

THE DUAL POROSITY MODEL

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DECLARATION

I, Siphokazi S. MANUNDU, hereby declare that the dissertation hereby submitted by me to the Institute for Groundwater Studies in the Faculty of Natural and Agricultural Sciences at the University of the Free State, in fulfilment of the degree of Magister Scientiae, is my own independent work. It has not previously been submitted by me to any other institution of higher education. In addition, I declare that all sources cited have been acknowledged by means of a list of references.

I furthermore cede copyright of the dissertation and its contents in favour of the University of the Free State.

Siphokazi S. MANUNDU

July 2021

DEDICATION

I would like to send my dedications as well as to thank my family, ooManundu amaNdila, ooJola namaMpondomise kwaye noNungu, amaYirha, ooMzondi nooZiyeka for their guidance, encouragement, love and support they have showed to me throughout the journey of my life, their lifelong lessons which have moulded me to be the person I am today. Especially, to my late grandmother, Nothobile Manundu, and late grandfather Maqebelana Jackson Manundu who valued the importance of education.

Ndiyabulela.

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Lastly, I would like to express thanks to every individual or an organisation that may be intentionally or unintentionally excluded from the above acknowledgements, which directly or indirectly contributed to the completion of this report. I Thank You.

ABSTRACT

In the past years, various researchers have devoted their attention to modelling groundwater flow in dual-porosity of aquifers. A system of partial differential equations has been suggested and applied in many scenarios with some limitations. In the past, researchers used fractional and classical derivatives as differential operators to model the problems. However, these models cannot depict the crossover effect and the randomness of the geological formations with great precision. In recent years, researchers suggested piecewise differential and integral operators which these operators are defined in a given interval. The passage of a process from one to another is called crossover. The concept of stochastic differential equations was then presented to include the effect of randomness into mathematical formulation. Partial equations depicting groundwater flow within a dual-porosity aquifer were modified. To achieve this, we considered fractional differential and integral operators with different kernels including power law, exponential decay and the generalized Mittag-Leffler functions. Furthermore, randomness was incorporated in the modified model to obtain a piecewise fractional stochastic system of partial differential equations. The numerical method based on the Newton polynomial interpolation was applied to solve the obtained systems of partial differential equations numerically. Matlab was then used towards the generation of simulations. For a fractional order starting from 0.95 to 0.7, the results displayed high flow velocity within fracture, this behaviour is known as long range dependency, also within the matrix soil, we also observed high flow meaning in this case the fractional derivative is expressing the flow within a matrix rock with high transmissivity. For fractional order ranging from 0.7 to 0.5, the results normal flow behaviours; water flows with average velocity within the matrix rock and moderate-high velocity within the fractures. And, from 0.5 to 0.2, we observed slow flow behaviour, the mathematical equations are exhibiting flow with low velocities, expressing the flow within a shale with limited or no transmissivity.

LIST OF GREEK NOTATIONS

α	Alpha
β	Beta
δ	Delta
ε	Epsilon
η	Eta
γ	Gamma
λ	Lambda
\mathcal{L}	Laplace Transform Operator
μ	Mu
ω	Omega
∂	Partial Differential Operator
φ	Phi
ϕ	Phi Variant
ψ	Psi
ρ	Rho

σ	Sigma
τ	Tau
θ	Theta
ξ	Xi

ABBREVIATIONS AND NOTATIONS

q	Darcy's flux
K	Hydraulic conductivity
h	Matric pressure head
D	Pore-water diffusivity
t	Time elapsed
z	Vertical distance
S	Storativity
T	Transmissivity
Δ	Delta

TABLE OF CONTENTS

DECLARATION.....	i
DEDICATION	ii
ACKNOWLEDGEMENTS	iii
ABSTRACT	iv
LIST OF GREEK NOTATIONS	v
ABBREVIATIONS AND NOTATIONS	vi
TABLE OF CONTENTS	vii
LIST OF FIGURES.....	ix
Chapter 1 Introduction.....	1
1.1. Background.....	2
Limitations.....	6
1.2. Problem statement.....	7
1.3. Aim and objectives.....	7
1.4. Research structure.....	8
Chapter 2 Piecewise concept and modified model of piecewise	9
2.1. Piecewise differential and integral operators.....	9
2.2. Modified model using piecewise concept.....	10
2.3. Analysis of the modified model.....	13
2.4. Numerical solution using Newton polynomial scheme	17
Chapter 3 : Stochastic model	24
3.1. Introduction	24
3.2. Stochastic versus deterministic.....	24
3.3. Modified model with stochastic approach.....	25
Chapter 4 : Application of caputo-Fabrizio and Caputo fractional derivatives to the piecewise Model.....	32
4.1. Introduction.....	32
4.2. Application of Caputo-Fabrizio and Caputo derivative	32

Chapter 5 : Numerical simulations.....	43
5.1. Introduction.....	43
5.2. Results and discussion.....	44
Chapter 6 : CONCLUSION.....	64
Chapter 7 : Bibliography.....	65

LIST OF FIGURES

Figure 1.1: Solid rock matrix exclusive of fractures.	3
Figure 1.2: Fractured block.	3
Figure 1.3.: Dual porosity model.	4
Figure 2.1. : Real-life dual porosity aquifer (Qingdong & Jun, 2015).....	10
Figure 2.2. : Model representing the dual-porosity aquifer (Qingdong & Jun, 2015)	11
Figure 2.3.: 2D model of the dual porosity model.	11
Figure 5.1: Contour plot representing h_2 in fractional order 0.95.....	45
Figure 5.2: Contour plot representing h_1 in fractional order 0.95.....	45
Figure 5.3: 3D plot representing h_2 in fractional order 0.95.	46
Figure 5.4: 3D plot representing h_1 in fractional order 0.95.	46
Figure 5.5: Contour plot representing h_2 in fractional order 0.9.....	47
Figure 5.6: Contour plot representing h_1 in fractional order 0.9.....	47
Figure 5.7: 3D plot representing h_2 in fractional order 0.9.....	48
Figure 5.8: 3D plot representing h_1 in fractional order 0.9.....	48
Figure 5.9: Contour plot representing h_2 in fractional order 0.8.....	49
Figure 5.10: Contour plot representing h_1 in fractional order 0.8.....	49
Figure 5.11: 3D plot representing h_2 in fractional order 0.8.....	50
Figure 5.12: 3D plot representing h_1 in fractional order 0.8.....	50
Figure 5.13: Contour plot representing h_2 in fractional order 0.7.....	51
Figure 5.14: Contour plot representing h_1 in fractional order 0.7.....	51
Figure 5.15: 3D plot representing h_2 in fractional order 0.7.....	52
Figure 5.16: 3D plot representing h_1 in fractional order 0.7.....	52
Figure 5.17: Contour plot representing h_2 in fractional order 0.6.....	53
Figure 5.18: Contour plot representing h_1 in fractional order 0.6.....	53
Figure 5.19: 3D plot representing h_2 in fractional order 0.6.....	54
Figure 5.20: 3D plot representing h_1 in fractional order 0.6.....	54
Figure 5.21: Contour plot representing h_2 in fractional order 0.5.....	55
Figure 5.22: Contour plot representing h_1 in fractional order 0.5.....	55
Figure 5.23: 3D plot representing h_2 in fractional order 0.5.....	56
Figure 5.24: 3D plot representing h_1 in fractional order 0.5.....	56
Figure 5.25: Contour plot representing h_2 in fractional order 0.4.....	57

Figure 5.26: Contour plot representing h_1 in fractional order 0.4.....	57
Figure 5.27: 3D plot representing h_2 in fractional order 0.4.....	58
Figure 5.28: 3D plot representing h_1 in fractional order 0.4.....	58
Figure 5.29: Contour plot representing h_2 in fractional order 0.3.....	59
Figure 5.30: Contour plot representing h_1 in fractional order 0.3.....	59
Figure 5.31: 3D plot representing h_2 in fractional order 0.3.....	60
Figure 5.32: 3D plot representing h_1 in fractional order 0.3.....	60
Figure 5.33: Contour plot representing h_2 in fractional order 0.2.....	61
Figure 5.34: Contour plot representing h_1 in fractional order 0.2.....	61
Figure 5.35: 3D plot representing h_2 in fractional order 0.2.....	62
Figure 5.36: 3D plot representing h_1 in fractional order 0.2.....	62
Figure 5.37: Diagram representing the crossover effect.....	63

CHAPTER 1 INTRODUCTION

Groundwater exists beneath the earth's surface in faults, fractures, and pores of soils (Freeze & Cherry, 1979). Areas where groundwater exists are referred to as aquifers (saturated fractured rock or sand, which usable volumes of groundwater can be pumped) (Freeze & Cherry, 1979). Aquifers are categorised into two types, namely the porous media and fractured rock aquifer (Babcock, 2004). The porous media aquifer is composed of discrete soil; some examples are gravel, silt or sand. Porous media aquifers can be further divided into unconsolidated porous media, whereby their grains are not intact to each other. For example, alluvial aquifer or consolidated porous media and consolidated porous media where the grains are cemented to each other e.g. sandstone. However, fractured aquifers are defined as geologic formations in which joints, cracks, and/or fractures act as conducive conduits in transporting groundwater. An example of a fractured rock aquifer is the Table Mountain Group.

Moreover, for both aquifers in the categories, aquifers can be further grouped into three types namely unconfined, confined, and leaky aquifers (Babcock, 2004). The unconfined aquifer may be defined as a geological layer with no confining layer above the aquifer. Whereas with the confined aquifer, the geologic layer is bounded by confining beds above and beneath it, resultant in the groundwater being subjected to pressure greater than that of the atmosphere. Lastly, the leaky aquifer in which the geologic layer is bounded between aquitards

1.1. Background

Over the years, mathematical models have been continuously developed to assist in the sustainable management and safeguarding of the world groundwater resources (Cordero et al., 2019). These include the dual-porosity models which assist in understanding aquifers with double porosity.

Dual media system

The dual-porosity media system theory is applied towards studying the response of fractured and double porosity aquifers. The assumption that the dual-porosity model makes is that the porous medium consists of two continua:

- one related to the matrix block and fracture network and,
- the other to a less permeable pore system of the matrix block (Knaust et al., 2020).

The fracture network in this system is characterised by high hydraulic conductivity (K) and low storage (S_s). These fractures are the main conduits of the system however their volume for storage capacity remains limited. Whereas the matrix block section of the system is characterised by low hydraulic conductivity (K) high storage (S_s) capability. These matrixes contribute a limited role in the transportation of fluids though they play a significant role in fluid storage. Often, the fluid in the matrix is drained through the fractures before it drains out of the aquifer (Elsowrth, 2015).

Existing mathematical models on dual media system

The dual-porosity model (Figure 1.3.) is governed by the assumption that the medium can be separated into two distinct pore systems, the system that is microscopically composed of soil particles (grey shaded area in Figure 1.1.) and the other system that is surrounded by fractures (grey horizontal and vertical lines in Figure 1.2.). The two distinct pore systems Both of which are treated as homogeneous media with separate hydraulic and solute transport properties. The dual-porosity medium is known to be a superposition of these two systems through the same volume (Warren & Root, 1963). This model has been further explained in Table 1.

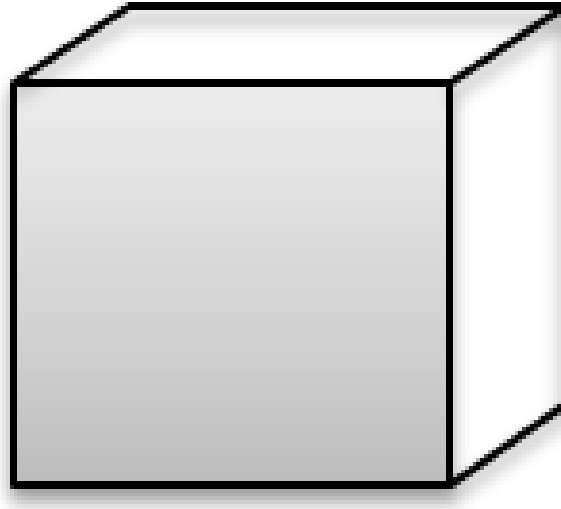


Figure 1.1: Solid rock matrix exclusive of fractures.

$$T_1 \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) = S_1 \frac{\partial^2 h_1}{\partial t} \quad (1.1)$$

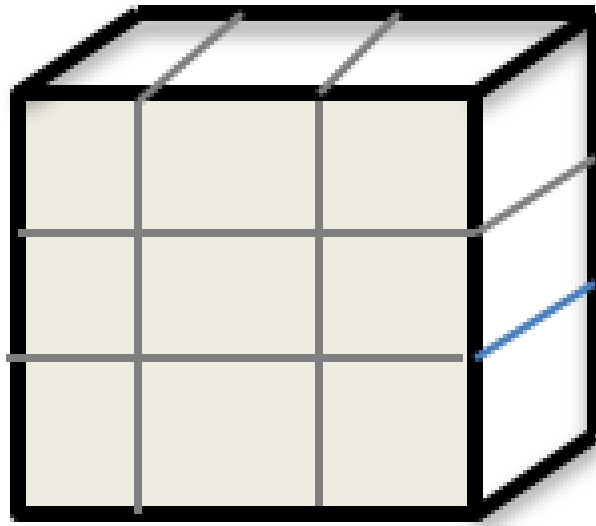


Figure 1.2: Fractured block.

$$T_2 \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right) = S_2 \frac{\partial^2 h_2}{\partial t} \quad (1.2)$$

T represents the transmissivity of the geologic layer, h represents the head, S is the Storativity of the aquifer, t represents time and a is the distance between fractures.

Equation (1.1.) and (1.2.) are further integrated to represent the dual-porosity model as indicated in equation (1.3.).

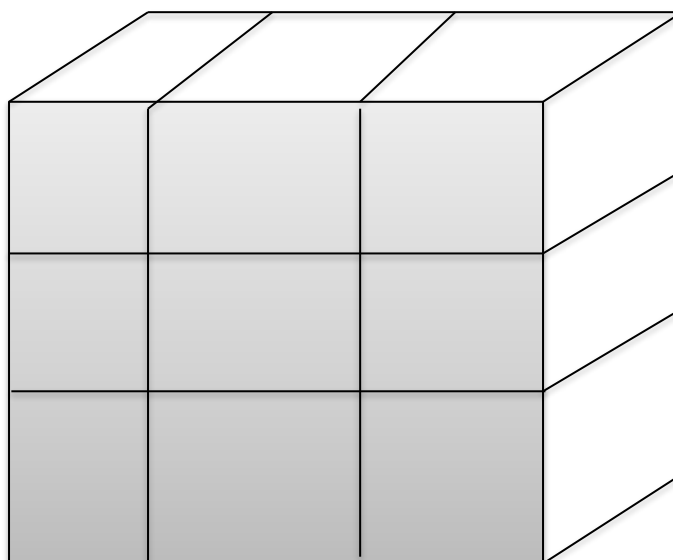


Figure 1.3.: Dual porosity model.

$$T \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) = S \frac{\partial h_1}{\partial t} \pm (h_1 - h_2) \pi a^2 \quad (1.3)$$

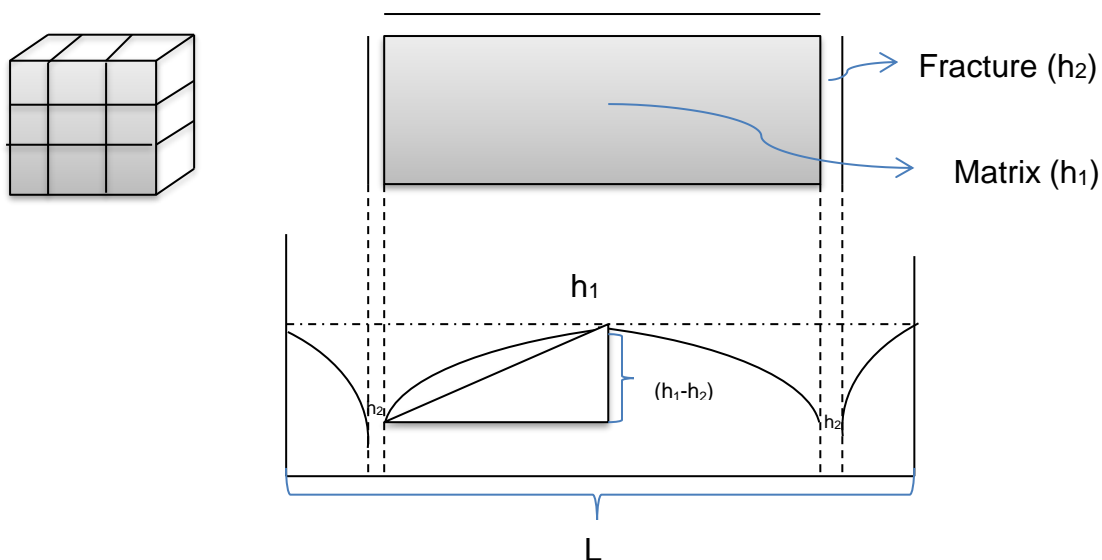
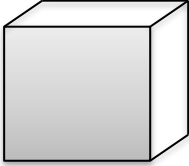
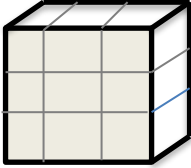
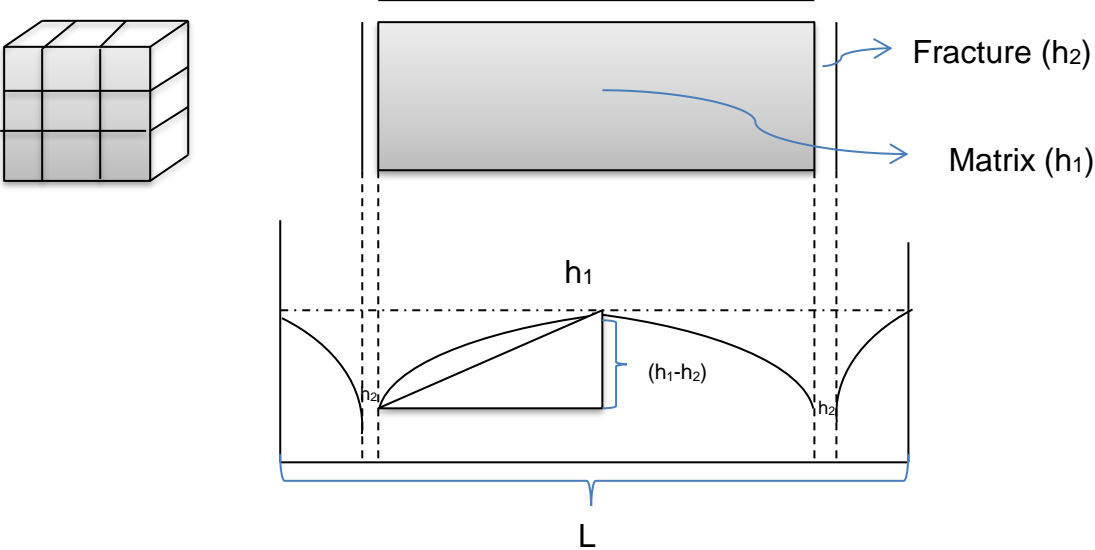


Table 1: Formulation of the dual porosity model.

1. Matrix (no fracture)	2. Fracture (impermeable matrix)
<p>Low Permeability High Storativity Water is stored here</p>  $T_1 \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) = S_1 \frac{\partial h_1}{\partial t} \quad (1.4)$	<p>High Permeability Low Storativity Fractures behave as effective conduits</p>  $T_2 \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right) = S_2 \frac{\partial h_2}{\partial t} \quad (1.5)$
Dual Porosity (2 and 1 combined)	
$T_1 \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) = S_1 \frac{\partial h_1}{\partial t} + (h_1 - h_2)\pi a^2 \quad (1.6)$	$T_2 \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right) = S_2 \frac{\partial h_2}{\partial t} - (h_1 - h_2)\pi a^2 \quad (1.7)$
	

Limitations

One of the most used concepts in modelling real world problems is perhaps the concept of differentiation that has been initiated by Leibniz in 1695 and applied in classical mechanics by Newton (Frappier, 2008). The concept of differentiation was introduced to explain the rate of change of a moving object between two points. One of the primary uses of mathematical equations is to replicate the observed facts, or collected data in order to predict future behaviours of the observed facts. In the last decades it was demonstrated in many published papers that, the concept of differentiation based on the rate of change is unable to capture some complex real world problems, for example Brownian motion, random walk, fractal shape, long range processes, processes with memories and many others that will not be listed here. To solve these issues, many notions were introduced including, stochastic differentiation, fractal differentiation and mapping, fractal derivative, fractional derivatives etc. These concepts have been used successfully in the last decades in many fields of science, technology and engineering. They have been applied to solving groundwater flow and pollution problems. For example, the flow of groundwater or groundwater pollution movement within fractures. The flow of sub-surface water within geological formations with elastic properties.

The mathematical model of groundwater flowing in a confined aquifer with double porosities presented in the literature uses classical differential operators, therefore cannot account for processes listed above. Nevertheless, even if the listed concepts have been used intensively to capture heterogeneity associated to the geological formation, it can still be argued that, the passage of water from different media follows a crossover process. While in literature, it is claimed that, exponential decay function and the generalized Mittag-Leffler function can exhibit crossover effects (Gómez-Aguilar, 2017), it is not clear when such crossover will take place. In addition, it worth noting that, the crossover associate to the generalized Mittag-Leffler function exhibits a passage from normal diffusion to sub-diffusion while the exponential decay function is able to capture the passage from usual to confined diffusion. However, when the crossover observed is different from the stated conditions, then both functions cannot be used and thus a different operator is needed. In real world problem, the collected data can give one a clear idea on where the crossover occurred and when it was observed. In the case of confined aquifers with double porosities, one would be able

to identify the point in the aquifer where there is change in geological formation. Thus, within the two intervals of time and space the model can be efficiently constructed using the adequate differential operators.

A piecewise derivative, a new concept suggested very recently by Atangana and Seda (2020), a differential operator defined in two or more intervals, seems to be the more ideal differential operator able to capture crossover. Thus, in this thesis, the concept of piecewise derivative is used to modify the model of groundwater flowing in a confined aquifer with double porosities.

1.2. Problem statement

Partial differential equations have been developed and applied in many scenarios. However, these cannot be applied in the mathematical formulation that considers, with great precision, random walk and crossover effects in geological formations.

1.3. Aim and objectives

This study seeks to modify a system of partial differential equations depicting the flow of subsurface water within a dual porosity system. This will be executed through the:

- Application of piecewise concept and modified model of piecewise,
- Analysis of modified model with stochastic (random walk incorporated)
- Application of Caputo-Fabrizio and Caputo fractional derivatives to the piecewise Model and
- Use of Numerical scheme based on Newtons Polynomial to solve the system of partial differentiation equation.

1.4. Research structure

This thesis consists of 6 chapters outlined below:

Chapter 1	Provides the background on different type of aquifers, dual-media systems, existing mathematical models of dual-media systems, limitations, problem statement, aim and objectives.
Chapter 2	Introduces the piecewise concept and modified model of piecewise.
Chapter 3	Provides the analysis of the modified model with stochastic.
Chapter 4	Provides the application of Caputo-Fabrizio and Caputo fractional derivatives on the piecewise model.
Chapter 5	Provides the numerical simulations with results and discussion.
Chapter 6	Provides the conclusion of the thesis.

CHAPTER 2 PIECEWISE CONCEPT AND MODIFIED MODEL OF PIECEWISE

2.1. Piecewise differential and integral operators

In the last decades, differential operators with integer and non-integer orders have been used intensively to replicate real-world problems. Let us note that, real-world problems following Markovian processes could be well replicated using classical differential operators, as their processes only depend on initial conditions and the generators. However, this operator cannot be used when some complexities are introduced, for example, the memory, crossover effects, fading memory, and many others. A clear example of the failure of this operator is the passage of subsurface water from the homogenous aquifer to heterogeneous one, or fractured rock aquifer. The flow process followed by subsurface water within these two media are not the same. It was then suggested that the classical differential operator should be extended to the concept of fractional differentiation, where the order of alpha can capture different scenarios. Fractional differentiation has contributed a very important role in modelling processes like fading memory, power-law and even crossover from fading memory to the power law. This is because, these differential operators are convolutions of classical differential operator and fractional kernels including the power-law kernel, exponential decay kernel and the generalized Mittag-Leffler kernel. These three kernels have properties that are observed in real-world situations, for example, the power process is comparable to the flow of subsurface water within fractures. While these differential operators have been praised in many fields of science, technology and engineering as they were able to depict very efficiently some complex problems. In 2017, a detailed study presented by Tateishi, Ribeiro, & Lenzi demonstrated the diffusion effects caused by different fractional derivatives. It was proven that power-law kernel had no crossover over in terms of means squared displacement, while exponential kernel has crossover from normal to confined diffusion and the generalized Mittag-Leffler kernel has a crossover from normal to sub-diffusion (Tateishi et al., 2017). These new properties gave both exponential kernel and power-law kernel ability to capture more anomalous diffusion. However, the concept of crossover is not limited to these two mentioned here as a more complex crossover can be observed in nature. Atangana (2020) argued that the world is so complex that it cannot be depicted using power law, exponential decay and the

generalised Mittag-Leffler based derivatives, as they cannot account for some complex crossover behaviours and suggested a new concept called piecewise derivative. This approach seems to be more efficient in modelling complex real-world problems with crossover effects. In this section, to accommodate readers that are not familiar with this new concept, some definitions of piecewise derivatives are thus presented.

2.2. Modified model using piecewise concept

In this section, the model is modified to include the mathematical formula of the effect of geological formation, especially the passage of water from the matrix soil to fractures. As discussed earlier, this passage cannot be depicted using existing differential operators especially the mathematical model describing the flow within fractures. However, to situate readers that are not aware of this concept, some definitions of piecewise differential operators are thus introduced.

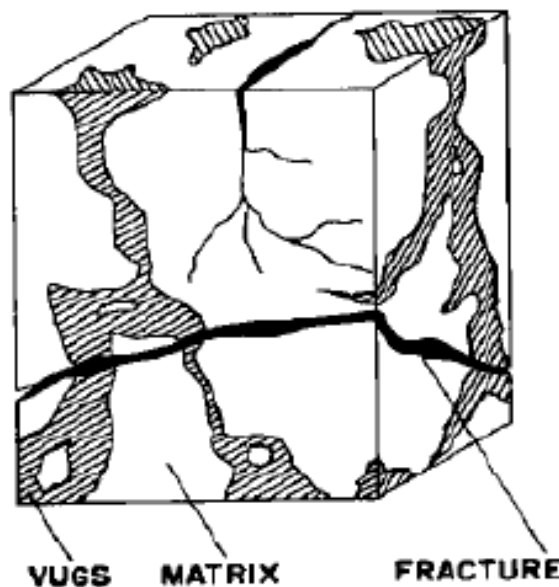


Figure 2.1. : Real-life dual porosity aquifer (Qingdong & Jun, 2015)

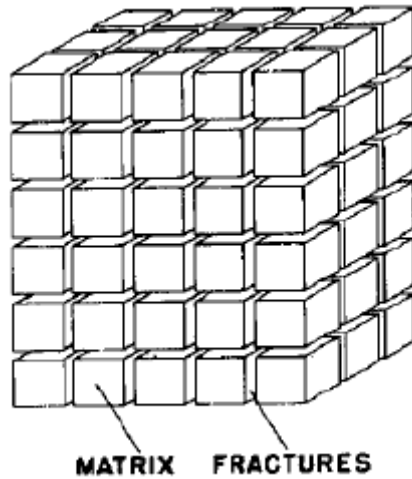


Figure 2.2. : Model representing the dual-porosity aquifer (Qingdong & Jun, 2015)

A piecewise derivative with close call and the global derivative is given as follows:

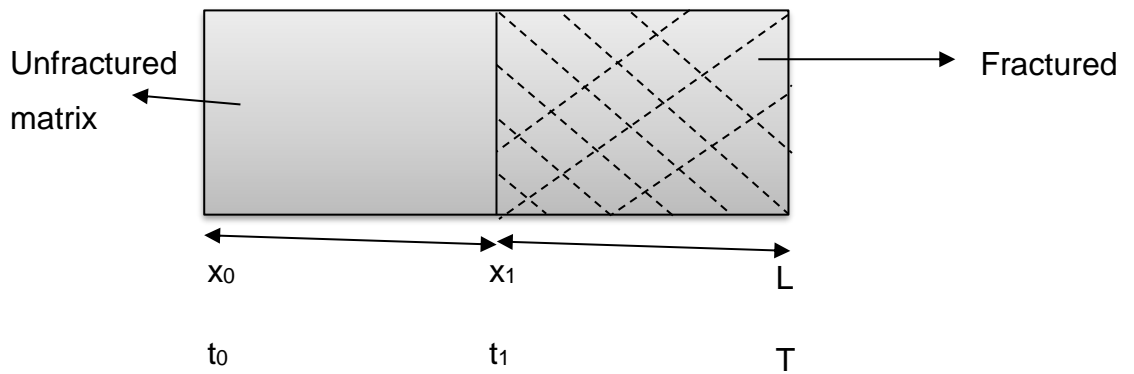


Figure 2.3.: 2D model of the dual porosity model.

$${}^P_0D_{g(t)}u(t) = \begin{cases} \frac{du(t)}{dt} & \text{if } 0 \leq t \leq t_1 \\ \frac{du(t)}{dt} g'(t) & \text{if } t_1 \leq t \leq T \end{cases} \quad (2.1)$$

Here $g'(t)$ is the non-negative and non-zero function. A piecewise power-law with global derivative is given as:

$${}^PCD_t^\alpha f(t) = \begin{cases} \frac{df(t)}{dt} & \text{if } 0 \leq t \leq t_1 \\ {}^CD_t^\alpha f(t) & \text{if } t_1 \leq t \leq T \end{cases} \quad (2.2)$$

or

$${}^PCD_t^\alpha f(t) = \begin{cases} {}^CD_t^\alpha f(t) & \text{if } 0 \leq t \leq t_1 \\ \frac{df(t)}{dt} & \text{if } t_1 \leq t \leq T \end{cases} \quad (2.3)$$

Differential operators can be used to modify the existing equations. If we consider that the confined aquifer has two different parallel media (Figure 2.3.) then, the mathematical model of groundwater flow in this system can be represented as:

$$\begin{cases} T_1 \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) = S_1 \frac{\partial h_1}{\partial t} + \pi a^2 (h_1 - h_2) & \text{if } 0 \leq t \leq t_1 \\ T_2 \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right) = S_2 \frac{\partial h_2}{\partial t} g'(t) - \pi a^2 (h_1 - h_2) & \text{if } t_1 \leq t < T \end{cases} \quad (2.4)$$

Here the function $g'(t)$ would be $\frac{t^{2-\alpha}}{2-\alpha}$ if $0 < \alpha \leq 1$. It is worth noting that, fractal derivatives are sub-classed of the global derivative. Below is presented the latest fractal derivative suggested by Atangana and Goufo (2020).

First, it is recalled that the classical differential operator defined as first principle is given as:

$$\lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{t \rightarrow t_1} \frac{f(t) - f(t_1)}{t - t_1} \quad (2.5)$$

If the function f is differentiable, the fractal derivative of the function f with fractal dimension $\alpha > 0$ is defined as follow:

$$\lim_{t \rightarrow t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha} = \frac{df(t)}{dt} \frac{t^{1-\alpha}}{\alpha} \quad (2.6)$$

As discussed earlier, Atangana and Goufo (2020) suggested that a fractal differential operator of a function f with fractal dimension $0 < \alpha \leq 1$ is given as:

$${}^F D_t^\alpha f(t) = \lim_{t \rightarrow t_1} \frac{f(t) - f(t_1)}{t^{2-\alpha} - t_1^{2-\alpha}} (2 - \alpha) \quad (2.7)$$

2.3. Analysis of the modified model

In this section, some analysis of equation 2.4 are presented. The first analysis will be to present the existence and uniqueness of the solution, to achieve this, the following steps should be followed:

$$S_1 \frac{\partial h_1(x, y, t)}{\partial t} = T_1 \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi a^2 (h_1 - h_2) \text{ if } t_1 \leq t \leq T, x_1 \leq x \leq L \quad (2.8)$$

Also,

$$g'(t) S_2 \frac{\partial h_2(x, y, t)}{\partial t} = T_2 \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right) + \pi a^2 (h_1 - h_2) \text{ if } t_1 \leq t \leq T, x_1 \leq x \leq L \quad (2.9)$$

Now, the first equation is transformed into Volterra equation as:

:

$$h_1(x, y, t) - h_1(x, y, 0) = \frac{1}{S_1} \int_0^t \left\{ T_1 \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi a^2 (h_1 - h_2) \right\} d\tau \quad (2.10)$$

For simplicity the following is placed:

$$f(x, t, h_1(x, t), h_2(x, t)) = \frac{T_1}{S_1} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi \frac{a^2}{S_1} (h_1 - h_2) \quad (2.11)$$

Then, we verify the Lipschitz condition and the growth condition

$$(i) \quad \forall (x, t) \in [0, x_1] \times [0, t_1] \quad (2.12)$$

$$|f(x, y, t, h_1, h_2)|^2 \leq k(1 + |h_1|^2)$$

$$(ii) \quad \forall (h_1, h_1') \in C\{[0, x_1] \times [0, y_1] \times [0, t_1]\} \quad (2.12)$$

$$|f(x, y, t, h_1, h_2) - f(x, y, t, h_1', h_2)|^2 < k|h_1 - h_1'|^2$$

Proof:

$$|f(x, y, t, h_1, h_2)|^2 = \left| \frac{T_1}{S_1} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \frac{\pi a^2}{S_1} (h_1 - h_2) \right|^2 \quad (2.13)$$

$$= \left| \frac{T_1}{S_1} \Delta^2 h_1 - \frac{\pi}{S_1} a^2 (h_1 - h_2) \right|^2$$

$$\leq 2 \left\{ \frac{T_1^2}{S_1^2} |\Delta^2 h_1|^2 + \frac{\pi^2}{S_2^2} a^4 |h_1 - h_2|^2 \right\} \quad (2.14)$$

$$\leq 2 \left\{ \frac{T_1^2}{S_1^2} \theta^2 |h_1|^2 + \frac{\pi^2}{S_2^2} a^4 (2|h_1|^2 + 2|h_2|^2) \right\} \quad (2.15)$$

$$\leq 2 \left\{ \frac{T_1^2}{S_1^2} \theta^2 + 2 \frac{\pi^2}{S_2^2} a^4 \right\} |h_1|^2 + 4 \frac{\pi^2}{S_2^2} a^4 |h_2|^2 \quad (2.16)$$

$$\leq 2 \frac{\pi^2}{S_1^2} a^4 \sup_{(t,x,y) \in [0,t] \times [0,x] \times [0,x]} |h_2(x, y, t)|^2 \frac{\left\{ 1 + \left(\frac{T_1^2 \theta^2}{S_1^2} + 2 \frac{\pi^2}{S_1^2} a^4 \right) |h_1|^2 \right\}}{\frac{2\pi^2}{S_1^2} a^4 |h_2|^2} \quad (2.17)$$

$$\text{if } \frac{\frac{T_1^2 \theta^2}{S_1^2} + 2 \frac{\pi^2}{S_1^2} a^4}{\frac{2\pi^2}{S_1^2} a^4 \|h_2\|_\infty^2} < 1 \text{ then} \quad (2.18)$$

$$\leq \frac{2\pi^2}{S_1^2} a^4 \|h_2\|_\infty^2 \{1 + |h_1|^2\} \quad (2.19)$$

$$\leq k \{1 + |h_1|^2\} \quad (2.20)$$

The condition under which the linear growth is satisfied, has now been presented. Further, it is presented here, the conditions under which the Lipschitz condition also holds. To do this, the following is considered.

$$\forall h_1, h_1^1$$

$$|f(x, y, t, h_1(x, y, t), h_2(x, y, t)) - f(x, y, t, h_1^1(x, y, t), h_2(x, y, t))|^2 \quad (2.21)$$

$$\left| \frac{T_1}{S_1} \Delta (h_1 - h_1^1) - \frac{a^2 \pi}{S_1} (h_1 - h_1^1) \right|^2 \leq \quad (2.22)$$

$$2 \left\{ \left(\frac{T_1}{S_1} \right)^2 \theta_1^2 |h_1 - h_1^1|^2 + \frac{a^4 \pi^2}{S_1^2} |h_1 - h_1^1|^2 \right\} \quad (2.23)$$

$$\leq \left(2 \frac{T_1^2}{S_1^2} \theta_1^2 + 2 \frac{a^4 \pi^2}{S_1^2} \right) |h_1 - h_1^1|^2 \quad (2.24)$$

$$\leq 2 \left(\frac{T_1^2}{S_1^2} \theta_1^2 + \frac{a^4 \pi^2}{S_1^2} \right) |h_1 - h_1^1|^2 \quad (2.25)$$

$$\leq k|h_1 - h_1^2|^2 \quad (2.26)$$

Therefore, under the conclusion that

$$\frac{T_1^2 \theta^2}{S_1^2} + \frac{2\pi^2 a^4}{S_1^2} < \frac{2\pi^2}{S_1^2} a^4 |h_2|^2 \quad (2.27)$$

Now is presented, the uniqueness and the existence solution within the following interval $(x_1, L) \times (t_1, T)$. In this region, it is assumed that the confined aquifer is fractured. The function $g'(t)$ will be selected to account for self-similarities in fractures.

$$\frac{\partial h_2(x, y, t)}{\partial t} g'(t) = \frac{T_2}{S_2} \left[\frac{\partial^2 h_2(x, y, t)}{\partial x^2} + \frac{\partial^2 h_2(x, y, t)}{\partial y^2} \right] + \frac{\pi a^2}{S_2} \text{ if } [h_1(x, y, t) - h_2(x, y, z)] \quad (2.28)$$

This is known from definition $g'(t) \neq 0$

$$\frac{\partial y_2(x, y, t)}{\partial t} = \frac{1}{g'(t)} \frac{T_2}{S_2} \left[\frac{\partial^2 h_2(x, y, t)}{\partial x^2} + \frac{\partial^2 h_2(x, y, t)}{\partial y^2} \right] + \frac{\pi a^2}{g'(t) S_2} [h_1(x, y, t) - h_2(x, y, z)] \quad (2.29)$$

Then is applied the integral operator on both sides to obtain

$$\begin{aligned} h_2(x, y, t) - h_2(x, y, t) & \quad (2.30) \\ &= \int_{t_1}^t \frac{1}{g'(l)} \left[\frac{T_2}{S_2} \left\{ \frac{\partial^2 h_2(x, y, l)}{\partial x^2} + \frac{\partial^2 h_2(x, y, l)}{\partial y^2} \right\} + \frac{\pi a^2}{S_2} (h_1(x, y, l) \right. \\ & \quad \left. - h_2(x, y, l)) \right] dl \end{aligned}$$

Firstly, is presented the condition of linear growth:

:

$$\begin{aligned} |f_2[x, y, t, h_1(x, y, t), h_2(x, y, t)]|^2 & \quad (2.31) \\ &= \left| \frac{1}{g'(t)} \left[\frac{T_2}{S_2} \Delta h_2(x, y, t) + \frac{\pi a^2}{S_2} [h_1(x, y, t) - h_2(x, y, t)] \right] \right|^2 \end{aligned}$$

$$= \left| \frac{1}{g'(t)} \right|^2 \left| \left[\frac{T_2}{S_2} \Delta h_2(x, y, t) + \frac{\pi a^2}{S_2} [h_1(x, y, t) - h_2(x, y, t)] \right] \right|^2 \quad (2.32)$$

$$\leq 2 \left| \frac{1}{g'(t)} \right|^2 \left\{ \left| \frac{T_2}{S_2} \Delta h_2(x, y, t) \right|^2 + \frac{\pi^2 a^4}{S_2^2} |h_1(x, y, t) - h_2(x, y, t)|^2 \right\} \quad (2.33)$$

$$\leq 2 \text{Sup}_{t_1 \leq t \leq T} \left| \frac{1}{g'(t)} \right|^2 \left\{ \frac{T_2^2}{S_2^2} \theta |h_2|^2 + \frac{\pi^2 a^4}{S_2^2} 2|h_1|^2 + 2|h_2|^2 \right\} \quad (2.34)$$

It is known that

$$\text{Sup}_{t_1 \leq t \leq T} \left| \frac{1}{g'} \right|^2 = \left\| \frac{1}{g} \right\| \quad (2.35)$$

$$\leq 2 \left\| \frac{1}{g'} \right\|_\infty^2 \left\{ \left(\frac{T_2^2 \theta}{S_2^2} + 1 \right) |h_2|^2 + 2 \frac{\pi^2 a^4}{S_2^2} |h_1|^2 \right\} \quad (2.36)$$

$$\leq 2 \left\| \frac{1}{g'} \right\|_\infty^2 \left\{ \left(\frac{T_2^2 \theta}{S_2^2} + 1 \right) |h_2|^2 + 4 \frac{\pi^2 a^4}{S_2^2} |h_1|^2 + 4 \frac{\pi^2 a^4 |h_1|^2}{S_2^2 \left\| \frac{1}{g} \right\|_\infty^2} \right\} \quad (2.37)$$

$$\leq \frac{4\pi^2 a^4 \|h_1\|_\infty^2}{S_2^2 \left\| \frac{1}{g'} \right\|_\infty^2} \left\{ 1 + \frac{|h_2|^2 \left\| \frac{1}{g} \right\|_\infty^2 \left(\frac{T_2^2 + 1}{S_2^2} \right)}{\frac{\pi^2 a^4 \|h_1\|_\infty^2}{2S_2^2 \left\| \frac{1}{g'} \right\|_\infty^2}} \right\} \quad (2.37)$$

$$\frac{T_2^2 \theta + 1}{S_2^2} \quad (2.38)$$

if $\frac{\pi^2 a^4 \|h_1\|_\infty^2}{2S_2^2 \left\| \frac{1}{g'} \right\|_\infty^2} < 1$

then

$$\leq \frac{4\pi^2 a^4 \|h_1\|_\infty^2}{S_2^2 \left\| \frac{1}{g'} \right\|_\infty^2} (1 + |h_2|^2) \quad (2.39)$$

$$\leq k(1 + |h_2|^2) \quad (2.40)$$

Now is then presented the Lipschitz condition

$$|f(x, y, t, h_1(x, y, z), h_2(x, y, t)) - f(x, y, t, h_1(x, y, t), h_1^1(x, y, t))|^2 \quad (2.41)$$

$$= \left| \frac{1}{g'(t)} \frac{T_2}{S_2} \Delta(h_2 - h_2^1) + \frac{\pi a^2}{S_2 g'(t)} (h_2 - h_2^1) \right|^2$$

$$\leq 2 \left| \frac{1}{g'(t)} \right|^2 \left(\frac{T_2}{S_2} \right)^2 \theta^2 |h_1 - h_2^1|^2 + 2 \frac{\pi^2 a^4}{S_2^2} \left| \frac{1}{g'(t)} \right|^2 |h_2 - h_2^1|^2 \quad (2.42)$$

$$\leq 2 \left| \frac{1}{g'(t)} \right|^2 \left(\frac{T_2^2}{S_2^2} \theta^2 + \frac{\pi^2 a^4}{S_2^2} \right) |h_2 - h_2^1|^2 \quad (2.43)$$

$$\leq \sup_{t_1 \leq t \leq T} \left| \frac{1}{g'(t)} \right|^2 \left(\frac{T_2^2}{S_2^2} \theta^2 + \frac{\pi^2 a^4}{S_2^2} \right) |h_2 - h_2^1|^2 \quad (2.44)$$

$$\leq 2 \left\| \frac{1}{g'} \right\|_{\infty}^2 \left(\frac{T_2^2 \theta^2}{S_2^2} + \frac{\pi^2 a^4}{S_2^2} \right) |h_2 - h_2^1|^2 \quad (2.45)$$

$$\leq k |h_2 - h_2^1|^2 \quad (2.46)$$

2.4. Numerical solution using Newton polynomial scheme

Mathematical models are largely used to depict real world problems. However, to depict this problem, one needs to obtain solution. It is ideal to have exact solutions by using analytical methods, however the use of these analytical methods is limited as some equations are more complex such that they cannot be solved analytically. In these cases, researchers relied on numerical methods. The literature contained several numerical schemes, one of the most used being multi-steps methods for example, the Adams-Bashforth numerical scheme, predictor-corrector method and Adams-Moulton method. These methods have been developed to solving mostly differential equations with classical derivative. An extension has been made to solving fractional differential equations. For example, there is a version of Adams-Bashforth that has been developed in the case of Caputo fractional derivative and its associate predictor-corrector. More versions have further been also developed. In the case of non-singular kernel many few numerical schemes have been developed, for example, Batogna and Atangana (2019) using Laplace transform to develop a new numerical scheme for fractional differential operators with non-singular kernels. Toufik and

Atangana (2017) suggested an alternative scheme for solving nonlinear fractional differential equations with Atangana-Baleanu fractional derivative. Using the Newton polynomial interpolation, Atangana and Seda (2020) developed a numerical scheme able to handle fractional and fractal-fractional differential equations. This section will therefore present the numerical solution using Newton polynomial scheme.

Firstly, is defined the general Euler's method of discretisation in the different derivatives

The general fist derivative with change in time is defined as follows:

$$\frac{df(t)}{dt} = \lim_{t_1 \rightarrow t} \frac{f(t_1) - f(t)}{t_1 - t} \quad (2.47)$$

$$= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \quad (2.48)$$

Therefore:

$$\left. \frac{df(t)}{dt} \right|_{t=t_j} \cong \frac{f(t_{j+1}) - f(t_j)}{\Delta t} \quad (2.49)$$

The first derivative (central) with change in space is generally given as follows:

$$\left. \frac{df(x)}{dx} \right|_{x=x_j} \cong \frac{f(x_{j+1}) - f(x_{j-1}))}{2\Delta x} \quad (2.50)$$

The general second derivative with change in time is given as follows:

$$\frac{\partial^2 f(t_j)}{\partial t^2} \cong \frac{f(t_{j+1}) - 2f(t_j) + f(t_{j-1}))}{(\Delta t)^2} \quad (2.51)$$

The proof can be found in the literature

$$\frac{d^2 f(t)}{dt^2} = \lim_{h \rightarrow 0} \frac{df(t+h) - df(t)}{h} \quad (2.52)$$

$$\frac{d^2 f}{dt^2} = \frac{d}{dt} \left(\frac{df}{dt} \right) \quad (2.53)$$

$$\frac{df(t)}{dt} = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \quad (2.54)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{f(h+t) - f(t)}{h} - \frac{f(t) - f(t-h)}{h}}{h} \quad (2.55)$$

$$= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t) - f(t) + f(t-h)}{h^2} \quad (2.56)$$

$$= \lim_{h \rightarrow 0} \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} \quad (2.57)$$

Now, let $\Delta t = h$ and $t = t_j$

$$= \lim_{h \rightarrow 0} \frac{f(t_j+h) - 2f(t_j) + f(t_j-h)}{h^2} \quad (2.58)$$

$$= \lim_{h \rightarrow 0} \frac{f(t_{j+1}) - 2f(t_j) + f(t_{j-1}))}{(\Delta t)^2} \quad (2.59)$$

Now is applied discretisation to the dual porosity equation using the Euler's method of discretisation.

If $t_0 \leq t \leq t_1$:

$$S_1 \frac{\partial h_1(x, y, t)}{\partial t} = T_1 \left(\frac{\partial^2 h_1(x, y, t)}{\partial x^2} + \frac{\partial h_1(x, y, t)}{\partial y^2} \right) - (h_1(x, y, z) - h_2(x, y, z)) \pi a^2 \quad (2.60)$$

$$\begin{aligned} S_1 \frac{h_1(x_i, y_k, t_{j+1}) - h_1(x_i, y_k, t_j)}{\Delta t} &= T_1 \left[\left(\frac{h_1(x_{i+1}, y_k, t_j) - 2h(x_i, y_k, t_j) + h_1(x_{i-1}, y_k, t_j)}{\Delta x^2} \right) \right. \\ &+ \left. \left(\frac{h_1(x_i, y_{k+1}, t_j) - 2h(x_i, y_k, t_j) + h_1(x_i, y_{k-1}, t_j)}{\partial y^2} \right) \right] - (h_1(x_i, y_k, z_j) \\ &- h_2(x_i, y_k, z_j)) \pi a^2 \end{aligned} \quad (2.61)$$

If $t_1 \leq t \leq T$:

$$S_2 \frac{\partial h_2}{\partial x} g'(t) = T_2 \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right) + (h_1 - h_2) \pi a^2 \quad (2.62)$$

$$S_2 \frac{\partial h_1(x, y, t)}{\partial t} \cdot \frac{1}{g'(t)} = T_2 \left(\frac{\partial^2 h_1(x, y, t)}{\partial x^2} + \frac{\partial h_1(x, y, t)}{\partial y^2} \right) + (h_1(x, y, t) - h_2(x, y, t))\pi a^2 \quad (2.63)$$

$$S_2 \frac{h_2(x_i, y_k, t_{j+1}) - h_2(x_i, y_k, t_j)}{\Delta t} \cdot \frac{g(t_{j+1}) - g(t_j)}{\Delta t} \quad (2.64)$$

$$= T_2 \left[\left(\frac{h_2(x_{i+1}, y_k, t_j) - 2h_2(x_i, y_k, t_j) + h_2(x_{i-1}, y_k, t_j)}{\Delta x^2} \right) \right.$$

$$\left. + \left(\frac{h_2(x_i, y_{k+1}, t_j) - 2h_2(x_i, y_k, t_j) + h_2(x_i, y_{k-1}, t_j)}{\Delta y^2} \right) \right] + (h_1(x_i, y_k, z_j) - h_2(x_i, y_k, z_j))\pi a^2$$

In this section, the Newton polynomial-based method is used to provide an alternative numerical solution to the model. To achieve this, firstly is presented the derivation of the method by considering, a general non-Linear Cauchy problem as indicated in the following equation. Thus later, the conversion is made to partial differential equation and apply it to the study system of equations.

$$\frac{dy(t)}{dt} = f(t, y(t)) \quad (2.65)$$

$$\int_0^t \frac{dy(\tau)}{d\tau} d\tau = \int_0^t f(\tau, y(\tau)) d\tau \quad (2.66)$$

$$y(t) - y(0) = \int_0^t f(\tau, y(\tau)) d\tau \quad (2.67)$$

$$t = t_{n+1} = (n + 1)\Delta t$$

$$y(t_{n+1}) - y(0) = \int_0^{t_{n+1}} f(\tau, y(\tau)) d\tau \quad (2.68)$$

$$y(t_n) - y(0) = \int_0^{t_n} f(\tau, y(\tau)) d\tau \quad (2.69)$$

Following the above, equation 3.6 is thus subtracted from equation 3.5.

$$(y(t_{n+1}) - y(0)) - (y(t_n) - y(0)) = \int_0^{t_{n+1}} f(\tau, y(\tau)) d\tau - \int_0^{t_n} f(\tau, y(\tau)) d\tau \quad (2.70)$$

$$y(t_{n+1}) - y(t_n) = \int_0^{t_{n+1}} f(\tau, y(\tau)) dx - \int_0^{t_n} f(\tau, y(\tau)) d\tau \quad (2.71)$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^0 f(\tau, y(\tau)) d\tau + \int_0^{t_{n+1}} f(\tau, y(\tau)) d\tau \quad (2.72)$$

$$= \int_{t_n}^{t_{n+1}} f(\tau, y(\tau)) dx \quad (2.73)$$

Within the interval $[t_n, t_{n+1}]$ we want to approximate the function $f(\tau, y(\tau))$

$$f(\tau, y(\tau)) \cong P_n(\tau) \quad (2.74)$$

$$\begin{aligned} &= f(t_{n-2}, y(t_{n-2})) + \frac{f(t_{n-1}, y(t_{n-1})) - f(t_{n-2}, y(t_{n-2})) (\tau - t_{n-2})}{\Delta t} \\ &+ \frac{f(t_n, y(t_n)) - 2f(t_{n-1}, y(t_{n-1})) + f(t_{n-2}, y(t_{n-2})) (\tau - t_{n-2})(\tau - t_{n-1})}{2(\Delta t)^2} \end{aligned}$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} P_n(\tau) d\tau \quad (2.75)$$

After integrating the following is obtained:

$$y(t_{n+1}) - y(t_n) = \frac{5}{12} f(t_{n-2}, y(t_{n-2})) \Delta t - \frac{4}{3} f(t_{n-1}, y(t_{n-1})) \Delta t + \frac{23}{12} f(t_n, y(t_n)) \Delta t \quad (2.76)$$

In our case

$$y(t_{n+1}) = h(x_i, y_j, t_{n+1})$$

$$f(t_n, y(t_n)) = f(x_i, y_j, h(x_i, y_j, t_n))$$

$$h(x_i, y_j, h(x_i, y_j, t_{n+1})) - h(x_i, y_j, h(x_i, y_j, t_n)) \quad (2.77)$$

$$\begin{aligned} &= \frac{5}{12} f(x_i, y_j, h(x_i, y_j, t_n)) \Delta t - \frac{4}{3} f(x_i, y_j, h(x_i, y_j, t_{n-1})) \\ &+ \frac{23}{12} f(x_i, y_j, h(x_i, y_j, t_{n-2})) \end{aligned}$$

From our equation, if $t_0 \leq t \leq t_1$:

$$\begin{aligned} &f_1(x_i, y_j, h(x_i, y_j, t_n)) \quad (2.78) \\ &= \frac{T_1}{S_1} \left(\frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial y^2} \right) \\ &- \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n)) \end{aligned}$$

$$\begin{aligned}
f_1(x_i, y_j, h(x_i, y_j, t_n)) & \quad (2.79) \\
&= \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \\
&\quad \left. + \frac{h_1(x_i, y_{j+1}, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \\
&\quad + \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n))
\end{aligned}$$

$$\begin{aligned}
h_1(x_i, y_j, h(x_i, y_j, t_{n+1})) - h_1(x_i, y_j, h(x_i, y_j, t_n)) & \quad (2.80) \\
&= \frac{5}{12} \left\{ \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_1(x_i, y_{j+1}, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n)) \right\} \Delta t \\
&\quad - \frac{4}{3} \left\{ \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_1(x_i, y_{j+1}, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1})) \right\} \\
&\quad + \frac{23}{12} \left\{ \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_{i-1}, y_j, t_{n-2})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_1(x_i, y_{j+1}, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_i, y_{j-1}, t_{n-2})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{n-2}) - h_2(x_i, y_j, t_{n-2})) \right\}
\end{aligned}$$

And if $t_1 \leq t \leq T$

$$\begin{aligned}
f_2(x_i, y_j, h(x_i, y_j, t_n)) & \quad (2.81) \\
&= \frac{T_2}{S_2} \left(\frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial y^2} \right) \\
&\quad + \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n))
\end{aligned}$$

$$\begin{aligned}
& f_2(x_i, y_j, h(x_i, y_j, t_n)) \tag{2.82} \\
&= \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_n) - 2h_2(x_i, y_j, t_n) + h_2(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \\
&+ \left. \frac{h_2(x_i, y_{j+1}, t_n) - 2h_2(x_i, y_j, t_n) + h_2(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \\
&+ \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n))
\end{aligned}$$

$$\begin{aligned}
& h_2(x_i, y_j, h(x_i, y_j, t_{n+1})) - h_2(x_i, y_j, h(x_i, y_j, t_n)) \tag{2.83} \\
&= \frac{5}{12} \left\{ \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_n) - 2h_2(x_i, y_j, t_n) + h_2(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_2(x_i, y_{j+1}, t_n) - 2h_2(x_i, y_j, t_n) + h_2(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \right. \\
&+ \left. \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n)) \right\} \Delta t \\
&- \frac{4}{3} \left\{ \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{n-1}) - 2h_2(x_i, y_j, t_{n-1}) + h_2(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_2(x_i, y_{j+1}, t_{n-1}) - 2h_2(x_i, y_j, t_{n-1}) + h_2(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \right. \\
&+ \left. \pi a^2 (h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1})) \right\} \\
&+ \frac{23}{12} \left\{ \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{n-2}) - 2h_2(x_i, y_j, t_{n-2}) + h_2(x_{i-1}, y_j, t_{n-2})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_2(x_i, y_{j+1}, t_{n-2}) - 2h_2(x_i, y_j, t_{n-2}) + h_2(x_i, y_{j-1}, t_{n-2})}{(\Delta y)^2} \right) \right. \\
&+ \left. \pi a^2 (h_1(x_i, y_j, t_{n-2}) - h_2(x_i, y_j, t_{n-2})) \right\}
\end{aligned}$$

CHAPTER 3 : STOCHASTIC MODEL

3.1. Introduction

Models are applied to provide qualitative description of natural phenomenon's using mathematical formulas. These models also greatly assist in solving complex real-world problems. Models can be classified into two different forms namely: stochastic models and deterministic models. A deterministic model does not consist of random outputs and the complete input and output characteristics of the model are conclusively determined. An example of a deterministic model would be a car being driven on a cruise control system. The pre-determined input values of the system such as speed and distance travelled over time provide one with a definite arrival time. This arrival time represents the output from the pre-determined set of circumstances (inputs) (Atangana, 2020). Whereas with stochastic models, the model consists of random outputs and the complete output characteristics of the model are randomly determined. One example of a stochastic model is a pumping test where head change is observed over time. In this case the head change is dependent on the aquifer characteristics at the given time which is also influenced by the geological conditions.

The following chapter will discuss the stochastic model. This will be done by first introducing the stochastic processes; secondly, a comparison between stochastic and deterministic processes will be conducted; thirdly, the modified model with stochastic approach will be presented and lastly; analysing the modified model with stochastic.

3.2. Stochastic versus deterministic

"Stochastic" is a phrase that originates from the Greek word $\sigma\acute{o}\chi\omicron\varsigma$ (stókhos) which is synonymous with the word random and chance. Of which its antonym is "deterministic." Stochastic processes possess some inherent randomness. The same parameter values and initial condition will lead to an assemble of different outputs. In deterministic models, the parameter values determine the output results of the model. Taylor (1998) applied an example of a coin tossed into the air which eventually returns to the ground at a particular point. Regardless of the coin landing heads or tails, the process is considered random and has no determined outcome. In the study by Taylor, a "fair" coin was considered to mean that the two alternatives, heads and tails are equally likely and, were thus each assigned the probability of outcome. It should be noted that the phenomena are not inherently stochastic or deterministic in and of

themselves. Taylor & Karlin, 1998, further stated that in order to model a phenomenon as stochastic or deterministic was the observer's choice. The purpose of what the observer seeks to accomplish is what make the choice. Taylor & Karlin, (1998), explains the controversial situation under which a proper choice in not clear. Using the coin scenario, Taylor and Karlin, show that the controversy can arise were after the coin having being tossed in the air fallen is quickly concealed so that the outcome is not determined giving a chance for randomising the situation, though the outcome can be considered by some people as now fixed or deterministic. Also, Taylor and Karlin (1998), state that regardless of the general agreement among many observers that for example in a large population that there are many events that contribute to the population fluctuating, these are often modelled deterministic.

3.3. Modified model with stochastic approach

In the following section, the function of random walk will be added to the study's equation. The general equation for random walk is given as follows:

$$\frac{dy(t)}{dt} = f(t, y(t)) + \bar{\sigma}G(t, y(t)) \quad (3.1)$$

Where $\bar{\sigma}$ represents the density of randomness

and G represents the random walk.

Random walk is then added to the dual porosity model for the matrix section ($t_0 \leq t \leq t_1$) and represented as follows:

$$\begin{aligned} \frac{\partial h_1(x_i, y_j, t_n)}{\partial t} = & \frac{T_1}{S_1} \left(\frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial y^2} \right) \\ & - \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \bar{\sigma}_1 h_1 B_1(x_i, y_j, t_n) \end{aligned} \quad (3.2)$$

$$\begin{aligned}
& h_1(x_i, y_j, t_n) - h_1(x_i, y_j, t_n) \tag{3.3} \\
& = \int_0^t \frac{T_1}{S_1} \left(\frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial y^2} \right) \\
& \quad - \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \sigma_1 h_1 dB_1(x_i, y_j, t_n)
\end{aligned}$$

At the point (x_i, y_j, t_{n+1}) and (x_i, y_j, t_n)

$$\begin{aligned}
& h_1(x_i, y_j, t_{n+1}) - h_1(x_i, y_j, t_n) \tag{3.4} \\
& = \int_{t_n}^{t_{n+1}} \frac{T_1}{S_1} \left(\frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_1(x_i, y_j, t_n)}{\partial y^2} \right) \\
& \quad - \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \int_{t_n}^{t_{n+1}} \sigma_1 h_1 dB_1(x_i, y_j, \tau)
\end{aligned}$$

The discretisation of the first integral is shown

$$\int_{t_n}^{t_{n+1}} \sigma_1 h_1(x_i, y_j, \tau), B_1(\tau) \tag{3.5}$$

It is assumed that $B_1(\tau)$ is differentiable:

$$dB(\tau) = B'(\tau)d\tau$$

Then equation (3.5) becomes:

$$\int_{t_n}^{t_{n+1}} \sigma_1 h_1(x_i, y_j, \tau) B'_1(\tau) d\tau \tag{3.6}$$

Again, we approximate

$$h_1(x_i, y_j, \tau) B'_1(\tau) \simeq p'_j(\tau) \tag{3.7}$$

Where $p'_j(\tau)$ is the Newton two steps polynomial.

$$\begin{aligned}
\sigma_1 \int_{t_n}^{t_{n+1}} p'_j(\tau) d\tau &= \frac{5}{12} \sigma_1 \Delta t h_1(x_i, y_j, t_{n-2}) B'_1(t_{n-2}) \tag{3.8} \\
&\quad - \frac{4}{3} \sigma_1 \Delta t h_1(x_i, y_j, t_{n-1}) B'_1(t_{n-1}) - \frac{23}{12} \sigma_1 \Delta t h_1(x_i, y_j, t_n) B'_1(t_n)
\end{aligned}$$

Where:

$$B'_1(t_{n-2}) = \frac{B_1(t_{n-1}) - B_1(t_{n-2})}{\Delta t} \tag{3.9}$$

$$B'_1(t_{n-1}) = \frac{B_1(t_n) - B_1(t_{n-1})}{\Delta t} \quad (3.10)$$

$$B'_1(t_n) = \frac{B_1(t_{n+1}) - B_1(t_n)}{\Delta t} \quad (3.11)$$

$$\begin{aligned} \mathfrak{b}_1 \int_{t_n}^{t_{n+1}} h_1(x_i, y_j, \tau) B'_1(\tau) d\tau &= \frac{5}{12} \mathfrak{b}_1 \Delta t h_1(x_i, y_j, t_{n-2}) \frac{B_1(t_{n-1}) - B_1(t_{n-2})}{\Delta t} \\ &\quad - \frac{4}{3} \mathfrak{b}_1 \Delta t h_1(x_i, y_j, t_{n-1}) \frac{B_1(t_n) - B_1(t_{n-1})}{\Delta t} \\ &\quad - \frac{23}{12} \mathfrak{b}_1 \Delta t h_1(x_i, y_j, t_n) \frac{B_1(t_{n+1}) - B_1(t_n)}{\Delta t} \end{aligned} \quad (3.12)$$

Therefore:

$$\begin{aligned}
& h_1(x_i, y_j, t_{n+1}) - h_1(x_i, y_j, t_n) \tag{3.13} \\
&= \frac{5}{12} \left\{ \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \right. \\
&- \left. \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \mathfrak{b}_1 h_1 B_1(x_i, y_j, t_n) \right\} \Delta t \\
&- \frac{4}{3} \left\{ \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \right. \\
&- \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1}) \right) + \mathfrak{b}_1 h_1 B_1(x_i, y_j, t_{n-1}) \right\} \\
&+ \frac{23}{12} \left\{ \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_{i-1}, y_j, t_{n-2})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_i, y_{j-1}, t_{n-2})}{(\Delta y)^2} \right) \right. \\
&- \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-2}) - h_2(x_i, y_j, t_{n-2}) \right) + \mathfrak{b}_1 h_1 B_1(x_i, y_j, t_{n-2}) \right\} \\
&+ \left(\frac{5}{12} \mathfrak{b}_1 \Delta t h_1(x_i, y_j, t_{n-2}) \frac{B_1(t_{n-1}) - B_1(t_{n-2})}{\Delta t} \right. \\
&- \frac{4}{3} \mathfrak{b}_1 \Delta t h_1(x_i, y_j, t_{n-1}) \frac{B_1(t_n) - B_1(t_{n-1})}{\Delta t} \\
&- \left. \frac{23}{12} \mathfrak{b}_1 \Delta t h_1(x_i, y_j, t_n) \frac{B_1(t_{n+1}) - B_1(t_n)}{\Delta t} \right)
\end{aligned}$$

In the following section, random walk is then introduced to the dual porosity model for the fractured section ($t_1 \leq t \leq T$) and represented as follows:

$$\begin{aligned}
\frac{\partial h_2(x_i, y_j, t_n)}{\partial t} &= \frac{T_2}{S_2} \left(\frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial y^2} \right) \tag{3.14} \\
&- \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \mathfrak{b}_2 h_2, B_2(x_i, y_j, t_n)
\end{aligned}$$

$$\begin{aligned}
& h_2(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \tag{3.15} \\
& = \int_0^t \frac{T_2}{S_2} \left(\frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial y^2} \right) \\
& \quad - \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \sigma_2 h_2 dB_2(x_i, y_j, t_n)
\end{aligned}$$

At the point (x_i, y_j, t_{n+1}) and (x_i, y_j, t_n)

$$\begin{aligned}
& h_2(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, t_n) \tag{3.16} \\
& = \int_{t_n}^{t_{n+1}} \frac{T_2}{S_2} \left(\frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 h_2(x_i, y_j, t_n)}{\partial y^2} \right) \\
& \quad - \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \int_{t_n}^{t_{n+1}} \sigma_2 h_2 dB_2(x_i, y_j, \tau)
\end{aligned}$$

The discretization of the first integral has been presented.

$$\int_{t_n}^{t_{n+1}} \sigma_2 h_2(x_i, y_j, \tau) dB_2(\tau) \tag{3.17}$$

It is assumed that $B_1(\tau)$ is:

$$dB(\tau) = B'(\tau)d\tau$$

Then equation (3.17.) becomes:

$$\int_{t_n}^{t_{n+1}} \sigma_2 h_2(x_i, y_j, \tau) B'_2(\tau) d\tau \tag{3.18}$$

Again, an approximation is carried out.

$$h_2(x_i, y_j, \tau) B'_2(\tau) \simeq p'_j(\tau) \tag{3.19}$$

Where $p'_j(\tau)$ represents the Newton two steps polynomial.

$$\begin{aligned}
\sigma_2 \int_{t_n}^{t_{n+1}} p'_j(\tau) d\tau &= \frac{5}{12} \sigma_2 \Delta t h_2(x_i, y_j, t_{n-2}) B'_2(t_{n-2}) \tag{3.20} \\
&\quad - \frac{4}{3} \sigma_2 \Delta t h_2(x_i, y_j, t_{n-1}) B'_2(t_{n-1}) - \frac{23}{12} \sigma_2 \Delta t h_2(x_i, y_j, t_n) B'_2(t_n)
\end{aligned}$$

Where:

$$B'_2(t_{n-2}) = \frac{B_2(t_{n-1}) - B_2(t_{n-2})}{\Delta t} \tag{3.21}$$

$$B'_2(t_{n-1}) = \frac{B_2(t_n) - B_2(t_{n-1})}{\Delta t} \tag{3.22}$$

$$B'_2(t_n) = \frac{B_2(t_{n+1}) - B_2(t_n)}{\Delta t} \quad (3.23)$$

Further represented as follows:

$$\begin{aligned} \sigma_2 \int_{t_n}^{t_{n+1}} h_2(x_i, y_j, \tau) B'_2(\tau) d\tau &= \frac{5}{12} \sigma_2 \Delta t h_2(x_i, y_j, t_{n-2}) \frac{B_2(t_{n-1}) - B_2(t_{n-2})}{\Delta t} \\ &\quad - \frac{4}{3} \sigma_2 \Delta t h_2(x_i, y_j, t_{n-1}) \frac{B_2(t_n) - B_2(t_{n-1})}{\Delta t} \\ &\quad - \frac{23}{12} \sigma_2 \Delta t h_2(x_i, y_j, t_n) \frac{B_2(t_{n+1}) - B_2(t_n)}{\Delta t} \end{aligned} \quad (3.24)$$

Therefore:

$$\begin{aligned}
& h_2(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, t_n) \tag{3.25} \\
&= \frac{5}{12} \left\{ \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_n) - 2h_2(x_i, y_j, t_n) + h_2(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_2(x_i, y_{j+1}, t_n) - 2h_2(x_i, y_j, t_n) + h_2(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \right. \\
&- \left. \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) + \mathfrak{b}_2, h_2, B_2(x_i, y_j, t_n) \right\} \Delta t \\
&- \frac{4}{3} \left\{ \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{n-1}) - 2h_2(x_i, y_j, t_{n-1}) + h_2(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_2(x_i, y_{j+1}, t_{n-1}) - 2h_2(x_i, y_j, t_{n-1}) + h_2(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \right. \\
&- \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1}) \right) + \mathfrak{b}_2, h_2, B_2(x_i, y_j, t_{n-1}) \right\} \\
&+ \frac{23}{12} \left\{ \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{n-2}) - 2h_2(x_i, y_j, t_{n-2}) + h_2(x_{i-1}, y_j, t_{n-2})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_2(x_i, y_{j+1}, t_{n-2}) - 2h_2(x_i, y_j, t_{n-2}) + h_2(x_i, y_{j-1}, t_{n-2})}{(\Delta y)^2} \right) \right. \\
&- \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-2}) - h_2(x_i, y_j, t_{n-2}) \right) + \mathfrak{b}_2, h_2, B_2(x_i, y_j, t_{n-2}) \right\} \\
&+ \left(\frac{5}{12} \mathfrak{b}_2 \Delta t h_2(x_i, y_j, t_{n-2}) \frac{B_2(t_{n-1}) - B_2(t_{n-2})}{\Delta t} \right. \\
&- \frac{4}{3} \mathfrak{b}_2 \Delta t h_2(x_i, y_j, t_{n-1}) \frac{B_2(t_n) - B_2(t_{n-1})}{\Delta t} \\
&- \left. \frac{23}{12} \mathfrak{b}_2 \Delta t h_2(x_i, y_j, t_n) \frac{B_2(t_{n+1}) - B_2(t_n)}{\Delta t} \right)
\end{aligned}$$

CHAPTER 4 : APPLICATION OF CAPUTO-FABRIZIO AND CAPUTO FRACTIONAL DERIVATIVES TO THE PIECEWISE MODEL

4.1. Introduction

To consider the effect of crossover, where the first section of the aquifer is considered to having fading memory flow while the second followed the power law process. In order to include this into a mathematical formulation, in the first equation the classical time differentiation is replaced by the Caputo-Fabrizio fractional derivative, while the second equation the time differential operator will be replaced by the Caputo fractional derivative. This section present some of the definitions stated.

4.2. Application of Caputo-Fabrizio and Caputo derivative

The Caputo derivative is given as follows:

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d}{d\tau} f(\tau) (t-\tau)^{-\alpha} d\tau \quad (4.1)$$

Where $0 < \alpha \leq 1$

The Caputo-Fabrizio fraction of derivative is given as

$${}_0^{CF} D_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t \frac{d}{d\tau} f(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau \quad (4.2)$$

The associate integrals are given below as follows:

Riemann-Liouville integral

$${}_0^{RL} D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) (t-\tau)^{\alpha-1} d\tau \quad (4.3)$$

Caputo-Fabrizio integral

$${}_0^{CF} D_t^\alpha f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_0^t (\tau) d\tau \quad (4.4)$$

To include into the study mathematical equation, the effect of fading memory and the power law, we modify the study system as follows:

With the application of Caputo-Fabrizio

$$\begin{aligned} \frac{M(\alpha)}{1-\alpha} \int_0^t \frac{\partial h_1(x, y, t)}{\partial \tau} \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau & \quad (4.5) \\ & = \frac{T_1}{S_1} \left[\frac{\partial^2 h_1(x, y, t)}{\partial x^2} + \frac{\partial^2 h_1(x, y, t)}{\partial y^2} \right] - \pi a^2 (h_1(x, y, t) - h_2(x, y, t)) \end{aligned}$$

With the application of Caputo

$$\begin{aligned} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial h_2(x, y, \tau)}{\partial \tau} (t-\tau)^{\alpha-1} d\tau & \quad (4.6) \\ & = \frac{T_2}{S_2} \left[\frac{\partial^2 h_2(x, y, t)}{\partial x^2} + \frac{\partial^2 h_2(x, y, t)}{\partial y^2} \right] + \pi a^2 (h_1(x, y, t) - h_2(x, y, t)) \end{aligned}$$

$${}^C D_t^\alpha h(x, y, t) = \frac{M(\alpha)}{1-\alpha} \int_0^t \frac{\partial h}{\partial \tau}(x, y, \tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau \quad (4.7)$$

Consider $t = tn + 1 = \Delta t(n + 1)$

x_i, y_i

$$\begin{aligned} {}^C D_t^\alpha h(x_i, y_i, t_{n+1}) & \quad (4.8) \\ & = \frac{M(\alpha)}{1-\alpha} \int_0^{n+1} \frac{\partial h}{\partial \tau}(x_i, y_i, \tau) \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right] d\tau \\ & = \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \frac{h(x_i, y_j, t_{j+1}) - h(x_i, y_j, t_j)}{\Delta t} \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right] d\tau \end{aligned}$$

$$\begin{aligned} {}^C D_t^\alpha h(x_i, y_i, t_{n+1}) & \quad (4.9) \\ & = \frac{M(\alpha)}{1-\alpha} \int_0^{n+1} \frac{\partial h}{\partial \tau}(x_i, y_i, \tau) \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right] d\tau \\ & = \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^n \frac{h(x_i, y_j, t_{j+1}) - h(x_i, y_j, t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right] d\tau \end{aligned}$$

$$\int_{t_j}^{t_{j+1}} \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right] d\tau \quad (4.10)$$

Let $\gamma = -\frac{\alpha}{1-\alpha}$, $y = t_{n+1} - \tau$, $dy = d(t_{n+1} - \tau) = dt_{n+1} - d\tau = -d\tau$

When $\tau = t_j$, $y = t_{n+1} - t_j$

And $\tau = t_{j+1}$, $y = t_{n+1} - t_{j+1}$

$$\int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} -\exp[-\gamma y] dy \quad (4.11)$$

It is known that:

$$\int_{\gamma}^{\beta} \exp(-ay) dy = \frac{1}{a} \exp[-ay] \quad (4.12)$$

Therefore:

$$- \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} \frac{-x}{-\gamma} \exp[-\gamma y] dy \quad (4.13)$$

$$\frac{1}{\gamma} \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} -\gamma \exp[-\gamma y] dy \quad (4.14)$$

$$(\exp(-\gamma y))' = -\gamma \exp(-\gamma y) \quad (4.15)$$

$$-\frac{1}{\gamma} \exp[-\gamma y]_{t_{n+1}-t_j}^{t_{j+1}-t_{j+1}} = \frac{1}{\gamma} [\exp(-\gamma(t_{n+1} - t_j)) - \exp(-\gamma(t_{n+1} - t_{j+1}))] \quad (4.16)$$

$$= \frac{1}{\gamma} [\exp(-\gamma(\Delta t(n+1) - \Delta t j))] - \exp[-\gamma(\Delta t(n+1) - \Delta t(j+1))] \quad (4.17)$$

$$= \frac{1}{\gamma} [\exp(-\gamma \Delta t(n+1-j))] - \exp[-\gamma \Delta t(n+j)] \quad (4.18)$$

$$= \frac{1-\alpha}{\alpha} \left[\exp\left[-\frac{\alpha}{1-\alpha} \Delta t(n-j)\right] + \exp\left[-\frac{\alpha}{1-\alpha} \Delta t(n+1-j)\right] \right] \quad (4.19)$$

With Caputo derivative, it is arrived at the following:

$${}_0^c D_t^\alpha h_2(x, y, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial \tau} h_2(x, y, t) (t-\tau)^{-\alpha} d\tau \quad (4.20)$$

at the point (x_i, y_j, t_{n+1}) , the above yields:

$${}^c D_t^\alpha h_2(x_i, y_j, t_{n+1}) \quad (4.21)$$

$$\begin{aligned} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \frac{\partial}{\partial \tau} h_2(x_i, y_j, \tau) (t_{n+1} - \tau)^{-\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \frac{h_2(x_i, y_j, t_{k+1}) - h_2(x_i, y_j, t_k)}{\Delta t} (t_{n+1} - \tau)^{-\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{h_2(x_i, y_j, t_{k+1}) - h_2(x_i, y_j, t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} (t_{n+1} - \tau)^{-\alpha} d\tau \end{aligned}$$

Here $\int_{t_k}^{t_{k+1}} (t_{n+1} - \tau)$, we put $y = y = t_{n+1} - \tau$, $dy = -d\tau$ or $d\tau = -dy$

When $\tau \rightarrow t_k$, $y \rightarrow t_{n+1} - t_k$

When $\tau \rightarrow t_{k+1}$, $y \rightarrow t_{n+1} - t_{k+1}$

Thus, the above integral is therefore presented as:

$$\int_{t_{n+1}-t_k}^{t_{n+1}-t_{k+1}} y^{-\alpha} \cdot d\tau = \frac{(t_{n+1} - t_{k+1})^{1-\alpha} - (t_{n+1} - t_k)^{1-\alpha}}{1-\alpha} \quad (4.22)$$

Replacing this back to the original equation, the following is arrived at:

$$\begin{aligned} &{}^c D_t^\alpha h_2(x_i, y_j, t_{n+1}) \quad (4.23) \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{(h_2(x_i, y_j, t_{k+1}) - h_2(x_i, y_j, t_k))}{\Delta t} \left[\frac{(t_{n+1} - t_k)^{1-\alpha} - (t_{n+1} - t_{k+1})^{1-\alpha}}{1-\alpha} \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{(h_2(x_i, y_j, t_{k+1}) - h_2(x_i, y_j, t_k))}{\Delta t} [(\Delta t(n+1) - k\Delta t)^{1-\alpha} \\ &\quad - (\Delta t(n+1) - \Delta t(k+1))^{1-\alpha}] \\ &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n (h_2(x_i, y_j, t_{k+1}) - h_2(x_i, y_j, t_k)) [(n+1-k)^{1-\alpha} - (n-k)^{1-\alpha}] \end{aligned}$$

With the above discretization in hand, the numerical of the study system as can now be represented:

$$\begin{aligned}
& \frac{M(\alpha)}{\alpha \Delta t} \sum_{k=0}^n \left(h_1(x_i, y_j, t_{k+1}) - h_1(x_i, y_j, t_k) \right) \left\{ \exp \left[-\frac{\alpha}{1-\alpha} \Delta t (n+k) \right] \right. \\
& \quad \left. - \exp \left[-\frac{\alpha}{1-\alpha} \Delta t (n+1-k) \right] \right\} \\
& = \frac{T_1}{S_1} \left\{ \frac{h_1(x_{i+1}, y_j, t_{n+1}) - 2h_1(x_i, y_j, t_{n+1}) + h_1(x_{i-1}, y_j, t_{n+1})}{(\Delta x)^2} \right. \\
& \quad \left. + \frac{h_1(x_i, y_{j+1}, t_{n+1}) - 2h_1(x_i, y_j, t_{n+1}) + h_1(x_i, y_{j-1}, t_{n+1})}{(\Delta y)^2} \right\} \\
& \quad - \pi a^2 \left(h_1(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, t_{n+1}) \right)
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
& \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n \left(h_2(x_i, y_j, t_{k+1}) - h_2(x_i, y_j, t_k) \right) \left[(n+1-k)^{1-\alpha} - (n-k)^{1-\alpha} \right] \\
& = \frac{T_2}{S_2} \left\{ \frac{h_2(x_{i+1}, y_j, t_{n+1}) - 2h_2(x_i, y_j, t_{n+1}) + h_2(x_{i-1}, y_j, t_{n+1})}{(\Delta x)^2} \right. \\
& \quad \left. + \frac{h_2(x_i, y_{j+1}, t_{n+1}) - 2h_2(x_i, y_j, t_{n+1}) + h_2(x_i, y_{j-1}, t_{n+1})}{(\Delta y)^2} \right\} \\
& \quad + \pi a^2 \left(h_1(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, t_{n+1}) \right)
\end{aligned} \tag{4.25}$$

$${}^{\text{CF}}_o D_t^\alpha h_1(x, y, t) = \frac{T_1}{S_1} \left[\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right] - \pi a^2 (h_1 - h_2) \tag{4.26}$$

The above to integral equation is now converted to have:

$${}^{\text{CF}}_o J_t^\alpha \left({}^{\text{CF}}_o D_t^\alpha h_1(x, y, t) \right) = {}^{\text{CF}}_o J_t^\alpha \left[\frac{T_1}{S_1} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi a^2 (h_1 - h_2) \right] \tag{4.27}$$

$$\begin{aligned}
& h_1(x, y, t) - h_1(x, y, 0) \\
& = \frac{1-\alpha}{M(\alpha)} \left[\frac{T_1}{S_1} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi a^2 (h_1 - h_2) \right] \\
& \quad + \frac{\alpha}{M(\alpha)} \int_0^t \left[\frac{T_1}{S_1} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi a^2 (h_1 - h_2) \right] d\tau
\end{aligned} \tag{4.28}$$

For simplification we let:

$$F(t, h_1, h_2) = \frac{T_1}{S_1} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) - \pi a^2 (h_1 - h_2) \tag{4.29}$$

$$h_1(x, y, t) - h_1(x, y, 0) = \frac{1 - \alpha}{M(\alpha)} F(t, h_1, h_2) + \frac{\alpha}{M(\alpha)} \int_0^t F(t, h_1, h_2) d\tau \quad (4.30)$$

Then at (x_i, y_j, t_{n+1}) and (x_i, y_j, t_n) can now have the following:

$$h(x_i, y_j, t_{n+1}) - h(x_i, y_j, 0) = \frac{1 - \alpha}{M(\alpha)} F(x_i, y_j, t_n) + \frac{\alpha}{M(\alpha)} \int_0^{t_{n+1}} F(x_i, y_j, \tau) d\tau \quad (4.31)$$

$$h_1(x_i, y_j, t_n) - h_1(x_i, y_j, 0) = \frac{1 - \alpha}{M(\alpha)} F(x_i, y_j, t_{n-1}) + \frac{\alpha}{M(\alpha)} \int_0^t F(x_i, y_j, \tau) d\tau \quad (4.32)$$

Equation (4.31.) – (4.32.) gives:

$$\begin{aligned} h_1(x_i, y_j, t_{n+1}) - h_1(x_i, y_j, t_n) & \quad (4.33) \\ &= \frac{1 - \alpha}{M(\alpha)} [F(x_i, y_j, t_n) - F(x_i, y_j, t_{n-1})] + \frac{\alpha}{M(\alpha)} \int_{t_n}^{t_{n+1}} F(x_i, y_j, \tau) d\tau \end{aligned}$$

Now is applied the Newton interpolation on $F(x_i, y_j, \tau)$ with the interval $[t_n, t_{n+1}]$ and get the following expression.

$$\begin{aligned} h_1(x_i, y_j, t_{n+1}) - h_1(x_i, y_j, t_n) & \quad (4.34) \\ &= \frac{1 - \alpha}{M(\alpha)} [F(x_i, y_j, t_n) - F(x_i, y_j, t_{n-1})] \\ &+ \frac{\alpha}{M(\alpha)} \left[\frac{5}{12} \Delta t F(x_i, y_j, t_{n-2}) - \frac{4}{3} \Delta t F(x_i, y_j, t_{n-1}) + \frac{23}{12} \Delta t F(x_i, y_j, t_n) \right] \end{aligned}$$

Where:

$$\begin{aligned} F(x_i, y_j, t_n) &= \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \\ &+ \left. \frac{h_1(x_i, y_{j+1}, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \\ &- \pi a^2 (h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n)) \end{aligned} \quad (4.35)$$

$$\begin{aligned}
F(x_i, y_j, t_{n-1}) &= \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. & (4.36) \\
&\quad \left. + \frac{h_1(x_i, y_{j+1}, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \\
&\quad - \pi a^2 (h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1}))
\end{aligned}$$

$$\begin{aligned}
F(x_i, y_j, t_{n-2}) &= \frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_{i-1}, y_j, t_{n-2})}{(\Delta x)^2} \right. & (4.37) \\
&\quad \left. + \frac{h_1(x_i, y_{j+1}, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_i, y_{j-1}, t_{n-2})}{(\Delta y)^2} \right) \\
&\quad - \pi a^2 (h_1(x_i, y_j, t_{n-2}) - h_2(x_i, y_j, t_{n-2}))
\end{aligned}$$

Therefore

$$\begin{aligned}
& h_1(x_i, y_j, t_{n+1}) - h_1(x_i, y_j, t_n) \tag{4.38} \\
&= \frac{1-\alpha}{M(\alpha)} \left\{ \left[\frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \right. \\
&- \left. \left. \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) \right] \right. \\
&- \left[\frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \right. \\
&- \left. \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1}) \right) \right] \right\} \\
&+ \frac{\alpha}{M(\alpha)} \left\{ \frac{5}{12} \Delta t \left[\frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_{i-1}, y_j, t_{n-2})}{(\Delta x)^2} \right. \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_{n-2}) - 2h_1(x_i, y_j, t_{n-2}) + h_1(x_i, y_{j-1}, t_{n-2})}{(\Delta y)^2} \right) \right. \\
&- \left. \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-2}) - h_2(x_i, y_j, t_{n-2}) \right) \right] \right. \\
&- \frac{4}{3} \Delta t \left[\frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_{i-1}, y_j, t_{n-1})}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_{n-1}) - 2h_1(x_i, y_j, t_{n-1}) + h_1(x_i, y_{j-1}, t_{n-1})}{(\Delta y)^2} \right) \right. \\
&- \left. \left. \pi a^2 \left(h_1(x_i, y_j, t_{n-1}) - h_2(x_i, y_j, t_{n-1}) \right) \right] \right\} \\
&+ \frac{23}{12} \Delta t \left[\frac{T_1}{S_1} \left(\frac{h_1(x_{i+1}, y_j, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_{i-1}, y_j, t_n)}{(\Delta x)^2} \right. \right. \\
&+ \left. \left. \frac{h_1(x_i, y_{j+1}, t_n) - 2h_1(x_i, y_j, t_n) + h_1(x_i, y_{j-1}, t_n)}{(\Delta y)^2} \right) \right. \\
&- \left. \left. \pi a^2 \left(h_1(x_i, y_j, t_n) - h_2(x_i, y_j, t_n) \right) \right] \right\}
\end{aligned}$$

$${}_0^c D_t^\alpha h_2(x, y, t) = \frac{T_2}{S_2} \left[\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right] - \pi a^2 (h_1 - h_2) \tag{4.39}$$

The study now applies here the Riemann-Liouville integral on both sides to obtain:

$$h_2(x, y, t) - h_2(x, y, 0) = \frac{1}{\Gamma(\alpha)} \int_0^t \left[\frac{T_2}{S_2} \left[\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right] - \pi a^2 (h_1 - h_2) \right] (t - \tau)^{\alpha-1} d\tau \quad (4.40)$$

$$F_1(x, y, t) = \int_0^t \left[\frac{T_2}{S_2} \left[\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right] - \pi a^2 (h_1 - h_2) \right] (t - \tau)^{\alpha-1} d\tau \quad (4.41)$$

$$h_2(x, y, t) - h_2(x, y, 0) = \frac{1}{\Gamma(\alpha)} \int_0^t F_1(x, y, \tau) (t - \tau)^{\alpha-1} d\tau \quad (4.42)$$

At (x_i, y_j, t_{n+1}) the following is arrived at:

$$h_2(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, 0) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n \int_{t_k}^{t_{k+1}} (t_{n+1} - \tau)^{\alpha-1} F(x_i, y_j, \tau) d\tau \quad (4.43)$$

Now the function $F(x_i, y_j, \tau) \simeq P_k(\tau)$ is approximated where $P_k(\tau)$ is the Newtons polynomial. Using the development done by Atangana and Seda, the following can be obtained:

$$\begin{aligned} h_2(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, 0) &= \frac{(\Delta t)^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^n F(t_{k-2}, x_i, y_j) \{(n - k + 1)^\alpha - (n - k)^\alpha\} \\ &+ \frac{(\Delta t)^\alpha}{\Gamma(2 + \alpha)} \sum_{k=2}^n [F(t_{k-1}, x_i, y_j) \\ &- F(t_{k-2}, x_i, y_j)] \{(n - k + 1)^\alpha (n - k + 3 + 2\alpha) - (n - k)^\alpha (n - k \\ &+ 3 + 3\alpha)\} \\ &+ \frac{(\Delta t)^\alpha}{2\Gamma(3 + \alpha)} [F(t_k, x_i, y_j) - 2F(t_{k-1}, x_i, y_j) \\ &+ F(t_{k-2}, x_i, y_j)] \{(n - k + 1)^2 (2(n - k)^2 + (3\alpha + 10)(n - j) + 2\alpha^2 \\ &+ 9\alpha + 12) \\ &- (n - k)^2 (2(n - k)^2 + (5\alpha + 10)(n - k) + 6\alpha^2 + 18\alpha + 12)\} \end{aligned} \quad (4.44)$$

Where:

$$\begin{aligned}
F(t_k, x_i, y_j) &= \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_k) - 2h_2(x_i, y_j, t_k) + h_2(x_{i-1}, y_j, t_k)}{(\Delta x)^2} \right. \\
&\quad \left. + \frac{h_2(x_i, y_{j+1}, t_k) - 2h_2(x_i, y_j, t_k) + h_2(x_i, y_{j-1}, t_k)}{(\Delta y)^2} \right) \\
&\quad - \pi a^2 (h_1(x_i, y_j, t_k) - h_2(x_i, y_j, t_k))
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
F(t_{k-1}, x_i, y_j) &= \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-1}) - 2h_2(x_i, y_j, t_{k-1}) + h_2(x_{i-1}, y_j, t_{k-1})}{(\Delta x)^2} \right. \\
&\quad \left. + \frac{h_2(x_i, y_{j+1}, t_{k-1}) - 2h_2(x_i, y_j, t_{k-1}) + h_2(x_i, y_{j-1}, t_{k-1})}{(\Delta y)^2} \right) \\
&\quad - \pi a^2 (h_1(x_i, y_j, t_{k-1}) - h_2(x_i, y_j, t_{k-1}))
\end{aligned} \tag{4.46}$$

$$\begin{aligned}
F(t_{k-2}, x_i, y_j) &= \frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_{i-1}, y_j, t_{k-2})}{(\Delta x)^2} \right. \\
&\quad \left. + \frac{h_2(x_i, y_{j+1}, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_i, y_{j-1}, t_{k-2})}{(\Delta y)^2} \right) \\
&\quad - \pi a^2 (h_1(x_i, y_j, t_{k-2}) - h_2(x_i, y_j, t_{k-2}))
\end{aligned} \tag{4.47}$$

Therefore:

$$\begin{aligned}
& h_2(x_i, y_j, t_{n+1}) - h_2(x_i, y_j, 0) \tag{4.48} \\
&= \frac{(\Delta t)^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^n \left[\frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_{i-1}, y_j, t_{k-2})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_2(x_i, y_{j+1}, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_i, y_{j-1}, t_{k-2})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{k-2}) - h_2(x_i, y_j, t_{k-2})) \right] \{(n - k + 1)^\alpha - (n - k)^\alpha\} \\
&\quad + \frac{(\Delta t)^\alpha}{\Gamma(2 + \alpha)} \sum_{k=2}^n \left[\frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-1}) - 2h_2(x_i, y_j, t_{k-1}) + h_2(x_{i-1}, y_j, t_{k-1})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_2(x_i, y_{j+1}, t_{k-1}) - 2h_2(x_i, y_j, t_{k-1}) + h_2(x_i, y_{j-1}, t_{k-1})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{k-1}) - h_2(x_i, y_j, t_{k-1})) \right] \\
&\quad - \left[\frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_{i-1}, y_j, t_{k-2})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_2(x_i, y_{j+1}, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_i, y_{j-1}, t_{k-2})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{k-2}) - h_2(x_i, y_j, t_{k-2})) \right] \left[\{(n - k + 1)^\alpha (h - k + 3 + 2\alpha) - (n - k)^\alpha (n - k \right. \right. \\
&\quad \left. \left. + 3 + 3\alpha)\} \right. \\
&\quad \left. + \frac{(\Delta t)^\alpha}{2\Gamma(3 + \alpha)} \left[\frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_k) - 2h_2(x_i, y_j, t_k) + h_2(x_{i-1}, y_j, t_k)}{(\Delta x)^2} \right. \right. \right. \\
&\quad \left. \left. + \frac{h_2(x_i, y_{j+1}, t_k) - 2h_2(x_i, y_j, t_k) + h_2(x_i, y_{j-1}, t_k)}{(\Delta y)^2} \right) - \pi a^2 (h_1(x_i, y_j, t_k) - h_2(x_i, y_j, t_k)) \right] \\
&\quad - 2 \left[\frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-1}) - 2h_2(x_i, y_j, t_{k-1}) + h_2(x_{i-1}, y_j, t_{k-1})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_2(x_i, y_{j+1}, t_{k-1}) - 2h_2(x_i, y_j, t_{k-1}) + h_2(x_i, y_{j-1}, t_{k-1})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{k-1}) - h_2(x_i, y_j, t_{k-1})) \right] \\
&\quad + \left[\frac{T_2}{S_2} \left(\frac{h_2(x_{i+1}, y_j, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_{i-1}, y_j, t_{k-2})}{(\Delta x)^2} \right. \right. \\
&\quad \left. \left. + \frac{h_2(x_i, y_{j+1}, t_{k-2}) - 2h_2(x_i, y_j, t_{k-2}) + h_2(x_i, y_{j-1}, t_{k-2})}{(\Delta y)^2} \right) \right. \\
&\quad \left. - \pi a^2 (h_1(x_i, y_j, t_{k-2}) - h_2(x_i, y_j, t_{k-2})) \right] \left[\{(n - k + 1)^2 (2(n - k)^2 + (3\alpha + 10)(n - j) \right. \right. \\
&\quad \left. \left. + 2\alpha^2 + 9\alpha + 12) - (n - k)^2 (2(n - k)^2 + (5\alpha + 10)(n - k) + 6\alpha^2 + 18\alpha + 12)\} \right]
\end{aligned}$$

CHAPTER 5 : NUMERICAL SIMULATIONS

5.1. Introduction

In the previous chapter, the study devoted attention to numerical analysis of the suggested models with different type of differential and integral operators. The aim was to include into mathematical formula the effect of non-localities that could be found within the geological formation. For example, the study uses the derivative with the generalized Mittag-Leffler function to include into the mathematical formula or the mathematical model depicting the flow within a dual media geological formation the effect of crossover. Besides, such crossover has some limitations. For example, Atangana and Seda argued that such crossover cannot depict the passage from power law to fading memory. Also, one cannot really use this to identify in the field the time at which such crossover will take place. The study uses the fractional derivative with exponential decay law to express the fading memory effect of the flow with the geological formation. This differential is also able to account for the passage from a Gaussian process to non-Gaussian with a steady state. The Caputo derivative was also used, which is known to depicting power law processes due to its power law kernel. But this power law kernel cannot exhibit processes like fading memory or even a crossover. Atangana and Seda suggested then a new concept called piecewise derivative, a differential operator that is defined within a well-defined interval. To perform simulations, the study used the numerical scheme based on the Newton polynomial to solve numerically the suggested models for different differential operators. The obtained numerical schemes were coded using MatLab. The study used some theoretical parameters to see the behaviour of the model, for example the study used $T_1=200$ and $T_2=400$ where T_1 and T_2 are the transmissivity of the matrix rock and the fracture respectively. Further, it used $S_1=0.001$ and $S_2= 0.0001$, where S_1 and S_2 are storativity of rock matrix and fractures respectively. The parameter $\alpha=0.5$ the numerical simulation was performed for different fractional orders, here the study chose the initial condition to be a constant 0.1 for both initial heads. The obtained results are depicted in the next section.

5.2. Results and discussion

Using the Matlab code, the study obtained the following simulations, which are depicted in figures below. The model with power law kernel was used to see the effect of non-locality. The figures were obtained for different value of fractional orders. In these figures, the x represents distance in kilometre, and t represents time in hour. For a fractional order starting from 0.95 to 0.8, the obtained figures show high flow within fracture. This behaviour is known as long range dependency. Also within the matrix soil, the study also observed high flow meaning in this case the fractional derivative is expressing the flow within a matrix rock with high transmissivity such as sand/sandstones. This model is therefore depicting the flow behaviour in a sandy/sandstone aquifer. So, from the mathematical equations, the function $h_1(x, t)$ and $h_2(x, t)$ represent the hydraulic heads in dolomite especially, the first function expresses the flow in rock matrix and the second function expresses the flow in fractures. This behaviour has been recognised in the literature as fast flow or even super flow. Also, it is shown from these figures that, more water will be found within fracture than in matrix rock. These scenarios are depicted in Figure 5.1 to Figure 5.12 for contour plots cases and x - t 3 dimensional phases. For fractional order ranging from 0.7 to 0.5, the obtained figures show normal flow behaviours, water flows with normal velocity within the matrix rock and moderate high velocity within the fractures. These simulations are depicted in Figure 5.13 to Figure 5.24 for contour plots and x - t 3 dimensional phases. Finally, from 0.4 to 0.2, the study observed slow flow behaviour, the mathematical equations are exhibiting flow with low velocities, expressing the flow within a shale with limited or no transmissivity. These simulations are depicted in Figure 5.25 to Figure 5.36. Furthermore, Figure 5.37 represent the crossover effect from the matrix to the fracture over distance x . The fracture is characterised by high turbulent velocities.

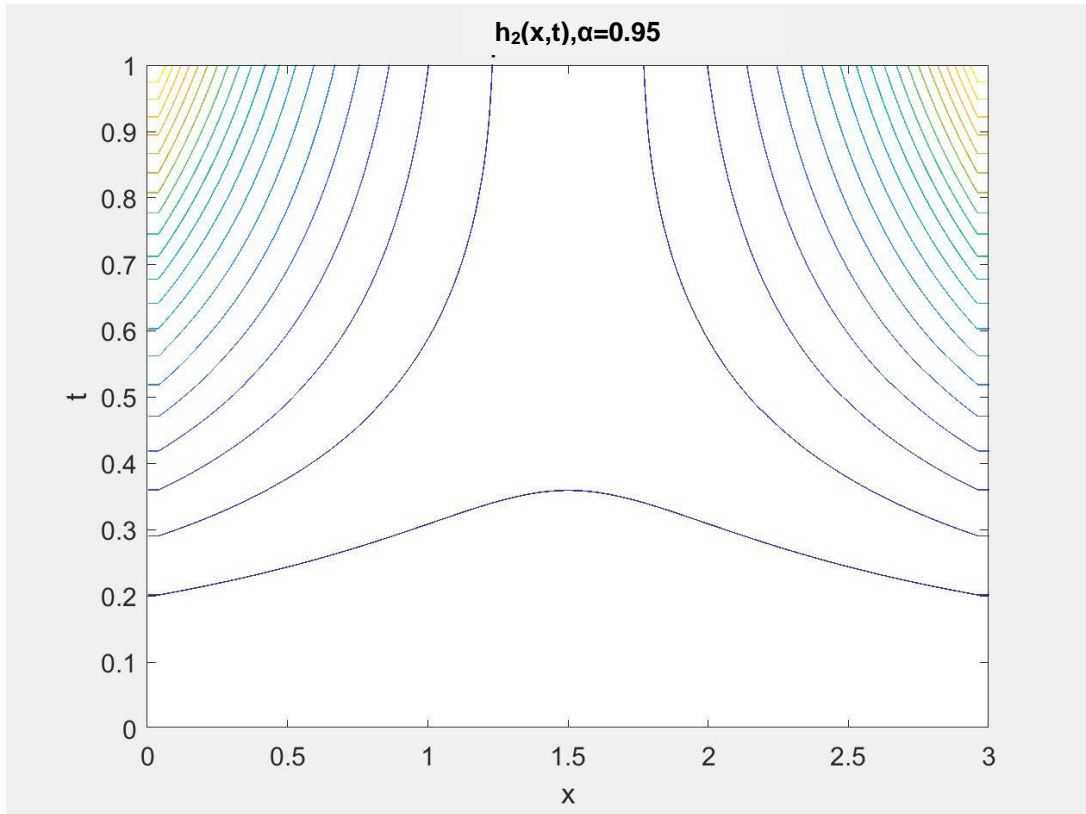


Figure 5.1: Contour plot representing h_2 in fractional order 0.95.

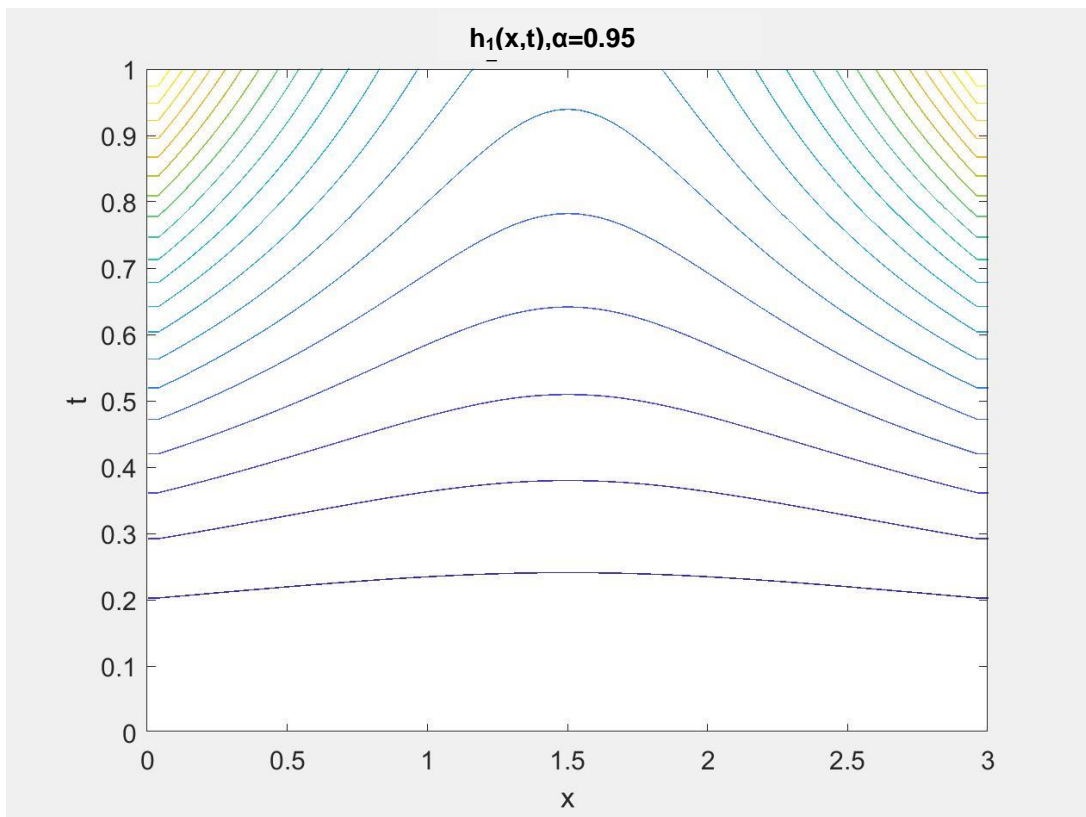


Figure 5.2: Contour plot representing h_1 in fractional order 0.95.

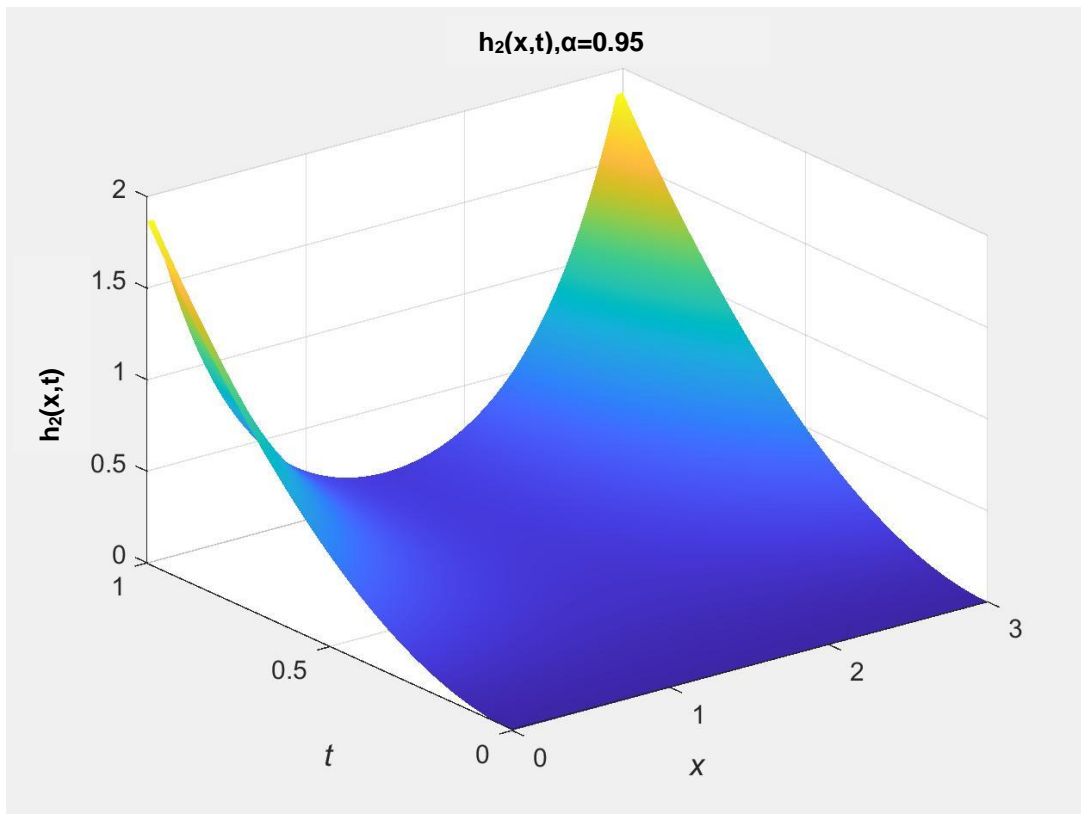


Figure 5.3: 3D plot representing h_2 in fractional order 0.95.

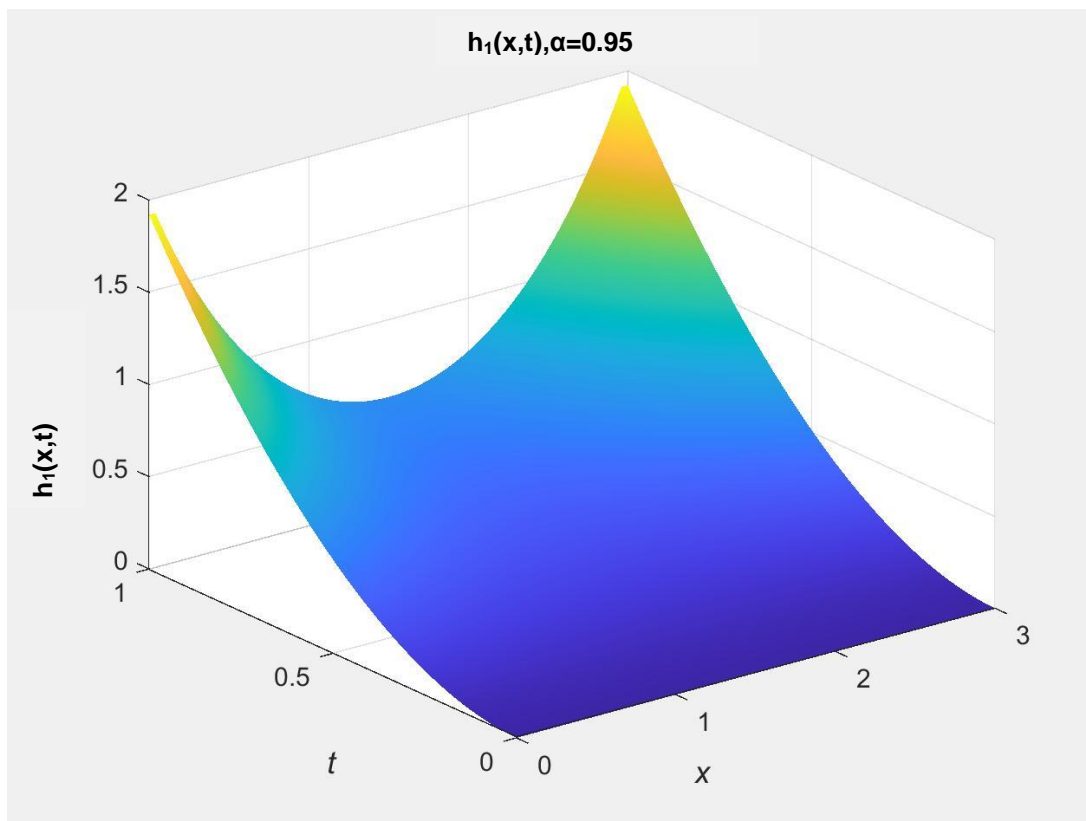


Figure 5.4: 3D plot representing h_1 in fractional order 0.95.

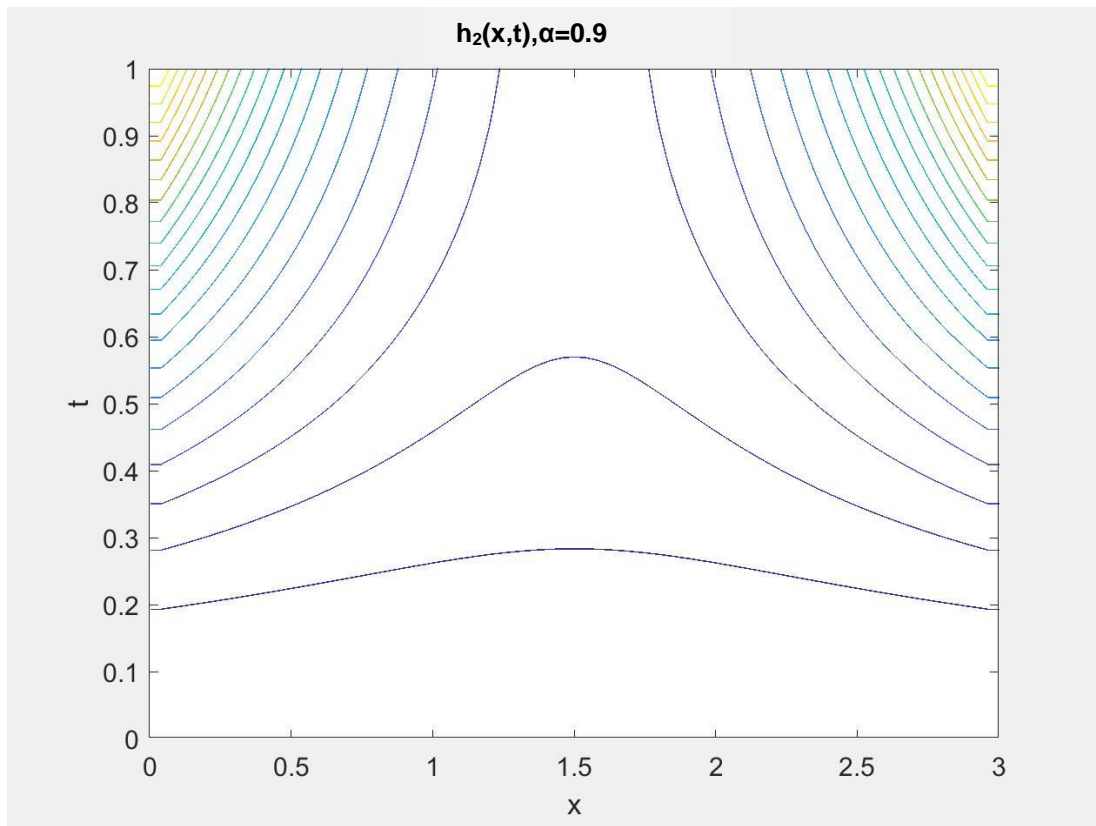


Figure 5.5: Contour plot representing h_2 in fractional order 0.9

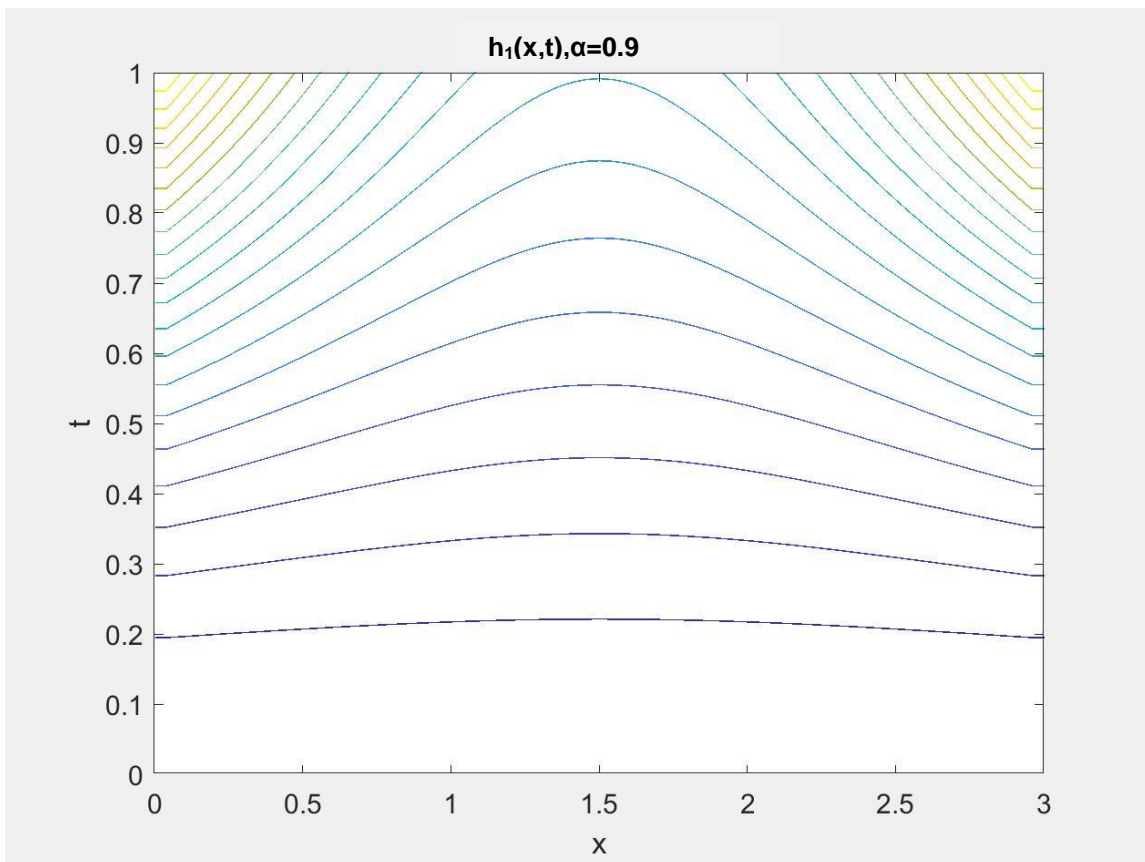


Figure 5.6: Contour plot representing h_1 in fractional order 0.9

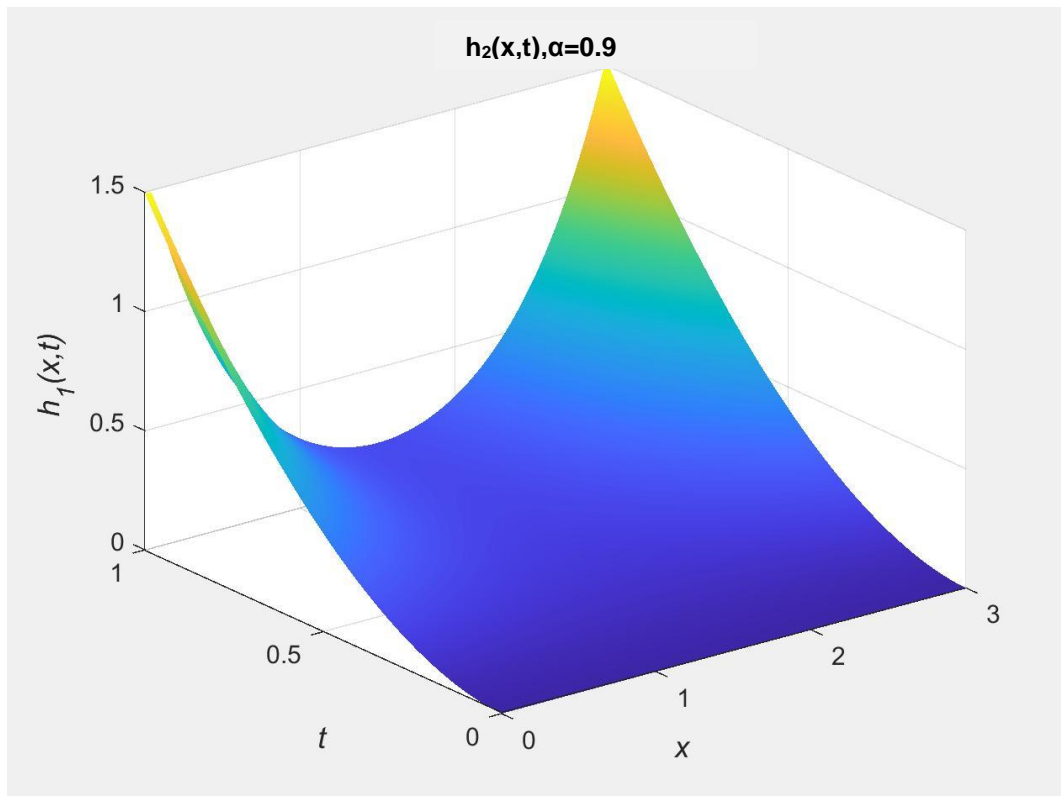


Figure 5.7: 3D plot representing h_2 in fractional order 0.9

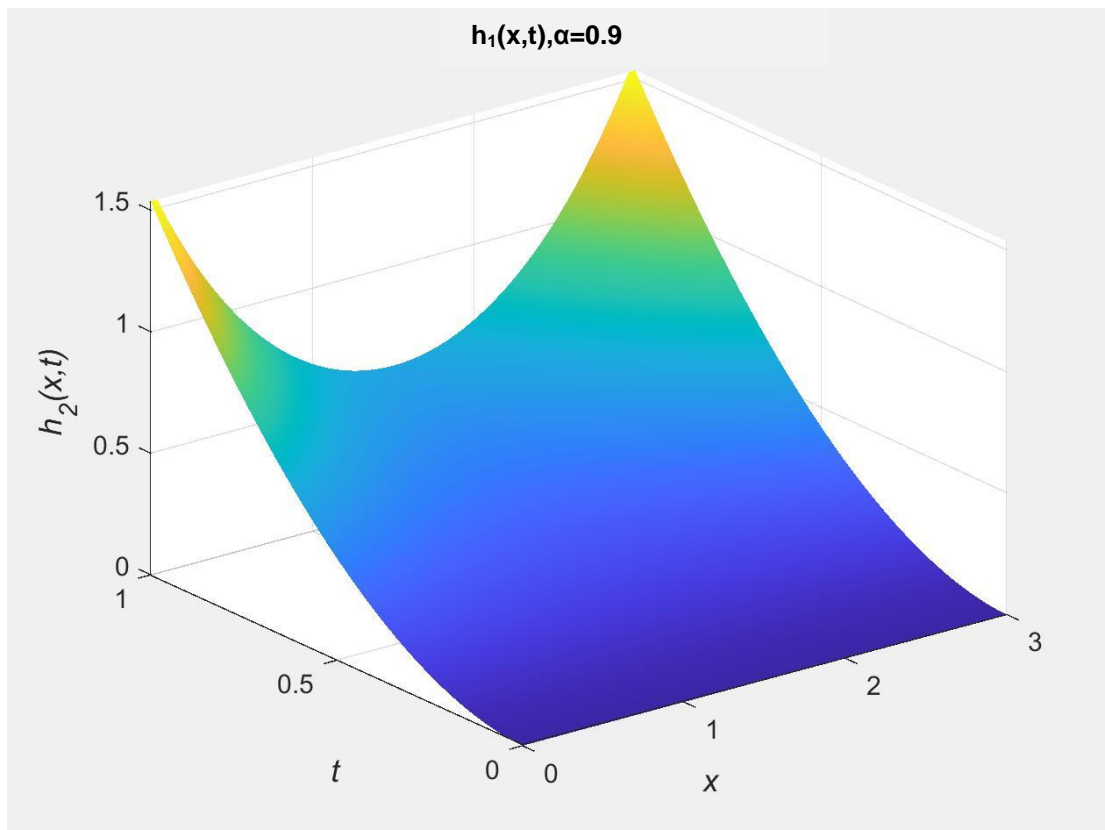


Figure 5.8: 3D plot representing h_1 in fractional order 0.9

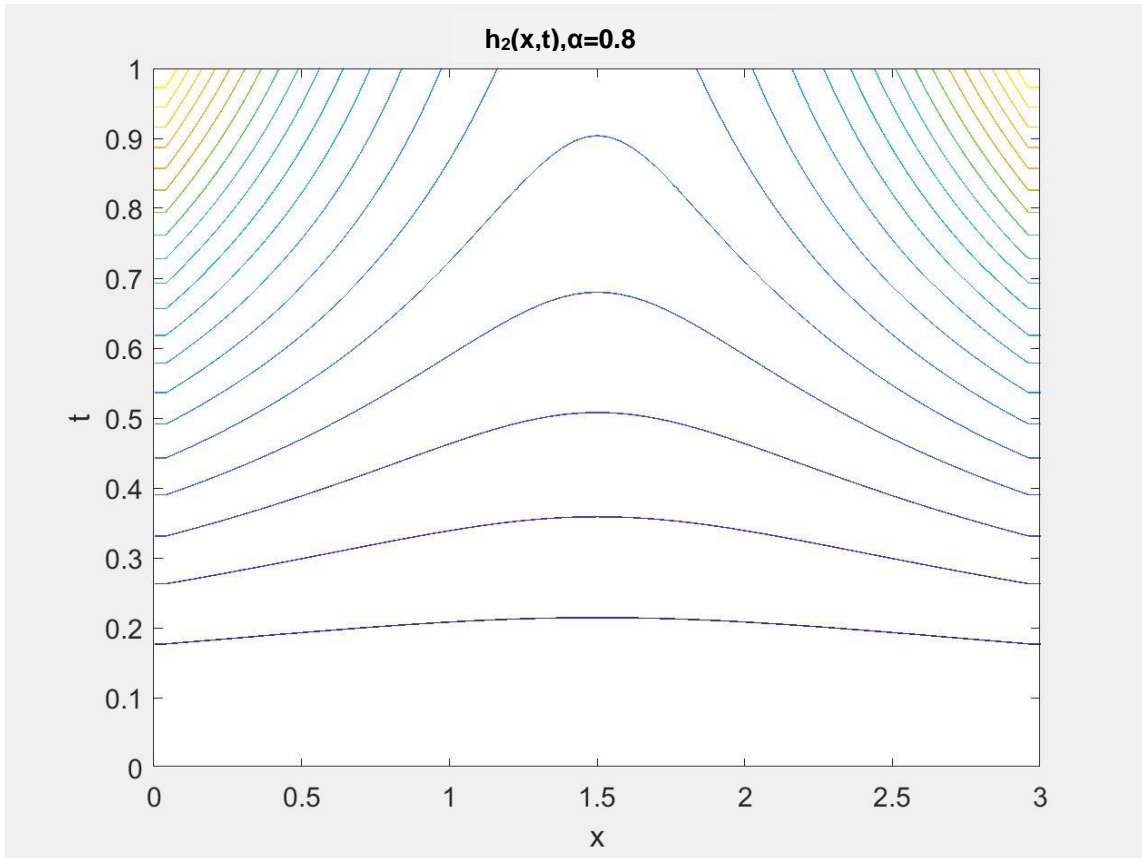


Figure 5.9: Contour plot representing h_2 in fractional order 0.8.

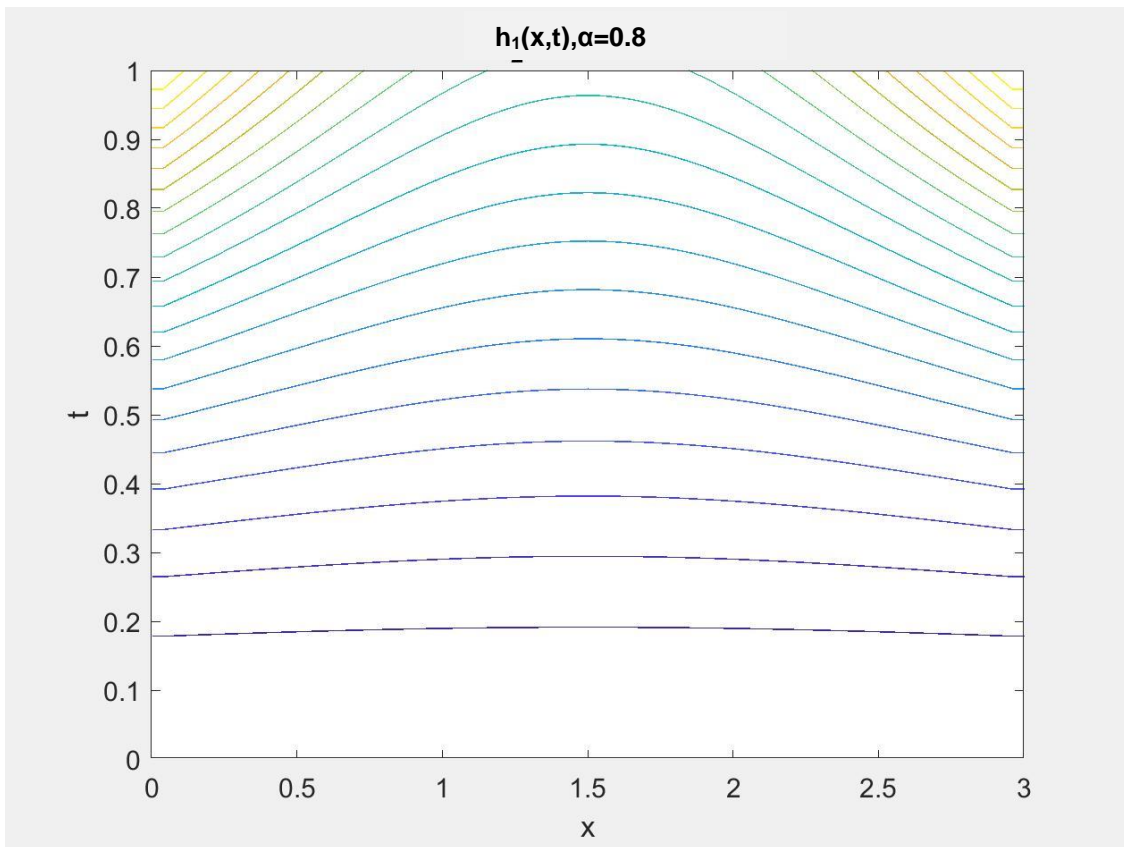


Figure 5.10: Contour plot representing h_1 in fractional order 0.8.

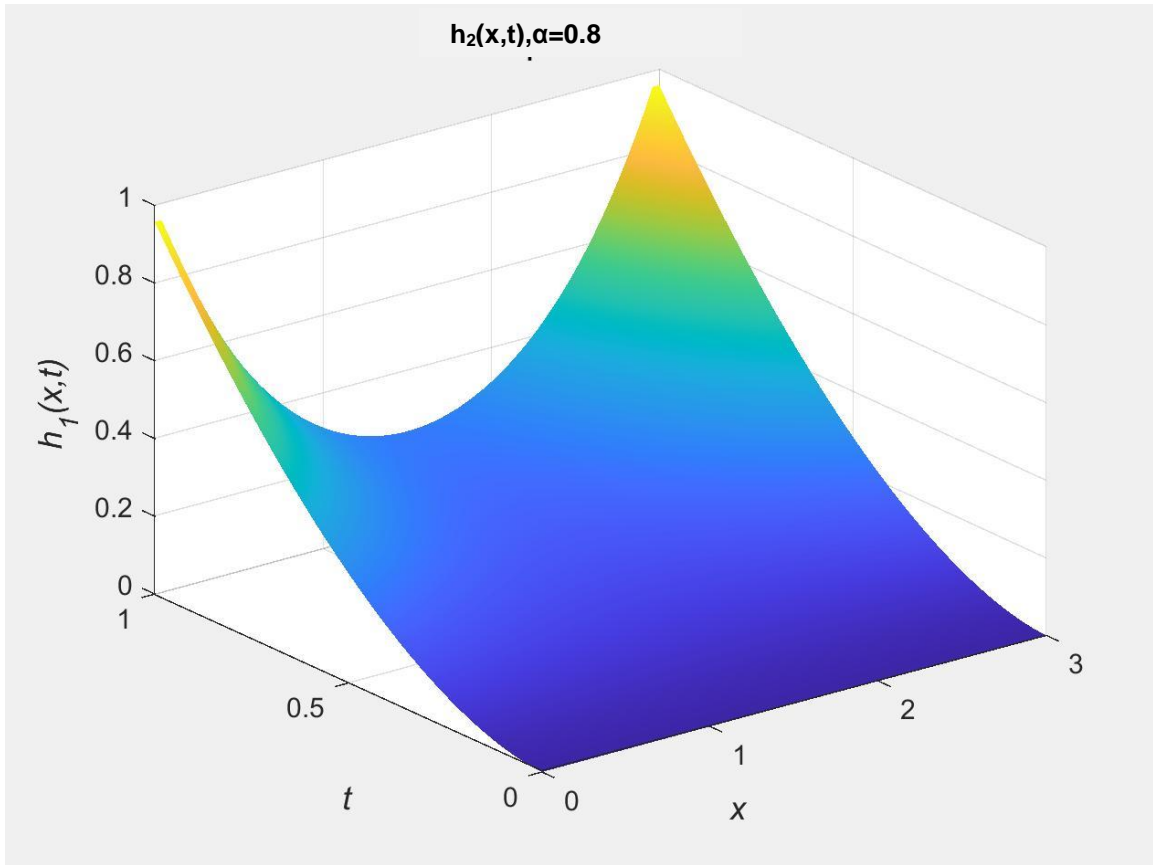


Figure 5.11: 3D plot representing h_2 in fractional order 0.8.

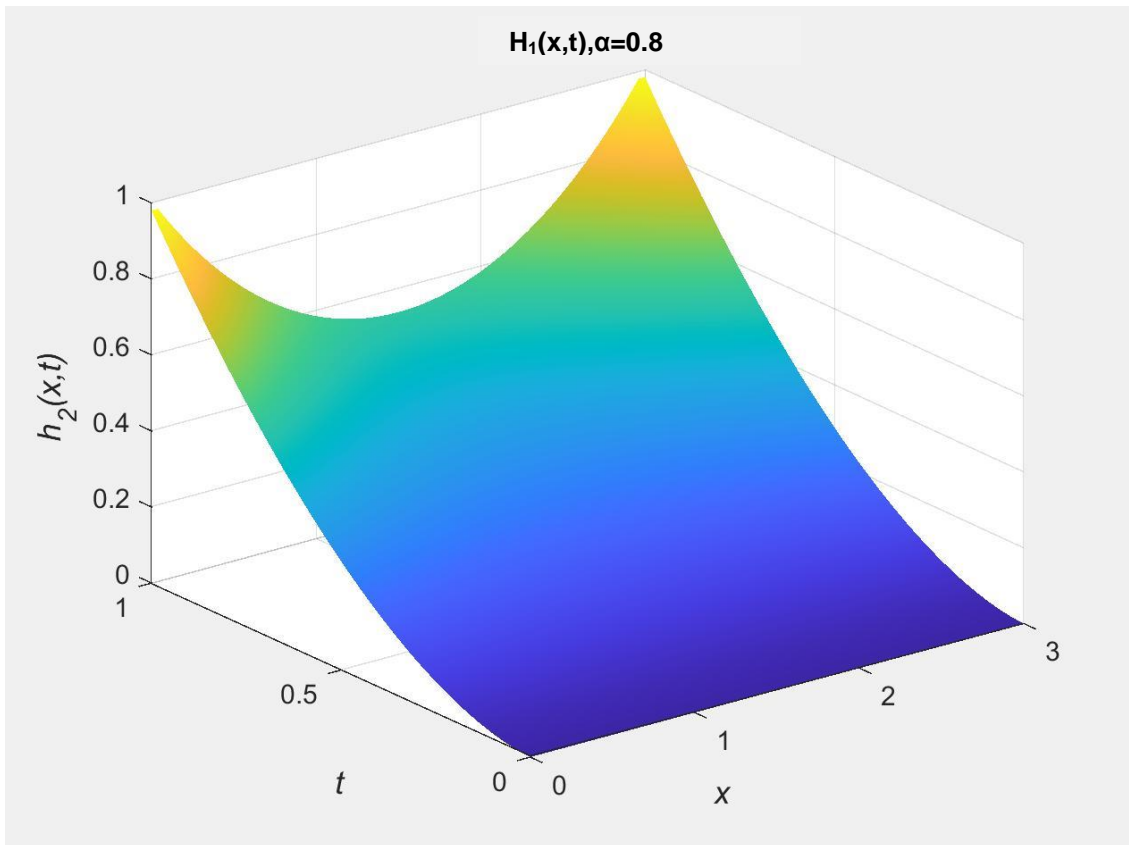


Figure 5.12: 3D plot representing h_1 in fractional order 0.8.

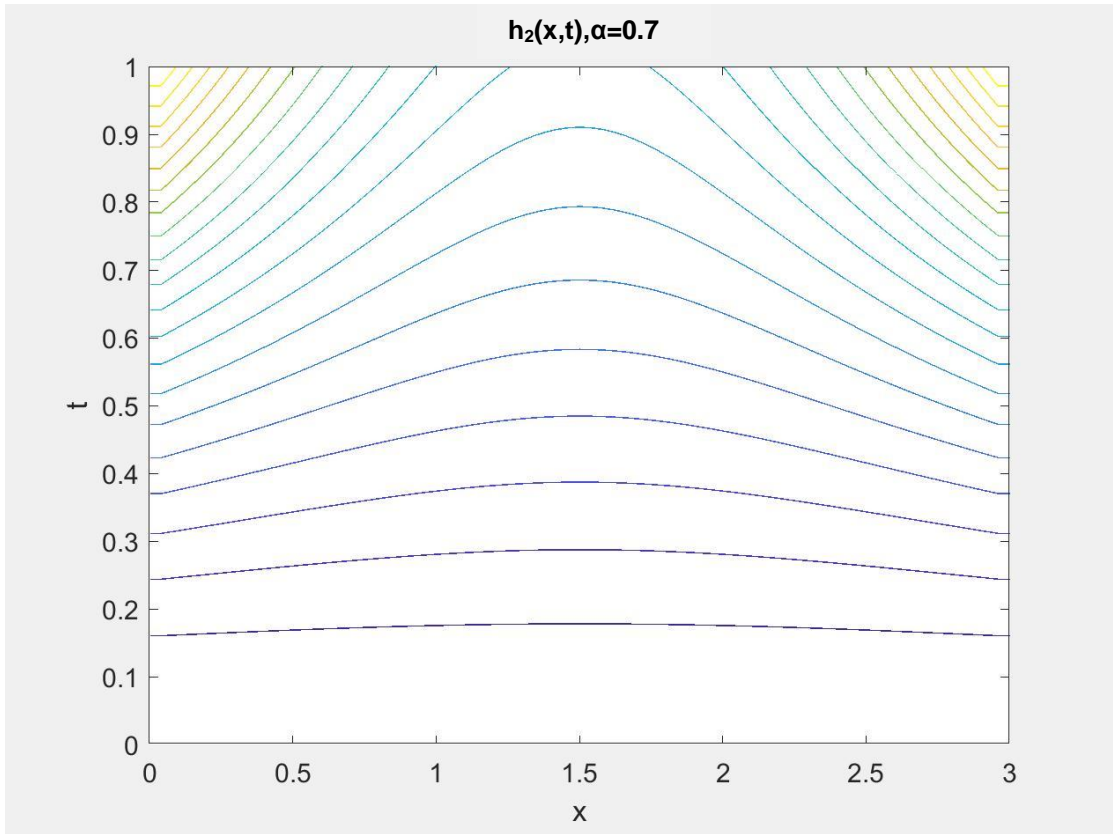


Figure 5.13: Contour plot representing h_2 in fractional order 0.7.

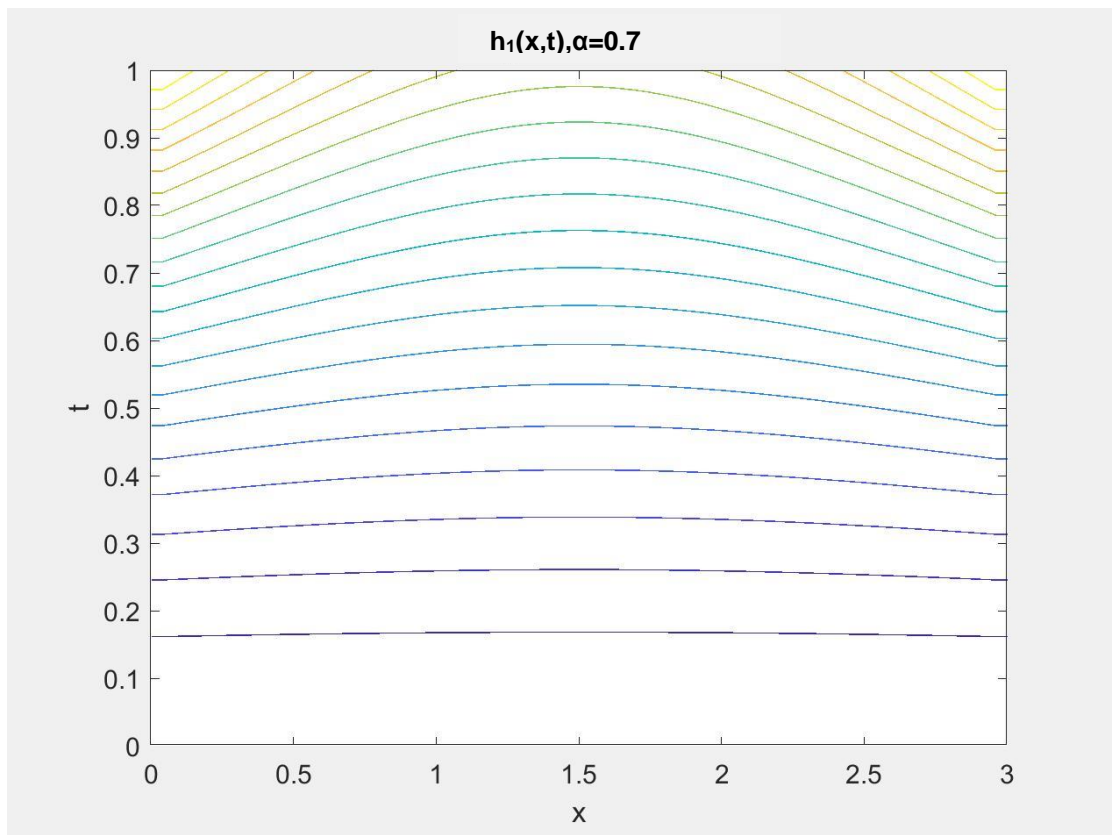


Figure 5.14: Contour plot representing h_1 in fractional order 0.7.

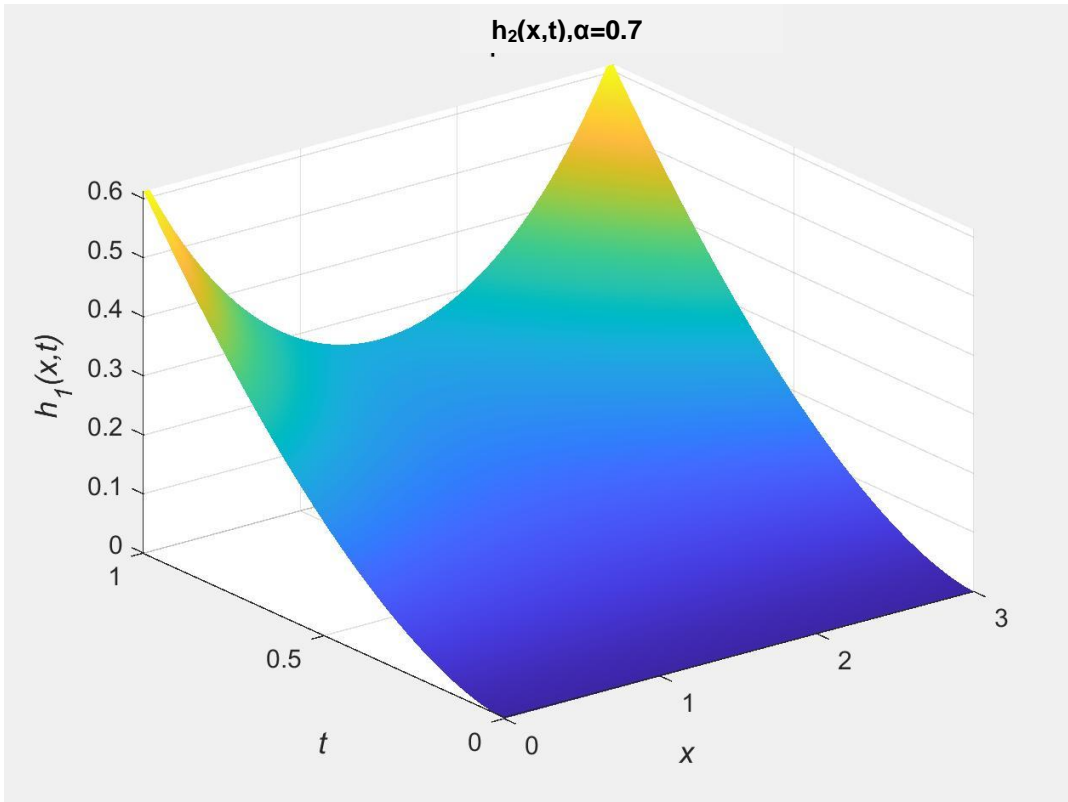


Figure 5.15: 3D plot representing h_2 in fractional order 0.7.

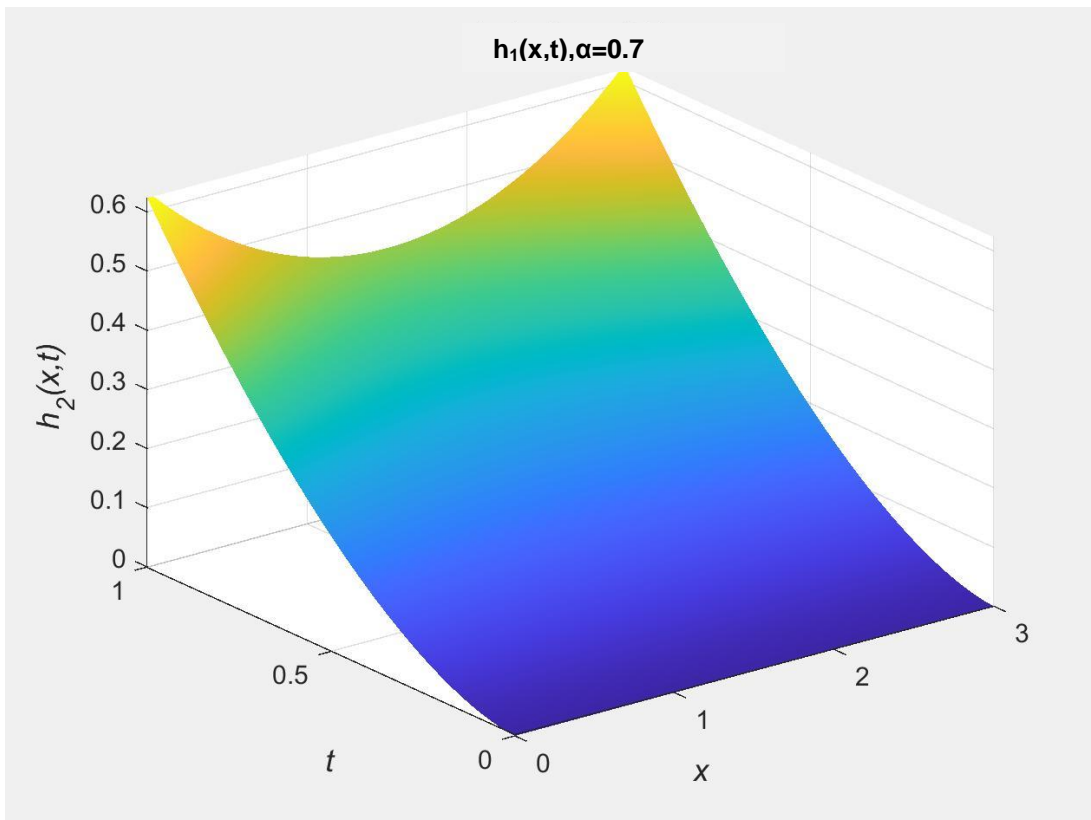


Figure 5.16: 3D plot representing h_1 in fractional order 0.7.

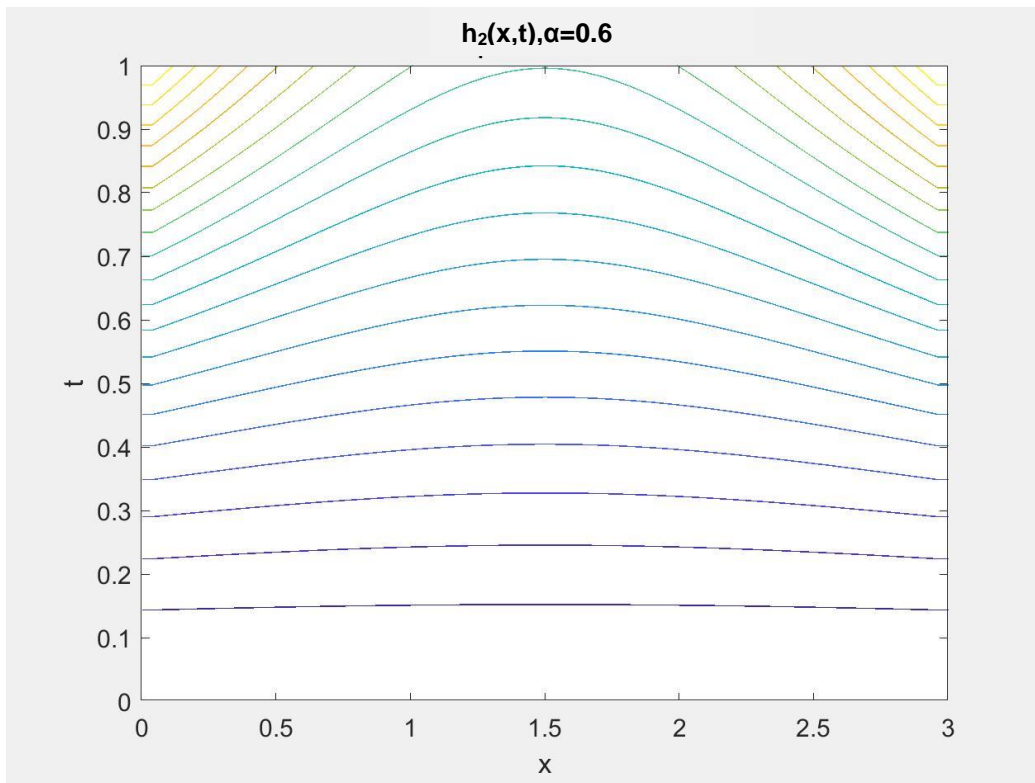


Figure 5.17: Contour plot representing h_2 in fractional order 0.6.

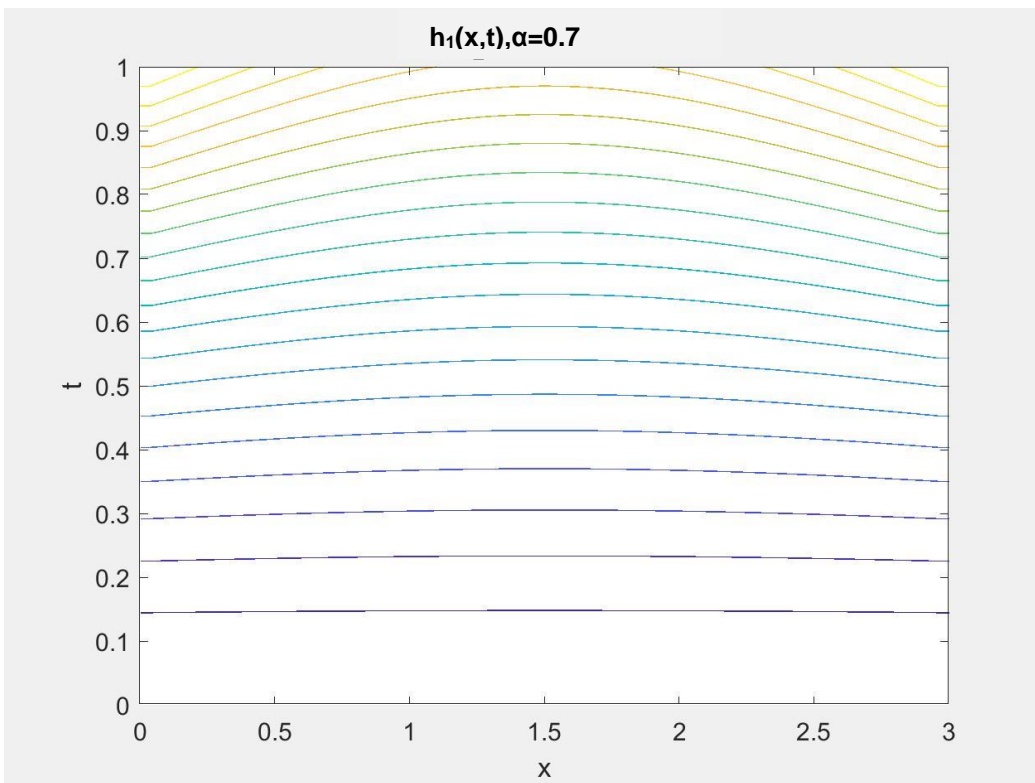


Figure 5.18: Contour plot representing h_1 in fractional order 0.6.

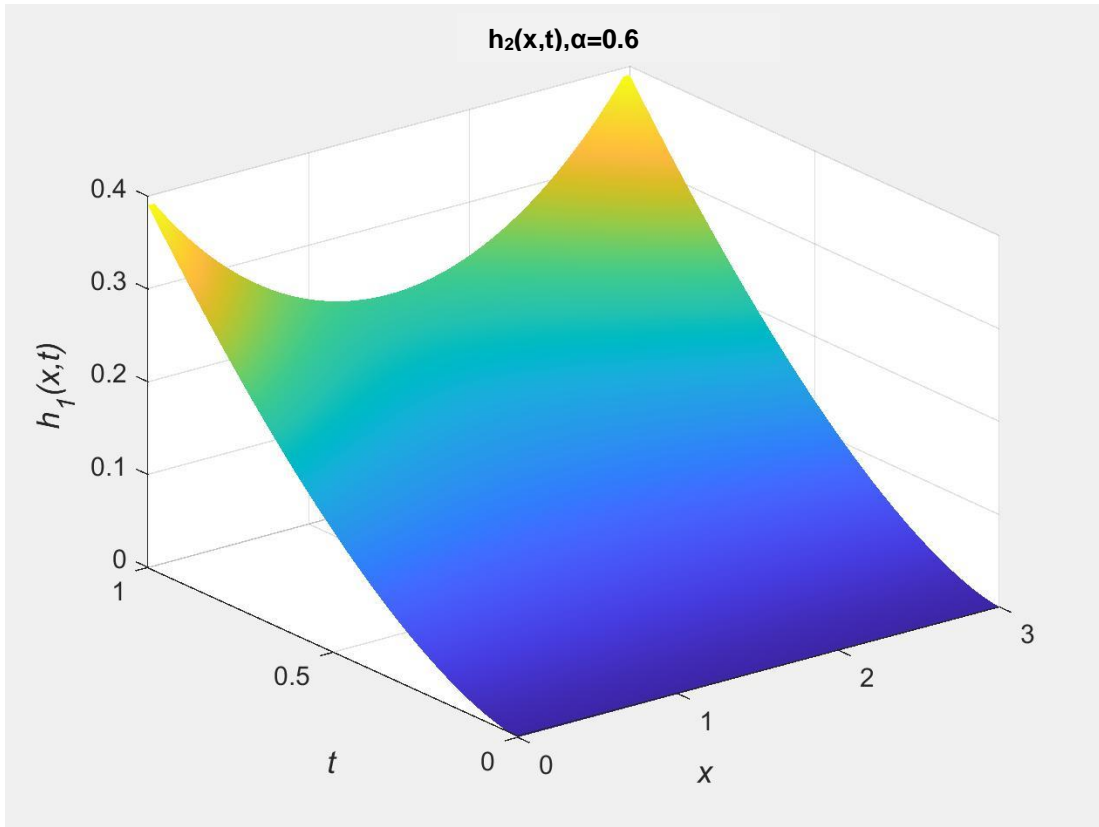


Figure 5.19: 3D plot representing h_2 in fractional order 0.6.

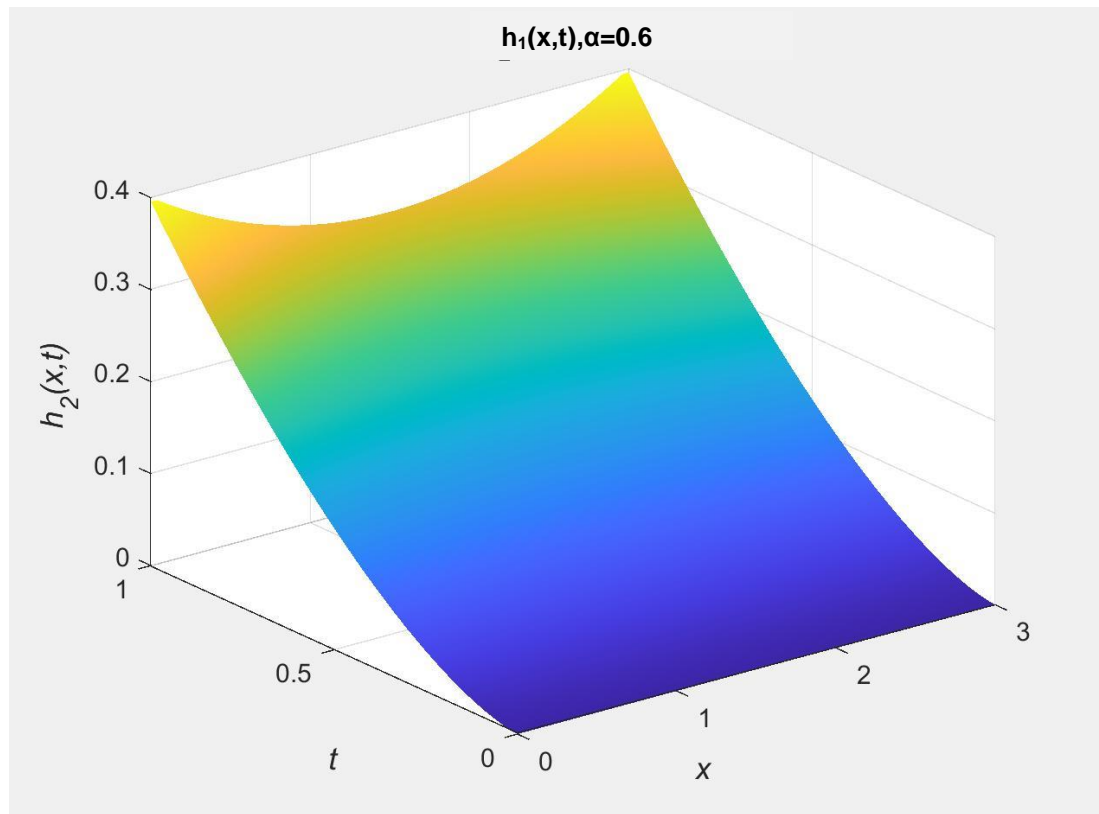


Figure 5.20: 3D plot representing h_1 in fractional order 0.6.

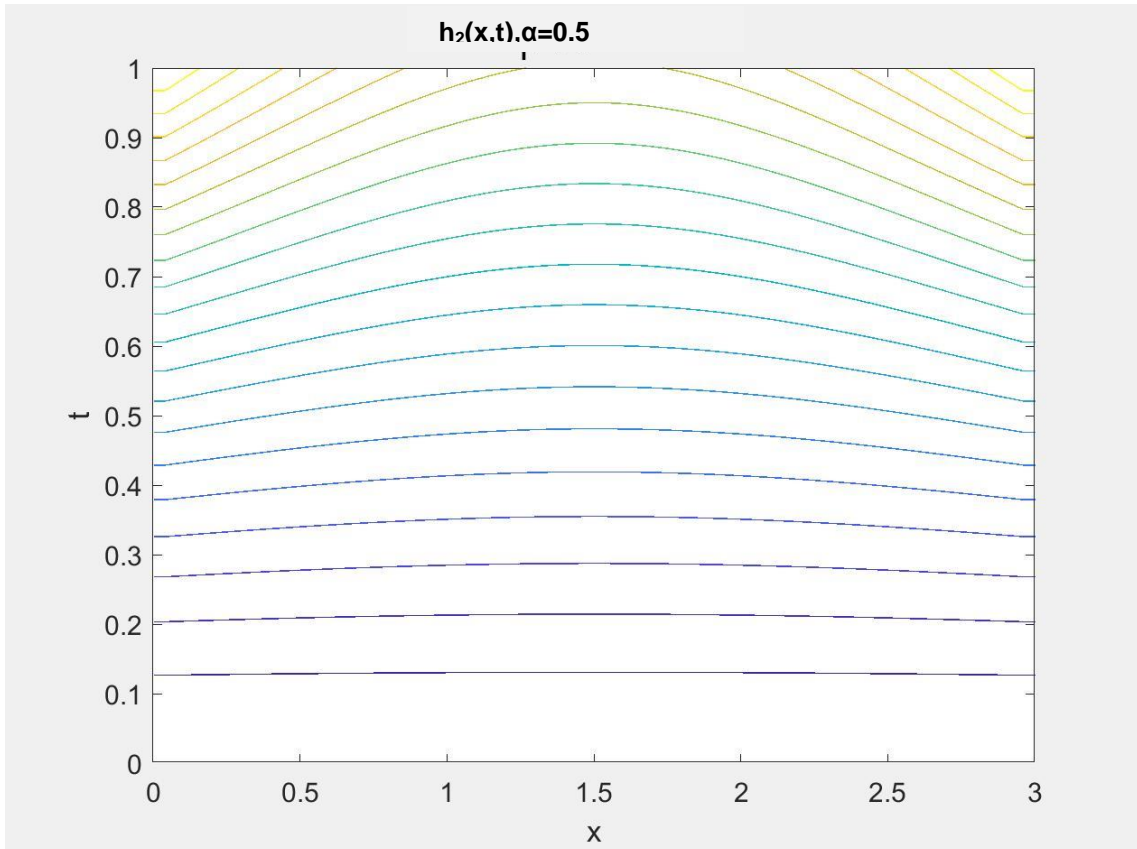


Figure 5.21: Contour plot representing h_2 in fractional order 0.5.

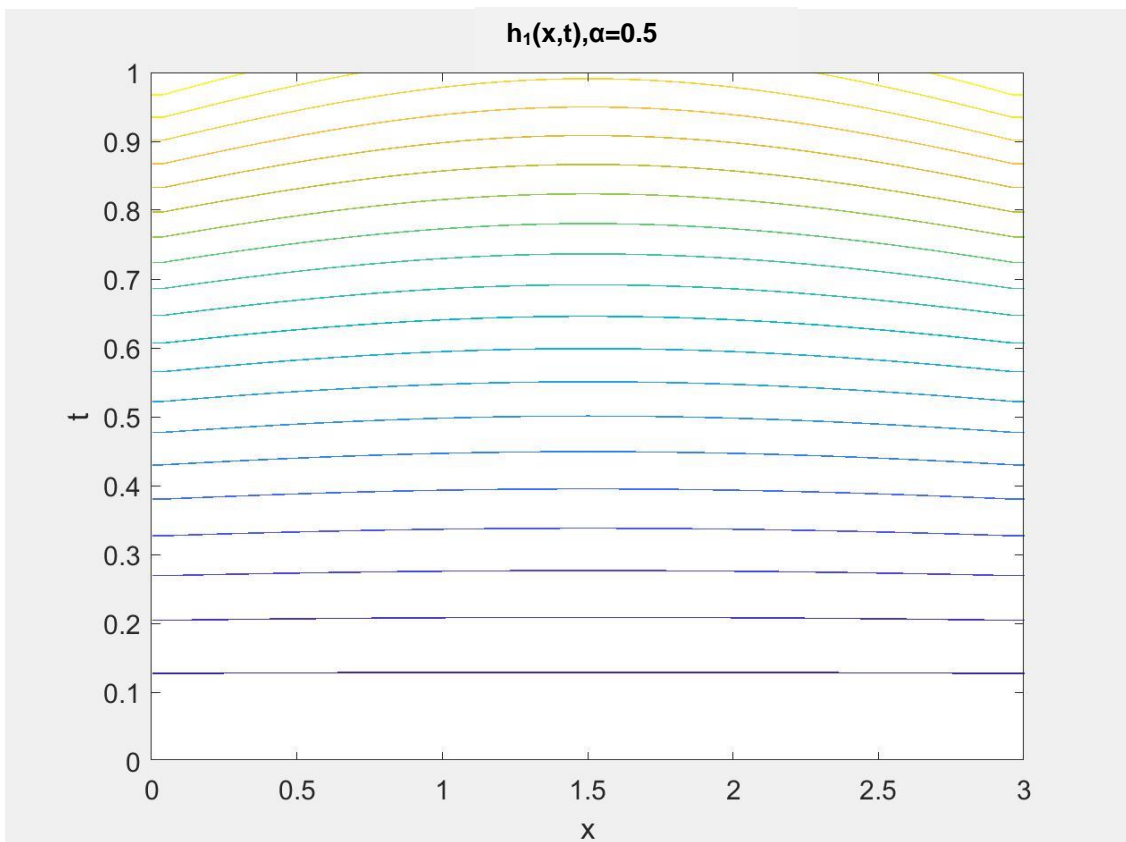


Figure 5.22: Contour plot representing h_1 in fractional order 0.5.

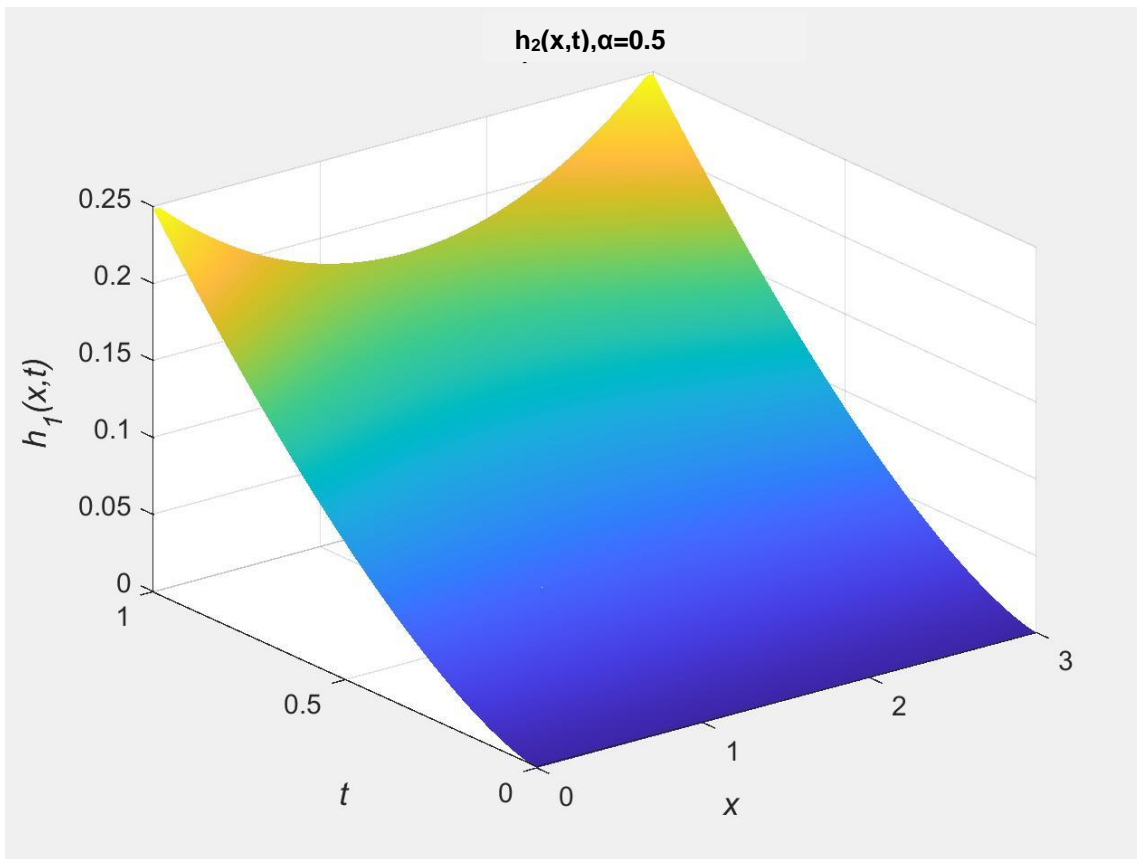


Figure 5.23: 3D plot representing h_2 in fractional order 0.5.

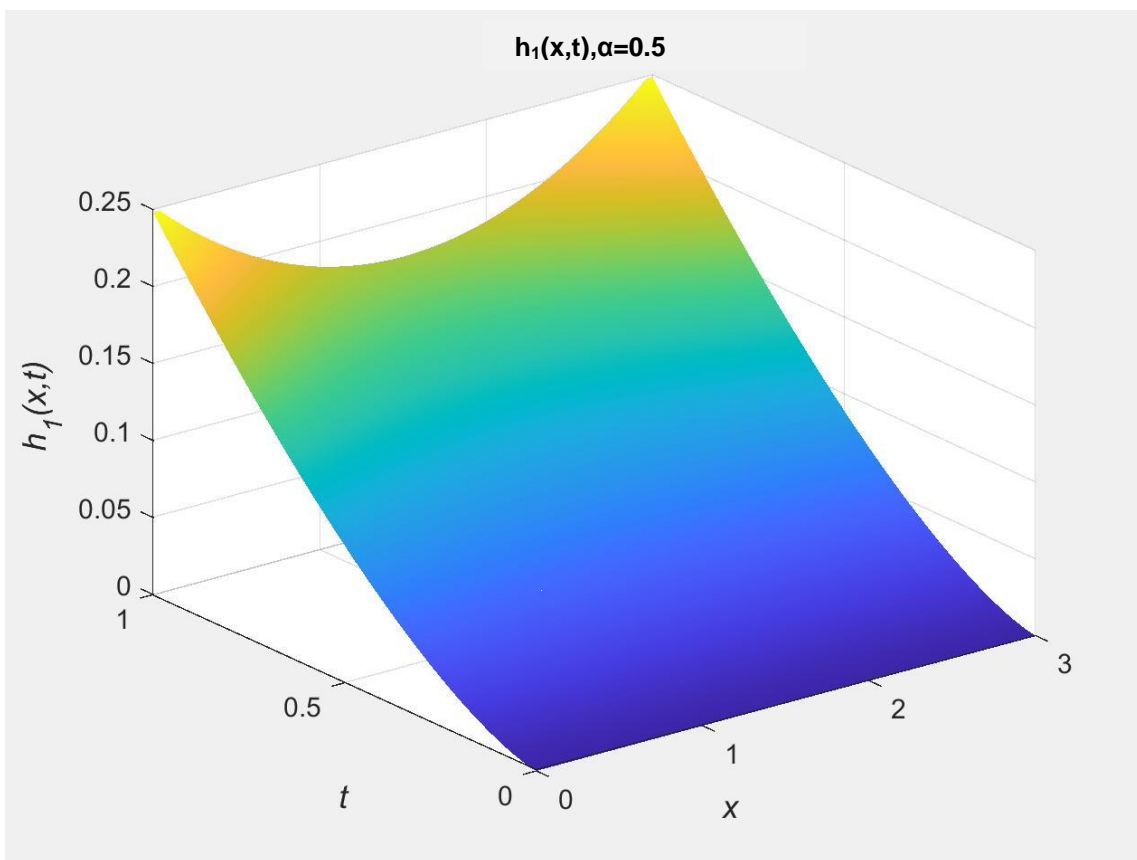


Figure 5.24: 3D plot representing h_1 in fractional order 0.5.

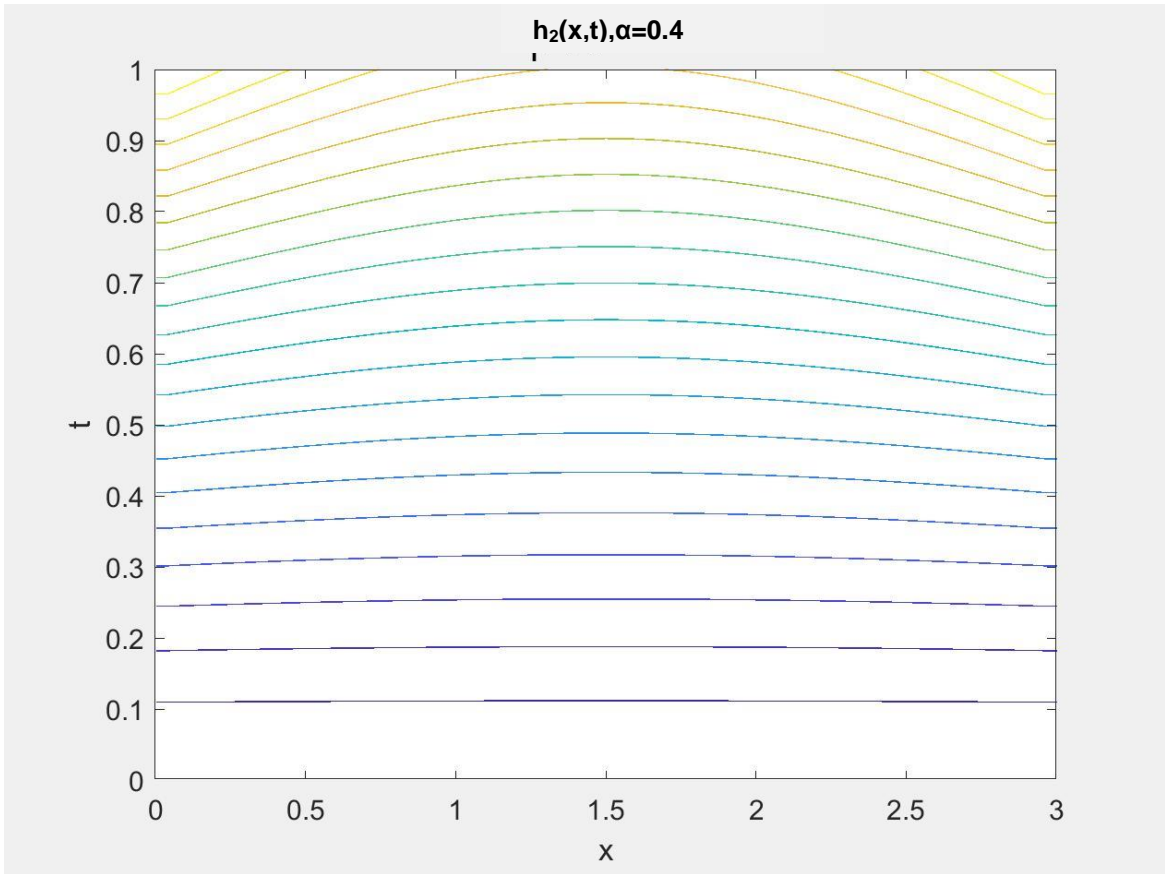


Figure 5.25: Contour plot representing h_2 in fractional order 0.4.

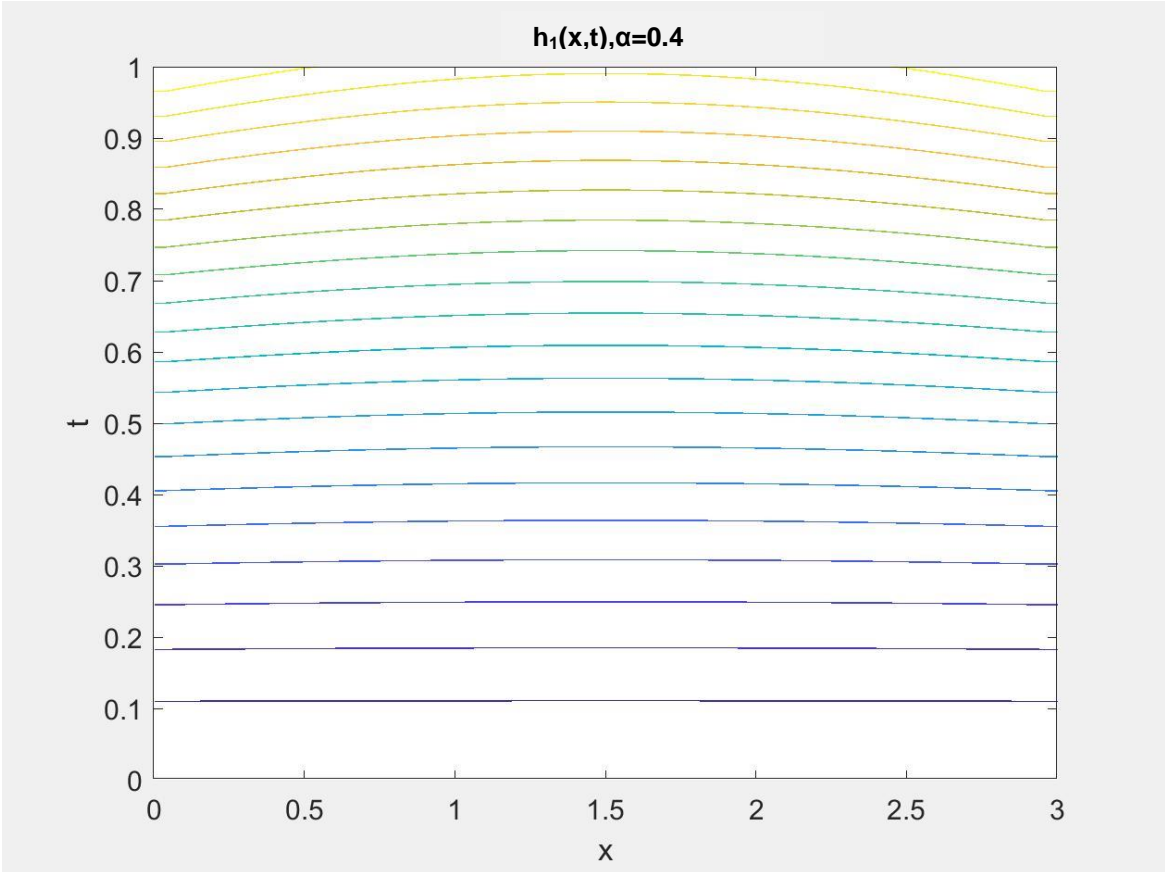


Figure 5.26: Contour plot representing h_1 in fractional order 0.4.

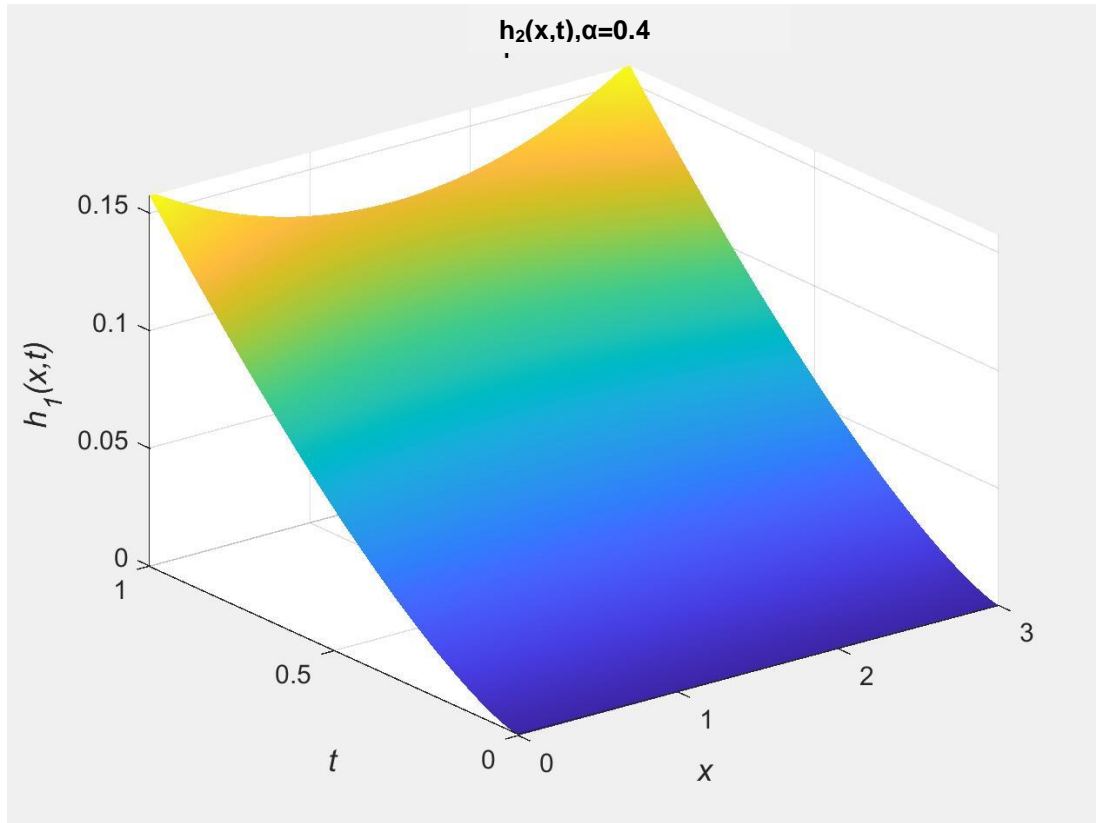


Figure 5.27: 3D plot representing h_2 in fractional order 0.4.

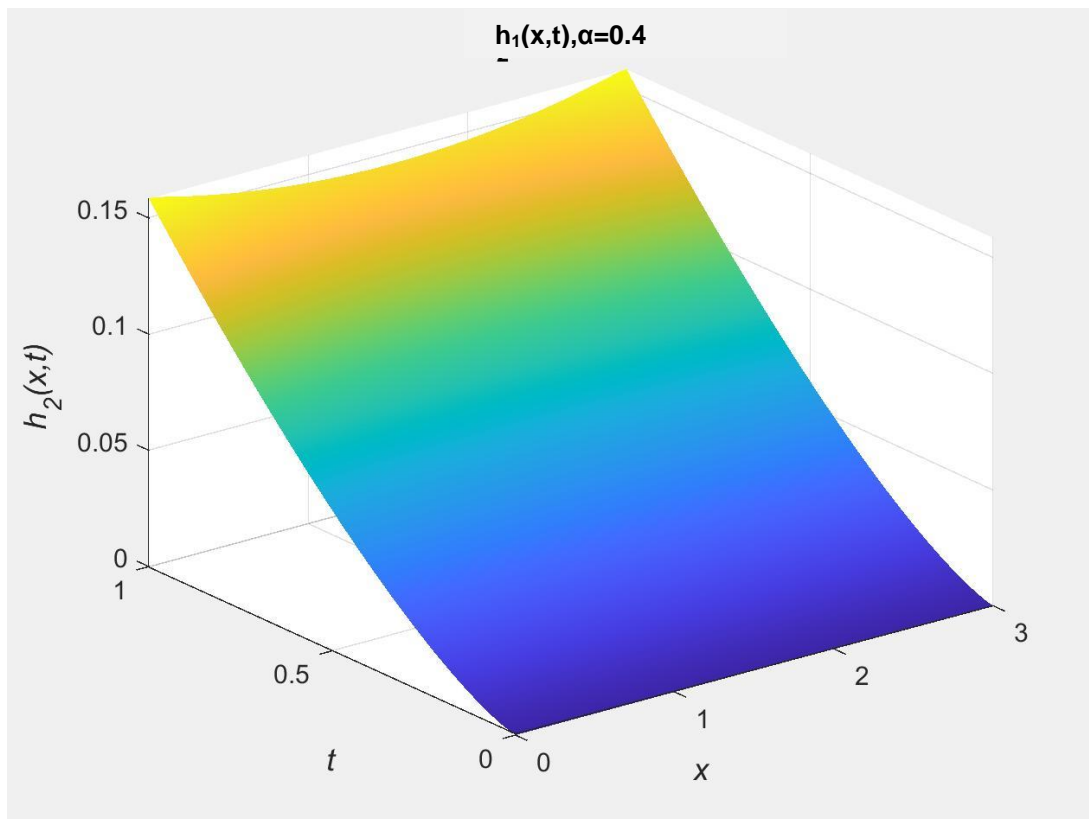


Figure 5.28: 3D plot representing h_1 in fractional order 0.4.

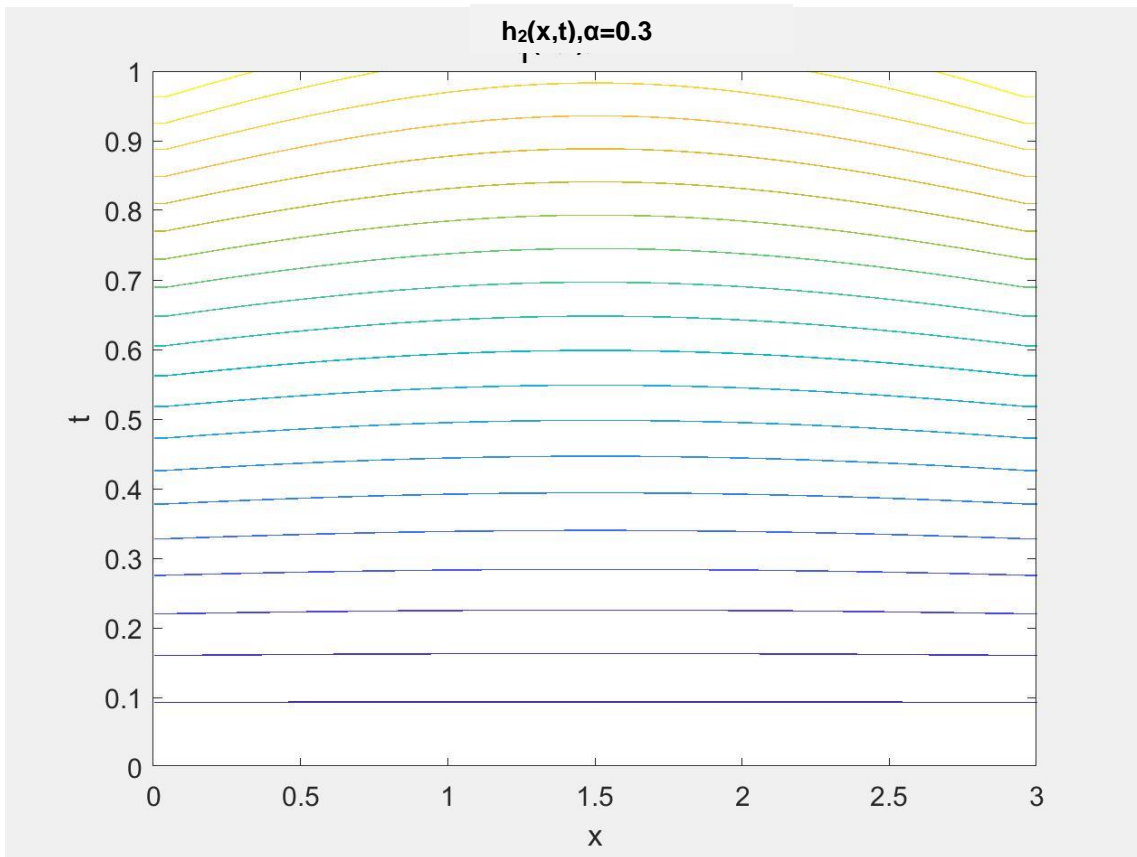


Figure 5.29: Contour plot representing h_2 in fractional order 0.3.

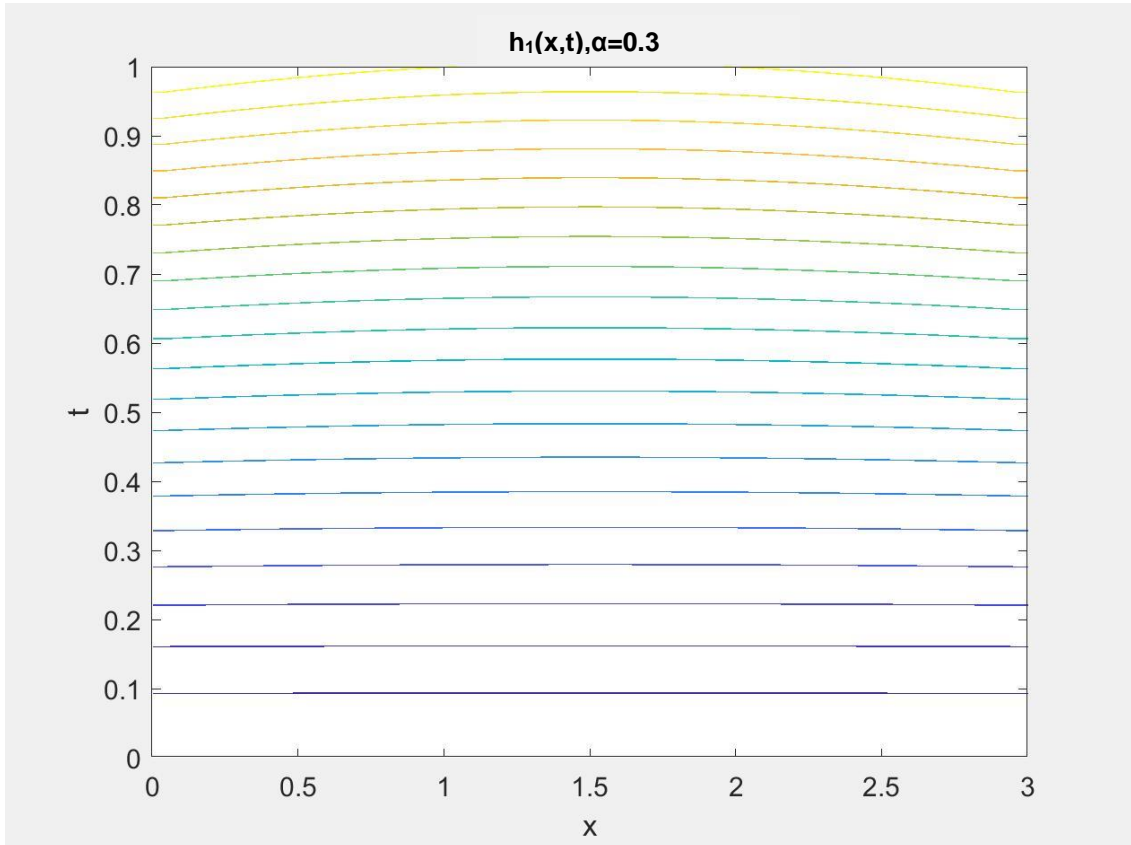


Figure 5.30: Contour plot representing h_1 in fractional order 0.3.

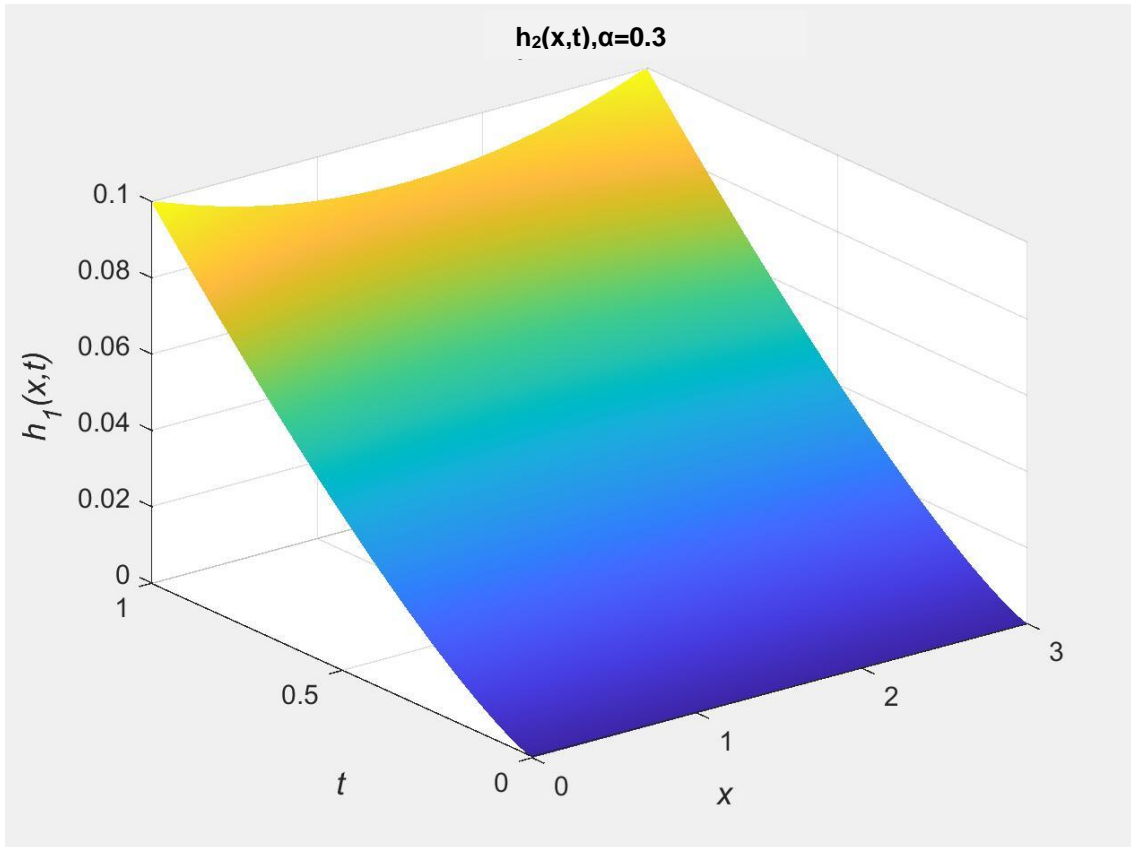


Figure 5.31: 3D plot representing h_2 in fractional order 0.3.

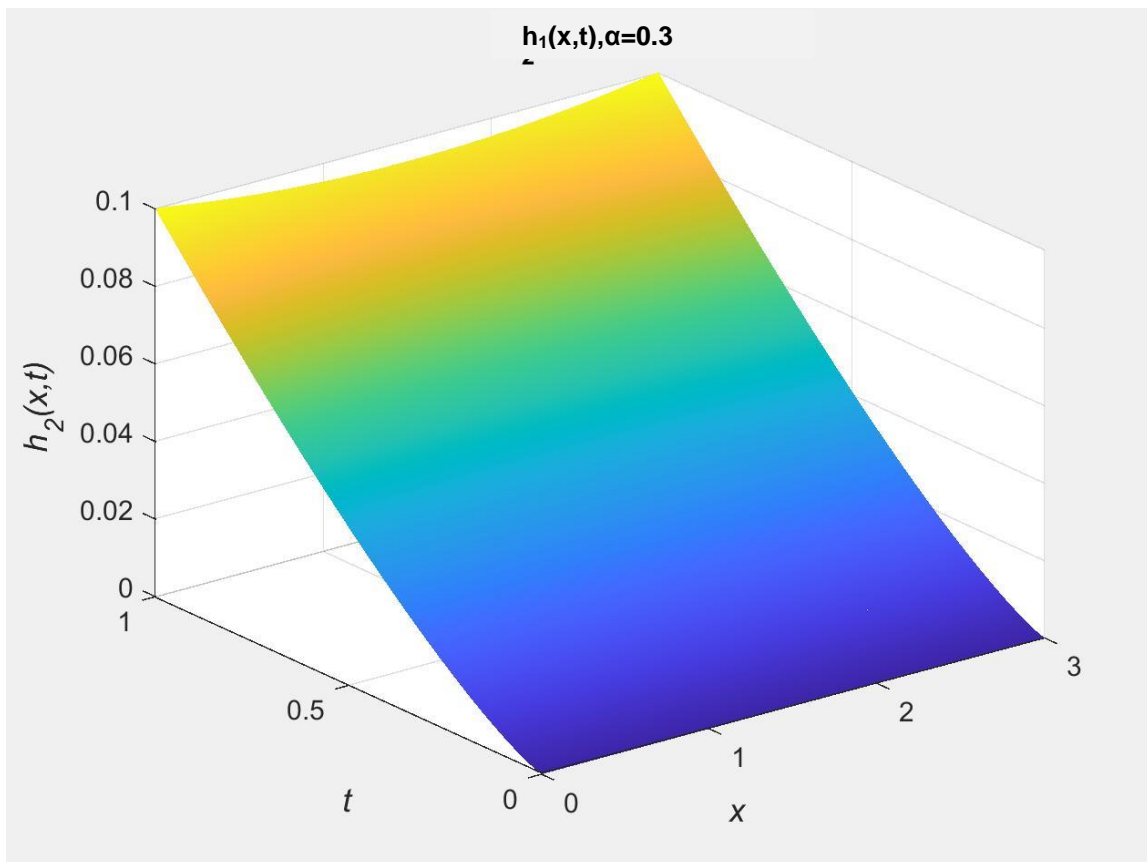


Figure 5.32: 3D plot representing h_1 in fractional order 0.3.

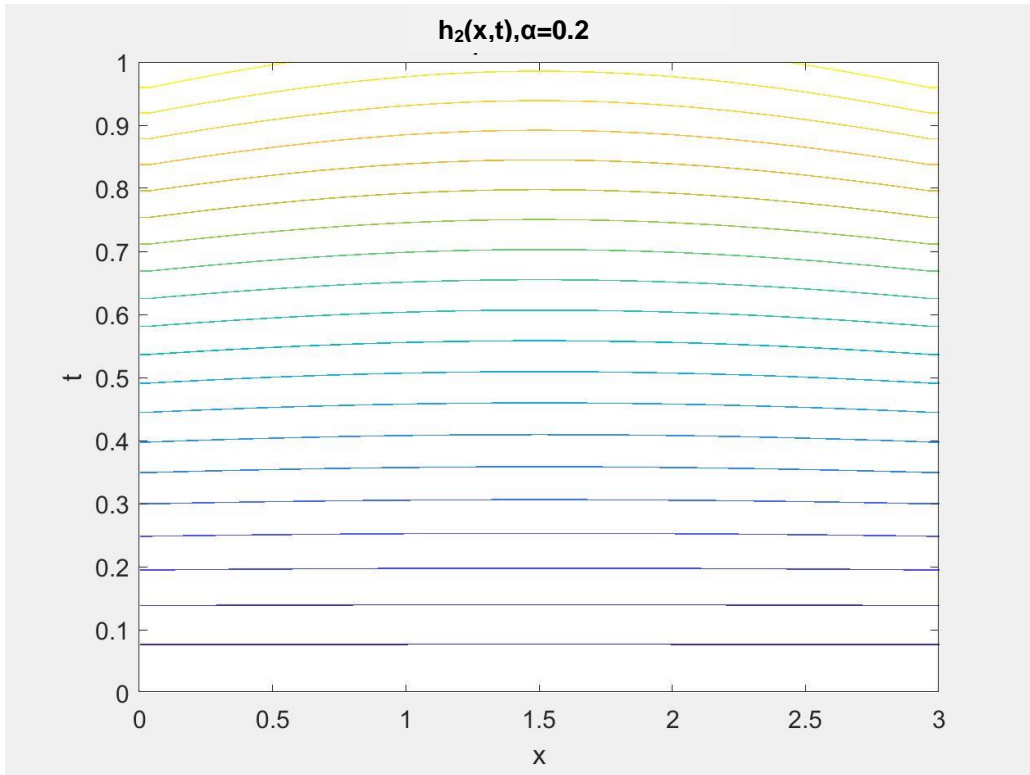


Figure 5.33: Contour plot representing h_2 in fractional order 0.2.

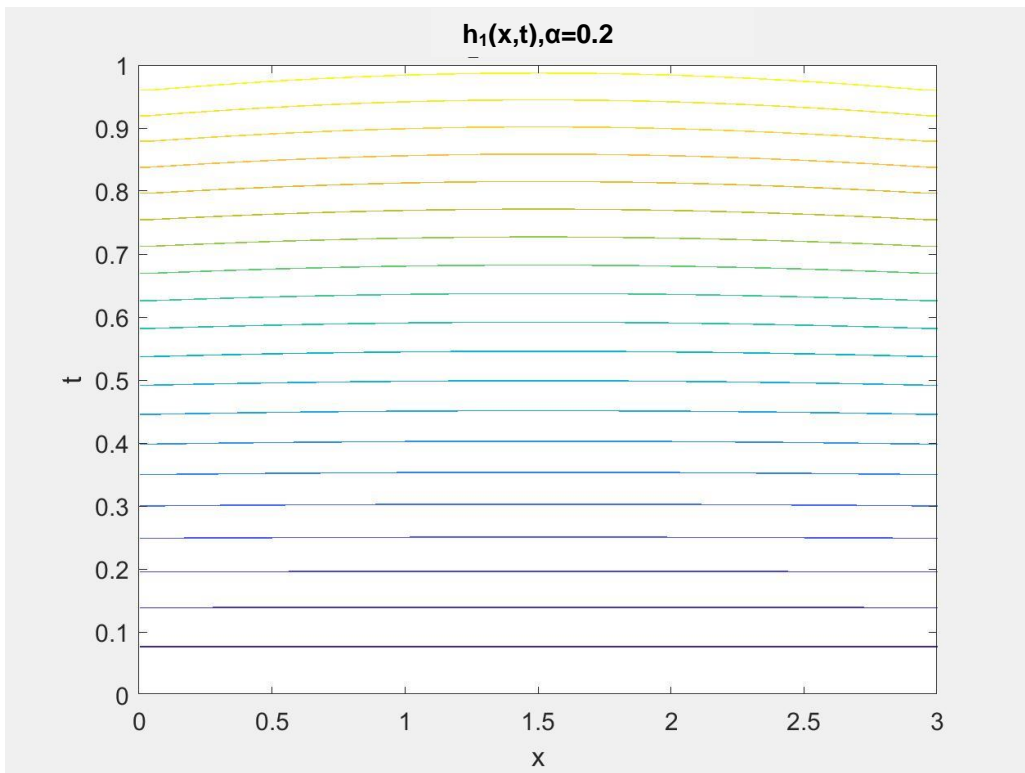


Figure 5.34: Contour plot representing h_1 in fractional order 0.2.

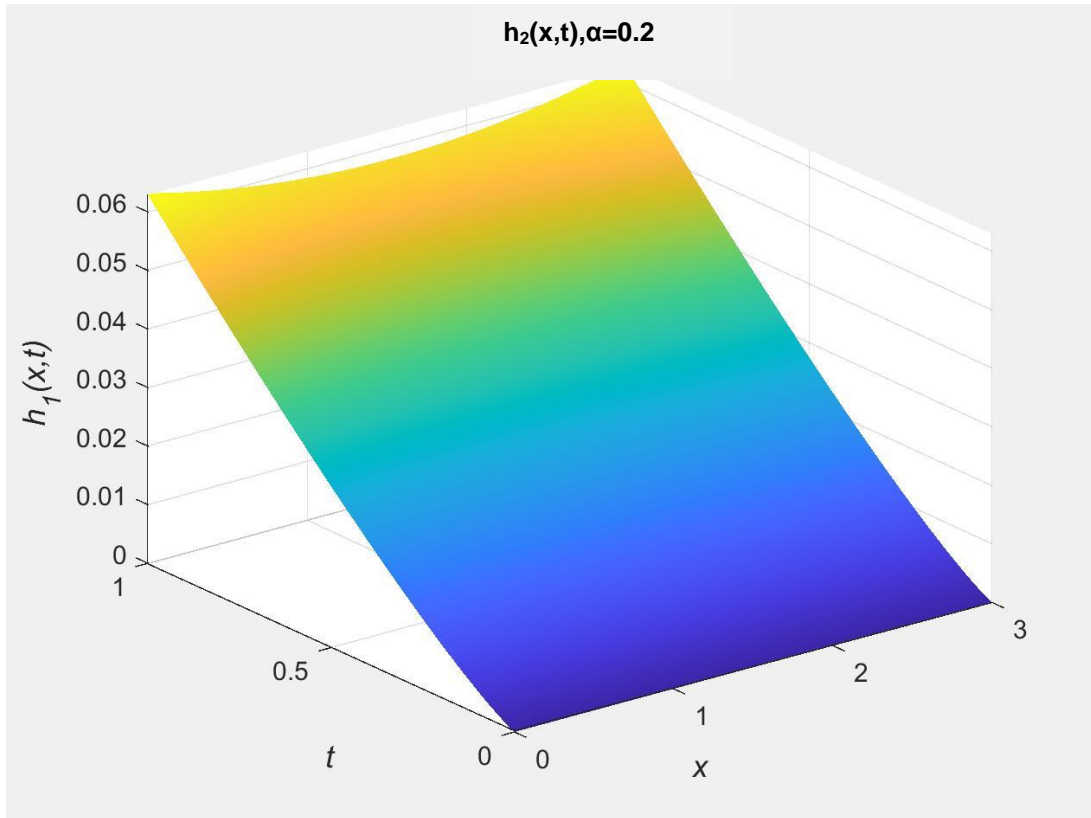


Figure 5.35: 3D plot representing h_2 in fractional order 0.2.

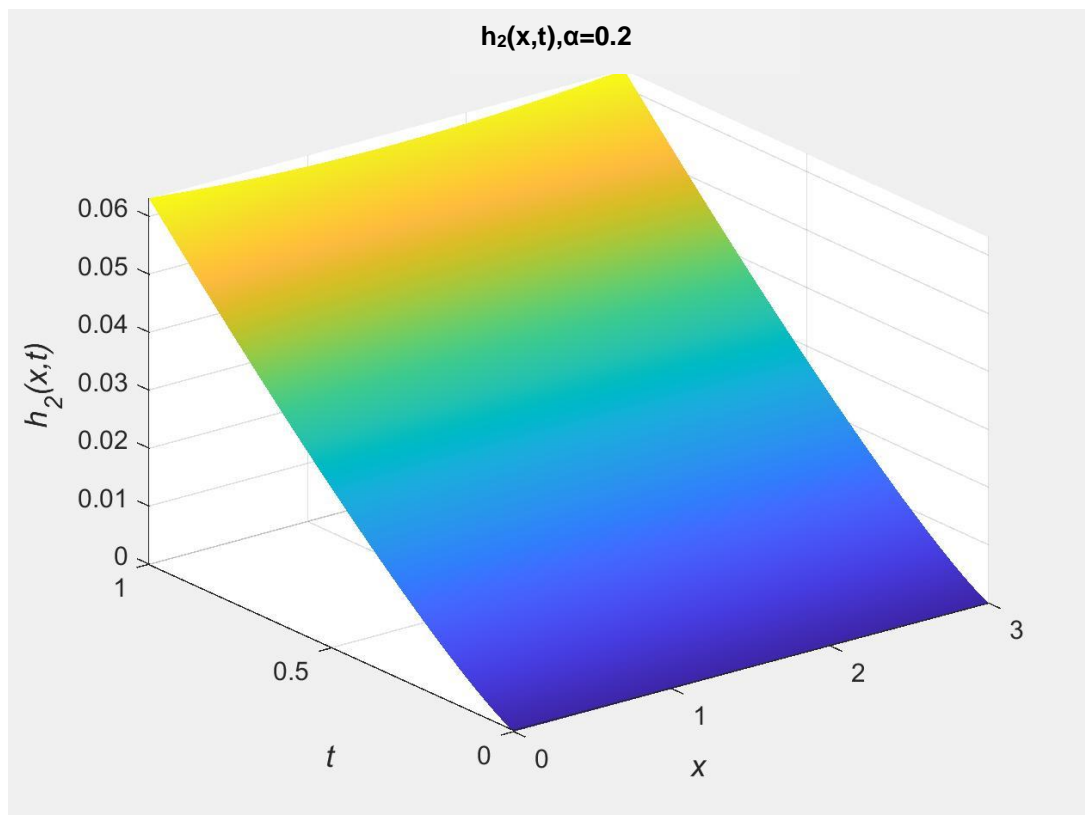


Figure 5.36: 3D plot representing h_1 in fractional order 0.2.

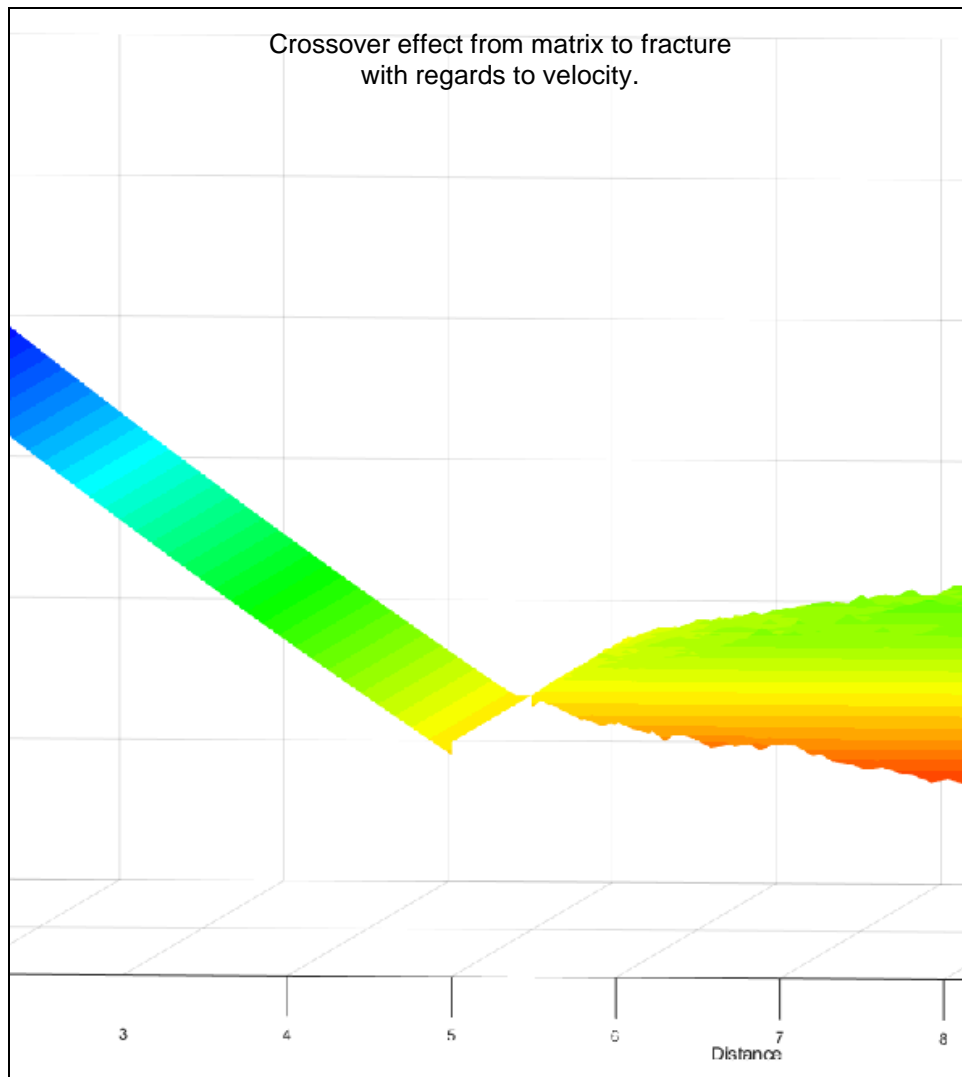


Figure 5.37: Diagram representing the crossover effect.

CHAPTER 6 : CONCLUSION

A dual porosity media has been considered in this study. In the last decades, many researchers have devoted attention to modelling the flow of water within dual porosity type of aquifers. Some authors suggested the use of two hydraulic head, where the first head accounts for the flow within the matrix rock and the second head the flow within the fracture. A system of partial differential equations has been suggested and applied in many scenarios with some limitations. The differential operators used in the past to model this situation are classical derivative and fractional derivatives. While the results obtained by these models were useful, one will still argue that the models cannot depict with great precision the crossover effect and the randomness of the geological formations. The two properties cannot be accounted when using classical derivative, even fractional derivatives. Very recently, piecewise differential and integral operators were suggested, these operators are defined in a given interval. The first part (head), account for a given process and the second one for another process. The passage from one process to another is called crossover. In addition, the concept of stochastic differential equations had been suggested to include into mathematical formulation of the effect of randomness. In this study, a system of partial differential equations depicting the flow of subsurface water within a dual porosity system was modified. To achieve this, the study considered fractional differential and integral operators with different kernels including power law, exponential decay and the generalized Mittag-Leffler functions. Further, the study included into the modified model randomness to obtain a piecewise fractional stochastic system of partial differential equations. The study presented an analysis to determine the condition under which the system has a unique system of solutions. Also, it made use of the numerical method based on the Newton polynomial interpolation to solve numerically the obtained systems of partial differential equations. The study used Matlab software to generate the figures in the case of power law.

CHAPTER 7 : BIBLIOGRAPHY

- Atangana, A. (2020, December 3). Prof. (S. Manundu, Interviewer)
- Atangana, A., & Goufo, E. F. (2020). Some misinterpretations and lack of understanding in differential operators with no singular kernels. *Open Physics*, 594-612.
- Atangana, A., & Seda, A. I. (2020). Extension of Atangana-Seda numerical method to partial differential equations with integer and non-integer order. *Alexandria Engineering Journal*, 2355-2370.
- Atangana, A., & Seda, A. I. (2020). New concept in Calculus: Piecewise differential and integral. *Chaos Solution and Fractal*.
- Babcock, M. (2004). *Michigan Tech*. Retrieved from Techa Live: Michigan Environmental Education Curriculum
- Batogna, G. R., & Atangana, A. (2019). Generalised class of Time Fractional Black Scholes equation and numerical analysis. *Discrete & Continuous Dynamical Systems* .
- Cordero, J. R., Eleazar, M. S., & Deane, R. (2019). Integrated discrete fracture and dual porosity-Dual permeability models for fluid flow in deformable fractured media. *Journal of Petroleum Science and Engineering*, 175, 644-653.
- Elsowrth, D. (2015, June 11). *You Tube*. Retrieved from You Tube: https://www.youtube.com/watch?v=Dm4nDj_1j7Q&list=LL&index=6&t=1001s&ab_channel=DerekElsworth
- Frappier, C. (2008). A repeated Leibniz integral rule. *International Journal of Pure and Applied Mathematics*, 151.
- Freeze, A., & Cherry, J. (1979). *Groundwater*, 2.
- Gómez-Aguilar, J. F. (2017). Irving–Mullineux oscillator via fractional derivatives with Mittag-Leffler kernel. *Chaos, Solitons & Fractals*, 179-186.

- Knaust, D., Dorador, J., & Rodrigues-Tovar, F. (2020). Burrowed matrix powering dual porosity systems—A case study from the Maastrichtian chalk of the Gullfaks Field, Norwegian North Sea. *Marine and Petroleum Geology*, 113.
- Qingdong, Z., & Jun, Y. (2015). Numerical Simulation of Fluid-Solid Coupling in Fractured Porous Media with Discrete Fracture Model and Extended Finite Element Method. *Computation*.
- Skinner, B. J., Stephen, P. C., Jeffrey, J. P., & Harold, L. L. (2004). *Dynamic Earth: An introduction to physical geology*.
- Tateishi, A. A., Ribeiro, H. V., & Lenzi, E. K. (2017). The role of fractional time-derivative operators on anomalous diffusion. *Frontiers in Physics*, 52.
- Taylor, H. M., & Karlin, S. (1998). *An Introduction to Stochastic Modeling*. California, USA: Academic Press Limited, 1-5.
- Toufik, M., & Atangana, A. (2017). New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models. *The European Physical Journal Plus*, 1-16.
- Warren, J. E., & Root, P. J. (1963). The Behavior of Naturally Fractured Reservoirs. *Society of Petroleum Engineers Journal*, 245-255.