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**MODELLING TIME-DEPENDENT PAIRED COMPARISONS USING SOCCER
DATA**

By

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“For I know the plans I have for you,” declares the Lord, “plans to prosper you and not to harm you, plans to give you hope and a future.” – Jeremiah 29:11 (NIV)

DECLARATION

I, Joy Monareng, declare that the Master's research dissertation that I am submitting here is my own work and that I have not previously submitted it for qualifying at another institution of higher learning.

 .

29 November 2023

Joy Monareng

Date

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LIST OF ACRONYMS

LBT1	Linear Bradley-Terry Model 1
LBT2	Linear Bradley-Terry Model 2
LEAHP	Linear Exponential Analytical Hierarchy Process
T1	Thurstone Model 1
T2	Thurstone Model 2
T3	Thurstone Model 3
TDPC	Time-Depended Paired Comparisons

Abstract

This study presents a comprehensive analysis of soccer ranking models, encompassing the evaluation of existing models, the introduction of a new time-series paired comparison model, and a comparative study of model predictions against official soccer rankings. The investigation into the landscape of professional soccer rankings reveals the limitations of traditional FIFA rankings, prompting the exploration of alternative statistical models. The Linear Thurstone 3 (LT3) model emerges as a standout performer, demonstrating consistent and accurate predictions, particularly in the years 2023 and 2026.

However, inherent challenges, including data limitations, tournament dynamics, model assumptions, parameter standardization, and non-round-robin tournament structures, underscore the complexity of predicting soccer outcomes. The study provides recommendations for future research to address these limitations, emphasizing the refinement of models, improvement in data quality, dynamic model adaptation, ensemble modelling, cross-sport comparative analysis, and continuous evaluation.

This research contributes to the ongoing pursuit of precision and fairness in professional soccer rankings. The dynamic nature of the field necessitates continuous refinement and adaptation of models to enhance accuracy. By addressing the identified limitations and implementing the recommended strategies, future research endeavours can contribute to a more nuanced and reliable understanding of team rankings in the ever-evolving world of international soccer.

Key terms: Bradley-Terry, Paired Comparisons, Thurstone Model, Time-Depended, Correlation

Chapter 1: Introduction

1.1 Background of the Study

Aitchison (2020) defines paired comparison as the comparison of objects in pairs. One or more judges are shown many pairings of objects in a row. The judges in each pair are entrusted with deciding whether they favour one thing over the other based on a certain subjective trait, such as attractiveness or dominance. The preferences gathered serve as input data for the paired comparison model. The model then creates weights for each object, which are then used to rank the objects based on the feature of interest. These weights can even be used to forecast outcomes in some models (Aitchison, 2020).

Paired comparison data are commonly found in various fields such as marketing and psychology, where individuals compare two objects to judge an unmeasurable attribute. Another example of paired data can be found in sports, when teams or players compete in pairs. Time-dependent paired comparison data, which are repeatedly collected over time, may be influenced by the previous comparisons made, making modelling these data essential for understanding decision-making dynamics and predicting future choices (Smith & Haisley, 2018).

According to Sjölander (2011), the purpose for comparing objects is to avoid a concept called as sensory weariness, which is drawn from "Litvine (2004)." This method reveals a person's lack of concentration and perplexity when he or she is required to rate more than two objects at the same time. Comparing things in pairs facilitates decision-making. It lessens individuals' cognitive burden and makes tasks more bearable.

In the present investigation, the term "teams" will be employed to denote the objects being studied, specifically sports teams. The judgments in this context will be derived from the scores obtained during matches played between two teams. These matches, acting as judges, determine the preferred team (object) in each instance. As a result, the study intends to assign weights to teams (objects) and so rank them depending on their performance in the sport.

A "balanced paired-comparison experiment" is referred to in a larger meaning when each judge is assigned to undertake all feasible paired comparisons either once or a predefined number of times (David, 1963). In the world of sports, this is known as a Round Robin tournament, in

which each team competes against every other team once or twice in the case of a double Round Robin tournament. A Round Robin event is exemplified by the RBS Six Nations tournament.

Paired comparison data can be found in a wide range of settings. Bradley (1976), Davidson and Farquhar (1976), and David (1988) provided more references to a range of modern uses. The Bradley-Terry and Thurstone models, despite their simplicity, have found broad use. Choisel and Wickelmaier (2007) investigated paired sound assessments using a standard Bradley-Terry model, whereas Bäuml (1994) and Kissler and Bäuml (2000) offer applications adding facial attractiveness. Mazzucchi, Linzey, and Bruning (2008) used the standard Bradley-Terry approach to solve a reliability problem. To assess the likelihood of wire failure as a function of influencing factors in an aircraft environment, a panel of wiring experts is asked to determine which scenario is the riskier when many scenarios are compared pairwise. Stigler (1994) uses the traditional Bradley-Terry methodology to rank scientific publications, but Sham and Curtis (1995) used the same model in genetics.

Many apps feature multiple possible comparison outcomes. Henery (1992) extended the Thurstone model for chess player rating to three outcomes: victory, draw, and defeat. Böckenholt and Dillon (1997a) offer a five-category model for beverage taste testing and cigarette brand preferences. Dittrich, Hatzinger, and Katzenbeisser (2004) investigated motives for starting a Ph.D. program using the log-linear variation of the Bradley-Terry model.

It is frequently intriguing to investigate whether particular aspects influence comparison results. Ellermeier, Mader, and Daniel (2004) studied paired sound assessments using a Bradley-Terry model and integrate sound-related covariates such as roughness, sharpness, and so on to identify which of these contribute to the unpleasantness of sounds. Duineveld et al. (2000) use the log-linear formulation of the Bradley-Terry model to investigate consumer preference data on orange soft drinks, including an analysis of the factorial design for the drinks compared, whereas Francis et al. (2002) include subject-specific covariates in the analysis of value orientation of people in different European countries.

In zoological data, the Bradley-Terry model is used to investigate animal behavior traits while accounting for species-specific factors (Stuart-Fox et al., 2006; Whiting et al., 2006; Head et al., 2008). Agresti (2002) extended the Bradley-Terry model to account for the home advantage impact in baseball statistics in Chapter 10. Observational reliance is sometimes more realistic. In sports data, for example, object-specific random effects could be used to establish correlation

between comparisons with common things (Cattelan, 2009). When all judges perform all paired comparisons, random effects can induce correlation between preferences given by the same subject about a shared object, as demonstrated by Böckenholt and Tsai (2007) for the university preference data. When performing paired comparisons over long time periods, it may be necessary to account for it.

Because the goal is to predict the outcomes of future matches, McHale and Morton (2011) estimated a Bradley-Terry model in which distant tennis matches are down-weighted. More dynamic extensions for sports data were provided by Barry and Hartigan (1993), Fahrmeir and Tutz (1994), Knorr-Held (2000), and Cattelan, Varin, and Firth (2012). In tournaments, a player can win all of the comparisons in which he participates. In this case, a traditional Bradley-Terry or Thurstone model would estimate an infinity worth parameter for this team. Mease (2003) proposes a likelihood penalization to address this issue. In this scenario, Firth's (1993) method for decreasing the bias of maximum likelihood estimates gives an alternative strategy for producing limited estimates. Finally, Stern (2011) investigated the issue where the margin of victory in sporting contests is continuous rather than discrete. Dittrich et al. (2012) used the log-linear specification of the Bradley-Terry model to account for missing data in a study on the characteristics of a good teacher. Below we will look at the set of objectives for this study.

1.2 Aim and Objectives

Aim

The primary aim of this study is to develop, evaluate, and validate time-dependent paired comparison (TDPC) models for ranking soccer teams and players.

Objectives

- 1 To test the existing models using soccer data.
- 2 To create a new time-series paired comparison model and also test it on soccer data.
- 3 Check how the time-series models compare to each other to the official soccer rankings using predictions.

A brief explanation of Time-dependent paired comparison is necessary to understand how comparison can be time-dependent (1.3).

1.3 Time-Depended Paired Comparison

Time-dependent paired comparison (TDPC) is a statistical tool used in a variety of fields, including psychology, economics, and sports. The method involves presenting pairs of stimuli to participants at different time periods and asking them to choose which of the two stimuli they prefer. Each choice and time-period in which it occurred, are then used to estimate how the underlying preferences of the participants change over time.

TDPC assumes that preferences are time-dependent, meaning that they can change over time due to various factors such as mood, context, and experience. This assumption is in contrast to the traditional paired comparison method, which assumes that preferences are stable and do not change over time.

TDPC involves presenting pairs of stimuli at different points in time and asking participants to choose which of the two stimuli they prefer at each time point. The choices are then used to estimate the underlying preferences of the participants over time. The TDPC data can be analysed using a variety of statistical models, such as the BT model, the rank-based model, and choice-based conjoint analysis (Bradley and Terry, 1952).

One commonly employed approach for analysing the data obtained from TDPC is the use of statistical models. The Bradley-Terry (BT) model is one such model frequently applied in this context (Bradley and Terry, 1952). This model assumes that the probability of an individual choosing one option over another depends on the relative subjective value of the two options. The BT model has been frequently utilized in paired-comparison data processing and has been shown to be successful in capturing the temporal dynamics of reward preferences in TDPC experiments.

The rank-based model is another statistical model used in the analysis of TDPC data. This model considers the ranks of the chosen stimuli over time and aims to capture the ordinal nature of the choices made by participants. By focusing on the temporal ordering of preferences, the rank-based model provides insights into the stability and consistency of individuals' choices across different time points in TDPC.

Choice-based conjoint analysis is another statistical method utilized in the analysis of TDPC data. This approach involves assessing how different attributes of stimuli contribute to participants' choices. By considering the impact of various stimulus features on decision-

making, choice-based conjoint analysis allows researchers to explore the underlying factors influencing temporal discounting behaviour.

In summary, TDPC is a valuable tool for investigating temporal discounting of delayed rewards. The Bradley-Terry model, rank-based models, and choice-based conjoint analysis are among the statistical approaches that can be employed to analyse the data generated by TDPC experiments, providing insights into the temporal dynamics of individuals' preferences over time (Bradley and Terry, 1952).

1.4 Application of Time-Depended Paired Comparison

Time-Depended Paired Comparisons (TDPC) has been utilized in a wide range of applications; in this study we are going to apply TDPC in sports data. Sports: TDPC has been used to study the performance of athletes over time and to evaluate the effectiveness of different training programs.

The TDPC method assumes that preferences are time-dependent, and it involves presenting pairs of stimuli at different points in time to estimate underlying preferences. TDPC has been used in psychology, economics, sports, and marketing, among other fields, and it has the potential to provide valuable insights into the dynamics of decision-making and behaviour (Luce and Tukey, 1964).

Time-Dependent Paired Comparison (TDPC) stands as a potent statistical method widely employed across diverse disciplines to probe into the nuanced dynamics of preferences unfolding over time. The foundational premise of TDPC is grounded in the recognition that preferences are not static but inherently time-dependent. This methodology involves the strategic presentation of pairs of stimuli at various temporal junctures, tasking participants with selecting their preferred option at each instance. Through this process, researchers aim to derive a nuanced understanding of the underlying temporal preferences shaping decision-making and behaviour (Luce and Tukey, 1964).

In the realm of psychology, TDPC has proven to be an invaluable tool for investigating the temporal dynamics of decision-making processes. The method facilitates the exploration of how individuals weigh immediate rewards against delayed gratification, offering insights into the intricate interplay of cognitive, emotional, and motivational factors influencing preferences over time. The application of TDPC in psychological research contributes to a deeper

understanding of phenomena such as impulse control, self-regulation, and the cognitive processes involved in delayed reward discounting (Luce and Tukey, 1964).

Economically, TDPC has found utility in the study of intertemporal choices and financial decision-making. By analysing how individuals discount the value of future rewards over time, researchers gain valuable insights into economic behaviours such as savings patterns, investment decisions, and consumer choices. The temporal dimension introduced by TDPC enriches economic models, allowing for a more comprehensive exploration of decision-making dynamics in financial contexts (Luce and Tukey, 1964).

In the realm of sports, TDPC emerges as a valuable tool for understanding the preferences of athletes regarding immediate versus delayed rewards. This has implications for training regimens, performance strategies, and contractual negotiations. By elucidating how athletes evaluate short-term gains against long-term benefits, TDPC contributes to a deeper comprehension of the motivational factors driving decision-making in the sports arena (Luce and Tukey, 1964).

The application of TDPC extends to the field of marketing, where it provides a lens through which to examine the temporal dynamics of consumer preferences. By assessing how individuals discount the value of future product benefits over time, marketers can tailor strategies to align with shifting consumer behaviours. TDPC's contribution to marketing research lies in its ability to inform pricing models, product development, and advertising strategies by unravelling the temporal intricacies of consumer decision-making.

The seminal work of Luce and Tukey in 1964 laid the groundwork for TDPC as a robust methodological approach for studying preferences over time. Their contribution underscores the enduring significance of TDPC in scientific inquiry, highlighting its potential to yield valuable insights into the complex and dynamic nature of decision-making and behaviour across various domains (Luce and Tukey, 1964).

Modelling TDPC data provides a number of benefits, including the ability to capture temporal dynamics, predict future preferences, and test hypotheses. TDPC data are prevalent in a wide range of applications, and modelling these data can provide valuable insights into the dynamics of decision-making and behaviour. By using sophisticated statistical models, researchers and practitioners can gain a deeper understanding of how preferences change over time and how they can be influenced by various factors (Tong and Yao, 2017).

To illustrate the benefits of TDPC models, we will use an example of TDPC data collected from a consumer testing study. The study involved testing four different brands of cereal (O_1, O_2, O_3, O_4) over three consecutive days, with ten judges asked to indicate their preference for one of the brands for the various pairs of brands presented to them. The data from the study is shown in Table 1.1. In Table 1.2 the data is organised into a matrix form suitable for input into the TDPC model. Table 1.3 shows the rankings of the objects (cereal brands) for each day based on the preferences recorded in the TDPC model.

Table 1.1: TDPC Questionnaire Data from a Consumer Testing Study for one of the ten judges

Day	$O_1 O_2$	$O_1 O_3$	$O_1 O_4$	$O_2 O_3$	$O_2 O_4$	$O_3 O_4$
1	O_2	O_3	O_1	O_2	O_2	O_3
2	O_2	O_3	O_1	O_3	O_2	O_3
3	O_1	O_3	O_1	O_3	O_2	O_3

Table 1.2: TDPC Data in form to input into model from a Consumer Testing Study for one of the ten judges

Day 1					Day 2					Day 3				
	O_1	O_2	O_3	O_4		O_1	O_2	O_3	O_4		O_1	O_2	O_3	O_4
O_1	0	0	0	1	O_1	0	0	0	1	O_1	0	1	0	1
O_2	1	0	1	1	O_2	1	0	0	1	O_2	0	0	0	1
O_3	1	0	0	1	O_3	1	1	0	1	O_3	1	1	0	1
O_4	0	0	0	0	O_4	0	0	0	0	O_4	0	0	0	0

(1 if object on the left is preferred over object on the top)

Table 1.3: Model output ranking the objects from a Consumer Testing Study

Ranks	Day 1	Day 2	Day 3
O_1	3	3	2
O_2	1	2	3
O_3	2	1	1
O_4	4	4	4

To model this data, they used the dynamic Bradley-Terry model with a random effect for each participant. The model assumes that each participant has a latent preference score for each brand, which evolves over time according to a random walk process. The probability of participant i choosing brand A over brand B on day t is given by:

$$P(\text{judge } h \text{ chooses } O_i \text{ over } O_j \text{ on day } t) \quad (1.1)$$

$$= \exp(\theta_{hO_i}(t)) / (\exp(\theta_{hO_i}(t)) + \exp(\theta_{hO_j}(t))),$$

where θ_{hO_i} and θ_{hO_j} are the latent preference scores for judge h ($h = 1, \dots, 10$) for brands O_i and O_j respectively ($i, j = 1, \dots, 4$), on day t ($t = 1, 2, 3$).

1.5 Method of Paired Comparison

The method of paired comparisons was introduced by Fechner in 1860, who was a German philosopher and pioneer in psychophysics, studying human perception and establishing a connection between mental and material phenomena (Fechner, 1860; Hawkins, 2011). In 1927, Louis Leon Thurstone laid the foundation for the mathematical aspects of paired comparisons in psychometrics, particularly in his publication titled "A Law of Comparative Judgment." Thurstone proposed the concept of a psychological continuum in this work to measure and evaluate various stimuli in pairs, such as weights or handwriting examples (Thurstone, 1927).

According to Sjölander (2011), it is conceivable to combine time series analysis and the approach of paired comparisons. Sjölander created three new time series paired comparison models by incorporating time-dependent factors into existing models. These models were solved using the maximum likelihood method. Sjölander tested the models' performance using

both exact and randomly generated data, altering the time periods and number of teams involved. The models were evaluated by estimating missing temperature data, which involved removing observations from a temperature dataset for a specified region from 2006 to 2008. To approximate the missing data points, various sorts of time series models were used.

The paired comparison strategy is most usually used when the things to be compared can only be evaluated subjectively; that is, when taking appropriate measurements to determine which of two objects is preferred is difficult or impractical. As one might assume, psychometricians frequently use paired comparisons, and Thurstone (1927) was the first to introduce the approach. Taste testing, color comparisons, people rating, and general preference testing have been the most prevalent applications. Of course, there are other methods of sensory discernment, and it is not meant to go into detail about their individual advantages, especially since a number of concise reports have recently been presented. Examples include Jones and Bock (1957), Torgerson (1958), and Bliss (1958). When comparing different types of razors on a man's two cheeks, the approach of paired comparisons is sometimes the only viable experimental procedure. It is sometimes possible for a judge to compare multiple items at the same time, and if this is practicable, the judge should award ranks to all of these objects. When the differences between things are minor, it is ideal to make the comparison as free of external effects as possible, such as those provided by studying other objects concurrently. As a result, when a fine judgment is required, the strategy of paired comparisons will be used. Again, it is sometimes impossible for a judge to deal with more than two flavours in taste testing, and the addition of a third flavour can be exceedingly confounding.

Ranking will surely be faster when both paired comparisons and ranking are viable options for sorting numerous things in order of preference. In contrast, the approach of paired comparisons permits the judge to contradict himself; for example, he may choose A above B, B over C, and C over A. Because it is not impossible, Kendall refers to this circumstance as a cyclical triad. A classic example is the game of stone, scissors, and paper. It is self-evident that a judge who commits many more circular triads than another is a less consistent judge. As a result, we now have a solid foundation for creating a strategy for identifying qualified judges. A circular triad can be explained by the judge just guessing or by modifying the criterion on which his judgment is based when making the three comparisons. In other words, the preference scale might not be one-dimensional. A preference may be based on a variety of item features, which are presumed

weighted in some way in the judge's mind before making a decision. The weights granted by an unskilled judge may vary from comparison to comparison.

1.6 Time-Series Models

Sjölander's time series models are based on the Bradley-Terry model, hence we must first introduce the Bradley-Terry model in our literature study (Bradley & Terry, 1952). The Bradley-Terry model serves as a probabilistic framework for predicting comparison outcomes and can be viewed as a specific instance of the Linear model. Note that the word “linear” here refers to a type of Paired Comparisons model, not to a Time Series Paired Comparisons model that employs a linear function of time to parameters. In the literature review we will go through some of the time-series models and Paired Comparison models.

A time series is a sequence of time periods that are typically measured at uniformly spaced time intervals. Daily sales numbers for a commodity, the number of strikes each year in South Africa, the population of South Africa recorded at ten-year intervals, and daily exchange rates between currencies are all examples of time series. The process of analysing time series data in order to extract meaningful statistics and other data attributes is known as time series analysis. The employment of a model to estimate future values based on historical values is known as time series forecasting (Mabaso, 2012).

Time series analysis methodologies are classified as parametric or non-parametric. The parametric techniques are based on the assumption that the underlying stationary stochastic process has a specific distribution that can be represented using a set of parameters. In these approaches, the goal is to estimate the model parameters that define the stochastic process. Non-parametric approaches, on the other hand, explicitly estimate the process's covariance or spectrum without making any assumptions about the underlying distribution. Time series analysis methods are further classified as linear and non-linear, univariate and multivariate (Mabaso, 2012).

Time Series Model with Time Period as the Dependant Variable is possibly the most fundamental time series model. It can be based on either a basic linear relationship (trend) in the time series or a more sophisticated non-linear relationship, such as the quadratic trend (Mabaso, 2012). Consider a time series X_t and write $X_t = a + bt$, which is a simple linear regression form with parameters a and b and one independent variable time, t . In other words, we build or estimate a linear trend or relationship in the time series and forecast using

the fitted model. When estimating the parameters a and b , the Ordinary Least Squares (OLS) technique is utilized, just as it is in conventional linear regression. The parameters' OLS estimations are given by:

$$b = \frac{\sum_{i=0}^{T-1} (X_i - \bar{X})(i - \bar{i})}{\sum_{i=0}^{T-1} (i - \bar{i})^2}$$

$$a = \bar{X} - b\bar{i}; \quad \bar{X} = \frac{1}{n} \sum_{i=0}^{T-1} X_i, \quad i = 1, 2, \dots, T - 1,$$

where \bar{X} is the sample mean of the observations in the time series and i represent the time periods.

1.7 Research layout

This section outlines the major components and steps involved in the research process, ensuring a clear and systematic approach to achieving the study's objectives. The **introduction** presents the background of paired comparisons and the application. The **literature review** covers existing theories, models, and research related to time-dependent paired comparisons (TDPC) and their applications, particularly in sports, providing a comprehensive overview of previous work and highlighting gaps that the current study aims to fill. The **methodology** section details the research design, data collection methods, and analytical techniques used, explaining how soccer data is collected and how existing and new TDPC models are tested and validated. The **data analysis** involves applying the selected statistical models to the soccer data, including the results of the analysis, comparisons between different models, and a discussion of the findings in relation to the study's objectives. The **results and discussion** section presents the results of the data analysis in detail, discussing the performance of the models, comparing their predictions with official soccer rankings, and exploring the implications of the findings. The **conclusion** summarizes the key findings of the study, highlighting the most important insights and their significance, and provides recommendations for future research, suggesting ways to address limitations and improve the accuracy and applicability of TDPC models in sports.

In Chapter 2 we are going to discuss in depth the paired comparisons and time-series models. Using formulas to explain other models and also sharing other authors methodologies based on paired comparisons. After chapter 2, we will then move to chapter 3, where we will be applying our real soccer data (World Cup 1930 – 2022) to our model of choice (Linear Exponential AHP; Linear Bradley-Terry Model and Linear Thurstone model which is the new model, and

extension from the normal Thurstone model). Then in Chapter 4 we are going to perform a hypothesis testing in the model using Kendall tau; Pearson and Spearman correlation methods; chapter 5 will be our conclusion and lastly our recommendation and limitation of the study.

Chapter 2: Literature Review

2.1 Introduction

Paired comparison is the comparison of two objects. One or more judges are shown many pairings of objects in a row. The judges in each pair are entrusted with deciding whether they favour one thing over the other based on a certain subjective trait, such as attractiveness or dominance. The preferences gathered serve as input data for the paired comparison model. The model then creates weights for each object, which are then used to rank the objects based on the feature of interest. These weights can even be used to forecast outcomes in some models (Aitchison, 2020).

The bibliography produced by Davidson and Farquhar (1976), which contains nearly 350 papers on paired comparison data, demonstrates the widespread interest in this type of data. This fascination endures, and model extensions for paired comparison data have been developed. This section focuses on contemporary extensions of the two traditional models, Thurstone (1927) and Bradley-Terry (Bradley and Terry, 1952), particularly those following Bradley's (1976) review and David's (1988) monograph, as well as work in the statistical and psychometric field.

Paired comparison data are commonly found in various fields such as marketing and psychology, where individuals compare two objects to judge an unmeasurable attribute. Another application of paired comparisons data can be seen in sports, where teams or players compete in pairs. Time-dependent paired comparison data, which are repeatedly collected over time, may be influenced by the previous comparisons made, making modelling these data essential for understanding decision-making dynamics and predicting future choices (Smith & Haisley, 2018).

Fechner introduced the paired comparisons method in 1860. The originator of psychophysics and a German philosopher, he studied human perception. Fechner proposed that the material (or physical) facts and the psychical (or mental) facts have a functional relationship (Fechner, 1860; Hawkins, 2011). In 1927, Louis Leon Thurstone's psychometric publications, including "A Law of Comparative Judgment," gave the field of paired comparisons a mathematical foundation.

According to Sjölander (2011), the purpose of comparing objects is to avoid a concept called as sensory weariness, which is borrowed from "Litvine (2004)." This method implies a lack of concentration and confusion that is said to occur when a person needs to rate more than two objects at the same time. Comparing objects in pairs simplifies the decision-making process. It reduces the cognitive load on individuals and it also make tasks more manageable.

In the present investigation, the term "teams" will be employed to denote the objects being studied, specifically sports teams. The judgments in this context will be derived from the scores obtained during matches played between two teams. These matches, acting as judges, determine the preferred team (object) in each instance. As a result, the study aims to determine the weights and thus the rankings of the teams (objects) based on their performance in the sport.

A "balanced paired-comparison experiment" is referred to in a larger meaning when each judge is assigned to undertake all feasible paired comparisons either once or a predefined number of times (David, 1963). In the world of sports, this is known as a Round Robin tournament, in which each team competes against every other team once, or twice in the case of a double Round Robin tournament. A Round Robin event is exemplified by the RBS Six Nations tournament. In section 2.2 we will now look at some Paired Comparison Models used in the typical Paired Comparison scenario as well as in Sport Statistics.

2.2 Some Paired Comparison Models

Paired comparison models are widely used in various domains, including decision-making, preference analysis, and ranking tasks. This literature review aims to provide an overview of some prominent paired comparison models proposed in the literature. The review highlights the underlying principles and applications of these models. By examining and comparing these models, this review offers insights into their suitability for different scenarios and identifies potential areas for future research and development (Smith, 2020).

Thurstone's Law of Comparative Judgment proposes that individuals make pairwise comparisons based on latent psychological scales. This model assumes that the preferences follow a normal distribution and estimates the item's location on the latent scale. Thurstone's model has been utilized in various domains such as psychophysics, market research, and social sciences. However, it requires a large number of pairwise comparisons to accurately estimate the latent scales, limiting its practicality in certain situations (Thurstone, 1927).

The Bradley-Terry model is one of the earliest and most widely used paired comparison models. It assumes that the probability of one item being preferred over another depends on the relative strengths of the two items. This model has been extensively applied in sports rankings, elections, and marketing research. It provides a simple and intuitive framework for modelling preferences but assumes independence of comparisons and does not consider uncertainty in the estimates (Bradley & Terry, 1952).

The Analytic Hierarchy Process is a decision-making method that incorporates pairwise comparisons to determine the relative importance of criteria and alternatives. It employs a hierarchical structure to organize the decision problem and elicit preferences. AHP has been extensively used in multi-criteria decision analysis, project management, and supplier selection. The model's strength lies in its ability to handle complex decision problems involving multiple criteria, but it relies heavily on subjective judgments and can be sensitive to inconsistent pairwise comparisons (Mabaso, 2012).

Paired comparison models offer valuable tools for ranking, decision-making, and preference analysis. The Bradley-Terry model, Thurstone's Law, AHP, and Haines Litvine model represent some of the prominent models discussed in this literature review. Each model has its strengths and limitations, making it suitable for specific contexts and applications. Understanding the underlying principles and assumptions of these models can guide researchers and practitioners in selecting the most appropriate model for their specific needs. Further research is needed to explore extensions, hybrid models, and integration of these models with other decision-making frameworks (Bradley & Terry, 1952).

2.2.1 Thurstone-Mosteller Models

Thurstone-Mosteller models are a class of statistical models widely employed in psychometrics and social sciences for measuring latent traits and analysing categorical data. This paper provides a comprehensive review of Thurstone-Mosteller models, exploring their theoretical foundations, historical development, and practical applications. Thurstone-Mosteller models, named after the eminent psychometricians L.L. Thurstone and R.D. Mosteller, have gained significant prominence in psychometrics and related fields. These models offer a robust framework for analysing data with categorical responses and estimating latent traits.

Thurstone-Mosteller models offer several advantages, such as their ability to handle categorical data, their focus on latent traits, and their potential for providing meaningful measurement

invariance across different groups. However, they also have certain limitations, including assumptions of local independence and one-dimensionality. Thurstone-Mosteller models have found applications in various fields, such as education, psychology, marketing research, and public opinion studies. In this section we shall apply Thurstone-Mosteller on “Soccer Data”.

Sjölander, (2011) outlines the main assumptions of the Thurstone-Mosteller model as follows:

1. There are N stimuli (from N objects O_i) that can be placed on a subjective continuum.
2. When a stimulus is presented to a person, the subject feels a sensation.
3. A population of individuals' feeling distribution from a specific stimulus is normal.
4. Stimuli are presented to a person in pairs or to a group of people, resulting in a sensation for each stimulus.
5. The individual compares these sensations and indicates which is the strongest.
6. These paired sensations are most likely connected.
7. With the exception of a linear translation, the goal is to rank the stimuli (what the sensation signifies).

Method

The people who make the comparisons are known as judges.

Let variables Y_i and Y_j represent sensations with respect to stimuli of objects i and j (O_i and O_j), respectively. The Thurstone-Mosteller model is based on the following mathematical assumptions:

$$E(Y_i) = V_i, \quad i = 1, 2, \dots, N$$

$$V(Y_i) = \sigma^2, \quad i = 1, 2, \dots, N$$

$$\rho(Y_i, Y_j) = \rho, \quad i, j = 1, 2, \dots, N$$

where ρ stands for the correlation coefficient.

Clearly, $V(Y_i - Y_j) = 2\sigma^2(1 - \rho)$, but we can set $V(Y_i - Y_j) = 1$ without losing generality, because the weights V_i are specified subject to an arbitrary linear transformation, and this convention will have no observable effect.

The Thurstone-Mosteller model's preference probabilities can be calculated as follows:

$$\begin{aligned}
 \pi_{ij} &= P\{O_j \rightarrow O_i\} \\
 &= P\{Y_i - Y_j > 0\} \\
 &= \Phi(V_i - V_j) \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{V_i - V_j} \exp\left(-\frac{x^2}{2}\right) dx,
 \end{aligned} \tag{2.1}$$

where $\Phi(\cdot)$ is said to be the cumulative distribution function (CDF) of the Standard Normal distribution.

As required by the principle 6, the objective is to estimate the V_i 's, which are the true weights of the objects (except for a linear transformation).

Since the origin of the scale for V_i may be selected arbitrarily, we add for certainty the following condition, which will guarantee the uniqueness of the estimates:

$$\sum_{i=1}^N V_i = 0.$$

$$\text{Let } \delta_{ij} = V_i - V_j \quad (i, j = 1, 2, \dots, N; j \neq i),$$

Then $\pi_{ij} = \Phi(\delta_{ij})$ and we have:

$$\sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij} = (N - 1)V_i - \sum_{\substack{j=1 \\ j \neq i}}^N V_j \Leftrightarrow V_i = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}, \tag{2.2}$$

Assume that the data set is balanced, and that n is the number of comparisons for each pair. We shall denote $p_{ij} = \frac{x_{ij}}{n}$, where x_{ij} represents the number of comparisons in which O_i was preferred over O_j in n comparisons. The variables d_{ij} are defined by equations:

$$\Phi(d_{ij}) = p_{ij}.$$

Thus, the estimated unknown weights:

$$v_i = \frac{1}{N} \sum_{j=1}^N d_{ij} \text{ for } i = 1, 2, \dots, N, \quad (2.3)$$

where $d_{ij} = \Phi^{-1}(p_{ij})$.

Computational procedure

The approach begins with constructing the sample preference matrix P, which consists of p_{ij} = the relative frequency of i being preferred to j out of the n judgments $i, j = 1, 2, \dots, N$.

As a result, this matrix changed into D, which has elements d_{ij} determined as values of function Φ^{-1} at positions p_{ij} .

Finally, we compute the unknown weight estimates v_i 's as the averages of the rows of the matrix D (diagonal components of D are set to zero).

The v_i 's values are used to order the objects.

When $p_{ji} = 1$ and $p_{ij} = 0$, some challenges in defining the matrix P arise. Such values are only achievable if the object O_i was favoured over the object O_j in all comparisons. In such circumstances, d_{ij} cannot be determined because

$$d_{ij} = \Phi^{-1}(p_{ij}) = \Phi^{-1}(0) = -\infty,$$

and

$$d_{ji} = \Phi^{-1}(p_{ji}) = \Phi^{-1}(1) = \infty.$$

To remedy this problem, David suggests replacing the original value of p_{ij} with $\frac{1}{2n}$, while p_{ji} is replaced with $1 - \frac{1}{2n}$, where n is the total number of comparisons for this pair.

2.2.2 Linear Trend Thurstone-Mosteller Models

The Linear Trend's Fundamental Principles of the Thurstone-Mosteller model is as follows when $t = 0, 1, \dots, T - 1$:

1. A subjective continuum is defined by a set of N stimuli (from N objects O_i).

2. When a stimulus is offered to an individual, the subject experiences a sensation.
3. The distribution of sensations from a specific stimulus is normal for a population of individuals.
4. Stimuli are offered to a person in pairs n times or to n individuals, resulting in a sensation for each stimulus. The person compares these sensations and reports which is stronger.
5. It's probable that these paired sensations are related.
6. Except for a linear translation, the aim is to space the inputs (the sensory means)

Method

The people who make the comparisons are typically known as judges or experts.

Let variables Y_i^t and Y_j^t represent sensations with respect to stimuli of objects i and j (O_i and O_j) for each time $t = 0, 1, \dots, T - 1$. The Thurstone-Mosteller model is based on the following mathematical assumptions:

$$E(Y_i^t) = V_i^t, \quad i = 1, 2, \dots, N$$

$$V_i^t = A_i + B_i t \text{ for each time } t = 0, 1, \dots, T - 1$$

$$V(Y_i) = \sigma^2, \quad i = 1, 2, \dots, N$$

$$\rho(Y_i, Y_j) = \rho, \quad i, j = 1, 2, \dots, N,$$

where ρ stands for the correlation coefficient.

$V(Y_i^t - Y_j^t) = 2\sigma^2(1 - \rho)$, but we can set $V(Y_i^t - Y_j^t) = 1$ without losing generality, because the weights V_i^t are specified subject to an arbitrary linear transformation, and this convention will have no obvious effect.

The Thurstone-Mosteller model's preference probabilities can be calculated as follows: $\pi_{ij}^t =$

$$\begin{aligned}
 &P\{O_j \rightarrow O_i \text{ at time } t\} \\
 &= P\{Y_i^t - Y_j^t > 0\} \\
 &= \Phi(V_i^t - V_j^t) \tag{2.4}
 \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{V_i^t - V_j^t} \exp\left(-\frac{x^2}{2}\right) dx.$$

where $\Phi(\cdot)$ is said to be a cumulative distribution function (CDF) of the Standard Normal distribution.

As required by principle 6, the objective is to estimate the V_i^t 's, which are the true weights of the objects (except for a linear transformation) for each time $t = 0, 1, \dots, T - 1$.

Since the origin of the scale (at time 0) for V_i^t may be selected arbitrarily and how each V_i^t 's differ over time is relative to how the V_i^t 's differs over time, we add for certainty the following 2 conditions, which will guarantee the uniqueness of the estimates:

$$\sum_{i=1}^N A_i = 0, \quad \sum_{i=1}^N B_i = 0 \quad (\text{Thus } \sum_{i=1}^N V_i^t = 0 \text{ for each time } t = 0, 1, \dots, T - 1.)$$

$$\text{Let } \delta_{ij}^t = V_i^t - V_j^t \quad (i, j = 1, 2, \dots, N; j \neq i. t = 0, 1, \dots, T - 1.)$$

Then $\pi_{ij}^t = \Phi(\delta_{ij}^t)$ and we have:

$$\sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = (N - 1)V_i^t - \sum_{\substack{j=1 \\ j \neq i}}^N V_j^t \Leftrightarrow V_i^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t)$$

$$\text{assuming } \sum_{i=1}^N A_i = 0 \text{ and } \sum_{i=1}^N B_i = 0 \therefore \sum_{i=1}^N V_i^t = 0$$

$$V_i^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t) \Leftrightarrow A_i + B_i t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t)$$

We now use the least squares approach (multivariate multiple regression) to estimate the A_i 's and B_i 's, which yield the following results:

$$A_i \approx \frac{2(2T-1)}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t - \frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \delta_{ij}^t$$

$$\text{i.e. } A_i \approx \frac{2(2T-1)}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t) - \frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \Phi^{-1}(\pi_{ij}^t)$$

$$B_i \approx -\frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t + \frac{12}{(T-1)T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \delta_{ij}^t$$

$$\text{i.e. } B_i \approx -\frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t) + \frac{12}{(T-1)T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \Phi^{-1}(\pi_{ij}^t)$$

Suppose the data set is balanced and n represents the number of comparisons for each pair. We will designate $p_{ij}^t = \frac{x_{ij}^t}{n}$ where x_{ij}^t stands for the number of comparisons when the O_i was preferred to O_j , in n comparisons for each time $t = 0, 1, \dots, T - 1$. The variables d_{ij}^t are defined from equations:

$$\Phi(d_{ij}^t) = p_{ij}^t$$

Thus, the estimates of the unknown weights of each object i at time t are $v_i^t = a_i + b_i t$, where:

$$a_i = \frac{2(2T - 1)}{T(T + 1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N d_{ij}^t - \frac{6}{T(T + 1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t d_{ij}^t$$

$$b_i = -\frac{6}{T(T + 1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N d_{ij}^t + \frac{12}{(T - 1)T(T + 1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t d_{ij}^t$$

for $i = 1, 2, \dots, N$ for each time $t = 0, 1, \dots, T - 1$, where $d_{ij}^t = \Phi^{-1}(p_{ij}^t)$.

Computation Procedure

The computational procedure begins with compiling the sample preference matrices P^t , consisting of p_{ij}^t = relative frequency of O_i being preferred to O_j out of the n judgments at each time $t = 0, 1, \dots, T - 1$ for $i, j = 1, 2, \dots, N$.

Consequently, these matrices are transformed into the matrices D^t , which have elements d_{ij}^t calculated as values of function Φ^{-1} at points p_{ij}^t .

Define matrices $E^t = tD^t$, and let matrices SD and SE be the sum of matrices D^t and E^t respectively.

$$\text{Let } c = \frac{2(2T-1)}{T(T+1)}, \text{ d} = -\frac{6}{T(T+1)} \text{ and } e = \frac{12}{(T-1)T(T+1)}.$$

Finally, we calculate the a_i 's and b_i 's of (and thus estimates of the unknown weights at each time t as $v_i^t = a_i + b_i t$) as:

$c \times$ average of rows of matrix $SD + d \times$ average of rows of matrix SD and $d \times$ average of rows of matrix $SD + e \times$ average of rows of matrix SD respectively. (We set the diagonal elements of SD and SE to zero).

The v_i^t 's are used to rank the objects at each time $t = 0, 1, \dots, T - 1$

Certain difficulties in defining the matrices P_{ji}^t arise when $p_{ji}^t = 1$ and $p_{ij}^t = 0$. Such values are only possible if the object O_i was preferred to object O_j in all comparisons between them at time t . In such cases d_{ij} cannot be calculated, because

$$d_{ij} = \Phi^{-1}(p_{ij}^t) = \Phi^{-1}(0) = -\infty,$$

and

$$d_{ji} = \Phi^{-1}(p_{ji}^t) = \Phi^{-1}(1) = \infty.$$

To resolve this difficulty, David recommends replacing the original value of p_{ij}^t with $\frac{1}{2n}$, while p_{ji}^t is replaced respectively with $1 - \frac{1}{2n}$, where n is the total number of comparisons for this pair.

2.2.3 Bradley-Terry Model

The Bradley-Terry model (Bradley & Terry, 1952) is a probability model that can forecast the outcome of a comparison. It is also a one-of-a-kind instance of the linear model. There will be N comparisons in a paired comparison experiment with N objects O_1, O_2, \dots, O_N .

The model is defined as follows:

$$P(Y_{ij} = 1 | p_i, p_j) = \frac{p_i}{p_i + p_j} \quad P(Y_{ij} = 0 | p_i, p_j) = \frac{p_j}{p_i + p_j} \quad (2.5)$$

where $Y_{ij} = 1$ and $Y_{ij} = 0$ imply the event that objects O_i is preferred to O_j and O_j is preferred to O_i respectively. When the p_i 's are unknown, non-negative weights approximate the position of the items on the relative scale to calculate the ranking (Bradley & Terry, 1952).

Starting with a linear model (Bradley & Terry, 1952), the density function $h(x)$ is allowed to be the density function of the squared hyperbolic secant distribution (which is very similar to the normal distribution), and $p_i > 0$ as $V_i = \log(p_i)$ for $i = 1, 2, \dots, N$. This linear model

is then used to generate the Bradley-Terry model. The preference probabilities are described as follows:

$$P_{ij} = \int_{v_j - v_i}^{\infty} h(x) dx = \int_{\log(p_j) - \log(p_i)}^{\infty} \frac{1}{4} \operatorname{sech}^2 \frac{x}{2} dx = \frac{p_i}{p_i + p_j}. \quad (2.6)$$

Bradley and Terry (1952) ranked the teams O_1, O_2, \dots, O_N after estimating the weights p_i using the maximum probability technique. If $X_{ij} = \sum_n Y_{ij\gamma}$, then X_{ij} represents the number of times team O_i has won over team O_j in its n comparisons. As a result, $X_{ij} \sim \text{Binomial}(n, p_{ij})$, where $p_{ij} = \frac{p_i}{p_i + p_j}$ is obtained from (6). To estimate $p_i + p_j$ the weights p_i ($i = 1, 2, \dots, N$), we can employ the maximum likelihood method.

2.3 Existing Time Series Models for Paired Comparisons

In this research study, we draw results using time series models for paired comparison developed by Sjölander (2011) as part of Sjölander's thesis. The Bradley-Terry and Linear models are two continuous models for paired comparisons, with the Bradley-Terry model being a subset of the Linear model. Sjölander extended the Bradley-Terry model with time series models that allowed the weights to change in a variety of ways throughout time. As a result, the time series models Linear-Trend Bradley-Terry, Sinusoidal Bradley-Terry, and Log Linear-Trend Bradley-Terry were developed. Sjölander and Litvine forecasted exchange rates and estimated missing temperature data using their models. In the following subsections, we will go over each of the three models in further detail.

2.3.1 Linear Trend Bradley-Terry Model

The Linear Trend Bradley-Terry Model is a variant of the Bradley-Terry model, which was first introduced in 1952 by Bradley and Terry. This model is commonly used for modeling paired comparison data, which compares two or more items with the purpose of estimating the relative strengths or preferences of these items. The Linear Trend Bradley-Terry Model adds a linear trend component to the original model, allowing for the analysis of time-dependent paired comparison data.

In the context of sports analytics, Hunter (2004) extended the Bradley-Terry model by introducing a linear trend component. This addition allowed for the modelling of time-

dependent changes in team strengths or player preferences over a season. The Linear Trend Bradley-Terry Model has been extensively applied in sports analytics to track changes in team strengths or player abilities over time. Researchers like Baio and Blangiardo (2010) used this model to analyse football league data.

By adding a time-dependent component, the Linear Trend Bradley-Terry Model enhances the original Bradley-Terry model. It recognizes that an entity's capabilities or strengths may alter over time as a result of a variety of variables, including adjustments in the team's makeup, injuries, coaching techniques, and more. It integrates a linear trend component into the latent abilities to model various temporal trends:

$$\theta_A(t) = \theta_A(0) + \beta_A * t, \quad (2.7)$$

where:

- $\theta_A(t)$ represents the latent ability of entity A at time t .
- $\theta_A(0)$ is the initial ability of entity A.
- β_A is the rate of change in the ability of entity A over time.
- t is the time variable.

To apply the Linear Trend Bradley-Terry model to real data estimate the model parameters, which include the initial $(\theta_A(0), \theta_B(0))$ and the rate of change of the parameters (β_A, β_B) for each entity. Finding the most accurate values for these parameters often entails using historical paired comparison data and statistical methods like maximum likelihood estimation or Bayesian inference.

The Linear Trend Bradley-Terry Model provides a more complex and realistic picture of how entities' abilities grow over time by including this time-dependent feature into the Bradley-Terry model. This makes it especially helpful for studying sports data, where performance might alter dynamically as a result of different circumstances, enabling better insights and forecasts.

Sjölander (2011) introduced the Linear-Trend Bradley-Terry Model. It is a time series version of the Bradley-Terry model (detailed in the preceding section), presuming that each judge evaluates each pair of items once where $t= 0, 1, \dots, T - 1$.

Sjölander (2011) considered $Y_{ij\gamma}^t(i, j = 1, 2, \dots, N; i \neq j; \gamma = 1, 2, \dots, n; t = 0, 1, \dots, T - 1)$ to be a random variable that takes on the value 1 if O_i is preferred in the t th comparison (by for example judge number γ if each judge judges each pair of items once each time period) of objects O_i and O_j at time t , and 0 if O_j is preferred in this comparison, then the extension process is done as follows:

$X_{ij}^t = \sum_{\gamma=1}^n Y_{ij\gamma}^t$ where X_{ij}^t is the number of times t that item O_i gets preferred in n comparisons with item O_j at time t . Therefore, $X_{ij}^t \sim \text{Binomial}(n, p_{ij}^t)$, where $p_{ij}^t = \frac{p_i^t}{p_i^t + p_j^t}$.

Sjölander (2011) defined Linear-Trend Bradley-Terry Model by allowing the object weights to change linearly over time, that is, $p_i^t = a_i + b_i t$, and in this case:

$$X_{ij}^t \sim \text{Binomial}(n, p_{ij}^t) \text{ where } p_{ij}^t = \frac{p_i^t}{p_i^t + p_j^t} = \frac{a_i + b_i t}{a_i + b_i t + a_j + b_j t}$$

$$\text{and } p(x_{ij}^t) = \binom{n}{x_{ij}^t} \left(\frac{a_i + b_i t}{a_i + b_i t + a_j + b_j t} \right)^{x_{ij}^t} \left(\frac{a_j + b_j t}{a_i + b_i t + a_j + b_j t} \right)^{n - x_{ij}^t}.$$

In order to determine the maximum probability estimates of the a_i 's and b_i 's (abbreviated as A_i 's and B_i 's) the following equations must be solved:

$$\sum_{t=0}^{T-1} (f_k^t - g_k^t) = 0 \text{ and } \sum_{t=0}^{T-1} t(f_k^t - g_k^t) = 0 \text{ for } k = 1, \dots, N,$$

where,

$$f_k^t = \frac{\sum_{j=1, j \neq i}^N x_{kj}^t}{A_k + B_k t} \text{ and } g_k^t = \sum_{j=1, j \neq k}^N \frac{n}{A_k + B_k t + A_j + B_j t}.$$

This approach was utilized by Sjölander (2011) to forecast the lack of temperature data. His research findings indicated that there were other data models that performed better than the Linear-Trend Bradley-Terry Model.

Another approach to this model, assume that judges will compare N items, O_1, \dots, O_N , n times. Let $Y_{ij\gamma}(i, j = 1, 2, \dots, N, i \neq j; \gamma = 1, 2, \dots, n)$ be a random variable that has a value of 1 if item O_i is preferred in this comparison and a value of 0 if object O_j is favoured. Assume that every comparison is statistically independent, with the exception of $Y_{ij} + Y_{ji} = 1$.

Suppose there's no judge impact (and no replication effect if some judges judge the same thing many times), therefore $p_{ij} = p_{ij\gamma}$ for all $i, j = 1, 2, \dots, n$. Let $P(Y_{ij\gamma} = 1) = p_{ij\gamma}$, with the only restrictions being that $0 \leq p_{ij} \leq 1$ and $p_{ij} = 1 - p_{ji}$.

Teams O_i, O_j and O_k can therefore be depicted as a straight line, with the distance between them on the line representing the preference probability between the two teams. This is possible for all teams. We can create a linear model by stretching the line to make sure that identical lengths on the line correspond to equivalent preference probabilities.

A linear model is said to be satisfied if the preference probabilities can be expressed in terms of a symmetrical cumulative distribution function (CDF) (David, 1963; Friskin, 2001). Let the merit of the teams be V_1, V_2, \dots, V_N . These genuine merits can be expressed on a scale by N points. The continuous random variable y_i can be used to represent the observed merit of O_i , which will vary for each observation, on the same scale, where $-\infty < Y_i < \infty$. O_i will be preferred to O_j if $Y_i > Y_j$.

Let $Z_i = Y_i - V_i$ for every $i, j = 1, 2, \dots, N$. $P(Z_i - Z_j(x)) = H(x)$ is how the cumulative distribution of $Z_i - Z_j$ should be written. (1963 David; 2001 Friskin). $Z_i - Z_j$ and $Z_j - Z_i$ will have the same distribution if Z_i is independent and equally distributed. Allow the cumulative distribution to run be represented by $(Z_i - Z_j(x)) = H(x)$ for $Z_i - Z_j$. P_{ij} therefore equals f_{ji} . in which $h(x)$ is the density formula.

2.3.2 Linear Exponential Analytical Hierarchy Process Model

The Linear Exponential Analytical Hierarchy Process (LE-AHP) Model is a development of the Analytical Hierarchy Process (AHP), a multi-criteria decision-making approach that is frequently applied in disciplines like operations research, decision analysis, and management. The LE-AHP model alters the fundamental AHP by applying a linear-exponential transformation, which can increase its adaptability to handle decision-making processes and better capture decision-makers' preferences.

According to Litvine (2004), the Haines-Litvine model for paired comparisons is a continuous stochastic model that was inspired by the AHP. The main difference between the AHP and the Haines-Litvine Model is that the latter assumes an exponential distribution while the former is non-parametric.

According to Litvine (2004): “The results of the comparisons are presented by the judgement matrix $\mathbf{A} = [a_{ij}]_{t \times t}$, which is also reciprocal:

$$a_{ij} = \frac{1}{a_{ji}} \text{ and } a_{ii} = 1, i, j = 1, \dots, t$$

The values a_{ij} ($i < j$) are regarded as random variables. Specifically, if the true weights of the items are α_i , then a_{ij} are distributed as the ratios

$$\frac{x_i}{x_j},$$

where $x_i, i = 1, \dots, t$ are exponentially distributed random variables with parameters

$$\lambda_i = \frac{1}{\alpha_i}$$

respectively (that is $E(x_i) = \alpha_i$). Also, all a_{ij} ($i < j$) are assumed to be independent.

The judges estimate how many times one of the two items is superior to the other just as it is in the case of the AHP model. However, in this case an underlying distribution which is the exponential distribution is assumed. The compliance with this distribution will ensure that to some degree the judgement matrix remains consistent. This model is said to yield equal weights to the AHP for an approximately consistent matrix.

The density function of the random variable a_{ij} which is a ratio of two independent exponentially distributed random variables is given by:

$$f(a_{ij}) = \frac{(\alpha_i \alpha_j)}{(\alpha_i + \alpha_j a_{ij})^2}, a_{ij} > 0$$

The likelihood function is therefore given by:

$$\begin{aligned} L(\alpha_1, \dots, \alpha_t) &= \prod_{i < j} \frac{(\alpha_i \alpha_j)}{(\alpha_i + \alpha_j a_{ij})^2} \\ &= \prod_{i < j} \frac{\alpha_i}{(\alpha_i + \alpha_j a_{ij})} \frac{\alpha_j}{(\alpha_i + \alpha_j / a_{ji})}, \quad \text{since } a_{ij} = \frac{1}{a_{ji}} \end{aligned}$$

$$\begin{aligned}
&= \prod_{i < j} a_{ji} \prod_{i < j} \frac{\alpha_i}{(\alpha_i + \alpha_j a_{ij})} \frac{\alpha_j}{(\alpha_i + \alpha_j a_{ji})} \\
&= \prod_{i < j} a_{ji} \prod_{i \neq j} \frac{\alpha_j}{(\alpha_i + \alpha_j a_{ij})} \\
&= \prod_{i < j} a_{ji} \prod_{i \neq j} \frac{1}{(\alpha_i/\alpha_j + a_{ij})}.
\end{aligned}$$

And the logarithmic likelihood function is:

$$\log L(.) = \sum_{i < j} a_{ji} - \sum_{i \neq j} \log \left(\frac{\alpha_i}{\alpha_j} + a_{ij} \right) \quad (2.8)$$

Taking the derivative of the log-likelihood function and equating it to zero give the following result:

$$\sum_{j=1}^n \frac{\alpha_i - \alpha_j a_{ij}}{\alpha_i + \alpha_j a_{ij}} = 0, i = 1, \dots, t.$$

This system takes us to two important conclusions. Firstly, the system is invariant with respect to a stretch and therefore has infinitely many solutions. Secondly, if the matrix \mathbf{A} is consistent, that is, there exist weights w_1, \dots, w_t such that $a_{ij} = w_i/w_j$, then these weights w_1, \dots, w_t solve the above system (each summand of the system is zero). The Haines-Litvine Model would therefore have the same solution as the AHP.

The system can be presented in the following form:

$$\begin{aligned}
\sum_{j=1}^n \frac{\alpha_i}{\alpha_i + \alpha_j a_{ij}} &= \sum_{j=1}^n \frac{\alpha_j a_{ij}}{\alpha_i + \alpha_j a_{ij}} \\
\alpha_i \sum_{j=1}^n \frac{1}{\alpha_i + \alpha_j a_{ij}} &= \sum_{j=1}^n \frac{\alpha_j a_{ij}}{\alpha_i + \alpha_j a_{ij}} \\
\alpha_i &= \sum_{j=1}^n \frac{\alpha_j a_{ij}}{\alpha_i + \alpha_j a_{ij}} \left(\sum_{j=1}^n \frac{1}{\alpha_i + \alpha_j a_{ij}} \right)^{-1} \quad i = 1, \dots, t. \quad (2.9)
\end{aligned}$$

The above formula serves as a basis for a simple iterative algorithm: starting with a certain initial approximation $\alpha_1^0, \dots, \alpha_t^0$, the right-hand side of equation (10) is substituted with the

approximation to obtain the next approximation on the left-hand side, which will be re-used in the next iteration.

In cases where a score is zero, and the ratio cannot be calculated, the same treatment as in other paired comparison models which do not allow zero scores is applied. Respective ratios recorded into the judgement matrix are set to

$$a_{ij} = \frac{1}{2n-1} \text{ and } a_{ji} = 2n - 1$$

respectively, where n is the total number of comparisons between the items”.

2.4 Soccer

2.4.1 Introduction to soccer

The Federation Internationale de Football Association (FIFA) is the regulatory organization that determines all professional soccer regulations and rankings. FIFA and Coca-Cola created the FIFA rankings system, which allows for the release of senior national team rankings. The rankings are determined by a number of factors, including match outcome, date of play, kind of match, home field advantage, number of goals scored, and points for contesting teams' regional strength. For each match, the two competing teams obtain a score based on the aforementioned criteria. A team's rating is calculated by its overall score from all games played during the eight-year ranking period (Hallinan, 2005).

The Federation Internationale de Football Association (FIFA), which governs soccer, now rates national soccer teams, according to Hallinan (2005). FIFA's system is thorough and takes into account a wide range of variables, although many of the weights used in the system's computations are fairly arbitrary. In this section, it is investigated whether adopting a statistical model would provide a more realistic comparison of the teams for ranking purposes. By approaching each game as a paired comparison experiment and using the Bradley-Terry model as a starting point, some relevant models are presented. When estimating the effect of a match on a team's ranking, the final model's ability to discern between friendly and competitive matches is critical. The posterior distributions of the rating parameters are computed, and the rankings and results from each model are compared to one another and to the FIFA rankings.

The Federation Internationale de Football Association (FIFA) is the sole entity that sets all regulations and standings in the world of professional soccer. The senior national teams of all nations can be ranked publicly thanks to the FIFA rankings system, which was created by FIFA and Coca-Cola. The outcomes of the matches, the dates and types of the matches, home field advantage, the number of goals scored, and the teams' points for regional strength all go into the rankings. For each match, the two competing teams each obtain a score based on the aforementioned criteria. A team's ranking is determined by the total points it has earned in all of its contests over the ranking period, which spans the previous eight years (Hallinan, 2005).

Paired comparisons are used to build ranking systems in various competitive sports. Stefani (1997) provided an overview of the main sports rating systems used globally, including soccer and tennis. He outlined the three common stages—weighting, summarization, and averaging—

for each rating system. For a more sophisticated examination of contests, statistical models for ordered paired comparisons have been presented over the past few decades.

In a variety of competitive sports, ranking systems are created using paired comparisons. Stefani (1997) gave a summary of the most popular sports rating systems in use around the world, including tennis and soccer. He described the three common steps for each rating system—weighting, summary, and average. Over the past few decades, statistical models for ordered paired comparisons have been introduced for a more in-depth analysis of competitions.

The creation of an efficient ranking system is becoming increasingly popular in professional sports. The Bradley-Terry model has shown to be a successful strategy for sports with two-way results. However, a well-known drawback of traditional Bradley-Terry models is that they cannot be used for sports that permit ties at the end of normal play. Numerous techniques have been investigated over the past few decades to model soccer matches in literature.

As indicated by the research of Moon (1968), Daniels (1969), Rubenstein (1980), Goddard (1983), Stob (1985), and Slutzki and Volij (2005), the theory of tournament rankings is a legitimate and consistent issue, within which a specific region - round robin tournaments - has received significant attention. Peterson (1993) and Stern (1995) have attempted to answer the question of who is genuinely No. 1 and how statistics can influence college football competition rankings or how it is picked in this regard. According to the research of Moon (1968), Daniels (1969), Rubenstein (1980), Goddard (1983), Stob (1985), and Slutzki and Volij (2005), the theory of tournament rankings is a legitimate and consistent issue, with a particular focus on round robin tournaments. Peterson (1993) and Stern (1995) attempted to answer the topic of who is truly No. 1 and how statistics can influence college football competition rankings or how they are chosen in this regard.

While the majority of the studies conducted to investigate and recommend improvements to tournament rankings had a strong quantitative component, there were some attempts to consider the ranking's qualitative dimension as well. As a result, Lebovic and Sigelman (2003) identified inertia and constraint, rank elevators, and the passage of time as predictors of league ranking positioning. They also discovered that lower-ranked teams move up in the rankings faster than higher-ranked teams, but moving up in the rankings after a victory is an incremental process — much more incremental than moving down after a loss. Boscá et al. (2009) discovered that enhancing offensive efficiency (in the Primera Division) and defensive

efficiency (in the Serie A) is more essential than improving overall ranking when comparing the Spanish and Italian leagues.

According to Da Silva, Matsushita, and Silveira (2013), sports are stylized confrontations that may follow power laws comparable to wars, individual fights, and terrorist actions. Their investigation of the top fifteen European football leagues indicated the presence of hidden power rules based on the substantial correlation between rank and points achieved by the teams involved.

In the realm of football, Hvattum and Arntzen (2010) delved into the significance of attributing ratings to teams derived from their historical performances. Their aim was to forecast match outcomes in association football. They introduced a scoring system, influenced by the ELO rating system, where a victory is assigned a score of 1, a draw receives 0.5, and a loss is given a score of 0. They also proposed a potential enhancement by adjusting the scores using a coefficient k , which hinges on the goal difference. Glickman and Jensen (2005) highlighted the significance of the paired comparison design in tournaments or league schedules. They noted that teams could be ranked using either prior competition outcomes or expert judgment when structuring a tournament. Additionally, they put forth a methodology tailored for devising paired comparison experiments, particularly in the context of tournament scheduling. In the next part (2.4.2) we will look at various statistical models for soccer.

2.4.2 Various statistical models used in analysing Soccer Data

A new extended Bradley-Terry model with a variety of covariates was developed by Wang Vandebroek (2013) in their research to provide a more sophisticated ranking system for international soccer clubs based on their matches played within the previous four years. Standard likelihood (ML) estimation was used initially. For additional enhancement, a weighted likelihood (WML) estimation approach was used. To finally capture the dynamic trend in performance during significant events, an evolutionary model was created. This paper looked at how their ranking differs from the current FIFA ranking. They also had examined the 2012 UEFA European Football Championships.

In the world of professional sports, a prevalent challenge involves recruiting athletes with limited financial resources. The general manager of the Oakland Athletics has become synonymous with the "Moneyball" concept, emphasizing quantitative analysis in the baseball player market. His strategy focuses on identifying players with exceptional technical skills

but undervalued in the market. This study extends the Moneyball approach to the domain of multiple criterion decision aid (MCDA). The innovation lies in applying four algorithms investigated through the composition of probabilistic preferences (CPP) - namely ranking, classification, dynamic evaluation, and regularity analysis - to assess soccer player performance.

This novel model was employed to evaluate the transfer of a recent left-back soccer player to Europe. Among the 32 left and right-back players analysed, the study identified 12 prospects with better investment potential. However, two years later, the same player's value significantly decreased, as he had a limited impact during the 2018-2019 season, participating in only five games without scoring or assisting. In contrast, players pinpointed as more promising by the CPP-MB model exhibited higher performance levels and market values (Gavião, Sant'Anna, Alves Lima, & de Almada Garcia, 2020).

Passing behaviour plays a crucial role in achieving optimal team sports performance. However, prior research has predominantly concentrated on notational data, such as overall passing frequencies, which offer limited insights into the characteristics of successful passing behaviour. This limitation has hindered the effective translation of research findings into practical applications. In response, we present two distinct approaches for evaluating passing effectiveness in elite soccer, focusing on scenarios where a team has the majority and the control of space in front of the goal. The determination of majority scenarios involves assessing the number of defenders between the ball carrier and the goal. The control of space is approximated using Voronoi diagrams based on player positions on the field. Employing a comprehensive big data approach, both methodologies were applied to position data from 103 German First Division games spanning the 2011/2012, 2012/2013, and 2014/2015 seasons. The results indicate a significant association between both indicators and successful gameplay, as measured by the number of goals scored and the probability of winning a game. Moreover, the data suggests that passes originating from the midfield into the attacking area tend to be more effective on average. These innovative measures of passing efficiency open up new possibilities for future applications in soccer and other sports disciplines. Importantly, they remain practical for integration into tactical training routines or game performance analyses (Rein, Raabe, & Memmert, 2017).

In the Netherlands, Koning (2000) estimated an ordered probit model for soccer results. The outcome of a game was considered to be influenced by home field advantage and quality

distinctions between competing teams. The model's parameters were used to determine if the balance of competition in Dutch professional soccer has altered over time. Contrary to popular opinion, they discover that the equilibrium has been rather stable since the mid-1970s.

McHale and Scarf (2011) developed models aimed at predicting the number of goals scored by opposing teams in international soccer matches. Their approach involved utilizing marginal distributions and a dependent copula to define bivariate discrete distributions. The use of a copula allowed for a flexible modelling of dependency in the bivariate distribution by selecting an appropriate family of copula functions and fitting it to the bivariate data through maximum likelihood estimation. Match covariates were incorporated to model marginal means. They introduced the theory that the strength of dependence is contingent on the competitive balance of a match, given the complex nature of dependence in the number of goals scored. Their analysis revealed that overall dependence was low for games between closely matched teams, and as the competitiveness of a match declined, the dependence became increasingly negative. In this framework, they established a connection between reliance and competitive balance, proposing a method for measuring the latter. Notably, McHale and Scarf suggested that their models might offer more accurate forecasts compared to match result models with independent marginal distributions. This underscores the importance of considering the interdependence of goal-scoring behaviour in soccer matches, especially in relation to the competitive dynamics of the teams involved.

2.4.3 Paired Comparisons models used in analysing Soccer Data

The bulk of well-known paired comparison models (including the regression models and their modifications, the Bradley-Terry model, and the Thurstone-Mosteller model) presuppose that the underlying distribution is continuous. However, the scoring system used in sporting events is by design discrete. The observation suggests that continuous models are insufficient for a precise statistical analysis of the data generated from these contests or tournaments. The essay that follows makes a special effort to use discrete well-motivated distributions to gauge soccer teams' capabilities. We can also explain these observations by using a mathematical model (Hilliard-Lomas, Litvine, 1998).

A strategy is presented by Leonard (1977) to supplement the approach used by Davidson and Solomon (1973). Incorporating previous knowledge of the likelihood of parameters that are

associated in an autoregressive manner or alternatively depend on some explanatory factors is not an easy task when using Davidson and Solomon's Bradley-Terry model. Leonard introduces methods based on logistic transformations and makes use of multivariate normal distributions for the transformed parameters in order to take these options into account.

In scenarios involving sequential paired comparisons over time, such as in chess competitions, it is reasonable to assume that underlying talents vary over time. Traditional methods employed fixed updating schemes, where increments and decrements of underlying abilities were determined by fixed functions. The parameters for these functions had to be predefined and were based on logical assumptions. Recognizing the dynamic nature of talents in sequential paired comparisons, this study introduces a transition model incorporating both local linear trend models and random walk models. Together, these elements form a non-Gaussian state-space model. Unlike previous approaches, this model doesn't rely on fixed updating schemes. The study provides recursive posterior mode estimation algorithms, analysing their relationship with earlier methods in light of recent insights. Simulated results and an application to German Bundesliga soccer data demonstrate the effectiveness of this approach. By allowing for changing underlying talents over time without the need for predefined parameters, this strategy provides a more adaptive and realistic representation of the evolving dynamics in sequential paired comparisons (Fahrmeir, Tutz, 1994).

In his 1970 study, Davidson expanded the Bradley-Terry model for paired comparisons to encompass situations where individuals have the option to indicate no preference. He introduced a novel model and compared its performance to Rao and Kupper's model. Through an iterative process that converges monotonically to the solution of the likelihood equations, Davidson determined the maximum likelihood estimates of the parameters. It is noteworthy that, in a balanced paired comparison experiment, the ranking derived from maximum probability estimates aligns with that determined by a scoring system awarding two points for a win, one point for a tie, and zero points for a loss. Davidson demonstrated that the likelihood ratio test for the equal preferences hypothesis in his model exhibits the same asymptotic efficiency as the Rao-Kupper model. The study included a scenario with data from an unbalanced paired comparison experiment. Initial results from the goodness-of-fit test suggest that the proposed model provides a reasonable representation of real-world experimentation (Davidson, 1970).

2.5 Paired Comparison models used in analysing data of other types of sport

In their 2015 study, Silva, Manage, and Swartz (2015) delved into the dynamics of the powerplay phase in one-day cricket. Given the evolution of rules governing the powerplay over the years, the primary aim of the article was to assess its impact on scoring. The researchers employed a "what if" methodology, wherein powerplay outcomes were replaced with hypothetical scenarios to examine what might have occurred if there had been no powerplay. This approach created a paired comparisons setup involving actual matches and parallel hypothetical matches, with outcomes imputed for the powerplay phase. Several key findings emerged from Silva, Manage, and Swartz's research, including (a) the varied forms of the powerplay adopted over the years, each yielding different effects, (b) recent versions of the powerplay providing a batting advantage, (c) a higher occurrence of wickets during the powerplay compared to scenarios without it, and (d) observable effects on run production associated with the over where the powerplay was initiated. The study also delved into the performances of individual batsmen and bowlers during the powerplay, providing a comprehensive analysis of their contributions and impacts during this critical phase in one-day cricket.

The Analytic Hierarchy Process (AHP) serves as a decision-making support system, aiding in the selection of a solution from a set of options based on various evaluation criteria. In recent years, this method has found application in assessing athletes' competitive potential, considering physical, technical, and mental components. The aim of the study conducted by Aoki, Uehara, Kato, and Hirahara (2016) was to develop an evaluation index for the psychological-competitive capacity and mental characteristics of rugby players. To achieve this, the researchers administered the Diagnostic Inventory of Psychological-Competitive Ability (DIPCA.3) to 67 male university rugby players. Subsequently, a rugby coach participated in an interview to rank five DIPCA criteria. The DIPCA scores, the coach's ranking, and the AHP were all integrated to calculate the total evaluation values for each rugby player. Players were then categorized into high or low groups based on their DIPCA.3 scores, and within each group, they were further divided based on whether they had a high or low total evaluation value from AHP. Upon comparing the high and low groups from the AHP total assessment values, it was observed that the proportion of rugby players in the high-level group, who were regular players with extensive experience in national competitions, was significantly higher compared to the low group. According to the study's findings, the

overall evaluation value derived using AHP proved to be more effective than the total scores of DIPCA.3 in assessing the mental elements of rugby players (Aoki et al., 2016).

The researchers aimed to investigate the impact of post-activation potentiation (PAP) on swim start performance (time to 15 m) in a group of international sprint swimmers. The study, approved by the university ethics committee, involved nine volunteers (7 men and 2 women) who provided informed consent. The swimmers underwent a countermovement jump (CMJ) on a portable force platform (FP) at baseline and at various intervals (15 seconds, 4, 8, 12, and 16 minutes) after a PAP stimulus (1 set of 3 repetitions at 87% 1 repetition maximum [RM]) to determine the recovery time needed for enhanced muscle performance. On two separate days, the swimmers performed a swim start to 15 m under 50-m freestyle race conditions, preceded by either their customized race-specific warm-up or a PAP stimulus. Two cameras recorded both trials, with camera 1 at the start and camera 2 at the 15-m mark. Peak vertical force (PVF) and peak horizontal force (PHF) were measured from a portable FP on top of the swim block during all swim starts. A repeated measures analysis of variance revealed a significant time effect for power output (PO) and jump height (JH), with a paired comparison showing a substantial increase in PO and JH after an 8-minute recovery from the PAP stimulus. PHF and PVF also increased significantly after the PAP stimulus compared to the swim-specific warm-up during the swim start. However, the time to 15 m was similar when both starts were compared. The study's findings suggest that muscle function during a CMJ improves after a PAP stimulus with appropriate recovery time (8 minutes) between activities. Moreover, when compared to a unique race-specific warm-up, swimmers performed similarly well in terms of time to 15 m, indicating that PAP could be a valuable addition to a warm-up routine before races. However, the researchers emphasize the need for further research to fully understand the significance of PAP in swimming performance (Kilduff et al., 2011).

The study conducted by Batchelder and Bershad in 1979 formalized and presented static and dynamic estimators for a scaling model designed for chess player ratings. This model drew inspiration from the work of Arpad Elo, the creator of the widely adopted chess rating system employed by both US and international chess federations. The model can be conceptualized as a Thurstone Case V model incorporating draws (ties). The study also explored a similar approach based on a linear approximation. In the context of chess, the model aimed to estimate potentially changing ability parameters sequentially from sparse data structures.

These structures often contained far fewer than $M(M-1)/2$ observations on the M rated players. This stands in contrast to psychological applications of paired-comparison scaling, where models without a draw provision are typically used to estimate static parameters from deliberately collected data structures, such as a replicated "round robin" including all M entities to be scaled. The study presented and assessed static and sequential estimators for various types of data, constructing their sampling theory. The application of rating systems to monitor temporally changing ability factors was considered valuable not only in chess but also in diverse areas of psychology (Batchelder and Bershad, 1979).

Dahl (2012) introduced a ranking methodology for sports like tennis that incorporates the pair-wise comparison approach. This methodology relies on linear algebra and involves computing a score for each player by solving a specific linear system of equations derived from a set of matches and weights representing their importance. The goal is to determine the ranking of tennis players based on this approach. Building upon this idea, Baker and McHale (2014) extended the models for paired comparisons to situations with numerous competitors engaged in a series of head-to-head competitions. Their extension takes into account the dynamic nature of competitor strengths over time. This adaptation allows for a more nuanced understanding of performance, acknowledging that the relative strengths of competitors can change over the course of multiple competitions.

2.6 Conclusion

In conclusion, the realm of professional soccer rankings, governed by the Federation Internationale de Football Association (FIFA), has evolved significantly over the years. The FIFA rankings system, developed in collaboration with Coca-Cola, has been the traditional method for assessing and comparing national teams. This system considers various factors such as match outcomes, match type, home field advantage, and regional strength. However, the arbitrary weights assigned to these variables have led to discussions about the system's accuracy and fairness.

The exploration of statistical models, particularly the Bradley-Terry model, Thurstone model and Haines-Litvine model, offers a promising alternative to the current FIFA rankings. Researchers have delved into the intricacies of these models, attempting to create more sophisticated and nuanced approaches to evaluating team performances. The consideration of ties and the differentiation between friendly and competitive matches are crucial aspects that these models address, providing a more accurate representation of team strengths.

The study of tournament rankings, especially in the context of round robin tournaments, has garnered attention from scholars seeking to determine the true hierarchy of teams. The research by Moon (1968), Daniels (1969), Rubenstein (1980), Goddard (1983), Stob (1985), Slutzki (2005), and Volij (2005), as well as the inquiries of Peterson (1993) and Stern (1995), underscores the ongoing effort to refine the understanding of tournament rankings across various sports.

Moreover, recent studies have incorporated both quantitative and qualitative dimensions in assessing rankings. Factors such as inertia, constraint, rank elevators, and the passage of time have been identified as predictors of league ranking positioning. Mathematical optimization methods have been employed to compare leagues, revealing the importance of offensive and defensive efficiency in influencing rankings.

In the broader context of sports, researchers like Da Silva (2013), Matsushita (2013), Silveira (2013), Dahl (2012), Baker (2014), and McHale (2014) have applied diverse methodologies, including power laws and linear algebra, to assess rankings and performance in sports like tennis and football. These alternative approaches provide valuable insights into the dynamics of sports competitions.

In essence, the evolution of ranking methodologies in professional soccer reflects an ongoing pursuit of precision and fairness. As researchers continue to explore statistical models, tournament rankings, and multidimensional factors influencing team performance, the landscape of soccer rankings is likely to see further refinement. The quest for a more accurate and comprehensive ranking system remains a dynamic and evolving aspect of the world of professional soccer.

Chapter 3: Application of the models to FIFA World Cup Soccer Data

3.1 The Federation International de Football Association (FIFA) World Cup Soccer Data

The FIFA World Cup, held every four years, stands as one of the most prestigious and widely watched sporting events globally. Spanning from 1930 to 2022 (before the Qatar World Cup), this period encapsulates the rich history and evolution of the tournament. The data used for the analysis in this chapter was a secondary data which we obtained from Kaggle (<https://www.kaggle.com/datasets/piterfm/fifa-football-world-cup>). The data was not from a round robin tournament, which was one of the limitations, meaning that our results might not be a true reflection of the future prediction. We took the top 20 teams according to the world ranking for the “World Cup 2022”. The main objective for taking the 20 top teams was to check if we used our three (3) models, LEAHP, LBT and Thurstone which one would give the most accurate results for 2023 and 2026 top 20 teams. The table below shows the top 20 FIFA World Cup teams.

Table 3.1: The top 20 FIFA World Cup

1	Argentina
2	France
3	Brazil
4	England
5	Belgium
6	Croatia
7	Netherlands
8	Portugal
9	Italy
10	Spain
11	United States
12	Mexico
13	Morocco
14	Switzerland
15	Germany
16	Colombia
17	Uruguay
18	Denmark
19	Japan
20	Senegal

3.2 Linear Exponential Analytic Hierarchy Process (LEAHP)

3.2.1 Introduction

In this section, we will employ the Haines-Litvine Model, also known as the Exponential Analytic Hierarchy Process, as a model for paired comparisons. This model has been made time-dependent based on insights from Mabaso (2012). Mabaso (2012) utilized this innovative time series model for paired comparisons in the context of exchange rate forecasting. To substantiate the findings, he conducted simulations and provided practical examples to illustrate the results, thereby verifying the effectiveness of the newly proposed time-dependent model.

The Haines-Litvine Model, as introduced in Chapter 2, finds application in scenarios where judges assess how many times one of two items being compared is deemed superior to the other. The outcome is then expressed as a ratio of the two items, making it a suitable choice for forecasting exchange rates over time. The primary objective was to extend this model into a time series framework, allowing historical observations to inform predictions of future observations. This extension enables the incorporation of temporal patterns and trends, enhancing the model's capability to capture and forecast changes in the paired comparison data over successive time points. One way to extend this model is to assume that each judge makes a judgement on each pair of items at each time $t = 0, 1, \dots, T - 1$, where t is a predefined time interval e.g. days, months, years.

Let X_{ijt} ($i, j = 1, 2, \dots, N, i \neq j; t = 0, 1, \dots, T - 1$) be a random variable that represents a ratio between two independent exponentially distributed random variables X_{it} and X_{jt} . With the Haines-Litvine Model, if $X_{ij} = \frac{X_i}{X_j}$ is a measure of how many times X_i is superior to X_j then it has a density function given by:

$$f(X_{ij}) = \frac{p_i p_j}{(p_i + p_j X_{ij})^2}; X_{ij} > 0, X_{ij} = \frac{1}{X_{ji}}. \quad (3.1)$$

To extend this function to a time series form, we set $p_i = \alpha_i + \beta_i t$, $p_j = \alpha_j + \beta_j t$ and X_{ij} is replaced with X_{ijt} . We let the weights of the items change linearly over time ($w_i = \alpha_i + b_i t$) hence the Linear-Trend Haines Litvine Model. We will therefore have $2N$ parameters α_i and b_i , for $i = 1, 2, \dots, N$. As in the case of the Haines-Litvine Model, the weights will have to be positive for all t . For uniqueness of the weights, we will set the constraints $\sum_{i=1}^N \alpha_i = 1$ and $\alpha_i > 0$ for all i . It is possible to observe negative values of b_i especially in cases where the weights decrease as time proceeds, but this should not violate the condition of non-negative weights. In the rest of this chapter, we will test the model on our soccer dataset.

3.2.2 Test on soccer data ($T = 10.25; T = 11$ and $N = 20$)

LEAHP also known as the model that works with ratio data, it cannot work with zeros. In our model we transformed our soccer data to ratio data, we took the exponential power of the scores $E^{X_{ijt}}$ and divided these two numbers by each other to get out $X_{ijt}'s$ which was then input into the model. The $X_{ijt}'s$ will just be $\frac{1}{X_{ij}}$. We programmed the model using Mathematica. The

default X_{ijt} was set to one (1) because if team i did not play against team j we would treat it like a draw as if they have the same scores and the ratio of the exponents for the scores would be 1. Then where we had scored the ratio would be $\frac{1}{X_{ij(scores)}}$. We then run our maximum likelihood equations and continued to solve it how the LEAHP is normally solved. With starting values, we said all the a_i 's to 1 and the b_i 's to 0 and iteratively solved the equations. We standardized our a_i 's before finding our b_i 's by letting our a_i 's add up to 1's. The maximum likelihood solutions the a_i 's and b_i 's eventually converged then we found prediction for 2023 and 2026. This was because our data was for every four years from top 20 ranked teams for the last 10 time periods, and each time period is every four years from 1986 to 2023. We then predicted 2023 we will then be time period 10.25 and 2026 will be time period 11.

Table 3.2: LEAHP

Actual Rank		2023	Teams	2026	Teams
1	Argentina	0,0678019	France	0,0685453	France
2	France	0,0563073	Argentina	0,0564864	Argentina
3	Brazil	0,0538064	Netherlands	0,0539002	Netherlands
4	England	0,0526901	Germany	0,0526901	Germany
5	Belgium	0,0517468	Colombia	0,0517468	Colombia
6	Croatia	0,051247	Portugal	0,051247	Portugal
7	Netherlands	0,0512412	Mexico	0,0512412	Mexico
8	Portugal	0,0511286	Senegal	0,0511286	Senegal
9	Italy	0,0505388	Switzerland	0,0505388	Switzerland
10	Spain	0,0483879	Spain	0,0482988	Spain
11	United States	0,0480776	England	0,0479884	England
12	Mexico	0,0474822	Uruguay	0,0473932	Uruguay
13	Morocco	0,0466587	Italy	0,0465098	Italy
14	Switzerland	0,0454132	United States	0,0451476	United States
15	Germany	0,0442753	Japan	0,0439152	Japan
16	Colombia	0,0405587	Belgium	0,0402627	Belgium
17	Uruguay	0,0378067	Croatia	0,037094	Croatia
18	Denmark	0,0377902	Morocco	0,0370861	Morocco
19	Japan	0,037206	Denmark	0,0364986	Denmark

20	Senegal	0,0342468	Brazil	0,0333569	Brazil
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In Table 3.2 we can noticed that according to the actual data “Argentina” is number 1, but according to our future prediction Argentina is number 2 for both 2023 and 2026 prediction. France was predicted to be number 1 for 2023 and 2026 while the original rank for the 2022 world cup France was number 2. Notice a big change in the predicted data, “Uruguay” went from position 18 to position 12 which is a huge change and this could be as a result of the teams not playing against each other team on every turn. In the original ranking Brazil was rated number 3, in the predicted rankings for 2023 and 2026 it was then ranked number 20. The results of this analysis could be showing an inconsistency in the results because of the data used or it could be of the four-year gap between the games.

3.2.3 Conclusion

In conclusion, our application of the LEAHP (Linear Exponential Analytical Hierarchy Process) to soccer data involved a transformation of the scores into ratio data, considering the exponential power of the scores. This approach allowed us to adapt the model to the nature of soccer scores, addressing the challenge that LEAHP faces with zeros. Utilizing Mathematica, we implemented the model, setting the default values appropriately, and solving the maximum likelihood equations iteratively.

Our standardization of the parameters (a_i 's) before determining the other parameters (b_i 's) ensured meaningful interpretations of the results. The convergence of the maximum likelihood solutions for the a_i 's and b_i 's facilitated predictions for 2023 and 2026. The data used for this analysis encompassed the top 20 ranked teams over ten time periods, each spanning four years from 1986 to 2023.

Upon examining the predictions in Table 3.2, discrepancies between our forecasted rankings and the actual data became apparent. Notably, Argentina, initially ranked first, dropped to the second position according to our predictions for both 2023 and 2026. France, on the other hand, was forecasted to ascend to the top spot, surpassing its second-place ranking in the 2022 World Cup. A substantial change was observed in Uruguay's predicted position, jumping from 18th to 12th, indicating potential fluctuations in team rankings due to factors like limited match opportunities.

One striking deviation was Brazil's original third-place ranking in the 2022 World Cup, contrasting with our predictions ranking it at 20th in 2023 and 2026. This notable inconsistency might stem from the unique characteristics of the data or the four-year gap between games, which could influence team dynamics and performance.

These disparities suggest the need for careful consideration when applying the LEAHP model to soccer data, acknowledging the nuances of the sport and potential limitations in the predictive capacity of the model. The results underscore the importance of periodic re-evaluation and refinement of models to ensure their relevance and accuracy in forecasting future outcomes in the dynamic world of soccer rankings. In the next chapter we are going to analyse the soccer data using the Linear Bradley-Terry Model.

3.3. Linear Bradley-Terry Model

3.3.1 Introduction

Sjölander (2011) introduced the Linear-Trend Bradley-Terry Model. It is a time series form of the Bradley-Terry model (which we discussed in chapter 2) that assumes that each judge makes a judgement of each pair of items at each time $t = 0, 1, \dots, T - 1$.

If we let $Y_{ij\gamma}^t (i, j = 1, 2, \dots, N; i \neq j; \gamma = 1, 2, \dots, n; t = 0, 1, \dots, T - 1)$ be a random variable which takes on the value 1 if O_i gets preferred in the γ^{th} comparison (by for example judge number γ if each judge judges each pair of items once each time period) of objects O_i and O_j at time t , and 0 if O_j gets preferred in this comparison, then the extension process is done as follows:

Define $X_{ij}^t = \sum_{\gamma=1}^n Y_{ij\gamma}^t$ then X_{ij}^t is the number of times that item O_i gets preferred in its n comparisons with item O_j at time t . Thus, $X_{ij}^t \sim Binomial(n, p_{ij}^t)$, where $p_{ij}^t = \frac{p_i^t}{p_i^t + p_j^t}$.

Let's assume that the weights of the objects change linearly over time, that is, $p_i^t = a_i + b_i t$, and so define the Linear-Trend Bradley-Terry Model where:

$$X_{ij}^t \sim Binomial(n, p_{ij}^t) \text{ where } p_{ij}^t = \frac{p_i^t}{p_i^t + p_j^t} = \frac{a_i + b_i t}{a_i + b_i t + a_j + b_j t}$$

$$\text{and } p(x_{ij}^t) = \binom{n}{x_{ij}^t} \left(\frac{a_i + b_i t}{a_i + b_i t + a_j + b_j t} \right)^{x_{ij}^t} \left(\frac{a_j + b_j t}{a_i + b_i t + a_j + b_j t} \right)^{x_{ji}^t}.$$

To find the maximum likelihood estimates of the a_i 's and b_i 's (denoted as A_i 's and B_i 's), we must solve the following equations:

$$\sum_{t=0}^{T-1} (f_k^t - g_k^t) = 0 \text{ and } \sum_{t=0}^{T-1} t (f_k^t - g_k^t) = 0 \text{ for } k = 1, \dots, N,$$

where

$$f_k^t = \frac{\sum_{j=1}^N x_{kj}^t}{A_k + B_k t} \text{ and } g_k^t = \sum_{\substack{j=1 \\ j \neq k}}^N \frac{n}{A_k + B_k t + A_j + B_j t}. \quad (3.3)$$

Sjölander (2011) applied the technique, possibly referring to the Linear-Trend Bradley-Terry Model, to forecast missing temperature data. However, the outcomes presented in his paper indicated that there were alternative models that exhibited superior performance compared to the Linear-Trend Bradley-Terry Model for the specific dataset. This suggests that, in the context of forecasting missing temperature data, there are models other than the Linear-Trend Bradley-Terry Model that may provide more accurate predictions or better capture the underlying patterns in the temperature dataset.

In this study, we will employ the Linear-Trend Bradley Terry Model on our soccer data for each time period. We will conduct a paired comparison analysis on our actual soccer data using the specific model developed by Sjolander (2011). The aim is to explore whether the Bradley-Terry model introduced by Sjolander (2011) is a suitable fit for predicting the future top 20 world cup soccer teams. Notably, the linear Bradley-Terry model, incorporating ties, is expected to yield improved results compared to the previous model, as it encompasses essential features of soccer. The inclusion of ties is crucial for any soccer model aspiring to be deemed credible, given the nature of competition in the sport.

The model remains incomplete as it does not adequately address several critical issues in soccer analysis. One notable limitation is its assumption that all games played at any given moment are equally weighted, overlooking the fact that not all types of games hold equal significance. Additionally, the model simplifies the treatment of home-field advantage, assuming the presence of a home team in every game. However, this assumption does not align with the reality of international soccer, where a substantial portion of games is played at neutral grounds rather than with a designated home team. Addressing these issues is crucial for enhancing the model's accuracy and relevance in capturing the complexities of soccer dynamics.

3.3.2 Test on Soccer data ($T = 10.25; T = 11$ and $N = 20$)

The linear Bradley-Terry 1 (LBT 1): In this approach we set default $X_{ij}^t = \frac{1}{2}$ and then we overwrote this with our data where data was available. Now the data was 1 if team X_i won against team X_j at time t and 0 if team X_i lost against team X_j . X_{ij}^t would automatically be $1 - X_{ij}^t$. We then ran the usual Bradley-Terry Model at each time-period t . We used the maximum likelihood approach to find our weights on $a_i + b_i t$ and thus found our predicted a_i 's and b_i 's. We could not just apply the model at each time period and regress the model, because that is

not how the model works out. We have to find the maximum likelihood estimates, to do that we actually used the Linear Exponential AHP a_i 's and b_i 's as the starting values using Mathematica. We added the condition that some of the a_i 's must be 1 and some of the a_i 's must be 0. We then found our weights $a_i + b_i t$ for time-period 2023 and 2026.

The linear Bradley-Terry 2 (LBT 2): In this approach we set the default values of X_{ij}^t 's equal to a $\frac{1}{2}$ but now overwrote them with data as follows: the X_{ij}^t 's was set for team i 's score over team j at time t over the sum of team i 's and team j 's scores when playing against each other at time t . This is the same as saying the comparisons are not games but scoring opportunities in the games. For example, if the score was 1:2 between team i and team j at time t we would set $X_{ij}^t = \frac{2}{3}$. There were 3 times where somebody scored, and the scoring opportunities were the paired comparisons. If a draw takes place, the X_{ij}^t would be $\frac{1}{2}$, for example 3:3 would result in $\frac{3}{3+3} = \frac{1}{2}$. We then ran the usual Bradley-Terry model at each time-period t . We regressed these weights on $a_i + b_i t$ and thus found our predicted a_i 's and b_i 's. We then found our weights $a_i + b_i t$ for time-period 2023 and 2026.

Table 3.3: Linear Bradley-Terry Model 1

Actual Team Ranks		2023	Teams	2026	Teams
1	Argentina	0,060342	France	0,060971	France
2	France	0,057608	Netherlands	0,058482	Netherlands
3	Brazil	0,055225	Germany	0,055571	Germany
4	England	0,052749	Belgium	0,053343	Belgium
5	Belgium	0,051819	Uruguay	0,052419	Uruguay
6	Croatia	0,051339	Japan	0,051715	Japan
7	Netherlands	0,05105	Colombia	0,051227	Colombia
8	Portugal	0,0502	Brazil	0,050159	Portugal
9	Italy	0,049993	Portugal	0,049856	Morocco
10	Spain	0,04973	Morocco	0,049751	Brazil
11	United States	0,04921	Italy	0,048855	Italy
12	Mexico	0,048634	Mexico	0,048587	Mexico
13	Morocco	0,048159	Croatia	0,047879	Croatia
14	Switzerland	0,047588	Spain	0,047217	Spain
15	Germany	0,046786	England	0,046599	England
16	Colombia	0,046541	United States	0,046315	United States
17	Uruguay	0,046254	Denmark	0,046059	Denmark
18	Denmark	0,046081	Argentina	0,045218	Argentina
19	Japan	0,045367	Switzerland	0,04503	Switzerland

20	Senegal	0,045323	Senegal	0,044745	Senegal
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Table 3.3 showed a change in the ranking positions, in 2022 Argentina was ranked number 1, but after applying the first process of the Linear Bradley-Terry Model Argentina was ranked number 18. Germany was number 15, but after applying LBT 1 it moved to number 3 for both 2023 and 2026 rankings. This huge indifference could be because its not a round robin tournament or it could be because we had to go far back in time with the data, and also the four-year gap between each world cup. With France being ranked number 1 after applying the linear Bradley-Terry model 1 method, shows a better prediction because the actual ranking is position 2, therefore it would not be misleading to predict that it will win the world cup for 2023 and 2026.

Table 3.4: Linear Bradley-Terry Model Vs. Actual soccer ranking

Actual Team Ranks		2023	Teams	2026	Teams
1	Argentina	0,055631	Germany	0,056239	Netherlands
2	France	0,055574	Netherlands	0,056139	Germany
3	Brazil	0,054568	France	0,0548	France
4	England	0,052321	Colombia	0,052565	Colombia
5	Belgium	0,051341	Uruguay	0,051804	Uruguay
6	Croatia	0,050483	Brazil	0,050262	Brazil
7	Netherlands	0,050037	Morocco	0,050192	Morocco
8	Portugal	0,050024	Spain	0,049941	Spain
9	Italy	0,0498	Portugal	0,049813	Portugal
10	Spain	0,049589	Japan	0,049755	Japan
11	United States	0,049541	England	0,049442	England
12	Mexico	0,049182	Italy	0,048927	Belgium
13	Morocco	0,0489	Belgium	0,048924	Italy
14	Switzerland	0,048689	Mexico	0,048644	Mexico
15	Germany	0,048597	Croatia	0,048394	Croatia
16	Colombia	0,048124	United States	0,048166	United States
17	Uruguay	0,047932	Denmark	0,047792	Denmark
18	Denmark	0,047854	Argentina	0,047201	Argentina
19	Japan	0,046579	Switzerland	0,046364	Switzerland

20	Senegal	0,045234	Senegal	0,044636	Senegal
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The Table 3.4 method is similar to the predictions for LBT 1. LBT 2 predicted that France would be ranked number 3 for year 2023 and 2026. Argentina moved from position 1 to position 18. Colombia moved from position 16 to position 4. The results of this method are not promising, logically it would make sense if a team moved a point or two up or down but the results show a big shift either up or down which could mean that we have a limitation in our study because of our missing data – it was not a round robin tournament where each team plays against each other the same amount of time.

3.3.3 Conclusion

In conclusion, the application of the Linear Bradley-Terry 1 (LBT 1) and Linear Bradley-Terry 2 (LBT 2) models to soccer data has provided interesting insights into the dynamics of team rankings. These models, based on the Bradley-Terry approach, incorporated variations in their default settings and data handling processes, leading to diverse predictions for 2023 and 2026.

For LBT 1, the initial rankings after the 2022 World Cup underwent substantial changes. Argentina, initially ranked first, dropped to 18th, indicating potential limitations in the model's predictive accuracy. Germany, on the other hand, exhibited a significant jump from 15th to 3rd for both 2023 and 2026, emphasizing the sensitivity of the model to different factors such as tournament structure and the time gap between World Cups. The prediction of France as the top-ranked team aligns closely with actual performance, enhancing the credibility of the model.

LBT 2, which considered scoring opportunities rather than direct match outcomes, echoed some similarities with LBT 1 predictions. However, notable disparities emerged, with Argentina plummeting to 18th and Colombia surging from 16th to 4th. The magnitude of these shifts raises questions about the reliability of the model, suggesting potential limitations arising from missing data or the non-round robin nature of the tournament.

The observed discrepancies in rankings underscore the challenges inherent in predicting soccer outcomes, especially when confronted with variations in tournament formats, time

gaps, and incomplete data. While certain predictions closely aligned with actual results, the models' limitations were evident in instances of significant shifts in team rankings.

In summary, the Linear Bradley-Terry models offer valuable tools for understanding team dynamics and predicting future outcomes in soccer. However, careful consideration of model parameters, data completeness, and the unique characteristics of each tournament is essential to enhance the accuracy and reliability of predictions. Future research may focus on refining these models, addressing data limitations, and adapting to evolving trends in the dynamic world of soccer.

3.4. Linear Thurstone Model

3.4.1 Introduction

Sjölander, (2011) outlines the basic principles of the Linear Trend Thurstone-Mosteller model are, at each time $t = 0, 1, \dots, T - 1$:

1. Set of N Stimuli: There is a collection of N stimuli, represented by N objects (O_i), which can be placed along a subjective continuum.
2. Sensations: Each stimulus, when presented to an individual, elicits a sensation in that individual.
3. Normal Distribution: The distribution of sensations resulting from a specific stimulus for a population of individuals is assumed to be normal.
4. Paired Presentations: Stimuli are presented in pairs to an individual either n times or to n individuals. This process generates a sensation for each stimulus. The individual then compares these sensations and indicates which one is perceived as greater.
5. Correlation: There is a recognition that the paired sensations may exhibit correlation, indicating that the judgments made on one stimulus may influence judgments on another.
6. Spacing of Stimuli: The objective is to arrange or space the stimuli along the subjective continuum, with the possibility of applying a linear transformation.
7. Judges or Experts: Individuals participating in the paired comparisons are referred to as judges or experts.
8. Let variables Y_i^t and Y_j^t stand for sensations with respect to stimuli of objects number i and j (O_i and O_j) respectively for each time $t = 0, 1, \dots, T - 1$.

This experimental design aims to understand subjective perceptions and preferences by analyzing the comparative judgments made by individuals. The statistical analysis of these paired comparisons allows for the inference of underlying relationships and the positioning of stimuli along a subjective scale.

3.4.2 Derivation

Sjölander, (2011) in mathematical terms the Thurstone-Mosteller model assumes the following:

$$E(Y_i^t) = V_i^t, \quad i = 1, 2, \dots, N$$

$$V_i^t = A_i + B_i t \text{ for each time } t = 0, 1, \dots, T - 1$$

$$V(Y_i) = \sigma^2, \quad i = 1, 2, \dots, N$$

$$\rho(Y_i, Y_j) = \rho, \quad i, j = 1, 2, \dots, N$$

where ρ stands for the correlation coefficient.

- Evidently $V(Y_i^t - Y_j^t) = 2\sigma^2(1 - \rho)$, but without loss of generality we can set $V(Y_i^t - Y_j^t) = 1$, since the weights V_i^t are defined subject to an arbitrary linear transformation, and this convention will have no visible effect.
- The preference probabilities for the Thurstone-Mosteller model may be evaluated as follows:

$$\begin{aligned} \pi_{ij}^t &= P\{O_j \rightarrow O_i \text{ at time } t\} \\ &= P\{Y_i^t - Y_j^t > 0\} \\ &= \Phi(V_i^t - V_j^t) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{V_i^t - V_j^t} \exp\left(-\frac{x^2}{2}\right) dx. \end{aligned}$$

where $\Phi(\cdot)$ is a cumulative distribution function (CDF) of the Standard Normal distribution.

- The objective is to estimate the V_i^t 's, which are the true weights of the objects (except for a linear transformation) for each time $t = 0, 1, \dots, T - 1$.
- Since the origin of the scale (at time 0) for V_i^t may be selected arbitrarily and how each V_i^t 's differ over time is relative to how the V_i^t 's differ over time, we add for

certainly the following 2 conditions, which will guarantee the uniqueness of the estimates:

$$\sum_{i=1}^N A_i = 0, \quad \sum_{i=1}^N B_i = 0 \quad (\text{Thus } \sum_{i=1}^N V_i^t = 0 \text{ for each time } t = 0, 1, \dots, T - 1.),$$

- Let $\delta_{ij}^t = V_i^t - V_j^t$ ($i, j = 1, 2, \dots, N; j \neq i. t = 0, 1, \dots, T - 1.$),
- Then $\pi_{ij}^t = \Phi(\delta_{ij}^t)$ and we have:

$$\sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = (N - 1)V_i^t - \sum_{\substack{j=1 \\ j \neq i}}^N V_j^t \Leftrightarrow V_i^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t),$$

$$\text{assuming } \sum_{i=1}^N A_i = 0 \text{ and } \sum_{i=1}^N B_i = 0 \therefore \sum_{i=1}^N V_i^t = 0,$$

$$V_i^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t) \Leftrightarrow A_i + B_i t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t).$$

- We now use the least squares approach (multivariate multiple regression) to estimate the A_i 's and B_i 's, which yield the following results:

$$A_i \approx \frac{2(2T-1)}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t - \frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \delta_{ij}^t.$$

$$\text{i.e. } A_i \approx \frac{2(2T-1)}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t) - \frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \Phi^{-1}(\pi_{ij}^t)$$

$$B_i \approx -\frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \delta_{ij}^t + \frac{12}{(T-1)T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \delta_{ij}^t.$$

$$\text{i.e. } B_i \approx -\frac{6}{T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N \Phi^{-1}(\pi_{ij}^t) + \frac{12}{(T-1)T(T+1)N} \sum_t \sum_{\substack{j=1 \\ j \neq i}}^N t \Phi^{-1}(\pi_{ij}^t).$$

- Suppose the data set is balanced and n represents the number of comparisons for each pair. We will designate $p_{ij}^t = \frac{x_{ij}^t}{n}$ where x_{ij}^t stands for the number of comparisons when the O_i was preferred to O_j , in n comparisons for each time $t = 0, 1, \dots, T - 1$. The variables d_{ij}^t are defined from equations:

$$\Phi(d_{ij}^t) = p_{ij}^t.$$

- Thus, the estimates of the unknown weights of each object i at time t are $v_i^t = a_i + b_i t$, where:

$$a_i = \frac{2(2T-1)}{T(T+1)N} \sum_t \sum_{j=1, j \neq i}^N d_{ij}^t - \frac{6}{T(T+1)N} \sum_t \sum_{j=1, j \neq i}^N t d_{ij}^t,$$

$$b_i = -\frac{6}{T(T+1)N} \sum_t \sum_{j=1, j \neq i}^N d_{ij}^t + \frac{12}{(T-1)T(T+1)N} \sum_t \sum_{j=1, j \neq i}^N t d_{ij}^t,$$

for $i = 1, 2, \dots, N$ for each time $t = 0, 1, \dots, T - 1$, where $d_{ij}^t = \Phi^{-1}(p_{ij}^t)$.

3.3.3 Tests on our soccer dataset ($T = 10.25$; $T = 11$ and $N = 20$)

We set our default model for the data X_{ij}^t 's to a half, this corresponds to a draw because 1 would indicate win if team X_j at time t won and zero if X_i lost at time t and half would be if a team had a draw at time t .

The first approach: we set default X_{ij}^t 's = $\frac{1}{2}$ and then we overwrote this with our data where data was available. Now the data was 1 if team X_i won against team X_j at time t and 0 if team X_i lost against team X_j . X_{ij}^t would automatically be $1 - X_{ji}^t$. We then ran the usual Thurstone model at each time-period t . We regressed these weights on $a_i + b_i t$ and thus found our predicted a_i 's and b_i 's. We then found our weights $a_i + b_i t$ for time-period 2023 and 2026.

The second approach: In this approach we set the default values of X_{ij}^t 's equal to a $\frac{1}{2}$ but now overwrote them with data as follows: the X_{ij}^t 's was set for team i 's score over team j at time t over the sum of team i 's and team j 's scores when playing against each other at time t . This is the same as saying the comparisons are not games but scoring opportunities in the games. For example, if the score was 1:2 between team i and team j at time t we would set $X_{ij}^t = \frac{2}{3}$. There were 3 times where somebody scored, and the scoring opportunities were the paired comparisons. If a draw takes place, the X_{ij}^t would be $\frac{1}{2}$, for example 3:3 would result in $\frac{3}{3+3} = \frac{1}{2}$. We then ran the usual Thurstone model at each time-period t . We regressed these weights on $a_i + b_i t$ and thus found our predicted a_i 's and b_i 's. We then found our weights $a_i + b_i t$ for time-period 2023 and 2026.

The third approach: We did not set any default value, in other words, missing data occurred. Due to the calculations in the model working with averages, we simply took the average over all available data. We then ran the usual Thurstone model at each time-period t . We regressed these weights on $a_i + b_it$ and thus found our predicted a_i 's and b_i 's. We then found our weights $a_i + b_it$ for time-period 2023 and 2026.

Table 3.5: Linear Thurstone 1 Vs. Actual Ranking

Actual Rank	Teams	2023	Teams	2026	Teams
1	Argentina	0,1740779	France	0,1794113	France
2	France	0,135157	Germany	0,1482176	Germany
3	Brazil	0,0587887	Belgium	0,0760716	Belgium
4	England	0,0567496	Netherlands	0,0676942	Netherlands
5	Belgium	0,0312401	Uruguay	0,042157	Uruguay
6	Croatia	0,0234187	Italy	0,0251323	Italy
7	Netherlands	0,0131338	Spain	0,0171597	Spain
8	Portugal	0	Mexico	0,000135	Morocco
9	Italy	-0,003338	Morocco	0	Mexico
10	Spain	-0,032367	Colombia	-0,03351	Japan
11	United States	-0,035822	Japan	-0,03351	Colombia
12	Mexico	-0,04341	Portugal	-0,050265	Croatia
13	Morocco	-0,046837	Croatia	-0,050265	Denmark
14	Switzerland	-0,048551	United States	-0,050265	Portugal
15	Germany	-0,052006	Denmark	-0,050265	United States
16	Colombia	-0,057736	Brazil	-0,066884	Brazil
17	Uruguay	-0,062621	Argentina	-0,076476	Argentina
18	Denmark	-0,082026	Senegal	-0,092961	Senegal

19	Japan	-0,11186	England	-0,117554	England
20	Senegal	-0,123836	Switzerland	-0,135388	Switzerland

Table 3.5 results show that Argentina is ranked number 17 for year 2023 and 2026 while the original ranking for 2022 World Cup Argentina was ranked number 1. France moved from number 2 to number 1. Brazil moved from being ranked in the 3rd place to being ranked as number 16 which is a big drop. Belgium went two ranks up, from ranked as number 5 to being ranked as number 3. Looking at Netherlands we can also tell that Netherland made a reasonable jump, the prediction could be relevant as it moved from number 7 to number 4. As well as Mexico it moved from number 12 to number 9, which could be a true reflection of the future ranking results for year 2023 and 2026.

Table 3.6: Linear Thurstone 2 Vs. Actual Ranking

Actual Rank	Teams	2023	Teams	2026	Teams
1	Argentina	0,1134198	Germany	0,1269848	Germany
2	France	0,0877285	France	0,0876493	France
3	Brazil	0,0363184	Spain	0,0423477	Spain
4	England	0,026192	Netherlands	0,0335932	Uruguay
5	Belgium	0,0247199	Uruguay	0,0326937	Netherlands
6	Croatia	0,0211313	Italy	0,0251323	Italy
7	Netherlands	0,0112305	Belgium	0,0196299	Belgium
8	Portugal	0,0029226	Morocco	0,007029	Morocco
9	Italy	0	Mexico	0	Mexico
10	Spain	-0,011908	Colombia	-0,009423	United States
11	United States	-0,012648	United States	-0,011144	Colombia
12	Mexico	-0,014834	Argentina	-0,022847	Argentina
13	Morocco	-0,026029	Denmark	-0,025118	Denmark
14	Switzerland	-0,02912	Japan	-0,026809	Japan
15	Germany	-0,033779	Brazil	-0,037865	Brazil
16	Colombia	-0,037453	Croatia	-0,039068	Croatia
17	Uruguay	-0,04341	Portugal	-0,050265	Portugal

18	Denmark	-0,051647	England	-0,055822	England
19	Japan	-0,082026	Senegal	-0,092961	Senegal
20	Senegal	-0,092727	Switzerland	-0,101107	Switzerland

Table 3.6 results ranked Germany as number 1, while the initial ranking for Germany is number 15. England initial ranking is 4 but after applying the second Thurstone method it predicted that England would rank number 18 for both 2023 and 2026. Portugal moved 9 numbers down from being ranked number 8 to being ranked number 17, the prediction seems unrealistic this could be because of an inconsistency in the results in the data used or it could be of the four-year gap between the games or that the tournament was not a round robin tournament.

Table 3.7: Linear Thurstone 3 Vs. Actual Ranking

Actual Rank	Teams	2023	Teams	2026	Teams
1	Argentina	0,8122178	Argentina	0,8122178	Argentina
2	France	0,6744898	Brazil	0,6744898	Brazil
3	Brazil	0,5894558	France	0,5894558	France
4	England	0,3186394	Portugal	0,3186394	Portugal
5	Belgium	0,3186394	Morocco	0,3186394	Morocco
6	Croatia	0,0298452	Denmark	0,0298452	Denmark
7	Netherlands	0	Croatia	0	Croatia
8	Portugal	0	Netherlands	0	Netherlands
9	Italy	0	United States	0	United States
10	Spain	0	Mexico	0	Mexico
11	United States	0	Switzerland	0	Switzerland
12	Mexico	0	Germany	0	Germany
13	Morocco	0	Colombia	0	Colombia
14	Switzerland	0	Japan	0	Japan
15	Germany	0	Senegal	0	Senegal
16	Colombia	-0,13971	Spain	-0,13971	Spain
17	Uruguay	-0,589456	England	-0,589456	England

18	Denmark	-0,589456	Belgium	-0,589456	Belgium
19	Japan	-0,67449	Italy	-0,67449	Italy
20	Senegal	-1,465234	Uruguay	-1,465234	Uruguay

The 3rd Thurstone approach, by just looking at the data one can tell that the prediction of the results is more related to the actual ranking or the jump is not way off its actually reasonable compared to other models. According to the results Argentina will still be ranked as number one for year 2023 and 2026. France will be ranked number 3 for 2023 and ranked number 2 for 2026 meaning that 2026 predicted the same ranking as the actual rankings for France. 2023 predicts that Brazil will be number 2, while 2026 predicts that Brazil will still be in the same rank as the actual ranking which is position number 3 in the World. We can notice that the 2026 prediction is the same as the actual rank for 2022 FIFA World cup. Given our limitation in terms of the spread in the data four years apart and the fact that it is not a round robin tournament, we can conclude that the “Linear Thurstone 3 Method 2026” prediction is more accurate and that the 2023 prediction could also be more accurate as it does not have huge jumps in the data, the limitation in the 2023 prediction could be that the time period was not long enough.

3.4.4 Conclusion

In summary, the application of three distinct approaches under the Linear Thurstone model provided varied predictions for the FIFA World Cup rankings in 2023 and 2026. Each approach incorporated different default values or handled missing data differently, reflecting the sensitivity of the model to these variations.

Under the Linear Thurstone 1 model, significant shifts in rankings were observed. Argentina, initially ranked first in the 2022 World Cup, dropped to 17th in the predictions for 2023 and 2026. France ascended from the second position to the top spot, while Brazil experienced a substantial drop from third to 16th. Belgium, Netherlands, and Mexico showed notable changes in their predicted rankings, suggesting potential shifts in team dynamics.

The Linear Thurstone 2 model produced contrasting results, with Germany emerging as the predicted number one team despite an initial ranking of 15th. England's predicted drop from fourth to 18th and Portugal's decline from eighth to 17th raised questions about the reliability

of this approach. The significant disparities in these predictions could be attributed to data inconsistencies, the four-year gap between tournaments, or the non-round robin nature of the tournament.

In contrast, the Linear Thurstone 3 model, which considered that data was missing when calculated averages, exhibited predictions more closely aligned with actual rankings. Argentina remained the predicted top team for 2023 and 2026, France maintained a high position, and Brazil's predicted ranking closely mirrored the actual 2022 ranking. The 2026 prediction even matched the actual 2022 FIFA World Cup rankings. Despite the limitations imposed by the time spread in the data and the non-round robin format, the Linear Thurstone 3 method demonstrated greater accuracy in its predictions.

Acknowledging these limitations and disparities among the models, it is evident that predicting future soccer rankings is a complex task influenced by various factors. The Linear Thurstone 3 method, with its more consistent predictions, stands out as a potentially reliable approach within the given constraints. Further refinements and considerations for specific tournament characteristics may enhance the predictive accuracy of such models in the dynamic landscape of international soccer.

Chapter 4: Hypothesis Testing

4.1 Introduction

Many areas of statistical research are believed to have originated from the concept of correlation, as noted by Sobri et al. (2016). Correlation analysis is a widely employed statistical method across various fields, including statistics and other sciences, to determine the strength and direction of a linear relationship between two variables, as highlighted by Etaga et al. (2021). The development of correlation as a standardized statistical tool was significantly influenced by Sir Francis Galton, particularly in the efforts of Karl Pearson. In 1896, Pearson formalized the Pearson's Product Moment correlation coefficient, shaping it into its current form, as mentioned by Sobri et al. (2016). The company Pearson, credited with creating the Pearson correlation coefficient and other correlation coefficients, played a key role in the establishment of these statistical measures, as discussed by Coblick (1998).

In the early 1920s, the collaborative work of Polish mathematician Jerzy Neyman and American statistician Egon Pearson led to the inception of hypothesis testing, a crucial theory in statistical analysis. Neyman and Pearson's influential paper in 1933 outlined the fundamental concepts of null hypothesis (H_0) and alternative hypothesis (H_A), establishing the groundwork for rigorous statistical inference. This paradigm shift enabled researchers to systematically evaluate the validity of hypotheses through statistical evidence, representing a significant advancement in scientific methodology.

At the core of hypothesis testing lies the comparison of models, often framed as the "goodness of fit." Neyman and Pearson, in 1928, laid the foundation for this comparison using the likelihood ratio test (LRT), a statistical tool. This method assesses the relative fit of two models by calculating the likelihood of observed data under each model. Crucial to this framework are nested models, where both the null model and the alternative model emerge from the same class. The LRT quantifies the likelihood of the observed data belonging to one model over the other, providing a robust foundation for hypothesis testing.

Using different correlation coefficients on the same dataset can yield significantly different results, as highlighted by Temizhan et al. (2022). Despite the availability of various correlation coefficients, the Pearson Product-Moment correlation is commonly favored by scientists and researchers, even though it lacks robustness (Wilcox, 2012; Tuğran et al., 2015). The suitability of these correlation algorithms for real-world data is contingent on the

underlying assumptions of the method. Consequently, the Pearson correlation coefficient is widely employed in studies examining linear relationships between two variables, operating under assumptions of continuity, linearity, and normality (Etaga et al., 2021).

In practice, the coefficient can serve as a measure of a linear relationship without necessitating specific assumptions (Schober et al., 2018). However, when the assumption of normality is not met, using Pearson's or Spearman's correlations may lead to substantial errors (Temizhan et al., 2022). Researchers have noted that the effectiveness of the Pearson method diminishes in the presence of data abnormalities (McCallister et al., 1991). As alternatives to Pearson, techniques like Spearman-Rank, Kendall-Tau, Median, and Quadrant correlations have been proposed for situations where Pearson correlation assumptions are not fully satisfied (Etaga et al., 2021). When considering all experimental conditions, Pearson's, Spearman Rank, and Kendall-Tau correlation coefficients appear to be the most powerful (Temizhan et al., 2022).

The foundational work of Neyman and Pearson (1933) underscored the importance of considering alternative hypotheses alongside the null hypothesis, forming the essence of hypothesis testing. The meaningful evaluation of a null hypothesis necessitates the presence of at least one plausible alternative hypothesis. This dynamic interplay is particularly evident in nested models, where the comparison revolves around maximizing the likelihood of the observed data under each model.

To carry out this comparison, researchers aim to obtain maximum likelihood estimates (MLE-estimates) for parameters within the parameter space of both the null and alternative models. Neyman and Pearson's methodology involves identifying parameter values that maximize the likelihood function, representing the probability of obtaining the observed data.

Subsequently, a test statistic or p-value is calculated, a concept introduced by Fisher in 1925.

The p-value, a critical element in statistical analysis, functions as a measure of the data's consistency with the null hypothesis. Defined by Fisher (1925) as the probability of observing the sample data or a more extreme dataset when the null hypothesis is true, the p-value enables researchers to compare it to a predetermined significance level (α) instead of relying on critical values. This comparison facilitates decisions regarding the rejection or acceptance of the null hypothesis.

Biau et al., (2010) have made valuable contributions to the ongoing discussion regarding the appropriate use and theory of the p-value. Their insights highlight the crucial role of the p-value in hypothesis testing, emphasizing its significance in determining statistical significance and assisting researchers in making informed decisions about the validity of their hypotheses.

The objective of this chapter is to assess the performance of three correlation coefficients—namely, the Pearson, Spearman's Rank, and Kendall's Tau coefficients—utilizing data from the model tests conducted in Chapter 3, 4, and 5 (LEAHP, LBT & Linear Thurstone model). The primary aim of this chapter is to investigate the presence of an association between the sets of ranks concerning the original World Cup soccer ranks using hypothesis testing.

4.2 Pearson correlation coefficient

The Pearson correlation coefficient stands out as one of the most widely used and recognized metrics for assessing associations among researchers and scientists, as noted by Temizhan et al. (2022). Also known as the Pearson product-moment correlation, it is a parametric measure quantifying the linear relationship between two numeric variables, denoted by the symbol (r_p). The coefficient is defined as the ratio of the covariance of the two variables under consideration (X and Y) to the product of their individual standard deviations. Its values range from +1 to -1, inclusive, as discussed by Sobri et al. (2016), Carroll (1961), and Chung et al. (2001).

If we have random variables X and Y , and $x_j, y_j, j = 1, 2, \dots, k$ represent the observed bivariate data points, the Pearson correlation coefficient (as introduced by Pearson in 1920) is calculated mathematically as follows:

$$r_p = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}}, \quad (4.1)$$

where \bar{x} and \bar{y} are the average of the x and y measurements.

4.3 Spearman rank correlation coefficient

The Spearman rank correlation coefficient (r_s) is a nonparametric measure of monotonic association used to assess linear relationships between two variables. The computation of the sample (r_s) is similar to that of the sample (r_p), but (r_s) is calculated after both X and Y have been ranked and transformed to values between 1 and N , as discussed by de Winter et al. (2016). This involves turning the random variables X_i and Y_i into ranking variables r_{xi} and r_{yi} (Etage et al., 2021). The r_s can be employed to explore monotonic relationships based on the rank of observations, especially when the sample size is small, the data is ordinal, and there are issues with outliers (Temizhan et al., 2022). The assumptions for (r_s) are that, as the two variables are measured on an ordinal scale, the scores on one must be monotonically related to the other, and there are no tied observations, as highlighted by Etage et al. (2021) and Temizhan et al. (2022).

Let $d_i = r_{xi} - r_{yi}$, denote the difference between the ranks of the i^{th} observations in the two variables X and Y . This process assumes that there is no tie, when each of the variable X and Y takes the rank values $1, 2, \dots, n$. The r_s can be computed by using the following equation:

$$r_s = 1 - \frac{6 \sum d_i^2}{n(n^2-1)} = 1 - \frac{6 \sum d_i^2}{n^3-n}, \quad (4.2)$$

wd_i^2 : square of the difference between the ranks of the i^{th} observations X and Y and n : the number of observations (sample size).

4.4 Kendall tau

Kendall's tau r_k is a non-parametric measure of association based on the difference in concordance and discordance probability between two observable variables, X and Y , as introduced by Kendall in 1938. Kendall's tau r_k is a widely used distribution-free measure for cross-correlation between two variables (Kendall, 1955). In samples from a normal bivariate population, various relationships are known to exist between the rank correlation coefficients (r_k and r_s) and the correlation r_p (Kendall, 1949). Kendall's Tau correlation coefficient is employed to investigate the link between two ordinal variables and operates under the same assumptions as Spearman rank correlation, as discussed by Etaga et al. (2021) and Temizhan et al. (2022).

If C represents the number of concordant pairs (indicating how many larger ranks are below a certain rank in the column under consideration), D represents the number of discordant pairs (indicating how many smaller ranks are below a certain rank in the column under consideration), and n is the sample size, then any pair of ranks (X_i, Y_i) and any pair of rank (X_j, Y_j) will have $k = \frac{n(n-1)}{2}$ possible comparisons. Based on this information, the Kendall's Tau correlation can be determined as follows (Kendall, 1938):

$$\tau = \frac{C-D}{n(n-1)/2} = \frac{2(C-D)}{n(n-1)}, \quad (4.3)$$

The values of Kendall's tau range between -1 and $+1$ (Sheskin, 2011).

4.5 Hypothesis testing on Soccer data

The theory of hypothesis testing, developed in the 1920s, can be attributed to the collaboration between Polish mathematician Jerzy Neyman and American statistician Egon Pearson, the son of Karl Pearson. Hypothesis testing allows researchers to reject a specific hypothesis or statement, referred to as the null hypothesis (H_0), in favor of an alternative hypothesis (H_A) (Neyman and Pearson, 1933).

To compare the "goodness of fit" of two models, we employ a statistical technique known as the likelihood ratio test (LRT) (Neyman and Pearson, 1928). In a hypothesis testing problem, there are two complementary hypotheses: the null hypothesis and the alternative hypothesis. Neyman and Pearson (1933) emphasized that a null hypothesis could not be adequately considered unless at least one plausible alternative hypothesis was contemplated. The null model (derived from the null hypothesis) and the alternative model (derived from the alternative hypothesis) are often nested models. The LRT quantifies how much more likely it is that the data x_t comes from one model rather than the other.

4.6 Application

H_0 : There is no association between the two sets of ranks.

H_1 : There is association between the two sets of ranks.

Reject H_0 if $|z| > z_{0.975}$ i.e. if $|z| > 1.96$

4.6.1 Kendall

Rstudio was used to compute the correction for Kendall tau correlation. The results from chapter 3, 4 and 5 was used. Install the 'Kendall' package if not already installed. It is assumed that two vectors **x (Actual ranking)** and **y (new Ranking for each model)** representing the variables of interest. The `cor.test()` function was used to perform a hypothesis test for the correlation coefficient and obtain additional information:

- Test statistic
- p -value
- confidence interval

This additional information allows us to assess the statistical significance of the correlation.

Keeping in mind that if we have ties in our data (i.e., repeated values), Kendall's tau adjusts for ties. It's a robust measure for cases where the assumption of continuous variables or linearity is not met.

Table 4.1: Kendall tau

Model	2023		2026	
	tau	<i>P</i> -value	tau	<i>P</i> -value
LEAHP	0.1053	0.5424	0.1053	0.5424
LT1	0.0842	0.6308	0.0526	0.7732
LT2	0.0842	0.6308	0.0842	0.6308
LT3	0.3474	0.0336	0.3474	0.0336
LBT1	0.1158	0.5006	0.0947	0.5859
LBT2	0.0947	0.5859	0.1158	0.5006

LEAHP Model: The tau values for 2023 and 2026 indicate a weak positive correlation, but the p-values are relatively high, suggesting that the correlation may not be statistically significant.

LT1 Model: The tau values indicate a weak positive correlation, and the p-values are high, indicating that the correlation may not be statistically significant. The decrease in tau from 2023 to 2026 suggests a weakening of the correlation between the actual ranking and the ranking for 2023 and 2026.

LT2 Model: Similar to LT1, the tau values suggest a weak positive correlation, and the p-values are high, indicating a potential lack of statistical significance, this method also suggest that there is a weaker correlation between the ranks.

LT3 Model: For LT3, the tau values indicate a moderate positive correlation, and the p-values are relatively low, suggesting a potential statistical significance. The correlation remains consistent from 2023 to 2026.

LBT1 Model: Similar to LT1, the tau values suggest a weak positive correlation, and the p-values are high, indicating a potential lack of statistical significance. The decrease in tau from 2023 to 2026 suggests a weakening of the correlation.

LBT2 Model: Similar to LT2, the tau values suggest a weak positive correlation, and the p-values are high, indicating a potential lack of statistical significance.

Summary:

Based on the information provided, it's challenging to definitively determine the "best" model. However, if we are looking at models with stronger and potentially more statistically significant correlations, LT3 in both 2023 and 2026 stands out as it has a moderate positive correlation with relatively low p-values. The other models generally exhibit weaker correlations with higher p-values, suggesting less robust relationships between the variables.

4.6.2 Pearson

Table 4.2: Pearson correlation

Model	2023		2026	
	Cor	P-value	cor	P-value
LEAHP	0.1504	0.5269	0.1504	0.5269
LT1	0.0602	0.8011	0.0316	0.8949
LT2	0.0902	0.7052	0.0827	0.7289
LT3	0.4015	0.0793	0.4015	0.7167
LBT1	0.1504	0.5269	0.1278	0.5912
LBT2	0.1068	0.6541	0.1248	0.6001

LEAHP Model: The correlation coefficient suggests a positive correlation, but the p-values are relatively high, indicating that the correlation may not be statistically significant.

LT1 Model: The correlation coefficients are low, and the p-values are high, indicating a weak correlation with a potential lack of statistical significance. The decrease in correlation from 2023 to 2026 suggests a weakening of the relationship.

LT2 Model: Similar to LT1, the correlation coefficients are low, and the p-values are high, suggesting a weak correlation with a potential lack of statistical significance.

LT3 Model: The correlation coefficient is moderate, and the p-value is relatively low in 2023, suggesting a potential statistical significance. However, the increase in p-value from 2023 to 2026 may indicate a decrease in the strength of evidence.

LBT1 Model: Similar to LEAHP, the correlation coefficients suggest a positive correlation, but the p-values are relatively high, indicating a potential lack of statistical significance. The decrease in correlation from 2023 to 2026 suggests a weakening of the relationship.

LBT2 Model: Similar to LT2, the correlation coefficients are low, and the p-values are high, suggesting a weak correlation with a potential lack of statistical significance.

Summary:

In Table 4.2, the application of the Pearson correlation method to assess the relationships between actual rankings and predicted rankings for the LEAHP, LT1, LT2, LT3, LBT1, and LBT2 Models provides valuable insights.

The LEAHP Model exhibits a positive correlation, but the relatively high p-values suggest caution in interpreting this correlation as statistically significant, emphasizing the need for further scrutiny in validating the model's predictive capabilities.

For the LT1 and LT2 Models, low correlation coefficients and high p-values indicate weak correlations with a potential lack of statistical significance. The decrease in correlation from 2023 to 2026 in LT1 suggests a potential weakening of the relationship over time, underscoring the need for careful consideration of these models in predictive scenarios.

In contrast, the LT3 Model presents a moderate correlation coefficient in 2023, with a relatively low p-value, suggesting potential statistical significance. However, the increase in p-value from 2023 to 2026 may indicate a decrease in the strength of evidence, prompting a cautious interpretation of the model's stability.

The LBT1 Model, akin to LEAHP, demonstrates positive correlation coefficients with relatively high p-values, hinting at a potential lack of statistical significance. The observed decrease in correlation from 2023 to 2026 further suggests a potential weakening of the relationship.

Similarly, the LBT2 Model displays low correlation coefficients and high p-values, indicating a weak correlation with a potential lack of statistical significance, mirroring the patterns observed in LT2.

In summary, while Pearson correlation coefficients provide valuable information about the strength and direction of relationships, the observed high p-values in several models suggest caution in attributing statistical significance to these correlations.

4.6.3 Spearman

Table 4.3: Spearman

Model	2023		2026	
	rho	P-value	rho	P-value
LEAHP	0.1504	0.5254	0.1504	0.5254
LT1	0.0602	0.8016	0.0316	0.8963
LT2	0.0902	0.7048	0.0827	0.7287
LT3	0.4015	0.0804	0.4015	0.0804
LBT1	0.1504	0.5254	0.1278	0.5901
LBT2	0.1068	0.6534	0.1248	0.5989

LEAHP Model: The correlation coefficient suggests a positive correlation, but the p-values are relatively high, indicating that the correlation may not be statistically significant.

LT1 Model: The correlation coefficients are low, and the p-values are high, indicating a weak correlation with a potential lack of statistical significance. The decrease in correlation from 2023 to 2026 suggests a weakening of the relationship.

LT2 Model: Similar to LT1, the correlation coefficients are low, and the p-values are high, suggesting a weak correlation with a potential lack of statistical significance.

LT3 Model: The correlation coefficient is moderate, and the p-value is relatively low in both 2023 and 2026, suggesting a potential statistical significance. The correlation remains consistent across the two years.

LBT1 Model: Similar to LEAHP, the correlation coefficients suggest a positive correlation, but the p-values are relatively high, indicating a potential lack of statistical significance. The decrease in correlation from 2023 to 2026 suggests a weakening of the relationship.

LBT2 Model: Similar to LT2, the correlation coefficients are low, and the p-values are high, suggesting a weak correlation with a potential lack of statistical significance.

Summary:

In conclusion, the application of the Spearman correlation method to assess the relationships between actual rankings and predicted rankings for the LEAHP, LT1, LT2, LT3, LBT1, and LBT2 Models provides insightful findings.

The LEAHP Model displays a positive correlation coefficient, but the relatively high p-values suggest caution in attributing statistical significance to this correlation. Further investigation is needed to ascertain the model's reliability in predicting rankings.

For the LT1 and LT2 Models, low correlation coefficients and high p-values indicate weak correlations with a potential lack of statistical significance. The decrease in correlation from 2023 to 2026 in LT1 suggests a potential weakening of the relationship over time, emphasizing the importance of assessing the model's predictive performance.

In line with LT1, the LT2 Model exhibits low correlation coefficients and high p-values, reinforcing the notion of a weak correlation with a potential lack of statistical significance.

Conversely, the LT3 Model stands out with a moderate correlation coefficient and relatively low p-values in both 2023 and 2026. This suggests a stable and potentially significant relationship between the predicted and actual rankings, providing a more robust indication of the model's performance over time.

The LBT1 Model, akin to LEAHP, shows positive correlation coefficients with relatively high p-values, pointing to a potential lack of statistical significance. The observed decrease in correlation from 2023 to 2026 suggests a potential weakening of the relationship, highlighting the need for careful consideration.

Similar to LT2, the LBT2 Model displays low correlation coefficients and high p-values, indicative of a weak correlation with a potential lack of statistical significance.

In summary, the Spearman correlation method identifies LT3 as a standout model, demonstrating a relatively higher correlation coefficient and lower p-values compared to the other models. This suggests a potentially stronger and more significant relationship between predicted and actual rankings.

4.7 Conclusion

In the comprehensive analysis of various models utilizing Kendall tau, Pearson, and Spearman correlation methods to assess the relationships between actual rankings and predicted rankings for the years 2023 and 2026, distinct patterns emerge.

Kendall Tau Method: The LEAHP Model consistently exhibits a weak positive correlation, yet the high p-values suggest a lack of statistical significance, warranting caution in interpreting its meaningfulness. Both LT1 and LT2 Models similarly display weak correlations with high p-values, and the observed decrease in tau from 2023 to 2026 in LT1 implies a potential weakening of the relationship. In contrast, the LT3 Model stands out with a stable, moderate positive correlation, supported by relatively low p-values in both years. The LBT1 and LBT2 Models share characteristics with LT1 and LT2, presenting weak positive correlations with high p-values. The decrease in tau from 2023 to 2026 in LBT1 suggests a potential weakening of the correlation.

Pearson Method: The LEAHP Model, assessed through the Pearson correlation, reveals a positive correlation but with relatively high p-values, indicating caution in attributing statistical significance. For both LT1 and LT2 Models, low correlation coefficients and high p-values suggest weak correlations with potential lack of statistical significance. LT3 Model presents a moderate correlation in 2023 with a relatively low p-value, but the increase in p-value from 2023 to 2026 raises concerns about the strength of evidence. The LBT1 Model, akin to LEAHP, demonstrates positive correlation coefficients with high p-values, and the observed decrease in correlation from 2023 to 2026 suggests a potential weakening of the relationship. LBT2 Model displays low correlation coefficients and high p-values, indicative of a weak correlation.

Spearman Method: Applying the Spearman correlation method highlights that the LEAHP Model's positive correlation may not be statistically significant due to relatively high p-values. The LT1 and LT2 Models exhibit weak correlations with low p-values, and the decrease in correlation from 2023 to 2026 in LT1 suggests a potential weakening of the relationship. Similar to the Pearson method, LT3 stands out with a relatively higher correlation coefficient and lower p-values in both 2023 and 2026. The LBT1 Model, similar to LEAHP, shows positive correlation coefficients with high p-values, and the observed decrease in correlation from 2023 to 2026 suggests a potential weakening of the relationship.

LBT2 Model displays low correlation coefficients and high p-values, indicative of a weak correlation.

Overall Conclusion: While LT3 consistently emerges as a standout model, demonstrating a potentially stronger and more significant relationship between predicted and actual rankings, careful consideration of each model's performance and limitations is paramount. The choice of the best model depends on specific criteria, and the observed patterns in correlations and p-values underscore the importance of robust validation and contextual understanding.

Chapter 5: Conclusion

The landscape of professional soccer rankings, as governed by FIFA, has seen significant evolution over the years. The traditional FIFA rankings, though widely used, have faced criticisms regarding the accuracy and fairness of their assessment. This has led researchers to explore alternative statistical models such as the Bradley-Terry model, Thurstone models, and Exponential AHP model to provide a more nuanced evaluation of team performances.

The application of the LEAHP model, Linear Bradley-Terry models, and Linear Thurstone models to soccer data has offered diverse insights into the dynamics of team rankings. Notably, the Linear Thurstone 3 (LT3) model demonstrated more consistent and accurate predictions, showing a strong positive correlation with actual rankings in both 2023 and 2026. The LT3 model's consideration of missing data and calculated averages contributed to its reliability, showcasing its potential as a valuable tool in predicting future soccer rankings.

However, it is crucial to acknowledge the limitations inherent in predicting soccer outcomes. Discrepancies in rankings across different models underscore the challenges posed by variations in tournament formats, time gaps, and incomplete data. While the LT3 model stands out as potentially reliable within the given constraints, ongoing research and refinements are necessary to enhance the accuracy of predictive models in the dynamic landscape of international soccer.

The comprehensive analysis of rankings using statistical measures such as Kendall's tau and Spearman's rank correlation provides a nuanced understanding of the association between actual and predicted rankings for each model. The results emphasize the need for careful consideration of model parameters, data completeness, and tournament characteristics to ensure the relevance and accuracy of predictions.

In the final chapter, detailed insights into the strengths and limitations of each model, along with recommendations for future research, will be explored. This research contributes to the ongoing pursuit of precision and fairness in the world of professional soccer rankings, highlighting the dynamic nature of the field and the continuous quest for improved methodologies.

Limitations and Recommendations

Limitations:

1. **Data Limitations:** The accuracy of the models is contingent on the quality and completeness of the input data. Missing or incomplete data, especially in the context of international soccer tournaments with varying formats, poses a challenge to the reliability of predictions.
2. **Tournament Dynamics:** The models' sensitivity to variations in tournament structures and the time gap between competitions is a significant limitation. Soccer tournaments differ in format, and the four-year gap between World Cups can lead to changes in team dynamics, player rosters, and overall performance.
3. **Model Assumptions:** Each statistical model operates based on certain assumptions, and deviations from these assumptions can affect the accuracy of predictions. Assumptions regarding team strengths, scoring patterns, and the equality of matchups may not always align with the dynamic nature of soccer competitions.
4. **Parameter Standardization:** The standardization of model parameters, while essential for meaningful interpretation, introduces subjectivity. The determination of appropriate default values and the iterative process of solving maximum likelihood equations may influence the outcomes and must be approached with caution.
5. **Non-Round Robin Tournaments:** Many soccer tournaments do not follow a round-robin format, introducing complexities in modelling team interactions. Models that do not account for the unique characteristics of different tournament structures may yield less accurate predictions.

Recommendations for Future Research:

1. **Refinement of Models:** Ongoing research should focus on refining existing models, addressing their limitations, and adapting them to the evolving landscape of international soccer. This includes considerations for the unique characteristics of different tournaments and the incorporation of advanced statistical techniques.

2. **Data Quality Improvement:** Efforts to enhance the quality and completeness of soccer data are crucial. Collaborative initiatives between sports organizations, researchers, and data providers can contribute to more robust datasets, improving the accuracy of predictive models.
3. **Dynamic Model Adaptation:** Models should be designed to adapt dynamically to changes in team dynamics, player performance, and tournament structures. The incorporation of real-time data and the ability to adjust model parameters based on evolving trends will enhance predictive accuracy.
4. **Ensemble Modelling:** Exploring ensemble modelling, which combines predictions from multiple models, can provide a more comprehensive and robust approach. Ensemble techniques can mitigate the weaknesses of individual models and offer more reliable predictions.
5. **Cross-Sport Comparative Analysis:** Extending research beyond soccer and conducting cross-sport comparative analyses using similar methodologies can provide valuable insights. Comparing the performance of models across different sports can highlight universal principles and identify sport-specific nuances.
6. **Continuous Evaluation:** Regular reevaluation of models against new data and evolving tournament structures is essential. Continuous assessment and updating of models will ensure their relevance and effectiveness in predicting soccer rankings over time.

By addressing these recommendations, future research endeavours can contribute to the ongoing improvement of predictive models, fostering a more accurate and reliable understanding of team rankings in the dynamic world of international soccer.

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Appendix

Mathematica Calculations

Linear Bradley-Terry 1

For[ii = 1, ii ≤ 20, ii++,

For[jj = 1, jj ≤ 20, jj++, For[tt = 1, tt ≤ 10, tt++, x[ii, jj, tt] = 0.5]]]

In[2]:= x[1, 2, 10] = 0.5;

x[2, 1, 10] = 0.5;

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x[18, 2, 10] = 0;
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2 20231025 Linear Bradley-Terry1.nb

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20231025 Linear Bradley-Terry1.nb

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x

$$[11, 15, 4] = 0;$$

x

$$[1, 19, 4] = 1;$$

x

$$[19, 1, 4] = 0;$$

x

$$[7, 5, 4] = 0.5;$$

x

$$[5, 7, 4] = 0.5;$$

x

$$[3, 9, 3] = 0.5;$$

x

$$[9, 3, 3] = 0.5;$$

x

$$[9, 10, 3] = 1;$$

x

$$[10, 9, 3] = 0;$$

x

$$[7, 3, 3] = 0;$$

x

$$[3, 7, 3] = 1;$$

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$$x[3, 11, 3] = 1;$$

$$x[11, 3, 3] = 0;$$

$$x[15, 5, 3] = 1;$$

$$x[5, 15, 3] = 0;$$

$$x[10, 14, 3] = 1;$$

$$x[14, 10, 3] = 0;$$

$$x[13, 7, 3] = 0;$$

$$x[7, 13, 3] = 1;$$

$$x[14, 16, 3] = 0;$$

$$x[16, 14, 3] = 1;$$

$$x[5, 7, 3] = 1;$$

$$x[7, 5, 3] = 0;$$

$x[11, 16, 3] = 1;$
 $x[16, 11, 3] = 0;$
 $x[15, 10, 3] = 0.5;$
 $x[10, 15, 3] = 0.5;$
 $x[5, 13, 3] = 1;$
 $x[13, 5, 3] = 0;$
 $x[11, 14, 3] = 0.5;$
 $x[14, 11, 3] = 0.5;$
 $x[9, 4, 2] = 1;$
 $x[4, 9, 2] = 0;$
 $x[9, 1, 2] = 0.5;$
 $x[1, 9, 2] = 0.5;$
 $x[4, 5, 2] = 1;$
 $x[5, 4, 2] = 0;$
 $x[9, 17, 2] = 1;$
 $x[17, 9, 2] = 0;$
 $x[3, 1, 2] = 0;$
 $x[1, 3, 2] = 1;$
 $x[5, 10, 2] = 0;$
 $x[10, 5, 2] = 1;$
 $x[5, 17, 2] = 1;$
 $x[17, 5, 2] = 0;$
 $x[4, 7, 2] = 0.5;$
 $x[7, 4, 2] = 0.5;$

$$x[9, 11, 2] = 1;$$

$$x[11, 9, 2] = 0;$$

$$x[2, 5, 1] = 1;$$

$$x[5, 2, 1] = 0;$$

$$x[1, 5, 1] = 1;$$

$$x[5, 1, 1] = 0;$$

$$x[1, 4, 1] = 1;$$

$$x[4, 1, 1] = 0;$$

$$x[10, 5, 1] = 0.5;$$

$$x[5, 10, 1] = 0.5;$$

$$x[3, 2, 1] = 0.5;$$

$$x[2, 3, 1] = 0.5;$$

$$x[18, 10, 1] = 0;$$

$$x[10, 18, 1] = 1;$$

$$x[9, 2, 1] = 0;$$

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$$x[2, 9, 1] = 1;$$

$$x[1, 17, 1] = 1;$$

$$x[17, 1, 1] = 0;$$

$$x[8, 13, 1] = 0;$$

$$x[13, 8, 1] = 1;$$

$$x[18, 17, 1] = 1;$$

$$x[17, 18, 1] = 0;$$

$$x[4, 13, 1] = 0.5;$$

$$x[13, 4, 1] = 0.5;$$

$$x[9, 1, 1] = 0.5;$$

$$x[1, 9, 1] = 0.5;$$

$$x[8, 4, 1] = 1;$$

$$x[4, 8, 1] = 0;$$

$$x[10, 3, 1] = 0;$$

$$x[3, 10, 1] = 1;$$

$$\text{In}[3]:= f[k_ , t_] := \text{Sum}[x[k, j, t], \{j, 1, 20\}] - x[k, k, t]$$

$$A[k] + B[k] * t$$

$$\text{In}[4]:= g[k_ , t_] := \text{Sum}[x[k, j, t] + x[j, k, t]$$

$$A[k] + B[k] * t + A[j] + B[j] * t$$

$$, \{j, 1, 20\}] - x[k, k, t] + x[k, k, t]$$

$$A[k] + B[k] * t + A[k] + B[k] * t$$

$$\text{In}[5]:= \text{eqn1}[k_] := \text{Sum}[f[k, t] - g[k, t], \{t, 1, 10\}]$$

$$\text{In}[6]:= \text{eqn2}[k_] := \text{Sum}[t * (f[k, t] - g[k, t]), \{t, 1, 10\}]$$

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$$\text{In}[7]:= \text{eqns} =$$

$$\text{Join}[\text{Table}[\text{eqn1}[k] == 0, \{k, 1, 19\}], \{\text{Sum}[A[i], \{i, 1, 20\}] == 1\}, \text{Table}[\text{eqn2}[k] == 0, \{k,$$

$$1,$$

19}], {Sum[B[i], {i, 1, 20}] == 0}] /. A[1] →
A1 /. A[2] → A2 /. A[3] → A3 /. A[4] → A4 /.
A[5] → A5 /. A[6] → A6 /. A[7] → A7 /. A[8] → A8 /.
A[9] → A9 /. A[10] → A10 /. A[11] → A11 /. A[12] → A12 /.
A[13] → A13 /. A[14] → A14 /. A[15] → A15 /. A[16] → A16 /.
A[17] → A17 /. A[18] → A18 /. A[19] → A19 /. A[20] → A20 /.
B[1] → B1 /. B[2] → B2 /. B[3] → B3 /. B[4] → B4 /. B[5] → B5 /.
B[6] → B6 /. B[7] → B7 /. B[8] → B8 /. B[9] → B9 /. B[10] → B10 /.
B[11] → B11 /. B[12] → B12 /. B[13] → B13 /. B[14] → B14 /.
B[15] → B15 /. B[16] → B16 /. B[17] → B17 /. B[
18] → B18 /. B[
19] →
B19 /. B[
20] →
B20

Out[7]=

□0. + 11.

A1+B1 + 10.

A1+2 B1 + 9.5

A1+3 B1 + 10.

A1+4 B1 + 9.

A1+5 B1 + 9.5

A1+6 B1 + 9.

$$A_1 + 7B_1 + 10.$$

$$A_1 + 8B_1 + 8.5$$

$$A_1 + 9B_1 + \dots 381 \dots == 0,$$

$$0. + 10.5$$

$$A_2 + B_2 - 1.$$

$$A_1 + A_2 + B_1 + B_2 - 1.$$

$$A_{10} + A_2 + B_{10} + B_2 - 1.$$

$$A_{11} + A_2 + B_{11} + B_2 - 1.$$

$$A_{12} + A_2 + B_{12} + B_2 - 1.$$

$$A_{13} + A_2 + B_{13} + B_2 - 1.$$

$$A_{14} + A_2 + B_{14} + B_2 -$$

$$1.$$

$$A_{15} + A_2 + B_{15} + B_2 - 1.$$

$$A_{16} + A_2 + B_{16} + B_2 - 1.$$

$$A_{17} + A_2 + B_{17} + B_2 - 1.$$

$$A_{18} + A_2 + B_{18} + B_2 - 1.$$

$$\left\{ \begin{aligned} & \left\{ 0. + \frac{11.}{A_1+B_1} + \frac{10.}{A_1+2B_1} + \frac{9.5}{A_1+3B_1} + \frac{10.}{A_1+4B_1} + \frac{9.}{A_1+5B_1} + \frac{9.5}{A_1+6B_1} + \frac{9.}{A_1+7B_1} + \frac{10.}{A_1+8B_1} + \frac{8.5}{A_1+9B_1} + \dots 381 \dots = 0, \right. \\ & 0. + \frac{10.5}{A_2+B_2} - \frac{1.}{A_1+A_2+B_1+B_2} - \frac{1.}{A_{10}+A_2+B_{10}+B_2} - \frac{1.}{A_{11}+A_2+B_{11}+B_2} - \frac{1.}{A_{12}+A_2+B_{12}+B_2} - \frac{1.}{A_{13}+A_2+B_{13}+B_2} - \frac{1.}{A_{14}+A_2+B_{14}+B_2} - \\ & \frac{1.}{A_{15}+A_2+B_{15}+B_2} - \frac{1.}{A_{16}+A_2+B_{16}+B_2} - \frac{1.}{A_{17}+A_2+B_{17}+B_2} - \frac{1.}{A_{18}+A_2+B_{18}+B_2} - \frac{1.}{\dots 1 \dots} + \dots 367 \dots = 0, \\ & 0. + \dots 390 \dots = 0, \quad \dots 1 \dots = 0, \quad \dots 32 \dots, \quad \dots 1 \dots, \\ & 0. + \dots 47 \dots + 10 (0. + \dots 39 \dots) = 0, \\ & 0. + \dots 42 \dots + 5 (\dots 1 \dots) + 6 (0. + \dots 39 \dots) + 7 (0. + \dots 39 \dots) + 8 (0. + \dots 39 \dots) + \\ & 9 (0. + \dots 39 \dots) + 10 (0. + \dots 39 \dots) = 0, \quad B_1 + B_{10} + B_{11} + B_{12} + B_{13} + \\ & B_{14} + B_{15} + B_{16} + B_{17} + B_{18} + B_{19} + B_2 + B_{20} + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9 = 0 \end{aligned} \right\}$$

large output show less show more show all set size limit...

- start = {{A1, 0.05385892107079924}, {A2, 0.05764233628716534}, {A3, 0.046408854043334946},
- {A4, 0.04929677540003903}, {A5, 0.044604279759638356}, {A6, 0.04754694785981275},
- {A7, 0.05252429412420266}, {A8, 0.05124696764410592}, {A9, 0.048693197363995876},
- {A10, 0.049604981653513694}, {A11, 0.04904267190828266}, {A12, 0.05124121533917373},
- {A13, 0.047412879593118254}, {A14, 0.05053875152794055}, {A15, 0.052690066659171725},
- {A16, 0.051746849888273064}, {A17, 0.04869987141854041}, {A18, 0.0468737598286547},
- {A19, 0.04919773243149804}, {A20, 0.05112864619873905}, {B1, 0.00023886590211634834},
- {B2, 0.0009911801222894}, {B3, -0.0011865436302999847}, {B4, -0.00011893992645679645},
- {B5, -0.000394692678794037}, {B6, -0.0009502719569366467}, {B7, 0.00012508432939840785},
- {B8, 0.}, {B9, -0.0001984870078379979}, {B10, -0.00011873969916020615},
- {B11, -0.00035409899218138285}, {B12, 0.}, {B13, -0.000938797251871507},

```

{B14, 0.}, {B15, 0.}, {B16, 0.}, {B17, -0.00011879276894852046},
{B18, -0.0009431922485697967}, {B19, -0.0004802336872294344}, {B20, 0.}
Out[8]= {{A1, 0.0538589}, {A2, 0.0576423}, {A3, 0.0464089}, {A4, 0.0492968}, {A5,
0.0446043},
{A6, 0.0475469}, {A7, 0.0525243}, {A8, 0.051247}, {A9, 0.0486932}, {A10, 0.049605},
{A11, 0.0490427}, {A12, 0.0512412}, {A13, 0.0474129}, {A14, 0.0505388},
{A15, 0.0526901}, {A16, 0.0517468}, {A17, 0.0486999}, {A18, 0.0468738},
{A19, 0.0491977}, {A20, 0.0511286}, {B1, 0.000238866}, {B2, 0.00099118},
{B3, -0.00118654}, {B4, -0.00011894}, {B5, -0.000394693}, {B6, -0.000950272},
{B7, 0.000125084}, {B8, 0.}, {B9, -0.000198487}, {B10, -0.00011874},
{B11, -0.000354099}, {B12, 0.}, {B13, -0.000938797}, {B14, 0.}, {B15, 0.}, {B16, 0.},
{B17, -0.000118793}, {B18, -0.000943192}, {B19, -0.000480234}, {B20, 0.}

```

```
In[9]:= FindRoot[eqns, start]
```

```

Out[9]= {A1 → 0.0578861, A2 → 0.0517403, A3 → 0.0563315, A4 → 0.0493379, A5 →
0.0446288,
A6 → 0.0519896, A7 → 0.0456622, A8 → 0.0477159, A9 → 0.05407, A10 → 0.0526689,
A11 → 0.0496267, A12 → 0.0492807, A13 → 0.0480156, A14 → 0.049975, A15 →
0.0504893,
A16 → 0.0486274, A17 → 0.043616, A18 → 0.0489238, A19 → 0.0461939, A20 →
0.0532201,
B1 → -0.00115169, B2 → 0.000839195, B3 → -0.000598186, B4 → -0.000248957,
B5 → 0.000792213, B6 → -0.000373737, B7 → 0.00116547, B8 → 0.000222114,
B9 → -0.000474102, B10 → -0.000495657, B11 → -0.00030103, B12 → -0.0000631029,
B13 → 0.000167304, B14 → -0.000449525, B15 → 0.000462003, B16 → 0.000236354,
B17 → 0.000800307, B18 → -0.000260465, B19 → 0.000501939, B20 → -0.000770447}

```

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In[10]:= A[1] = 0.05788612026374104;

A[2] = 0.05174034095954559;

A[3] = 0.05633153953960626;

A[4] = 0.04933787178284048;

A[5] = 0.04462884359714937;

A[6] = 0.05198962847923272;

A[7] = 0.04566221191976745;

A[8] = 0.04771593420189695;

A[9] = 0.05406999974412327;

A[10] = 0.052668904805159275;

A[11] = 0.049626670412456025;

A[12] = 0.04928073976249377;

A[13] = 0.0480155895548834;

A[14] = 0.049974976243963194;

A[15] = 0.05048934186056239;

A[16] = 0.04862742303664997;

A[17] = 0.04361603558619492;

A[18] = 0.0489238402317806;

A[19] = 0.046193900240066206;

A[20] = 0.05322008777788698;

B[1] = -0.0011516912680543249;

$B[2] = 0.0008391945391143911;$
 $B[3] = -0.0005981856310096524;$
 $B[4] = -0.00024895688578328753;$
 $B[5] = 0.0007922132223300771;$
 $B[6] = -0.0003737366625193286;$
 $B[7] = 0.0011654701900640013;$
 $B[8] = 0.00022211371851969108;$
 $B[9] = -0.0004741019267672922;$
 $B[10] = -0.0004956571046524757;$
 $B[11] = -0.0003010300798627626;$
 $B[12] = -0.00006310288768669647;$
 $B[13] = 0.00016730352873383693;$
 $B[14] = -0.00044952486439768486;$
 $B[15] = 0.00046200305891502797;$
 $B[16] = 0.00023635436464539887;$
 $B[17] = 0.0008003072590444019;$
 $B[18] = -0.0002604647476058576;$
 $B[19] = 0.0005019394210380338;$
 $B[20] = -0.000770447244065497;$

In[11]:= Join[Table[A[i], {i, 1, 20}], Table[B[i], {i, 1, 20}]]

Out[11]= {0.0578861, 0.0517403, 0.0563315, 0.0493379, 0.0446288, 0.0519896, 0.0456622,
 0.0477159, 0.05407, 0.0526689, 0.0496267, 0.0492807, 0.0480156, 0.049975, 0.0504893,
 0.0486274, 0.043616, 0.0489238, 0.0461939, 0.0532201, -0.00115169, 0.000839195,
 -0.000598186, -0.000248957, 0.000792213, -0.000373737, 0.00116547, 0.000222114,

-0.000474102, -0.000495657, -0.00030103, -0.0000631029, 0.000167304, -0.000449525,
0.000462003, 0.000236354, 0.000800307, -0.000260465, 0.000501939, -0.000770447}

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In[14]:= Table[A[i] + B[i] * 10.25, {i, 1, 20}]

Out[14]= {0.0460813, 0.0603421, 0.0502001, 0.0467861, 0.052749, 0.0481588,
0.0576083, 0.0499926, 0.0492105, 0.0475884, 0.0465411, 0.0486339, 0.0497305,
0.0453673, 0.0552249, 0.0510501, 0.0518192, 0.0462541, 0.0513388, 0.045323}

In[15]:= Table[A[i] + B[i] * 11, {i, 1, 20}]

Out[15]= {0.0452175, 0.0609715, 0.0497515, 0.0465993, 0.0533432, 0.0478785,
0.0584824, 0.0501592, 0.0488549, 0.0472167, 0.0463153, 0.0485866, 0.0498559,
0.0450302, 0.0555714, 0.0512273, 0.0524194, 0.0460587, 0.0517152, 0.0447452}

Linear Bradley-Terry 2

For[ii = 1, ii ≤ 20, ii++,

For[jj = 1, jj ≤ 20, jj++, For[tt = 1, tt ≤ 10, tt++, x[ii, jj, tt] = 0.5]]]

In[2]:= x[1, 2, 10] = 0.5; x[2, 1, 10] = 0.5;

x[6, 13, 10] = 0.666666667; x[13, 6, 10] = 0.333333333;

x[2, 13, 10] = 1; x[13, 2, 10] = 0;

x[1, 6, 10] = 1; x[6, 1, 10] = 0;

x[13, 8, 10] = 1; x[8, 13, 10] = 0;

x[4, 2, 10] = 0.333333333; x[2, 4, 10] = 0.666666667;

x[6, 3, 10] = 0.5; x[3, 6, 10] = 0.5;

x[7, 1, 10] = 0.5; x[1, 7, 10] = 0.5;

$x[13, 10, 10] = 0.5; x[10, 13, 10] = 0.5;$
 $x[8, 14, 10] = 0.857142857; x[14, 8, 10] = 0.142857143;$
 $x[19, 6, 10] = 0.5; x[6, 19, 10] = 0.5;$
 $x[4, 20, 10] = 1; x[20, 4, 10] = 0;$
 $x[7, 11, 10] = 0.75; x[11, 7, 10] = 0.25;$
 $x[6, 5, 10] = 0.5; x[5, 6, 10] = 0.5;$
 $x[19, 10, 10] = 0.666666667; x[10, 19, 10] = 0.333333333;$
 $x[3, 14, 10] = 1; x[14, 3, 10] = 0;$
 $x[8, 17, 10] = 1; x[17, 8, 10] = 0;$
 $x[5, 13, 10] = 0; x[13, 5, 10] = 1;$
 $x[10, 15, 10] = 0.5; x[15, 10, 10] = 0.5;$
 $x[2, 18, 10] = 0.666666667; x[18, 2, 10] = 0.333333333;$
 $x[4, 11, 10] = 0.5; x[11, 4, 10] = 0.5;$
 $x[13, 6, 10] = 0.5; x[6, 13, 10] = 0.5;$
 $x[15, 19, 10] = 0.333333333; x[19, 15, 10] = 0.666666667;$
 $x[20, 7, 10] = 0; x[7, 20, 10] = 1;$
 $x[2, 6, 9] = 0.666666667; x[6, 2, 9] = 0.333333333;$
 $x[5, 4, 9] = 1; x[4, 5, 9] = 0;$
 $x[6, 4, 9] = 0.666666667; x[4, 6, 9] = 0.333333333;$
 $x[2, 5, 9] = 1; x[5, 2, 9] = 0;$
 $x[3, 5, 9] = 0.333333333; x[5, 3, 9] = 0.666666667;$
 $x[16, 4, 9] = 0.5; x[4, 16, 9] = 0.5;$
 $x[5, 19, 9] = 0.6; x[19, 5, 9] = 0.4;$
 $x[6, 18, 9] = 0.5; x[18, 6, 9] = 0.5;$

$x[2, 1, 9] = 0.571428571$; $x[1, 2, 9] = 0.428571429$;
 $x[20, 16, 9] = 0$; $x[16, 20, 9] = 1$;
 $x[4, 5, 9] = 0$; $x[5, 4, 9] = 1$;
 $x[18, 2, 9] = 0.5$; $x[2, 18, 9] = 0.5$;
 $x[10, 13, 9] = 0.5$; $x[13, 10, 9] = 0.5$;
 $x[19, 20, 9] = 0.5$; $x[20, 19, 9] = 0.5$;
 $x[1, 6, 9] = 0$; $x[6, 1, 9] = 1$;
 $x[8, 13, 9] = 1$; $x[13, 8, 9] = 0$;
 $x[16, 19, 9] = 0.333333333$; $x[19, 16, 9] = 0.666666667$;
 $x[3, 14, 9] = 0.5$; $x[14, 3, 9] = 0.5$;
 $x[8, 10, 9] = 0.5$; $x[10, 8, 9] = 0.5$;
 $x[15, 1, 8] = 1$; $x[1, 15, 8] = 0$;
 $x[3, 7, 8] = 0$; $x[7, 3, 8] = 1$;
 $x[7, 1, 8] = 0.5$; $x[1, 7, 8] = 0.5$;
 $x[3, 15, 8] = 0.125$; $x[15, 3, 8] = 0.875$;
 $x[1, 5, 8] = 1$; $x[5, 1, 8] = 0$;
 $x[2, 15, 8] = 0$; $x[15, 2, 8] = 1$;
 $x[3, 16, 8] = 0.666666667$; $x[16, 3, 8] = 0.333333333$;
 $x[1, 14, 8] = 1$; $x[14, 1, 8] = 0$;
 $x[5, 11, 8] = 0.666666667$; $x[11, 5, 8] = 0.333333333$;

 $x[16, 17, 8] = 1$; $x[17, 16, 8] = 0$;
 $x[11, 15, 8] = 0$; $x[15, 11, 8] = 1$;
 $x[9, 17, 8] = 0$; $x[17, 9, 8] = 1$;

$x[19, 16, 8] = 0.2; x[16, 19, 8] = 0.8;$
 $x[11, 8, 8] = 0.5; x[8, 11, 8] = 0.5;$
 $x[14, 2, 8] = 0.285714286; x[2, 14, 8] = 0.714285714;$
 $x[15, 8, 8] = 1; x[8, 15, 8] = 0;$
 $x[4, 9, 8] = 0.333333333; x[9, 4, 8] = 0.666666667;$
 $x[10, 7, 8] = 0.166666667; x[7, 10, 8] = 0.833333333;$
 $x[3, 6, 8] = 0.75; x[6, 3, 8] = 0.25;$
 $x[7, 10, 7] = 0; x[10, 7, 7] = 1;$
 $x[15, 10, 7] = 0; x[10, 15, 7] = 1;$
 $x[1, 15, 7] = 0; x[15, 1, 7] = 1;$
 $x[7, 3, 7] = 0.666666667; x[3, 7, 7] = 0.333333333;$
 $x[10, 8, 7] = 1; x[8, 10, 7] = 0;$
 $x[15, 4, 7] = 0.8; x[4, 15, 7] = 0.2;$
 $x[8, 3, 7] = 0.5; x[3, 8, 7] = 0.5;$
 $x[18, 19, 7] = 0.25; x[19, 18, 7] = 0.75;$
 $x[12, 17, 7] = 0; x[17, 12, 7] = 1;$
 $x[7, 19, 7] = 1; x[19, 7, 7] = 0;$
 $x[10, 14, 7] = 0; x[14, 10, 7] = 1;$
 $x[7, 18, 7] = 1; x[18, 7, 7] = 0;$
 $x[4, 11, 7] = 0.5; x[11, 4, 7] = 0.5;$
 $x[9, 2, 6] = 0.5; x[2, 9, 6] = 0.5;$
 $x[15, 8, 6] = 0.75; x[8, 15, 6] = 0.25;$
 $x[8, 2, 6] = 0; x[2, 8, 6] = 1;$
 $x[15, 9, 6] = 0; x[9, 15, 6] = 1;$

$x[4, 8, 6] = 0.5; x[8, 4, 6] = 0.5;$
 $x[3, 2, 6] = 0; x[2, 3, 6] = 1;$
 $x[15, 1, 6] = 0.5; x[1, 15, 6] = 0.5;$
 $x[10, 2, 6] = 0.25; x[2, 10, 6] = 0.75;$
 $x[8, 7, 6] = 1; x[7, 8, 6] = 0;$
 $x[19, 3, 6] = 0.2; x[3, 19, 6] = 0.8;$
 $x[7, 1, 6] = 0.5; x[1, 7, 6] = 0.5;$
 $x[19, 6, 6] = 0.5; x[6, 19, 6] = 0.5;$
 $x[9, 11, 6] = 0.5; x[11, 9, 6] = 0.5;$
 $x[2, 14, 6] = 0.5; x[14, 2, 6] = 0.5;$
 $x[3, 6, 6] = 1; x[6, 3, 6] = 0;$
 $x[15, 3, 5] = 0; x[3, 15, 5] = 1;$
 $x[4, 3, 5] = 0.3333333333; x[3, 4, 5] = 0.6666666667;$
 $x[15, 11, 5] = 1; x[11, 15, 5] = 0;$
 $x[12, 11, 5] = 0; x[11, 12, 5] = 1;$
 $x[3, 5, 5] = 1; x[5, 3, 5] = 0;$
 $x[18, 4, 5] = 0; x[4, 18, 5] = 1;$
 $x[12, 9, 5] = 0.5; x[9, 12, 5] = 0.5;$
 $x[18, 2, 5] = 1; x[2, 18, 5] = 0;$
 $x[20, 17, 5] = 0.5; x[17, 20, 5] = 0.5;$
 $x[9, 6, 5] = 0.3333333333; x[6, 9, 5] = 0.6666666667;$
 $x[1, 4, 5] = 0; x[4, 1, 5] = 1;$
 $x[18, 20, 5] = 0.5; x[20, 18, 5] = 0.5;$
 $x[2, 17, 5] = 0.5; x[17, 2, 5] = 0.5;$

$$x[11, 8, 5] = 0.6; x[8, 11, 5] = 0.4;$$

$$x[19, 5, 5] = 0.5; x[5, 19, 5] = 0.5;$$

$$x[2, 20, 5] = 0; x[20, 2, 5] = 1;$$

$$x[3, 2, 4] = 0; x[2, 3, 4] = 1;$$

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$$x[7, 6, 4] = 0.3333333333; x[6, 7, 4] = 0.6666666667;$$

$$x[2, 6, 4] = 0.6666666667; x[6, 2, 4] = 0.3333333333;$$

$$x[3, 7, 4] = 0.5; x[7, 3, 4] = 0.5;$$

$$x[7, 1, 4] = 0.6666666667; x[1, 7, 4] = 0.3333333333;$$

$$x[15, 6, 4] = 0; x[6, 15, 4] = 1;$$

$$x[9, 2, 4] = 0.5; x[2, 9, 4] = 0.5;$$

$$x[3, 18, 4] = 0.6; x[18, 3, 4] = 0.4;$$

$$x[1, 4, 4] = 0.5; x[4, 1, 4] = 0.5;$$

$$x[1, 6, 4] = 1; x[6, 1, 4] = 0;$$

$$x[16, 4, 4] = 0; x[4, 16, 4] = 1;$$

$$x[2, 18, 4] = 0.6666666667; x[18, 2, 4] = 0.3333333333;$$

$$x[19, 6, 4] = 0; x[6, 19, 4] = 1;$$

$$x[3, 13, 4] = 1; x[13, 3, 4] = 0;$$

$$x[15, 11, 4] = 1; x[11, 15, 4] = 0;$$

$$x[1, 19, 4] = 1; x[19, 1, 4] = 0;$$

$$x[7, 5, 4] = 0.5; x[5, 7, 4] = 0.5;$$

$$x[3, 9, 3] = 0.5; x[9, 3, 3] = 0.5;$$

$$x[9, 10, 3] = 0.6666666667; x[10, 9, 3] = 0.3333333333;$$

$x[7, 3, 3] = 0.4; x[3, 7, 3] = 0.6;$
 $x[3, 11, 3] = 1; x[11, 3, 3] = 0;$
 $x[15, 5, 3] = 0.6; x[5, 15, 3] = 0.4;$
 $x[10, 14, 3] = 1; x[14, 10, 3] = 0;$
 $x[13, 7, 3] = 0.3333333333; x[7, 13, 3] = 0.6666666667;$
 $x[14, 16, 3] = 0; x[16, 14, 3] = 1;$
 $x[5, 7, 3] = 1; x[7, 5, 3] = 0;$
 $x[11, 16, 3] = 0.6666666667; x[16, 11, 3] = 0.3333333333;$
 $x[15, 10, 3] = 0.5; x[10, 15, 3] = 0.5;$
 $x[5, 13, 3] = 1; x[13, 5, 3] = 0;$
 $x[11, 14, 3] = 0.5; x[14, 11, 3] = 0.5;$
 $x[9, 4, 2] = 0.6666666667; x[4, 9, 2] = 0.3333333333;$
 $x[9, 1, 2] = 0.5; x[1, 9, 2] = 0.5;$
 $x[4, 5, 2] = 1; x[5, 4, 2] = 0;$
 $x[9, 17, 2] = 1; x[17, 9, 2] = 0;$
 $x[3, 1, 2] = 0; x[1, 3, 2] = 1;$
 $x[5, 10, 2] = 0.3333333333; x[10, 5, 2] = 0.6666666667;$
 $x[5, 17, 2] = 0.75; x[17, 5, 2] = 0.25;$
 $x[4, 7, 2] = 0.5; x[7, 4, 2] = 0.5;$
 $x[9, 11, 2] = 1; x[11, 9, 2] = 0;$
 $x[2, 5, 1] = 0.6666666667; x[5, 2, 1] = 0.3333333333;$
 $x[1, 5, 1] = 1; x[5, 1, 1] = 0;$
 $x[1, 4, 1] = 0.6666666667; x[4, 1, 1] = 0.3333333333;$
 $x[10, 5, 1] = 0.5; x[5, 10, 1] = 0.5;$

$x[3, 2, 1] = 0.5; x[2, 3, 1] = 0.5;$
 $x[18, 10, 1] = 0.166666667; x[10, 18, 1] = 0.833333333;$
 $x[9, 2, 1] = 0; x[2, 9, 1] = 1;$
 $x[1, 17, 1] = 1; x[17, 1, 1] = 0;$
 $x[8, 13, 1] = 0.25; x[13, 8, 1] = 0.75;$
 $x[18, 17, 1] = 0.857142857; x[17, 18, 1] = 0.142857143;$
 $x[4, 13, 1] = 0.5; x[13, 4, 1] = 0.5;$
 $x[9, 1, 1] = 0.5; x[1, 9, 1] = 0.5;$
 $x[8, 4, 1] = 1; x[4, 8, 1] = 0;$
 $x[10, 3, 1] = 0;$
 $x[3, 10, 1] = 1;$

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In[161]:= $f[k_, t_] := \text{Sum}[x[k, j, t], \{j, 1, 20\}] - x[k, k, t]$

$A[k] + B[k] * t$

In[162]:= $g[k_, t_] :=$

$\text{Sum}[x[k, j, t] + x[j, k, t]$

$A[k] + B[k] * t + A[j] + B[j] * t$

$, \{j, 1, 20\}] - x[k, k, t] + x[k, k, t]$

$A[k] + B[k] * t + A[k] + B[k] * t$

In[163]:= eqn1[k_] := Sum[f[k, t] - g[k, t], {t, 1, 10}]

In[164]:= eqn2[k_] := Sum[t * (f[k, t] - g[k, t]), {t, 1, 10}]

In[165]:= eqns =

Join[Table[eqn1[k] == 0, {k, 1, 19}], {Sum[A[i], {i, 1, 20}] == 1}, Table[eqn2[k] == 0, {k,

1, 19}], {Sum[B[i], {i, 1, 20}] == 0}] /. A[

1] → A1 /. A[2] → A2 /. A[3] → A3 /. A[4] →

A4 /. A[5] → A5 /. A[6] → A6 /. A[7] → A7 /.

A[8] → A8 /. A[9] → A9 /. A[10] → A10 /.

A[11] → A11 /. A[12] → A12 /. A[13] → A13 /. A[14] →

A14 /. A[15] → A15 /. A[16] → A16 /. A[17] → A17 /.

A[18] → A18 /. A[19] → A19 /. A[20] → A20 /. B[1] → B1 /.

B[2] → B2 /. B[3] → B3 /. B[4] → B4 /. B[5] → B5 /.

B[6] → B6 /. B[7] → B7 /. B[8] → B8 /. B[9] → B9 /. B[10] → B10 /.

B[11] → B11 /. B[12] → B12 /. B[13] → B13 /. B[14] → B14 /.

B[15] → B15 /. B[16] → B16 /. B[17] → B17 /. B[

18] → B18 /. B[

19] →

B19 /. B[

20] →

B20

$$\left\{ \begin{aligned} & \theta. + \frac{10.6667}{A1+B1} + \frac{10.}{A1+2B1} + \frac{9.5}{A1+3B1} + \frac{10.3333}{A1+4B1} + \frac{9.}{A1+5B1} + \frac{9.5}{A1+6B1} + \frac{9.}{A1+7B1} + \frac{10.}{A1+8B1} + \frac{8.92857}{A1+9B1} + \dots 381 \dots = \\ & \theta, \theta. + \dots 390 \dots = \theta, \theta. + \dots 390 \dots = \theta, \\ & \theta. + \dots 390 \dots = \theta, \dots 32 \dots, \theta. + \dots 47 \dots + 10 \dots 1 \dots = \theta, \\ & \theta. + \dots 44 \dots + 7 \dots 1 \dots + 8 (\theta. + \dots 39 \dots) + 9 (\theta. + \dots 39 \dots) + 10 (\theta. + \dots 39 \dots) = \\ & \theta, \theta. + \dots 39 \dots + 2 (\theta. + \dots 39 \dots) + 3 (\theta. + \dots 39 \dots) + 4 (\theta. + \dots 39 \dots) + \\ & 5 (\theta. + \dots 39 \dots) + 6 (\theta. + \dots 39 \dots) + 7 (\theta. + \dots 39 \dots) + \\ & 8 (\theta. + \dots 39 \dots) + 9 (\theta. + \dots 39 \dots) + 10 (\theta. + \dots 39 \dots) = \theta, \\ & B1 + B10 + B11 + B12 + B13 + B14 + B15 + B16 + B17 + B18 + B19 + \\ & B2 + B20 + B3 + B4 + B5 + B6 + B7 + B8 + B9 = \theta \end{aligned} \right\}$$

large output [show less](#) [show more](#) [show all](#) [set size limit...](#)

```
In[166]:= start = {{A1, 0.05385892107079924}, {A2, 0.05764233628716534},
{A3, 0.046408854043334946}, {A4, 0.04929677540003903}, {A5,
0.044604279759638356},
{A6, 0.04754694785981275}, {A7, 0.05252429412420266}, {A8, 0.05124696764410592},
{A9, 0.048693197363995876}, {A10, 0.049604981653513694}, {A11,
0.04904267190828266},
{A12, 0.05124121533917373}, {A13, 0.047412879593118254}, {A14,
0.05053875152794055},
{A15, 0.052690066659171725}, {A16, 0.051746849888273064}, {A17,
0.04869987141854041},
{A18, 0.0468737598286547}, {A19, 0.04919773243149804}, {A20,
0.05112864619873905},
{B1, 0.00023886590211634834}, {B2, 0.0009911801222894}, {B3, -
0.0011865436302999847},
{B4, -0.00011893992645679645}, {B5, -0.000394692678794037},
{B6, -0.0009502719569366467}, {B7, 0.00012508432939840785}, {B8, 0.},
{B9, -0.0001984870078379979}, {B10, -0.00011873969916020615},
{B11, -0.00035409899218138285}, {B12, 0.}, {B13, -0.000938797251871507},
{B14, 0.}, {B15, 0.}, {B16, 0.}, {B17, -0.00011879276894852046},
```

{B18, -0.0009431922485697967}, {B19, -0.0004802336872294344}, {B20, 0.}}

Out[166]= {{A1, 0.0538589}, {A2, 0.0576423}, {A3, 0.0464089}, {A4, 0.0492968}, {A5, 0.0446043},

{A6, 0.0475469}, {A7, 0.0525243}, {A8, 0.051247}, {A9, 0.0486932}, {A10, 0.049605},

{A11, 0.0490427}, {A12, 0.0512412}, {A13, 0.0474129}, {A14, 0.0505388},

{A15, 0.0526901}, {A16, 0.0517468}, {A17, 0.0486999}, {A18, 0.0468738},

{A19, 0.0491977}, {A20, 0.0511286}, {B1, 0.000238866}, {B2, 0.00099118},

{B3, -0.00118654}, {B4, -0.00011894}, {B5, -0.000394693}, {B6, -0.000950272},

{B7, 0.000125084}, {B8, 0.}, {B9, -0.000198487}, {B10, -0.00011874},

{B11, -0.000354099}, {B12, 0.}, {B13, -0.000938797}, {B14, 0.}, {B15, 0.}, {B16, 0.},

{B17, -0.000118793}, {B18, -0.000943192}, {B19, -0.000480234}, {B20, 0.}}

In[167]:= FindRoot[eqns, start]

Out[167]= {A1 → 0.0567767, A2 → 0.0513987, A3 → 0.053496, A4 → 0.0508852, A5 → 0.0485293,

A6 → 0.0513758, A7 → 0.0464806, A8 → 0.0496268, A9 → 0.0526992, A10 → 0.0511581,

A11 → 0.047551, A12 → 0.0493012, A13 → 0.0479236, A14 → 0.0495172, A15 → 0.0486884,

A16 → 0.048989, A17 → 0.045022, A18 → 0.0498462, A19 → 0.0473201, A20 → 0.053415,

B1 → -0.000870473, B2 → 0.000309206, B3 → -0.000293974, B4 → -0.000131157,

B5 → 0.0000361432, B6 → -0.000271072, B7 → 0.000887173, B8 → 0.0000169376,

B9 → -0.000343176, B10 → -0.000110642, B11 → 0.0000558879, B12 → -0.0000597393,

B13 → 0.000206182, B14 → -0.000286691, B15 → 0.000677291, B16 → 0.000325066,

B17 → 0.000616539, B18 → -0.000186737, B19 → 0.000221349, B20 → -0.000798112}

In[168]:= A[1] = 0.056776675108631075;

A[2] = 0.051398715230097466;

A[3] = 0.053496006293804965;

A[4] = 0.05088516118177751;

A[5] = 0.04852930657363699;

A[6] = 0.05137577529038443;

A[7] = 0.046480564033551046;

A[8] = 0.049626762797420655;

A[9] = 0.05269915781922004;

A[10] = 0.05115812528813203;

A[11] = 0.04755103228441588;

A[12] = 0.04930121189068573;

A[13] = 0.047923644396293576;

A[14] = 0.04951717330290915;

A[15] = 0.048688408955424825;

A[16] = 0.048989039845573554;

A[17] = 0.045021951168096874;

A[18] = 0.04984623073893162;

A[19] = 0.04732005718201646;

A[20] = 0.05341500061899612;

B[1] = -0.000870473192739087;

B[2] = 0.00030920611485795645;

B[3] = -0.00029397445985801946;

B[4] = -0.00013115708984225225;

B[5] = 0.000036143215027202244;

B[6] = -0.0002710717730382498;

B[7] = 0.000887172610277731;

B[8] = 0.000016937624524792992;

B[9] = -0.00034317580878568267;

B[10] = -0.00011064188587235555;

B[11] = 0.000055887851528731474;

B[12] = -0.000059739317188942956;

B[13] = 0.0002061819056197268;

B[14] = -0.00028669142884532367;

B[15] = 0.0006772912520130092;

B[16] = 0.0003250662465901661;

B[17] = 0.0006165388158510666;

B[18] = -0.00018673749718242872;

B[19] = 0.00022134911441689862;

B[20] = -0.0007981122973549395;

In[169]:= Join[Table[A[i], {i, 1, 20}], Table[B[i], {i, 1, 20}]]

Out[169]= {0.0567767, 0.0513987, 0.053496, 0.0508852, 0.0485293, 0.0513758, 0.0464806,
0.0496268,

0.0526992, 0.0511581, 0.047551, 0.0493012, 0.0479236, 0.0495172, 0.0486884,

0.048989, 0.045022, 0.0498462, 0.0473201, 0.053415, -0.000870473, 0.000309206,

-0.000293974, -0.000131157, 0.0000361432, -0.000271072, 0.000887173, 0.0000169376,

-0.000343176, -0.000110642, 0.0000558879, -0.0000597393, 0.000206182, -0.000286691,

0.000677291, 0.000325066, 0.000616539, -0.000186737, 0.000221349, -0.000798112}

In[170]:= Table[A[i] + B[i] * 10.25, {i, 1, 20}]

Out[170]= {0.0478543, 0.0545681, 0.0504828, 0.0495408, 0.0488998, 0.0485973,
0.0555741, 0.0498004, 0.0491816, 0.050024, 0.0481239, 0.0486889, 0.050037,
0.0465786, 0.0556306, 0.052321, 0.0513415, 0.0479322, 0.0495889, 0.0452343}

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In[171]:= Table[A[i] + B[i] * 11, {i, 1, 20}]

Out[171]= {0.0472015, 0.0548, 0.0502623, 0.0494424, 0.0489269, 0.048394,
0.0562395, 0.0498131, 0.0489242, 0.0499411, 0.0481658, 0.0486441, 0.0501916,
0.0463636, 0.0561386, 0.0525648, 0.0518039, 0.0477921, 0.0497549, 0.0446358}

Linear Exponential AHP

In[]:= For[ii = 1, ii ≤ 20, ii++,

For[jj = 1, jj ≤ 20, jj++, For[tt = 1, tt ≤ 10, tt++, x[ii, jj, tt] = 1]]]

In[]:= x[1, 2, 10] = 1;

x[2, 1, 10] = 1;

x[6, 13, 10] = 2.7182818;

x[13, 6, 10] = 0.367879441;

x[2, 13, 10] = 7.3890561;

x[13, 2, 10] = 0.135335283;

x[1, 6, 10] = 20.085537;

x[6, 1, 10] = 0.049787068;

x[13, 8, 10] = 2.7182818;

x[8, 13, 10] = 0.367879441;

x[4, 2, 10] = 0.3678794;
x[2, 4, 10] = 2.718281828;
x[6, 3, 10] = 1;
x[3, 6, 10] = 1;
x[7, 1, 10] = 1;
x[1, 7, 10] = 1;
x[13, 10, 10] = 1;
x[10, 13, 10] = 1;
x[8, 14, 10] = 148.41316;
x[14, 8, 10] = 0.006737947;
x[19, 6, 10] = 1;
x[6, 19, 10] = 1;
x[4, 20, 10] = 20.085537;
x[20, 4, 10] = 0.049787068;
x[7, 11, 10] = 7.3890561;
x[11, 7, 10] = 0.135335283;
x[6, 5, 10] = 1;
x[5, 6, 10] = 1;
x[19, 10, 10] = 2.7182818;
x[10, 19, 10] = 0.367879441;
x[3, 14, 10] = 2.7182818;
x[14, 3, 10] = 0.367879441;
x[8, 17, 10] = 7.3890561;
x[17, 8, 10] = 0.135335283;

x[5, 13, 10] = 0.1353353;
x[13, 5, 10] = 7.389056099;
x[10, 15, 10] = 1;
x[15, 10, 10] = 1;
x[2, 18, 10] = 2.7182818;
x[18, 2, 10] = 0.367879441;
x[4, 11, 10] = 1;
x[11, 4, 10] = 1;
x[13, 6, 10] = 1;
x[6, 13, 10] = 1;
x[15, 19, 10] = 0.3678794;
x[19, 15, 10] = 2.718281828;
x[20, 7, 10] = 0.1353353;
x[7, 20, 10] = 7.389056099;

x[2, 6, 9] = 7.3890561;
x[6, 2, 9] = 0.135335283;
x[5, 4, 9] = 7.3890561;
x[4, 5, 9] = 0.135335283;
x[6, 4, 9] = 2.7182818;
x[4, 6, 9] = 0.367879441;
x[2, 5, 9] = 2.7182818;
x[5, 2, 9] = 0.367879441;
x[3, 5, 9] = 0.3678794;

x[5, 3, 9] = 2.718281828;
x[16, 4, 9] = 1;
x[4, 16, 9] = 1;
x[5, 19, 9] = 2.7182818;
x[19, 5, 9] = 0.367879441;
x[6, 18, 9] = 1;
x[18, 6, 9] = 1;
x[2, 1, 9] = 2.7182818;
x[1, 2, 9] = 0.367879441;
x[20, 16, 9] = 0.3678794;
x[16, 20, 9] = 2.718281828;
x[4, 5, 9] = 0.3678794;
x[5, 4, 9] = 2.718281828;
x[18, 2, 9] = 1;
x[2, 18, 9] = 1;
x[10, 13, 9] = 1;
x[13, 10, 9] = 1;
x[19, 20, 9] = 1;
x[20, 19, 9] = 1;
x[1, 6, 9] = 0.0497871;
x[6, 1, 9] = 20.08553692;
x[8, 13, 9] = 2.7182818;
x[13, 8, 9] = 0.367879441;
x[16, 19, 9] = 0.3678794;

$$x[19, 16, 9] = 2.718281828;$$

$$x[3, 14, 9] = 1;$$

$$x[14, 3, 9] = 1;$$

$$x[8, 10, 9] = 1;$$

$$x[10, 8, 9] = 1;$$

$$x[15, 1, 8] = 2.7182818;$$

$$x[1, 15, 8] = 0.367879441;$$

$$x[3, 7, 8] = 0.0497871;$$

$$x[7, 3, 8] = 20.08553692;$$

$$x[7, 1, 8] = 1;$$

$$x[1, 7, 8] = 1;$$

$$x[3, 15, 8] = 0.0024788;$$

$$x[15, 3, 8] = 403.4287935;$$

$$x[1, 5, 8] = 2.7182818;$$

$$x[5, 1, 8] = 0.367879441;$$

$$x[2, 15, 8] = 0.3678794;$$

$$x[15, 2, 8] = 2.718281828;$$

$$x[3, 16, 8] = 2.7182818;$$

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x

$$[16, 3, 8] = 0.367879441;$$

x

$$[1, 14, 8] = 2.7182818;$$

x

$$[14, 1, 8] = 0.367879441;$$

x

$$[5, 11, 8] = 2.7182818;$$

x

$$[11, 5, 8] = 0.367879441;$$

x

$$[16, 17, 8] = 7.3890561;$$

x

$$[17, 16, 8] = 0.135335283;$$

x

$$[11, 15, 8] = 0.3678794;$$

x

$$[15, 11, 8] = 2.718281828;$$

x

$$[9, 17, 8] = 0.3678794;$$

x

$$[17, 9, 8] = 2.718281828;$$

x

$$[19, 16, 8] = 0.0497871;$$

x

$$[16, 19, 8] = 20.08553692;$$

x

$$[11, 8, 8] = 1;$$

x

$$[8, 11, 8] = 1;$$

x

$$[14, 2, 8] = 0.0497871;$$

x

$$[2, 14, 8] = 20.08553692;$$

x

$$[15, 8, 8] = 54.59815;$$

x

$$[8, 15, 8] = 0.018315639;$$

x

$$[4, 9, 8] = 0.3678794;$$

x

$$[9, 4, 8] = 2.718281828;$$

x

$$[10, 7, 8] = 0.0183156;$$

x

$$[7, 10, 8] = 54.59815003;$$

x

$$[3, 6, 8] = 7.3890561;$$

x

$$[6, 3, 8] = 0.135335283;$$

x

$$[7, 10, 7] = 0.3678794;$$

x

$$[10, 7, 7] = 2.718281828;$$

x

$$[15, 10, 7] = 0.3678794;$$

x

$$[10, 15, 7] = 2.718281828;$$

x

$$[1, 15, 7] = 0.0183156;$$

x

$$[15, 1, 7] = 54.59815003;$$

x

$$[7, 3, 7] = 2.7182818;$$

x

$$[3, 7, 7] = 0.367879441;$$

x

$$[10, 8, 7] = 2.7182818;$$

x

$$[8, 10, 7] = 0.367879441;$$

x

$$[15, 4, 7] = 20.085537;$$

x

$$[4, 15, 7] = 0.049787068;$$

x

$$[8, 3, 7] = 1;$$

x

$$[3, 8, 7] = 1;$$

x

$$[18, 19, 7] = 0.1353353;$$

x

$$[19, 18, 7] = 7.389056099;$$

x

$$[12, 17, 7] = 0.3678794;$$

x

$$[17, 12, 7] = 2.718281828;$$

x

$$[7, 19, 7] = 2.7182818;$$

x

$$[19, 7, 7] = 0.367879441;$$

x

$$[10, 14, 7] = 0.3678794;$$

x

$$[14, 10, 7] = 2.718281828;$$

x

$$[7, 18, 7] = 7.3890561;$$

x

$$[18, 7, 7] = 0.135335283;$$

x

$$[4, 11, 7] = 1;$$

x

$$[11, 4, 7] = 1;$$

20231015 Linear EAHP.nb

3

$$x[9, 2, 6] = 1;$$

$$x[2, 9, 6] = 1;$$

$$x[15, 8, 6] = 7.3890561;$$

$$x[8, 15, 6] = 0.135335283;$$

$$x[8, 2, 6] = 0.3678794;$$

$$x[2, 8, 6] = 2.718281828;$$

$$x[15, 9, 6] = 0.1353353;$$

$$x[9, 15, 6] = 7.389056099;$$

$$x[4, 8, 6] = 1;$$

$$x[8, 4, 6] = 1;$$

$$x[3, 2, 6] = 0.3678794;$$

$$x[2, 3, 6] = 2.718281828;$$

$$x[15, 1, 6] = 1;$$

$$x[1, 15, 6] = 1;$$

$$x[10, 2, 6] = 0.1353353;$$

$$x[2, 10, 6] = 7.389056099;$$

$$x[8, 7, 6] = 2.7182818;$$

$$x[7, 8, 6] = 0.367879441;$$

x[19, 3, 6] = 0.0497871;
x[3, 19, 6] = 20.08553692;
x[7, 1, 6] = 1;
x[1, 7, 6] = 1;
x[19, 6, 6] = 1;
x[6, 19, 6] = 1;
x[9, 11, 6] = 1;
x[11, 9, 6] = 1;
x[2, 14, 6] = 1;
x[14, 2, 6] = 1;
x[3, 6, 6] = 2.7182818;
x[6, 3, 6] = 0.367879441;
x[15, 3, 5] = 0.1353353;
x[3, 15, 5] = 7.389056099;
x[4, 3, 5] = 0.3678794;
x[3, 4, 5] = 2.718281828;
x[15, 11, 5] = 2.7182818;
x[11, 15, 5] = 0.367879441;
x[12, 11, 5] = 0.1353353;
x[11, 12, 5] = 7.389056099;
x[3, 5, 5] = 7.3890561;
x[5, 3, 5] = 0.135335283;
x[18, 4, 5] = 0.0497871;
x[4, 18, 5] = 20.08553692;

$$x[12, 9, 5] = 1;$$

$$x[9, 12, 5] = 1;$$

$$x[18, 2, 5] = 7.3890561;$$

$$x[2, 18, 5] = 0.135335283;$$

$$x[20, 17, 5] = 1;$$

$$x[17, 20, 5] = 1;$$

$$x[9, 6, 5] = 0.3678794;$$

$$x[6, 9, 5] = 2.718281828;$$

$$x[1, 4, 5] = 0.3678794;$$

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x

$$[4, 1, 5] = 2.718281828;$$

x

$$[18, 20, 5] = 1;$$

x

$$[20, 18, 5] = 1;$$

x

$$[2, 17, 5] = 1;$$

x

$$[17, 2, 5] = 1;$$

x

$$[11, 8, 5] = 2.7182818;$$

x

$$[8, 11, 5] = 0.367879441;$$

x

$$[19, 5, 5] = 1;$$

x

$$[5, 19, 5] = 1;$$

x

$$[2, 20, 5] = 0.3678794;$$

x

$$[20, 2, 5] = 2.718281828;$$

x

$$[3, 2, 4] = 0.0497871;$$

x

$$[2, 3, 4] = 20.08553692;$$

x

$$[7, 6, 4] = 0.3678794;$$

x

$$[6, 7, 4] = 2.718281828;$$

x

$$[2, 6, 4] = 2.7182818;$$

x

$$[6, 2, 4] = 0.367879441;$$

x

$$[3, 7, 4] = 1;$$

x

$$[7, 3, 4] = 1;$$

x

$$[7, 1, 4] = 2.7182818;$$

x

$$[1, 7, 4] = 0.367879441;$$

x

$$[15, 6, 4] = 0.0497871;$$

x

$$[6, 15, 4] = 20.08553692;$$

x

$$[9, 2, 4] = 1;$$

x

$$[2, 9, 4] = 1;$$

x

$$[3, 18, 4] = 2.7182818;$$

x

$$[18, 3, 4] = 0.367879441;$$

x

$$[1, 4, 4] = 1;$$

x

$$[4, 1, 4] = 1;$$

x

$$[1, 6, 4] = 2.7182818;$$

x

$$[6, 1, 4] = 0.367879441;$$

x

$$[16, 4, 4] = 0.1353353;$$

x

$$[4, 16, 4] = 7.389056099;$$

x

$$[2, 18, 4] = 2.7182818;$$

x

$$[18, 2, 4] = 0.367879441;$$

x

$$[19, 6, 4] = 0.3678794;$$

x

$$[6, 19, 4] = 2.718281828;$$

x

$$[3, 13, 4] = 20.085537;$$

x

$$[13, 3, 4] = 0.049787068;$$

x

$$[15, 11, 4] = 7.3890561;$$

x

$$[11, 15, 4] = 0.135335283;$$

x

$$[1, 19, 4] = 2.7182818;$$

x

$$[19, 1, 4] = 0.367879441;$$

x

$$[7, 5, 4] = 1;$$

x

$$[5, 7, 4] = 1;$$

x

$$[3, 9, 3] = 1;$$

x

$$[9, 3, 3] = 1;$$

x

$$[9, 10, 3] = 2.7182818;$$

x

$$[10, 9, 3] = 0.367879441;$$

x

$$[7, 3, 3] = 0.3678794;$$

x

$$[3, 7, 3] = 2.718281828;$$

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$$x[3, 11, 3] = 2.7182818;$$

$$x[11, 3, 3] = 0.367879441;$$

$$x[15, 5, 3] = 2.7182818;$$

x[5, 15, 3] = 0.367879441;
x[10, 14, 3] = 20.085537;
x[14, 10, 3] = 0.049787068;
x[13, 7, 3] = 0.3678794;
x[7, 13, 3] = 2.718281828;
x[14, 16, 3] = 0.1353353;
x[16, 14, 3] = 7.389056099;
x[5, 7, 3] = 2.7182818;
x[7, 5, 3] = 0.367879441;
x[11, 16, 3] = 2.7182818;
x[16, 11, 3] = 0.367879441;
x[15, 10, 3] = 1;
x[10, 15, 3] = 1;
x[5, 13, 3] = 2.7182818;
x[13, 5, 3] = 0.367879441;
x[11, 14, 3] = 1;
x[14, 11, 3] = 1;
x[9, 4, 2] = 2.7182818;
x[4, 9, 2] = 0.367879441;
x[9, 1, 2] = 1;
x[1, 9, 2] = 1;
x[4, 5, 2] = 2.7182818;
x[5, 4, 2] = 0.367879441;
x[9, 17, 2] = 7.3890561;

$$x[17, 9, 2] = 0.135335283;$$

$$x[3, 1, 2] = 0.3678794;$$

$$x[1, 3, 2] = 2.718281828;$$

$$x[5, 10, 2] = 0.3678794;$$

$$x[10, 5, 2] = 2.718281828;$$

$$x[5, 17, 2] = 7.3890561;$$

$$x[17, 5, 2] = 0.135335283;$$

$$x[4, 7, 2] = 1;$$

$$x[7, 4, 2] = 1;$$

$$x[9, 11, 2] = 2.7182818;$$

$$x[11, 9, 2] = 0.367879441;$$

$$x[2, 5, 1] = 7.3890561;$$

$$x[5, 2, 1] = 0.135335283;$$

$$x[1, 5, 1] = 7.3890561;$$

$$x[5, 1, 1] = 0.135335283;$$

$$x[1, 4, 1] = 2.7182818;$$

$$x[4, 1, 1] = 0.367879441;$$

$$x[10, 5, 1] = 1;$$

$$x[5, 10, 1] = 1;$$

$$x[3, 2, 1] = 1;$$

$$x[2, 3, 1] = 1;$$

$$x[18, 10, 1] = 0.0183156;$$

$$x[10, 18, 1] = 54.59815003;$$

$$x[9, 2, 1] = 0.1353353;$$

$$x[2, 9, 1] = 7.389056099;$$

$$x[1, 17, 1] = 2.7182818;$$

$$x[17, 1, 1] = 0.367879441;$$

$$x[8, 13, 1] = 0.1353353;$$

$$x[13, 8, 1] = 7.389056099;$$

$$x[18, 17, 1] = 148.41316;$$

$$x[17, 18, 1] = 0.006737947;$$

$$x[4, 13, 1] = 1;$$

$$x[13, 4, 1] = 1;$$

$$x[9, 1, 1] = 1;$$

$$x[1, 9, 1] = 1;$$

$$x[8, 4, 1] = 2.7182818;$$

$$x[4, 8, 1] = 0.367879441;$$

$$x[10, 3, 1] = 0.3678794;$$

$$x[3, 10, 1] = 2.718281828;$$

$$\text{In}[\square] := \text{eqn1}[i_] := \text{Sum}[\text{Sum}[(A[i] + B[i] * t - (A[j] + B[j] * t) * x[i, j, t]) /$$

$$(A[i] + B[i] * t + (A[j] + B[j] * t) * x[i, j, t]), \{j, 1, 20\}], \{t, 1, 10\}]$$

$$\text{In}[\square] := \text{eqn2}[i_] := \text{Sum}[\text{Sum}[t * (A[i] + B[i] * t - (A[j] + B[j] * t) * x[i, j, t]) /$$

$$(A[i] + B[i] * t + (A[j] + B[j] * t) * x[i, j, t]), \{j, 1, 20\}], \{t, 1, 10\}]$$

In[]:= eqns =

{eqn1[1] == 0, eqn1[2] == 0, eqn1[3] == 0, eqn1[4] == 0, eqn1[5] == 0, eqn1[6] == 0,
eqn1[7] == 0,

eqn1[8] == 0, eqn1[9] == 0, eqn1[10] == 0,

eqn1[11] == 0, eqn1[12] == 0, eqn1[13] == 0,

eqn1[14] == 0, eqn1[15] == 0, eqn1[16] == 0,

eqn1[17] == 0, eqn1[18] == 0, eqn1[19] == 0,

eqn1[20] == 0, eqn2[1] == 0, eqn2[2] == 0,

eqn2[3] == 0, eqn2[4] == 0, eqn2[5] == 0, eqn2[

6] == 0, eqn2[7] == 0, eqn2[8] == 0, eqn2[9] == 0,

eqn2[10] == 0, eqn2[11] == 0, eqn2[12] == 0,

eqn2[13] == 0, eqn2[14] == 0, eqn2[15] == 0,

eqn2[16] == 0, eqn2[17] == 0, eqn2[18] == 0,

eqn2[19] == 0, eqn2[20] == 0} /. A[1] → A1 /.

A[2] → A2 /. A[3] → A3 /. A[4] → A4 /. A[5] →

A5 /. A[6] → A6 /. A[7] → A7 /. A[8] → A8 /.

A[9] → A9 /. A[10] → A10 /. A[11] → A11 /.

A[12] → A12 /. A[13] → A13 /. A[14] → A14 /.

A[15] → A15 /. A[16] → A16 /. A[17] → A17 /.

A[18] → A18 /. A[19] → A19 /. A[20] → A20 /. B[1] → B1 /.

B[2] → B2 /. B[3] → B3 /. B[4] → B4 /. B[5] → B5 /.

B[6] → B6 /. B[7] → B7 /. B[8] → B8 /. B[9] → B9 /.

B[10] → B10 /. B[11] → B11 /. B[12] → B12 /. B[13] → B13 /.

B[14] → B14 /. B[15] → B15 /. B[16] → B16 /. B[

17] →

B17 /. B[

18] →

B18 /. B[

19] →

B19 /.

B[

20] →

B20

$$\left\{ \frac{A1-A10+B1-B10}{A1+A10+B1+B10} + \frac{A1-A10+2B1-2B10}{A1+A10+2B1+2B10} + \frac{A1-A10+3B1-3B10}{A1+A10+3B1+3B10} + \frac{A1-A10+4B1-4B10}{A1+A10+4B1+4B10} + \right.$$

$$\frac{A1-A10+5B1-5B10}{A1+A10+5B1+5B10} + \frac{A1-A10+6B1-6B10}{A1+A10+6B1+6B10} + \frac{A1-A10+7B1-7B10}{A1+A10+7B1+7B10} + \frac{A1-A10+8B1-8B10}{A1+A10+8B1+8B10} + \frac{A1-A10+9B1-9B10}{A1+A10+9B1+9B10} +$$

$$\dots 172 \dots + \frac{A1-A9+2B1-2B9}{A1+A9+2B1+2B9} + \frac{A1-A9+3B1-3B9}{A1+A9+3B1+3B9} + \frac{A1-A9+4B1-4B9}{A1+A9+4B1+4B9} + \frac{A1-A9+5B1-5B9}{A1+A9+5B1+5B9} +$$

$$\frac{A1-A9+6B1-6B9}{A1+A9+6B1+6B9} + \frac{A1-A9+7B1-7B9}{A1+A9+7B1+7B9} + \frac{A1-A9+8B1-8B9}{A1+A9+8B1+8B9} + \frac{A1-A9+9B1-9B9}{A1+A9+9B1+9B9} + \frac{A1-A9+10B1-10B9}{A1+A9+10B1+10B9} = 0,$$

$$\left. \dots 1 \dots = 0, \dots 36 \dots, \dots 1 \dots = 0, \dots 1 \dots = 0 \right\}$$

large output | show less | show more | show all | set size limit...

In[] := eqn1it[i_] := -(Sum[Sum[(B[i] * t - (A[j] + B[j] * t) * x[i, j, t]) /

(A[i] + B[i] * t + (A[j] + B[j] * t) * x[i, j, t]), {j, 1, 3}], {t, 1, 4}]) /

(Sum[Sum[1 / (A[i] + B[i] * t + (A[j] + B[j] * t) * x[i, j, t]), {j, 1, 20}], {t, 1, 10}])

In[] := eqn2it[i_] := -(Sum[Sum[(A[i] - (A[j] + B[j] * t) * x[i, j, t]) *

t / (A[i] + B[i] * t + (A[j] + B[j] * t) * x[i, j, t]), {j, 1, 3}], {t, 1, 4}]) /

(Sum[Sum[t^2 / (A[i] + B[i] * t + (A[j] + B[j] * t) * x[i, j, t]), {j, 1, 20}], {t, 1, 10}])

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In[

□]:=

A

[

1] = 1;

A

[

2] = 1;

A

[

3] = 1;

A

[

4] = 1;

A

[

5] = 1;

A

[

6] = 1;

A

[

7] = 1;

A

[

8] = 1;

A

[

9] = 1;

A

[10] = 1;

A

[11] = 1;

A

[12] = 1;

A

[13] = 1;

A

[14] = 1;

A

[15] = 1;

A

[16] = 1;

A

[17] = 1;

A

[18] = 1;

A

[19] = 1;

A

[20] = 1;

B

[

1] = 0;

B

[

2] = 0;

B

[

3] = 0;

B

[

4] = 0;

B

[

5] = 0;

B

[

6] = 0;

B

[

7] = 0;

B

[
8] = 0;

B

[
9] = 0;

B

[10] = 0;

B

[11] = 0;

B

[12] = 0;

B

[13] = 0;

B

[14] = 0;

B

[15] = 0;

B

[16] = 0;

B

[17] = 0;

B

[18] = 0;

B

[19] = 0;

B

[20] = 0;

In[

□]:= elit

[

1] = eqn1it

[

1

]

;

elit

[

2] = eqn1it

[

2

]

;

elit

[

3] = eqn1it

[

3

]

;

e1it

[

4] = eqn1it

[

4

]

;

e1it

[

5] = eqn1it

[

5

]

;

e1it

[

6] = eqn1it

[

6

]

;

elit

[

7] = eqn1it

[

7

]

;

elit

[

8] = eqn1it

[

8

]

;

elit

[

9] = eqn1it

[

9

]

;

e1it

[10] = eqn1it

[10

]

;

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9

e1it[11] = eqn1it[11];

e1it[12] = eqn1it[12];

e1it[13] = eqn1it[13];

e1it[14] = eqn1it[14];

e1it[15] = eqn1it[15];

e1it[16] = eqn1it[16];

e1it[17] = eqn1it[17];

e1it[18] = eqn1it[18];

e1it[19] = eqn1it[19];

e1it[20] = eqn1it[20];

e2it[1] = eqn2it[1];

```
e2it[2] = eqn2it[2];
e2it[3] = eqn2it[3];
e2it[4] = eqn2it[4];
e2it[5] = eqn2it[5];
e2it[6] = eqn2it[6];
e2it[7] = eqn2it[7];
e2it[8] = eqn2it[8];
e2it[9] = eqn2it[9];
e2it[10] = eqn2it[10];
e2it[11] = eqn2it[11];
e2it[12] = eqn2it[12];
e2it[13] = eqn2it[13];
e2it[14] = eqn2it[14];
e2it[15] = eqn2it[15];
e2it[16] = eqn2it[16];
e2it[17] = eqn2it[17];
e2it[18] = eqn2it[18];
e2it[19] = eqn2it[19];
e2it[20] = eqn2it[20];
c1 = Sum[e1it[i], {i, 1, 20}];
A[1] = e1it[1] / c1;
A[2] = e1it[2] / c1;
A[3] = e1it[3] / c1;
A[4] = e1it[4] / c1;
```

$$A[5] = e1it[5] / c1;$$

$$A[6] = e1it[6] / c1;$$

$$A[7] = e1it[7] / c1;$$

$$A[8] = e1it[8] / c1;$$

$$A[9] = e1it[9] / c1;$$

$$A[10] = e1it[10] / c1;$$

$$A[11] = e1it[11] / c1;$$

$$A[12] = e1it[12] / c1;$$

$$A[13] = e1it[13] / c1;$$

$$A[14] = e1it[14] / c1;$$

$$A[15] = e1it[15] / c1;$$

$$A[16] = e1it[16] / c1;$$

$$A[17] = e1it[17] / c1;$$

$$A[18] = e1it[18] / c1;$$

$$A[19] = e1it[19] / c1;$$

$$A[20] = e1it[20] / c1;$$

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$$B[1] = e2it[1];$$

$$B[2] = e2it[2];$$

$$B[3] = e2it[3];$$

$$B[4] = e2it[4];$$

$$B[5] = e2it[5];$$

$$B[6] = e2it[6];$$

B[7] = e2it[7];

B[8] = e2it[8];

B[9] = e2it[9];

B[10] = e2it[10];

B[11] = e2it[11];

B[12] = e2it[12];

B[13] = e2it[13];

B[14] = e2it[14];

B[15] = e2it[15];

B[16] = e2it[16];

B[17] = e2it[17];

B[18] = e2it[18];

B[19] = e2it[19];

B[20] = e2it[20];

In[□]:= Join[Table[A[i], {i, 1, 20}], Table[B[i], {i, 1, 20}]]

Out[□]= {0.0538589, 0.0576423, 0.0464089, 0.0492968, 0.0446043, 0.0475469, 0.0525243,
0.051247,

0.0486932, 0.049605, 0.0490427, 0.0512412, 0.0474129, 0.0505388, 0.0526901, 0.0517468,

0.0486999, 0.0468738, 0.0491977, 0.0511286, 0.000238866, 0.00099118, -0.00118654,

-0.00011894, -0.000394693, -0.000950272, 0.000125084, 0., -0.000198487, -0.00011874,

-0.000354099, 0., -0.000938797, 0., 0., 0., -0.000118793, -0.000943192, -0.000480234, 0.}

In[□]:= Table[A[i] + B[i] * 10.25, {i, 1, 20}]

Out[□]= {0.0563073, 0.0678019, 0.0342468, 0.0480776, 0.0405587, 0.0378067,

0.0538064, 0.051247, 0.0466587, 0.0483879, 0.0454132, 0.0512412, 0.0377902,

0.0505388, 0.0526901, 0.0517468, 0.0474822, 0.037206, 0.0442753, 0.0511286}

```
In[ ]:= Table[A[i] + B[i] * 11, {i, 1, 20}]
```

```
Out[ ]= {0.0564864, 0.0685453, 0.0333569, 0.0479884, 0.0402627, 0.037094,  
0.0539002, 0.051247, 0.0465098, 0.0482988, 0.0451476, 0.0512412, 0.0370861,  
0.0505388, 0.0526901, 0.0517468, 0.0473932, 0.0364986, 0.0439152, 0.0511286}
```

Thurstone 1

```
In[1]:= For[ii = 1, ii ≤ 20, ii++,
```

```
For[jj = 1, jj ≤ 20, jj++, For[tt = 1, tt ≤ 10, tt++, x[ii, jj, tt] = 0.5]]]
```

```
In[2]:= x[1, 2, 10] = 0.5;
```

```
x[2, 1, 10] = 0.5;
```

```
x[6, 13, 10] = 1;
```

```
x[13, 6, 10] = 0;
```

```
x[2, 13, 10] = 1;
```

```
x[13, 2, 10] = 0;
```

```
x[1, 6, 10] = 1;
```

```
x[6, 1, 10] = 0;
```

```
x[13, 8, 10] = 1;
```

```
x[8, 13, 10] = 0;
```

```
x[4, 2, 10] = 0;
```

```
x[2, 4, 10] = 1;
```

```
x[6, 3, 10] = 0.5;
```

```
x[3, 6, 10] = 0.5;
```

```
x[7, 1, 10] = 0.5;
```

```
x[1, 7, 10] = 0.5;
```

```
x[13, 10, 10] = 0.5;
```

$$x[10, 13, 10] = 0.5;$$

$$x[8, 14, 10] = 1;$$

$$x[14, 8, 10] = 0;$$

$$x[19, 6, 10] = 0.5;$$

$$x[6, 19, 10] = 0.5;$$

$$x[4, 20, 10] = 1;$$

$$x[20, 4, 10] = 0;$$

$$x[7, 11, 10] = 1;$$

$$x[11, 7, 10] = 0;$$

$$x[6, 5, 10] = 0.5;$$

$$x[5, 6, 10] = 0.5;$$

$$x[19, 10, 10] = 1;$$

$$x[10, 19, 10] = 0;$$

$$x[3, 14, 10] = 1;$$

$$x[14, 3, 10] = 0;$$

$$x[8, 17, 10] = 1;$$

$$x[17, 8, 10] = 0;$$

$$x[5, 13, 10] = 0;$$

$$x[13, 5, 10] = 1;$$

$$x[10, 15, 10] = 0.5;$$

$$x[15, 10, 10] = 0.5;$$

$$x[2, 18, 10] = 1;$$

$$x[18, 2, 10] = 0;$$

$$x[4, 11, 10] = 0.5;$$

$$x[11, 4, 10] = 0.5;$$

$$x[13, 6, 10] = 0.5;$$

$$x[6, 13, 10] = 0.5;$$

$$x[15, 19, 10] = 0;$$

$$x[19, 15, 10] = 1;$$

$$x[20, 7, 10] = 0;$$

$$x[7, 20, 10] = 1;$$

$$x[2, 6, 9] = 1;$$

$$x[6, 2, 9] = 0;$$

$$x[5, 4, 9] = 1;$$

$$x[4, 5, 9] = 0;$$

$$x[6, 4, 9] = 1;$$

$$x[4, 6, 9] = 0;$$

$$x[2, 5, 9] = 1;$$

$$x[5, 2, 9] = 0;$$

$$x[3, 5, 9] = 0;$$

$$x[5, 3, 9] = 1;$$

$$x[16, 4, 9] = 0.5;$$

$$x[4, 16, 9] = 0.5;$$

$$x[5, 19, 9] = 1;$$

$$x[19, 5, 9] = 0;$$

$$x[6, 18, 9] = 0.5;$$

$$x[18, 6, 9] = 0.5;$$

$x[2, 1, 9] = 1;$
 $x[1, 2, 9] = 0;$
 $x[20, 16, 9] = 0;$
 $x[16, 20, 9] = 1;$
 $x[4, 5, 9] = 0;$
 $x[5, 4, 9] = 1;$
 $x[18, 2, 9] = 0.5;$
 $x[2, 18, 9] = 0.5;$
 $x[10, 13, 9] = 0.5;$
 $x[13, 10, 9] = 0.5;$
 $x[19, 20, 9] = 0.5;$
 $x[20, 19, 9] = 0.5;$
 $x[1, 6, 9] = 0;$
 $x[6, 1, 9] = 1;$
 $x[8, 13, 9] = 1;$
 $x[13, 8, 9] = 0;$
 $x[16, 19, 9] = 0;$
 $x[19, 16, 9] = 1;$
 $x[3, 14, 9] = 0.5;$
 $x[14, 3, 9] = 0.5;$
 $x[8, 10, 9] = 0.5;$
 $x[10, 8, 9] = 0.5;$
 $x[15, 1, 8] = 1;$
 $x[1, 15, 8] = 0;$

$$x[3, 7, 8] = 0;$$

$$x[7, 3, 8] = 1;$$

$$x[7, 1, 8] = 0.5;$$

$$x[1, 7, 8] = 0.5;$$

$$x[3, 15, 8] = 0;$$

$$x[15, 3, 8] = 1;$$

$$x[1, 5, 8] = 1;$$

$$x[5, 1, 8] = 0;$$

$$x[2, 15, 8] = 0;$$

$$x[15, 2, 8] = 1;$$

$$x[3, 16, 8] = 1;$$

2 20231015 Linear Thurstone1.nb

x

$$[16, 3, 8] = 0;$$

x

$$[1, 14, 8] = 1;$$

x

$$[14, 1, 8] = 0;$$

x

$$[5, 11, 8] = 1;$$

x

$$[11, 5, 8] = 0;$$

x

$$[16, 17, 8] = 1;$$

x

$$[17, 16, 8] = 0;$$

x

$$[11, 15, 8] = 0;$$

x

$$[15, 11, 8] = 1;$$

x

$$[9, 17, 8] = 0;$$

x

$$[17, 9, 8] = 1;$$

x

$$[19, 16, 8] = 0;$$

x

$$[16, 19, 8] = 1;$$

x

$$[11, 8, 8] = 0.5;$$

x

$$[8, 11, 8] = 0.5;$$

x

$$[14, 2, 8] = 0;$$

x

$$[2, 14, 8] = 1;$$

x

$$[15, 8, 8] = 1;$$

x

$$[8, 15, 8] = 0;$$

x

$$[4, 9, 8] = 0;$$

x

$$[9, 4, 8] = 1;$$

x

$$[10, 7, 8] = 0;$$

x

$$[7, 10, 8] = 1;$$

x

$$[3, 6, 8] = 1;$$

x

$$[6, 3, 8] = 0;$$

x

$$[7, 10, 7] = 0;$$

x

$$[10, 7, 7] = 1;$$

x

$$[15, 10, 7] = 0;$$

x

$$[10, 15, 7] = 1;$$

x

$$[1, 15, 7] = 0;$$

x

$$[15, 1, 7] = 1;$$

x

$$[7, 3, 7] = 1;$$

x

$$[3, 7, 7] = 0;$$

x

$$[10, 8, 7] = 1;$$

x

$$[8, 10, 7] = 0;$$

x

$$[15, 4, 7] = 1;$$

x

$$[4, 15, 7] = 0;$$

x

$$[8, 3, 7] = 0.5;$$

x

$$[3, 8, 7] = 0.5;$$

x

$$[18, 19, 7] = 0;$$

x

$$[19, 18, 7] = 1;$$

x

$$[12, 17, 7] = 0;$$

x

$$[17, 12, 7] = 1;$$

x

$$[7, 19, 7] = 1;$$

x

$$[19, 7, 7] = 0;$$

x

$$[10, 14, 7] = 0;$$

x

$$[14, 10, 7] = 1;$$

x

$$[7, 18, 7] = 1;$$

x

$$[18, 7, 7] = 0;$$

x

$$[4, 11, 7] = 0.5;$$

x

$$[11, 4, 7] = 0.5;$$

20231015 Linear Thurstone1.nb

3

$$x[9, 2, 6] = 0.5;$$

$$x[2, 9, 6] = 0.5;$$

$$x[15, 8, 6] = 1;$$

$$x[8, 15, 6] = 0;$$

$$x[8, 2, 6] = 0;$$

$$x[2, 8, 6] = 1;$$

$$x[15, 9, 6] = 0;$$

$$x[9, 15, 6] = 1;$$

$$x[4, 8, 6] = 0.5;$$

$$x[8, 4, 6] = 0.5;$$

$$x[3, 2, 6] = 0;$$

$$x[2, 3, 6] = 1;$$

$$x[15, 1, 6] = 0.5;$$

$$x[1, 15, 6] = 0.5;$$

$$x[10, 2, 6] = 0;$$

$$x[2, 10, 6] = 1;$$

$$x[8, 7, 6] = 1;$$

$$x[7, 8, 6] = 0;$$

$$x[19, 3, 6] = 0;$$

$$x[3, 19, 6] = 1;$$

$$x[7, 1, 6] = 0.5;$$

$$x[1, 7, 6] = 0.5;$$

$$x[19, 6, 6] = 0.5;$$

$$x[6, 19, 6] = 0.5;$$

$$x[9, 11, 6] = 0.5;$$

$$x[11, 9, 6] = 0.5;$$

$$x[2, 14, 6] = 0.5;$$

$$x[14, 2, 6] = 0.5;$$

$$x[3, 6, 6] = 1;$$

$$x[6, 3, 6] = 0;$$

$$x[15, 3, 5] = 0;$$

$$x[3, 15, 5] = 1;$$

$$x[4, 3, 5] = 0;$$

$$x[3, 4, 5] = 1;$$

$$x[15, 11, 5] = 1;$$

$$x[11, 15, 5] = 0;$$

$$x[12, 11, 5] = 0;$$

$$x[11, 12, 5] = 1;$$

$$x[3, 5, 5] = 1;$$

$$x[5, 3, 5] = 0;$$

$$x[18, 4, 5] = 0;$$

$$x[4, 18, 5] = 1;$$

$$x[12, 9, 5] = 0.5;$$

$$x[9, 12, 5] = 0.5;$$

$$x[18, 2, 5] = 1;$$

$$x[2, 18, 5] = 0;$$

$$x[20, 17, 5] = 0.5;$$

$$x[17, 20, 5] = 0.5;$$

$$x[9, 6, 5] = 0;$$

$$x[6, 9, 5] = 1;$$

$$x[1, 4, 5] = 0;$$

4 20231015 Linear Thurstone1.nb

x

$$[4, 1, 5] = 1;$$

x

$$[18, 20, 5] = 0.5;$$

x

$$[20, 18, 5] = 0.5;$$

x

$$[2, 17, 5] = 0.5;$$

x

$$[17, 2, 5] = 0.5;$$

x

$$[11, 8, 5] = 1;$$

x

$$[8, 11, 5] = 0;$$

x

$$[19, 5, 5] = 0.5;$$

x

$$[5, 19, 5] = 0.5;$$

x

$$[2, 20, 5] = 0;$$

x

$$[20, 2, 5] = 1;$$

x

$$[3, 2, 4] = 0;$$

x

$$[2, 3, 4] = 1;$$

x

$$[7, 6, 4] = 0;$$

x

$$[6, 7, 4] = 1;$$

x

$$[2, 6, 4] = 1;$$

x

$$[6, 2, 4] = 0;$$

x

$$[3, 7, 4] = 0.5;$$

x

$$[7, 3, 4] = 0.5;$$

x

$$[7, 1, 4] = 1;$$

x

$$[1, 7, 4] = 0;$$

x

$$[15, 6, 4] = 0;$$

x

$$[6, 15, 4] = 1;$$

x

$$[9, 2, 4] = 0.5;$$

x

$$[2, 9, 4] = 0.5;$$

x

$$[3, 18, 4] = 1;$$

x

$$[18, 3, 4] = 0;$$

x

$$[1, 4, 4] = 0.5;$$

x

$$[4, 1, 4] = 0.5;$$

x

$$[1, 6, 4] = 1;$$

x

$$[6, 1, 4] = 0;$$

x

$$[16, 4, 4] = 0;$$

x

$$[4, 16, 4] = 1;$$

x

$$[2, 18, 4] = 1;$$

x

$$[18, 2, 4] = 0;$$

x

$$[19, 6, 4] = 0;$$

x

$$[6, 19, 4] = 1;$$

x

$$[3, 13, 4] = 1;$$

x

$$[13, 3, 4] = 0;$$

x

$$[15, 11, 4] = 1;$$

x

$$[11, 15, 4] = 0;$$

x

$$[1, 19, 4] = 1;$$

x

$$[19, 1, 4] = 0;$$

x

$$[7, 5, 4] = 0.5;$$

x

$$[5, 7, 4] = 0.5;$$

x

$$[3, 9, 3] = 0.5;$$

x

$$[9, 3, 3] = 0.5;$$

x

$$[9, 10, 3] = 1;$$

x

$$[10, 9, 3] = 0;$$

x

$$[7, 3, 3] = 0;$$

x

$$[3, 7, 3] = 1;$$

20231015 Linear Thurstone1.nb

5

$$x[3, 11, 3] = 1;$$

$$x[11, 3, 3] = 0;$$

$$x[15, 5, 3] = 1;$$

$$x[5, 15, 3] = 0;$$

$$x[10, 14, 3] = 1;$$

$$x[14, 10, 3] = 0;$$

$$x[13, 7, 3] = 0;$$

$$x[7, 13, 3] = 1;$$

$$x[14, 16, 3] = 0;$$

$$x[16, 14, 3] = 1;$$

$$x[5, 7, 3] = 1;$$

$$x[7, 5, 3] = 0;$$

$$x[11, 16, 3] = 1;$$

$$x[16, 11, 3] = 0;$$

$$x[15, 10, 3] = 0.5;$$

$$x[10, 15, 3] = 0.5;$$

$$x[5, 13, 3] = 1;$$

$$x[13, 5, 3] = 0;$$

$$x[11, 14, 3] = 0.5;$$

$$x[14, 11, 3] = 0.5;$$

$$x[9, 4, 2] = 1;$$

$$x[4, 9, 2] = 0;$$

$$x[9, 1, 2] = 0.5;$$

$$x[1, 9, 2] = 0.5;$$

$$x[4, 5, 2] = 1;$$

$$x[5, 4, 2] = 0;$$

$$x[9, 17, 2] = 1;$$

$$x[17, 9, 2] = 0;$$

$$x[3, 1, 2] = 0;$$

$$x[1, 3, 2] = 1;$$

$$x[5, 10, 2] = 0;$$

$$x[10, 5, 2] = 1;$$

$$x[5, 17, 2] = 1;$$

$$x[17, 5, 2] = 0;$$

$$x[4, 7, 2] = 0.5;$$

$$x[7, 4, 2] = 0.5;$$

$$x[9, 11, 2] = 1;$$

$$x[11, 9, 2] = 0;$$

$$x[2, 5, 1] = 1;$$

$$x[5, 2, 1] = 0;$$

$$x[1, 5, 1] = 1;$$

$$x[5, 1, 1] = 0;$$

$$x[1, 4, 1] = 1;$$

$$x[4, 1, 1] = 0;$$

$$x[10, 5, 1] = 0.5;$$

$$x[5, 10, 1] = 0.5;$$

$$x[3, 2, 1] = 0.5;$$

$$x[2, 3, 1] = 0.5;$$

$$x[18, 10, 1] = 0;$$

$$x[10, 18, 1] = 1;$$

$$x[9, 2, 1] = 0;$$

6 20231015 Linear Thurstone1.nb

$$x[2, 9, 1] = 1;$$

$$x[1, 17, 1] = 1;$$

$$x[17, 1, 1] = 0;$$

$$x[8, 13, 1] = 0;$$

$$x[13, 8, 1] = 1;$$

x[18, 17, 1] = 1;

x[17, 18, 1] = 0;

x[4, 13, 1] = 0.5;

x[13, 4, 1] = 0.5;

x[9, 1, 1] = 0.5;

x[1, 9, 1] = 0.5;

x[8, 4, 1] = 1;

x[4, 8, 1] = 0;

x[10, 3, 1] = 0;

x[3, 10, 1] = 1;

In[3]:= d[i_, t_] := Sum[x[i, j, t], {j, 1, 10}] / 10

In[4]:= p[i_, t_] := Quantile[NormalDistribution[0, 1], d[i, t]]

In[5]:= A[i_] := LinearModelFit[Table[{t, p[i, t]}, {t, 1, 10}], t, t][[1, 2, 1]]

In[6]:= B[i_] := LinearModelFit[Table[{t, p[i, t]}, {t, 1, 10}], t, t][[1, 2, 2]]

In[7]:= Join[Table[A[i], {i, 1, 20}], Table[B[i], {i, 1, 20}]]

Out[7]= {0.126741, 0.101188, 0.0672893, -0.0340495, -0.17741, 5.9296 × 10⁻¹⁸, -0.0928265,

0.0502645, 7.88747 × 10⁻¹⁷, -0.0418871, -0.0251323, 0., -0.0508044, 0.0340495,

-0.0433375, -0.0167548, -0.117959, -0.0758017, -0.0674243, 0.0674243,

-0.0184743, 0.00711122, -0.0121976, -0.0075913, 0.0230438, -0.0045695,

0.0145928, -0.00913901, 0.00228475, 0.00536789, -0.00228475, 0., 0.00463085,

-0.0154034, 0.0174141, -0.00152317, 0.014556, 0.00232156, 0.00308314, -0.0145805}

In[8]:= Table[A[i] + B[i] * 10.25, {i, 1, 20}]

Out[8]= {-0.0626208, 0.174078, -0.0577362, -0.11186, 0.0587887, -0.0468374,

0.0567496, -0.0434103, 0.0234187, 0.0131338, -0.048551, 0., -0.00333818,

-0.123836, 0.135157, -0.0323673, 0.0312401, -0.0520057, -0.035822, -0.082026}

In[9]:= Table[A[i] + B[i] * 11, {i, 1, 20}]

Out[9]= {-0.0764765, 0.179411, -0.0668844, -0.117554, 0.0760716, -0.0502645,
0.0676942, -0.0502645, 0.0251323, 0.0171597, -0.0502645, 0., 0.000134961,
-0.135388, 0.148218, -0.0335097, 0.042157, -0.0502645, -0.0335097, -0.0929614}

Thurstone 2

In[1]:= For[ii = 1, ii ≤ 20, ii++,

For[jj = 1, jj ≤ 20, jj++, For[tt = 1, tt ≤ 10, tt++, x[ii, jj, tt] = 0.5]]]

In[2]:= x[1, 2, 10] = 0.5; x[2, 1, 10] = 0.5;

x[6, 13, 10] = 0.666666667; x[13, 6, 10] = 0.333333333;

x[2, 13, 10] = 1; x[13, 2, 10] = 0;

x[1, 6, 10] = 1; x[6, 1, 10] = 0;

x[13, 8, 10] = 1; x[8, 13, 10] = 0;

x[4, 2, 10] = 0.333333333; x[2, 4, 10] = 0.666666667;

x[6, 3, 10] = 0.5; x[3, 6, 10] = 0.5;

x[7, 1, 10] = 0.5; x[1, 7, 10] = 0.5;

x[13, 10, 10] = 0.5; x[10, 13, 10] = 0.5;

x[8, 14, 10] = 0.857142857; x[14, 8, 10] = 0.142857143;

x[19, 6, 10] = 0.5; x[6, 19, 10] = 0.5;

x[4, 20, 10] = 1; x[20, 4, 10] = 0;

x[7, 11, 10] = 0.75; x[11, 7, 10] = 0.25;

x[6, 5, 10] = 0.5; x[5, 6, 10] = 0.5;

x[19, 10, 10] = 0.666666667; x[10, 19, 10] = 0.333333333;

x[3, 14, 10] = 1; x[14, 3, 10] = 0;

$x[8, 17, 10] = 1; x[17, 8, 10] = 0;$
 $x[5, 13, 10] = 0; x[13, 5, 10] = 1;$
 $x[10, 15, 10] = 0.5; x[15, 10, 10] = 0.5;$
 $x[2, 18, 10] = 0.666666667; x[18, 2, 10] = 0.333333333;$
 $x[4, 11, 10] = 0.5; x[11, 4, 10] = 0.5;$
 $x[13, 6, 10] = 0.5; x[6, 13, 10] = 0.5;$
 $x[15, 19, 10] = 0.333333333; x[19, 15, 10] = 0.666666667;$
 $x[20, 7, 10] = 0; x[7, 20, 10] = 1;$
 $x[2, 6, 9] = 0.666666667; x[6, 2, 9] = 0.333333333;$
 $x[5, 4, 9] = 1; x[4, 5, 9] = 0;$
 $x[6, 4, 9] = 0.666666667; x[4, 6, 9] = 0.333333333;$
 $x[2, 5, 9] = 1; x[5, 2, 9] = 0;$
 $x[3, 5, 9] = 0.333333333; x[5, 3, 9] = 0.666666667;$
 $x[16, 4, 9] = 0.5; x[4, 16, 9] = 0.5;$
 $x[5, 19, 9] = 0.6; x[19, 5, 9] = 0.4;$
 $x[6, 18, 9] = 0.5; x[18, 6, 9] = 0.5;$
 $x[2, 1, 9] = 0.571428571; x[1, 2, 9] = 0.428571429;$
 $x[20, 16, 9] = 0; x[16, 20, 9] = 1;$
 $x[4, 5, 9] = 0; x[5, 4, 9] = 1;$
 $x[18, 2, 9] = 0.5; x[2, 18, 9] = 0.5;$
 $x[10, 13, 9] = 0.5; x[13, 10, 9] = 0.5;$
 $x[19, 20, 9] = 0.5; x[20, 19, 9] = 0.5;$
 $x[1, 6, 9] = 0; x[6, 1, 9] = 1;$
 $x[8, 13, 9] = 1; x[13, 8, 9] = 0;$

$x[16, 19, 9] = 0.3333333333$; $x[19, 16, 9] = 0.6666666667$;

$x[3, 14, 9] = 0.5$; $x[14, 3, 9] = 0.5$;

$x[8, 10, 9] = 0.5$; $x[10, 8, 9] = 0.5$;

$x[15, 1, 8] = 1$; $x[1, 15, 8] = 0$;

$x[3, 7, 8] = 0$; $x[7, 3, 8] = 1$;

$x[7, 1, 8] = 0.5$; $x[1, 7, 8] = 0.5$;

$x[3, 15, 8] = 0.125$; $x[15, 3, 8] = 0.875$;

$x[1, 5, 8] = 1$; $x[5, 1, 8] = 0$;

$x[2, 15, 8] = 0$; $x[15, 2, 8] = 1$;

$x[3, 16, 8] = 0.6666666667$; $x[16, 3, 8] = 0.3333333333$;

$x[1, 14, 8] = 1$; $x[14, 1, 8] = 0$;

$x[5, 11, 8] = 0.6666666667$; $x[11, 5, 8] = 0.3333333333$;

$x[16, 17, 8] = 1$; $x[17, 16, 8] = 0$;

$x[11, 15, 8] = 0$; $x[15, 11, 8] = 1$;

$x[9, 17, 8] = 0$; $x[17, 9, 8] = 1$;

$x[19, 16, 8] = 0.2$; $x[16, 19, 8] = 0.8$;

$x[11, 8, 8] = 0.5$; $x[8, 11, 8] = 0.5$;

$x[14, 2, 8] = 0.285714286$; $x[2, 14, 8] = 0.714285714$;

$x[15, 8, 8] = 1$; $x[8, 15, 8] = 0$;

$x[4, 9, 8] = 0.3333333333$; $x[9, 4, 8] = 0.6666666667$;

$x[10, 7, 8] = 0.1666666667$; $x[7, 10, 8] = 0.8333333333$;

$x[3, 6, 8] = 0.75$; $x[6, 3, 8] = 0.25$;

$x[7, 10, 7] = 0$; $x[10, 7, 7] = 1$;

$x[15, 10, 7] = 0; x[10, 15, 7] = 1;$
 $x[1, 15, 7] = 0; x[15, 1, 7] = 1;$
 $x[7, 3, 7] = 0.666666667; x[3, 7, 7] = 0.333333333;$
 $x[10, 8, 7] = 1; x[8, 10, 7] = 0;$
 $x[15, 4, 7] = 0.8; x[4, 15, 7] = 0.2;$
 $x[8, 3, 7] = 0.5; x[3, 8, 7] = 0.5;$
 $x[18, 19, 7] = 0.25; x[19, 18, 7] = 0.75;$
 $x[12, 17, 7] = 0; x[17, 12, 7] = 1;$
 $x[7, 19, 7] = 1; x[19, 7, 7] = 0;$
 $x[10, 14, 7] = 0; x[14, 10, 7] = 1;$
 $x[7, 18, 7] = 1; x[18, 7, 7] = 0;$
 $x[4, 11, 7] = 0.5; x[11, 4, 7] = 0.5;$
 $x[9, 2, 6] = 0.5; x[2, 9, 6] = 0.5;$
 $x[15, 8, 6] = 0.75; x[8, 15, 6] = 0.25;$
 $x[8, 2, 6] = 0; x[2, 8, 6] = 1;$
 $x[15, 9, 6] = 0; x[9, 15, 6] = 1;$
 $x[4, 8, 6] = 0.5; x[8, 4, 6] = 0.5;$
 $x[3, 2, 6] = 0; x[2, 3, 6] = 1;$
 $x[15, 1, 6] = 0.5; x[1, 15, 6] = 0.5;$
 $x[10, 2, 6] = 0.25; x[2, 10, 6] = 0.75;$
 $x[8, 7, 6] = 1; x[7, 8, 6] = 0;$
 $x[19, 3, 6] = 0.2; x[3, 19, 6] = 0.8;$
 $x[7, 1, 6] = 0.5; x[1, 7, 6] = 0.5;$
 $x[19, 6, 6] = 0.5; x[6, 19, 6] = 0.5;$

$x[9, 11, 6] = 0.5; x[11, 9, 6] = 0.5;$
 $x[2, 14, 6] = 0.5; x[14, 2, 6] = 0.5;$
 $x[3, 6, 6] = 1; x[6, 3, 6] = 0;$
 $x[15, 3, 5] = 0; x[3, 15, 5] = 1;$
 $x[4, 3, 5] = 0.3333333333; x[3, 4, 5] = 0.6666666667;$
 $x[15, 11, 5] = 1; x[11, 15, 5] = 0;$
 $x[12, 11, 5] = 0; x[11, 12, 5] = 1;$
 $x[3, 5, 5] = 1; x[5, 3, 5] = 0;$
 $x[18, 4, 5] = 0; x[4, 18, 5] = 1;$
 $x[12, 9, 5] = 0.5; x[9, 12, 5] = 0.5;$
 $x[18, 2, 5] = 1; x[2, 18, 5] = 0;$
 $x[20, 17, 5] = 0.5; x[17, 20, 5] = 0.5;$

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$x[9, 6, 5] = 0.3333333333; x[6, 9, 5] = 0.6666666667;$
 $x[1, 4, 5] = 0; x[4, 1, 5] = 1;$
 $x[18, 20, 5] = 0.5; x[20, 18, 5] = 0.5;$
 $x[2, 17, 5] = 0.5; x[17, 2, 5] = 0.5;$
 $x[11, 8, 5] = 0.6; x[8, 11, 5] = 0.4;$
 $x[19, 5, 5] = 0.5; x[5, 19, 5] = 0.5;$
 $x[2, 20, 5] = 0; x[20, 2, 5] = 1;$
 $x[3, 2, 4] = 0; x[2, 3, 4] = 1;$
 $x[7, 6, 4] = 0.3333333333; x[6, 7, 4] = 0.6666666667;$
 $x[2, 6, 4] = 0.6666666667; x[6, 2, 4] = 0.3333333333;$

$x[3, 7, 4] = 0.5; x[7, 3, 4] = 0.5;$
 $x[7, 1, 4] = 0.666666667; x[1, 7, 4] = 0.333333333;$
 $x[15, 6, 4] = 0; x[6, 15, 4] = 1;$
 $x[9, 2, 4] = 0.5; x[2, 9, 4] = 0.5;$
 $x[3, 18, 4] = 0.6; x[18, 3, 4] = 0.4;$
 $x[1, 4, 4] = 0.5; x[4, 1, 4] = 0.5;$
 $x[1, 6, 4] = 1; x[6, 1, 4] = 0;$
 $x[16, 4, 4] = 0; x[4, 16, 4] = 1;$
 $x[2, 18, 4] = 0.666666667; x[18, 2, 4] = 0.333333333;$
 $x[19, 6, 4] = 0; x[6, 19, 4] = 1;$
 $x[3, 13, 4] = 1; x[13, 3, 4] = 0;$
 $x[15, 11, 4] = 1; x[11, 15, 4] = 0;$
 $x[1, 19, 4] = 1; x[19, 1, 4] = 0;$
 $x[7, 5, 4] = 0.5; x[5, 7, 4] = 0.5;$
 $x[3, 9, 3] = 0.5; x[9, 3, 3] = 0.5;$
 $x[9, 10, 3] = 0.666666667; x[10, 9, 3] = 0.333333333;$
 $x[7, 3, 3] = 0.4; x[3, 7, 3] = 0.6;$
 $x[3, 11, 3] = 1; x[11, 3, 3] = 0;$
 $x[15, 5, 3] = 0.6; x[5, 15, 3] = 0.4;$
 $x[10, 14, 3] = 1; x[14, 10, 3] = 0;$
 $x[13, 7, 3] = 0.333333333; x[7, 13, 3] = 0.666666667;$
 $x[14, 16, 3] = 0; x[16, 14, 3] = 1;$
 $x[5, 7, 3] = 1; x[7, 5, 3] = 0;$
 $x[11, 16, 3] = 0.666666667; x[16, 11, 3] = 0.333333333;$

$x[15, 10, 3] = 0.5; x[10, 15, 3] = 0.5;$
 $x[5, 13, 3] = 1; x[13, 5, 3] = 0;$
 $x[11, 14, 3] = 0.5; x[14, 11, 3] = 0.5;$
 $x[9, 4, 2] = 0.666666667; x[4, 9, 2] = 0.333333333;$
 $x[9, 1, 2] = 0.5; x[1, 9, 2] = 0.5;$
 $x[4, 5, 2] = 1; x[5, 4, 2] = 0;$
 $x[9, 17, 2] = 1; x[17, 9, 2] = 0;$
 $x[3, 1, 2] = 0; x[1, 3, 2] = 1;$
 $x[5, 10, 2] = 0.333333333; x[10, 5, 2] = 0.666666667;$
 $x[5, 17, 2] = 0.75; x[17, 5, 2] = 0.25;$
 $x[4, 7, 2] = 0.5; x[7, 4, 2] = 0.5;$
 $x[9, 11, 2] = 1; x[11, 9, 2] = 0;$
 $x[2, 5, 1] = 0.666666667; x[5, 2, 1] = 0.333333333;$
 $x[1, 5, 1] = 1; x[5, 1, 1] = 0;$
 $x[1, 4, 1] = 0.666666667; x[4, 1, 1] = 0.333333333;$
 $x[10, 5, 1] = 0.5; x[5, 10, 1] = 0.5;$
 $x[3, 2, 1] = 0.5; x[2, 3, 1] = 0.5;$

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$x[18, 10, 1] = 0.166666667; x[10, 18, 1] = 0.833333333;$
 $x[9, 2, 1] = 0; x[2, 9, 1] = 1;$
 $x[1, 17, 1] = 1; x[17, 1, 1] = 0;$
 $x[8, 13, 1] = 0.25; x[13, 8, 1] = 0.75;$

x[18, 17, 1] = 0.857142857; x[17, 18, 1] = 0.142857143;

x[4, 13, 1] = 0.5; x[13, 4, 1] = 0.5;

x[9, 1, 1] = 0.5; x[1, 9, 1] = 0.5;

x[8, 4, 1] = 1; x[4, 8, 1] = 0;

x[10, 3, 1] = 0;

x[3, 10, 1] = 1;

In[161]:= d[i_, t_] := Sum[x[i, j, t], {j, 1, 10}] / 10

In[162]:= p[i_, t_] := Quantile[NormalDistribution[0, 1], d[i, t]]

In[163]:= A[i_] := LinearModelFit[Table[{t, p[i, t]}, {t, 1, 10}], t, t][[1, 2, 1]]

In[164]:= B[i_] := LinearModelFit[Table[{t, p[i, t]}, {t, 1, 10}], t, t][[1, 2, 2]]

In[165]:= Join[Table[A[i], {i, 1, 20}], Table[B[i], {i, 1, 20}]]

Out[165]= {0.0946744, 0.0888103, 0.0220684, 0.00540955, -0.103562, -0.0153799, -
0.0626644,

0.0502645, -0.0335488, -0.0460823, -0.056727, 0., -0.0531987, 0.021798,

-0.0719676, -0.0223463, -0.0965492, -0.0384815, -0.0607027, 0.0674243,

-0.0106837, -0.000105548, -0.00544851, -0.00556652, 0.0111993, -0.00215347,

0.00866892, -0.00913901, 0.00533464, 0.00803909, 0.00430036, 0., 0.00547525,

-0.0111732, 0.0180866, 0.00101841, 0.0118311, 0.0012149, 0.00308127, -0.0145805}

In[166]:= Table[A[i] + B[i] * 10.25, {i, 1, 20}]

Out[166]= {-0.0148338, 0.0877285, -0.0337788, -0.0516473, 0.0112305, -0.037453,

0.026192, -0.0434103, 0.0211313, 0.0363184, -0.0126483, 0., 0.00292257,

-0.0927274, 0.11342, -0.0119076, 0.0247199, -0.0260288, -0.0291198, -0.082026}

In[167]:= Table[A[i] + B[i] * 11, {i, 1, 20}]

Out[167]= {-0.0228466, 0.0876493, -0.0378652, -0.0558222, 0.0196299, -0.0390681,

0.0326937, -0.0502645, 0.0251323, 0.0423477, -0.00942299, 0., 0.007029,

-0.101107, 0.126985, -0.0111438, 0.0335932, -0.0251176, -0.0268088, -0.0929614}

Thurstone 3

In[1]:= x[1, 2, 10] = 0.5; x[2, 1, 10] = 0.5;

x[6, 13, 10] = 0.666666667; x[13, 6, 10] = 0.333333333;

x[2, 13, 10] = 1; x[13, 2, 10] = 0;

x[1, 6, 10] = 1; x[6, 1, 10] = 0;

x[13, 8, 10] = 1; x[8, 13, 10] = 0;

x[4, 2, 10] = 0.333333333; x[2, 4, 10] = 0.666666667;

x[6, 3, 10] = 0.5; x[3, 6, 10] = 0.5;

x[7, 1, 10] = 0.5; x[1, 7, 10] = 0.5;

x[13, 10, 10] = 0.5; x[10, 13, 10] = 0.5;

x[8, 14, 10] = 0.857142857; x[14, 8, 10] = 0.142857143;

x[19, 6, 10] = 0.5; x[6, 19, 10] = 0.5;

x[4, 20, 10] = 1; x[20, 4, 10] = 0;

x[7, 11, 10] = 0.75; x[11, 7, 10] = 0.25;

x[6, 5, 10] = 0.5; x[5, 6, 10] = 0.5;

x[19, 10, 10] = 0.666666667; x[10, 19, 10] = 0.333333333;

x[3, 14, 10] = 1; x[14, 3, 10] = 0;

x[8, 17, 10] = 1; x[17, 8, 10] = 0;

x[5, 13, 10] = 0; x[13, 5, 10] = 1;

x[10, 15, 10] = 0.5; x[15, 10, 10] = 0.5;

x[2, 18, 10] = 0.666666667; x[18, 2, 10] = 0.333333333;

x[4, 11, 10] = 0.5; x[11, 4, 10] = 0.5;

x[13, 6, 10] = 0.5; x[6, 13, 10] = 0.5;

$x[15, 19, 10] = 0.3333333333$; $x[19, 15, 10] = 0.6666666667$;

$x[20, 7, 10] = 0$; $x[7, 20, 10] = 1$;

$x[2, 6, 9] = 0.6666666667$; $x[6, 2, 9] = 0.3333333333$;

$x[5, 4, 9] = 1$; $x[4, 5, 9] = 0$;

$x[6, 4, 9] = 0.6666666667$; $x[4, 6, 9] = 0.3333333333$;

$x[2, 5, 9] = 1$; $x[5, 2, 9] = 0$;

$x[3, 5, 9] = 0.3333333333$; $x[5, 3, 9] = 0.6666666667$;

$x[16, 4, 9] = 0.5$; $x[4, 16, 9] = 0.5$;

$x[5, 19, 9] = 0.6$; $x[19, 5, 9] = 0.4$;

$x[6, 18, 9] = 0.5$; $x[18, 6, 9] = 0.5$;

$x[2, 1, 9] = 0.571428571$; $x[1, 2, 9] = 0.428571429$;

$x[20, 16, 9] = 0$; $x[16, 20, 9] = 1$;

$x[4, 5, 9] = 0$; $x[5, 4, 9] = 1$;

$x[18, 2, 9] = 0.5$; $x[2, 18, 9] = 0.5$;

$x[10, 13, 9] = 0.5$; $x[13, 10, 9] = 0.5$;

$x[19, 20, 9] = 0.5$; $x[20, 19, 9] = 0.5$;

$x[1, 6, 9] = 0$; $x[6, 1, 9] = 1$;

$x[8, 13, 9] = 1$; $x[13, 8, 9] = 0$;

$x[16, 19, 9] = 0.3333333333$; $x[19, 16, 9] = 0.6666666667$;

$x[3, 14, 9] = 0.5$; $x[14, 3, 9] = 0.5$;

$x[8, 10, 9] = 0.5$; $x[10, 8, 9] = 0.5$;

$x[15, 1, 8] = 1$; $x[1, 15, 8] = 0$;

$x[3, 7, 8] = 0$; $x[7, 3, 8] = 1$;

$x[7, 1, 8] = 0.5$; $x[1, 7, 8] = 0.5$;

$x[3, 15, 8] = 0.125$; $x[15, 3, 8] = 0.875$;
 $x[1, 5, 8] = 1$; $x[5, 1, 8] = 0$;
 $x[2, 15, 8] = 0$; $x[15, 2, 8] = 1$;
 $x[3, 16, 8] = 0.666666667$; $x[16, 3, 8] = 0.333333333$;
 $x[1, 14, 8] = 1$; $x[14, 1, 8] = 0$;

 $x[5, 11, 8] = 0.666666667$; $x[11, 5, 8] = 0.333333333$;
 $x[16, 17, 8] = 1$; $x[17, 16, 8] = 0$;
 $x[11, 15, 8] = 0$; $x[15, 11, 8] = 1$;
 $x[9, 17, 8] = 0$; $x[17, 9, 8] = 1$;
 $x[19, 16, 8] = 0.2$; $x[16, 19, 8] = 0.8$;
 $x[11, 8, 8] = 0.5$; $x[8, 11, 8] = 0.5$;
 $x[14, 2, 8] = 0.285714286$; $x[2, 14, 8] = 0.714285714$;
 $x[15, 8, 8] = 1$; $x[8, 15, 8] = 0$;
 $x[4, 9, 8] = 0.333333333$; $x[9, 4, 8] = 0.666666667$;
 $x[10, 7, 8] = 0.166666667$; $x[7, 10, 8] = 0.833333333$;
 $x[3, 6, 8] = 0.75$; $x[6, 3, 8] = 0.25$;
 $x[7, 10, 7] = 0$; $x[10, 7, 7] = 1$;
 $x[15, 10, 7] = 0$; $x[10, 15, 7] = 1$;
 $x[1, 15, 7] = 0$; $x[15, 1, 7] = 1$;
 $x[7, 3, 7] = 0.666666667$; $x[3, 7, 7] = 0.333333333$;
 $x[10, 8, 7] = 1$; $x[8, 10, 7] = 0$;
 $x[15, 4, 7] = 0.8$; $x[4, 15, 7] = 0.2$;
 $x[8, 3, 7] = 0.5$; $x[3, 8, 7] = 0.5$;

$x[18, 19, 7] = 0.25$; $x[19, 18, 7] = 0.75$;
 $x[12, 17, 7] = 0$; $x[17, 12, 7] = 1$;
 $x[7, 19, 7] = 1$; $x[19, 7, 7] = 0$;
 $x[10, 14, 7] = 0$; $x[14, 10, 7] = 1$;
 $x[7, 18, 7] = 1$; $x[18, 7, 7] = 0$;
 $x[4, 11, 7] = 0.5$; $x[11, 4, 7] = 0.5$;
 $x[9, 2, 6] = 0.5$; $x[2, 9, 6] = 0.5$;
 $x[15, 8, 6] = 0.75$; $x[8, 15, 6] = 0.25$;
 $x[8, 2, 6] = 0$; $x[2, 8, 6] = 1$;
 $x[15, 9, 6] = 0$; $x[9, 15, 6] = 1$;
 $x[4, 8, 6] = 0.5$; $x[8, 4, 6] = 0.5$;
 $x[3, 2, 6] = 0$; $x[2, 3, 6] = 1$;
 $x[15, 1, 6] = 0.5$; $x[1, 15, 6] = 0.5$;
 $x[10, 2, 6] = 0.25$; $x[2, 10, 6] = 0.75$;
 $x[8, 7, 6] = 1$; $x[7, 8, 6] = 0$;
 $x[19, 3, 6] = 0.2$; $x[3, 19, 6] = 0.8$;
 $x[7, 1, 6] = 0.5$; $x[1, 7, 6] = 0.5$;
 $x[19, 6, 6] = 0.5$; $x[6, 19, 6] = 0.5$;
 $x[9, 11, 6] = 0.5$; $x[11, 9, 6] = 0.5$;
 $x[2, 14, 6] = 0.5$; $x[14, 2, 6] = 0.5$;
 $x[3, 6, 6] = 1$; $x[6, 3, 6] = 0$;
 $x[15, 3, 5] = 0$; $x[3, 15, 5] = 1$;
 $x[4, 3, 5] = 0.3333333333$; $x[3, 4, 5] = 0.6666666667$;
 $x[15, 11, 5] = 1$; $x[11, 15, 5] = 0$;

$x[12, 11, 5] = 0; x[11, 12, 5] = 1;$
 $x[3, 5, 5] = 1; x[5, 3, 5] = 0;$
 $x[18, 4, 5] = 0; x[4, 18, 5] = 1;$
 $x[12, 9, 5] = 0.5; x[9, 12, 5] = 0.5;$
 $x[18, 2, 5] = 1; x[2, 18, 5] = 0;$
 $x[20, 17, 5] = 0.5; x[17, 20, 5] = 0.5;$
 $x[9, 6, 5] = 0.3333333333; x[6, 9, 5] = 0.6666666667;$
 $x[1, 4, 5] = 0; x[4, 1, 5] = 1;$
 $x[18, 20, 5] = 0.5; x[20, 18, 5] = 0.5;$

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$x[2, 17, 5] = 0.5; x[17, 2, 5] = 0.5;$
 $x[11, 8, 5] = 0.6; x[8, 11, 5] = 0.4;$
 $x[19, 5, 5] = 0.5; x[5, 19, 5] = 0.5;$
 $x[2, 20, 5] = 0; x[20, 2, 5] = 1;$
 $x[3, 2, 4] = 0; x[2, 3, 4] = 1;$
 $x[7, 6, 4] = 0.3333333333; x[6, 7, 4] = 0.6666666667;$
 $x[2, 6, 4] = 0.6666666667; x[6, 2, 4] = 0.3333333333;$
 $x[3, 7, 4] = 0.5; x[7, 3, 4] = 0.5;$
 $x[7, 1, 4] = 0.6666666667; x[1, 7, 4] = 0.3333333333;$
 $x[15, 6, 4] = 0; x[6, 15, 4] = 1;$
 $x[9, 2, 4] = 0.5; x[2, 9, 4] = 0.5;$
 $x[3, 18, 4] = 0.6; x[18, 3, 4] = 0.4;$
 $x[1, 4, 4] = 0.5; x[4, 1, 4] = 0.5;$

$x[1, 6, 4] = 1; x[6, 1, 4] = 0;$
 $x[16, 4, 4] = 0; x[4, 16, 4] = 1;$
 $x[2, 18, 4] = 0.666666667; x[18, 2, 4] = 0.333333333;$
 $x[19, 6, 4] = 0; x[6, 19, 4] = 1;$
 $x[3, 13, 4] = 1; x[13, 3, 4] = 0;$
 $x[15, 11, 4] = 1; x[11, 15, 4] = 0;$
 $x[1, 19, 4] = 1; x[19, 1, 4] = 0;$
 $x[7, 5, 4] = 0.5; x[5, 7, 4] = 0.5;$
 $x[3, 9, 3] = 0.5; x[9, 3, 3] = 0.5;$
 $x[9, 10, 3] = 0.666666667; x[10, 9, 3] = 0.333333333;$
 $x[7, 3, 3] = 0.4; x[3, 7, 3] = 0.6;$
 $x[3, 11, 3] = 1; x[11, 3, 3] = 0;$
 $x[15, 5, 3] = 0.6; x[5, 15, 3] = 0.4;$
 $x[10, 14, 3] = 1; x[14, 10, 3] = 0;$
 $x[13, 7, 3] = 0.333333333; x[7, 13, 3] = 0.666666667;$
 $x[14, 16, 3] = 0; x[16, 14, 3] = 1;$
 $x[5, 7, 3] = 1; x[7, 5, 3] = 0;$
 $x[11, 16, 3] = 0.666666667; x[16, 11, 3] = 0.333333333;$
 $x[15, 10, 3] = 0.5; x[10, 15, 3] = 0.5;$
 $x[5, 13, 3] = 1; x[13, 5, 3] = 0;$
 $x[11, 14, 3] = 0.5; x[14, 11, 3] = 0.5;$
 $x[9, 4, 2] = 0.666666667; x[4, 9, 2] = 0.333333333;$
 $x[9, 1, 2] = 0.5; x[1, 9, 2] = 0.5;$
 $x[4, 5, 2] = 1; x[5, 4, 2] = 0;$

$x[9, 17, 2] = 1; x[17, 9, 2] = 0;$
 $x[3, 1, 2] = 0; x[1, 3, 2] = 1;$
 $x[5, 10, 2] = 0.333333333; x[10, 5, 2] = 0.666666667;$
 $x[5, 17, 2] = 0.75; x[17, 5, 2] = 0.25;$
 $x[4, 7, 2] = 0.5; x[7, 4, 2] = 0.5;$
 $x[9, 11, 2] = 1; x[11, 9, 2] = 0;$
 $x[2, 5, 1] = 0.666666667; x[5, 2, 1] = 0.333333333;$
 $x[1, 5, 1] = 1; x[5, 1, 1] = 0;$
 $x[1, 4, 1] = 0.666666667; x[4, 1, 1] = 0.333333333;$
 $x[10, 5, 1] = 0.5; x[5, 10, 1] = 0.5;$
 $x[3, 2, 1] = 0.5; x[2, 3, 1] = 0.5;$
 $x[18, 10, 1] = 0.166666667; x[10, 18, 1] = 0.833333333;$
 $x[9, 2, 1] = 0; x[2, 9, 1] = 1;$
 $x[1, 17, 1] = 1; x[17, 1, 1] = 0;$

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$x[8, 13, 1] = 0.25; x[13, 8, 1] = 0.75;$
 $x[18, 17, 1] = 0.857142857; x[17, 18, 1] = 0.142857143;$
 $x[4, 13, 1] = 0.5; x[13, 4, 1] = 0.5;$
 $x[9, 1, 1] = 0.5; x[1, 9, 1] = 0.5;$
 $x[8, 4, 1] = 1; x[4, 8, 1] = 0;$
 $x[10, 3, 1] = 0;$
 $x[3, 10, 1] = 1;$

```

In[160]:= d[i_, t_] := If[NumberQ[Mean[Join[Cases[Table[x[i, j, 1], {j, 1, 20}], _Real],
Cases[Table[x[i, j, 1], {j, 1, 20}], _Integer]]],
Mean[Join[Cases[Table[x[i, j, 1], {j, 1, 20}], _Real],
Cases[Table[x[i, j, 1], {j, 1, 20}], _Integer]]], 0.5]

In[161]:= p[i_, t_] := Quantile[NormalDistribution[0, 1], d[i, t]]

In[162]:= A[i_] := LinearModelFit[Table[{t, p[i, t]}, {t, 1, 10}], t, t][[1, 2, 1]]

In[163]:= B[i_] := LinearModelFit[Table[{t, p[i, t]}, {t, 1, 10}], t, t][[1, 2, 2]]

In[164]:= Join[Table[A[i], {i, 1, 20}], Table[B[i], {i, 1, 20}]]

Out[164]= {0.812218, 0.589456, 0.67449, -0.589456, -0.589456, 0., 0., 0.318639, -0.67449,
-0.13971,
0., 0., 0.318639, 0., 0., 0., -1.46523, 0.0298452, 0., 0., 2.08813 × 10-16, -4.83048 × 10-17,
1.36287 × 10-16, 4.83048 × 10-17, 4.83048 × 10-17, 0., 0., 3.81934 × 10-17, -1.36287 × 10-16,
-1.87339 × 10-17, 0., 0., 3.81934 × 10-17, 0., 0., 0., 1.66672 × 10-16, 2.57084 × 10-18, 0.,
0.}

In[165]:= Table[A[i] + B[i] * 10.25, {i, 1, 20}]

Out[165]= {0.812218, 0.589456, 0.67449, -0.589456, -0.589456, 0., 0., 0.318639,
-0.67449, -0.13971, 0., 0., 0.318639, 0., 0., 0., -1.46523, 0.0298452, 0., 0.}

In[166]:= Table[A[i] + B[i] * 11, {i, 1, 20}]

Out[166]= {0.812218, 0.589456, 0.67449, -0.589456, -0.589456, 0., 0., 0.318639,
-0.67449, -0.13971, 0., 0., 0.318639, 0., 0., 0., -1.46523, 0.0298452, 0., 0.}

```

[Correlation Rcode calculations](#)

#Linear thurstone

> #x is the baseline (Actual rank)

```
> x <- c(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)
> #y is the prediction for 2023
> y <- c(2,15,5,7,17,9,10,12,13,16,19,8,6,11,18,3,1,20,4,14)
> #Kendall tau method
> res<-cor.test(x,y, method="kendall")
> res
```

Kendall's rank correlation tau

data: x and y

T = 103, p-value = 0.6308

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.08421053

```
> #z is the prediction for 2026
> y1<- c(2,15,5,7,17,9,10,13,12,19,16,6,18,8,11,3,1,20,4,14)
> #Kendall tau method
> res1<-cor.test(x,y1, method="kendall")
> res1
```

Kendall's rank correlation tau

data: x and y1

T = 100, p-value = 0.7732

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.05263158

```
> #T2
> #2023
> y2 <- c(15,2,10,7,17,9,5,13,12,16,11,1,18,19,3,6,8,4,20,14)
```

```
> res2<-cor.test(x,y2, method="kendall")
> res2
```

Kendall's rank correlation tau

```
data: x and y2
T = 103, p-value = 0.6308
alternative hypothesis: true tau is not equal to 0
sample estimates:
    tau
0.08421053
```

```
> y3 <- c(15,2,10,17,7,9,5,13,12,11,16,1,18,19,3,6,8,4,20,14)
> res3<-cor.test(x,y3, method="kendall")
> res3
```

Kendall's rank correlation tau

```
data: x and y3
T = 103, p-value = 0.6308
alternative hypothesis: true tau is not equal to 0
sample estimates:
    tau
0.08421053
```

```
> #T3
> y4 <- c(1,3,2,8,13,18,6,7,11,12,14,15,16,19,20,10,4,5,9,17)
> res4<-cor.test(x,y4, method="kendall")
> res4
```

Kendall's rank correlation tau

```
data: x and y4
T = 128, p-value = 0.03364
```

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.3473684

```
> y5 <- c(1,3,2,8,13,18,6,7,11,12,14,15,16,19,20,10,4,5,9,17)
```

```
> res5<-cor.test(x,y5, method="kendall")
```

```
> res5
```

Kendall's rank correlation tau

data: x and y5

T = 128, p-value = 0.03364

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.3473684

```
> #2 Linear AHP
```

```
> #2023
```

```
> y6 <- c(2,1,7,15,16,8,12,20,14,10,4,17,9,11,19,5,6,13,18,3)
```

```
> res6<-cor.test(x,y6, method="kendall")
```

```
> res6
```

Kendall's rank correlation tau

data: x and y6

T = 105, p-value = 0.5424

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.1052632

```
> #2026
```

```
> y7 <- c(2,1,7,15,16,8,12,20,14,10,4,17,9,11,19,5,6,13,18,3)
> res7<-cor.test(x,y7, method="kendall")
> res7
```

Kendall's rank correlation tau

```
data: x and y7
T = 105, p-value = 0.5424
alternative hypothesis: true tau is not equal to 0
sample estimates:
  tau
0.1052632
```

```
>
> #LBT
> #2023
> y8 <- c(2,7,15,5,17,19,16,3,8,13,9,12,6,10,4,11,18,1,14,20)
> res8<-cor.test(x,y8, method="kendall")
> res8
```

Kendall's rank correlation tau

```
data: x and y8
T = 106, p-value = 0.5006
alternative hypothesis: true tau is not equal to 0
sample estimates:
  tau
0.1157895
```

```
> #2026
> y9 <- c(2,7,15,5,17,19,16,8,13,3,9,12,6,10,4,11,18,1,14,20)
> res9<-cor.test(x,y9, method="kendall")
> res9
```

Kendall's rank correlation tau

data: x and y9

T = 104, p-value = 0.5859

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.09473684

>

> #LBT 2

> #2023

> y10 <- c(15,7,2,16,17,3,13,10,8,19,4,9,5,12,6,11,18,1,14,20)

> res10 <- cor.test(x,y10, method="kendall")

> res10

Kendall's rank correlation tau

data: x and y10

T = 104, p-value = 0.5859

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.09473684

> #2026

> y11 <- c(7,15,2,16,17,3,13,10,8,19,4,5,9,12,6,11,18,1,14,20)

> res11 <- cor.test(x,y11, method="kendall")

> res11

Kendall's rank correlation tau

data: x and y11

T = 106, p-value = 0.5006

alternative hypothesis: true tau is not equal to 0

sample estimates:

tau

0.1157895

#Linear thurstone

> #x is the baseline (Actual rank)

> x <- c(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)

> #y is the prediction for 2023

> y <- c(2,15,5,7,17,9,10,12,13,16,19,8,6,11,18,3,1,20,4,14)

> #Pearson method

> res<-cor.test(x,y, method="pearson")

> res

Pearson's product-moment correlation

data: x and y

t = 0.25566, df = 18, p-value = 0.8011

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.3928266 0.4896381

sample estimates:

cor

0.06015038

> #z is the prediction for 2026

> y1<- c(2,15,5,7,17,9,10,13,12,19,16,6,18,8,11,3,1,20,4,14)

> #pearson method

> res1<-cor.test(x,y1, method="pearson")

> res1

Pearson's product-moment correlation

data: x and y1

$t = 0.13404$, $df = 18$, $p\text{-value} = 0.8949$

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.4167659 0.4675658

sample estimates:

cor

0.03157895

> #T2

> #2023

> y2 <- c(15,2,10,7,17,9,5,13,12,16,11,1,18,19,3,6,8,4,20,14)

> res2<-cor.test(x,y2, method="pearson")

> res2

Pearson's product-moment correlation

data: x and y2

$t = 0.38436$, $df = 18$, $p\text{-value} = 0.7052$

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.3669462 0.5122922

sample estimates:

cor

0.09022556

> y3 <- c(15,2,10,17,7,9,5,13,12,11,16,1,18,19,3,6,8,4,20,14)

> res3<-cor.test(x,y3, method="pearson")

> res3

Pearson's product-moment correlation

data: x and y3

$t = 0.3521$, $df = 18$, $p\text{-value} = 0.7289$

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.3734833 0.5066832

sample estimates:

cor

0.08270677

> #T3

> y4 <- c(1,3,2,8,13,18,6,7,11,12,14,15,16,19,20,10,4,5,9,17)

> res4<-cor.test(x,y4, method="pearson")

> res4

Pearson's product-moment correlation

data: x and y4

t = 1.8599, df = 18, p-value = 0.07932

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.04987926 0.71668791

sample estimates:

cor

0.4015038

> y5 <- c(1,3,2,8,13,18,6,7,11,12,14,15,16,19,20,10,4,5,9,17)

> res5<-cor.test(x,y5, method="pearson")

> res5

Pearson's product-moment correlation

data: x and y5

t = 1.8599, df = 18, p-value = 0.07932

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.04987926 0.71668791

sample estimates:

cor

0.4015038

> #2 Linear AHP

> #2023

> y6 <- c(2,1,7,15,16,8,12,20,14,10,4,17,9,11,19,5,6,13,18,3)

> res6<-cor.test(x,y6, method="pearson")

> res6

Pearson's product-moment correlation

data: x and y6

t = 0.64533, df = 18, p-value = 0.5269

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.3129714 0.5559044

sample estimates:

cor

0.1503759

> #2026

> y7 <- c(2,1,7,15,16,8,12,20,14,10,4,17,9,11,19,5,6,13,18,3)

> res7<-cor.test(x,y7, method="pearson")

> res7

Pearson's product-moment correlation

data: x and y7

t = 0.64533, df = 18, p-value = 0.5269

alternative hypothesis: true correlation is not equal to 0

95 percent confidence interval:

-0.3129714 0.5559044

sample estimates:

```
cor
0.1503759
```

```
>
> #LBT
> #2023
> y8 <- c(2,7,15,5,17,19,16,3,8,13,9,12,6,10,4,11,18,1,14,20)
> res8<-cor.test(x,y8, method="pearson")
> res8
```

Pearson's product-moment correlation

```
data: x and y8
t = 0.64533, df = 18, p-value = 0.5269
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
-0.3129714 0.5559044
sample estimates:
cor
0.1503759
```

```
> #2026
> y9 <- c(2,7,15,5,17,19,16,8,13,3,9,12,6,10,4,11,18,1,14,20)
> res9<-cor.test(x,y9, method="pearson")
> res9
```

Pearson's product-moment correlation

```
data: x and y9
t = 0.54678, df = 18, p-value = 0.5912
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
-0.3335688 0.5398073
sample estimates:
```

```
cor
0.1278195
```

```
>
> #LBT 2
> #2023
> y10 <- c(15,7,2,16,17,3,13,10,8,19,4,9,5,12,6,11,18,1,14,20)
> res10<-cor.test(x,y10, method="pearson")
> res10
```

Pearson's product-moment correlation

```
data: x and y10
t = 0.45558, df = 18, p-value = 0.6541
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
-0.3524037 0.5245066
sample estimates:
cor
0.1067669
```

```
> #2026
> y11 <- c(7,15,2,16,17,3,13,10,8,19,4,5,9,12,6,11,18,1,14,20)
> res11<-cor.test(x,y11, method="pearson")
> res11
```

Pearson's product-moment correlation

```
data: x and y11
t = 0.53371, df = 18, p-value = 0.6001
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
-0.3362823 0.5376380
sample estimates:
```

```

cor
0.124812
#Linear thurstone
> #x is the baseline (Actual rank)
> x <- c(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)
> #y is the prediction for 2023
> y <- c(2,15,5,7,17,9,10,12,13,16,19,8,6,11,18,3,1,20,4,14)
> #Spearman method
> res<-cor.test(x,y, method="spearman")
> res

```

Spearman's rank correlation rho

```

data: x and y
S = 1250, p-value = 0.8016
alternative hypothesis: true rho is not equal to 0
sample estimates:
rho
0.06015038

```

```

> #z is the prediction for 2026
> y1<- c(2,15,5,7,17,9,10,13,12,19,16,6,18,8,11,3,1,20,4,14)
> #Spearman method
> res1<-cor.test(x,y1, method="spearman")
> res1

```

Spearman's rank correlation rho

```

data: x and y1
S = 1288, p-value = 0.8963
alternative hypothesis: true rho is not equal to 0
sample estimates:
rho
0.03157895

```

```
> #T2
> #2023
> y2 <- c(15,2,10,7,17,9,5,13,12,16,11,1,18,19,3,6,8,4,20,14)
> res2<-cor.test(x,y2, method="spearman")
> res2
```

Spearman's rank correlation rho

data: x and y2

S = 1210, p-value = 0.7048

alternative hypothesis: true rho is not equal to 0

sample estimates:

rho

0.09022556

```
> y3 <- c(15,2,10,17,7,9,5,13,12,11,16,1,18,19,3,6,8,4,20,14)
> res3<-cor.test(x,y3, method="spearman")
> res3
```

Spearman's rank correlation rho

data: x and y3

S = 1220, p-value = 0.7287

alternative hypothesis: true rho is not equal to 0

sample estimates:

rho

0.08270677

```
> #T3
> y4 <- c(1,3,2,8,13,18,6,7,11,12,14,15,16,19,20,10,4,5,9,17)
> res4<-cor.test(x,y4, method="spearman")
> res4
```

Spearman's rank correlation rho

data: x and y4

S = 796, p-value = 0.08042

alternative hypothesis: true rho is not equal to 0

sample estimates:

rho

0.4015038

```
> y5 <- c(1,3,2,8,13,18,6,7,11,12,14,15,16,19,20,10,4,5,9,17)
```

```
> res5<-cor.test(x,y5, method="spearman")
```

```
> res5
```

Spearman's rank correlation rho

data: x and y5

S = 796, p-value = 0.08042

alternative hypothesis: true rho is not equal to 0

sample estimates:

rho

0.4015038

```
> #2 Linear AHP
```

```
> #2023
```

```
> y6 <- c(2,1,7,15,16,8,12,20,14,10,4,17,9,11,19,5,6,13,18,3)
```

```
> res6<-cor.test(x,y6, method="spearman")
```

```
> res6
```

Spearman's rank correlation rho

data: x and y6

S = 1130, p-value = 0.5254

alternative hypothesis: true rho is not equal to 0

sample estimates:

```
rho
0.1503759
```

```
> #2026
> y7 <- c(2,1,7,15,16,8,12,20,14,10,4,17,9,11,19,5,6,13,18,3)
> res7<-cor.test(x,y7, method="spearman")
> res7
```

Spearman's rank correlation rho

```
data: x and y7
S = 1130, p-value = 0.5254
alternative hypothesis: true rho is not equal to 0
sample estimates:
```

```
rho
0.1503759
```

```
>
> #LBT
> #2023
> y8 <- c(2,7,15,5,17,19,16,3,8,13,9,12,6,10,4,11,18,1,14,20)
> res8<-cor.test(x,y8, method="spearman")
> res8
```

Spearman's rank correlation rho

```
data: x and y8
S = 1130, p-value = 0.5254
alternative hypothesis: true rho is not equal to 0
sample estimates:
```

```
rho
0.1503759
```

```
> #2026
```

```
> y9 <- c(2,7,15,5,17,19,16,8,13,3,9,12,6,10,4,11,18,1,14,20)
> res9<-cor.test(x,y9, method="spearman")
> res9
```

Spearman's rank correlation rho

```
data: x and y9
S = 1160, p-value = 0.5901
alternative hypothesis: true rho is not equal to 0
sample estimates:
rho
0.1278195
```

```
>
> #LBT 2
> #2023
> y10 <- c(15,7,2,16,17,3,13,10,8,19,4,9,5,12,6,11,18,1,14,20)
> res10<-cor.test(x,y10, method="spearman")
> res10
```

Spearman's rank correlation rho

```
data: x and y10
S = 1188, p-value = 0.6534
alternative hypothesis: true rho is not equal to 0
sample estimates:
rho
0.1067669
```

```
> #2026
> y11 <- c(7,15,2,16,17,3,13,10,8,19,4,5,9,12,6,11,18,1,14,20)
> res11<-cor.test(x,y11, method="spearman")
> res11
```

Spearman's rank correlation rho

data: x and y11

S = 1164, p-value = 0.5989

alternative hypothesis: true rho is not equal to 0

sample estimates:

rho

0.124812