quad , \quad k = 1 , 2 , \ldots , n , \tag{4.14}$$

where $0 < h \leq 1$, $\eta > 0$ and

$$n = \begin{cases} h^{-(1/(\eta-1))} & \eta > 1 \\ \infty & \eta \leq 1 \end{cases}.$$

From relation (4.10), the alternative failure is given by,

$$\lambda_a(k) = -\log\left[\left(1 - \left(h k^{\eta-1}\right)\right)\right]. \tag{4.15}$$

The corresponding failure rate plots for different values of $\eta < 1$ and $\eta > 1$ with $h = 0.5$ are shown, respectively, in Fig.16 and Fig.17.

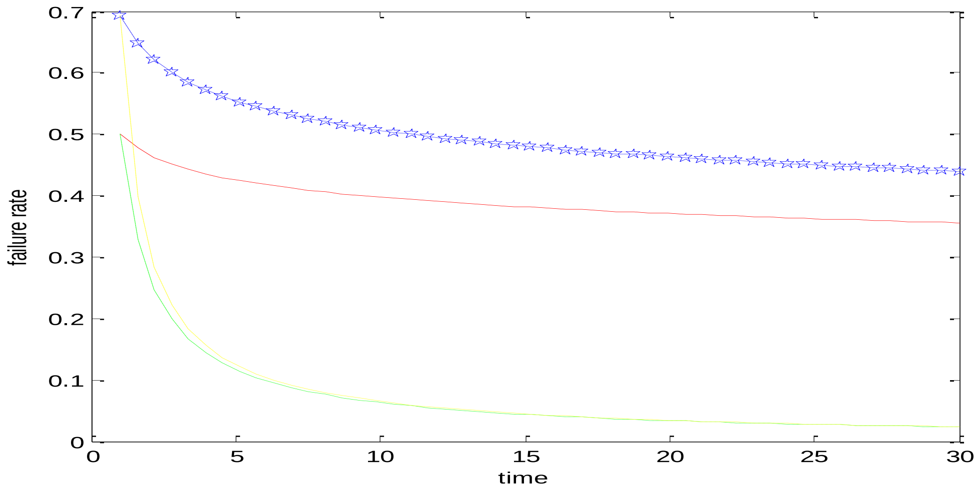

Fig. 16: A plot of $\lambda(k)$ and $\lambda_a(k)$ for different values of $\eta < 1$ and $h = 0.5$.

Obviously from Fig.16, $\lambda(k)$ and $\lambda_a(k)$, decreases as a functions of k for values of $\eta < 1$. On the other hand, when $\eta > 1$, $\lambda(k)$ is monotonically increasing (IFR) and the corresponding $\lambda_a(k)$ initially increases sharply to a certain maximum point then monotonically decreases, (UBT). See Fig. 17.

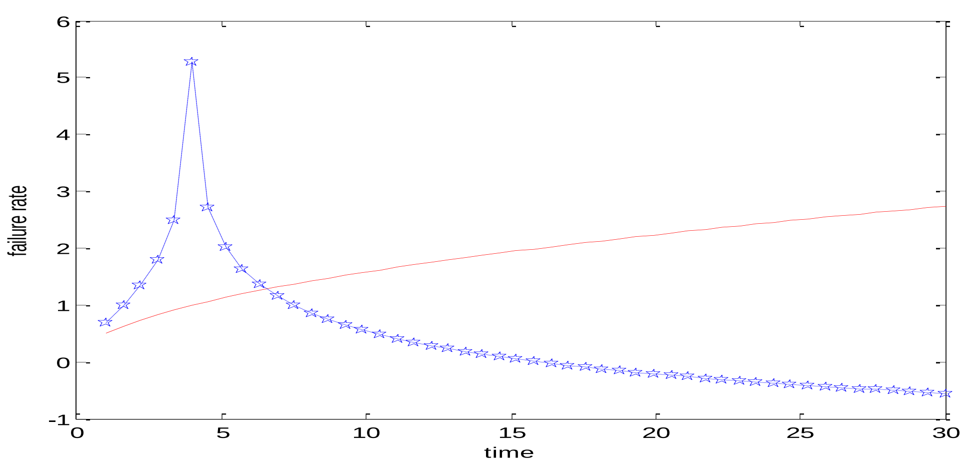

Fig.17: A plot of $\lambda(k)$ and $\lambda_a(k)$ for $\eta = 1.5$ and $h = 0.5$.

Type III Discrete Weibull Distribution

This distribution is an extension of the type I Weibull model introduced by the authors of reference [244] and was advanced further in reference [223]. Its failure rate is given by,

$$\lambda(k) = 1 - \exp\left(-\beta k^{\eta}\right), \tag{4.16}$$

where, $\beta \in \Re^{+}$ and $\eta \in \Re$. Using relation (4.10), we obtain the alternative failure rate,

$$\lambda_a(k) = \log\left[1 - \left(1 - \exp\left(-\beta k^{\eta}\right)\right)\right] = \beta k^{\eta} \ , \tag{4.17}$$

The corresponding plots of the failures rates (4.16) and (4.17) for different values of η and $\beta = 1$ are shown subsequently in Fig. 18 and Fig.19. As can be noted from these figures, the failure rates are increasing for $\eta > 0$ (see Fig. 18) and decreasing for $\eta < 0$ (see Fig. 19). Whereas, from (4.16) and (4.17), it can, also, be seen that the failure rates become constant for $\eta = 0$ and the Type III discrete Weibull distribution reduces to a geometric distribution with constant failure rate.

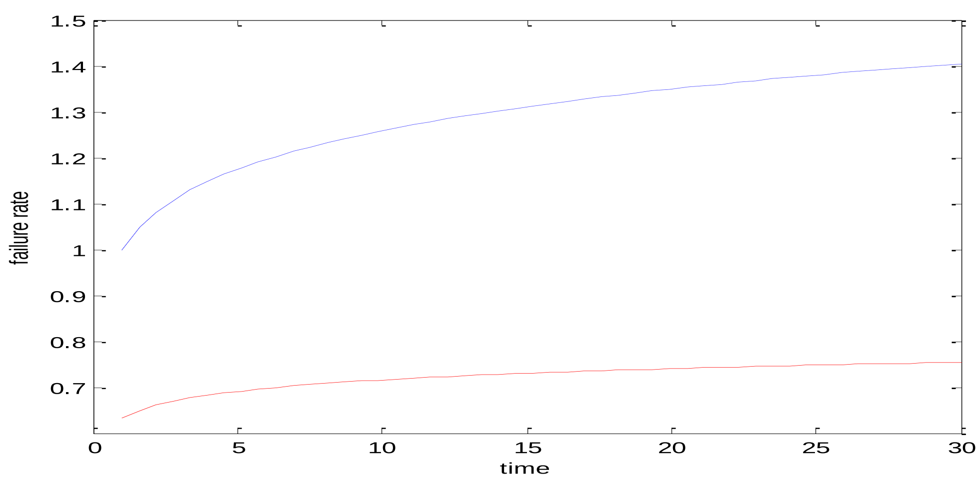

Fig. 18: A plot of $\lambda(k)$ and $\lambda_a(k)$ for $\eta = 0.1$ and $\beta = 1$.

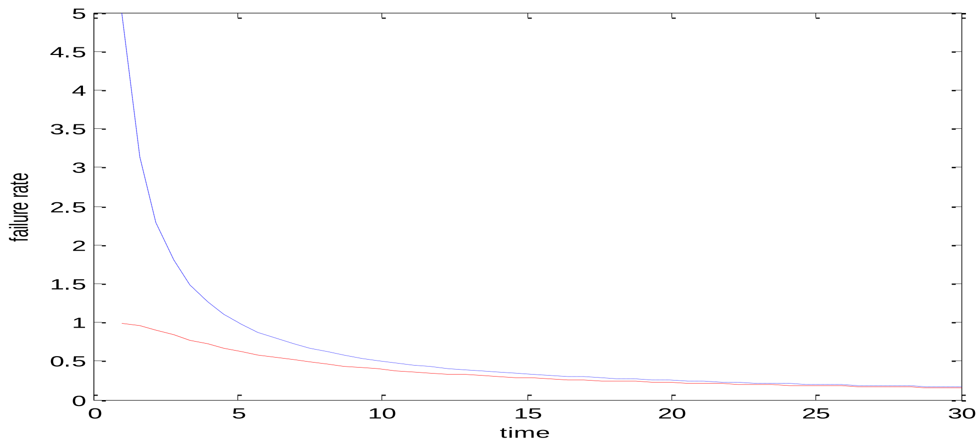

Fig. 19: A plot of $\lambda(k)$ and $\lambda_a(k)$ for $\eta=-1$ and $\beta=5$

The Discrete Inverse Weibull Distribution

The survival function of this distribution is given by

$$S(k)=1-v^{k^{-\eta}},$$

and the failure rate,

$$\lambda(k)=\frac{v^{k^{-\eta}}-v^{(k-1)^{-\eta}}}{1-v^{(k-1)^{-\eta}}}.$$

The corresponding failure rate plot for different values of v and η is shown on Fig. 20 below,

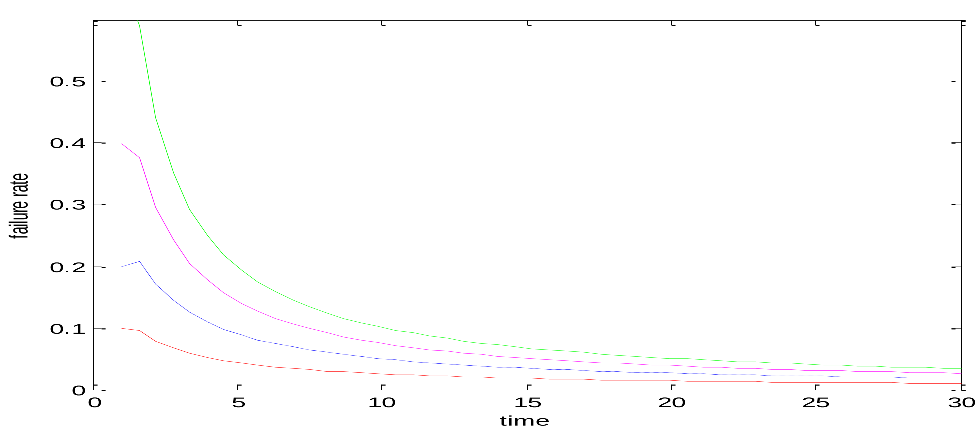

Fig. 20: A plot of $\lambda(k)$ for different values of v and η.

On the other hand, accordingly with (4.10), the corresponding alternative failure rate is given by,

$$\lambda_a(k) = \log\left(\frac{1-v^{(k-1)^{-\eta}}}{1-v^{k^{-\eta}}}\right),$$

with the plots for different values of v and η given in Fig.21 below. As may be noted from these figures in both cases the failure rates are decreasing (DFR).

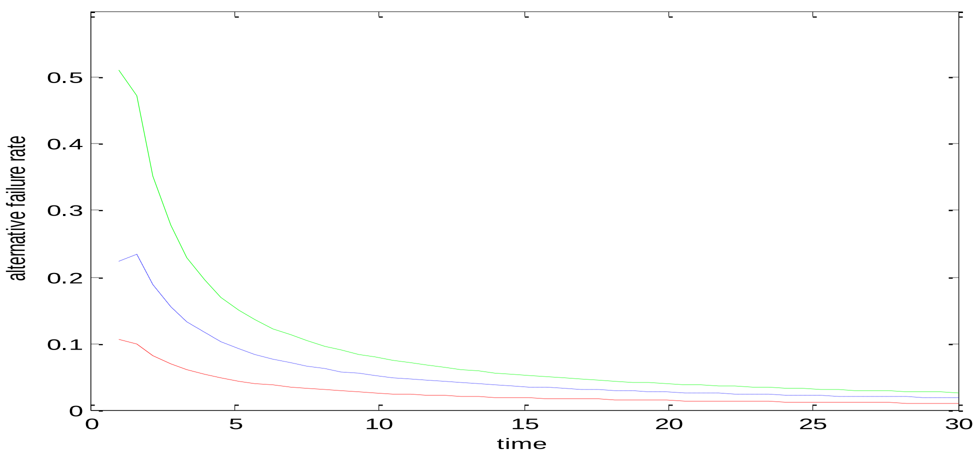

Fig. 21: A plot of $\lambda_a(k)$ for different values of v and η.

Discrete Modified Weibull Distribution

This distribution is the discrete analogue of the modified Weibull distribution. The survival function is given by,

$$S(k) = v^{k^{\eta}\beta^{k}}.$$

The corresponding failure rate is,

$$\lambda(k) = 1 - \frac{v^{k^{\eta}\beta^{k}}}{v^{(k-1)^{\eta}\beta^{k-1}}}.$$

The failure rates plots for different values of of η, β and $v = 0.5$, are shown on Fig. 22 below.

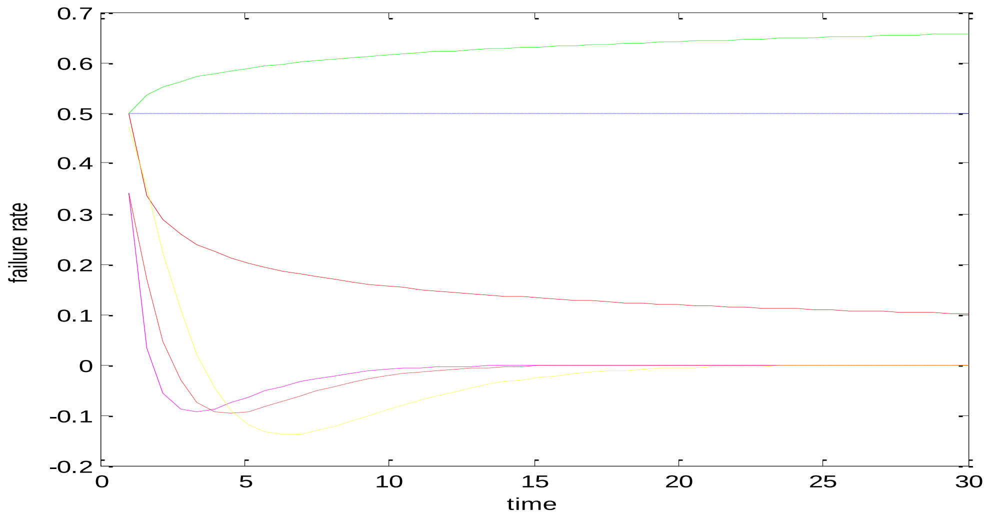

Fig. 22: A plot of $\lambda(k)$ for different values of η, β and $v = 0.5$.

Correspondingly with (4.10), the alternative failure rate is given by,

$$\lambda_a(k) = \log\left\{\left(v^{(k-1)^\eta \beta^{k-1}}\right) / \left(v^{k^\eta \beta^k}\right)\right\}.$$

The failure rates plots for different values of of η, β and $v = 0.5$, are shown on Fig. 23 below.

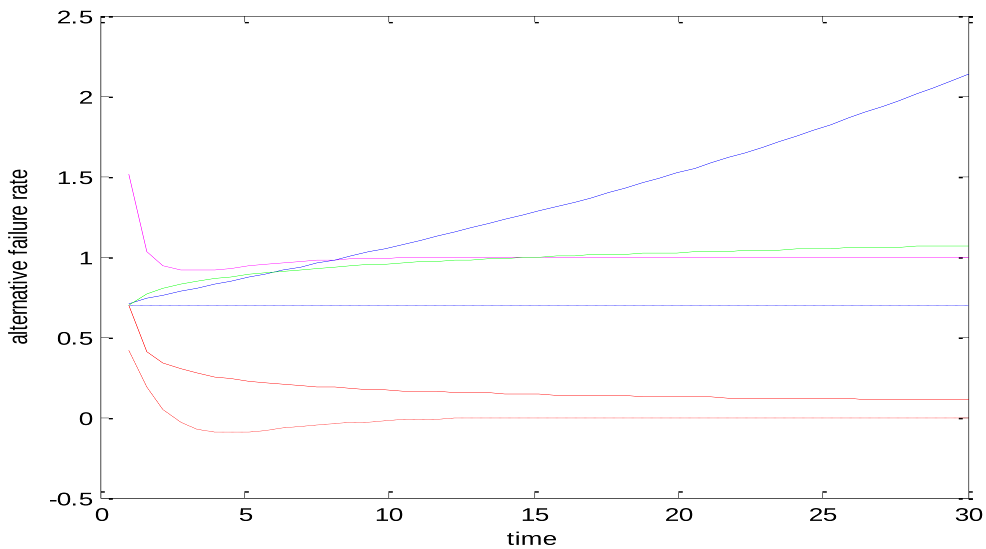

Fig. 23: A plot of $\lambda_a(k)$ for different values of η, β and $v = 0.5$.

As can be noted from Fig. 22 and Fig. 23, various shapes of the failure rate are exhibited for different pairs of η and β for the discrete modified Weibull distribution. For the values of $\eta < 1$ and $\beta = 1$, the failure rates are decreasing (DFR) and are increasing (IFR) for values of $\eta = 1$, when $\beta > 1$ (see the blue in Fig.23 and green curve in Fig. 22). When $\eta = 1$ for $\beta = 1$ the failure rate is constant and the modified discrete Weibull distribution reduces to the geometric distribution with constant failure rate (see dashed blue curve). On the other hand, the failure rate initially decreases to a certain minimum point and then increases (BT) for pairs of values: $\eta < 1$ and $\beta < 1$ (see the magnata curve); $\eta > 1$ and $\beta = 1$ (see dashed red curve) and finally when $\eta > 1$ and $\beta > 1$ (see dashed yellow curve).

In most of the specific cases considered here, the failure rate (4.4) and the alternative failure rate (4.9) exhibit the same monotocity properties. However, for the Type II Weibull distribution, different monotocity properties are exhibited for the same parameter values: the failure rate (4.4) is increasing (IFR) while the failure rate (4.9) is of the UBT type. This apparent difference should be taken into account in practical applications. It means that the alternative failure rate, $\lambda_a(k)$ may be an appropriate choice in the modeling and analysis of various aging characteristics as compared to $\lambda(k)$.

We, firstly present and briefly discuss some results on the behavior of the failure rate of the mixture of two distributions, and then further investigate this behavior for the mixture of some specific lifetime distributions in the subsequent section.

4.2. Failure rate of a mixture of two discrete distributions

Continuous mixtures are usually more flexible and suitable for modeling heterogeneity in practical settings. However, as noted earlier, there are several situations in which discrete lifetimes do arise in practical settings. Some special and relevant examples that involve 'shock processes in a natural scale' are also considered in reference, [3]. We analyze the failure rate for the mixture of two lifetime distributions $F_1(k)$ and $F_2(k)$ with the corresponding pdfs, $f_1(k)$ and $f_2(k)$, where $\lambda_1(k)$ and $\lambda_2(k)$ are the respective failure rates for the two subpopulations. Under this setting, the general mixture models are represented as follow; see e.g. references, [66], [82], and [86]:

The mixture survival and density functions are given as

$$S_m(k) = p\ S_1(k) + q\ S_2(k), \quad ; \quad f_m(k) = p\ f_1(k) + q\ f_2(k), \tag{4.18}$$

respectively, where the masses p and $q = 1 - p$ define discrete mixture of distribution. The corresponding mixture failure rate is, therefore, given by:

$$\lambda_m(k) = \frac{p\ f_1(k) + q\ f_2(k)}{p\ S_1(k) + q\ S_2(k)}. \tag{4.19}$$

The relation (4.19) may also be represented as:

$$\lambda_m(k) = p\ (k)\ \lambda_1(k) + q(k)\ \lambda_2(k), \tag{4.20}$$

where, $\lambda_j(k)$, $j = 1\ ,\ 2$ are the corresponding failure rates and the time-dependent probabilities are,

$$p(k) = \frac{p\ S_1(k)}{p\ S_1(k) + q\ S_2(k)} \text{ and } q(k) = \frac{q\ S_2(k)}{p\ S_1(k) + q\ S_2(k)}. \tag{4.21}$$

4.2.1. Some important results

The above relations are useful for studying the properties of the failure rate of the mixture of two distributions[2]. For instance, the authors of reference [107] utilized the relation (4.21) and showed that, "$\min\{\lambda_1(k)\ ,\ \lambda_2(k)\} \le \max\{\lambda_1(k)\ ,\ \lambda_2(k)\}$", see e.g. references [66] and [71]. Specifically, as pointed out in the forgoing references, if the failure rates $\lambda_1(k)$ and $\lambda_2(k)$ are ordered such that $\lambda_1(k) \le \lambda_2(k)$, then the mixture failure rate lies between these failure rates,

[2] Albeit, may also be extended to a case of more than two distributions.

$$\lambda_1(k) \le \lambda_m(k) \le \lambda_2(k). \tag{4.22}$$

For example, consider a specific case

$$\lambda_1(k) = 1.2 - \exp(-1.2\,k) + 0.01\,k\,,$$

and

$$\lambda_2(k) = 1.2 + 1.14\exp(-0.08\,k) + 0.01\,k\,,$$

where, $p = p_1(0) = 0.60$ ($q = 1 - p = 0.40$), then according to (4.20), the mixture failure rate is given by

$$\lambda_m(k) = 1.2 - 0.144\exp(-1.28\,k) + 0.12\,k$$

The corresponding failure rate plots for $\lambda_1(k)$, $\lambda_2(k)$ and $\lambda_m(k)$ are reflected on Fig. 24 below.

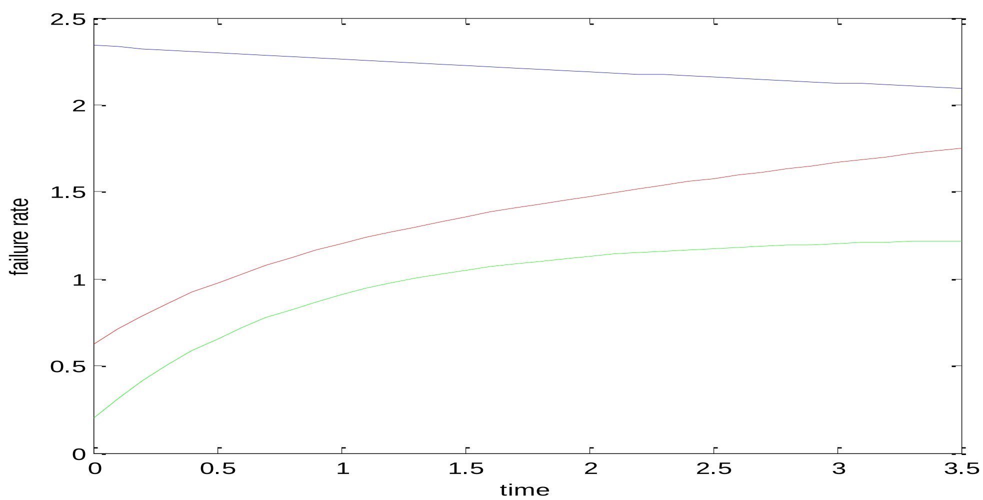

Fig. 24: A plot of $\lambda_1(k)$, $\lambda_2(k)$ and $\lambda_m(k)$ for different values of k and $p = p_1(0) = 0.60$.

From Fig. 24, the failure rate, $\lambda_1(k)$ is increasing (see the green curve) and the failure rate, $\lambda_2(k)$ is decreasing (see the blue curve). On the other hand, the mixture failure rate, $\lambda_m(k)$ (see the red curve) lies between the $\lambda_1(k)$ and $\lambda_2(k)$.

For the specific case above, the mixture failure rate is increasing, whereas "on differentiating (4.20), the following is obtained", [86 p.48] and [136 p.5]:

$$\lambda_m^{'}(k) = p(k)\,\lambda_1^{'}(k) + q(k)\,\lambda_2^{'}(k) - p(k)\,q(k)\,(\lambda_1(k) - \lambda_2(k))^2. \qquad (4.23)$$

As, $\lambda_j^{'}(k) \le 0 \; j = 1, 2$, it follows from (4.23) that the mixture failure rate, in (4.20) is decreasing, [66]. Specifically, as pointed by the authors of the forgoing reference, for $S_j(0),\; j = 1, 2$, the initial value of the mixture failure rate, (at $k = 0$) is defined by $\lambda(0) = p\,\lambda_1(0) + q\,\lambda_2(0)$. This, already, means that for $k > 1$ the conditional probabilities $p(k)$ and $1 - p(k)$ are increasing (decreasing), which can be observed when dividing the numerator and denominator in first equation (4.18), respectively, by $S_1(k)$. This effect, which can be explained as “the weakest items are dying out first” principle means that the proportion of the survived up to t items in the mixed population is increasing.

As a result,

$$\lambda_m(k) < p(k)\,\lambda_1(k) + q(k)\,\lambda_2(k). \qquad (4.24)$$

This means that, $\lambda_m(k)$ is always smaller than the expectation $p\,\lambda_1(k) + q\,\lambda_2(k)$. On the other hand, the “sufficient condition for the mixture failure rate to initially (at least, for small t) decrease” is, [58]:

$$\lambda_1^{'}(k) + q\,\lambda_2^{'}(k) - p\,(1-p)\,(\lambda_1(0) - \lambda_2(0))^2 < 0, \qquad (4.25)$$

where the derivatives are obtained at $k = 0$. In fact, “the mixture failure rate is initially decreasing not matter how fast the failure rates $\lambda_1(k)$ and $\lambda_2(k)$ are increasing in the neighborhood of 0 when $|\lambda_1(0) - \lambda_2(0)|$ is very large”, [66].

Mixtures of discrete distributions, which are not necessarily derived from the corresponding continuous lifetime distributions, are known in literature. For instance, a discussion of mixtures of discrete distributions can, already be found in chapter 4 of reference [238]. However, less well-known are the mixtures of the recently discretized lifetime distributions. Therefore, hereafter, under the defined settings we analyze the corresponding shapes of the failure rates of the mixtures of some selected (i.e. specifically discretized) lifetime distributions.

4.2.2. Mixture of two geometric distributions

The geometric distribution with the pdf,

$$f(k) = \theta(1-\theta)^{k-1} \text{ for } k = 1, 2, \ldots, \; 0 < \theta \leq 1,$$

arise as the discretized version of the continuous exponential distribution (hence also known as the discrete exponential distribution, when $\theta = \exp(-\lambda)$). Its distribution function is given by

$$F(k) = 1-(1-\theta)^k,$$

where, $S(k) = 1 - F(k)$ is the corresponding survival function. The failure rate, in accordance with (4.3) is

$$\lambda(k) = \frac{f(k)}{S(k-1)} = \frac{\theta(1-\theta)^{k-1}}{(1-\theta)^{k-1}} = \theta,$$

which is, constant. It is well-known in literature that continuous mixtures of exponential distributions are DFR, could it then be the case that the mixture of geometric distributions has a decreasing failure rate (DFR)?

Suppose, two subpopulations with lifetimes described by geometric distributions with the distributions functions, $F_1(k)$ and $F_2(k)$, survival functions $S_1(k)$ and $S_2(k)$ and mass functions $f_1(k)$ and $f_2(k)$, respectively. Therefore, the corresponding mixture Cdf and pdf are given, respectively, by relations in (4.18). On the other hand, in accordance with relation (4.20) and (4.21), the failure rate of the mixture of the two subpopulations is given be

$$\lambda_m(k) = \frac{p\,(1-\theta_1)^k}{p(1-\theta_1)^k + q(1-\theta_2)^k}\theta_1 + \frac{q\,(1-\theta_2)^k}{p\,(1-\theta_1)^k + q(1-\theta_2)^k}\theta_2 = \frac{p\,\theta_1(1-\theta_1)^k + q\,\theta_2(1-\theta_2)^k}{p\,(1-\theta_1)^k + q(1-\theta_2)^k}.$$

The corresponding plot of $\lambda_m(t)$ for different values of k and $\theta_1, \theta_2 > 0$ is shown on Fig. 25 below.

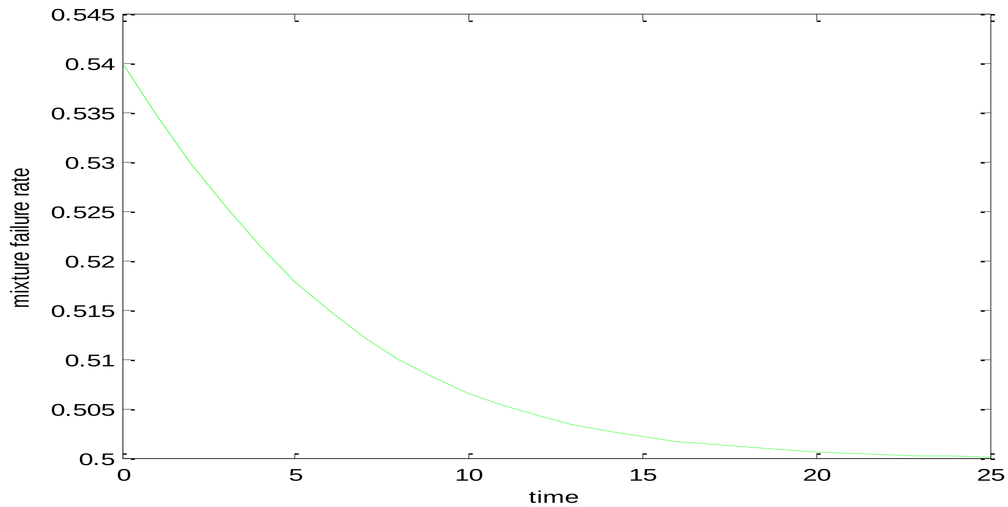

Fig. 25: A plot of $\lambda_m(k)$ for different values of k, $p = 0.6(q = 0.4)$ and $\theta_1, \theta_2 > 0$.

As can be noted from Fig. 25, similar to the failure rate of a mixture of two continuous exponential distributions, which is DFR, the failure rate of the mixture of two geometric distributions is also decreasing and even tends to zero as time increases. Thus the geometric distribution is an equivalent of the continuous exponential distribution in the discrete setting.

4.2.3. Mixture of the geometric distribution and discrete Weibull distributions

Several discrete Weibull lifetime distributions arise as discretized versions of their continuous counterparts. We have compared the failure rate (4.4) with the failure rate (4.9) for some of these lifetime distributions. In this section, we focus on analyzing the failure rate of the mixture of these distributions with the geometric distribution. We are particularly interested in the case, when the failure rate is increasing (IFR), BT or UBT, as these somehow indicate some kind of deterioration (aging) of an item (object) with time. On the other hand, as pointed out in the previous section, the failure rate of the geometric distribution reflect the non-aging property.

Mixture of geometric distribution with the Type I discrete Weibull distribution

The survival function and failure rate for type I discrete Weibull distribution are given in (4.11) and (4.12), respectively, whereas the survival function and the failure rate for discrete geometric distribution are given in section 4.2.2. Therefore, under the defined settings and utilizing the relations (4.20) and (4.21), we have:

$$\lambda_m(k) = \frac{p\, v^{k^\eta}}{p\, v^{k^\eta} + q(1-\theta)^k}\left[1 - v^{k^\eta - (k-1)^\eta}\right] + \frac{q(1-\theta)^k}{p\, v^{k^\eta} + q(1-\theta)^k}\theta = \frac{q\theta(1-\theta)^k + p v^{k^\eta}\left[1 - v^{k^\eta - (k-1)^\eta}\right]}{p v^{k^\eta} + q(1-\theta)^k}$$

where, the type I discrete Weibull distribution is the baseline distribution. The following plots show the mixture failure rate for the case $\eta > 1$ (Fig. 26) and $\eta < 1$ (Fig. 27).

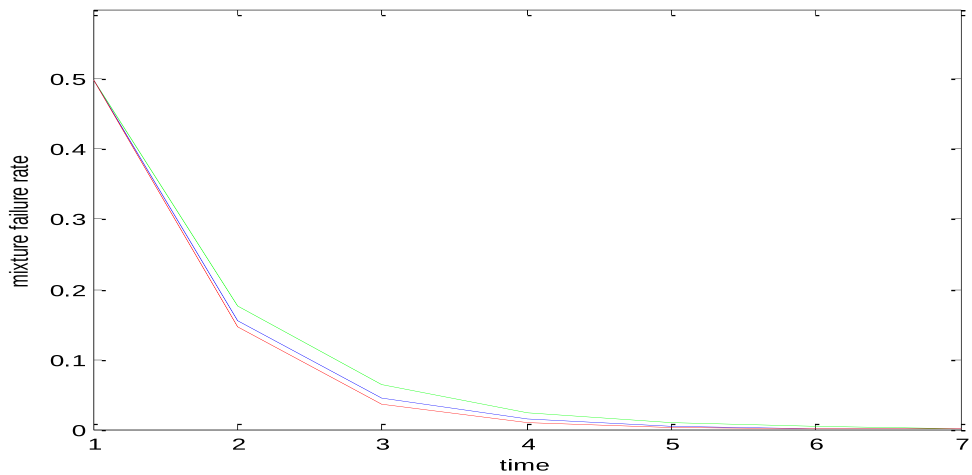

Fig. 26: A plot of $\lambda_m(k)$ for different values of k, $p = 0.6(q = 0.4), \theta = v = 0.5$ and $\eta > 1$.

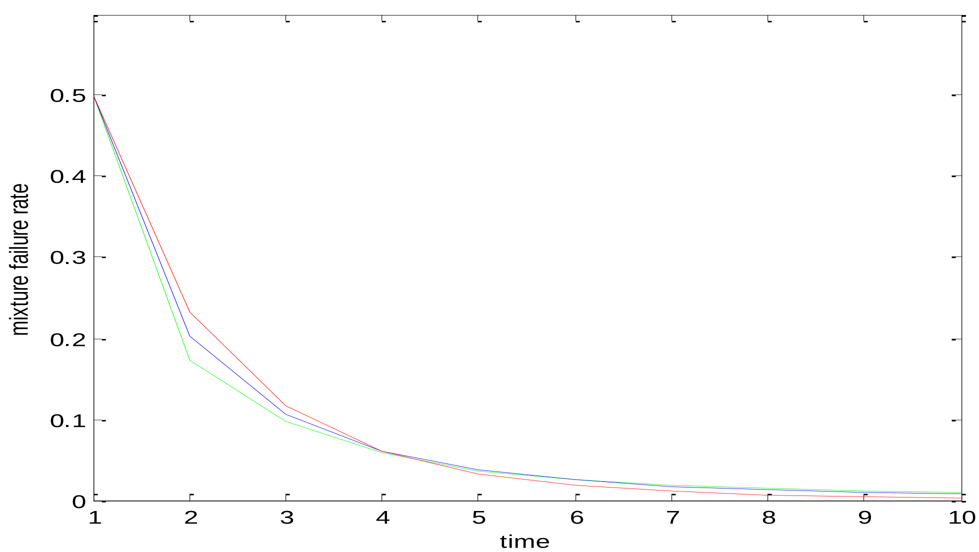

Fig. 27: A plot of $\lambda_m(k)$ for different values of k and $p = 0.6(q = 0.4), \theta = v = 0.5$ $\eta < 1$.

We showed in section 4.1., that the failure rate of the type I discrete Weibull distribution is decreasing for values of $\eta < 1$ (Fig. 14), constant for $\eta = 1$, whereas it is increasing for values of $\eta > 1$ (Fig. 15), and the failure rate of the geometric distribution is constant. As can be noted, the failure of the mixture of these distributions preserve the DFR-property for values $\eta < 1$, whereas the corresponding aging changes significantly for values of $\eta > 1$, e.g. the failure rate is increasing (IFR), whereas the mixture failure rate is decreasing (DFR) and even tends to zero as time increases. On the other hand the mixture failure rate reduces to a constant for $k = 1$ and $\eta = 1$, which is an evident property of a geometric distribution.

The latter non-aging property is also preserved for the mixture of geometric distribution with both the type II discrete Weibull distribution and type III discrete Weibull distribution (e.g. when $\eta = 1$), whereas it is increasing (decreasing) for the values of $\eta > 1$.

Mixture of geometric distribution and the discrete modified Weibull distribution

The discrete analogue of the modified Weibull distribution, which was introduced by the authors of reference, [49] and was shown to exhibit various shapes of the failure rate for different pairs of η and β (see e.g. Fig. 22 and Fig. 23) is mixed with the geometric distribution. Hence, in accordance with (4.20) and (4.21), the failure rate of the mixture these distributions is given by

$$\lambda_m(k) = \frac{p v^{k^\eta \beta^k}}{p v^{k^\eta \beta^k} + q(1-\theta)^k}\left[1 - \frac{v^{k^\eta \beta^k}}{v^{(k-1)^\eta \beta^{k-1}}}\right] + \frac{q(1-\theta)^k}{p v^{k^\eta \beta^k} + q(1-\theta)^k}\theta = \frac{\theta q(1-\theta)^k + p v^{k^\eta \beta^k}}{p v^{k^\eta \beta^k} + q(1-\theta)^k}\left[1 - \frac{v^{k^\eta \beta^k}}{v^{(k-1)^\eta \beta^{k-1}}}\right].$$

The corresponding mixture failure rate plot for $\eta \geq 1$ and $\beta \geq 1$ is shown on Fig. 28, whereas for $\eta \leq 1$ and $\beta \leq 1$ is reflected on Fig. 29 below.

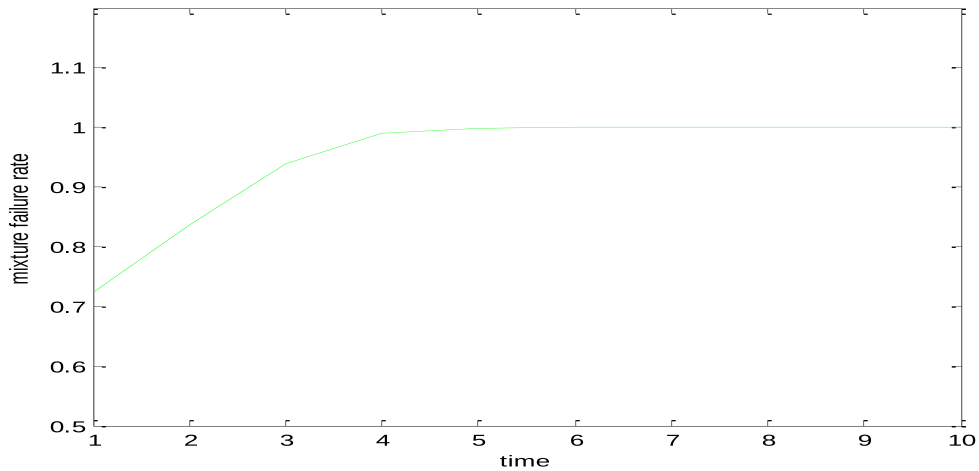

Fig. 28: A plot of $\lambda_m(k)$ for different values of k, $\eta \geq 1$ and $\beta \geq 1$, $p = 0.6(q = 0.4), \theta = v = 0.5$.

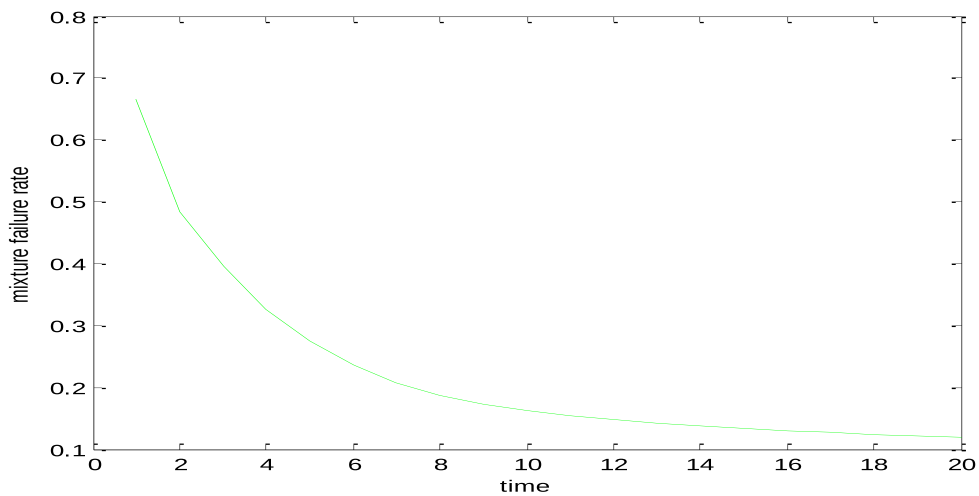

Fig. 29: A plot of $\lambda_m(k)$ for different values of k, $\eta \leq 1$ and $\beta \leq 1$, $p = 0.6(q = 0.4), \theta = v = 0.5$.

From Fig. 28, the mixture failure rate for different pairs of the values of parameters (especially when $\eta > 1$ and $\beta = 1$; $\eta > 1$ and $\beta > 1$; $\eta = 1$ and $\beta > 1$) is increasing (IFR) approaching one from below, whereas for different pairs of the values of parameters (especially when $\eta < 1$ and $\beta = 1$; $\eta < 1$ and $\beta < 1$) it is decreasing (DFR) and even tends to zero as time increases. As may be noted when $\eta = 1$ and $\beta = 1$, it reduces to a constant. These results show that the aging can change as a result of mixing. For instance, we have showed for the discrete modified distribution that when $\eta > 1$ and $\beta = 1$; $\eta > 1$ and $\beta > 1$; $\eta < 1$ and $\beta < 1$, the failure rate is BT, whereas for first two cases the aging changes from BT to IFR, and in the latter the aging changes from BT to DFR., respectively under the operation of mixing.

In fact, it can easily be shown, using (4.20) and assuming, $p = 0.6(q = 0.4)$ that for the type II discrete Weibull distribution and the type III discrete Weibull distribution when mixed with the geometric distribution the failure rate is increasing for the values of $\eta > 1$, whereas for the corresponding values $\eta < 1$ is decreasing and reduces to the geometric distribution, which exhibits the non-aging property when $\eta = 1$. It is perhaps, also interesting to note that, the considered here discrete Weibull distributions when mixed with the Lindley distribution, with the failure rate,

$$\lambda(k) = 1 - v - \frac{v\lambda}{1 + \lambda(k+1)},$$

leads to increasing mixture failure rate.

4.2.4. Mixture of the discrete gamma distribution and discrete Weibull distribution

Recall: the two-parameter standard gamma distribution has the pdf,

$$f(t) = \frac{Z\lambda^{\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{for} \quad t \geq 0,$$

where, $Z = \exp(-\lambda t)$ and λ is the scale parameter with α being the shape parameter (all positive). The discrete analogue of this distribution is obtained via discretization of both α and t. For instance, the discrete gamma distribution will, (for consistency of notation) be defined by the parameters α and w in this case, with the pdf given by,

$$f(k) = R k^{\alpha-1} w^k (1-w)^{\alpha}, 0 < w < 1 \quad \text{and } \forall \alpha \in N^+,$$

where, $w = \exp(-\lambda)$ and

$$R = \frac{1}{w \sum_{j=0}^{\alpha-2} A_{\alpha-1,\,j} w^j} \quad \text{for } \alpha \geq 2,$$

where $A_{n,m}$ is the Euler number given by

$$A_{n,m}=\binom{n}{m}=\sum_{i=0}^{m}(-1)^i(m+1-i)^n\binom{n+1}{i}.$$

Thus, for the specific case $\alpha = 2$, $A_{1,0}=1$ and $R=w^{-1}$, which means the pdf of the discrete gamma distribution reduces to,

$$f(k)=(1-w)^2 w^{k-1}\ ,\ 0<w<1.$$

The Cdf is given by

$$F(k)=1-w^k[1-k[1-w]],$$

where as usual, $S(k)=1-F(k)$. Therefore, in accordance with (4.3), the corresponding failure rate is

$$\lambda(k)=\frac{f(k)}{S(k-1)}=\frac{k(1-w)^2 w^{k-1}}{w^{k-1}[k(1-w)+w]}=\frac{k(1-w)^2}{k(1-w)+w}.$$

As can be noted the failure rate similar to the continuous case is increasing in this case. In accordance with (4.20) and (4.21), the failure rate of mixture of the discrete gamma distribution and the type I discrete Weibull distribution is

$$\lambda_m(k)=\frac{p\,v^{k^\eta}\left[1-v^{k^\eta-(k-1)^\eta}\right]}{p\,v^{k^\eta}+q\,w^k[1+k(1-w)]}+\frac{q\,w^k[1+k(1-w)]}{p\,v^{k^\eta}+q\,w^k[1+k(1-w)]}\left(\frac{k(1-w)^2}{k(1-w)+w}\right).$$

The corresponding plots for different values of k, where $p=0.6\ (q=0.4)$, $v=w=0.5$ and $\eta>1$ are shown in Fig. 30, whereas for $\eta<1$ is reflected in Fig. 31.

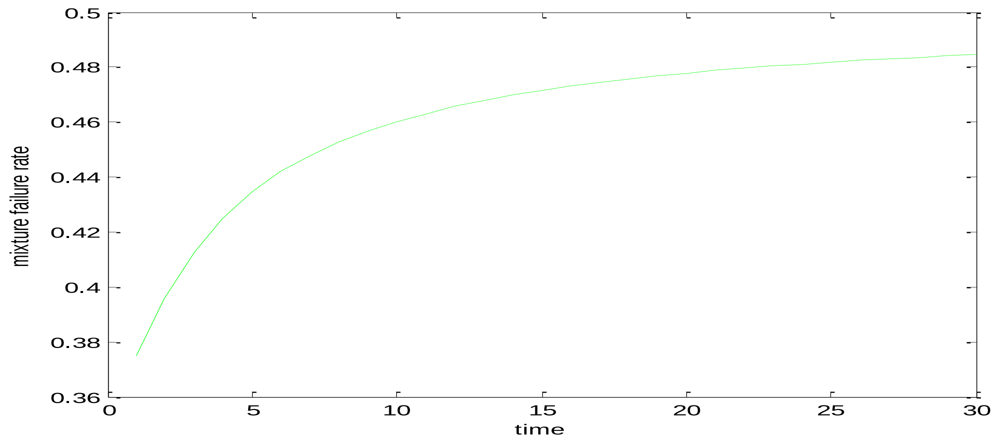

Fig. 30: A plot of $\lambda_m(k)$ for different values of k, $p = 0.6(q = 0.4), v = w = 0.5$ and $\eta > 1$.

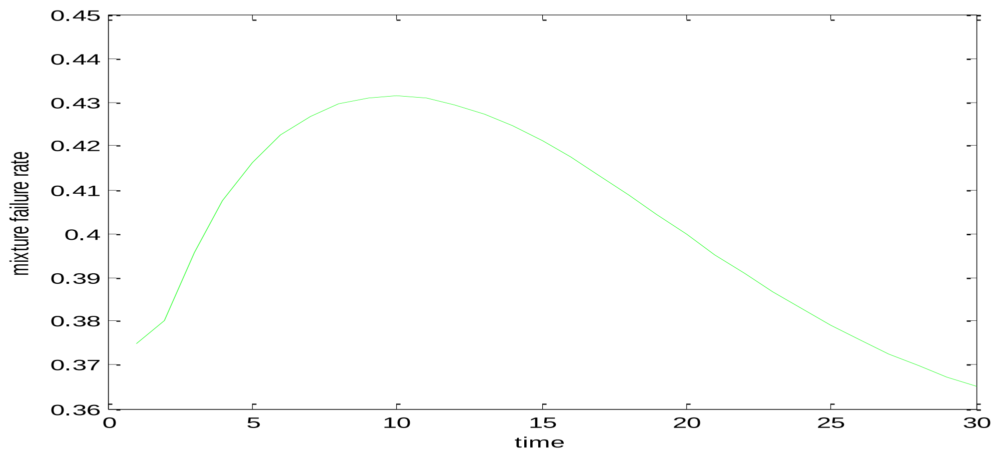

Fig. 31: A plot of $\lambda_m(k)$ for different values of k, $p = 0.6(q = 0.4)$, $v = w = 0.5$ and $\eta < 1$.

It can be seen from Fig. 30, the mixture failure rate is increasing (IFR) for values of $\eta > 1$, whereas from Fig. 31, it is observed to be UBT for values of $\eta < 1$. This means that the IFR property is preserved for values $\eta > 1$, whereas it changes significantly from IFR to UBT for values of $\eta < 1$.

On the other hand, the failure rate of the mixture of discrete gamma distribution and the discrete modified Weibull distribution, in accordance with (4.20) and (4.21) is given by

$$\lambda_m(k) = \frac{p\,v^{k^\eta \beta^k}}{p\,v^{k^\eta \beta^k} + q\,w^k[1 + k(1-w)]}\left[1 - \frac{p\,v^{k^\eta \beta^k}}{v^{(k-1)\beta^{k-1}}}\right] + \frac{q\,w^k[1 + k(1-w)]}{p\,v^{k^\eta \beta^k} + q\,w^k[1 + k(1-w)]}\left(\frac{k(1-w)^2}{k(1-w) + w}\right).$$

The corresponding plot of for different values of k, where $p = 0.6\ (q = 0.4)$, $v = w = 0.5$, and different values of the pairs $\eta < 1$ and $\beta < 1$ is shown on Fig. 32, whereas for $\eta \geq 1$ and $\beta \geq 1$ is shown on Fig. 33.

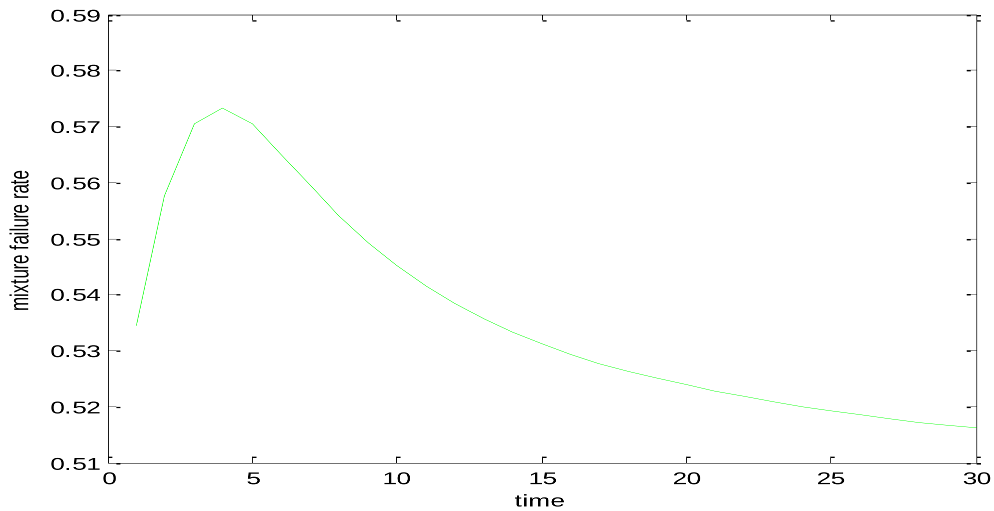

Fig. 32: A plot of $\lambda_m(k)$ for different values of k and pairs $\eta < 1$ and $\beta < 1$, $p = 0.6(q = 0.4)$, $v = w = 0.5$.

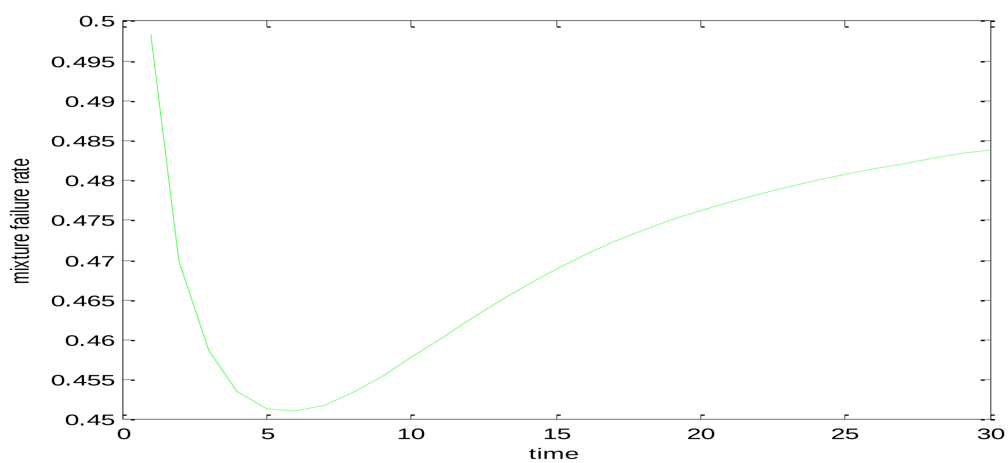

Fig. 33: A plot of $\lambda_m(k)$ for different values of k, $\eta \geq 1$ and $\beta \geq 1$, $p = 0.6(q = 0.4)$, $v = w = 0.5$.

For the discrete modified Weibull distribution, the failure rate is of BT type shape for pairs of values: $\eta < 1$ and $\beta < 1$ (and $\eta > 1$ and $\beta = 1$; $\eta > 1$ and $\beta > 1$) and the failure rate of discrete gamma distribution is increasing. On the other hand, the mixture failure rate is UBT for pairs of values, $\eta < 1$ and $\beta < 1$, whereas it is of BT shape for pairs of values $\eta \geq 1$ and $\beta \geq 1$.

In fact, it is worth noting that for the mixture of the discrete gamma distribution and the type II discrete Weibull distribution the failure is increasing (IFR) when $\eta \leq 1$, whereas for this mixture the failure rate is decreasing for values of $\eta > 1$. The IFR property is also preserved for mixtures of the considered here discrete Weibull distributions and the discrete Lindley distribution with increasing failure rate.

4.3. Some general results on Discrete mixture failure rate modeling

In what follows we present some important results on the properties of a discrete mixture failure rates and illustrate via some examples the corresponding applications.

4.3.1. Some properties of discrete mixture failure rates

Let the Cdf of the discrete random variable K be indexed by the continuous non-negative random variable W with support in $[0,\infty)$. Then,

$$\Pr(K \leq k) = \Pr(K \leq k \mid W = w) = F(k \mid w),\ k = 1, 2, \ldots, \tag{4.26}$$

with the corresponding survival function $S(k \mid w) = 1 - F(k \mid w)$. The failure rate that corresponds to the subpopulation indexed by $W = w$ is, therefore, given by:

$$\lambda(k \mid w) = \frac{F(k-1 \mid w) - F(k \mid w)}{F(k-1 \mid w)},\ k = 1, 2, \ldots \tag{4.27}$$

The mixture (population) Cdf is defined by:

$$F_m(k) = \int_a^b F(k \mid w)\, g(w) dw, \tag{4.28}$$

where, $g(w)$ is the pdf of W. The corresponding mixture survival function is given by:

$$S_m(k)=\int_0^\infty S(k\mid w)\,g(w)dw. \tag{4.29}$$

Consequently, from (4.4) and using relation (4.7), the corresponding mixture failure rate is obtained as,

$$\lambda_m(k)=\frac{\int_0^\infty \lambda(k\mid w)\,B(w)dw}{\int_0^\infty B(w)dw}, \tag{4.30}$$

where,

$$B(w)=\prod_{j=1}^{k-1}\left[1-\lambda(j\mid w)\right]g(w) \text{ and } \prod_{j=1}^{k-1}[\cdot]\equiv 1 \text{ for } k=1.$$

In particular, the mixture failure rate (4.30) may be written as the conditional expectation,

$$\lambda_m(k)=\frac{\int_0^\infty \lambda(k\mid w)\,B(w)dw}{\int_0^\infty B(w)dw}=E_{W\mid T>k-1)}[\lambda(k\mid W)], \tag{4.31}$$

where, $E_{(W\mid T>k-1)}[\cdot]$ denotes expectation with respect to the distribution of $(W\mid T>k-1)$, which is given by (the pdf),

$$g(w\mid k-1)\equiv\frac{B(w)}{\int_0^\infty B(w)dw}.$$

We have shown in the continuous setting that the mixture failure rate tends to be bent down when compared with a specific form of our model (2.1). We use a similar reasoning for comparing the shape of the population failure rate $\lambda_m(k)$ and the shape of subpopulation's failure rates $\lambda(k\mid w)$. The corresponding unconditional expectation of the random failure (hazard) rate, $\lambda(k\mid w)$ is,

$$\lambda_p(k)\equiv E\left[\lambda(k\mid W)\right]=\int_0^\infty \lambda(k\mid w)g(w)dw. \tag{4.32}$$

This function captures the monotonicity properties to the subpopulation failure rates in the mixtures. If $\lambda(k\mid w)$ is differentiable with respect to w and the failure rates are ordered such that $\lambda(k \mid w_1) < \lambda(k \mid w_2)$ when $w_1 < w_2$ for all $k \geq 1$, then $\lambda_P(k) > \lambda_m(k)$ for all $k \geq 1$ (the failure rate is bend down). If additionally $\frac{\partial}{dw}\lambda(k \mid w) \geq \frac{\partial}{\partial w}\lambda(k-1 \mid w)$ for all $k \geq 2$, then $\lambda_P(k) - \lambda_m(k)$ is strictly increasing in k. In particular, for the ordered failures rates, the ratio $\frac{g(w \mid k)}{g(w \mid k-1)}$ is decreasing in w. The implication of this is that, $(W \mid K > k-1) \underset{lr}{\leq} W$ and $(W \mid T > k-1)$ is decreasing in k in the sense of the likelihood ratio order. Consequently,

$$\lambda_p(k) \equiv E[\lambda(k \mid W)] > E_{(W \mid K > k-1)}[\lambda(k \mid W) = \lambda_m(t)].$$

4.3.2. Additive and proportional hazards models in the discrete setting

Let $\lambda_b(k)$ be the baseline failure rate for the discrete lifetime K (where $f_b(k)$ and $S_b(k)$ are respectively the probability mass function and the survival function), e.g.

$$\lambda_b(k) = \frac{f_b(k)}{S_b(k-1)} = \frac{S_b(k-1) - S_b(k)}{S_b(k)}. \tag{4.33}$$

Then, the corresponding equivalents of models (2.15) and (2.19) in the discrete setting (i.e. when K is indexed by the continuous non-negative random variable W) are given, respectively, by:

The additive model:

$$\lambda(k \mid w) = 1 - \Pr(K_w > k \mid K_w > k-1) = 1 - \exp(-w)\{1 - \lambda_b(k)\}. \tag{4.34}$$

On the other hand, for the multiplicative model:

$$\lambda(k \mid w) = 1 - \Pr(K_w > k \mid K_w > k-1) = 1 - \{1 - \lambda_b(k)\}^w, \tag{4.35}$$

where w is a realization of the random variable W. The corresponding conditional survival probabilities, are respectively,

$$\Pr(K_w > k \mid K_w > k-1) = \exp(-w)\left(\frac{\Pr(K_0 > k)}{\Pr(K_0 > k-1)}\right) = \exp(-w)\{1 - \lambda_b(k)\}, \tag{4.36}$$

and,

$$\Pr\left(K_w > k \mid K_w > k-1\right) = \left(D(w)\right)^{-1} \prod_{j=1}^{k}\left[1-\lambda_b\left(j\right)\right] = \left(1-\lambda_b\left(k\right)\right)^w, \tag{4.37}$$

where, $D\left(w\right) = \prod_{j=1}^{k-1}\left[1-\lambda_b\left(j\right)\right]$. It can easily be shown that these results also hold for the multiplicative and additive models defined via the alternative failure rate. For instance, let the baseline alternative failure rate be defined as,

$$\lambda_{a_{(b)}}(k) = -\ln[1-\lambda_b(k)]. \tag{4.38}$$

From, the fact that $\lambda_a\left(k \mid w\right) = \lambda_{a_{(b)}}\left(k\right) + w$ and using result (4.11), the additive model is given by,

$$\lambda\left(k \mid w\right) = 1 - \exp\left[-\lambda_{a_{(b)}}\left(k\right) - w\right] = 1 - \exp\left(-w\right)\left(1-\lambda_b\left(k\right)\right). \tag{4.39}$$

Using relation (4.12) and from the fact that, $\lambda_a\left(k \mid w\right) = w\lambda_{a_{(b)}}\left(k\right)$ the multiplicative model is obtained as:

$$\lambda\left(k \mid w\right) = 1 - \exp\left[-\lambda_{a_{(b)}}\left(k \mid w\right)\right] = 1 - \exp\left(-\lambda_{a_{(b)}}\left(k\right)\right)^w = 1 - \left(1-\lambda_b\left(k\right)\right)^w \tag{4.40}$$

Clearly, (4.39) and (4.40), which are, respectively similar to (4.34) and (4.35), are important in the corresponding analysis of the behavior of the failure rate in the discrete setting. The multiplicative and additive models (in these forms), provide equivalent descriptions of the behavior of the failure rate when using the alternative failure rate.

4.4. MRL in discrete setting

Let K be a discrete random lifetime with support in, $N^+ = \{1, 2, \ldots\}$. Then in relation to (2.28), the mean remaining (residual) lifetime at time k is given by,

$$m\left(k\right) = E\left[K - k \mid K > k\right] \qquad \text{for } k > 0, \tag{4.41}$$

where, the mean lifetime of an item, (e.g. $E\left[K\right]$ with the support N) is,

$$E[K] = S(0) + S(1) + S(3) + \ldots = \sum_{k=0}^{\infty} S(k). \tag{4.42}$$

Using (4.6), the equation (4.42) could also be represented via the following relation,

$$E[K] = \sum_{k=0}^{\infty} \prod_{j=0}^{k} [1 - \lambda(j)], \tag{4.43}$$

where, $\lambda(0) = 0$. The mean residual lifetime (MRL) can also be defined in terms of the failure in the following way, [157],

$$m(k) = E[K - k \mid K > k - 1] = \sum_{j=k}^{\infty} \frac{S(j)}{S(k-1)} = \int_{0}^{\infty} \frac{S(t)dt}{S(k-1)} = \frac{\int_{t_j}^{\infty} \exp\left[- \int_{0}^{t} \prod_{0 \le t_j \le t} [1 - \lambda(t_j)] \right]}{\prod_{0 \le t_j \le t_{j-1}} [1 - \lambda(t_j)]} . \tag{4.44}$$

On the other hand, using (4.42), MRL is given as:

$$m(k) = \int_{k}^{\infty} \frac{S(t)dt}{S(k)} = \sum_{j=k}^{\infty} \frac{S(j)}{S(k)} \qquad \text{for } j = \{0, 1, 2, \ldots\}. \tag{4.45}$$

Thus, the discrete MRL would have similar properties as its continuous counterpart. Therefore, the following relation may easily be employed for studying the properties of MRL via the corresponding failure rate:

$$\lambda(k) = 1 - \frac{m(k)}{1 + m(k+1)} . \tag{4.46}$$

4.5. Concluding Remarks

In this chapter, the properties of the failure rate are generalized to the discrete case. The failure rates that correspond to the continuous and discrete distributions have some important differences. These differences are highlighted and discussed, especially in case of the corresponding mixtures.

The shapes of the failure rate are investigated in detail for some discrete Weibull-type distributions. For instance, for the type II discrete Weibull distribution, the classical failure rate increases (IFR), whereas the alternative failure rate is of the UBT type. This striking difference was not, however, discussed in the literature before. Obviously this is important in practical applications. It means that the alternative failure rate, may be an appropriate choice in the modeling and analysis of various aging characteristics as compared to the usual ("classical") failure rate. It is, also, interesting to explore further this behavior for other discrete lifetime distributions.

Some important results on the failure rate of a mixture of two discrete distributions are presented and briefly discussed. We show, for instance, that the discrete mixture failure rate also bends down when compared with the expectation of the conditional failures rates.

Specifically, some selected discrete lifetime distributions are studied. We show that, under the defined settings, the corresponding failure rate of the mixture of the geometric distribution and the Type I discrete Weibull distribution is decreasing for values of $\eta < 1$ ($\eta > 1$) and reduces to a geometric distribution with constant failure rate when, $k = 1$ and $\eta = 1$. The latter non-aging property is also preserved for the mixture of geometric distribution with both the type II discrete Weibull distribution and the type III discrete Weibull distribution (e.g. when $\eta = 1$), whereas it is increasing (decreasing) for the values of $\eta > 1$ ($\eta < 1$). On the other hand, the failure rate of a mixture of geometric distribution and the modified discrete Weibull distribution is initially increasing then after some time it decreases (UBT), in particular for values of $\beta \geq 1.25$ and $\eta \geq 1.25$, otherwise the mixture failure rate is increasing (IFR).

On the other hand, the IFR property is preserved when the discrete gamma distribution is mixed with the type I discrete Weibull distribution, as well as when mixed with the modified discrete Weibull distribution for $\eta > 1$ and pairs $\eta < 1$($\beta < 1$), respectively. It is UBT for $\eta < 1$ in the prior and in the latter is BT for $\eta > 1$ and $\beta > 1$. Whereas, for Type II discrete Weibull distribution the IFR property is preserved for values of $\eta \leq 1$ and decreasing for $\eta > 1$.

These results already show that the corresponding aging changes significantly for the Type I discrete Weibull distribution (when $\eta > 1$), e.g. the failure rate is increasing (IFR), and the mixture failure rate is decreasing (DFR) and even tends to zero. A similar result is obtained for the mixture of discrete gamma distribution and the Type II discrete Weibull distributions. On the other hand, for the mixture of the geometric distribution and the discrete modified Weibull distribution the mixture failure rate is UBT. This property is also reflected for some values of the parameters when the latter distribution is mixed with the discrete gamma distribution whereas it shows the reversed pattern (BT) for other values. This means that the proportion of surviving items (objects) in the mixed population is increasing, e.g., the population lifetime is improving somehow as the "weakest subpopulations are dying out first".

Finally, some results on the *general properties* of discrete mixture failure rates are discussed and some simple models of heterogeneity are presented. In the final section of this chapter, we also define the MRL function in the discrete setting and highlight some useful relations with the corresponding failure rate.

CHAPTER 5: Shocks and heterogeneity

5.1. Brief Overview

"We understand, a shock as some instantaneous, potentially harmful event", [33]. We consider modeling the impacts of shocks via some extreme shock models. These models, are useful for studying the aging characteristics of lifetime distributions under variable environmental conditions. In this case, "produced items may result in a mixed population with a certain proportion of items with normal lifetimes and defective items (with shorter lifetimes)", see e.g. reference, [39]. Usually, burn-in (often performed in accelerated environments) is considered to weed out the defective items from a heterogeneous population. Alternatively, these may be eliminated via introducing high levels of some form of environmental stresses, e.g. shocks. Intuitively, the weaker items are 'killed' first by shocks than stronger ones. In this context, "a shock performs a kind of a burn-in operation", [193] in heterogeneous populations.

Henceforth, we consider a stochastically ordered heterogeneous population. The shapes of mixture failure rate for this population under some shock settings are analyzed. In particular, we compare the mixture failure rate prior to and after a shock: when the frailty W is either a discrete or continuous random variable.

We also consider a specific increasing mortality rate process induced by the non-homogeneous Poisson process of shocks in section 5.3. The shape of the observed (marginal) failure rate is analyzed. Specifically, the population mortality (failure) rate for some specific cases is compared with the corresponding "sample paths of the unconditional mortality rate process, which are monotonically increasing", see reference, [4]. Further, we discuss some results relating to some simple failure (mortality) rate with a single change point in section 5.4. See, e.g., also references [44], [81], [122] and [225] to name a few for some other relevant discussions and results on these topics.

5.2. Mixture models under some shock settings

5.2.1. Continuous mixture models

Consider the general mixing model (2.9) and (2.11) for a heterogeneous population. Under this setting: $g(w)$ is the pdf of a random frailty parameter W prior to a shock. Denote the corresponding mixture failure rate for a population without a shock: $\lambda_m(t)$ for $t \geq 0$ and the corresponding failure rate after the occurrence of an instantaneous shock at by $\lambda_{m(s)}(t)$, $t \geq 0$. The frailty and its distribution after the shock are respectively denoted by W_s and $g_s(w)$. Therefore, from (2.11) the mixture failure rate after a shock is given by,

$$\lambda_{\mathrm{m(s)}}(t)=\int_0^\infty \lambda\,(t\,|\,w)\;g_{\mathrm{s}}\;(w|t)dw_{\mathrm{s}} \qquad t \geq 0\,, \tag{5.1}$$

where, $g_s\,(w\,|\,t)$ is defined as in (2.12), whereas the corresponding density $g_s\,(w)$ as in (2.14) is given by:

$$g_{\mathrm{s}}\;(w)=\frac{u\;(w)\;g\,(w)}{\int_0^\infty u\;(w)\;g\,(w)dw}\,, \tag{5.2}$$

where, $u(w)$ may be interpreted as the survival probability of an item with frailty w after the shock. Intuitively, the more frail subpopulations with larger failure rates are killed first by the shocks. Henceforth, the function $u(w)$ is assumed to be decreasing. Therefore, the function $\frac{g_s(w)}{g\,(w)}$ is also decreasing and we can conclude that, $\lambda_{m(s)}\;(t) < \lambda_m(t)$. In particular, this result holds, whenever, the population frailties prior to and after a shock are ordered in the sense of (3.10): i.e. $W \geq_{lr} W_s$, where the corresponding failure rates are also ordered in the sense of (3.11). This may be explained as the effect of a shock eliminating out those subpopulations with higher failure rates, which may be due to production irregularities, among other things.

Specifically, the difference between the failure rates for $t = 0$, can be defined as

$$D = \int_0^\infty \lambda(0\,|\,w)(g\;(w) - g_s\,(w))dw, \tag{5.3}$$

where, $D = \lambda_m(0) - \lambda_{m(s)}(0)$. From, relation (3.10), the usual stochastic ordering for lifetimes prior to a shock, T_w and after the shock, $T_{w(s)}$ also hold in this case. It means that the burn-in via shocks improves the population lifetime to a certain extend. However, the usual stochastic ordering (3.4), does not lead to the forgoing ordering of lifetimes if the frailties are not ordered in the sense of (3.10). Consequently, relation (3.11) also does not hold. This means that, "the likelihood ratio ordering is sufficient to ensure ordering of mixture failure rates", [83].

5.2.2. Discrete mixture models with shocks

Let, $F_1(t)$, $f_1(t)$ and $\lambda_1(t)$ denote, respectively: the Cdf., and pdf., as well the corresponding failure rate for a lifetime of a strong subpopulation. Whereas, $F_2(t)$, $f_2(t)$ and $\lambda_2(t)$ are, respectively, the corresponding characteristics for a lifetime of a weak subpopulation.

Let, the mixing proportion be:

$$g(w) = \begin{cases} p & w = w_1 \\ 1-p & w = w_2 \end{cases}, \tag{5.4}$$

where w_1 and w_2 describe the strong and the weak subpopulations, respectively, [66]. Assume further, that the subpopulations are ordered accordingly as: $\lambda_1(t) \le \lambda_2(t), t \ge 0$. Therefore, the survival functions are also ordered accordingly: i.e. $S_1(t) \ge S_2(t), t \ge 0$, where $S_i(t) = 1 - F_i(t), i = 1,2$. Then, $w_1 < w_2$, in this case. Under this setting $W = (w_1, w_2)$ can be considered a discrete random variable ("frailty"). The corresponding mixture survival and density functions are respectively defined by (4.18). From relation (4.20), the corresponding mixture failure rate is given by:

$$\lambda_m(t) = g(w_1 \mid t)\lambda_1(t) + g(w_2 \mid t)\,\lambda_2(t), \tag{5.5}$$

where now, the time dependent probabilities are given by:

$$g(w_1 \mid t) = \frac{p\,S_1(t)}{p\,S_1(t) + (1-p)S_2(t)}; \quad g(w_2 \mid t) = \frac{p\,S_2(t)}{p\,S_1(t) + (1-p)S_2(t)}. \tag{5.6}$$

This is a general setting before a shock. We denote, the corresponding frailties after the shock (say after, $t = 0$), $W_s = (w_{s(1)}, w_{s(2)})$ with the corresponding pdf, $g_s(w)$. The corresponding mixture failure rate, $\lambda_{m(s)}(t)$ is given by a similar expression as (5.5) with

$$g_s(w) = \begin{cases} p_s \equiv \dfrac{(1-v_1)p}{(1-v_1)p + (1-v_2)p} & w = w_{s(1)} \\ 1 - p_s \equiv \dfrac{(1-v_2)(1-p)}{(1-v_1)p + (1-v_2)(1-p)} & w = w_{s(2)} \end{cases} . \tag{5.7}$$

where, v_1 and v_2 are the corresponding probabilities of failure due to a shock, [33], which affects the whole population (with the frailties, $w = w_{s(1)}$ and $w = w_{s(2)}$). As the failure rates and the corresponding survival functions are assumed to be ordered, then we can conclude that if $v_1 \le v_2$, then $\lambda_{m(s)}(t) < \lambda_m(t)$ and, $S_{m(s)}(t) \ge S_m(t)$. Furthermore, we note that, if the population frailties prior to and after a shock are ordered in the sense of (3.10): i.e. $W \ge_{lr} W_s$, then the foregoing results are justified. This means that, the failure rate before the shock is greater than the one after it as was in the continuous case.

5.3. Failure rate processes governed by shocks

Let, $(N(t), t \ge 0)$ be a stochastic point process (i.e. an external shock process), where $N(t)$ represents the number of total shocks experienced by the system at time, t. We consider λ_t (model (2.2)) as a specific increasing stochastic process i.e. the failure rate process, $\lambda_t, t \ge 0$ and analyze the shape of the observed (marginal) failure rate. In particular, we model the stochastic aging for each realization of this process utilizing the baseline failure rate process, ($\lambda_t, t \ge 0$), see reference, [4]:

$$\lambda_t = \lambda_b(t) + \omega N(t), \tag{5.8}$$

where, $\lambda_b(t)$ defines the baseline failure rate (which, of course can be assumed deterministic). Whereas, ω is a jump on the failure rate from each shock experienced by an item. Model (5.8) is just a simple generalization of model (2.15) when there is no ordering of lifetimes in some probabilistic sense (see section 3.5).

The corresponding conditional failure rate process in this case is,

$$\lambda_t \mid T > t = \lambda_b(t) + \omega(N(t) \mid T > t). \tag{5.9}$$

This process can, already, describe damage accumulation in systems exposed to external shocks. In this case, $\lambda_b(t)$ defines the failure rate in the baseline environment (i.e. with no external shocks). The impacts (damage accumulated) from the external shocks are reflected as jumps, ω in the failure rate. "The stochastic term $\omega N(t)$ in (5.8), will be crucial in the subsequent analysis", [4]. We assume $\lambda_b(t) = 0$ or to be the background constant ($\lambda_b(t) = \lambda_b$). In this case, the shocks' history, ($N(u) = n(u)$, $0 \le u \le t$) could already be specified by the corresponding joint distribution of T_1, T_2 ... $T_{N(t)}$, $N(t)$ defining the sequential arrival of shocks in a NHPP ($N(t)$):

$$f_{T_1,\ T_2, \ldots, T_{N(t)}, N(t)}(t_{1,} t_{2,} \ldots t_n, n) = \left\{ \prod_{i=0}^{n} \lambda\ (t_i) \exp\left(-\int_0^t \lambda\ (v)\ dv \right) \right\}. \tag{5.10}$$

Therefore, the probability of no effect from a shock is given by $\exp\left(-\int_0^t \lambda\ (v)\ dv \right)$, whereas, the effects of arriving shocks are accounted by $\lambda\ (t_i),\ i = 1,\ 2,\ 3, \ldots$

From (5.8), the survival function of an item given shocks history is given by

$$P(T > t \mid T_1, T_2, \ldots, T_{N(t),}\ N(t)) = \exp\left(-\int_0^t \lambda\ (v)\ dv \right) \exp\left\{ -\sum_{i=}^{N(t)} \omega\ (t - T_i) \right\}. \tag{5.11}$$

Using relations (5.10) and (5.11), we obtain the distribution of $((T > t)\ ,\ N(t))$ by integrating out $T_1,\ T_{2, \ldots,} T_n$:

$$P(T>t,\ N(t)=n)=\exp\left\{-\int_0^t \lambda_0(u)du\right\}\exp\left\{-\int_0^t \lambda_b(u)du\right\}\frac{\left\{\int_0^t \exp(-\omega(t-v)\lambda(v)dv)\right\}^n}{n!}\ , \tag{5.12}$$

where

$$\int_0^t \ldots \int_0^{t_3}\int_0^{t_2}\prod_{i=1}^{n}\lambda(t_i)\exp\{-\omega(t-t_i)\}dt_1 dt_2, \ldots, dt_n = \frac{\left\{\int_0^t \exp\{-\omega(t-v)\lambda(v)dv\}\right\}^n}{n!}.$$

Considering the above relations, the survival function, in this case could then be written in the form:

$$P(T>t)=\exp\left\{-\int_0^t \lambda_0(u)du\right\}\exp\left\{-\int_0^t \lambda_b(u)du\right\}\exp\left\{\int_0^t \exp(-\omega(t-v)\lambda(v)dv)\right\}\ . \tag{5.13}$$

Using (5.12) and (5.13), the conditional distribution of $(N(t)=n \mid T>t)$ for each $t>0$ is derived as:

$$P(N(t)=n \mid T>t)=\exp\left\{-\int_0^t \exp\{-\omega(t-v)\}\lambda(v)dv\right\} \times \frac{\left\{\int_0^t \exp\{-\omega(t-v)\lambda(v)dv\}\right\}^n}{n!}, \quad n=0,1,2\ldots\ , \tag{5.14}$$

which is, therefore, given by the Poisson distribution with mean $\int_0^t \exp\{-\omega(t-v)\lambda(v)dv\}$. We study the corresponding shapes of the marginal (observed) failure rate. In relation to model (5.10) taking expectation on both sides, yields the population failure rate model,

$$\lambda_p(t)=\lambda_b(t)+\omega E[N(t) \mid T>t], \tag{5.15}$$

which, from (5.8) and (5.14), can be written in the following form,

$$\lambda_p(t) = \lambda_b(t) + \omega \int_0^t \exp\{-\omega(t-v)\,\lambda(v)\,dv\}. \tag{5.16}$$

In what follows, we analyzed the shape of the population failure rate when compared with the shape of sample paths of the unconditional failure rate process (which, of course, are intuitively monotonically increasing) for some specific cases. For simplicity, we refer to this as examples, however, the corresponding findings can be considered as independent results.

Example 5.1

Suppose that, $\lambda_b(t) = \lambda(1+t)$. In this case, the expectation of the number of events in the corresponding nonhomogeneous Poisson process, $\{N(t), t \geq 0\}$ is given by

$$E[N(t)] = \lambda t\left(1 + \frac{t}{2}\right),$$

whereas, the conditional expectation, which defines the shape of the population failure rate in accordance with (5.15) is:

$$\begin{aligned}
\lambda_p(t) &= \omega E[N(t \mid T > t)] = \omega \int_0^t \exp\{-\omega(t-v)\}\,dv \\
&= \omega \int_0^t \lambda\,(1+t)\exp\{-\omega(t-v)\}dv \\
&= \lambda\,\omega\exp(-\omega t)\int_0^t (1+t)\exp(\omega v)\,dv
\end{aligned} \tag{5.17}$$

Firstly, consider the integrand in the last equation (5.17),

$$\int_0^t (1+t)\exp(\omega v)dv, \tag{5.18}$$

and let,

$$u = 1+t \Rightarrow du = dt \text{ and } dz = \exp(\omega v) \Rightarrow z = \omega^{-1}[\exp(wt) - 1].$$

Then, integrating (5.16) by parts we obtain:

$$
\begin{aligned}
\int_0^t u\,dz &= (1+t)\left(\omega^{-1}\exp(\omega t)-\omega^{-1}\right)-\int_0^t\left[\omega^{-1}\left(\exp(\omega t)-1\right)\right]dt \\
&= \omega^{-1}\exp(\omega t)-\omega^{-1}+t\,\omega^{-1}\exp(\omega t)-t\,\omega^{-1}-\omega^{-1}\int_0^t \exp(\omega t)dt+\omega^{-1}\int_0^t dt \\
&= \omega^{-1}\exp(\omega t)-\omega^{-1}+t\,\omega^{-1}\exp(\omega t)-t\,\omega^{-1}+t\,\omega^{-1}-\omega^{-1}\int_0^t \exp(\omega t)dt \\
&= \omega^{-1}\exp(\omega t)-\omega^{-1}+t\,\omega^{-1}\exp(\omega t)-\omega^{-2}\exp(\omega t)+\omega^{-2}
\end{aligned}
\quad . \qquad (5.19)
$$

From (5.16), it can be seen that (5.19) reduces to,

$$
\begin{aligned}
&= \lambda\,\omega\exp(-\omega t)\left[\omega^{-1}\exp(\omega t)-\omega^{-1}+t\,\omega^{-1}\exp(\omega t)-\omega^{-2}\exp(\omega t)+\omega^{-2}\right] \\
&= \lambda-\lambda\exp(-\omega t)-\lambda\,\omega^{-1}+\lambda\,\omega^{-1}\exp(-\omega t)+\lambda t
\end{aligned}
\quad .
$$

Therefore,

$$
\lambda_p(t)=\left(\lambda-\lambda\omega^{-1}\right)\left[1-\exp(-\omega t)\right]+\lambda t \,.
$$

The corresponding shapes of $E[N(t)]$ and $\lambda_p(t)$ are shown on Fig. 12 below,

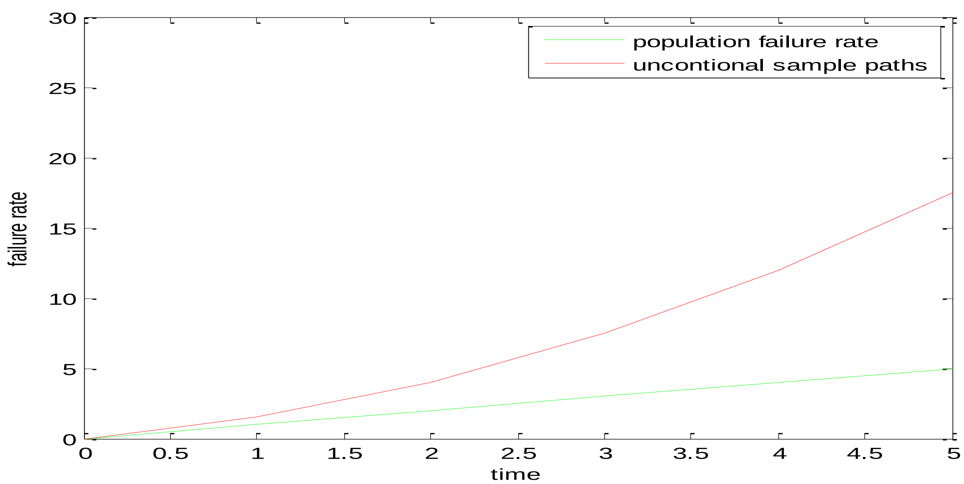

Fig. 28: Plot of $E[N(t)]$ and, $\lambda_p(t)$ for $\lambda=\omega=1$

From Fig. 12 above (red curve); the sample paths of the process $\{N(t), t \geq 0\}$ with $E[N(t)] = \lambda t\left(1 + \frac{t}{2}\right)$ are increasing (as intuitively would have been expected). Assuming $\lambda_b(t) = 0$ in the model (5.15), the shape of the population failure rate (green curve) is decreasing as compared to the sample paths of the process, $\{N(t), t \geq 0\}$. This curve monotonically increases approaching the plateau as, $t \to \infty$. Therefore, the failure rate can be described by the composition of the strongest surviving population, as the "weakest populations" are eliminated first by the external shocks.

Example 5.2

Now if we let, $\lambda_b(t) = \lambda\{1 - \exp(-t)\}$, then the expectation of the number of events in the corresponding nonhomogeneous Poisson process, $\{N(t), t \geq 0\}$ is given by:

$$E[N(t)] = \lambda t + \lambda\{\exp(-t) - 1\},$$

while the conditional expectation (5.15) is:

$$\begin{aligned}
\lambda_p(t) = \omega E[N(t \mid T > t)] &= \omega \int_0^t \exp\{-\omega(t - v)\} dv \\
&= \lambda\omega(1 - \exp(-t))\exp(-\omega t)\int_0^t \exp(\omega v) dv \\
&= \lambda\omega[\exp(-\omega t) - \exp\{-t(\omega + 1)\}][\omega^{-1}(\exp(\omega t) - 1)]. \\
&= \lambda - \lambda\exp(-\omega t) - \lambda\exp(-t) + \lambda\exp\{-t(\omega + 1)\} \\
&= \lambda[(1 - \exp(-\omega t))] - \lambda\exp(-t)[(\exp(\omega + 1) - 1)]
\end{aligned} \tag{5.20}$$

The plots of $E[N(t)]$ and $\lambda_p(t)$ are shown on Fig. 13 below,

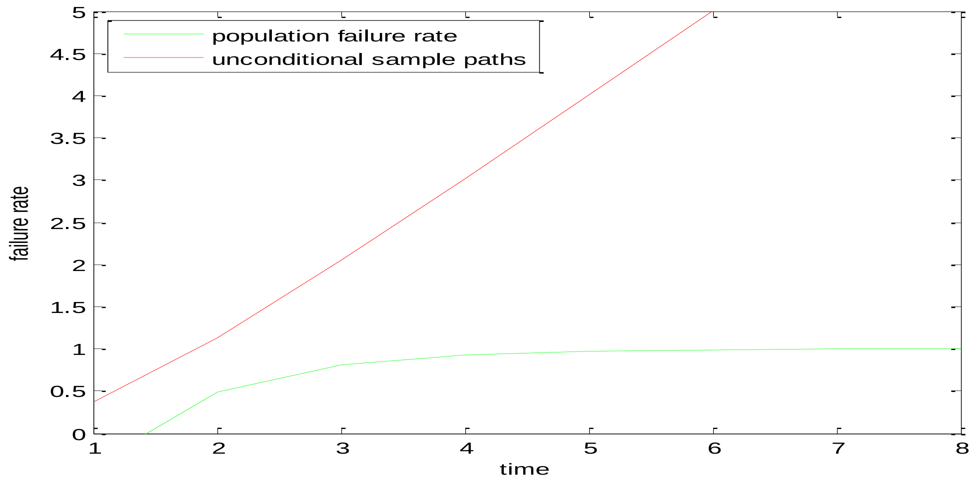

Fig. 29: Plot of $E[N(t)]$ and, $\lambda_p(t)$ for $\lambda = \omega = 1$

Similar to example 5.1, the sample paths of the process $\{N(t), t \geq 0\}$ are increasing in this case (see the red curve in Fig. 29). Whereas, the corresponding population failure rate is decreasing when compared with the unconditional sample paths of the process, $\{N(t), t \geq 0\}$, under the assumption $\lambda_b(t) = 0$. In particular, the population failure rate tends to a plateau as time increases.

Example 5.3

Assume, $\lambda_b(t) = \lambda t^2$ then it follows that $E[N(t)] = (1/3)\lambda t^3$ in this case and the conditional expectation, $\lambda_p(t)$, which defines the population failure rate and in accordance with (5.15) is obtained as:

$$\begin{aligned} \lambda_p(t) &= \omega \int_0^t \lambda t^2 \exp\left(-\omega(t+v)\right) dv \\ &= \omega \lambda \exp(-\omega t) \int_0^t t^2 \exp(\omega v) dv \end{aligned} . \tag{5.21}$$

Consider first, $\int_0^t t^2 \exp(\omega v)\, dv$ and let:

$$u=t^2 \Rightarrow du=2t\,dt \text{ and, } dz=\exp\left(\omega v\right)\Rightarrow z=\frac{1}{\omega}\left[\exp\left(\omega t\right)-1\right].$$

Hence,

$$\begin{aligned}\int_0^t u dz &= \frac{t^2}{\omega}\left[\exp\left(\omega t\right)-1\right]+\frac{2}{\omega}\int_0^t t\,dt-\frac{2}{\omega}\int_0^t t\exp\left(\omega t\right)dt \\ &= \frac{t^2}{\omega}\exp\left(\omega t\right)-\frac{2}{\omega}\int_0^t t\exp\left(\omega t\right)dt\end{aligned}. \tag{5.22}$$

Now consider, $\int_0^t t\,\exp\left(\omega t\right)dt$ and let:

$$u=t \Rightarrow du=dt \text{ and } dv=\exp\left(\omega t\right)\Rightarrow v=\frac{1}{\omega}\left[\exp\left(\omega t\right)-1\right].$$

Hence,

$$\int_0^t u dv=\frac{t}{\omega}\exp\left(\omega\, t\right)-\frac{1}{\omega^2}\exp\left(\omega\, t\right)-\frac{1}{\omega^2}. \tag{5.23}$$

Now from (5.22) and (5.23) we have,

$$\int_0^t u dz=\frac{t^2}{\omega^2}\exp\left(\omega\, t\right)+\frac{2}{\omega^2}\exp\left(\omega\, t\right)+\frac{2}{\omega^3}\exp\left(\omega\, t\right)+\frac{2}{\omega^3}. \tag{5.24}$$

Considering now (5.21), we finally have,

$$\begin{aligned}\lambda_p(t) &= \lambda t^2-\frac{2\lambda t}{\omega}+\frac{2\lambda}{\omega^2}+\frac{2\lambda}{\omega^2}\exp\left(-\omega\, t\right) \\ &= \lambda\, t\left(t+2\omega^{-1}\right)+2\lambda\omega^{-2}\left(1+\exp\left(-\omega\, t\right)\right)\end{aligned}. \tag{5.25}$$

The corresponding plots of unconditional sample paths and (5.25), are shown on Fig. 14 below.

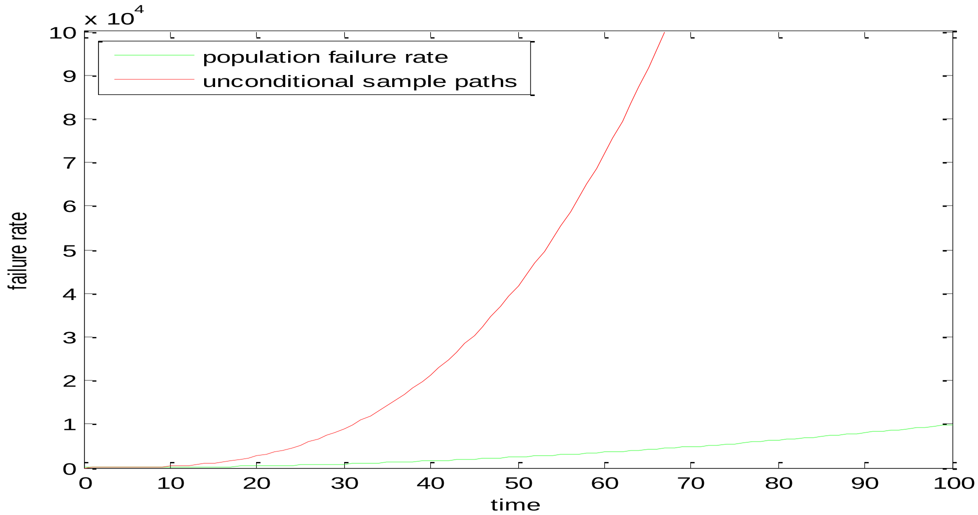

Fig.30: Plot of $E[N(t)]$ and, $\lambda_p(t)$ for $\lambda = 1$ and $\omega = 2$

As can be noted from Fig. 14, the population failure rate is, again, decreasing as compared to the unconditional sample paths of the process, $\{N(t), t \geq 0\}$) . It also decelerate and reaches a "plateau" as time increases.

In general, it is plausible that the population failure rate exhibits the same monotocity properties for, $\lambda_b(t) = \lambda t^k$, where $k = 3$, 4 ... , as $t \to \infty$ and we conclude that shocks do perform some kind of burn-in in heterogeneous populations. This leads to the improvement of the surviving populations somehow.

5.4. Models for the failure (mortality) rate with change point

5.4.1. Some general aspects of the change point model

Consider an item (object) starting to operate (born) at time ($t = 0$) in some specific environment. Suppose that the object's lifetime, under this baseline environment is described by, $F_1(t)$ with the failure rate, $\lambda_1(t)$. If the environment changes, say after some time at, e.g., $t = w$, the failure rate of the object can be described by employing the simplest failure rate models with a change point [116] when w is fixed. In this case, the corresponding failure rate, due to [130] is given by

$$\lambda(t \mid w) = \lambda_1(t) I_1 + \lambda_2(t) I_2 \; ; \; t \geq 0, \tag{5.26}$$

where $\lambda_1(t)$ is the failure rate before the change point and $\lambda_2(t)$ is the failure rate after this change point and $I_1 \equiv I(t < w)$; $I_2 \equiv I(t \geq w)$ represent the corresponding indicators, i.e.,

$$I_1 = \begin{cases} 1, & t < w \\ 0, & t \geq w \end{cases}, \qquad I_2 = \begin{cases} 0, & t < w \\ 1, & t \geq w \end{cases}.$$

Thus, the failure rate of the object, in this case, is defined by model (5.26), i.e. $\lambda(t \mid w)$ with the Cdf, $F(t \mid w)$. The survival function, $S(t \mid w) \equiv 1 - F(t \mid w)$ is given by

$$S(t, w) = S_1(t) I(t < w) + S_1(w) \frac{S_2(t)}{S_2(w)} I(t \geq w), \tag{5.27}$$

where, $S_2(t)/S_2(w) = \exp\left\{ -\int_z^t \lambda_2(u) du \right\}$ is the conditional survival probability in $[w, t)$ for an object with the Cdf $F_2(t)$. Alternatively, these models can be generalized to a case when the failure rate after the change point depends on w; particularly for model (5.26) we have:

$$\lambda(t \mid w) = \lambda_1(t) I_1 + \lambda_2(t \mid w) I_2 \; ; t \geq 0, \tag{5.28}$$

Although, we may know (from the data) $\lambda_1(t)$ and $\lambda_2(t)$ (and/or the functional form of $\lambda_2(t \mid w)$), the random variable W is usually unknown. Therefore, the random change point, somehow, introduces some kind of heterogeneity in the population. Henceforth, consider model (5.26) and let W be the random change point variable with a pdf, $g(w)$ having the support in $[0, \infty)$, then [115] the corresponding hazard rate process (which already defines the random failure rate model (2.1)) in this case is defined as,

$$\lambda(t \mid w) = \lambda_1(t) I_1 + \lambda_2(t \mid w) I_2, \tag{5.29}$$

where, now $I_1 \equiv I(t < w)$; $I_2 \equiv I(t \geq w)$. Thus, the random change point model (5.29) can already be considered as a specific case of mixing and W is the mixing parameter. Therefore, some general results on mixture failure rates can also be applied in this case. Under this setting, the corresponding mixture Cdf. is given by,

$$F_m(t) = E[F(t \mid W)] = \int_0^{\infty} F(t \mid w)\, g(w) dw. \tag{5.30}$$

On the other hand, for the general, the mixture failure is defined by (2.11),

$$\lambda_m(t) = \int_0^{\infty} \lambda(t \mid w) g(w \mid t) dw, \tag{5.31}$$

where, the conditional pdf $g(w \mid t)$ is accordingly defined by (2.12), i.e.

$$g(w \mid t) = g(w) \frac{S(t \mid w)}{\int_0^{\infty} S(t \mid w) g(w) dw}. \tag{5.32}$$

Under this setting, model (5.32) can be written as:

$$g(w \mid t) = \frac{g(w)}{\int_0^{\infty} S(t \mid w) g(w) dw} \begin{cases} S_1(t) & t < w \\ S_1(t) & t \geq w \end{cases}. \tag{5.33}$$

Whereas, now the corresponding mixture failure rate is given by,

$$\lambda_m(t) = \frac{\lambda_1(t) S_1(t) \int_0^{\infty} g(w) dw + \lambda_2(t) S_2(t) \int_0^{t} \frac{S_1(w)}{S_2(w)} g(w) dw}{S_1(t) \int_t^{\infty} g(w) dw + S_2(t) \int_0^{t} \frac{S_1(w)}{S_2(w)} g(w) dw}. \tag{5.34}$$

Alternatively, considering model (5.28) we have,

$$\lambda_m(t) = \frac{\lambda_1(t) S_1(t) \int_0^{\infty} g(w) dw + \int_0^{t} \lambda_2(t \mid w) S_2(t \mid w) \frac{S_1(w)}{S_2(w \mid w)} g(w) dw}{S_1(t) \int_t^{\infty} g(w) dw + \int_0^{t} S_2(t \mid w) \frac{S_1(w)}{S_2(w)} g(w) dw}. \tag{5.35}$$

Our interest is on the analysis of the shapes of the failure (mortality) rate. We consider model (5.34) which, although rather cumbersome could already be utilized in some specific cases. For instance, reference [66] consider the specific case when $\lambda_1(t) = \lambda_1; \lambda_2(t) = \lambda_2$ with a joint pdf

$g(w)$ following an exponential distribution with parameter, λ_s: i.e. $g(w) = \lambda_s \exp\{-\lambda_s w\}$. Then from (5.34), it is clear that,

$$\lambda_m(t) = \frac{\lambda_1 + \lambda_2 C\ (1-\exp\{-Bt)}{1+C(1-\exp\{-Bt)}, \tag{5.36}$$

where, $B = \lambda_2 - \lambda_1 - \lambda_s$ and $C = \lambda_s / (\lambda_2 - \lambda_1 - \lambda_s)$. If, $\lambda_2 > (\lambda_1 + \lambda_s)$, then from (5.36), we have,

$$\lim_{t\to\infty} \lambda_m(t) = \lambda_1 + \lambda_s. \tag{5.37}$$

In this case, $\lambda_1 < \lambda(\infty) < \lambda_2$. Consequently for sufficiently small, λ_s, the pdf, $g(w)$ gives more weight to smaller values of t, where $\lambda_m(t) = \lambda_1$ with a larger probability. It can be shown [66] that $\lambda_m{}'(t) > 0,\ \forall t \geq 0$ by differentiating the right hand side of (5.36). This result already indicates that "$\lambda_m(t)$ monotonically increases from the level λ_1 to the level, $\lambda_1 + \lambda_s$, as $t \to \infty$", see, e.g., reference, [116]. If, $\lambda_1 < \lambda_2 < (\lambda_1 + \lambda_s)$, then,

$$\lim_{t\to\infty} \lambda_m(t) = \lambda_2. \tag{5.38}$$

This result is reasonable because the probability distribution in this case gives more weight to larger values of, t, where $\lambda_m(t) = \lambda_2$ with a larger probability. As the limit (5.38) also holds for the case when, $\lambda_2 < \lambda_1$, these results show that, the change point eventually bends down the failure (mortality) curve and reflect the effect of "the weakest populations are dying out first".

In particular, the impact of changing environment can be modeled via the relation (2.18). In relation to this model, the change point model (5.28) could also be written in the following form:

$$\lambda(t \mid w) = \Psi(t \mid w)\ \lambda_1(t), \tag{5.39}$$

where,

$$\Psi(t \mid w) = \begin{cases} 1 & t < w \\ \dfrac{\lambda_2(t)}{\lambda_1(t)} & t \geq w \end{cases}.$$

If we let $\Psi(t \mid w) = w$, then it can be seen that the value $w = 1$ then it can be seen that the value $w = 1$ corresponds with the baseline failure rate, i.e. model (5.39) reduces to,

$$\lambda(t \mid w) = \lambda_1(t). \tag{5.40}$$

Then obviously, if our item is aging, this result implies that the failure rate ($\lambda_1(t)$) in the baseline environment is increasing, i.e. for $t < w$. As the baseline failure rate is increasing in this case, for $t \geq w$, then $\Psi(t \mid w) = \lambda_2(t)/\lambda_1(t)$ is decreasing. This result already shows that the failure rate tends to be decreasing eventually.

5.5. Variability characteristics in heterogeneous populations

We investigate the behavior of the variance of the conditional random variable, $W \mid t \equiv (W \mid T > t)$. The coefficient of variation is another useful measure, which we consider. Henceforth, the ordered subpopulations are not necessarily the same.

5.1.1. Mixing distributions with different variances

Mixing distributions offer convenient tools for the analysis of heterogeneous populations. For instance, consider two random variables W_1 and W_2 with realizations w_1 and w_2 respectively. Denote, the corresponding mixing distribution functions, $G_1(w)$ and $G_2(w)$. When the expectations of these random variables are equal, the respective mixture failure rates are equal. In particular, for the multiplicative model (2.18): the initial failure rates are equal whenever the mixing distributions have equal means. However, there are many situations in practice when the means may be equal but the corresponding variances are not equal. In particular, it is proved for the specific case of the multiplicative model, e.g. when the mixing parameter follows a gamma distribution: when

$$Var(W_1) > Var(W_2),$$

the corresponding mixture failures rates are ordered as:

$$\lambda_{m1}(t) < \lambda_{m2}(t), \tag{5.41}$$

See, e.g., reference [66]. Our, ordered heterogeneous population is improving: “the weakest populations are dying out first”. In fact, it was further shown by the forgoing authors, that the ordering of the variances is the sufficient condition for ordering the corresponding mixture failure rates.

The conditional variance of a subpopulation of items that survived the operational interval, $[0,t)$, i.e. for the random variable, $W \mid t \equiv (W \mid T > t)$ is of interest. This, already, provides an additional information on the random variable, W in relation to surviving subpopulation items.

5.1.2. Variance of $W \mid t$: Discrete mixtures setting

Consider the mixture setting of section 5.2.2. It turns out, that the failure rate of surviving items, may also be described by the corresponding random failure rate, say $\lambda^*(t)$:see e.g. references, [29] and [33],

$$\lambda^*(t) = \begin{cases} \lambda_1(t) \\ \lambda_2(t) \end{cases}, \tag{5.42}$$

where, $\lambda_1(t)$ and $\lambda_2(t)$ are the realizations of $\lambda^*(t)$ with probabilities $p(t)$ and $q(t)$, respectively. In fact, when relation (3.11) holds and the corresponding survival functions are also ordered as, $S_1(t) \geq S_2(t), t \geq 0$, where $S_i(t) = 1 - F_i(t)$, $i = 1,2$ ($w_1 < w_2$), we can, then, conclude from relations (4.21) and relation (5.5): the proportion of strong items, $p(t)$ is increasing ($q(t)$ is decreasing) as t increases. Whereas, also, from the relation (4.25) and (5.42), it can be seen that $\lambda_m(t)$ is decreasing because, $\lambda_j(t) \leq 0,\ j = 1,2$.

We consider, a specific case of a more general model (2.2): that is, $\lambda^*(t)$ defined via the respective conditional probabilities, $p(t)$ and $q(t)$: i.e. model (5.42). The differences in the corresponding failure rates, are of interest and can be represented by an equivalent random variable[3]: see e.g. reference, [66],

$$\lambda_D(t) = \begin{cases} \lambda_2(t) - \lambda_1(t) & with\ probability \quad p(t) \\ 0 & with\ probability \quad q(t) \end{cases}, \tag{5.43}$$

From the fact that ordering (3.11) holds in this case, it is plausible that as "the weakest subpopulations are dying first", and $\lambda_D(t)$ is decreasing.

When, $\lambda^*(t)$ is defined, analogously to, $\lambda(t \mid w) \equiv \lambda((t \mid w) \mid T > t)$, via (5.43), then, for model (2.1), the mixture failure rate, $\lambda_m(t)$ in (4.20) can, also, be considered as "the expectation of the random variable, $\lambda_D(t)$:

$$\lambda_m(t) = E[\lambda_D(t)], \tag{5.44}$$

where, the corresponding conditioning is via, $p(t)$ and $q(t)$", see e.g., reference [29]. As $w_1 < w_2$ and ordering (3.11) also hold in this case, in accordance with relation (5.42), $p(t)$ is increasing, whereas $q(t)$ is decreasing. Therefore, $\lambda_m(t)$ is decreasing as well and the heterogeneous population is improving as the weakest subpopulations are eliminated first from this population. The corresponding variance of $\lambda_D(t)$ is given by: see e.g. reference, [33],

$$Var(\lambda_D(t)) = (\lambda_2(t) - \lambda_1(t))^2\ p(t)\,q(t), \tag{5.45}$$

As a specific case describing, e.g. the shape of the variance of the random variable, $\lambda_D(t)$, the foregoing authors, consider the simplest mixture of two exponential distributions, where the failure rates are, $\lambda_2(t) = \lambda_2 > \lambda_1(t) = \lambda_1$ and show that the corresponding mixture failure rate, $\lambda_m(t)$ and $Var(\lambda_D(t))$ strictly decreases as time increases. For this particular case, model (5.45) reduces to the simplest model,

$$Var(\lambda_D(t)) = (\lambda_2 - \lambda_1)^2\ p(t)q(t). \tag{5.46}$$

From (4.25),

[3]Where the subscript D signify the difference between the failure rates.

$$\lambda_m^{'}(t) = -(\lambda_2 - \lambda_1)^2\ p(t)q(t) = -Var(\lambda_D(t)). \tag{5.47}$$

In accordance with relations (5.43) and (5.47) it can be shown, [66], that $p(t)q(t)$ is strictly deceasing in t from, $p(0)q(0)$, when the proportion of strong subpopulation is larger than 50% of the remaining population. Consequently, the variance is decreasing as well. Whereas, when this proportion is less than or equal 50% of the remaining population, (where (5.43) increasing, then $p(t)q(t)$ firstly increases and then decreases) the corresponding $Var(\lambda^*(t))$ initially strictly increases. Hence, if (5.46) increases (decreases), then $Var(\lambda^*(t))$ would initially be strictly increasing to a certain maximum point before it eventually strictly decreases.

5.1.3. Variance of the conditional random variable, $W \mid t$: Continuous mixtures

Consider a subpopulation indexed via the unobservable frailty parameter, W with pdf, $g(w)$, $w \in [0, \infty)$. The mixture failure rate is given by (2.11), with the conditional pdf (2.12). The distribution functions of the unconditional random variables, W and the conditional one, $W \mid t$ and $W \mid 0 = W$ are given by the respective relations in (2.13). For, the specific case of the multiplicative (frailty) model (2.19): $\lambda(t \mid w) = w\ \lambda_b(t)$ and, w is the realization of the random variable, W , the corresponding random failure rate is,

$$\lambda^*(t) = W\ \lambda_b(t), \tag{5.48}$$

where, $W \equiv W \mid t$. Considering definition (2.11) and (5.44), we can see that (5.48) in a way reduces to (2.20): i.e.

$$\lambda_m(t) = E[\lambda^*(t)] = \lambda_b(t)\int_0^\infty w\ g(w \mid t)dw = \lambda_b(t)\ E\ [W \mid t]. \tag{5.49}$$

As, $E[W \mid 0] \equiv E[W]$, the conditional expectation, $E[W \mid t]$ is decreasing as $E^{'}[W \mid t]$ is a decreasing function of t. This result further implies that the relation (5.49) is decreasing as well and

$$Var(\lambda^*(t)) = (\lambda_b(t))^2\, Var(W) = (\lambda_b(t))^2\, Var(W \mid t). \tag{5.50}$$

Thus, from (2.18) and considering (5.49) as well (5.50), we have,

$$\lambda_m^{'}(t) = \lambda_b^{'} E[W \mid t] - (\lambda_b(t))^2 Var(W \mid t). \quad (5.51)$$

The mixture failure rate, $\lambda_m(t)$, is evidently decreasing as a function of time, and the corresponding variance, $Var(\lambda^*(t))$ is decreasing as a function of time as well. For a mixture of exponential distributions, the relation (5.51) reduces to: $\lambda_m^{'}(t) = -(\lambda_b(t))^2 Var(W \mid t)$. As another example, consider, the distribution of W to be a gamma distribution: i.e.

$$g(w) = \frac{1}{\Gamma(\alpha)} \beta^{-2} w^{\alpha-1} \exp\left\{-\frac{w}{\beta}\right\} \ , \ w \ , \alpha \ , \beta > 0 \, ,$$

where, the baseline survival function is $S(t) = \exp(-V)$, $t \geq 0$, and $V = \int_0^t \lambda_b(u) du$. In this case, it can easily be shown that:

$$E[z \mid t] = \alpha \beta / (1 + V). \quad (5.52)$$

On the other hand, the corresponding variance is given by,

$$Var\, \lambda^*(t) = \alpha \beta^2 / (1 + V)^2 . \quad (5.53)$$

The expectation (5.52) is decreasing and the corresponding variance (5.53), is decreasing as well. The distribution of “the changing in time composition of our heterogeneous population is improving with time”, [29].

5.1.4. Coefficient of Variation of the random variable, $W \mid t$

The coefficient of variation (CV) being the ratio of the standard deviation and the mean, (i.e. $CV = \sigma^2 / \mu$) reflects the relative variability of the reliability measures from what otherwise may be their expected values. In the context of our setting, it is given by,

$$CV(\lambda^*(t)) = \sqrt{Var\,(\lambda^*(t)) / \lambda_m(t)} . \quad (5.54)$$

Thus, in relation to the specific case of model (5.42), and from (4.20), the coefficient of variation is given by (cf. reference, [29])

$$CV\left(\lambda^*(t)\right)=\frac{\left(\lambda_2(t)-\lambda_1(t)\right)\sqrt{p(t)q(t)}}{p(t)\,\lambda_1(t)+q(t)\,\lambda_2(t)}. \tag{5.55}$$

Hence, for the specific case of the mixture of two exponential distributions, we have,

$$CV\left(\lambda^*(t)\right)=\frac{\left(\lambda_2-\lambda_1\right)\sqrt{p(t)q(t)}}{p(t)\,\lambda_1+q(t)\lambda_2}. \tag{5.56}$$

From the corresponding discussion just subsequent to relation (5.44) and (5.49), it is clear that derivative (i.e. $p'(t)$) is positive, which means the proportion of surviving subpopulations. Consequently, from this result the derivative of (5.56) is also positive, which further implies: $(\lambda_2/\lambda_1)>(p(t)/q(t))$. In particular, it is shown by the forgoing authors that (5.56) may be IFR for this case, whereas it is of UBT type, in particular when $(\lambda_2/\lambda_1)=(p(t)/q(t))$ and $(\lambda_2/\lambda_1)>(p(0)/q(0))$. However, it monotonically decreases when the inequality in the latter expression is reversed. It was also shown by the forgoing authors, for a specific case of continuous mixtures of exponentials with the conditional failure rate, $\lambda(t\,|\,w)$ and the distribution, $g(w)=\theta\exp(-\theta w)$: the covariance of the random failure rate, $\lambda^*(t)$ is constant, whereas the corresponding variance is decreasing.

As another example, consider the distribution of W follows an inverse Gaussian distribution:

$$g(w)=(1/2\ a\,p\,w)^{1/2}\exp\left\{-(b\,w-1)^2/2\,a\,w\right\}\ ,\ \ w\ ,\alpha\ ,\ \beta>0\,.$$

It can easily be shown that,

$$E[w\,|\,t]=1/\left[a\left(b^2+2\,a\,V\right)^{1/2}\right],$$

and the corresponding variance and the coefficient of variation are, respectively,

$$Var\left(\lambda^*(t)\right)==a\left(b^2+2\,a\,V\right)^{-3/2}\text{ and, }CV\left(\lambda^*(t)\right)=a^{1/2}\left(b^2+2\,a\,V\right)^{1/4}.$$

The variance and the coefficient of variation are, respectively, decreasing in this case. The coefficient of variation of the random variable, $W\,|\,t$ for the Gamma distribution considered in section 5.5.3 (e.g. $CV\left(\lambda^*(t)\right)=\alpha^{-1/2}$), is also decreasing.

5.6. Concluding Remarks

We consider stochastically ordered heterogeneous populations. The shapes of mixture failure rate for these population under some shock settings are analyzed for two specific cases. In particular, we compare the mixture failure rate before and after a shock. When the frailty W is a continuous random variable, we show that the failure rate after a shock is smaller than the one without a shock.

Therefore, shocks under some assumptions can improve the probabilities of survival for a heterogeneous population. These results are, also, extended to the case, when frailty is a discrete random variable. Shocks as an alternative kind of burn-in is theoretically justified in these cases. Whereas, the conventional burn-in is premised on the failure rate being initially decreasing or bathtub, in the considered case, burn-in can be performed even for increasing failure rates.

In section 5.3., a specific increasing mortality (hazard) rate process induced by the non-homogeneous Poisson process of shocks is considered. The shape of the observed (marginal) failure rate is analyzed in this case. In particular, we show for some specific cases: the population failure/mortality rate decreases with age and, even tend to reach a plateau. This result is obtained, when compared with sample paths of the unconditional mortality rate process, which are monotonically increasing.

Our model can be used to model and analyze the damage accumulated by organisms experiencing external shocks. In this case, the cumulated damage is reflected by jumps on the failure rate. An overview of results on mortality rate processes with a single change point are also presented and discussed in final section 5.4.

There are situations where the mixing distributions have equal means but the variances are different. Henceforth, the ordered subpopulations are not necessarily the same. We show, for some specific cases, that the corresponding mixing can lead to the corresponding ordering of mixture failure rate in, $[t\ ,\infty)$.

For a subpopulation of items that survived the operational interval, $[0,t)$, the variance of the conditional random variable, $W \mid t$ is analyzed. Two specific cases are considered: the case, when the random variable frailty, W is discrete and/or continuous. When, the failure rates are ordered, $E[W \mid t]$ is decreasing and $\lambda_m(t)$ is also decreasing as functions of time. We use, simple but meaningful examples to illustrate our results. Specifically, it is shown that the corresponding variance is not monotone, e.g. it may decrease or be UBT shaped.

Another, useful measure, which was considered in section 5.5, is the coefficient of variation of the random variable, $W \mid t$. Whereas the variance may be decreasing, we show for some specific cases, that the coefficient of variation, may be decreasing or exhibit other shapes: e.g., constant or UBT shape.

The focus in literature has been mostly on the study of expectations, however, the obtained in this section results show that the variability characteristics in heterogeneous populations may also change dynamically. Therefore, along with the expectations~~,~~ this should also be considered in practical applications.

Chapter 6: Final Concluding Remarks

We firstly discussed some general reliability notions, which are relevant to our study and presented a brief introductory literature survey in chapter one, whereas the more detailed analysis of specific references are also conducted throughout the text at appropriate places while discussing relevant issues. In particular, we devoted a considerable attention to describing various (aging) classes of distributions, as these are important and useful for discussing the statistical modeling for mixtures of distributions.

Heterogeneity in populations (of items) is often induced by changing environment conditions and/or other random effects. We focused, on describing the corresponding aging characteristics for heterogeneous populations. The notion of a random failure rate, which is particularly important for the corresponding analysis of the mixture failure rates is considered and briefly discussed.

We focused on some initial frailty (mixture) models and discussed some results that describe the shape of the mixture failure rate. Specifically, a meaningful case of a population which consists of two subpopulations, which we believe was not sufficiently studied in the literature is considered. The corresponding, properties describing the shape of the failure rate under these mixtures are analyzed. It is shown that the mixture failure rate can decrease or be UBT for some specific cases. Another specific case, which also explicitly illustrate some further applications of these models to a case, where the mixing parameter, is the initial (usual unknown) random age is considered. It is shown, in this case, that this type of "mixing" can change the aging properties of an object, e.g., for certain values of parameters, the mixture failure rate may either preserve the IFR property or have a bathtub (BT) shaped: initially decreasing to some minimum point and eventually increasing as $t \to \infty$.

The mean residual life also plays a central role in characterizing lifetime distributions. We presented, some useful general results on the properties of MRL and obtained the corresponding 'shape properties' for the corresponding mixtures. The MRL 'shape properties' are analyzed for some specific cases and some relations with the failure rate are also highlighted. Especially, we show that for the inverse Weibull baseline distribution with the UBT shaped failure rate results in the corresponding decreasing MRL for certain values of parameters, whereas it is UBT for other values. These results already show the flexibility of the inverse Weibull model in describing different aging phenomena.

Some general and essential aspects of stochastic orderings, which are relevant to our study, are presented and discussed. We consider ordering of mixing distributions in the sense of the likelihood ratio. Specifically, some relevant and useful results for the case of two frailties are discussed. It turns out that the mixture failure rates are ordered as functions of time in $[0,\infty)$, when the mixing distributions are ordered in the sense of the likelihood ratio.

Some findings on the bending properties of the mixture failure rates are presented. It follows from conditioning on survival in the past interval of time that the mixture failure rate is majorized by the unconditional one. Hence, the mixture failure rate bends down in a weak sense or a strong sense as time increases.

These results are extended to other main reliability indices. Specifically, we show, that the MRL function under the operation of mixing bends up either in a strong sense or weak sense as time increases. The reversed failure rate is also bending down either in a weak sense or a strong sense, whereas the corresponding mean inactivity (waiting) time exhibits the reverse behavior. These results are shown to hold for the multiplicative model. However, the meaning impacts of the observed increasing mixture reversed failure rates in some literature, for lifetime modeling requires some further independent study.

Some relevant results on failure (mortality) rate when the corresponding parameters are randomized are presented. In this case, it turns out that randomization of parameters may lead to the strictly decreasing population failure rate. Under this setting, we also discuss some relevant results with respect to the vitality modeling.

Some useful results on relative aging of the mentioned main reliability indices are discussed. Specifically, ordering of lifetimes in terms of monotonicity properties of the ratio of the mean waiting (inactivity) times is proposed and some conditions for a random variable X to be aging faster than the lifetime random variable Y in the relative mean inactivity order are established.

It is also proved that if X is increasing in the sense of the mean inactivity time (IMIT), then Y is also IMIT when the corresponding random variables are ordered accordingly.

In addition to other existing measures, e.g., of the “rate of aging” considered in some literature, it is also interesting in future to consider other measures, which account for the impacts of heterogeneity on population’s lifetime distributions.

We have generalized the properties of the failure rate to the discrete case. In particular, some major and important differences between the failure rates that correspond to the continuous and discrete distributions are highlighted and discussed. The shapes of the failure rate are investigated in detail for some discrete Weibull-type distributions. For instance, for the type II discrete Weibull distribution, the classical failure rate increases (IFR), whereas the alternative failure rate is of the UBT type. This striking difference, was not, however, discussed in the literature before. Obviously this is important in practical applications. It means that the alternative failure rate, may be an appropriate choice in the modeling and analysis of various aging characteristics as compared to the usual (classical) failure rate. It is, also, interesting to explore further this behavior for other discrete lifetime distributions.

We present and discuss some important results on the failure rate of a mixture of two distributions. We show, for instance that the mixture failure rate bends down when compared with the expectation of the conditional failures rates. Specifically, some selected discrete lifetime distributions (e.g. with increasing failure rate (IFR), BT or UBT shaped failure rates) are studied. For instance, we show that, the failure rate of a mixture of geometric distribution with the type I discrete Weibull distribution, tends to decrease and even tends to zero for certain values of parameters, whereas it is increasing (decreasing) for the certain values of parameters, for the type II discrete Weibull distribution and the type III discrete Weibull distribution, respectively.

Under the defined settings, the IFR property is also preserved for mixtures of the considered here discrete Weibull distributions and the Lindley distribution with increasing failure rate.
Further, the failure rate of the mixture of the type I discrete Weibull distribution and discrete gamma distribution is of UBT shape type for certain values of the parameters. This property is also reflected for some values of the parameters when the latter distribution is mixed with the discrete gamma distribution, otherwise it shows the reversed pattern (BT) for other values.

Finally, some results on the *general properties* of discrete mixture failure rates are discussed and some simple models of heterogeneity are presented. In the final section of this chapter, we also define the MRL function in the discrete setting and highlight some useful relations with the corresponding failure rate.

Stochastically ordered heterogeneous populations are considered. The shapes of mixture failure rate for these populations under some shock settings are analyzed for two specific cases. In particular, we compared the mixture failure rate before and after a shock. When the frailty W is a continuous random variable, we show that the failure rate after the shock is smaller than the one without a shock.

Therefore, shocks under some assumptions can improve the probabilities of survival for a heterogeneous population. These results are also extended to the case when frailty is a discrete random variable.

Shocks, as an alternative kind of burn-in, is theoretically justified in these cases. On the other hand, while the conventional burn-in is premised on the failure rate being initially decreasing or bathtub, in this case, burn-in can be performed even for increasing failure rates.

We consider a specific increasing mortality (hazard) rate process induced by the non-homogeneous Poisson process of shocks. The shape of the observed (marginal) failure rate is analyzed in this case. In particular, we show for some specific cases: the population failure/mortality rate decreases with age and, even tend to reach a plateau. This result is obtained, when compared with sample paths of the unconditional mortality rate process, which are monotonically increasing.

Our model can, be used to model and analyze the damage accumulated by organisms experiencing external shocks. In this case, the cumulated damage is reflected by jumps on the failure rate. An overview of results on mortality rate processes with a single change point are also presented and discussed.

We show, for some specific cases, that the corresponding mixing, e.g. when for example, the ordered subpopulations are not necessarily the same (i.e. with equal means, but different variances) can lead to the corresponding ordering of mixture failure rate in, $[t, \infty)$.

For a subpopulation of items that survived the operational interval $[0,t)$, the variance of the conditional random variable, $W \mid t$ is analyzed. Two specific cases are considered: the case, when the random variable (frailty) W is discrete and/or continuous. When, the failure rates are ordered, $E[W \mid t]$ is decreasing and $\lambda_m(t)$ is also decreasing as functions of time. We use, simple but meaningful examples to illustrate our results. Specifically, it is shown that the corresponding variance is not monotone, e.g. it may decrease or be UBT shaped.

Another, useful measure, which we consider is the coefficient of variation of the random variable, $W \mid t$. Whereas the variance may be decreasing, we show for some specific cases, that the coefficient of variation, may be decreasing or exhibit other shapes: e.g., constant or UBT shape.

The focus in the literature has been mostly on the study of expectations, however, the obtained in this section results show that the variability characteristics in heterogeneous populations may change dynamically. Therefore, along with the expectations~~,~~ this should also be considered in practical applications.

The obtained results on bending down of the mixture failure rates with time are well justified for human populations (Gompertz law of mortality for baseline distributions in mixing models) from developed countries where there is gradual availability of validated mortality data on centerians and supercentarians. It is also interesting to investigate this phenomenon utilizing data from less developed countries.

Moreover, the results obtained in this thesis, clearly show that the proportion of surviving items (objects) in the mixed populations is increasing, e.g., the population lifetime is improving somehow as the “weakest subpopulations are dying out first”. Thus, mixture of distributions provides a useful tool for modeling and analysis of heterogeneous populations.

Bibliography

1. **Cha J. H., and Finkelstein M. S. 2018:** *Point Processes for reliability-Shocks and Repairable Systems*. © Springer International Publishing AG 2018.
2. **Bouaziz O., and Nuel G. 2017:** *A change point model for detecting heterogeneity for ordered survival responses*. Statistical Methods in Medical Research, SAGE Publications, 2017. <hal01287075v2>.
3. **Cha J.H. and Finkelstein M.S. 2016a:** *On some properties of shock processes in a 'natural' scale*. Reliability Engineering and System Safety **145,** pp. 104–110. © 2015 Published by Elsevier Ltd.
4. **Cha J.H. and Finkelstein M.S. 2016b:** *On some mortality rate processes and mortality deceleration with age,* Journal of Mathematical Biology, **72** pp. 331-342. © Springer-Verlag Berlin Heidelberg 2015.
5. **Cha J. H. and Pulcini G. 2016:** *Optimal burn-in procedure for mixed populations based on the device degradation process history*. European Journal of Operational Research **251** pp 988–998. ©2015Elsevier B.V.
6. **Kayid M., Izadkhah S., and Almufarrej A. 2016:** *General proportional mean residual model***.** Applications of Mathematics, **61** (5), pp.607-622. DOI .10.1007/s10492-016-0149-3.
7. **Kayid M., Izadkhah S., and Mesfioui M. 2016:** *Relative stochastic comparisons of additive frailty models.* Journal of Inequalities and Applications, 2016:158. © 2016 Kayid et al. DOI 10.1186/s13660-016-1103-3.
8. **Li C., and Li X. 2016:** *Relative ageing of series and parallel systems with statistically independent and heterogeneous component lifetimes*. **99**, pp. 1–8. IEEE Transactions on Reliability.
9. **Szymkowiak M. and Iwińska M. 2016:** *Characterizations of Discrete Weibull related distributions,* Statistics and Probability Letters **111** pp. 41-48. © 2016 Elsevier B.V.
10. **Chakraborty S. 2015:** *Generating discrete analogues of continuous probability distributions-A survey of methods and constructions,* Journal of Statistical distributions and applications, **2**(6). DOI: 10:1186/s40488-015-0028-6.

11. **Cha J.H. and Finkelstein M.S. 2015:** *Environmental stress screening modelling, analysis and optimization*. Reliability Engineering and System Safety **139** (2015) pp149–155. © 2016 Elsevier L.td
12. **Gupta P. L. 2015:** *Properties of Reliability Functions of Discrete Distributions*. Communications in Statistics - Theory and Methods, 44:19, 4114-4131, DOI: 10.1080/03610926.2014.982827. © Taylor & Francis Group, LLC.
13. **Kayid, M., Izadkhah, S., and Zuo, M.J. 2015:** *Some results on the relative ordering of two frailty models*. Statistical Papers pp. 1–15.
14. **Misra, N. and Francis, J. 2015:** *Relative ageing of (n − k + 1)-out-of-n systems*. Statistics and Probability Letters **106**, pp. 272–280. © 2016 Elsevier L.td.
15. **Rezaei M., Gholizadeh B. 2015:** *On the Mixture Proportional Mean Residual Life Model*, Communications in Statistics - Theory and Methods, **44** (20), pp. 4263-4277, DOI: 10.1080/03610926.2013.810262. © Taylor & Francis Group, LLC.
16. **Rezaei M., Gholizadeh B., and Izadkhah S. 2015:** *On Relative Reversed Hazard Rate Order*, Communications in Statistics - Theory and Methods, **44** (2), pp. 300-308, DOI: 10.1080/03610926.2012.745559. © Taylor & Francis Group, LLC.
17. **Wrycza T. F., Missov T. I., and Baudisch A. 2015:** *Quantifying the Shape of Aging.* PLoS ONE 10(3): e0119163.doi:10.1371/journal.pone.0119163.
18. **Almalki S.J and Nadarajah S. 2014:** *Modifications of the Weibull distribution: A review,* Reliability Engineering and System Safety **124** pp. 32-55. © 2013 Elsevier Ltd.
19. **Cha J.H. and Finkelstein M.S. 2014a:** *Some notes on unobserved parameters (frailties) in reliability modeling.* Reliability Engineering and System Safety **123**(2014) pp 99–103. © 2013Elsevier Ltd.
20. **Cha J.H. and Finkelstein M.S. 2014b:** *on some conditional characteristics of the hazard rate induced by external shocks*, Operations Research Letters, **42** (2014) pp 325-331. ©2015 Elsevier B.V.
21. **Cha J.H. and Finkelstein M.S. 2014c:** *Burn-in for Eliminating Weak Items in Heterogeneous Populations, Communications in Statistics* - Theory and Methods, 43:24, pp 5115-5129, DOI: 10.1080/03610926.2012.744054.

22. **Chen X., and Baron M. 2014:** *Change-Point Analysis of Survival Data with Application in Clinical Trials*. Open Journal of Statistics, 2014, 4, 663-677. © 2014 by authors and Scientific Research Publishing Inc.

23. **Finkelstein M**.S. **and Vaupel J. 2014:** *On random age and remaining lifetime for populations of items,* Applied Stochastic Models for Bussiness and Industry. © 2014 John Wiley & Sons, Ltd. DOI: 10.1002/asmb.2073.

24. **Hanagal D. D., and Pandey A. 2014:** *Gamma shared frailty model based on reversed hazard rate for bivariate survival data*. Statistics and Probability Letters. 88, pp 190-196. © 2014ElsevierB.V.

25. **Kayid M. and Izadkhah S. 2014:** *Mean Inactivity Time Function, Associated Orderings, and Classes of Life Distributions*. IEEE Transactions on Reliability, Vol. 63, No.2, June 2014.

26. **Kayid M., Izadkhah S., and Alhalees H. 2014:** *Reliability Analysis of the Proportional Mean Residual Life Order.* Hindawi Publishing Corporation Mathematical Problems in Engineering, Volume 2014, Article ID 142169, 8 pages, Mathematical Problems in Engineering.

27. **Parsa M., Motashami Borzadaran G.R., and Rezaei Roknabadi A.H. 2014:** *On the change points of mean residual life and failure rate for some generalized Gamma type distributions.* International Journal of Meterology, Quality and Engineering, **5**, 102 (2014). ©EDP Sciences 2014.

28. **Veres-Ferrer E. J., and Jose M. Pavía J. M. 2014:** *On the relationship between the reversed hazard rate and elasticity*. Stat Papers (2014) **55**, pp. 275–284. DOI 10.1007/s00362-012-0470-1.

29. **Cha J. and Finkelstein M.S. 2013a**: *The failure rate dynamics in heterogeneous populations.* Reliability Engineering and System Safety **112** (2013) pp 120–128. © 2012 Elsevier Ltd.

30. **Cha J. and Finkelstein M.S. 2013b**: *On history-dependent shock models.* Operations Research Letters **41 (**2013) pp 232–237. © 2013 Elsevier B.V.

31. **Elbatal I. and Ahsanullah M. 2013:** *Preservation of Classes of Discrete Distributions under Reliability Operations*. Journal of Statistical Theory and Applications, **Vol. 12**, No. 1 (May 2013), pp. 1-10.

32. **Finkelstein M. S. 2013:** On *some comparisons of lifetime for reliability analysis.* Short communication, Reliability Engineering and Systems Safety. Elsevier Ltd.
33. **Finkelstein M.S. and Cha J. 2013:** *Stochastic Modeling for Reliability: Shocks, Burn-in and Heterogeneous Populations*-Springer London Heidelberg New York Dordrecht, **ISBN 978-1-4471-5027-5.**
34. **Khorashadizadeh M., Rezaei Roknabadi A. H., and Mohtashami Borzadaran G. R. 2013:** *Characterization of Life Distributions Using Logodds Rate in Discrete Aging*. Communications in Statistics - Theory and Methods, 42:1, 76-87, DOI: 10.1080/03610926.2011.575512. © Taylor & Francis Group, LLC.
35. **Lai C.D. 2013:** *Issues concerning constructions of discrete lifetime models,* Quality Technology and Quantitative Management, **10** (2) pp. 251-262.
36. **Nanda A. K. and Das S. 2013:** *On dynamic proportional mean residual life model*. Probability in the Engineering and Informational Sciences, **27**, pp. 553–588. Doi: 10.1017/S0269964813000259. ©Cambridge University Press 2013.
37. **Bebbington M., Lai C. D., Wellington M., and Zitikis R. 2012:** *The discrete additive Weibull distribution: A bathtub-shaped hazard for discontinuous failure data*. Reliability Engineering and System Safety 106 (2012), pp. 37–44. ©2012 Elsevier Inc.
38. **Block H.W., Langberg N.A. and Savits T.H. 2012:** *A mixture of exponential and IFR Gamma distributions having upside-down bathtub failure rate.* Probability in the Engineering and Informational Sciences, **26**, pp. 573-580. DOI: 10.1017/S026996481000204.
39. **Cha J.H. and Finkelstein M. S. 2012:** *Shocks as Burn-In in Heterogeneous Populations.* Communications in Statistics—Theory and Methods, **41** pp 325–340. . ©2012 Taylor & Francis Group, LLC. (**NB:** see, DOI: 10.1080/03610926.2010.526742).
40. **Elbatal, A.N. Ahmed I and Ahsanullah M. 2012:** *Characterization of continuous distribution by their mean inactivity times,* Pakistan Journal of Statistics, **28** (3), pp. 279-292.
41. **Finkelstein M. S. 2012:** *On ordered subpopulations and population mortality at advanced ages.* Theoretical Population Biology, **81**, pp. 292-299. ©2012 Elsevier Inc.
42. **Goliforushan S, Asadi M., and Balakrishnan N. 2012:** *On the residual and inactivity times of the components of a used coherent systems*. J. Appl. Prob. 49, pp. 384-404. © Applied Probability Trust 2012.

43. **Cha J. H., and Finkelstein M.S. 2011:** *on new classes of extreme shock models and some generalizations*. Journal of Applied Probability, Vol. 48, No. 1 (MARCH 2011), pp. 258-270. © Applied Probability Trust 2011.
44. **Finkelstein M. S. 2011a:** *On the 'rate of aging' in heterogeneous populations.* Mathematical Biosciences, **232**, pp. 20-23. ©2011 Elsevier Inc.
45. **Finkelstein M. S. 2011b:** *On Ordered Subpopulations and Population Mortality at Advanced Age.* MPIDR WORKING PAPER WP 2011-022.
46. **Gavrilov L.A. and Gavrilova N. S. 2011:** *Mortality Measurement at Advanced Ages- A Study of the Social Security Administration Death Master File*. N Am Actuar. J. 2011; 15(3): 432–447.
47. **Missov T. I. and Finkelstein M.S. 2011:** *Admissible mixing distributions for a general class of mixture survival models with known asymtotics,* Theoretical population Biology, **80**, pp 64-70. ©2011 Elsevier Inc. (**NB:** see also MPIDR WP 2011-004).
48. **Nanda A. K. and Nanda S. D. 2011:** *Dynamic proportional hazard rate and reversed hazard rate models*. Journal of Statistical Planning 141, pp. 2108-2119. ©2011 Elsevier B.V.
49. **Nooghabi M. S., Rezaei A. H., Reza R. G., and Morzadaran M. 2011:** *Discrete modified Weibull distribution*. METRON - International Journal of Statistics 2011, vol. LXIX, n. 2, pp. 207-222.
50. **Block H.W., Langberg N.A and Wang J. 2010:** *Continuous mixtures of exponentials and IFR Gammas having bathtub-shaped failure rates***.** Journal of Applied Probability, **47**, pp 899-907.
51. **Cha J.H. and Finkelstein M. S. 2010:** *Burn-in by environmental shocks for two ordered subpopulations*. European Journal of Operational Research **206** (2010) pp 111–117. © 2010 Elsevier B.V.
52. **Finkelstein M. S., and Marais F. 2010:** *On terminating Poisson processes in some shock models*. Reliability Engineering and System Safety **95** (2010), pp. 874–879. ©2008 Elsevier Ltd.
53. **Jiang H., Xie M., and Tang L. C. 2010:** *On MLEs of the parameters of a modified Weibull distribution for progressively type-2 censored samples*. Journal of Applied Statistics. **37**, pp. 617–27.

54. **Li X., Da G. and Zhao P. 2010:** *On reversed hazard rate in general mixture models,* Statistics and Probability Letters, **80**, pp 654-661. © 2009 Elsevier B.V.
55. **Cha J. H., and Finkelstein M. S. 2009:** *On a Terminating Shock Process with Independent Wear Increments*. Journal of Applied Probability, Vol. 46, No. 2 (JUNE 2009), pp. 353-362. ©Applied Probability Trust.
56. **Curto J. D., and Pinto J. C. 2009:** '*The coefficient of variation asymptotic distribution in the case of non-iid random variables'.* Journal of Applied Statistics, **36** (1), pp. 21- 32.
57. **Finkelstein M. S. 2009a:** *On damage accumulation and biological aging.* Journal of Statistical Planning and Inference, **139** (2009), pp 1643-1648. ©2008 Elsevier B.V.
58. **Finkelstein M. S. 2009b:** *understanding the shape of mixture failure rate (with engineering and demographic applications),* MPIDR working paper. Max Plank Institute for Demographic Research**, 26**, Article 9, pp. 191-206.
59. **Gorjian N., Ma L., Mittinty M., Yarlagadda P., and Sun, Y. 2009:** *A review on degradation models in reliability analysis*. Proceedings of the 4th World Congress on Engineering Asset Management, 28-30 September 2009, Marriott Athens Ledra Hotel, Athens. © Copyright 2009 Springer.
60. **Krishna H., and Pundir P.S. 2009:** *Discrete Burr and discrete Pareto distributions,* Statistical Methodology, **6** (2) pp. 177-188.
61. **Li T., and Anderson J. J. 2009:** *The vitality model -A way to understand population survival and demographic heterogeneity*. Theoretical Population Biology 76, pp. 118-131.
62. **Ortega E. M. 2009:** *A Note on Some Functional Relationships Involving the Mean Inactivity Time Order*, IEEE Transactions on Reliability, Vol. 58, No1, March 2009. ©IEEE.
63. **Righter R., Shaked M., and Shanthikumar J. G. 2009:** *Intrinsic aging and classes of nonparametric distributions*. Probab. Eng. Inform. Sci. **23**(4), pp. 563–582.
64. **Zhao X., Wu X., and Zhou X. 2009:** *A change point model for survival data with long-term survivors*. Statistica Sinica, Vol. 19, No. 1 (January 2009), pp. 377-390. ©Institute of Statistical Science, Academia Sinica.
65. **Block H.W., Li Y., Savits T.H. and Wang J. 2008:** *Continuous mixtures with bathtub-shaped failure rates*. Journal of Applied Probability**, 45**, pp. 260-270.

66. **Finkelstein M. S. 2008:** *Failure Rate Modeling for Reliability and Risk*, Springer Series in Reliability Engineering, **ISSN 1614-7839; ISBN 978-1-84800-985-1**.

67. **Gavrilov L.A. and Gavrilova N. S. 2008:** *Mortality Measurement at Advanced Ages: A Study of the Social Security Administration Death Master File.* Presented at the Living to 100 and Beyond Symposium Orlando, Fla. January 7-9, 2008. © 2008 by the Society of Actuaries.

68. **Ghitany M.E., Atieh B., and Nadarajah S. 2008:** *Lindley distribution and its application.* Mathematics and Computers in Simulation **78**, pp. 493–506. © 2007 IMACS. Published by Elsevier B.V.

69. **Goliforushani S., and Asadi M. 2008:** *On the discrete mean past lifetime*. Metrika (2008) 68:209–217. DOI 10.1007/s00184-007-0153-4.

70. **Li X., and Li Z. 2008:** *A Mixture Model of Proportional Reversed Hazard Rate*. Communications in Statistics—Theory and Methods, 37: 2953–2963, 2008. Copyright © Taylor & Francis Group, LLC ISSN: 0361-0926 print/1532-415X online DOI: 10.1080/03610920802050935.

71. **Navarro J. and Hernandez P. J. 2008:** *Mean residual life functions of finite mixtures, order statistics and coherent systems***.** Metrika (2008) **67**, pp. 277–298. DOI 10.1007/s00184-007-0133-8. © Springer-Verlag 2007.

72. **Wang J. 2008:** *Bathtub failure rates of mixtures in reliability and the simes inequality under dependence in multiple testing*, University of Pittsburgh, 2008 (PhD Thesis).

73. **Bebbington M., Lai C.D and Zitikis R. 2007:** *Modeling human mortality using mixtures of bathtub shaped failure distributions*, Journal of Theoretical Biology**, 245**, pp 528-538, Elsevier Ltd.

74. **Finkelstein M. S. 2007a:** *Aging: Damage accumulation versus increasing mortality rate*, Mathematical Biosciences, pp 104-112. ©2006 Elsevier Inc.

75. **Finkelstein M. S. 2007b:** *Shocks in homogeneous and heterogeneous populations.* Reliability Engineering and System Safety **92** (2007) pp 569–574. © 2006 Elsevier Ltd. (**NB:** see also MPIDR WP 2005-024).

76. **Gupta R. C. and Gupta R. D. 2007:** *proportional reversed hazard rate model and its applications.* Journal of Statistical Planning and Inference, 137 (2007) pp 3525-3536. © 2007 Elsevier B.V.

77. **Marshall A. W., and Olkin I. 2007:** *Life Distributions, structure of nonparametric, semiparametric and parametric families.* © 2007 Springer Science+Business Media, LLC. ISBN-13: 978-0-387-20333-1 (e-ISBN-13: 978-0-387-68477-2)
78. **Nanda A. K., Battacharjee S., and Alam S. S. 2007:** *Properties of aging intensity function***.** Statistics & Probability Letters, **77**, pp. 365-373.
79. **Nakagawa T. 2007:** *Shock and damage models in reliability theory*. London: Springer.
80. **Shaked M., and Shantikumar J. G. 2007:** *Stochastic Orders*. © 2007 Springer Science+Business Media, LLC.
81. **Esaulova V. 2006:** *Failure rates modelling for heterogeneous populations.* University of Magdeburg. (PhD Thesis).
82. **Finkelstein M. S. 2006:** *On relative ordering of mean residual lifetime functions.* Statistics & Probability Letters 76 (2006), pp. 939–944. © 2005 Elsevier B.V.
83. **Finkelstein M. S and Esaulova V. 2006:** *On Mixture Failure Rates Ordering.* Communications in Statistics—Theory and Methods, **35**, pp 1943–1955. DOI: 10.1080/03610920600762871. © Taylor & Francis Group, LLC.
84. **Gavrilov L.A. and Gavrilova N. S. 2006:** *Reliability Theory of Aging and Longevity*; In: “handbook of the biology of aging” sixth edition, Masoro E.J. &Austad S.N (eds.), San Diego, CA, USA: Elsevier Academic Press, pp. 3-42.
85. **Gupta R. C., Gupta R. D., and Gupta P. L. 2006:** *Monotonicity of the (reversed) hazard rate of the (maximum) minimum in bivariate distributions.* Metrika (2006) 63, pp. 223–241. DOI 10.1007/s00184-005-0014-y.
86. **Lai C. D. and Xie M. 2006:** *Stochastic aging and dependence,* © 2006 Springer Science and Business Media, Inc. ISBN-10: 0-387-29742-1 (ISBN-13: 978-0387-29742-2).
87. **Nanda A. K., Bhattacharjee S., and Alam S. S. 2006:** *properties of proportional mean residual life model*. Statistics & Probability Letters 76 (2006), pp. 880–890. © 2005 Elsevier B.V.
88. **Steinsaltz D., and Wachter K. W. 2006:** *Understanding mortality rate deceleration and heterogeneity.* Mathematical Population Studies, **13**, pp. 19–37.
89. **Ahmad, I. A. and Kayid, M. 2005:** *Characterizations of the RHR and MIT orderings and the DRHR and IMIT classes of life distributions*. Probability in the Engineering and Information Sciences, 19, 447-461.

90. **Ahmad, I., Kayid, M., and Pellerey, F. 2005:** *Further results involving the MIT order and the IMIT clas*s. Probability in the Engineering and Informational Science. 19, 377- 395.
91. **Chandra and Roy 2005:** *Classification of Distributions Based on Reversed hazard Rate*. Calcutta Statistical Association Bulletin, Vol. 56, (Special 5-th Triennial Proceeding Volume), 2005, Nos. 221-224.
92. **Finkelstein M. S. 2005a:** *Aging: Damage accumulation versus increasing mortality rate*, MDR, WP 2005-018
93. **Finkelstein M. S. 2005b:** *On some reliability approaches to human aging.* The International Journal of Reliability, Quality and Safety Engineering, **12**, pp 1-10.
94. **Finkelstein M. S. 2005c:** *Why mixture failure rates decrease?* South African Statistical Journal**, 39**, pp. 23-33.
95. **Finkelstein M. S. 2005d:** *Lifesaving explains mortality decline with time,* Mathematical Biosciences, **196**, pp. 187–197. ©2005 Elsevier Inc.
96. **Finkelstein M. S. 2005e:** *On stochastic comparisons of population densities and life expectancies.* **13** (6), pp. 143-162. © 19 October, 2005 Max-Planck-Gesellschaft. Max Planck Institute for Demographic Research.
97. **Gupta R. C., and Lvin S. 2005:** *Monotonicity of failure rate and mean residual life function of a gamma-type model*. Applied Mathematics and Computation **165** (2005), pp. 623–633. ©2004 Elsevier Inc.
98. **Gut A., and Hüsler J. 2005:** *Realistic variation of shock models.* Statistics & Probability Letters 74 (2005), pp. 187–204. ©2005 Elsevier B.V.
99. **Ahmad I. A. 2004:** *Some properties of life distributions with unknown age*. Statistics & Probability Letters, **69**, pp. 333-342. ©2004 Elsevier B.V.
100. **Ahmad I. A., and Mugdadi A. R. 2004:** *Further moments inequalities of life distributions with hypothesis testing applications: the IFRA, NBUC and DMRL classes*. Journal of Statistical Planning and Inference, 120, pp. 1-12.
101. **Finkelstein M. S. 2004:** *Minimal Repair in Heterogeneous Populations*. Journal of Applied Probability, Vol. 41, No. 1 (Mar., 2004), pp. 281-286. ©Applied Probability Trust.
102. **Kayid M., and Ahmad I. A. 2004:** *On the mean inactivity time ordering with reliability applications.* Probability in the Engineering and Informational Sciences / Volume 18 / Issue

03 / July 2004, pp. 395 - 409 DOI: 10.1017/S0269964804183071, Published online: 01 July 2004.

103. **Kemp A.W. 2004:** *Classes of discrete lifetime distributions,* Communications in Statistics: Theory and Methods, **33**(12) pp. 3069-3093.

104. **Lai C. D., Zhang L., and XIE M. 2004:** *Mean residual life and other properties of weibull related bathtub failure rate distributions*. International Journal of Reliability, Quality and Safety Engineering. Vol. 11, No. 2 (2004) pp. 113-1.32. ©2004 World Scientific Publishing Company.

105. **Li X., and Zuo M. 2004:** *Stochastic comparisons of residual life and inactivity time at random time*. Stochastic Models, vol. 20, pp. 229–235, 2004.

106. **Murthy D.N.P., Xie M. and Jiang R. 2004:** *Weibull Models*. ©2004 by John Wiley & Sons, INC. Hoboken, New Jersy.

107. **Navarro J., and Hernandez P. J. 2004:** *How to obtain bathtub-shaped failure rate models from normal mixtures*. Probability in the Engineering and Informational Sciences, 18, pp 511-531 doi: 10.1017/S0269964804184076. © 2004 Cambridge University Press.

108. **Rausand M. and Hoyland A. 2004:** *System Reliability Theory: Models, Statistical Methods, and Applications.* 2nd Ed. Wiley Series in Probability and Mathematical Statistics (Applied Probability and Statistics). © 2004 by John Wiley & Sons, Inc.

109. **Steinsaltz and Evans, (2004):** *Markov mortality models: implications of quasi-stationary and varying initial distributions*. Theoretical Population Biology **65**, pp. 319-337. © 2004 Elsevier Inc.

110. **Wondmagegnehu E.T 2004:** *On the behavior and shape of mixture failure rates from a family of IFR weibull distributions*. Wiley Periodicals, Inc. Naval Research Logistics, **51,** pp. 491-500.

111. **Xie M., Goha T. N. and Tang Y. 2004:** *On changing points of mean residual life and failure rate function for some generalized Weibull distributions*. Reliability Engineering and System Safety 84 (2004) pp 293–299.

112. **Block H. W., Savits T. H., and Wondmagegnehu E. T. 2003:** *Mixtures of Distributions with Increasing Linear Failure Rates*. Journal of Applied Probability, Vol. 40, No. 2 (Jun., 2003), pp. 485-504. © Applied Probability Trust.

113. **Badia F.G., Berrade M.D., Clemente A and Campos C.A., 2003:** *Why do failure rates decrease?* Monografias del Semin. Matem. Garcia de Galdeano**. 27,** pp. 97-104.

114. **Bracquemond C. and Gaudion O. 2003:** *A survey of discrete lifetime distributions.* International Journal of reliability, Quality and Safety Engineering, **10** (69): DOI: http://dx.doi.org/10.1142/S0218539303001007.

115. **Bradley D. M., and Gupta R.C. 2003:** *Limiting behavior of the mean residual life*. Annals of Institute of Statistical Mathematics, **55 (**1), pp. 217-226. © 2003 The Institute of Statistical Mathematics.

116. **Finkelstein M. S. 2003**: *Modeling Failure (Mortality) Rate with a Change Point.* MPIDR WORKING PAPER WP 2003-041 December 2003.

117. **Fox C. W., and Moya-Laraño J. 2003:** *Why organisms show late-life mortality plateaus: a null model for comparing patterns of mortality.* Evolutionary Ecology Research, 2003, **5**: pp. 999–1009.

118. **Gupta R.C., and Bradley D. M. 2003:** *Representing the Mean Residual Life in Terms of the Failure Rate.* Mathematical and Computer Modelling **0** (2003) 1-0, PERGAMON. © 2003 Elsevier Science Ltd.

119. **Lai C.D. and Xie M. 2003:** *Relative Ageing for Two Parallel Systems and Related Problems.* Mathematical and Computer Modelling **38,** pp. 1339-1345. © 2003 Elsevier Ltd.

120. **Lawless J. F. 2003:** *Statistical Methods for lifetime Data,* 2nd edition. John Wiley & Sons, Inc., Hoboken, New Jersey.

121. **Li X., and Lu J. 2003:** *Stochastic comparisons on residual life and inactivity time of series and parallel systems*. Probability in the Engineering and Informational Sciences, **17**, pp. 267–275.

122. **Mallor F., and Santos J. 2003:** *Classification of shock models in system reliability,* Monograf˜ıas del Semin. Matem. Garc´ıa de Galdeano. **27,** pp. 405–412.

123. **Nanda A. K., Singh H., Misra N., and Paul P. 2003:** *Reliability properties of reversed residual lifetime*. Communications in Statistics: Theory and Methods, **32** (10), pp. 2031–2042.

124. **Suresh R. P. 2003:** *On the inter-relationships between some classes of life distributions*. Calcutta. Statistical Association Bulletin. **54**, pp. 215-216.

125. **Abu-Youssef S. E. 2002:** A *moment inequality for decreasing (increasing) mean residual life distributions with hypothesis testing applications*. Statistics and Probability letters, **57**, pp. 171-177.

126. **Bassan B., and Rinott Y., and Yehuda Vardi Y. 2002:** *On stochastic comparisons of excess times*. Discussion paper number 302. Center for the Study of Rationality, Hebrew University, Jerusalem.

127. **Finkelstein M.S. 2002a:** *On the reversed hazard rate*. Reliability Engineering and System Safety, **78**, pp. 71–75. Elsevier Science Ltd.

128. **Finkelstein M.S. 2002b:** *Modeling Lifetimes with initial unknown age*, Reliability Engineering and **Sy**stem Safety **76**, pp. 75-80.

129. **Finkelstein M.S. 2002c:** *On the shape of the mean residual lifetime function.* Applied stochastic Models in Business and Industry, **18**, pp. 135–146 (DOI: 10.1002/asmb.461). © 2002 John Wiley & Sons, Ltd.

130. **Patra K., and Dey D. K. 2002:** *A General Class of Change Point and Change Curve Modeling for Life Time Data*. Ann. Inst. Statist. Meth. **54** (3), pp. 517-530. © 2002 The Institute of Statistical Mathematics.

131. **Xie M., Gaudion O. and Bracquemond C. 2002:** *Redefining the failure rate function for discrete distributions*. International Journal of Reliability, Quality and Safety Engineering, **9** (3), pp. 275-285. ©World Scientific Publishing Company.

132. **Aalen O. O., and Gjessing H. K. 2001:** *Understanding the Shape of the Hazard Rate- A Process Point of View.* Statistical Science, **16** (1), pp. 1-14. ©Institute of Mathematical Statistics.

133. **Bloch-Mercier S. 2001:** *Monotone Markov Processes with Respect to the Reversed Hazard Rate Ordering: An Application to Reliability*. Journal of Applied Probability, **38** (1), pp. 195-208.

134. **Bracquemond C., Gaudon O., Roy D., and Xie M. 2001:** *on some discrete notions of aging*. July 27, 2001, WPSC/Guidelines- Discrete aging (Chapter 1).

135. **Chandra N. K., and Roy D. 2001:** *Some results on reversed failure rate***.** Probability in the Engineering and Informational Sciences, **15**, pp. 95–102. Printed in the U.S.A.

136. **Finkelstein M. S and Esaulova V. 2001a:** *Modeling a failure rate for a mixture of distributions***.** Probability in the Engineering and Informational Sciences, **15,** pp. 383-400.

137. **Finkelstein M. S and Esaulova V. 2001b:** *Why the mixture failure rates decrease?* Reliability Engineering and System Safety, **71**, pp. 173-177. Elsevier Science, Ltd.

138. **Gupta R. C. and Warren R. 2001:** *Determination of change points of non-monotonic failure rates,* Communications in Statistics - Theory and Methods, **30**:8-9, 1903-1920, DOI: 10.1081/STA-100105704.

139. **Gupta R. C. and Wu H. 2001:** *Analysing survival data by proportional reversed hazard rate model*. International Journal of Reliability and Applications. **2** (1), pp. 1-26.

140. **Gupta R. D., and Nanda A. K. 2001:** *Some results on reversed hazard rate ordering*, Communications in Statistics (proceedings of the Maine Conference).

141. **Gut A. 2001:** *Mixed shock models*. Bernoulli, **7** (3), pp. 541-555. International Statistical Institute (ISI) and the Bernoulli Society for Mathematical Statistics and Probability. © 2001 ISI/BS.

142. **Lai C. D., Xie M., and Murthy D. N. P. 2001:** *Bathtub-Shaped Failure Rate Life Distributions*. N. Balakrishnan and C. R. Rao, eds., Handbook of Statistics, **20**. © 2001 Elsevier Science B.V.

143. **Shaked M. and Spizzichino F. 2001:** *Mixtures and monotonicity of failure rate functions*. In: Balakrishnan N, Rao CR (eds.) Advances in Reliability. Elsevier Science, Amsterdam, **20**, pp. 185-197.

144. **Weitz J. S., and Fraser H. B. 2001:** *Explaining mortality rate plateaus*. Proc. Nat. Acad. Sci. (PNAS), **98** (26), pp. 15383-6.

145. **Anderson J. J. 2000:** *A Vitality-Based Model Relating Stressors and Environmental Properties to Organism Survival***.** Ecological Monographs, **70** (3), pp. 445-470. ©2000 by the Ecological Society of America.

146. **Chen J., and Gupta A. K. 2000**: *Parametric statistical change point analysis*. Birkhauser.

147. **Di Crescenzo A. 2000:** *Some results on the proportional reversed hazards model*. Statistics & Probability Letters **50**, pp. 313–321

148. **Finkelstein M. S., and Zarudnij V. I. 2000:** *A shock process with a non-cumulative damage*. Reliability Engineering and System Safety **71**, pp. 103–107. ©2001 Elsevier Science Ltd.

149. **Yashin A. L., Ivan A. Iachine I. A., and Begun A. S. 2000:** *Mortality modeling: A review.* Mathematical Population Studies, **8** (4), pp. 305-332. DOI: 10.1080/08898480009525489.

150. **Willmot G. E., and Cai J. 2000:** *On classes of lifetime distributions with unknown age***.** Probability in Engineering and Informational Sciences, **14**, pp. 473-484.

151. **Di Crescenzo A. 1999:** *Dual stochastic orderings describing aging properties of devices of unknown age***.** Communications in Statistics—Stochastic Models, **15**, pp. 561-576.

152. **Ghai G. L. and Mi J., (1999):** *Mean residual life and its association with the failure rate***.** IEEE Transaction on reliability, **48** (3). © 1999 IEEE.

153. **Lynch J. D. 1999:** *On conditions for mixtures of increasing failure rate of distributions to have an increasing failure rate.* Probability in the Engineering and Informational Sciences, **13**, pp. 33-36.

154. **Manton K.G. 1999:** *Dynamic Paradigms for Human Mortality and Aging.* Journal of Gerontology: Biological Sciences 1999, **54A** (6), B247-B254.

155. **Sengupta D., and Nanda A. K, 1999:** *Log concave and log convex distributions in reliability*. *Nav. Res. Logistics Quart.* **46**(4), pp. 419–433.

156. **Thatcher R. A. 1999:** *The Long-Term Pattern of Adult Mortality and the Highest Attained Age*. Journal of the Royal Statistical Society. Series A (Statistics in Society), **162** (1), pp. 5-43.

157. **Tang L.C, Lu Y, and Chew E.P., 1999:** *Mean Residual Life of Lifetime Distributions*. IEEE Transactions on Reliability, **48** (1). ©1999 IEEE

158. **Abouammoh A. M., and Ahmad A. N. 1998:** T*he new better than used failure rate class of distributions*. Advances in applied probability, **20**, pp. 237-240.

159. **Block H.W., Savits T.H., and Singh H. 1998:** *The reversed hazard rate function*. Probability Engineering and Information Science, **12**, pp 69-90.

160. **Gupta R. C., Gupta R. D. and Gupta P. L. 1998:** *Modeling failure time data by Lehman alternatives*. Communication in Statistics-Theory and Methods, **27**(4), pp. 887-904. © 1998 by Marcel Dekker Inc.

161. **Gupta R. C. and Kirmani S, N. U. A. 1998:** *On the Proportional Mean Residual Life Model and its Implications*. Statistics, **32** (2), pp 175-187. DOI: 10.1080/02331889808802660.

162. **Jiang T. R, Murphy D.N.P, 1998:** *Mixture of Weibull distributions-parametric characterization of failure rate function.* Appl. Stochastic Models Data Anal. **14**, pp 47-65. ©1998 John Wiley & Sons, Ltd.

163. **Block H.W and Joe H. 1997:** *Tail behavior of the failure rate functions of mixtures*. Lifetime Data Anal, **3**, pp. 269–288.

164. **Gupta P.L., Gupta R.C., and Tripathi R.C. 1997:** *on monotonic properties of discrete failure rates,* Journal of Statistical Planning and Inference, **65,** pp. 255-268.

165. **Beirlant J., Teugels J. L., and Vynckier P. 1996:** *Practical Analysis of Extreme Values*. Leuven University Press, 1996.

166. **Mi J. 1996:** *Limiting Behavior of Mixtures of Discrete Lifetime Distributions*. Naval Research Logistics. **43**, pp. 365-380 (1996). ©1998 John Wiley & Sons, Inc.

167. **Mitrta M., and Basu S. K. 1996:** *On some properties of bathtub failure rate family of life distributions*. Micro electron. Reliab. **36** (5), pp. 679-684. ©1996 Elsevier Science Ltd.

168. **Mueller L., and Rose M. R. 1996:** *Evolutionary theory predicts late-life mortality plateaus.* Proc. Natl. Acad. Sci. USA. 1996; 93, pp. 15249–15253.

169. **Ross S. M. 1996:** *Stochastic Processes*. ©1996 John Wiley, New York.

170. **Salivia A. A. 1996:** *Some Results on Discrete Mean Residual Life*. IEEE Transactions on Reliability, **45** (3). ©1996 IEEE.

171. **Abrahams P. A., and Ludwig D. 1995:** *Optimality theory, Gompertz's law and disposable soma theory of senescene*. Evolution **49**, pp. 1055-1066.

172. **Aalen, O. O. 1995:** *Phase Type Distributions in Survival Analysis*. Scandinavian Journal of Statistics, **22** (4), pp. 447-463.

173. **Beirlant J., Broniatowski M., Teugels J.L and Vynckier P. 1995:** *The mean residual life function at great age. Applications to tail estimation.* Journal of Statistical Planning and Inference (1995). pp. 21-48.

174. **Eeckhoudt L., and Gollier C. 1995:** *Demand for Risky Assets and the Monotone Probability Ratio Order*. Journal of Risk and Uncertainty, **11** (2), pp. 113-122. © 1995 Kluwer Academic Publishers.

175. **Finkelstein M. S. 1995:** *Succession-Dependent Shock Models*. Microelectron. Reliab., **36** (2), pp. 195-197. ©1996 Elsevier Science Ltd.

176. **Gupta, R.C. and Akman, H.O. 1995:** *Mean residual life function for certain types of non-monotonic aging*. Stochastic models, **11**, pp. 219-225.

177. **Gurland J, Sethuraman J. 1995**: *How pooling failure data may reverse increasing failure rate*. Journal of American Statistical Association, **90**, pp. 1416-1423.

178. **Jiang T. R, Murphy D.N.P, 1995:** *Modelling failure data by mixture of two Weibull distributions*. IEEE Transactions on Reliability, **44** (3), pp 477-488. © 1995 IEEE.

179. **Johnson N. L., Kotz S., and Balakrishinah N. 1995:** *Continuous univariate distributions*. Vol. 2 (2nd edition), Wiley, New York.

180. **Johnson N. L., Kemp A. W., and Kotz S. 1995:** *Univariate discrete distributions*. John Wiley & Sons.

181. **Mi J. 1995:** *Bathtub failure rate and upside down mean residual life*. IEEE Transactions on Reliability, **44** (3), pp. 388-391.

182. **Mitra M. and Basu S. 1995:** *Change point estimation in non-monotonic aging models*. Ann. Inst. Statist. Math. Vol. 47, No. 3, 483-491 (1995).

183. **Sengupta D., Chatterjee A., and Chakraborty B. 1995:** *Reliability bounds and other inequalities for discrete life distributions*. Microelectron. Reliab. **35**, (12), pp. 1473 1478, 1995. © 1995 Elsevier Science Ltd.

184. **Shaked M., Shantikumar J.G., and Valdez-Torres J.B. 1995:** *Discrete hazard rate functions.* Computers Operations Research, **22**(4) pp. 391-402 ©1995 Elsevier Science Ltd.

185. **Singpurwalla N. D., and Wilson S. P. 1995:** *The exponential formula of reliability and survival analysis: does it always hold?* Lifetime Data Analysis, **1**, pp. 187-194.

186. **Block H. W., Mi J., and Savits T. H. 1994:** *some results on burn-in*. Statistica Sinica, **4** (1994), pp. 525-533.

187. **Alzaid A. A. 1994:** *Aging concepts for items of unknown ages*. Communications in Statistics-Stochastic Models, **10**, pp. 649-659.

188. **Gurland J. and Sethuraman J. 1994:** *How pooling failure data may reverse increasing failure rates,* March 1994. FSU Technical Report M890 (USARO Technical Report D-134), November 1993.

189. **Ruiz J. M., and Navarro J. 1994:** *Characterization of Distributions by Relationships between Failure Rate and Mean Residual Life*. IEEE Transactions on Reliability, **43** (4), 1994 December. © 1994 IEEE.

190. **Sengupta D. 1994:** *Another look at moments bounds in reliability*. Journal of Applied Probability, **31**, pp. 777-787.

191. **Sengupta D. and Deshpande J. V. 1994:** *Some Results on the Relative Ageing of Two Life Distributions*. Journal of Applied Probability, Vol. **31**, No. 4 (Dec., 1994), pp. 991-1003. © Applied Probability Trust 1994.

192. **Shaked, M., Shanthikumar, J.G., 1994:** *Stochastic Orders and their Applications.* Academic Press, New York.

193. **Block H.W., Mi J., and Savits T.H. 1993:** *Burn-in and Mixed populations. Journal of applied probability*, **30**, pp. 692-702**.** © 1993 Applied Probability Trust.

194. **Mi J. 1993:** *Discrete Bathtub Failure Rate and Upside-Down Bathtub Mean Residual Life*. Naval Research Logistics, Vol. 40, pp. 361-371 (1993) © 1993 by John Wiley & Sons, Inc.

195. **Carey J. R., Liedo P., Orozco D., and Vaupel J. W. 1992:** *Slowing of mortality rates at old ages in large medfly cohorts***.** Science, New Series Vol. 258 No. 5081 (Oct 16 1992) pp. 457-461.

196. **Doksum K. A., and Hoyland A. 1992:** *Models for Variable-Stress Accelerated Life Testing Experiments Based on Wiener Processes and the Inverse Gaussian distribution*. Technometrics, Vol. 34, No. 1 (Feb., 1992), pp. 74-82. Published by Taylor & Francis, Ltd. on behalf of American Statistical Association and American Society for Quality.

197. **Johnson, N. L., Kotz, S. and Kemp, A. W. 1992:** *Univariate Discrete Distributions*, Second Edition. Wiley, New York.

198. **Roy D., and Gupta, R. P. 1992:** *Classification of discrete live*s. Microelectron. Reliab., Vol. 32, No. 10, pp. 1459-1473. © 1992 Pergamon Press Ltd.

199. **Wei X., 1992:** *Relative Mean Residual Life: Theory and Related topics*. Microelectron Reliab. **32** (9), pp 1319-1326. © 1992 Pergamon Press Ltd.

200. **Abouammoh A. M. 1991:** *On the mean remaining life of discrete distributions*. Journal of Information and Optimization Sciences, **12** (1) pp. 145-158,
DOI: 10.1080/02522667.1991.10699057

201. **Bhattachejee M. C. 1991:** *Some generalized variability orderings among life distributions with reliability applications*. Journal of Probability, **28**, pp. 314-383.

202. **Wong L. K. 1991:** *"Physical basis for the roller-coaster hazard rate curve for electronics"*, Quality and Reliability Engineering Vol. 7, pp. 489-495.

203. **Zahedi H. 1991:** *Proportional mean remaining life model*. Journal of Statistical Planning and Inference **29**, pp. 221-228. North-Holland.

204. **Deshpande J. V., and Suresh R. P. 1990:** *Non-Monotonic Ageing* Scandinavian Journal of Statistics, Vol. 17, No. 3 (1990), pp. 257-262. Published by Wiley on behalf of Board of the Foundation of the Scandinavian Journal of Statistics.

205. **Finch C. E., Pike M. C., and Witten M. 1990:** *Slow Mortality Rate Accelerations During Aging in Some Animals Approximate That of Humans*. Science, New Series, Vol. 249, No. 4971 (Aug. 24, 1990), pp. 902-905. ©American Association for the Advancement of Science.

206. **Cheng K., and He Z. F. 1989:** *Reliability bounds in NBUE and NWUE life distributions*. Acta Mathematicae Applacatac Sinica, **5**, pp. 81-88.

207. **Hitha N., and Nair N. U. 1989:** *Characterization of some discrete models by properties of residual life function*, Vol. 38, No. 3-4, pp. 219-224. Doi: org/10: 1177/008068319890310.

208. **Khalique A. 1989:** *On Discrete Failure-Time Distributions*. Reliability Engineering and System Safety **25**, pp. 99-107.

209. **Khan A.M.S., Khalique A., Abouammoh A.M., 1989:** *On estimating parameters in a discrete Weibull distribution,* **38** (3) pp 348–350. IEEE Transactions on Reliability.

210. **Guess F., and Proschan F. 1988:** *Mean Residual Life- Theory and Applications*. FSU Statistics Report M702. AFOSR Technical Report No. 85-178. The Florida State University.

211. **Guess F. M. and Parker D.H. 1988:** *Modeling Discrete Bathtub and Upside Down Bathtub Mean Residual Life Functions*. Statistical Technical Report NO.134 (62N5- 14-AFOSR-TR-88-0198.

212. **Rajarshi, S. and Rajarshi M.B. 1988:** *Bathtub distributions: a review*. Communications in Statistics- Theory and Methods, **17**(8), pp. 2597-2621.

213. **Abouammoh A. M., and El-Neweihi E. 1986:** *Closure of the NBUE and DMRL classes under the formation of parallel systems*. Statistics and Probability letters, 4, pp. 223-225.

214. **Basu A. P., and Ebrahimi N. 1986:** *HNBUE and HNWUE distributions- A survey*. In A. P Basu (Editor), Reliability and Quality Control, pp33-46, North-Holland, Amsterdam.

215. **Deshpande J. V., Kochar S. C., and Singh H. 1986:** *Aspects of positive ageing*. Journal of Applied Probability **23**, pp. 748– 758.

216. **Ebrahimi N. 1986:** *Classes of Discrete Decreasing and Increasing Mean-Residual-Life Distributions*. IEEE Transactions on Reliability, **R-35**, (4), 1986 October.

217. **Guess F., Hollander M., and Proschan F. 1986:** *Testing exponential versus a trend change in mean residual life*. Annals of Statistics, 14 pp. 1388-1398.

218. **Hollander M., Park H. D., and Proschan F. 1986:** *A class of life distributions for aging*. Journal of American Statistical Association, 81, pp. 91-95.

219. **Kalashnikov, V.V. and Rachev, S.T., 1986:** *Characterization of queuing models and their stability*. In: Prochorov, Y.K. (Ed.), Probability Theory and Mathematical Statistics, **2**, pp. 37–53. Amsterdam.

220. **Leemis L. M. 1986:** *Lifetime Distribution Identities*. IEEE Transactions on Reliability, **R-35** (2), 1986 June. ©1986 IEEE.

221. **Whitmore G. A. 1986:** *First-Passage-Time Models for Duration Data: Regression Structures and Competing Risks*. Journal of the Royal Statistical Society. Series D (The Statistician), Vol. 35, No. 2, Special Issue: Statistical Modelling (1986), pp. 207-219. Published by Wiley for the Royal Statistical Society.

222. **Aarset, M. V. 1985:** *The null distribution for a test of constant versus bathtub failure rate*. Scandinavian J. Statist. **12**, pp. 55-61.

223. **Padgett W.J. and Spurrier J.D. 1985:** *on discrete failure models*, **34** (3), pp. 253-256. IEEE Transactions on Reliability.

224. **Sigh H., and Deshpande J. V. 1985:** *On some new ageing properties.* Scandinavian Journal of Statistics, **12**, 213-220.

225. **Sumita, U. and Shanthikumar, J. G. 1985:** *A class of correlated cumulative shock models.* Advanced Applied Probability, **17**, pp 347-366.

226. **Hollander M., and Proschan F. 1984:** *Nonparametric Concepts and Methods in Reliability*. Handbook of Statistics, Volume 4, Nonparametric Methods (Krishnaiah, P. R. and Sen, P. K., eds.) pp. 613-655. © Elsevier Science Publishers (1984), North-Holland, Amsterdam.

227. **Nguyen H. T., Rogers G. S., and Walkern E. A. 1984:** *Estimation in change-point hazard rate models*. Biomtlrika (1984), 71, 2, pp. 299-304.

228. **Stein W. E., and Dattero R. 1984:** *A New Discrete Weibull Distribution*. IEEE Transactions on Reliability, **R-33** (2), June 1984.

229. **Govil K. K., and Aggrawal K. K. 1983:** *Mean residual life function for normal, gamma and lognormal densities.* Reliability Engineering, **5**, pp 47-51. Applied Science Publishers Ltd, England. **Applied Science** Publishers

230. **Patel J. K. 1983:** *Hazard rate and other classifications of distributions*. In encyclopedia in Statistical Sciences, **3**, pp. 590-594.

231. **Xelakaki E. 1983:** *Hazard functions and life distributions in discrete time*. Communications in Statistics-Theory and Methods, **12** (21), pp. 2503-2509.

232. **Bhattchrjee M. C. 1982:** *The class of mean residual lives and some consequences*. SIAM Journal on Algebraic Discrete Mathematics (1992), **3** pp 56-65.

233. **Keilson J., and Sumita U. 1982:** *Uniform Stochastic Ordering and Related Inequalities*. The Canadian Journal of Statistics / La Revue Canadienne de Statistique, Vol. 10 No. 3 (Sep., 1982), pp. 181-198. Published by: Statistical Society of Canada.

234. **Klefsjö B. 1982:** *The HNBUE and HNWUE classes of life distributions*. Naval Research Logistics Quarterly 29, 331–344.

235. **Lawless J. F. 1982:** *Statistical Models and Methods for lifetime data*. ©John Wiley, New York.

236. **Salvia A. A., and Bollinger R. C. 1982:** *On Discrete Hazard Functions*. IEEE Transactions on Reliability, **R-31** (5), pp. 458-459, December 1982.

237. **Barlow R. E., and Proschan F. 1981:** *Statistical Theory of Reliability and Life testing*. To Begin With, Silver Springs, Maryland.

238. **Hall W. J., and Wellner J. A. 1981:** *Mean Residual Life. In Statistics and related topics*, Csorgo M, Dawson D.A, Rao V. N. K, Saleh A.K (eds), North-Holland: Amsterdam, pp169-184.

239. **Kalbfleish J. D., and Prentice R. L. 1980:** *The statistical analyses of failure time data*. ©1980 Wiley, New York.

240. **Vaupe J. W. Manton K.G., and Stallard E. 1979:** *The impact of heterogeneity in individual frailty on mortality dynamics.* Demography, Vol. 16 (3), pp. 439-454.

241. **Folks J. L., and Chhikara R. S. 1978:** *The Inverse Gaussian distribution and its Statistical Application-A Review.* J. R. Statist. Soc. B (1978), **40**, No. 3, pp. 263-289.

242. **Block H. W., and Savits T. H. 1976:** *The IFRA closure theorem*. Annals of Probability, **4**, pp. 1020-1032.

243. **Barlow R.E., and Proschan F. 1975:** *Statistical Theory of reliability and life testing*. New York: Holt, Rinehart and Winston.

244. **Nakagawa T., and Osaki, S. 1975:** *The discrete Weibull distribution*, **24 (**5) pp 300–301. IEEE Transactions on Reliability.

245. **Rolski T. 1975:** *Mean residual life*. Bulletin of International Statistical Institute, 46, pp. 266-270.

246. **Esary J. D., Marshall A.W., and Proschan F. 1973:** *shock models and wear processes.* The Annals of Probability, Vol. 1, No.4, pp. 627-649.

247. **Meiljson I. 1972:** *Limiting properties of the mean residual lifetime function*. The Annals of Mathematical Statistics, Vol. 43, No. 1 (Feb., 1972), pp. 354-357.

248. **Bryson M. C., and Siddiqui M. M. 1969**: *Some criteria for aging*. Journal of the American Statistical Association, **64**, pp. 1472-1483.

249. **Birnbaum Z. W., Esary J. D., and Marshall A. W. 1966:** *A stochastic characterization of wearout for components and systems*. The Annals of mathematical Statistics, **37**, pp. 816-825.

250. **Barlow R.E., Marshall A. W., and Proschan F. 1963:** *Properties of probability distribution with monotone hazard rate*. The Annals of Mathematical Statistics, **34**, pp. 375-389.

251. **Beard R.E. 1959:** *Note on some mathematical mortality models*. In: G.E.W. Wolstenholme and MÇonor (eds). The lifespan of animals. Ciba Foundation Colloquium on aging. Little, Brown, Boston, pp. 302-311.