

Analysis of Option Pricing within the Scope of Fractional Calculus.

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Declaration

I, Rodrigue Gnitchogna Batogna, declare that the thesis hereby submitted by me for the degree of Doctor of Philosophy at the University of the Free State (Department of mathematics and Applied Mathematics), is my own independent work and has not previously been submitted by me at any other institution.

I further declare that all sources cited or quoted are indicated and acknowledged by means of a list of references.

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List of Publications

Peer-Reviewed Journal Articles

1. R. B. Gnitchogna, A. Atangana, Comparison of two iteration methods for solving nonlinear fractional partial differential equations, *Int. J. Math. Models Methods Appl. Sci.* vol. 9 (2015) pp. 105–113.
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2. R. B. Gnitchogna, A. Atangana, Laplace Transform Adam-Bashforth Method for fractional P.D.E with non local and non singular kernel of the Atangana-Baleanu type, (2017)

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1. Numerical solution of a Time Fractional Black Scholes Equation with Caputo-Fabrizio operator, for European option.
2. Numerical solution of American time fractional Black Scholes Equation with Caputo-Fabrizio operator, for American option.

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Dedication

To my wonderful parents gone too soon, my mother *Helene Ngnintchongna*, my father *Gabriel Ngnintchogna*, where you are, I pray God that you see and find in this effort an appreciation of the sacrifices you both made for your children. I was privileged by birth, to be your son. If anything could ever honour in any manner the gifts of yourselves, the lives you lived and what you thought us, I pray to God that some of it is found in this achievement.

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List of Abbreviations

AB: Atangana-Baleanu.

ABC: Atangana-Baleanu in the sense of Caputo.

ABR: Atangana-Baleanu the sense of Riemann.

ATM: At The Money.

BS: Black-Scholes.

BSM: Black-Scholes-Merton.

CF: Caputo-Fabrizio.

CGMY: Carr-Geman-Madan-Yor.

Conv.Order: Convergence Order.

DO: Down and Out.

FMLS: Finite Moment Log Stable.

FPDE: Fractional Partial Differential Equation.

FPDEs: Fractional Partial Differential Equations.

ITM: In The Money.

KoBoL (CGMY): Koponen, Bouchaud&(Potters or Matacz), Boyarchenko&Levendorski.

ODE: Ordinary Differential Equation.

OUT: Out of The Money.

Max-error: Maximum error.

PDE: Partial Differential Equation.

PDEs: Partial Differential Equations.

RL: Riemann Liouville fractional operator.

TFBS: Time Fractional Black Scholes.

TFBSE: Time Fractional Black Scholes Equation.

TFBSEs: Time Fractional Black Scholes Equations.

UO: Up and Out.

Abstract

This work opens new and promising avenues of investigation in pricing mechanisms of financial derivatives. A notorious problem when pricing an option with the Black Scholes model is the poor long-term prediction, and the failure to capture the large jumps that often occur over small time intervals. This is mostly due to the fact that in his classic version, the Black Scholes model assumption for the change in price of the underlying asset is that prices are subjected to a Brownian motion type of process. As Gaussian Markovian processes proved incapable of satisfactorily give account of those occurrences, various ways of incorporating memory as well as models with jumps were considered as a remediation. Based on the fact that diffusion equations with fractional derivatives have been efficient in describing some very complex anomalous diffusion systems, fractional operators were introduced in mathematical finance. The rationale of our exploratory analysis is to exploit memory properties, non-locality, non-singularity, ‘globalness’, of fractional differentiation operators, to develop and analyse a new class of Time Fractional Black Scholes Equations (TFBSEs). Some recent fractional operators like, the Caputo-Fabrizio, the Atangana-Baleanu in the sense of Riemann and the Atangana-Baleanu in the sense of Caputo, show crossover properties and behaviours of some basic measures and indicators. Some related statistics that are not scale invariant, but crossing over from ordinary to sub-diffusion properties, with waiting time distributions, moving from Gaussian to non-Gaussian distributions, etc.

In the first part of this PhD, we generalise a double barrier knock out Black Scholes diffusion equation to five Fractional Partial Differential Equations (FPDEs) that we will analyse. The fractional differential operator is successively defined in the sense of Caputo, Riemann Liouville, Caputo-Fabrizio, Atangana-Baleanu in the Riemann sense and Atangana-Baleanu in the sense of Caputo. To the best of our knowledge no single appearance of any of the last three equations can be found in the literature. With the given boundary conditions, we establish the existence and uniqueness of solutions to the five equations. We develop six new numerical scheme solutions to the TFBSEs on one side, five semi-analytical solutions to our new TFBSEs using Laplace transform and Sumudu transform, on the other side. We assess the convergence of the numerical scheme solutions derived with Caputo and Riemann Liouville fractional derivative operator, to compare with the very scarce similar literature. With the obtained numerical scheme solution from the

Caputo-Fabrizio TFBSE, we proceed to price a double barrier knock out call option with specific parameter values, and look at the price behaviours if they seem to reflect some of the properties we ambition to capture.

Additionally, due to the fact with FPDEs there is usually a very cumbersome and tricky to handle summation term that appears in the numerical scheme solutions, making the stability analysis a considerable challenge, we develop a complete novel method to tackle Partial Differential Equations (PDEs) and FPDEs. The new method causes the incriminated summation term to disappear, when it is used with some fractional differential operators. The stability and error analysis of the new method are also presented. The method is conceived from a skilful combination of higher order accuracy Adam-Bashforth method and Laplace transform. To illustrate the potential of the method, we present some general applications on PDE and FPDEs. We then use the method to derive a numerical scheme solution the TFBSE with ABC fractional derivative.

In the last part of the thesis we comment the results we obtained, conclude our analysis and share our outlook on fractional operators and Black Scholes models in option pricing.

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1. Introduction

A financial derivative is an instrument whose value depends on the value of some other traded item, that of an other financial entity or variable, usually called the underlying asset. The quest of rigorous methods to price financial derivatives is permanent. As Options remain one of the most popular financial derivatives, the problem of option pricing has been at the core questions of Mathematical Finance. Louis Bachelier in his PhD's work in 1900 "Theorie de la Speculation" [1], is credited with the very first attempt to price an option. He simply described the price of an option as following a Brownian motion. Further work by Paul Samuelson [2] showed that geometric Brownian motion was giving better approximations than Bachelier's preliminary Brownian motion. That historical landscape led to the seminal work by Fisher Black, Myron Scholes and Robert Merton [3, 4]. Black-Scholes-Merton models provided the basis on which almost all the work done on the question has evolved. These models are unanimously considered as acceptably accurate approximations. However, well-known limitations of Black-Scholes Models include the failure to capture substantial variations in financial markets over small time steps Peter Carr [5]. In a continuous effort to address those models' shortcomings, fractional Black-Scholes models have come to the forth, building on the early introduction of fractional Brownian in 1940 by Kolmogorov [6], and the representation of Mandelbrot and Van Ness [7] in 1968. A Geometric Brownian Motion used in the classic Black-Scholes-Merton model is replaced by a fractional Brownian to capture the property of long-range dependence in financial markets [8]. Also in a bid to capture changes not incorporated by Gaussian models, several models were proposed on the assumption that the dynamics of equity prices follow Jump-diffusion processes or infinite activity Levy processes. Noticeable improvements were noted. Among some of the most popular financial models are KoBoL processes [9], Carr Geman Madan Yor CGMY processes [10], and Finite Moment Log Stable process FMLS processes [5]. In [11] A. Carpinteri and F. Mainardi, use differential equations involving fractional derivatives for the study of fractal geometry and fractal dynamics. Similarities of fractal geometry, fractal dynamics in general, with dynamics of financial modelling, stochastic processes and models, prompted the introduction of fractional derivatives and integrals in financial modelling and theory [12, 13]. W. Wyss [12] used a time fractional Black-Scholes model to price a European Call Option. The work by A. Carlea and D. del-castillo-Negrete [14] showed that, for

Black-Scholes models obtained on the assumption that the dynamics of equity price, follow Jump-diffusion processes or infinite activity levy processes, the price of financial derivatives satisfies a space Fractional Partial Differential Equations. They derived three popular space fractional derivative Black Scholes Equations. G. Jumarie used Taylor's expansion formula with fractional order derivative to achieve a time and space fractional Black Scholes model [15, 16].

Based on the connection of the fractal structure and the diffusion process of an option Li [17] obtained a time fractional Black-Scholes-Merton Equation. Hsuan-Ku Liu and Jui-Jane Chang [18] investigated the valuation of European option, with transactions costs under the fractional Black-Scholes model; the model is obtained when using a fractional Brownian motion [2]. Generally we should take note of the fact that Fractional Black-Scholes Equations arise in two categories, space and/or time fractional. Based on the assumptions that the underlying asset is following a fractional SDE (stochastic differential equation), or it is obeying a fractal transmission system for changes in the option price, whereas the underlying asset is still described by a geometric Brownian motion. The first results in both space and time fractional derivatives Liang et al. [19], when the second only has a time fractional derivative. However the time fractional Black-Scholes equation here differs from that of Jumarie G. [15, 16, 20], with a time-dependent volatility. The later is obtained from defining the stock exchange fractional dynamics as fractional exponential growths subjected to Gaussian white noise.

This research aims at broadening the spectrum of the proposed time fractional Black-Scholes-Merton models in the literature so far. In this work we will define and analyse a new class of Time-Fractional Black-Scholes Equations. We will extensively make use of the range of new-and never before used in mathematical finance-time fractional derivatives, enriching the conceptual framework offered by instruments of Fractional Calculus. There is no financial institution, financial markets, financial instruments that can operate without clear mathematical models, which are built on tools developed by researchers. Just like in other domains of knowledge with applied branches, the practice of finance has been immensely influenced by innovations. These innovations are the vitality's pulse of scientific research and breakthroughs. Discrepancies between actual observations and theoretical expectations are the fuel of scientific research. Quite often, uncertainties analysis and approximations are incorporated in many forms, to explain mismatches between actual obtained results and anticipated ones. However many mathematical models

still fall short of describing accurately the observed reality, even after proper model calibration and uncertainty quantification have been taken into account. This is simply because the instruments and tools of investigation that were used might not necessary be the most appropriate. It is possible that fundamental concepts or ideas underlying our model, for example the concept of differentiation is deficient. It could be that a slight twist of perception will reveal results never observed before. To broaden the spectrum of Black Scholes Merton models in this investigation, we will consider solve and analyse Time Fractional Black-Scholes models with Riemann-Liouville derivative, Caputo derivative, Atangana-Baleanu derivative with non-local and non-singular kernel [21], the later in both Riemann and Caputo sense.

The overall objective of this work is to improve and expand our understanding of Fractional Black Scholes Equations, and show that their potential in responding to the challenges we face in traditional option pricing theory with variations of models based on standard Black Scholes Merton can be fully, and satisfactorily addressed. The thesis primarily focuses on developing numerical solutions, and innovative numerical methods.

The work is organized as follows: in chapters two and three, we deal with some relevant literature pertaining to the subject of our study. We present a brief history of fractional differentiation and main definitions of fractional derivatives in chapter two. In chapter three we give an extensive literature review of option pricing theory from Black Scholes models to time Fractional Black Scholes equations. In chapters four and five respectively, we present the derivation of a Time Fractional Black Scholes Equation, from that of its standard version, and we formulated our new Time Fractional Black Scholes Equations, we discuss the existence and uniqueness of their respective solutions. In chapters six and seven respectively, we present semi analytical solution using Laplace transform and/or Sumudu transform, and numerical solutions of our new TFBSEs. In chapter eight, in order to circumvent some of the challenges we had in the handling of our numerical solutions, we successfully develop a total novel numerical method to solve PDE and FPDE, and show some applications on PDEs and FPDEs. Chapter nine gives our conclusion perspectives and outlook.

2. Brief History of Fractional Derivatives

Fractional derivatives made a relatively recent irruption in the modelling of real world problems. A fractional differential equation is a generalization of classical differential equations of integer order. Fractional derivatives and integrals have been successfully applied in problems in engineering, including fluid and Continuum Mechanics [22], anomalous diffusion problems including super-diffusion and non-Gaussian diffusion [23-25]. The non-locality, of fractional derivatives overall provide very powerful tools for the description of memory [5, 6, 26]. Recently, new and exciting avenues of investigations were opened with the advent of fractional derivatives with non-local and non-singular kernel [21]. The progress in theories of Fractional Calculus and applications in various sciences are now playing a major role in understanding mechanisms of complicated nonlinear physical phenomena. We believe these innovations can shed a new light on how understanding of option pricing via Time Fractional Black Scholes Equation. With the development of analytical and numerical techniques to solve fractional differential equations [27-40], fractional differential equations are widely used to describe various complex phenomena in fluid flow, signal processing, control theory, systems identification, finance and other areas. We present here definitions of some of the most used fractional order derivatives.

2.1. Definitions of some fractional operators

1. The Caputo derivative of order α is defined by:

$${}_0^C\mathcal{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, \quad n-1 < \alpha \leq n. \quad (2.1)$$

2. The Riemann-Liouville derivative of order α is given by:

$${}_0^{RL}\mathcal{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt. \quad (2.2)$$

3. Guy Jumarie proposed a simple alternative definition to the Riemann-Liouville derivative:

$$\mathcal{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t) - f(0)\} dt. \quad (2.3)$$

4. The Weyl fractional derivative of order α is defined by:

$$\mathcal{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^{+\infty} (x-t)^{n-\alpha-1} f(t) dt. \quad (2.4)$$

5. The Erdelyi-Kober fractional derivative is defined by:

$$\mathcal{D}_{0,\sigma,\eta}^\alpha(f(x)) = x^{-\eta\sigma} \left(\frac{1}{\sigma x^{\sigma-1}} \frac{d}{dx} \right)^n x^{\sigma(n+\eta)} I_{0,\sigma,(\eta+\sigma)}^{n-\alpha}(f(x)), \sigma > 0, \quad (2.5)$$

with

$$I_{0,\sigma,(\eta+\sigma)}^{n-\alpha}(f(x)) = \frac{\sigma x^{-\sigma(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \frac{t^{\sigma\eta+\sigma-1} f(t)}{(t^\sigma - x^\sigma)^{1-\alpha}} dt.$$

6. The Hadamard fractional derivative is defined by:

$$\begin{aligned} \mathcal{D}_0^\alpha(f(x)) = & -\frac{1}{2 \cos\left(\frac{\alpha\pi}{2}\right)} \left\{ \frac{1}{\Gamma(\alpha)} \left(\frac{d}{dx}\right)^m \left(\int_{-\infty}^x (x-t)^{m-\alpha-1} f(t) dt \right. \right. \\ & \left. \left. + \int_x^{+\infty} (t-x)^{m-\alpha-1} f(t) dt \right) \right\}. \end{aligned} \quad (2.6)$$

7. The Grünwald-Letnikov fractional derivative is defined by:

$$\mathcal{D}^\alpha f(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{0 \leq m < +\infty} (-1)^m \binom{\alpha}{m} f(x + (\alpha - m)h). \quad (2.7)$$

8. The Caputo-Fabrizio fractional derivative is defined by:

$${}^{CF}_a \mathcal{D}_t^\alpha (f(t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(x) \exp \left[-\alpha \frac{t-x}{1-\alpha} \right] dx, \quad (2.8)$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

If the function does not belong to $H^1(a, b)$, then the derivative can be reformulated as

$$\mathcal{D}_t^\alpha (f(t)) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(x)) \exp \left[-\alpha \frac{t-x}{1-\alpha} \right] dx. \quad (2.9)$$

For our applications except where a different mention is made the function $M(\alpha)$ is given as:

$$M(\alpha) = \frac{2}{2-\alpha}, 0 \leq \alpha \leq 1.$$

9. The Atangana-Baleanu derivative in the sense of Riemann Liouville (ABR fractional derivative) is defined as:

$${}^{ABR}_a \mathcal{D}_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx, \alpha \in [0,1], \quad (2.10)$$

where $B(\alpha)$ a normalization function such that $B(0) = B(1) = 1$.

For our applications, except where a different mention is made the function $B(\alpha)$ is given as:

$$B(\alpha) = 1 + \frac{\alpha(1-\alpha)}{1+\alpha} = \frac{1+2\alpha-\alpha^2}{1+\alpha}, 0 \leq \alpha \leq 1.$$

10. The Atangana-Baleanu derivative in the sense of Caputo (ABC fractional derivative) is defined as:

$${}^{ABC}_a \mathcal{D}_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx, \alpha \in [0,1], \quad (2.11)$$

where $B(\alpha)$ has the same properties as in Caputo and Fabrizio case.

For our applications, except where a different mention is made, the function $B(\alpha)$ is given as:

$$B(\alpha) = 1 + \frac{\alpha(1-\alpha)}{1+\alpha} = \frac{1+2\alpha-\alpha^2}{1+\alpha}, 0 \leq \alpha \leq 1.$$

2.2. Brief description and main difference between the fractional operators

Fractional differentiation operators can be roughly grouped into three main categories, the distinction being made on the type of kernel of the fractional derivative. We can distinguish kernels based on the power law function. Of the definitions given above the derivatives from one to six, fall in that category: Caputo fractional derivative, Riemann-Liouville, Guy-Jumarie, Weyl, Erdelyi-Kober, and Hadamard. Then we have kernels based on the exponential function, that is number eight of our given definitions the Caputo-Fabrizio fractional derivative. Finally, kernels based on the Mittag-Leffler function: the Atangana-Baleanu fractional derivatives. Take note that for the first group, power law based fractional derivatives, both their range is restricted, and there exists possibilities of singularities in the kernels. The exponential based kernels, are non-singular but remain local, due to the local range of the exponential function. The Mittag-Leffler kernels, that is, the Atangana-Baleanu derivatives are the only non-local and non-singular fractional operator. As far as modelling applications are concerned fractional derivatives overall, improve on instruments of classical calculus [22]. However power law based fractional operators seem to have a noticeably limited scope of applications and are outperformed by their non-singular, and/or non-local counterparts. When it comes to describing complex processes, like for instance diffusion in inhomogeneous or heterogeneous medium, sub-diffusion or hyper-diffusion [21, 22, 26], power law fractional operators are less suitable than their non-singular/local counterparts. The Riemann-Liouville and Caputo derivatives, arguably the most commonly used fractional operators having non Gaussian probability distributions, will be incapable to describe some random processes, but will be suitable candidates for long tailed problems in financial risk theory for example. Stochastic simulations show that some statistical properties (like the mean squared displacements) of power law based derivatives are scale invariant, whereas the Caputo-Fabrizio fractional derivative is crossover via a steady state, and the Atangana-Baleanu fractional derivatives are remarkably crossover from Gaussian to non Gaussian processes without a steady state; with the change happening solely, on the range of the order of differentiation α . Interesting properties of fractional operators include the various transforms and their convolution properties. We will make use of them as we develop solutions to our equations.

3. Literature Review

It is essential for whichever trading activity to rightly price any product, good or services that is being provided. Rightly in this case refers to the value that can be attached to the product. The beliefs on the value of the product are the main factors, enticing an investor into buying a given stock at a certain price, or selling it for another one. In 1973, Fisher Black and Myron Scholes in their pioneering paper [3] formulated what is still seen as the corner stone of option pricing theory. From an equilibrium price principle, they derived what should be the price of an option, in order to eliminate opportunities of making a certain profit by simply constructing an adequate portfolio. An option is a security- a financial instrument or simply a contract-giving the right, but not the obligation to buy or sell an asset within a given period of time or at a given date, for a specified price. If the option can be exercised at any time within a given time period, that is, the buying or selling of the asset can happen at any time within a given period, we have an **American** style option. If the option can only be exercised at a prescribed date in the future, that is the buying or selling of the asset can only happen on a given date in the future, we have a **European** style option. The date in question is called the **expiration date** of the option. A **Call Option** gives the right but not the obligation to the owner to buy a given stock, at a specified price called the **strike price** or **exercise price**. A **Put Option** gives the right but not the obligation to the owner to sell an asset at a specified price, **strike price** or **exercise price**, within the **expiration date** for American option or at the **expiration date** for *European style option*. The stock or asset that is being sold is referred to as the underlying asset of the option. The owner of the option is generally referred to as the holder, and the first seller of the option is referred to as the writer. Writing an option creates the obligation for the writer to either sell (for a Call Option) or buy (for a Put Option), the underlying asset at the strike price in accordance with the expiration date, should the holder chooses to exercise the option. Simple Call and Put options are often referred to as Vanilla Options. Fischer Black and Myron Scholes fundamental paper of option pricing theory [3] focuses on European Vanilla Options, although the theory can be extended other contingent claim assets (These are simply assets whose values are dependent on other future uncertain event). We will review majors contributions to the question of option pricing, privileging European type option as the corpus of this research work restrict itself on this type of option. In obtaining the price of an option Black and Scholes made the following seven assumptions:

1. The short term interest rate is known and constant
2. The stock prices are continuous and follow a lognormal distribution
3. The stock does not pay dividend
4. The option is European and can only be exercised at the maturity date
5. There are no transactions costs
6. The underlined security is perfectly divisible
7. There are no penalties for short selling (short selling refers to the selling of a security that was borrowed by the seller for a fee.)

The studies that followed there were variations of the models, primarily grounded on studying the validity and impact of the assumptions made in deriving their Option price. Robert C. Merton [4, 41, 42] assessed the robustness of the results by relaxing the assumptions made in deriving the equilibrium prices. He established that no single assumption is vital to the obtained analytical results. The methodological approach and techniques remain valid even if the assumptions specifying stock and option are relaxed. Merton [41] also generalised the model to stochastic interest rates. He argued [42], that the solution in the considered case of continuous trading is the asymptotic limit of the solution obtained when we assume discrete trading. The effect of the tax regime on the solution, both capital gain and income taxes was studied by Ingersoll [43], in its application of the model for the valuation of dual-purpose funds. Thorp [44] analysed what are the effects of restricting the use short sales proceeds. Both Merton and Thorpe [4, 44] studied the modified version of the model in which the underlying stock pays dividends. Robert C. Merton [42], Cox and Ross [45] also considered the case in which the movements of stock prices are not continuous. Black and Scholes argue that their equilibrium price solution may be extended to the pricing of other contingent claim asset. A contingent claim asset can simply be seen as a derivative whose payout depends on the realization on some uncertain future event. For example, the equilibrium pricing technique could be used to value the equity of a levered firm. They argued that the purchaser of a Call is similar to the position of a stockholder, when the writer of a Call is equivalent in position to a bondholder. By paying the face value of the bonds to bondholders, stockholders can exercise their right to buy the firm. Merton [41], Galai and Masulis [46] apply the same model to assess the impact of risk on the valuation of corporate debts for the first, and to investigate the effects of a range of financial operations, mergers, acquisitions, scale

expansions and spin-offs, on the relative value of the debt and equity claim of a firm, as specified by Smith [47]. Further application of the model can be found in Black [48] for the valuation of commodity options, forward and future contracts.

Because of observed empirical deficiencies of Black-Scholes researchers developed approaches that were a little more than mere modifications the Black Scholes model. Those can be categorized in two major groups, simulation and non-simulation based approaches. Among the first group some of the most visible works include Levy [49], who obtained a closed form analytical solutions to approximate the value of European options, using the arithmetic mean of future foreign exchange rates. Erling D. Andersen and Damgaard included transaction costs, and assumed an environment with more than one risky security, to compute the reservation price of an option [50]. To obtain more accurate estimates of the volatility in the model, Muzzioli and Torricelli [51] introduced probability distributions. Reynaerts and Vanmaele [52] showed that unlike in the continuous Black Scholes model, the price of an option is not necessarily a strictly continuous function of the volatility. They performed a sensitivity analysis of the option price to the volatility, in a binary tree model. In 2004, Wu [53] applied fuzzy set theory to the Black Scholes formula. By computing continuous averages over the full lifetime of the option, Reynaerts et al [54] obtained new accurate lower and upper bounds for the price of a European style Asian option, using a discrete-time binary tree model. They proved that contrary to Chalasani et al model [55] whose price intervals do not always lie within the Black and Scholes intervals, their bounds converge to the equivalent Black-Scholes ones.

The idea of looking into tools and instruments of fractional calculus to address the shortfalls of classic Black-Scholes-Merton-Models appeared around 1999. The books by Oldham et al, Podlubny and Hilfer [22, 23, 24] showed the suitability of fractional derivatives to describe a wide range of problems in science and engineering, including fractal dynamics and anomalous diffusion problems. Their memory property their ‘globalness’ the fact that they are non local, and even the crossover properties of some fractional operator make capable of describing phenomena following both Gaussian and non Gaussian models [56]. Enrico Scalas et al [57] studied a tick-by-tick dynamics in financial markets where long time behaviour of waiting time probability density are described in terms of fractional differentiation operators. Several models aiming at incorporating changes that are not described by Gaussian models were introduced. On the

assumption that the dynamics of equity prices follow Jump-diffusion processes or infinite activity levy processes. Among some of the most popular financial models are KoBoL processes [9], Carr Geman Madan Yor CGMY processes [10], and Finite Moment Log Stable process FMLS processes [5]. In [11] A. Carpinteri and F. Mainardi, use differential equations involving fractional derivatives for the study of fractal geometry and fractal dynamics. Generally the classic Brownian motion is unreliable to describe some transport processes with diffusion rate compatible to long memory property. Researchers will replace in such cases, classic diffusion equation, by time fractional diffusion equations. Diego A. Murio [58] developed an implicit unconditionally stable method, to solve a one dimensional linear time fractional diffusion equation, formulated with Caputo's fractional derivative. W. Wyss [12] used a time fractional Black-Scholes model to price a European Call Option. The work by A. Cartea and D. del-castillo-Negrete [14] showed that, for Black-Scholes models obtained on the assumption that the dynamics of equity price, follow Jump-diffusion processes or infinite activity levy processes, the price of financial derivatives satisfies a space Fractional Partial Differential Equations. They derived three popular space fractional derivative Black Scholes Equations. G. Jumarie used Taylor's expansion formula with fractional order derivative to achieve a time and space fractional Black Scholes model [15, 16]. Wen Chen and Song Wang [59] proposed a power penalty method for fractional order partial derivative arising in the valuation of American options, when the underlying stock prices follow a geometric Levy process. Based on the connection of the fractal structure and the diffusion process of an option Li [17] obtained a time fractional Black-Scholes-Merton Equation. Generally we should take note of the fact that Fractional Black-Scholes Equations arise in two categories, space and/or time fractional. Based on the assumptions that the underlying asset is following a fractional SDE (stochastic differential equation), or it is obeying a fractal transmission system for changes in the option price, whereas the underlying asset is still described by a geometric Brownian motion. The first results in both space and time fractional derivatives Liang et al. [18], when the second only has a time fractional derivative. However the time fractional Black-Scholes equation here differs from that of Jumarie G. [15, 16, 19], with a time-dependent volatility. The later is obtained from defining the stock exchange fractional dynamics as fractional exponential growths subjected to Gaussian white noise. The following chronological table adapted from Shahbandarzadeh et al [60] summarizes some of the main studies on European options from 1973 until 2010.

Table 1: Chronological table of main studies of European call option

	Authors	Reference	Description
1	Black and Scholes	Black & Scholes, 1973	Basic principle of general equilibrium pricing of European options.
2	Robert C. Merton	R.C. Merton, 1973,	Generalised the model to stochastic interest rates, incorporated dividends.
3	Robert C. Merton	R. C. Merton, 1974	Valuation of corporate debts
4	John C. Cox, Stephen A. Ross	John C. Cox, Stephen A. Ross, 1976	Movements of stock prices are not continuous
5	Dan Galai, Ronald W. Masulis	Dan Galai, Ronald W. Masulis, 1976	Assess the impact of risk on the valuation of corporate debts for the first, and to investigate the effects of firm operations,
6	Garman-Kohlhagen model	Garman & Kohlhagen, 1983	Closed-form solution of a European currency options pricing model.
7	Levy	Levy, 1992	Analyses the sensitivity of pricing European options.
8	Rogers and Shi	Rogers & Shi, 1995	Determines lower and upper bounds for the price of a European-style Asian option.
9	Malz	Malz, 1996	Uses jump-diffusion model for estimating the realignment probabilities of option pricing
10	Chalasani et al	Chalasani, & Varikooty, 1998	Determines accurate lower and upper bounds for the price of a European-style Asian option.
11	Andersen and Damgaard	Andersen & Damgaard, 1999	Suggests an approach to compute the reservation price of an option in an economy with multiple risks.
12	Kaas et al.	Kaas, Dhaene, & Goovaerts, 2000	Uses random variables to determine bounds of call options.
13	W. Wyss	W. Wyss, 2000	Introduced a Time fractional Black Scholes Equations
14	Zmeskal	Zmeskal, 2001	Applies a new fuzzy stochastic model to valuing European call option.
15	Yoshida	Yoshida, 2003	Evaluates European options in uncertain environment based on randomness and fuzziness.
16	A. Cartea & Negrete	Á. Cartea & D. Negrete, 2007	Derived a space fractional Black Scholes Equations
17	Andreou et al.	Andreou, Charalambous, & Martzoukos, 2008	Combines artificial neural networks and parametric models in order to pricing European options.
18	W. Li	W. li, 2009	Provided numerical solution of fractional order equation in financial models
19	Xu et al.	Xu, Li, & Zhang, 2010	Proposes a fuzzy model based on Greek letters.

20	Cassidy, D et al	Cassidy, Hamp, & Ouyed, 2010	Presented a Gosset formula for pricing European options with a log Student's t-distribution.
21	H. Liu & Chang	H. Liu & JJ. Chang, 2013	Presented approximate solution for FBSE with transaction cost.

As mentioned earlier, an alternative main approach in option pricing is a simulation-based approach. The vast majority of simulation-based option pricing methods in the literature involves American style option. As American options constitute most of the existing applications and since they are not part of our investigative work, we will just mention here a few of the most preeminent results on simulation-based option pricing techniques. Phelim Boyle et al [61] discussed the use of Monte Carlo method for pricing securities, with emphasis on efficiency. Longstaff and Schwartz [62] simulated the price of American option through from Least square approach to estimate the conditional expected payoff. Broadie and Glasserman [63] used two estimators for confidence intervals, to develop an algorithm for the pricing of American style securities through simulations. Other interesting contributions include but are not limited to Peter Carr et al. [64], Maidanov [65], Rompolis [66], Klar and Jacobson [67].

4. Time Fractional Black-Scholes Equation

4.1 Preliminaries

4.1.1. Payoff Functions

Let us recall that a European call option gives to its holder the right, but not the obligation to purchase the underlying asset at a given strike price and at a given expiration date.

A European Put option gives its holder the right, but not the obligation to sell the underlying asset at a given strike price at the date of expiration.

An option is said to be in-the-money (ITM) if exercising it will lead to a positive cash flow for its holder.

An option is said to be at-the-money (ATM) if exercising it will lead to a zero cash flow for its holder.

An option is said to be out-of-the money (OTM) if exercising it will lead to a negative cash flow for its holder.

The payoff of a European option at the expiration date is determined by the price of the underlying stock. The payoff a European option represents its value of at the expiration time, as a function of the underlying stock price.

The profit of a European option at the expiration date is payoff of the option decreased by its discounted price value.

Let:

C denote a European call option price,

P denote a European put option price,

S denote the underlying stock price of the asset,

K denote the exercise or strike price,

T denote the expiration date.

The payoff at T of a European call option is: $\max(S - K, 0)$. A European call option is ITM, if the stock price is greater than the strike price that is $S > K$.

The payoff at T of a European put option is: $\max(K - S, 0)$. A European call option is ITM, if the stock price is less than the strike price that is $S < K$.

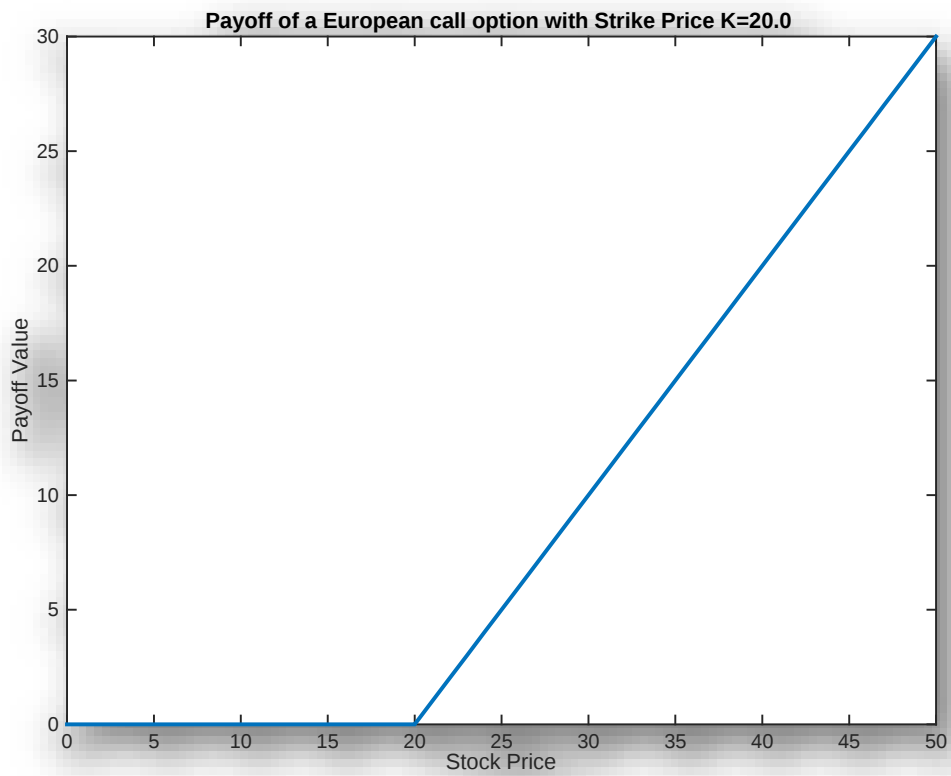


Figure 1: Payoff of a European Call Option

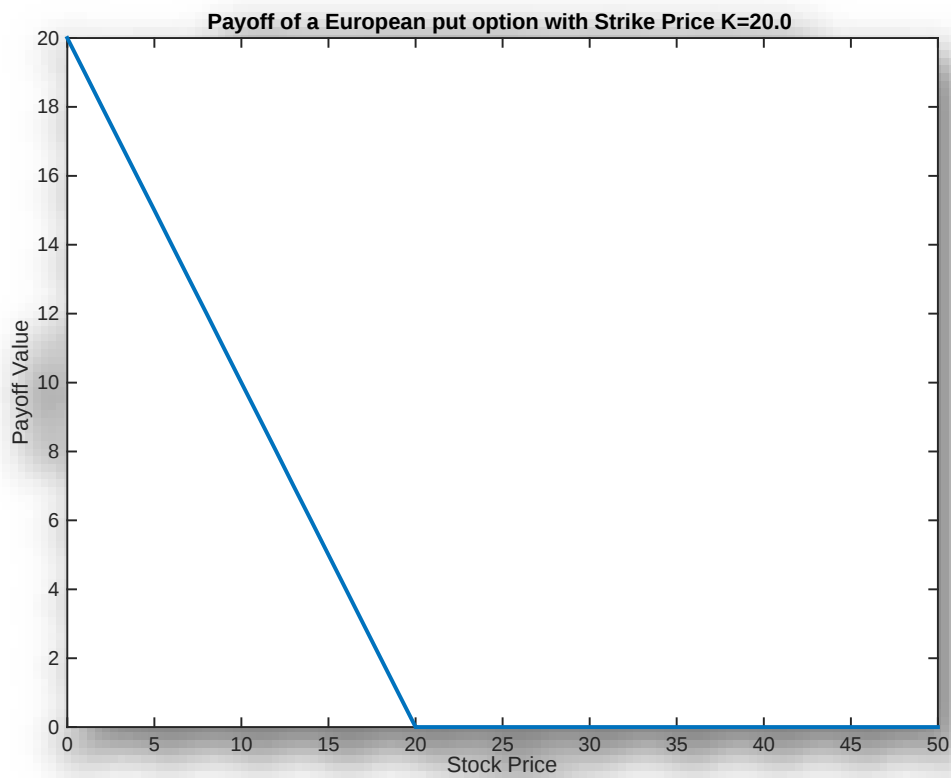


Figure 2: Payoff of a European Put Option

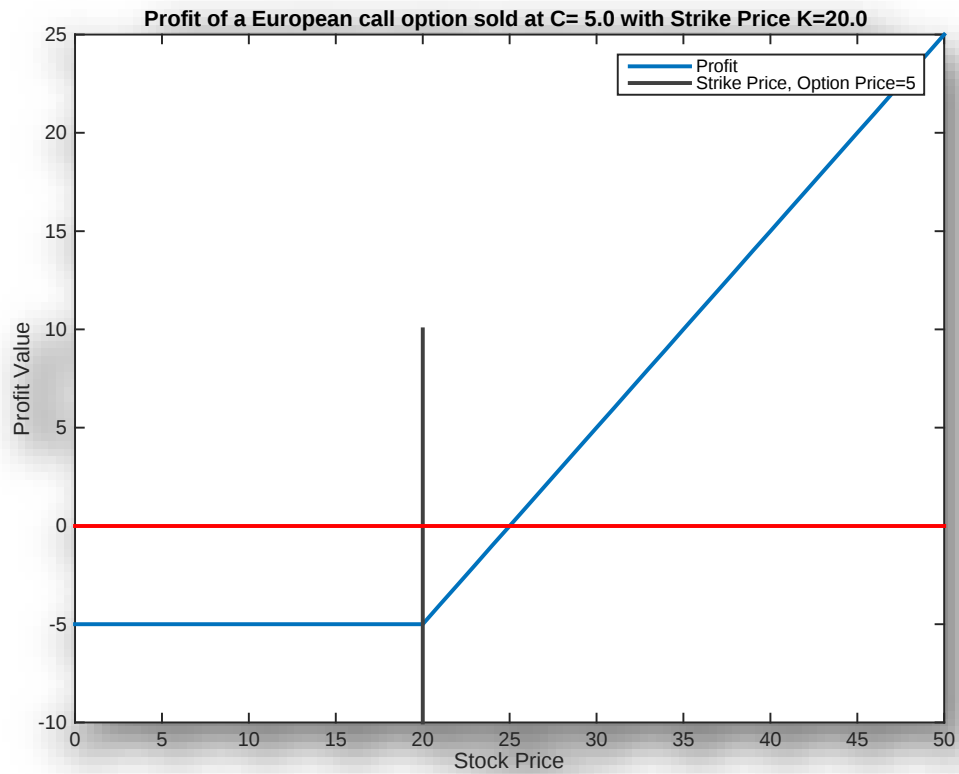


Figure 3: Profit of a European Call Option

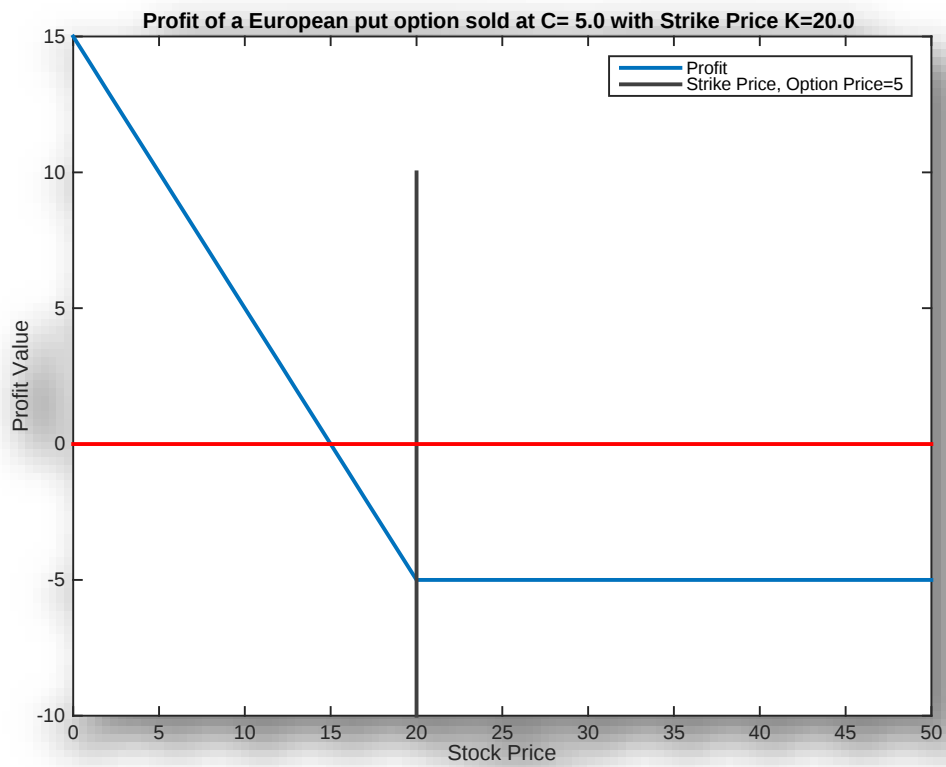


Figure 4: Profit of a European Put Option

As the equations that we will consider later on are all for double barriers option type, let us also briefly describe Barrier options. Barrier options are a case of what is generally called exotic options, which are options with a usually more complex structure in terms of calculations of their payoff. For standard options the payoff depends on the strike level, while for barrier options the payoff depends on both the strike and the barrier level. In addition to the features of standard vanilla options, a barrier option is characterized by a barrier level, and sometimes a cash rebate if the barrier level is crossed by the stock price during the lifetime of the option. The cash rebate is not always included with all types of barrier option. For the purpose of this brief depiction, we will assume it to be always zero. An up barrier is a price level above the current stock price, and a down barrier is a price level below the current stock price.

We distinguish two main types of barrier options *in options and out options*.

When the barrier level is crossed, an in barrier option payoff is still calculated like a standard option payoff. They are called **knock-in options**. The option needs to expire ITM, and the barrier level crossed for the option holder to receive his payoff. Similarly, **knock-out options** are out options and they will pay off if the option expires ITM and the barrier level is never crossed. Barrier options could be: *down-and-in, down-and-out, up-and-in, up-and-out*. The payoffs and variations of barrier options are given in the following table.

Table 2: Description of Barrier options and their Payoff.

Option	Type	Barrier		Payoff
		Location	Barrier crossed	Barrier not crossed
Call	<i>Down-and-In</i>	Below spot	Standard Call Payoff	Zero
	<i>Down-and-Out</i>	Below spot	Zero	Standard Call Payoff
	<i>Up-and-In</i>	Above Spot	Standard Call Payoff	Zero
	<i>Up-and-Out</i>	Above Spot	Zero	Standard Call Payoff
Put	<i>Down-and-In</i>	Below spot	Standard Put Payoff	Zero
	<i>Down-and-Out</i>	Below spot	Zero	Standard Put Payoff
	<i>Up-and-In</i>	Above Spot	Standard Put Payoff	Zero
	<i>Up-and-Out</i>	Above Spot	Zero	Standard Call Payoff

The equations we will be studying are that of double barrier **knock out** Call options, with a lower and an upper barrier. In 1973, Fisher Black, Myron Scholes and Robert C. Merton introduced the Black–Scholes-Merton model, [3, 4]. The model became broadly recognized as one that gives good approximations and the fact that, a closed-form analytical solution known as the Black Scholes formula could be easily derived, paved the way for Black Scholes models to be used as benchmark to other models. It provided at the same time, the much-needed scientific legitimation to option pricing and trading, and as a consequence led to an expansion of the market. Rephrasing words of Robert Merton himself, just as the model helped shape the markets, the markets in turn helped shape the evolving model. Fractional Black Scholes models arrived within the same line of thought, which is striving to address shortcomings and perfect existing models. Addressing well-known discrepancies such as the failure of capturing the “volatility smile” of the financial market [68].

In this work we will introduce a class of new Time-fractional Black Scholes equations. As pointed by Wenting Chen et al. [26], Black Scholes equations with single time fractional derivatives are receiving a growing attention [12, 69, 70], despite the fact that no plausible reasons are provided yet for their adoption in trading practice. We share nonetheless the conviction that, the assumption of the underlying following a fractal transmission system one side, together with heuristic arguments in [22, 71, 72] combined to advances made in fractional derivatives [21] have not yet revealed the full spectrum of knowledge on Black Scholes equations.

We will consider for this work time fractional Black Scholes equations with a single time fractional derivatives in the Riemann Liouville sense, the Caputo fractional order derivative, the Caputo-Fabrizio Fractional derivatives in the Caputo sense, the Atangana-Baleanu fractional derivative in the Caputo sense and the Atangana-Baleanu fractional derivative in the Riemann Liouville sense. To the best of our knowledge the late three equations have no appearance in the literature so far.

4.2 The Standard Black Scholes Merton Equation

We briefly present here a derivation of the Standard Black Scholes Equation, as inspired by Fisher Black, Myron Scholes and Robert Merton [3, 4]. Let $f(S_t, t)$ denote the price of an option with S_t being the underlying asset and t being the current time. Under the classical Black Scholes model, we assumed the dynamics of stock Price follow a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad (4.1)$$

where μ represents the percentage drift and σ represents the volatility, both are constants.

If $dS_t = \mu S_t dt + \sigma S_t dW_t$, and $f: (S_t, t) \rightarrow \mathbb{R}$, we would like to determine df .

By Ito's lemma

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2. \quad (4.2)$$

Substituting in $dS_t = \mu S_t dt + \sigma S_t dW_t$, we have

$$df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S_t \frac{\partial f}{\partial S} dW. \quad (4.3)$$

Taking into account the no arbitrage assumption, we are specifically interested in the infinitesimal change of a mixture of a call option and a quantity of assets. Let us denote the quantity by Δ .

$$d(f + \Delta S) = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \Delta \mu S \right) dt + \Delta S \left(\frac{\partial f}{\partial S} + \Delta \right) dW. \quad (4.4)$$

If we choose the quantity of asset Δ to be equal to $\Delta = -\frac{\partial f}{\partial S}$, we have

$$d(f + \Delta S) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt. \quad (4.5)$$

For such a choice of Δ we have eliminated the only term associated with some element of randomness dW . Therefore the growth rate of the portfolio should be that of the risk free interest rate to eliminate any arbitrage opportunity. Thus

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r \left(f - S \frac{\partial f}{\partial S} \right). \quad (4.6)$$

Rearranging this equation we obtain

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0. \quad (4.7)$$

The equation is a second order linear partial differential equation; we will need to associate a boundary condition in terms of the payoff functions to be able to solve it. This last Partial Differential Equation is known as the classical Black Scholes (BS) or Black Scholes Merton Equation or model (BSM).

4.3 The Time Fractional Black-Scholes Equation

In [26] Wenting et al. presented a derivation of a Time Fractional Black Scholes Equation. We also assume here that the change with time in option price follows a fractal transmission, while the underlying asset still follows a Geometric Brownian motion like in the classical Black Scholes equation.

Let $V(S, t)$ denote the price of an option, S is the price of the underlying asset and that t is the current time. The total flux rate of the option price $\bar{Y}(s, t)$ per unit time from the current time t to expiry date T and the option price $V(S, t)$ should satisfy

$$\int_t^T \bar{Y}(s, t') dt' = S^{d_f-1} \int_t^T H(t' - t) [V(S, t') - V(S, T)] dt', \quad (4.8)$$

where $H(t)$ is the transmission function and d_f is the Hausdorff dimension of the fractal transmission system. In [19] J.R. Liang et al. argued that equation (4.8) is a conservation equation with an explicit reference to the history of diffusion process of the option price on the fractal structure. We consider now the transmission function $H(t)$ of the diffusion sets, which ones are also assumed to be underlying fractal.

$$H(t) = \frac{A_\alpha}{\Gamma(1 - \alpha)t^\alpha}$$

where A_α is a constant and α is a transmission exponent. By differentiating equation (4.8) with respect to t we have

$$-\bar{Y}(s, t) = S^{d_f-1} \frac{d}{dt} \left(\int_t^T H(t' - t) [V(S, t') - V(S, T)] dt' \right). \quad (4.9)$$

Now from the classical Black-Scholes equation we have

$$\bar{Y}(s, t) = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - D) S \frac{\partial f}{\partial S} - rf,$$

combining this with (4.9) we have

$$A_\alpha S^{d_f-1} \frac{\partial^\alpha f}{\partial t^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - D) S \frac{\partial f}{\partial S} - r f = 0, \quad (4.10)$$

where $\frac{\partial^\alpha f}{\partial t^\alpha}$ is defined as

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial t^n} \int_t^T \frac{f(S, t') - f(S, T)}{(t' - t)^{\alpha+1-n}} dt', \text{ for } n - 1 \leq \alpha < n. \quad (4.11)$$

Equation (4.10) is a Time Fractional Black Scholes equation where the time fractional derivative defined in the sense of Guy Jumarie (2.2).

Let us take note of the consistency argument with the classical Black Scholes equation, as if we assume $A_\alpha = d_f = 1$, then

$$\lim_{\alpha \rightarrow 1} \frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(2 - 1)} \frac{\partial^2}{\partial t^2} \int_t^T \frac{f(S, t') - f(S, T)}{(t' - t)^{1+1-2}} dt' = \frac{\partial f}{\partial t}. \quad (4.12)$$

This limit result of consistency holds true for all the time Fractional derivatives listed in the subsection 2.1 of definitions. This allows us to define for the sake of this investigation, different types of Time Fractional Black Scholes Equations by setting the time fractional derivative $\frac{\partial^\alpha f}{\partial t^\alpha}$ in (4.10), to be sequentially of Caputo time fractional derivative, the Riemann Liouville type, the Caputo-Fabrizio type, the Atangana-Baleanu, both in the sense of Riemann Liouville and of Caputo type later on.

5. Existence and Uniqueness of solutions of TFBSE

Throughout this chapter we assume the function $C(S, t)$ is continuously differentiable with respect to S , its the partial derivatives are bounded. This is a fairly realistic assumption in framework of the Black Scholes, and it will imply that the partial derivatives with respect to the space variable are Lipschitz.

5.1 Existence and Uniqueness of the solution of TFBSE with Caputo Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given in Caputo Sense, equation (2.1).

$$\begin{cases} {}^C_0D_t^\alpha C(S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + rS \frac{\partial C(S, t)}{\partial S} - rC(S, t) = 0, & (S, t) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.1)$$

where $0 < \alpha \leq 1$, T is the expiry time, r is the risk free rate $\sigma \geq 0$ is the volatility of returns

$${}^C_0D_t^\alpha C(S, t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial C(S, \tau)}{\partial \tau} d\tau, & 0 < \alpha < 1, \\ \frac{\partial C(S, t)}{\partial t}, & \alpha = 1. \end{cases}$$

We will use the well know Picard-Lindelof theorem to prove the existence and uniqueness of the solution to the equation.

Let

$$f(t, C(S, t)) = rC(S, t) - rS \frac{\partial C(S, t)}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2}.$$

To prove the existence and uniqueness of the solution we will truncate the original unbounded domain to a finite interval.

Let $C_{a,b} = \overline{I_a(t_0)} \times \overline{B_b(S)}$ where

$\overline{I_a(t_0)} = [t_0 - a, t_0 + a]$, $\overline{B_b(S)} = [C_0 - b, C_0 + b]$ this is a compact cylinder where f is defined and $M = \sup_{C_{a,b}} ||f||$.

Let us establish that the function $f(t, C(S, t))$ is Lipschitz continuous in y . This is there exists a constant k such that

$$\begin{aligned}
& \left| f(t, C(S, t)) - f(t_0, C_0(S, t_0)) \right| \\
&= \left| rC(S, t) - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t_0)}{\partial S^2} - rS \frac{\partial C(S, t_0)}{\partial S} - rC_0(S, t_0) \right. \\
&\quad \left. + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_0(S, t_0)}{\partial S^2} + rS \frac{\partial C_0(S, t_0)}{\partial S} \right| \\
&\leq r|C(S, t) - C_0(S, t_0)| + \frac{1}{2}\sigma^2 S^2 \left| \frac{\partial^2}{\partial S^2} (C(S, t) - C(S, t_0)) \right| \\
&\quad + rS \left| \frac{\partial}{\partial S} (C(S, t) - C_0(S, t_0)) \right|.
\end{aligned}$$

It follows Lipschitz properties of the partial derivatives that

$$\begin{aligned}
& \left| f(t, C(S, t)) - f(t_0, C_0(S, t_0)) \right| \\
&\leq r|C(S, t) - C_0(S, t_0)| + \frac{1}{2}\sigma^2 S^2 k_1 |C(S, t) - C_0(S, t_0)| \\
&\quad + rS k_2 |C(S, t) - C_0(S, t_0)| \\
&\leq \left(r + \frac{1}{2}\sigma^2 S^2 k_1 + rS k_2 \right) |C(S, t) - C_0(S, t_0)| \\
&\leq \left(r + \frac{1}{2}\sigma^2 b^2 k_1 + r b k_2 \right) |C(S, t) - C_0(S, t_0)|.
\end{aligned}$$

$A = r + \frac{1}{2}\sigma^2 b^2 k_1 + r b k_2$ is independent of t . Therefore $f(t, C(S, t))$ is Lipschitz continuous in y .

Now let us define $\mathcal{T}: \mathcal{C}(I_a(t_0), B_b(S)) \rightarrow \mathcal{C}(I_a(t_0), B_b(S))$ and prove that Γ is a contraction.

$$\mathcal{T}\varphi(t) = C_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t f(s, \varphi(s))(t-s)^{\alpha-1} ds,$$

First we impose $\mathcal{T}$ is well defined that is

$$\begin{aligned} \|\mathcal{T}\varphi(t) - C_0\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_{t_0}^t f(s, \varphi(s))(t-s)^{\alpha-1} ds \right\| \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \|f(s, \varphi(s))(t-s)^{\alpha-1}\| ds \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \|f(s, \varphi(s))\| |t-s|^{\alpha-1} ds \leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^t |t-s|^{\alpha-1} ds \\ &\leq \frac{M|t-t_0|^\alpha}{\alpha\Gamma(\alpha)} \leq \frac{Ma^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Now given two functions $\varphi_1, \varphi_2 \in \mathcal{C}(I_a(t_0), B_b(S))$, in order to apply the Banach fixed point theorem, we want

$$\|\mathcal{T}\varphi_1 - \mathcal{T}\varphi_2\|_\infty \leq q \|\varphi_1 - \varphi_2\|_\infty,$$

with $q < 1$. Let t be such that

$$\|\mathcal{T}\varphi_1 - \mathcal{T}\varphi_2\|_\infty = \|(\Gamma\varphi_1 - \Gamma\varphi_2)(t)\|.$$

Using the definition of $\mathcal{T}$

$$\begin{aligned} \|(\mathcal{T}\varphi_1 - \mathcal{T}\varphi_2)(t)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} (f(s, \varphi_1(s)) - f(s, \varphi_2(s))) ds \right\| \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \|(t-s)^{\alpha-1} (f(s, \varphi_1(s)) - f(s, \varphi_2(s)))\| ds \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t |t-s|^{\alpha-1} \|f(s, \varphi_1(s)) - f(s, \varphi_2(s))\| ds, \end{aligned}$$

since f is lipstchiz continuous

$$\begin{aligned} ||(\mathcal{T}\varphi_1 - \mathcal{T}\varphi_2)(t)|| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^t |t-s|^{\alpha-1} k_1 ||\varphi_1(s) - \varphi_2(s)|| ds \\ &\leq \frac{k_1 b}{\Gamma(\alpha)} \int_{t_0}^t a^{\alpha-1} ||\varphi_1(s) - \varphi_2(s)|| ds \leq \frac{k_1 b a^{\alpha-1}}{\Gamma(\alpha)} |\varphi_1 - \varphi_2|. \end{aligned}$$

$\mathcal{T}$ is a contraction if

$$\frac{k_1 b a^{\alpha-1}}{\Gamma(\alpha)} < 1,$$

this is if

$$a < \left(\frac{\Gamma(\alpha)}{k_1 b} \right)^{\frac{1}{\alpha}}.$$

We then impose the condition

$$\frac{k_1 b a^{\alpha-1}}{\Gamma(\alpha)} < 1.$$

Thus $\mathcal{T}$ is a contraction. It follows that the Time Fractional Black Scholes Equation with Caputo fractional derivative (5.1) has a unique solution.

5.2 Existence and Uniqueness of the solution of TFBSE with RL Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given in Riemann Liouville equation (2.2)

$$\begin{cases} {}^{RL}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.2)$$

where $0 < \alpha \leq 1$, T is the expiry time, r is the risk free rate $\sigma \geq 0$ is the volatility of returns

$${}^{RL}_0 D_t^\alpha C(S, t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} C(S, \tau) d\tau.$$

$${}^{RL}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0 \Rightarrow$$

$${}^{RL}_0 D_t^\alpha C(S, t) = rC(S, t) - rS \frac{\partial C(S, t)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2}.$$

Let

$$f(S, t, C(S, t)) = rC(S, t) - rS \frac{\partial C(S, t)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} ;$$

$${}^{RL}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + rS \frac{\partial C(S, t)}{\partial S} - rC(S, t) = 0,$$

$$\Leftrightarrow {}^{RL}_0 D_t^\alpha C(S, t) = f(S, t, C(S, t))$$

$${}^{RL}_0 D_t^\alpha C(S, t) = f(S, t, C(S, t)) \Rightarrow C(S, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau.$$

The time Fractional Black Scholes Equation (5.2) with Riemann Liouville fractional derivative is equivalent to a generalised Volterra fractional equation

$$C(S, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau. \quad (5.3)$$

We will give a proof of the existence and uniqueness of the solution to the generalised fractional Volterra equation (5.3), inspired by Atangana A. and Necdet Bildik in [73]. From subsection 5.1 we can directly establish that

$$f(S, t, C(S, t)) = rC(S, t) - rS \frac{\partial C(S, t)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2},$$

is Lipschitz in z with Lipschitz constant independent of t and S . In fact

$$\begin{aligned} & \left| f(S, t, C(S, t)) - f(S, t_0, C_0(S, t_0)) \right| \\ &= \left| rC(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_0)}{\partial S^2} - rS \frac{\partial C(S, t_0)}{\partial S} - rC_0(S, t_0) \right. \\ & \quad \left. + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C_0(S, t_0)}{\partial S^2} + rS \frac{\partial C_0(S, t_0)}{\partial S} \right|. \end{aligned}$$

It follows from Lipschitz properties, that

$$\begin{aligned}
& \left| f(S, t, C(S, t)) - f(S, t_0, C_0(S, t_0)) \right| \\
& \leq r \left| C(S, t) - C_0(S, t_0) \right| + \frac{1}{2} \sigma^2 S^2 \left\| \frac{\partial^2}{\partial S^2} (C(S, t) - C(S, t_0)) \right\| \\
& \quad + rS \left\| \frac{\partial}{\partial S} (C(S, t) - C_0(S, t_0)) \right\|, \\
& \leq r \left| C(S, t) - C_0(S, t_0) \right| + \frac{1}{2} \sigma^2 S^2 k_1 \left| C(S, t) - C_0(S, t_0) \right| \\
& \quad + rS k_2 \left| C(S, t) - C_0(S, t_0) \right|, \\
& \leq \left(r + \frac{1}{2} \sigma^2 S^2 k_1 + rS k_2 \right) \left| C(S, t) - C_0(S, t_0) \right|, \\
& \leq \left(r + \frac{1}{2} \sigma^2 b^2 k_1 + r b k_2 \right) \left| C(S, t) - C_0(S, t_0) \right|,
\end{aligned}$$

where $A = r + \frac{1}{2} \sigma^2 b^2 k_1 + r b k_2$ is independent of t . Therefore $f(S, t, C(S, t))$ is Lipschitz continuous in z . The kernel $f(S, t, C(S, t))$ is continuous. Now let us define the iteration

$$C_n(S, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C_{n-1}(S, \tau)) d\tau, \quad (5.4)$$

and

$$\begin{aligned}
\delta_n(S, t) &= C_n(S, t) - C_{n-1}(S, t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [f(S, \tau, C_{n-1}(S, \tau)) - f(S, \tau, C_{n-2}(S, \tau))] d\tau,
\end{aligned}$$

Since f is Lipschitz continuous in z there is an independent constant k such that

$$\left| f(S, \tau, C_{n-1}(S, \tau)) - f(S, \tau, C_{n-2}(S, \tau)) \right| \leq k \left| C_{n-1}(S, \tau) - C_{n-2}(S, \tau) \right|,$$

$$\begin{aligned}
\left| \delta_n(t) \right| &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [f(S, \tau, C_{n-1}(S, \tau)) - f(S, \tau, C_{n-2}(S, \tau))] d\tau \right\|, \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} k \left| C_{n-1}(S, \tau) - C_{n-2}(S, \tau) \right| d\tau,
\end{aligned}$$

$$\leq \frac{k}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \|\delta_{n-1}(S, \tau)\| d\tau,$$

This is $\|\delta_n(S, t)\| \leq \frac{k}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \|\delta_{n-1}(S, \tau)\| d\tau$ it follows that

$$\|\delta_n(S, t)\| \leq \max_{0 \leq t \leq T} \frac{(k^{-\alpha} t)^{n\alpha}}{\Gamma(1 + n\alpha)}.$$

Therefore the function

$$C(S, t) = \sum_{i=0}^n \delta_i(S, t), \quad (5.5)$$

exists and it is a continuous function. However, to prove that the above function is the solution of the generalized Volterra fractional integral equation (5.3), we let

$$C(S, t) = C_n(S, t) + P_n(S, t).$$

From the iteration we have the following,

$$C_n(S, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C_{n-1}(S, \tau)) d\tau.$$

This implies that

$$C(S, t) - P_n(S, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau) - P_{n-1}(S, \tau)) d\tau,$$

and therefore

$$\begin{aligned} C(S, t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau \\ = P_n(S, t) \\ - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [f(S, \tau, C(S, \tau) - P_{n-1}(S, \tau)) - f(S, \tau, C(S, \tau))] d\tau. \end{aligned}$$

By applying the norm and making use of the triangular inequality, we have the following:

$$\begin{aligned}
& \left\| C(S, t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau \right\| \\
&= \left\| P_n(S, t) \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [f(S, \tau, C(S, \tau)) - P_{n-1}(S, t)] d\tau \right\| \\
&\leq \|P_n(S, t)\| + \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [f(S, \tau, C(S, \tau)) - P_{n-1}(S, t)] d\tau \right\|.
\end{aligned}$$

(Recalling our kernel function is Lipschitz continuous in z we have, with Lipschitz constant k)

$$\begin{aligned}
&\leq \|P_n(S, t)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} k \|P_{n-1}(S, t)\| d\tau, \\
&\leq \|P_n(S, t)\| + \frac{k \|P_{n-1}(S, t)\|}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau, \\
&\leq \|P_n(S, t)\| + \frac{kt}{\Gamma(\alpha + 1)} \|P_{n-1}(S, t)\|.
\end{aligned}$$

Applying the limit on both sides of the above inequality when n tends to infinity, the right-hand side tends to zero, then, $C(S, t)$ in (5.5) satisfies

$$C(S, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau,$$

and therefore is a unique solution and to the time Fractional Black Scholes Equation with Riemann Liouville time fractional derivative.

5.3 Existence and Uniqueness of the solution of TFBSE with Caputo-Fabrizio Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given by the Caputo-Fabrizio definition equation (2.8).

$$\begin{cases} {}^{CF}D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.6)$$

with

$${}^{CF}_a \mathcal{D}_t^\alpha (C(S, t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{\partial C(S, \tau)}{\partial \tau} \exp \left[-\alpha \frac{t-\tau}{1-\alpha} \right] d\tau,$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. However, if the function does not belongs to $H^1(a, b)$ then, the derivative can be reformulated as:

$${}^{CF}_a \mathcal{D}_t^\alpha (C(S, t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t (C(S, t) - C(S, x)) \exp \left[-\alpha \frac{t-x}{1-\alpha} \right] dx.$$

To prove the existence and uniqueness of the solution to the time fractional Black Scholes Equation (5.6), we will present first some general results.

Let us consider the Banach Space $X = C[a, b]$, of continuous real functions defined over the closed interval $[a, b]$. X contains the sub-norm and let Z be the shaft defined as :

$$Z = \{C \in X, C(S, t) \geq 0, 0 < t \leq T\}.$$

Definition 5.1

Let E be a real Banach space with a cone H . H initiates a restricted order $\leq$ in E in the succeeding approach

$$x \leq y \Rightarrow y - x \in H.$$

For every $x, y \in E$, the order interval is defined as: $\langle a, b \rangle = \{f \in E: a \leq f \leq b\}$.

A cone K is denoted normal if one can find a positive constant j such that

$$h, d \in K, \Phi < h < d \Rightarrow ||h|| \leq j ||d||, \text{Where } \Phi \text{ denotes the zero element of } K.$$

Theorem 5.1

Let H be a closed set subspace of a Banach space of D . Let T be a contraction mapping with Lipschitz constant $g < 1$ from H to H . Thus, T possesses a fixed-point t^* in H . In addition, if t_0 is a random point in H and (t_n) is a sequence defined by:

$$t_{n+1} = T t_n \quad n = 0, 1, 2, \dots$$

then for a large number n , t_n tends to t^* in H and

$$d(t_n, t^*) \leq \frac{g^n}{1-g} d(t_1, t_0).$$

Let us consider now the time fractional Black Scholes equation (5.6) with time fractional derivative given by the Caputo Fabrizio definition (2.8).

If we apply to Caputo-Fabrizio fractional integral to (5.6) we have the following:

$$I_{\alpha}^t({}^{CF}D_t^{\alpha}C(S, t)) = I_{\alpha}^t(f(S, t, C(S, t))),$$

where,

$$f(S, t, C(S, t)) = rC(S, t) - rS \frac{\partial C(S, t)}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2}.$$

$$I_{\alpha}^t({}^{CF}D_t^{\alpha}C(S, t)) = I_{\alpha}^t[f(S, t, C(S, t))] \Rightarrow$$

$$C(S, t) - C(S, t_0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(S, t, C(S, t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(S, \tau, C(S, \tau)) d\tau. \quad (5.7)$$

Lemma 5.1 The mapping $T: H \rightarrow H$ is defined as

$$TC(S, t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(S, t, C(S, t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(S, \tau, C(S, \tau)) d\tau,$$

Lemma 5.2 Let $M \subset H$ be bounded, implying we can find $n > 0$ such that if

$$||u(S, t_2) - u(S, t_1)|| \leq n|t_2 - t_1|, \forall u \in M.$$

Then $\overline{T(M)}$ is compact.

Proof. Let

$$N = \max_{0 \leq u(x, t) \leq P} \left\{ \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + f(S, t, u(S, t)) \right\}.$$

For $C(x, t) \in M$, then we have the following

$$||TC(x, t)|| = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} ||f(S, t, C(S, t))|| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t ||f(S, \tau, C(S, \tau))|| d\tau$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} N + \frac{2\alpha N}{(2-\alpha)M(\alpha)} t.$$

We will consider now $C(x, t) \in M, t_1, t_2$, and $t_1 < t_2$; then for a given $\varepsilon > 0$, if $|t_1 - t_2| < \delta$. Then

$$\begin{aligned} & ||TC(S, t_2) - TC(S, t_1)|| \\ & \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} ||f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1))|| \\ & \quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \left\| \int_0^{t_2} ||f(S, \tau, C(S, \tau))|| d\tau - \int_0^{t_1} ||f(S, \tau, C(S, \tau))|| d\tau \right\|, \\ & \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} ||f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1))|| \\ & \quad + \frac{2\alpha P}{(2-\alpha)M(\alpha)} \left\{ \int_0^{t_2} d\tau - \int_0^{t_1} d\tau \right\}, \\ & ||TC(S, t_2) - TC(S, t_1)|| \\ & \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} ||f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1))|| \\ & \quad + \frac{2\alpha P}{(2-\alpha)M(\alpha)} (t_2 - t_1). \end{aligned} \tag{5.8}$$

Let us consider now $||f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1))||$.

$$\begin{aligned} & ||f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1))|| \\ & = \left\| rC(S, t_2) - rS \frac{\partial C(S, t_2)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_2)}{\partial S^2} - rC(S, t_1) + rS \frac{\partial C(S, t_1)}{\partial S} \right. \\ & \quad \left. + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_1)}{\partial S^2} \right\| \\ & \leq r ||C(S, t_2) - C(S, t_1)|| + r \max(S_2, S_1) \left\| \frac{\partial C(S, t_2)}{\partial S} - \frac{\partial C(S, t_1)}{\partial S} \right\| \\ & \quad + \frac{1}{2} \sigma^2 \max(S_2, S_1) \left\| \frac{\partial^2 C(S, t_2)}{\partial S^2} - \frac{\partial^2 C(S, t_1)}{\partial S^2} \right\| \\ & \leq \left(r + r \max(S_2, S_1) + \frac{1}{2} \sigma^2 \max(S_2, S_1) \right) ||C(S, t_2) - C(S, t_1)|| \end{aligned}$$

setting $K_1 = \left(r + r \max(S_2, S_1) + \frac{1}{2} \sigma^2 \max(S_2, S_1) \right)$,

$$\left| \left| f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1)) \right| \right| \leq K_1 \|C(S, t_2) - C(S, t_1)\|,$$

now Using lemma 2 since $C \in M$

$$\left| \left| f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1)) \right| \right| \leq K_1 n |t_2 - t_1|,$$

$$\left| \left| f(S, t_2, C(S, t_2)) - f(S, t_1, C(S, t_1)) \right| \right| \leq L.$$

Regrouping in equation (5.8) we have

$$\left| \left| TC(S, t_2) - TC(S, t_1) \right| \right| \leq \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} |t_2 - t_1| + \frac{2\alpha P}{(2-\alpha)M(\alpha)} (t_2 - t_1)$$

This is

$$\left| \left| TC(S, t_2) - TC(S, t_1) \right| \right| \leq \frac{2(1-\alpha)L + 2\alpha P}{(2-\alpha)M(\alpha)} |t_2 - t_1|.$$

Taking

$$\delta = \frac{(2-\alpha)M(\alpha)\varepsilon}{2(1-\alpha)L + 2\alpha P} \text{ we will have } \left| \left| TC(S, t_2) - TC(S, t_1) \right| \right| < \varepsilon \text{ satisfied.}$$

So $\overline{T(M)}$ is equi-continuous and by Arzela–Ascoli theorem, $\overline{T(M)}$ is compact.

Uniqueness of solution

Assume we have two solutions $C_1(S, t)$ and $C_2(S, t)$ that satisfy the equation,

$$\begin{aligned}
& ||TC_1(S, t) - TC_2(S, t)|| \\
&= \left\| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(S, t, C_1(S, t)) \right. \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{t_1} f(S, \tau, C_1(S, \tau)) d\tau - f(S, t, C_2(S, t)) \\
&\quad \left. - \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{t_2} f(S, \tau, C_2(S, \tau)) d\tau \right\| \\
&\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} ||f(S, t, C_1(S, t)) - f(S, t, C_2(S, t))|| \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_{t_1}^{t_2} ||f(S, t, C_1(S, t)) - f(S, \tau, C_2(S, \tau))|| d\tau \\
&\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} K_1 ||C_1(S, t) - C_2(S, t)|| \\
&\quad + \frac{2\alpha K_1}{(2-\alpha)M(\alpha)} \int_0^t ||C_1(S, \tau) - C_2(S, \tau)|| d\tau \\
&\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} K_1 ||C_1(S, t) - C_2(S, t)|| + \frac{2\alpha K_1 T}{(2-\alpha)M(\alpha)} ||C_1(S, \tau) - C_2(S, \tau)|| \\
&\leq \left(\frac{(2(1-\alpha) + 2\alpha T)K_1}{(2-\alpha)M(\alpha)} \right) ||C_1(S, t) - C_2(S, t)||.
\end{aligned}$$

We impose the condition

$$\frac{2(1-\alpha)K_1 + 2\alpha K_1 T}{(2-\alpha)M(\alpha)} < 1.$$

Therefore, the mapping T is a contraction, which implies a fixed point, and thus the Time Fractional Black Scholes Equation (5.6) with Caputo-Fabrizio fractional derivative has a unique solution.

5.4 Existence and Uniqueness of the solution of a TFBSE with ABR Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given by the Atangana-Baleanu derivative definition in the Riemann Liouville sense equation (2.10). the proof of existence and uniqueness of the solution to the Time Fractional Black Scholes Equation with the Atangana-Baleanu derivative in the Caputo sense is similar and will not be presented here.

$$\begin{cases} {}^{ABR}D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.9)$$

with

$${}^{ABR}D_t^\alpha (C(S, t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t C(S, t) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx, \alpha \in [0, 1].$$

To give a condition for a unique global solution for (eq 5.9) we assume the Atangana Baleanu Derivative of the function $C(S, t)$, ${}^{ABR}D_t^\alpha C(S, t)$ is in the space of summable function $L(a, b)$ in a finite interval $[a, b]$ of the real line $\mathbb{R}$. We need to establish the following Lemma.

Lemma 5.3: The fractional integral operator

$${}^{AB}I_t^\alpha (f(t)) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(y)(t-y)^{\alpha-1} dy,$$

is bounded in $L(a, b)$:

$$\| {}^{AB}I_t^\alpha (f(t)) \| \leq \left(\frac{(1-\alpha)}{B(\alpha)} + \frac{T^\alpha}{B(\alpha)\Gamma(\alpha)} \right) \| f(t) \|.$$

Proof

Let f be a bounded function on $\mathbb{R}$, we define

$$\| f \| = \max_{[0, T]} |f(t)|.$$

$$\begin{aligned}
\left| {}^{AB}I_t^\alpha(f(t)) \right| &= \left| \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(y)(t-y)^{\alpha-1}dy \right| \\
&\leq \frac{1-\alpha}{B(\alpha)}\|f(t)\| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t \|f(y)(t-y)^{\alpha-1}\|dy \\
&\leq \frac{(1-\alpha)\|f(t)\|}{B(\alpha)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t \|f(y)\| |t-y|^{\alpha-1}dy \\
&\leq \frac{(1-\alpha)\|f(t)\|}{B(\alpha)} + \frac{\alpha\|f(t)\|}{B(\alpha)\Gamma(\alpha)} \int_0^t |t-y|^{\alpha-1}dy \\
&\leq \frac{(1-\alpha)\|f(t)\|}{B(\alpha)} + \frac{\|f(t)\|T^\alpha}{B(\alpha)\Gamma(\alpha)} \\
&\leq \left(\frac{(1-\alpha)}{B(\alpha)} + \frac{T^\alpha}{B(\alpha)\Gamma(\alpha)} \right) \|f(t)\|.
\end{aligned}$$

Theorem 5.2

Let $\alpha > 0, n = -[-\alpha]$. Let G be an open set in $\mathbb{R}$ and let $f: (a, b] \times G \rightarrow \mathbb{R}$ be a function such that $f: (a, b] \times G \rightarrow \mathbb{R}$ be a function such that $f(x, y) \in L(a, b)$ for any $y_1, y_2 \in G$, $|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|$, $A > 0$ where A does not depend on $x \in [a, b]$. Then there is a unique solution to equation (5.9) in the space of function in finite interval $[a, b]$, whose fractional derivative of order α is summable.

Equation (5.9) is equivalent to the Volterra Equation

$${}^{AB}I_t^\alpha({}^{ABR}D_t^\alpha C(S, t)) = {}^{AB}I_t^\alpha(f(S, t, C(S, t))),$$

$$C(S, t) = \frac{1-\alpha}{B(\alpha)}f(S, t, C(S, t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(S, \tau, C(S, \tau))(t-\tau)^{\alpha-1}d\tau. \quad (5.10)$$

To prove the existence of a unique solution to $C(S, t)$ we apply a method for Volterra integral equation by first proving the result on a part of the interval $[0, T]$. In any interval $[0, t_1] \subset [0, T]$, $L(0, t_1)$, is clearly a complete metric space with the distance d .

$$d(C_1(S, t_1), C_2(S, t_2)) = \|C_1(S, t_1) - C_2(S, t_2)\| = \int_0^{t_1} |C_1(S, \tau) - C_2(S, \tau)|d\tau.$$

If we impose the condition for $[0, t_1] \subset [0, T]$

$$\frac{(1-\alpha)B(\alpha) + t_1^\alpha}{B(\alpha)\Gamma(\alpha)} < 1,$$

and let

$$\gamma C(S, t) = \frac{1-\alpha}{B(\alpha)} f(S, t, C(S, t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_1} f(S, \tau, C(S, \tau))(t-\tau)^{\alpha-1} d\tau,$$

then for any $C_1, C_2 \in L(0, t_1)$

$$||\gamma C_1(S, t_1) - \gamma C_2(S, t_2)|| \leq \frac{(1-\alpha)B(\alpha) + t_1^\alpha}{B(\alpha)\Gamma(\alpha)} ||C_1(S, t_1) - C_2(S, t_2)||.$$

therefore there exists a unique Solution in $L(0, t_1)$ to equation (5.10). Next we consider the interval $[t_1, t_2]$, where $t_2 = t_1 + h, h > 0, t_2 < T$, and reformulate (5.9) as follows

$$\begin{aligned} C(s, t) &= \frac{1-\alpha}{B(\alpha)} f(S, t, C(S, t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_1} f(S, \tau, C(S, \tau))(t-\tau)^{\alpha-1} d\tau \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{t_1}^t f(S, \tau, C(S, \tau))(t-\tau)^{\alpha-1} d\tau. \end{aligned}$$

Since $C(S, t)$ is uniquely defined on $[0, t_1)$, we can consider the second integral as the unknown function and rewrite

$$C(s, t) = C_{01}(S, t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{t_1}^t f(S, \tau, C(S, \tau))(t-\tau)^{\alpha-1} d\tau,$$

where

$$C_{01}(S, t) = \frac{1-\alpha}{B(\alpha)} f(S, t, C(S, t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_1} f(S, \tau, C(S, \tau))(t-\tau)^{\alpha-1} d\tau,$$

is the known function. Using the same arguments as above we derive that there exists a unique solution $C^*(s, t)$ within $L(t_1, t_2)$, on the interval $[t_1, t_2]$. Taking a next interval $[t_2, t_3]$, where $t_3 = t_2 + h_2, h_2 > 0, t_3 < T$, and repeating the process, we conclude that there exists a unique solution $C(S, t) \in L(0, T)$ for the equation (5.9) on the interval $[0, T]$.

6. Semi Analytical Solutions of a TFBS Equation

We give here essential topical definitions Laplace trans form, Sumudu transform and illustrate the how they can be used to derive semi analytical solutions of our TFBSEs.

6.1 Preliminaries

6.1.1. Laplace Transform Of Some Operators And Properties

Definition 6.1

Let $f(t)$ be a function defined for $t \geq 0$. The Laplace transform of $f(t)$, that we will denote by $F(s)$ or $\mathcal{L}\{f(t)\}$ is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{+\infty} e^{-st} f(t) dt,$$

if the improper integral exists as a number, that is if the improper integral is convergent.

Theorem 6.1

Suppose that f is a piecewise continuous function on an interval $0 \leq t \leq T$ for any $T > 0$. If $|f(t)| \leq H e^{at}$ when $t \geq G$, for any real constant a , and some positive constants H and G , then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$.

Definition 6.2

The convolution of two positively defined functions $f(t)$ and $g(t)$ is a function of t , denoted by $(f * g)(t)$ and defined as :

$$(f * g)(t) = \int_0^t f(t-x)g(x)dx.$$

Theorem 6.2

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

This is

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} &= \int_0^{+\infty} e^{-st}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)dt, \\
&= \int_0^{+\infty}\left(f(\tau)\int_\tau^{+\infty} e^{-st}g(t-\tau)dt\right)d\tau, \\
&= \int_0^{+\infty}\left(f(\tau)\int_0^{+\infty} e^{-(\tau+u)s}g(u)du\right)d\tau, \\
&= \int_0^{+\infty}\left(e^{-\tau s}f(\tau)\int_0^{+\infty} e^{-us}g(u)du\right)d\tau = \int_0^{+\infty}(e^{-\tau s}f(\tau)G(s))d\tau, \\
&= G(s)\int_0^{+\infty}(e^{-\tau s}f(\tau))d\tau = F(s)G(s).
\end{aligned}$$

Laplace transform of the ABR derivative

$${}^{ABR}_0\mathcal{D}_t^\alpha(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t f(t)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx, \alpha \in [0,1].$$

$$\begin{aligned}
\mathcal{L}\{{}^{ABR}_0\mathcal{D}_t^\alpha f(t)\} &= \mathcal{L}\left\{\frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t f(t)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx\right\}, \\
&= \frac{B(\alpha)}{1-\alpha}\mathcal{L}\left\{\frac{d}{dt}\int_0^t f(t)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx\right\}, \\
&= \frac{B(\alpha)}{1-\alpha}\mathcal{L}\left\{\frac{d}{dt}\left[f(t)*E_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\right]\right\}, \\
&= \frac{B(\alpha)}{1-\alpha}s\mathcal{L}\left\{\left[f(t)*E_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\right]\right\}, \\
&= \frac{B(\alpha)}{1-\alpha}s\mathcal{L}\left\{\left[f(t)*E_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\right]\right\}, \\
&= \frac{B(\alpha)}{1-\alpha}s\mathcal{L}\{f(t)\}\mathcal{L}\left\{E_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\right\}, \\
&= \frac{B(\alpha)}{1-\alpha}s\frac{s^{\alpha-1}}{s^\alpha + \frac{\alpha}{\alpha-1}}\mathcal{L}\{f(t)\}, \\
&= \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha}{s^\alpha + \frac{\alpha}{\alpha-1}}\mathcal{L}\{f(t)\}.
\end{aligned}$$

6.1.2. Sumudu Transform Of Some Operators And Properties

Definition 6.3

Sumudu Transform was introduced by Gamage K. Watugala in 1993, [74]. The Sumudu transform $F(u)$ of any function $f(t)$ can be defined by the following formula as:

$$F(u) = \mathbb{S}[f(t)] = \frac{1}{u} \int_0^{+\infty} \exp\left(-\frac{t}{u}\right) f(t) dt.$$

provided the integral exists for some u .

In fact the transform is defined over the set of functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, +\infty) \right\}.$$

The former definition can also be written as:

$$F(u) = \mathbb{S}[f(t)] = \int_0^{+\infty} f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2).$$

Theorem 6.3: Sumudu Transform Convolution Theorem

If $\mathbb{S}\{f(t)\} = F(u)$ and $\mathbb{S}\{g(t)\} = G(u)$ then

$$\mathbb{S}\{(f * g)(t)\} = uF(u)G(u).$$

This simply means that

$$\mathbb{S}(f * g) = u \mathbb{S}(f(t)) \mathbb{S}(g(t)).$$

Sumudu Transform the ABR derivative of a function $f(t)$

$$\mathbb{S}({}^{ABR}_0 D_t^\alpha f(t)) = \frac{AB(\alpha) \alpha \Gamma(\alpha + 1)}{1 - \alpha} E_\alpha \left(-\frac{1}{1 - \alpha} u^\alpha \right) \mathbb{S}(f(t)) - \frac{AB(\alpha) f(0) E_\alpha(0)}{1 - \alpha} \frac{1}{u}.$$

6.2 Semi-Analytical solution of RL TFBSE

6.2.1. Semi Analytical solution of RL TFBSE with Laplace Transform

We consider the time fractional Black-Scholes equation (5.2) where the time fractional derivative is given in Riemann Liouville (2.2).

$$\begin{cases} {}^{RL}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.2)$$

where $0 < \alpha \leq 1$, T is the expiry time, r is the risk free rate $\sigma \geq 0$ is the volatility of returns

$$\begin{aligned} {}^{RL}_0 D_t^\alpha C(S, t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} C(S, \tau) d\tau, \\ {}^{RL}_0 D_t^\alpha C(S, t) &= r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S}. \end{aligned}$$

Let

$$\begin{aligned} f(S, t, C(S, t)) &= r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S}, \\ \mathcal{L}({}^{RL}_0 D_t^\alpha C(S, t)) &= \mathcal{L}(f(S, t, C(S, t))). \end{aligned}$$

$$\text{Let } v(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad u(t) = C(S, t),$$

$$\begin{aligned} {}^{RL}_0 D_t^\alpha C(S, t) &= \frac{d}{dt} (v(t) * u(t)), \\ &= \frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} C(S, \tau) d\tau \right), \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} C(S, \tau) d\tau, \end{aligned}$$

$$\begin{aligned} \mathcal{L}({}^{RL}_0 D_t^\alpha C(S, t)) &= \mathcal{L} \left(\frac{d}{dt} (v(t) * u(t)) \right), \\ &= p \mathcal{L}(v(t) * u(t)) - v * u|_{t=0}, \\ &= p \mathcal{L}(v(t)) \times \mathcal{L}(u(t)), \quad \text{since } v * u|_{t=0} = 0, \\ &= p \mathcal{L} \left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right) \times \mathcal{L}(C(S, t)), \\ &= p \frac{1}{\Gamma(1-\alpha)} \mathcal{L}(t^{-\alpha}) \times \mathcal{L}(C(S, t)), \\ &= p \frac{1}{\Gamma(1-\alpha)} p^{\alpha-1} \Gamma(1-\alpha) \times \mathcal{L}(C(S, t)), \\ &= p^\alpha \times \mathcal{L}(C(S, t)), \quad 0 < \alpha \leq 1. \end{aligned}$$

This means that

$$p^\alpha \times \mathcal{L}(C(S, t)) = \mathcal{L}(f(S, t, C(S, t))) \Rightarrow \mathcal{L}(C(S, t)) = \frac{1}{p^\alpha} \mathcal{L}(f(S, t, C(S, t))),$$

$$C(S, t) = \mathcal{L}^{-1} \left\{ \frac{1}{p^\alpha} \mathcal{L}(f(S, t, C(S, t))) \right\}.$$

From the above we introduce an iteration on $C(S, t)$ and we obtain the following

$$C(S, t_{n+1}) = \mathcal{L}^{-1} \left\{ \frac{1}{p^\alpha} \mathcal{L}(f(S, t_n, C(S, t_n))) \right\}.$$

For ease of notation we let

$$C(S, t_n) = C_n.$$

The former becomes

$$C_{n+1} = \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L}(f(S, t_n, C_n)) \right\}. \quad (6.1)$$

- **Stability of the Iterative Scheme**

Considering the TFBSE equation (5.2) and the iterative scheme solution equation (6.1).

$$C_{n+1} = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau,$$

therefore

$$\|C_{n+1} - C_n\| = \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau \right. \\ \left. - \int_0^{t_n} (t_n - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau \right\|.$$

In subsection 5.2 we established that

$$\begin{aligned}
\|\delta_n(t)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [f(S, \tau, C_{n-1}(S, \tau)) - f(S, \tau, C_{n-2}(S, \tau))] d\tau \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} k \|C_{n-1}(S, \tau) - C_{n-2}(S, \tau)\| d\tau \\
&\leq \frac{k}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\delta_{n-1}(S, \tau)\| d\tau.
\end{aligned}$$

This is

$$\|\delta_n(S, t)\| \leq \frac{k}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\delta_{n-1}(S, \tau)\| d\tau,$$

it follows that

$$\|\delta_n(S, t)\| \leq \max_{0 \leq t \leq T} \frac{(k^{-\alpha} t)^{n\alpha}}{\Gamma(1 + n\alpha)}.$$

Showing the iterative scheme solution is stable.

6.3 Semi-Analytical solution of Caputo TFBSE

6.3.1. Semi Analytical solution of the TFBSE with Caputo fractional operator

Let us consider the following time fractional Black-Scholes equation where the time fractional derivative is given in Caputo Sense (2.1)

$$\begin{cases} {}_0^c D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (S, t) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.1)$$

where $0 < \alpha \leq 1$, T is the expiry time, r is the risk free rate $\sigma \geq 0$ is the volatility of returns and

$$\begin{aligned}
{}_0^c D_t^\alpha C(S, t) &= \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial C(S, \tau)}{\partial \tau} d\tau, & 0 < \alpha < 1, \\ \frac{\partial C(S, t)}{\partial t}, & \alpha = 1. \end{cases} \\
{}_0^c D_t^\alpha C(S, t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial C(S, \tau)}{\partial \tau} d\tau, \\
{}_0^c D_t^\alpha C(S, t) &= r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S},
\end{aligned}$$

Let

$$f(S, t, C(S, t)) = rC(S, t) - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - rS \frac{\partial C(S, t)}{\partial S},$$

$$\mathcal{L}({}_0^C D_t^\alpha C(S, t)) = \mathcal{L}\left(f(S, t, C(S, t))\right).$$

Let

$$v(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad u(t) = \frac{\partial C(S, \tau)}{\partial \tau},$$

$${}_0^C D_t^\alpha C(S, t) = v(t) * u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial C(S, \tau)}{\partial \tau} d\tau,$$

$$\mathcal{L}({}_0^C D_t^\alpha C(S, t)) = \mathcal{L}(v(t) * u(t)) = \mathcal{L}(v(t)) \times \mathcal{L}(u(t)),$$

$$\mathcal{L}({}_0^C D_t^\alpha C(S, t)) = \mathcal{L}\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right) \times \mathcal{L}\left(\frac{\partial C(S, t)}{\partial t}\right),$$

$$\begin{aligned} \mathcal{L}({}_0^C D_t^\alpha C(S, t)) &= \frac{1}{\Gamma(1-\alpha)} \mathcal{L}(t^{-\alpha}) \times \mathcal{L}\left(\frac{\partial C(S, t)}{\partial t}\right), \\ &= \frac{1}{\Gamma(1-\alpha)} p^{\alpha-1} \Gamma(1-\alpha) \times \left(p\mathcal{L}(C(S, t)) - C(S, t_0)\right), \\ &= p^\alpha \times \mathcal{L}(C(S, t)) - p^{\alpha-1} C_0, \quad 0 < \alpha \leq 1. \end{aligned}$$

This means that

$$p^\alpha \times \mathcal{L}(C(S, t)) - p^{\alpha-1} C_0 = \mathcal{L}\left(f(S, t, C(S, t))\right) \Rightarrow$$

$$\mathcal{L}(C(S, t)) = \frac{1}{p^\alpha} \mathcal{L}\left(f(S, t, C(S, t))\right) + \frac{1}{p} C_0 \Rightarrow$$

$$C(S, t) = \mathcal{L}^{-1} \left\{ \frac{1}{p^\alpha} \mathcal{L}\left(f(S, t, C(S, t))\right) + \frac{1}{p} C_0 \right\}$$

$$C(S, t) = \mathcal{L}^{-1} \left\{ \frac{1}{p^\alpha} \mathcal{L}\left(f(S, t, C(S, t))\right) \right\} + C(S, t_0),$$

From the above we introduce an iteration on $C(S, t)$ and we obtain the following

$$C(S, t_{n+1}) = \mathcal{L}^{-1} \left\{ \frac{1}{p^\alpha} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) \right\} + C(S, t_0).$$

For ease of notation we let

$$C(S, t_n) = C_n.$$

The former becomes

$$C_{n+1} = \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L}(f(S, t_n, C_n)) \right\} + C_0. \quad (6.2)$$

- **Stability of the Iterative Scheme**

Notice that the semi analytical solution of to the Caputo TFBSE is similar to the RL TFBSE semi-analytical solution, accounting for the constant term $C_0 = C(S, t_0)$.

Considering the TFBSE equation (5.1) and the iterative scheme solution equation (6.2).

$$C_{n+1} = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau + C_0.$$

Therefore

$$\begin{aligned} \|C_{n+1} - C_n\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau \right. \\ &\quad \left. - \int_0^{t_n} (t_n - \tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau \right\|. \end{aligned}$$

In subsection 5.2 we established that

$$\begin{aligned} \|\delta_n(t)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [f(S, \tau, C_{n-1}(S, \tau)) - f(S, \tau, C_{n-2}(S, \tau))] d\tau \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} k \|C_{n-1}(S, \tau) - C_{n-2}(S, \tau)\| d\tau \\ &\leq \frac{k}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \|\delta_{n-1}(S, \tau)\| d\tau \end{aligned}$$

This is $\|\delta_n(S, t)\| \leq \frac{k}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \|\delta_{n-1}(S, \tau)\| d\tau$ it follows that

$$\|\delta_n(S, t)\| \leq \max_{0 \leq t \leq T} \frac{(k^{-\alpha} t)^{n\alpha}}{\Gamma(1 + n\alpha)},$$

showing that the iterative scheme solution is stable.

6.4 Semi-Analytical solution of Caputo-Fabrizio TFBSE

6.4.1. Semi Analytical solution of the TFBSE with Caputo-Fabrizio fractional operator

Let us have the following time fractional Black-Scholes equation where the time fractional derivative is given by the Caputo-Fabrizio definition in equation (2.8).

$$\begin{cases} {}^{CF}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.6)$$

with

$${}^{CF}_a D_t^\alpha (C(S, t)) = \frac{M(\alpha)}{1 - \alpha} \int_a^t \frac{\partial C(S, \tau)}{\partial \tau} \exp \left[-\alpha \frac{t - \tau}{1 - \alpha} \right] d\tau,$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. However, if the function does not belongs to $H^1(a, b)$ then, the derivative can be reformulated as:

$$\begin{aligned} {}^{CF}_a D_t^\alpha (C(S, t)) &= \frac{M(\alpha)}{1 - \alpha} \int_a^t (C(S, t) - C(S, x)) \exp \left[-\alpha \frac{t - x}{1 - \alpha} \right] dx. \\ {}^{CF}_0 D_t^\alpha C(S, t) &= r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S}. \end{aligned}$$

Let

$$\begin{aligned} f(S, t, C(S, t)) &= r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S}, \\ \mathcal{L}({}^{CF}_0 D_t^\alpha C(S, t)) &= \mathcal{L}(f(S, t, C(S, t))), \end{aligned}$$

Let

$$v(t) = \frac{M(\alpha)}{1 - \alpha} \exp \left(-\frac{\alpha}{1 - \alpha} t \right), \quad u(t) = \frac{\partial C(S, \tau)}{\partial \tau},$$

$$\begin{aligned} {}^{CF}_0 D_t^\alpha C(S, t) &= v(t) * u(t) \\ &= \frac{M(\alpha)}{1 - \alpha} \int_a^t \frac{\partial C(S, \tau)}{\partial \tau} \exp \left[-\alpha \frac{t - \tau}{1 - \alpha} \right] d\tau. \end{aligned}$$

$$\mathcal{L}({}^{CF}_0 D_t^\alpha C(S, t)) = \mathcal{L}(v(t) * u(t)) = \mathcal{L}(v(t)) \times \mathcal{L}(u(t)).$$

$$\begin{aligned}
\mathcal{L}({}^{CF}_0 D_t^\alpha C(S, t)) &= \mathcal{L}\left(\frac{M(\alpha)}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha} t\right)\right) \times \mathcal{L}\left(\frac{\partial C(S, \tau)}{\partial \tau}\right), \\
&= \frac{M(\alpha)}{1-\alpha} \frac{1}{p + \frac{\alpha}{1-\alpha}} \left(p\mathcal{L}(C(S, t)) - C(S, t_0)\right).
\end{aligned}$$

This means that

$$\begin{aligned}
\frac{M(\alpha)}{1-\alpha} \frac{1}{p + \frac{\alpha}{1-\alpha}} \left(p\mathcal{L}(C(S, t)) - C(S, t_0)\right) &= \mathcal{L}\left(f(S, t, C(S, t))\right) \Rightarrow \\
p\mathcal{L}(C(S, t)) &= \frac{(1-\alpha)}{M(\alpha)} \left(p + \frac{\alpha}{1-\alpha}\right) \mathcal{L}\left(f(S, t, C(S, t))\right) + C(S, t_0) \Rightarrow \\
\mathcal{L}(C(S, t)) &= \frac{(1-\alpha)}{M(\alpha)p} \left(p + \frac{\alpha}{1-\alpha}\right) \mathcal{L}\left(f(S, t, C(S, t))\right) + \frac{1}{p} C(S, t_0) \\
C(S, t) &= \mathcal{L}^{-1} \left\{ \frac{(1-\alpha)}{M(\alpha)p} \left(p + \frac{\alpha}{1-\alpha}\right) \mathcal{L}\left(f(S, t, C(S, t))\right) + \frac{1}{p} C(S, t_0) \right\}.
\end{aligned}$$

From the above we introduce an iteration on $C(S, t)$ and we obtain the following

$$C(S, t_{n+1}) = \mathcal{L}^{-1} \left\{ \frac{(1-\alpha)}{M(\alpha)p} \left(p + \frac{\alpha}{1-\alpha}\right) \mathcal{L}\left(f(S, t_n, C(S, t_n))\right) \right\} + C(S, t_0).$$

For ease of notation we let

$$C(S, t_n) = C_n.$$

The former becomes

$$C_{n+1} = \mathcal{L}^{-1} \left\{ \frac{(1-\alpha)}{M(\alpha)p} \left(p + \frac{\alpha}{1-\alpha}\right) \mathcal{L}\left(f(S, t_n, C(S, t_n))\right) \right\} + C_0.$$

$$C_{n+1} = \mathcal{L}^{-1} \left\{ \left(\frac{1-\alpha}{M(\alpha)} \mathcal{L}\left(f(S, t_n, C(S, t_n))\right) + \frac{(1-\alpha)}{M(\alpha)} \frac{\alpha}{1-\alpha} \frac{1}{p} \mathcal{L}\left(f(S, t_n, C(S, t_n))\right) \right) \right\} + C_0. \quad (6.3)$$

- **Stability of the Iterative Scheme**

Considering the TFBSE equation (5.6) and the iterative scheme solution equation (6.3).

$$C_{n+1} = \mathcal{L}^{-1} \left\{ \left(\frac{1-\alpha}{M(\alpha)} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) + \frac{(1-\alpha)}{M(\alpha)} \frac{\alpha}{1-\alpha} \frac{1}{p} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) \right) \right\} + C_0$$

$$C_{n+1} = \frac{1-\alpha}{M(\alpha)} f(S, t_n, C(S, t_n)) + \frac{\alpha}{M(\alpha)} \mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) \right\} + C_0.$$

Therefore

$$\begin{aligned} \|C_{n+1} - C_n\| &= \left\| \frac{1-\alpha}{M(\alpha)} \left(f(S, t_n, C(S, t_n)) - f(S, t_{n-1}, C(S, t_{n-1})) \right) \right. \\ &\quad + \frac{\alpha}{M(\alpha)} \left(\mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) \right\} \right. \\ &\quad \left. \left. - \mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_{n-1}, C(S, t_{n-1})) \right) \right\} \right) \right\| \\ &< \frac{1-\alpha}{M(\alpha)} \|f(S, t_n, C(S, t_n)) - f(S, t_{n-1}, C(S, t_{n-1}))\| \\ &\quad + \frac{\alpha}{M(\alpha)} \left\| \mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) \right\} \right. \\ &\quad \left. \left. - \mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_{n-1}, C(S, t_{n-1})) \right) \right\} \right\| \right\|. \end{aligned}$$

In Chapter 5 subsection 5.3 where we discussed the existence and uniqueness of the solution to the TFBSE with Caputo-Fabrizio fractional derivative, lemma 5.2 showed that $\|f(S, t_n, C(S, t_n)) - f(S, t_{n-1}, C(S, t_{n-1}))\|$ is bounded, and with the contraction we constructed together with Arzela–Ascoli theorem,

$\left\| \mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_n, C(S, t_n)) \right) \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{p} \mathcal{L} \left(f(S, t_{n-1}, C(S, t_{n-1})) \right) \right\} \right\|$ is also bounded , and therefore our iterative semi analytical solution is stable.

6.5 Semi-Analytical solution of ABR TFBSE

6.5.1. Semi-Analytical solution of ABR TFBSE with Laplace transform

For this application we use the following Time Fractional Black Scholes Equation, where the time fractional derivative is the Atangana-Baleanu derivative in the Riemann Liouville sense.

$$\begin{cases} {}^{ABR}_0D_t^\alpha C(S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - rC(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.9)$$

with

$${}^{ABR}_0D_t^\alpha (C(S, t)) = \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t C(S, t) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx, \alpha \in [0, 1].$$

Using the preliminaries in subsection 6.1.1, and applying the Laplace transform operator on equation (5.9), while letting at the same time

$$f(S, t, C(s, t)) = rC(S, t) - r S \frac{\partial C(S, t)}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2}.$$

We have

$$\mathcal{L}\{{}^{ABR}_0D_t^\alpha (C(S, t))\} = \mathcal{L}\{f(S, t, C(s, t))\} \Rightarrow \frac{AB(\alpha)}{1-\alpha} \frac{s^\alpha}{s^\alpha + \frac{\alpha}{\alpha-1}} \mathcal{L}\{C(S, t)\} = \mathcal{L}\{f(S, t, C(s, t))\},$$

this is

$$\begin{aligned} \mathcal{L}\{C(S, t)\} &= \left(\frac{1-\alpha}{AB(\alpha)} + \frac{\alpha}{s^\alpha AB(\alpha)} \right) \mathcal{L}\{f(S, t, C(s, t))\}, \\ C(S, t) &= \mathcal{L}^{-1} \left\{ \left(\frac{1-\alpha}{AB(\alpha)} + \frac{\alpha}{s^\alpha AB(\alpha)} \right) \mathcal{L}\{f(S, t, C(s, t))\} \right\}, \end{aligned}$$

From the above we introduce iteration on $C(S, t)$ and we obtain the following

$$C(S, t_{n+1}) = C(S, t_n) + \mathcal{L}^{-1} \left\{ \left(\frac{1-\alpha}{AB(\alpha)} + \frac{\alpha}{s^\alpha AB(\alpha)} \right) \mathcal{L}\{f(S, t, C(s, t_n))\} \right\}.$$

For ease of notation we let

$$C(S, t_n) = C_n.$$

the former becomes

$$C_{n+1} = C_n + \mathcal{L}^{-1} \left\{ \left(\frac{1-\alpha}{AB(\alpha)} + \frac{\alpha}{s^\alpha AB(\alpha)} \right) \mathcal{L}\{f(S, t_n, C_n)\} \right\}.$$

Further the iterative scheme can be written as:

$$C_{n+1} = C_n + \frac{1-\alpha}{AB(\alpha)} f(S, t_n, C_n) + \frac{\alpha}{AB(\alpha)} \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L}\{f(S, t_n, C_n)\} \right\}. \quad (6.4)$$

- **Stability of the Iterative Scheme**

Considering the TFBSE equation (5.9) and the iterative scheme solution equation (6.4).

$$\begin{aligned} C_{n+1} &= C_n + \frac{1-\alpha}{AB(\alpha)} f(S, t_n, C_n) + \frac{\alpha}{AB(\alpha)} \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L}\{f(S, t_n, C_n)\} \right\}, \\ &= C_n + \frac{1-\alpha}{AB(\alpha)} f(S, t_n, C_n) + \frac{\alpha}{AB(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t-\tau)^{\alpha-1} f(S, \tau, C(S, \tau)) d\tau. \end{aligned}$$

Recall theorem 5.3, and the operator γ defined in subsection 5.4 we have that

$$\gamma C(S, t) = \frac{1-\alpha}{B(\alpha)} f(S, t, C(S, t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_1} f(S, \tau, C(S, \tau)) (t-\tau)^{\alpha-1} d\tau,$$

and that for any , $C_n, C_{n-1} \in L(0, t_n)$

$$||C_{n+1} - C_n|| = ||\gamma C(S, t_n) - \gamma C(S, t_{n-1})|| \leq \frac{(1-\alpha)B(\alpha) + t_n^\alpha}{B(\alpha)\Gamma(\alpha)} ||C_n - C_{n-1}||,$$

therefore the iterative scheme is stable by setting

$$\frac{(1-\alpha)B(\alpha) + t_n^\alpha}{B(\alpha)\Gamma(\alpha)} < 1.$$

6.5.2. Semi-Analytical solution of ABR TFBSE with Sumudu Transform

Similarly to subsection 6.5.1, we use here the Time Fractional Black Scholes Equation (5.9), where the time fractional derivative is the Atangana-Baleanu derivative in the Riemann Liouville sense.

$$\begin{cases} {}^{ABR}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.9)$$

with

$${}^{ABR}_0 \mathcal{D}_t^\alpha (C(S, t)) = \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t C(S, t) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx, \alpha \in [0, 1].$$

Using the preliminaries in subsection 6.1.2, and applying the Sumudu transform operator on equation (5.9), while letting at the same time

$$f(S, t, C(s, t)) = r C(S, t) - r S \frac{\partial C(S, t)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2},$$

we have

$$\mathbb{S}\{{}^{ABR}_0 \mathcal{D}_t^\alpha (C(S, t))\} = \mathbb{S}\{f(S, t, C(s, t))\} \Rightarrow$$

$$\frac{AB(\alpha) \alpha \Gamma(\alpha + 1)}{1-\alpha} E_\alpha \left(-\frac{1}{1-\alpha} u^\alpha \right) \mathbb{S}(C(S, t)) - \frac{AB(\alpha)}{1-\alpha} \frac{C(S, t_0) E_\alpha(0)}{u} = \mathbb{S}\{f(S, t, C(s, t))\},$$

$$\Rightarrow \mathbb{S}(C(S, t)) = \frac{\mathbb{S}\{f(S, t, C(s, t))\} + \frac{AB(\alpha)}{1-\alpha} \frac{C(S, t_0)}{u}}{\frac{AB(\alpha) \alpha \Gamma(\alpha + 1)}{1-\alpha} E_\alpha \left(-\frac{1}{1-\alpha} u^\alpha \right)},$$

$$\begin{aligned}
\Rightarrow \mathbb{S}(C(S, t)) &= \frac{1 - \alpha}{\alpha \Gamma(\alpha + 1) AB(\alpha) E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} \mathbb{S}\{f(S, t, C(S, t))\} \\
&\quad + \frac{E_\alpha(0)}{u \alpha \Gamma(\alpha + 1) E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} C(S, t_0), \\
\Rightarrow C(S, t) &= \mathbb{S}^{-1} \left\{ \frac{1 - \alpha}{\alpha \Gamma(\alpha + 1) AB(\alpha) E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} \mathbb{S}\{f(S, t, C(S, t))\} \right. \\
&\quad \left. + \frac{1}{u \alpha \Gamma(\alpha + 1) E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} C(S, t_0) \right\},
\end{aligned}$$

Since $E_\alpha(0) = 1$

$$\begin{aligned}
C(S, t) &= \frac{1 - \alpha}{\alpha \Gamma(\alpha + 1) AB(\alpha)} \mathbb{S}^{-1} \left\{ \frac{1}{E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} \mathbb{S}\{f(S, t, C(S, t))\} \right\} \\
&\quad + \frac{C(S, t_0)}{\alpha \Gamma(\alpha + 1)} \mathbb{S}^{-1} \left\{ \frac{1}{u E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} \right\}.
\end{aligned}$$

Next we introduce an iteration on $C(S, t)$ while also letting $C(S, t_n) = C_n$ for ease of notation, we have the iterative scheme defined as follows:

$$\begin{aligned}
C_{n+1} &= C_n + \frac{1 - \alpha}{\alpha \Gamma(\alpha + 1) AB(\alpha)} \mathbb{S}^{-1} \left\{ \frac{1}{E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} \mathbb{S}\{f(S, t, C_n)\} \right\} \\
&\quad + \frac{C_0}{\alpha \Gamma(\alpha + 1)} \mathbb{S}^{-1} \left\{ \frac{1}{u E_\alpha\left(-\frac{1}{1 - \alpha} u^\alpha\right)} \right\}. \tag{6.5}
\end{aligned}$$

- **Stability of the Iterative Scheme**

Considering the TFBSE (5.9) and the iterative scheme solution (6.5).

$$\begin{aligned}
C_{n+1} &= C_n + \frac{1-\alpha}{\alpha\Gamma(\alpha+1)AB(\alpha)} \mathbb{S}^{-1} \left\{ \frac{1}{E_\alpha\left(-\frac{1}{1-\alpha}u^\alpha\right)} \mathbb{S}\{f(S, t, C_n)\} \right\} \\
&\quad + \frac{C_0}{\alpha\Gamma(\alpha+1)} \mathbb{S}^{-1} \left\{ \frac{1}{uE_\alpha\left(-\frac{1}{1-\alpha}u^\alpha\right)} \right\}, \\
||C_{n+1} - C_n|| &= \left\| \left\| \frac{1-\alpha}{\alpha\Gamma(\alpha+1)AB(\alpha)} \mathbb{S}^{-1} \left\{ \frac{1}{E_\alpha\left(-\frac{1}{1-\alpha}u^\alpha\right)} \mathbb{S}\{f(S, t, C_n)\} \right\} \right. \right. \\
&\quad \left. \left. + \frac{C_0}{\alpha\Gamma(\alpha+1)} \mathbb{S}^{-1} \left\{ \frac{1}{uE_\alpha\left(-\frac{1}{1-\alpha}u^\alpha\right)} \right\} \right\| \right\| \\
&\leq \frac{1-\alpha}{\alpha\Gamma(\alpha+1)AB(\alpha)} \left\| \left\| \mathbb{S}^{-1} \left\{ \frac{1}{E_\alpha\left(-\frac{1}{1-\alpha}u^\alpha\right)} \mathbb{S}\{f(S, t, C_n)\} \right\} \right\| \right\| \\
&\quad + \frac{C_0}{\alpha\Gamma(\alpha+1)} \left\| \left\| \mathbb{S}^{-1} \left\{ \frac{1}{uE_\alpha\left(-\frac{1}{1-\alpha}u^\alpha\right)} \right\} \right\| \right\| \\
&\leq \frac{1-\alpha}{\alpha\Gamma(\alpha+1)AB(\alpha)}.
\end{aligned}$$

If $\frac{1-\alpha}{\alpha\Gamma(\alpha+1)AB(\alpha)} < 1$, then the iterative scheme is stable.

7. Numerical Analysis of TFBSE

This Chapter is devoted to the derivation of numerical scheme solutions to the Time Fractional Black Scholes equations (TFBSEs), with fractional operators in the sense of Caputo, Riemann Liouville, Caputo-Fabrizio, and the two Atangana-Baleanu operators. We discuss the convergence of the Caputo and RL numerical scheme solutions, to benchmark their rate against the existing literature. Additionally we present a Crank-Nicholson solution scheme of the Caputo TFBSE.

7.1 Numerical solution of TFBSE with Caputo Derivative

We will present here a numerical scheme to approximate the Caputo-TFBSE

$$\begin{cases} {}^c_0D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (S, t) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.1)$$

where

$${}^c_0D_t^\alpha C(S, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial C(S, \tau)}{\partial \tau} d\tau, \quad 0 < \alpha < 1.$$

Just as H. Zhang et al in [75] we will truncate the original space domain to make a finite interval $[B_a, B_u]$.

Let us adopt the following notation

$$t_i = i \Delta t, \quad t_{i+1} = (i+1)\Delta t, \quad \Delta t = t_{i+1} - t_i,$$

$$x_j = j \Delta x, \quad x_{j+1} = (j+1)\Delta x, \quad \Delta x = x_{j+1} - x_j; \quad U(x_j, t_i) = U_j^i.$$

With the notation above, the time fractional Caputo derivative in equation (5.1) can now be written as:

$${}^c_0D_t^\alpha C(S, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial C(S, \tau)}{\partial \tau} d\tau,$$

$$\begin{aligned}
{}_0^C D_t^\alpha C(S_i, t_n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial C(S_i, \tau)}{\partial \tau} (t_n - \tau)^{-\alpha} d\tau, \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \frac{C(S_i, t_{j+1}) - C(S_i, t_j)}{\Delta t} (t_n - \tau)^{-\alpha} d\tau, \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^n \frac{C(S_i, t_{j+1}) - C(S_i, t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{-\alpha} d\tau, \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^n \frac{C_i^{j+1} - C_i^j}{\Delta t} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{-\alpha} d\tau, \\
{}_0^C D_t^\alpha C(S_i, t_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^n \frac{C_i^{j+1} - C_i^j}{\Delta t} \frac{1}{1-\alpha} \left((t_n - t_j)^{1-\alpha} - (t_n - t_{j+1})^{1-\alpha} \right), \\
&= \frac{1}{1-\alpha} \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^n \frac{C_i^{j+1} - C_i^j}{\Delta t} \left((n-j)^{1-\alpha} - (n-j-1)^{1-\alpha} \right) (\Delta t)^{1-\alpha}, \\
&= \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n \frac{C_i^{j+1} - C_i^j}{\Delta t} \left((n-j)^{1-\alpha} - (n-j-1)^{1-\alpha} \right).
\end{aligned}$$

Let

$$\delta_{n,j}^\alpha = (n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}.$$

$${}_0^C D_t^\alpha C(S_i, t_n) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial C(S_i, \tau)}{\partial \tau} (t_n - \tau)^{-\alpha} d\tau = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha.$$

Moreover discretizing the space derivatives in equation (5.1) we have

$$\begin{aligned}
\frac{\partial C(S_i, t_n)}{\partial S} &= \frac{C(S_{i+1}, t_n) - C(S_{i-1}, t_n)}{2\Delta x}, \\
\frac{\partial^2 C(S_i, t_n)}{\partial S^2} &= \frac{C(S_{i+1}, t_n) - 2C(S_i, t_n) + C(S_{i-1}, t_n)}{(\Delta x)^2}.
\end{aligned}$$

For simplicity let $a = \frac{1}{2} \sigma^2$ equation (4.1) can be rewritten as

$$\begin{aligned}
& \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha \\
&= r C(S_i, t_n) - a S_i^2 \frac{C(S_{i+1}, t_n) - 2C(S_i, t_n) + C(S_{i-1}, t_n)}{(\Delta x)^2} \\
&\quad - r S_i \frac{C(S_{i+1}, t_n) - C(S_{i-1}, t_n)}{2\Delta x},
\end{aligned}$$

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha = r C_i^n - a S_i^2 \frac{C_i^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2} - r S_i \frac{C_{i+1}^n - C_{i-1}^n}{2\Delta x}.$$

By letting $d = (\Delta t)^\alpha \Gamma(2-\alpha)$, $h = \Delta x$,

we can write

$$\sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha = r d C_i^n - \frac{ad}{h^2} S_i^2 (C_{i+1}^n - 2C_i^n + C_{i-1}^n) - \frac{rd}{2h} S_i (C_{i+1}^n - C_{i-1}^n),$$

$$\sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha = r d C_i^n + 2 \frac{ad}{h^2} S_i^2 C_i^n - \frac{ad}{h^2} S_i^2 C_{i+1}^n - \frac{ad}{h^2} S_i^2 C_{i-1}^n + \frac{rd}{2h} S_i C_{i-1}^n - \frac{rd}{2h} S_i C_{i+1}^n,$$

$$\begin{aligned}
& \sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha \\
&= -\frac{rd}{2h} S_i C_{i+1}^n - \frac{ad}{h^2} S_i^2 C_{i+1}^n + r d C_i^n + 2 \frac{ad}{h^2} S_i^2 C_i^n - \frac{ad}{h^2} S_i^2 C_{i-1}^n + \frac{rd}{2h} S_i C_{i-1}^n,
\end{aligned}$$

$$\begin{aligned}
& (C_i^{n+1} - C_i^n) \delta_{n,n}^\alpha \\
&= -\left(\frac{rd}{2h} S_i + \frac{ad}{h^2} S_i^2\right) C_{i+1}^n + \left(rd + 2 \frac{ad}{h^2} S_i^2\right) C_i^n + \left(\frac{rd}{2h} S_i - \frac{ad}{h^2} S_i^2\right) C_{i-1}^n \\
&\quad - \sum_{j=0}^{n-1} (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha,
\end{aligned}$$

$$\begin{aligned}
& (C_i^{n+1} - C_i^n) \delta_{n,n}^\alpha \\
& = - \left(\frac{rd}{2h} S_i + \frac{ad}{h^2} S_i^2 \right) C_{i+1}^n + \left(rd + 2 \frac{ad}{h^2} S_i^2 \right) C_i^n + \left(\frac{rd}{2h} S_i - \frac{ad}{h^2} S_i^2 \right) C_{i-1}^n \\
& \quad - C_i^n \delta_{n,(n-1)}^\alpha + C_i^{n-1} \delta_{n,(n-1)}^\alpha - \sum_{j=0}^{n-2} (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha, \\
& \delta_{n,n}^\alpha C_i^{n+1} = \left(rd + 2 \frac{ad}{h^2} S_i^2 - \delta_{n,(n-1)}^\alpha + \delta_{n,n}^\alpha \right) C_i^n - \left(\frac{rd}{2h} S_i + \frac{ad}{h^2} S_i^2 \right) C_{i+1}^n \\
& \quad + \left(\frac{rd}{2h} S_i - \frac{ad}{h^2} S_i^2 \right) C_{i-1}^n - \sum_{j=0}^{n-2} (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha. \tag{7.1}
\end{aligned}$$

7.1.1. Numerical Applications

To assess the accuracy of our solution, we choose a time fractional model with non-homogenous boundary condition and for which unlike a TFBSE, there exists an exact solution. We compare convergence rate and errors of the numerical approximations of the solution scheme (7.1). We benchmark our results for the same equation with H. Zhang et al [75]. The equation is:

$$\begin{cases} {}_0^c D_\tau U(x, \tau) = a \frac{\partial^2 U(x, \tau)}{\partial x^2} + b \frac{\partial U(x, \tau)}{\partial x} - c \partial U(x, \tau) + f(x, \tau), \\ U(0, \tau) = 0, \quad U(1, \tau) = 0, \quad U(x, 0) = x^2(1 - x^2), \end{cases} \tag{7.2}$$

where

$$\begin{aligned}
f(x, \tau) &= \left(\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)} \right) x^2(1-x) \\
&\quad - (t+1)^2 (a(2-6x) + b(2x-3x^2) - cx^2(1-x)),
\end{aligned}$$

and the exact solution is

$$U(x, t) = (t+1)^2 (x^3 + x^2 + 1).$$

We will first do the calculations by fixing the time step and varying the space step, and thereafter by fixing the space step and varying the time step.

Fixing the time step we have the following parameters:

the time step $\Delta t = \frac{1}{1000}$;

the order of convergence is calculated as: $conv.order = \frac{\ln(\frac{error_1}{error_2})}{\ln 2}$.

The results are summarised by table 3 below:

Table 3: Approximations errors for (7.2) from solution scheme (7.1)

$h = \Delta x$	$\ \cdot\ - error$	<i>Conv. Order</i>	<i>Max - error</i>	<i>Conv. Order</i>
1/4	0.0023		0.0028	
1/8	4.668×10^{-4}	2.30	5.607×10^{-4}	2.32
1/16	8.906×10^{-5}	2.39	1.055×10^{-4}	2.41
1/32	1.563×10^{-5}	2.51	1.839×10^{-5}	2.52
1/64	1.420×10^{-6}	3.46	2.236×10^{-6}	3.04

Fixing the space step we have the following parameters:

the space step $h = \frac{1}{100}$;

the order of convergence is calculated as $conv.order = \frac{\ln(\frac{error_1}{error_2})}{\ln 2}$.

The results are summarised by table 4 below:

Table 4: Approximations errors for (7.2) from solution scheme (7.1)

Δt	$\ \cdot\ - error$	<i>Conv. Order</i>	<i>Max - error</i>	<i>Conv. Order</i>
1/10	0.0024		0.0096	
1/20	8.368×10^{-4}	1.52	3.465×10^{-3}	1.47
1/40	3.106×10^{-4}	1.43	1.234×10^{-3}	1.49
1/80	1.098×10^{-4}	1.50	4.332×10^{-4}	1.51
1/180	3.776×10^{-5}	1.54	1.500×10^{-4}	1.53

From tables 3 and 4 we can see that our numerical scheme solution delivers satisfactory order of convergence. Approximate solutions are close enough to the exact solution. The solution of our numerical scheme solution (7.1) in time converges slightly faster than that of [75]. The convergence rate in the space parameter is quite similar to that of [75].

7.1.2. Discretization with Crank Nicholson Scheme

Applying Crank-Nicholson Scheme we will approximate the derivatives as follows:

$$\begin{aligned}\frac{\partial C(S_i, t_n)}{\partial S} &= \frac{1}{2} \left(\frac{C(S_{i+1}, t_n) - C(S_{i-1}, t_n)}{2\Delta x} + \frac{C(S_{i+1}, t_{n+1}) - C(S_{i-1}, t_{n+1})}{2\Delta x} \right), \\ \frac{\partial C(S_i, t_n)}{\partial S} &= \frac{1}{2} \left(\frac{C_{i+1}^n - C_{i-1}^n}{2h} + \frac{C_{i+1}^{n+1} - C_{i-1}^{n+1}}{2h} \right), \\ \frac{\partial^2 C(S_i, t_n)}{\partial S^2} &= \frac{1}{2} \frac{C(S_{i+1}, t_n) + C(S_{i+1}, t_{n+1}) - 2(C(S_i, t_n) + C(S_i, t_{n+1})) + C(S_{i-1}, t_n) + C(S_{i-1}, t_{n+1})}{(\Delta C)^2}, \\ &= \frac{1}{2} \left(\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{h^2} + \frac{C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}}{h^2} \right).\end{aligned}$$

By letting $d = (\Delta t)^\alpha \Gamma(2 - \alpha)$, $h = \Delta x$, $a = \frac{1}{2} \sigma^2$.

The Time Fractional Black Scholes Equation (5.1) with Caputo derivative becomes

$$\begin{aligned}\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha \\ = r C_i^n - a S_i^2 \frac{1}{2} \left(\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{h^2} + \frac{C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}}{h^2} \right) \\ - r S_i \frac{1}{2} \left(\frac{C_{i+1}^n - C_{i-1}^n}{2h} + \frac{C_{i+1}^{n+1} - C_{i-1}^{n+1}}{2h} \right).\end{aligned}$$

$$\begin{aligned}
& \sum_{j=0}^n (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha \\
&= rC_i^n - \frac{aS_i^2}{2h^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n + C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}) \\
&\quad - \frac{rS_i}{4h} (C_{i+1}^n - C_{i-1}^n + C_{i+1}^{n+1} - C_{i-1}^{n+1}).
\end{aligned}$$

This implies that

$$\begin{aligned}
& (C_i^n - C_i^{n-1}) \delta_{n,n-1}^\alpha + (C_i^{n+1} - C_i^n) \delta_{n,n}^\alpha \\
&= rC_i^n - \frac{aS_i^2}{2h^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n + C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}) \\
&\quad - \frac{rS_i}{4h} (C_{i+1}^n - C_{i-1}^n + C_{i+1}^{n+1} - C_{i-1}^{n+1}) - \sum_{j=0}^{n-2} (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha,
\end{aligned}$$

and

$$\begin{aligned}
& \delta_{n,n}^\alpha C_i^{n+1} - \frac{aS_i^2}{h^2} C_i^{n+1} \\
&= rC_i^n + \frac{aS_i^2}{h^2} C_i^n - \delta_{n,n-1}^\alpha C_i^n + \delta_{n,n}^\alpha C_i^n - \frac{aS_i^2}{2h^2} C_{i+1}^n - \frac{rS_i}{4h} C_{i+1}^n - \frac{aS_i^2}{2h^2} C_{i-1}^{n+1} \\
&\quad + \frac{rS_i}{4h} C_{i-1}^{n+1} - \frac{rS_i}{4h} C_{i+1}^{n+1} - \frac{aS_i^2}{2h^2} C_{i+1}^{n+1} + \frac{rS_i}{4h} C_{i-1}^n - \frac{aS_i^2}{2h^2} C_{i-1}^n + \delta_{n,n-1}^\alpha C_i^{n-1} \\
&\quad - \sum_{j=0}^{n-2} (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha.
\end{aligned}$$

The numerical scheme solution can therefore be obtained as:

$$\begin{aligned}
& (\delta_{n,n}^\alpha - \frac{aS_i^2}{h^2}) C_i^{n+1} \\
&= \left(r + \frac{aS_i^2}{h^2} - \delta_{n,n-1}^\alpha + \delta_{n,n}^\alpha \right) C_i^n - \left(\frac{aS_i^2}{2h^2} + \frac{rS_i}{4h} \right) C_{i+1}^n + \left(\frac{rS_i}{4h} - \frac{aS_i^2}{2h^2} \right) C_{i-1}^{n+1} \\
&\quad - \left(\frac{aS_i^2}{2h^2} + \frac{rS_i}{4h} \right) C_{i+1}^{n+1} + \left(\frac{rS_i}{4h} - \frac{aS_i^2}{2h^2} \right) C_{i-1}^n + \delta_{n,n-1}^\alpha C_i^{n-1} \\
&\quad - \sum_{j=0}^{n-2} (C_i^{j+1} - C_i^j) \delta_{n,j}^\alpha. \tag{7.3}
\end{aligned}$$

As Crank Nicholson order of convergence for diffusion equations is well studied problem [80, 81]-the solution in most cases being generally unconditionally stable and convergent of order 2, in both space and time-we did not provide numerical approximations for the solution scheme (7.3).

7.2 Numerical Solution of TFBSE with Riemann Liouville Derivative

We will present here a numerical scheme to approximate the TFBSE

$$\begin{cases} {}^{RL}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (S, t) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.2)$$

where

$${}^{RL}_0 D_t^\alpha C(S, t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} C(S, \tau) d\tau, \quad 0 < \alpha < 1.$$

Let us start by presenting a discretization of the Riemann-Liouville Fractional derivative ${}^{RL}_0 D_t^\alpha C(S, t)$. The approach choose here has been suggested by [76-78] and used by Atangana A and J.F Gomez-Aguilar in [79].

If we let

$$E(S, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} C(S, \tau) d\tau,$$

$${}^{RL}_0 D_t^\alpha C(S, t) = \frac{d}{dt} E(S, t),$$

which we can discretize as

$$\frac{d}{dt} E(S, t) \approx \frac{E(S, t_{j+1}) - E(S, t_j)}{\Delta t},$$

$$E(S, t_{j+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} (t_{j+1} - \tau)^{-\alpha} C(S, \tau) d\tau,$$

$$E(S, t_j) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j - \tau)^{-\alpha} C(S, \tau) d\tau,$$

Further we give the following full discretization without loss of generality

$$\begin{aligned}
E(S, t_{j+1}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} (t_{j+1} - \tau)^{-\alpha} C(S, \tau) d\tau, \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} (t_{j+1} - \tau)^{-\alpha} \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} d\tau + o(\Delta t), \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} \int_{t_k}^{t_{k+1}} (t_{j+1} - \tau)^{-\alpha} d\tau + o(\Delta t), \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} \frac{1}{1-\alpha} [(t_{j+1} - t_k)^{1-\alpha} \\
&\quad - (t_{j+1} - t_{k+1})^{1-\alpha}] + o(\Delta t), \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^j \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} [(t_{j+1} - t_k)^{1-\alpha} \\
&\quad - (t_{j+1} - t_{k+1})^{1-\alpha}] + o(\Delta t).
\end{aligned}$$

Likewise we can write

$$\begin{aligned}
E(S, t_j) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j - \tau)^{-\alpha} C(S, \tau) d\tau, \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \frac{(C(S, t_k) + C(S, t_{k-1}))}{2} [(t_j - t_{k-1})^{1-\alpha} - (t_j - t_k)^{1-\alpha}] \\
&\quad + o(\Delta t), \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^j \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} [(t_j - t_k)^{1-\alpha} - (t_j - t_{k+1})^{1-\alpha}] + o(\Delta t).
\end{aligned}$$

It then follows that

$$\begin{aligned}
& \frac{d}{dt} E(S, t) \\
&= \frac{1}{\Delta t \Gamma(2 - \alpha)} \left(\sum_{k=0}^j \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} [(t_{j+1} - t_k)^{1-\alpha} - (t_{j+1} - t_{k+1})^{1-\alpha}] \right. \\
&\quad \left. - \sum_{k=1}^j \frac{(C(S, t_{k+1}) + C(S, t_k))}{2} [(t_j - t_k)^{1-\alpha} - (t_j - t_{k+1})^{1-\alpha}] \right) \\
&\quad + R_{\alpha, j},
\end{aligned} \tag{7.4}$$

where

$$R_{\alpha, j} = \frac{1}{\Delta t \Gamma(1 - \alpha)} \left\{ \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{C(S, \tau) - C(S, t_{k+1})}{(t_{j+1} - \tau)^\alpha} d\tau - \sum_{k=0}^{j-1} \int_{t_{k-1}}^{t_k} \frac{C(S, \tau) - C(S, t_k)}{(t_j - \tau)^\alpha} d\tau \right\}.$$

Theorem 7.1

Let f be a function not necessary differentiable within an interval $[0, T]$, then the fractional derivative of f of order α in the Riemann Liouville sense is given as

$$\begin{aligned}
{}^{RL}_0 D_t^\alpha \{f(t)\} &= \frac{1}{\Delta t \Gamma(2 - \alpha)} \left(\sum_{k=0}^j \frac{(f(t_{k+1}) + f(t_k))}{2} [(t_{j+1} - t_k)^{1-\alpha} - (t_{j+1} - t_{k+1})^{1-\alpha}] \right. \\
&\quad \left. - \sum_{k=1}^j \frac{(f(t_{k+1}) + f(t_k))}{2} [(t_j - t_k)^{1-\alpha} - (t_j - t_{k+1})^{1-\alpha}] \right) + R_{\alpha, j},
\end{aligned}$$

where there is a A such that

$$|R_{\alpha, j}| \leq A \times (t_{j+1}^{1-\alpha} - t_j^\alpha),$$

and

$$R_{\alpha, j} = \frac{1}{\Delta t \Gamma(1 - \alpha)} \left\{ \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{f(\tau) - f(t_{k+1})}{(t_{j+1} - \tau)^\alpha} d\tau - \sum_{k=0}^{j-1} \int_{t_{k-1}}^{t_k} \frac{f(\tau) - f(t_k)}{(t_j - \tau)^\alpha} d\tau \right\}.$$

Proof

Since we have

$$|R_{\alpha,j}| = \left| \frac{1}{\Delta t \Gamma(1-\alpha)} \left\{ \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{f(\tau) - f(t_{k+1})}{(t_{j+1} - \tau)^\alpha} d\tau - \sum_{k=0}^{j-1} \int_{t_{k-1}}^{t_k} \frac{f(\tau) - f(t_k)}{(t_j - \tau)^\alpha} d\tau \right\} \right|,$$

$$\begin{aligned} & \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{f(\tau) - f(t_{k+1})}{(t_{j+1} - \tau)^\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{(f(\tau) - f(t_{k+1}))(\tau - t_{k+1})}{(\tau - t_{k+1})(t_{j+1} - \tau)^\alpha} d\tau, \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} f'(\lambda_k) \frac{\tau - t_{k+1}}{(t_{j+1} - \tau)^\alpha} d\tau, \quad \tau < t < t_{j+1}, \end{aligned}$$

$$\begin{aligned} & \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} f'(\lambda_k) \frac{\tau - t_{k+1}}{(t_{j+1} - \tau)^\alpha} d\tau \right| \leq \frac{t_{j+1}}{\Gamma(1-\alpha)} \max_{0 \leq t < t_{j+i}} |f'(t)| \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{1}{(t_{j+1} - \tau)^\alpha} d\tau \\ & \leq \frac{t_{j+1}}{\Gamma(1-\alpha)} \max_{0 \leq t < t_{j+i}} |f'(t)| \sum_{k=0}^j \left[-\frac{1}{1-\alpha} (t_{j+1} - \tau)^{1-\alpha} \right]_{t_k}^{t_{k+1}} \\ & \leq \frac{1}{1-\alpha} \frac{t_{j+1}}{\Gamma(1-\alpha)} \max_{0 \leq t < t_{j+i}} |f'(t)| \sum_{k=0}^j \left((t_{j+1} - t_k)^{1-\alpha} - (t_{j+1} - t_{k+1})^{1-\alpha} \right) \\ & \leq \frac{(\Delta t)^{1-\alpha} t_{j+1}}{\Gamma(2-\alpha)} \max_{0 \leq t < t_{j+i}} |f'(t)| \sum_{k=0}^j ((j+1-k)^{1-\alpha} - (j-k)^{1-\alpha}) \\ & \leq \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \max_{0 \leq t < t_{j+i}} |f'(t)| (j+1)^{1-\alpha} t_{j+1}. \end{aligned}$$

Letting

$$A = \frac{1}{\Gamma(2-\alpha)} \max_{0 \leq t < t_{j+i}} |f'(t)|,$$

we obtain the following

$$|R_{\alpha,j}| < \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \max_{0 \leq t < t_{j+1}} |f'(t)| (j+1)^{1-\alpha} t_{j+1}$$

$$|R_{\alpha,j}| < A(t_{j+1}^\alpha - t_j^\alpha).$$

Theorem 7.2 proved the discretization of the Riemann Liouville time fractional derivative is stable.

Now before we proceed to discretize the TFBSE (5.2), let us recall the discretization of the space partial derivatives

$$\begin{aligned}\frac{\partial C(S_i, t_n)}{\partial S} &= \frac{C(S_{i+1}, t_n) - C(S_{i-1}, t_n)}{2\Delta x}, \\ \frac{\partial^2 C(S_i, t_n)}{\partial S^2} &= \frac{C(S_{i+1}, t_n) - 2C(S_i, t_n) + C(S_{i-1}, t_n)}{(\Delta x)^2}.\end{aligned}$$

Continuing and incorporating into equation (5.2) we have the following discretized version of the TFBSE with the Riemann Liouville derivative.

$$\begin{aligned}& \frac{1}{\Delta t \Gamma(2-\alpha)} \left(\sum_{k=0}^{n-1} \frac{(C(S_i, t_{k+1}) + C(S_i, t_k))}{2} [(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}] \right. \\ & \quad \left. - \sum_{k=1}^n \frac{(C(S_i, t_{k+1}) + C(S_i, t_k))}{2} [(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}] \right) \\ & = rC(S_i, t_n) - \frac{1}{2}\sigma^2 S_i^2 \frac{C(S_{i+1}, t_n) - 2C(S_i, t_n) + C(S_{i-1}, t_n)}{(\Delta x)^2} \\ & \quad - rS_i \frac{C(S_{i+1}, t_n) - C(S_{i-1}, t_n)}{2\Delta x}.\end{aligned}$$

This is

$$\begin{aligned}
& \frac{1}{\Delta t \Gamma(2-\alpha)} \left(\sum_{k=0}^{n-1} \frac{(C_i^{k+1} + C_i^k)}{2} [(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}] \right. \\
& \quad \left. - \sum_{k=1}^n \frac{(C_i^{k+1} + C_i^k)}{2} [(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}] \right) \\
& = rC_i^n - \frac{1}{2} \sigma^2 S_i^2 \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2} \\
& \quad - rS_i \frac{C_{i+1}^n - C_{i-1}^n}{2\Delta x}. \tag{7.5}
\end{aligned}$$

By letting $d = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}$, $h = \Delta x$, $a = \frac{1}{2} \sigma^2$ equation (7.5) becomes

$$\begin{aligned}
& \frac{1}{\Delta t \Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{(C_i^{k+1} + C_i^k)}{2} [(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}] \\
& \quad - \sum_{k=1}^n \frac{(C_i^{k+1} + C_i^k)}{2} [(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha}] \\
& = rC_i^n - \frac{aS_i^2}{h^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n) - \frac{rS_i}{2h} (C_{i+1}^n - C_{i-1}^n), \\
& \Rightarrow (\Delta t)^{1-\alpha} \frac{1}{\Delta t \Gamma(2-\alpha)} \left\{ \sum_{k=0}^{n-1} \frac{(C_i^{k+1} + C_i^k)}{2} [(n-k)^{1-\alpha} \right. \\
& \quad \left. - (n-k-1)^{1-\alpha}] - \sum_{k=1}^n \frac{(C_i^{k+1} + C_i^k)}{2} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}] \right\} \\
& = rC_i^n - \frac{aS_i^2}{h^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n) - \frac{rS_i}{2h} (C_{i+1}^n - C_{i-1}^n),
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left\{ \frac{C_i^1 + C_i^0}{2} (n^{1-\alpha} - (n-1)^{1-\alpha}) \right. \\
&\quad + \sum_{k=1}^{n-1} \frac{(C_i^{k+1} + C_i^k)}{2} [(n-k)^{1-\alpha} \\
&\quad - (n-k-1)^{1-\alpha}] - \sum_{k=1}^{n-1} \frac{(C_i^{k+1} + C_i^k)}{2} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}] \\
&\quad \left. + \frac{C_i^{n+1} + C_i^n}{2} (-1)^{2-\alpha} \right\} \\
&= rC_i^n - \frac{aS_i^2}{h^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n) - \frac{rS_i}{2h} (C_{i+1}^n - C_{i-1}^n), \\
&\Rightarrow \left\{ \frac{C_i^1 + C_i^0}{2} (n^{1-\alpha} - (n-1)^{1-\alpha}) + \frac{C_i^{n+1} + C_i^n}{2} (-1)^{2-\alpha} \right\} \\
&= \frac{r}{d} C_i^n - \frac{aS_i^2}{dh^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n) - \frac{rS_i}{2dh} (C_{i+1}^n - C_{i-1}^n), \\
&\Rightarrow \{(C_i^1 + C_i^0)(n^{1-\alpha} - (n-1)^{1-\alpha}) + (C_i^{n+1} + C_i^n)(-1)^{2-\alpha}\} \\
&= \frac{2r}{d} C_i^n - \frac{2aS_i^2}{dh^2} (C_{i+1}^n - 2C_i^n + C_{i-1}^n) - \frac{rS_i}{dh} (C_{i+1}^n - C_{i-1}^n), \\
&\Rightarrow (-1)^{2-\alpha} C_i^{n+1} \\
&= \frac{2r}{d} C_i^n - \frac{2aS_i^2}{dh^2} C_{i+1}^n + \frac{4aS_i^2}{dh^2} C_i^n - \frac{2aS_i^2}{dh^2} C_{i-1}^n - \frac{rS_i}{dh} C_{i+1}^n + \frac{rS_i}{dh} C_{i-1}^n \\
&\quad - (C_i^1 + C_i^0)(n^{1-\alpha} - (n-1)^{1-\alpha}) - (-1)^{2-\alpha} C_i^n, \\
&\Rightarrow (-1)^{2-\alpha} C_i^{n+1} \\
&= \frac{2r}{d} C_i^n + \frac{4aS_i^2}{dh^2} C_i^n - \frac{2aS_i^2}{dh^2} C_{i+1}^n - \frac{rS_i}{dh} C_{i+1}^n - \frac{2aS_i^2}{dh^2} C_{i-1}^n + \frac{rS_i}{dh} C_{i-1}^n \\
&\quad - (C_i^1 + C_i^0)(n^{1-\alpha} - (n-1)^{1-\alpha}) - (-1)^{2-\alpha} C_i^n, \\
&(-1)^{2-\alpha} C_i^{n+1} = \left(\frac{2r}{d} + \frac{4aS_i^2}{dh^2} \right) C_i^n - \left(\frac{2aS_i^2}{dh^2} + \frac{rS_i}{dh} \right) C_{i+1}^n + \left(\frac{rS_i}{dh} - \frac{2aS_i^2}{dh^2} \right) C_{i-1}^n \\
&\quad - (C_i^1 + C_i^0)(n^{1-\alpha} - (n-1)^{1-\alpha}) \\
&\quad - (-1)^{2-\alpha} C_i^n. \tag{7.6}
\end{aligned}$$

7.2.1. Numerical Applications

Similarly to what was done in the subsection 7.1.1, we assess the convergence of the numerical scheme solution (7.6) by comparing its approximations to the diffusion equation (7.2) whose exact solution is known. The parameterization and methodology is the same as in the subsection 7.1.1. The results are summarized in tables 5 and 6 below:

Fixing the time step we have the following parameters

$$\text{Time step } \Delta t = \frac{1}{1000};$$

$$\text{The order of convergence is calculated as: } conv.order = \frac{\ln\left(\frac{error_1}{error_2}\right)}{\ln 2}.$$

The results are summarised by table 5 below:

Table 5: Approximations errors for (7.2) from solution scheme (7.6)

$h = \Delta x$	$\ \cdot \ - error$	<i>Conv. Order</i>	<i>Max - error</i>	<i>Conv. Order</i>
1/4	0.0025		0.0030	
1/8	5.555×10^{-4}	2.17	6.575×10^{-4}	2.19
1/16	1.160×10^{-4}	2.26	1.354×10^{-4}	2.28
1/32	2.108×10^{-5}	2.46	2.443×10^{-5}	2.47
1/64	2.528×10^{-6}	3.06	3.865×10^{-6}	2.66

Fixing the space step we have the following parameters

$$\text{The space step } h = \frac{1}{100};$$

$$\text{The order of convergence is calculated as: } conv.order = \frac{\ln\left(\frac{error_1}{error_2}\right)}{\ln 2}.$$

The results are summarised by table 6 below:

Table 6: Approximations errors for (7.2) from solution scheme (7.6)

Δt	$\ \cdot\ - error$	<i>Conv. Order</i>	<i>Max - error</i>	<i>Conv. Order</i>
1/10	0.0026		0.0086	
1/20	9.921×10^{-4}	1.39	3.304×10^{-3}	1.38
1/40	3.974×10^{-4}	1.32	4.778×10^{-3}	1.40
1/80	1.538×10^{-4}	1.37	1.798×10^{-4}	1.41
1/180	5.746×10^{-5}	1.42	6.672×10^{-5}	1.43

From tables 5 and 6 we can also say that the numerical scheme solution (7.6) delivers a satisfactory order of convergence. The Approximate solutions seem to converge a little bit slower than the approximations obtained in subsection 7.1.1 where the scheme solution is derived from the time Caputo fractional derivative.

7.3 Numerical Solution of TFBSE with Caputo-Fabrizio Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given by the Caputo-Fabrizio definition in equation (2.8).

$$\begin{cases} {}^{CF}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (S, t) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.6)$$

with

$${}^{CF}_a D_t^\alpha (C(S, t)) = \frac{M(\alpha)}{1 - \alpha} \int_a^t \frac{\partial C(S, \tau)}{\partial \tau} \exp \left[-\alpha \frac{t - \tau}{1 - \alpha} \right] d\tau,$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. However, if the function does not belongs to $H^1(a, b)$ then, the derivative can be reformulated as:

$${}^C_a D_t^\alpha (C(S, t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t (C(S, t) - C(S, x)) \exp \left[-\alpha \frac{t-x}{1-\alpha} \right] dx.$$

The proof of existence and uniqueness of a solution was already presented in subsection 4.3. While discretizing equation (5.6) we proposed an analytical solution the equation. We also discuss its stability and the convergence of the derived numerical scheme.

Let us adopt the following notation

$$t_l = l \Delta t, \quad t_{l+1} = (l+1)\Delta t, \quad \Delta t = t_{l+1} - t_l,$$

$$x_j = j \Delta x, \quad x_{j+1} = (j+1)\Delta x, \quad \Delta x = x_{j+1} - x_j,$$

$$U(x_j, t_l) = U_j^l.$$

With the notation above, let $0 = t_0 < t_1 < t_2 < \dots < t_n = t$ the Caputo-Fabrizio time fractional derivative in equation (5.6) can now written as:

$$\begin{aligned} {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \frac{\partial C(S_l, \tau)}{\partial \tau} \exp \left[-\frac{\alpha(t_n - \tau)}{1-\alpha} \right] d\tau, \\ {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^n \left(\frac{C(S_l, t_{j+1}) - C(S_l, t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} \exp \left[-\frac{\alpha(t_n - \tau)}{1-\alpha} \right] d\tau \right), \\ {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^n \left(\frac{C_l^{j+1} - C_l^j}{\Delta t} \int_{t_j}^{t_{j+1}} \exp \left[-\frac{\alpha(t_n - \tau)}{1-\alpha} \right] d\tau \right), \\ {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^n \left(\frac{C_l^{j+1} - C_l^j}{\Delta t} \frac{1-\alpha}{\alpha} \exp \left[-\frac{\alpha(t_n - \tau)}{1-\alpha} \right]_{t_j}^{t_{j+1}} \right), \\ {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{1-\alpha} \frac{1-\alpha}{\alpha} \frac{1}{\Delta t} \sum_{j=0}^n \left((C_l^{j+1} - C_l^j) \exp \left[-\frac{\alpha(t_n - \tau)}{1-\alpha} \right]_{t_j}^{t_{j+1}} \right), \\ {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{\alpha \Delta t} \sum_{j=0}^n (C_l^{j+1} - C_l^j) \left(\exp \left(-\frac{\alpha(t_n - t_{j+1})}{1-\alpha} \right) - \exp \left(-\frac{\alpha(t_n - t_j)}{1-\alpha} \right) \right), \\ {}^C_0 D_t^\alpha C(S_l, t_n) &= \frac{M(\alpha)}{\alpha \Delta t} \sum_{j=0}^n (C_l^{j+1} - C_l^j) \left(\exp \left(-\frac{\alpha(n-j-1)\Delta t}{1-\alpha} \right) \right. \\ &\quad \left. - \exp \left(-\frac{\alpha(n-j)\Delta t}{1-\alpha} \right) \right), \end{aligned}$$

$${}^C D_t^\alpha C(S_l, t_n) = \frac{M(\alpha)}{\alpha \Delta t} \left(\exp\left(\frac{\alpha \Delta t}{1-\alpha}\right) - 1 \right) \sum_{j=0}^n (C_l^{j+1} - C_l^j) \exp\left(-\frac{\alpha(n-j)\Delta t}{1-\alpha}\right).$$

If we let $\delta_0 = \exp\left(\frac{-\alpha \Delta t}{1-\alpha}\right)$ and $\delta_{n,j} = (\delta_0)^{n-j} = \exp\left(-\frac{\alpha(n-j)\Delta t}{1-\alpha}\right)$, then

$${}^C D_t^\alpha C(S_l, t_n) = \frac{M(\alpha)}{\alpha \Delta t} \left(\frac{1}{\delta_0} - 1 \right) \sum_{j=0}^n (C_l^{j+1} - C_l^j) \delta_{n,j}, \quad (7.7)$$

$${}^C D_t^\alpha C(S_l, t_n) = \frac{M(\alpha)}{\alpha \Delta t} (1 - \delta_0) \sum_{j=0}^n (C_l^{j+1} - C_l^j) \delta_0^{n-j-1}. \quad (7.8)$$

Moreover by discretizing the space derivatives in (5.6) we have:

$$\begin{aligned} \frac{\partial C(S_l, t_n)}{\partial S} &= \frac{C(S_{l+1}, t_n) - C(S_{l-1}, t_n)}{2\Delta x} = \frac{C_{l+1}^n - C_{l-1}^n}{2\Delta x}, \\ \frac{\partial^2 C(S_l, t_n)}{\partial S^2} &= \frac{C(S_{l+1}, t_n) - 2C(S_l, t_n) + C(S_{l-1}, t_n))}{(\Delta x)^2} = \frac{C_{l+1}^n - 2C_l^n + C_{l-1}^n}{(\Delta x)^2}. \end{aligned}$$

For simplicity let $a = \frac{1}{2}\sigma^2; h = \Delta x; L_\alpha = \frac{M(\alpha)}{\alpha \Delta t}$ making use of this notation and substituting in (7.7), equation (5.6) can be rewritten as

$$\begin{aligned} (1 - \delta_0)L_\alpha \sum_{j=0}^n (C_l^{j+1} - C_l^j) \delta_0^{n-j-1} \\ = r C_l^n - \frac{a}{h^2} S_l^2 (C_{l+1}^n - 2C_l^n + C_{l-1}^n) - \frac{r S_l}{2h} (C_{l+1}^n - C_{l-1}^n), \\ (1 - \delta_0)L_\alpha \sum_{j=0}^n (C_l^{j+1} - C_l^j) \delta_0^{n-j-1} \\ = r C_l^n - \frac{a}{h^2} S_l^2 (C_{l+1}^n - 2C_l^n + C_{l-1}^n) - \frac{r S_l}{2h} (C_{l+1}^n - C_{l-1}^n), \end{aligned}$$

this implies

$$\begin{aligned} (1 - \delta_0)L_\alpha (C_l^{n+1} - C_l^n) \delta_0^{-1} \\ = r C_l^n - \frac{a}{h^2} S_l^2 C_{l+1}^n + \frac{2a}{h^2} S_l^2 C_l^n - \frac{a}{h^2} S_l^2 C_{l-1}^n - \frac{r}{2h} S_l C_{l+1}^n + \frac{r}{2h} S_l C_{l-1}^n \\ - (1 - \delta_0)L_\alpha \sum_{j=0}^{n-1} (C_l^{j+1} - C_l^j) \delta_0^{n-j-1}. \end{aligned}$$

Expanding and grouping likewise terms of the iteration we have

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha C_l^{n+1} \\
&= \left(\frac{1}{\delta_0} - 1\right) L_\alpha C_l^n + r C_l^n + \frac{2a}{h^2} S_l^2 C_l^n - \frac{a}{h^2} S_l^2 C_{l+1}^n - \frac{r}{2h} S_l C_{l+1}^n + \frac{r}{2h} S_l C_{l-1}^n \\
&\quad - \frac{a}{h^2} S_l^2 C_{l-1}^n - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (C_l^{j+1} - C_l^j) \delta_0^{n-j-1}.
\end{aligned}$$

The iterative scheme can then be fully obtained as

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha C_l^{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 \right) C_l^n - \left(\frac{r}{2h} S_l + \frac{a}{h^2} S_l^2 \right) C_{l+1}^n \\
&\quad + \left(\frac{r}{2h} S_l - \frac{a}{h^2} S_l^2 \right) C_{l-1}^n - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (C_l^{j+1} - C_l^j) \delta_0^{n-j-1}. \quad (7.9)
\end{aligned}$$

7.3.1. Stability Analysis of the Numerical Scheme

We follow von Neumann stability analysis to investigate the stability of the numerical scheme. We let

$$\begin{aligned}
C(S_l, t_n) &= C_l^n = \hat{C}(t_n) e^{if S_l} = \hat{C}_n e^{if l \Delta x} = \hat{C}_n e^{if l h}, \\
C_l^{n+1} &= \hat{C}_{n+1} e^{if l h}; \quad C_l^n = \hat{C}_n e^{if l h}; \quad C_{l+1}^n = \hat{C}_n e^{if (l+1) h}; \quad C_{l-1}^n = \hat{C}_n e^{if (l-1) h}.
\end{aligned}$$

With the above equation (7.9) becomes

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} e^{if l h} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 \right) \hat{C}_n e^{if l h} - \left(\frac{r}{2h} S_l + \frac{a}{h^2} S_l^2 \right) \hat{C}_n e^{if (l+1) h} \\
&\quad + \left(\frac{r}{2h} S_l - \frac{a}{h^2} S_l^2 \right) \hat{C}_n e^{if (l-1) h} - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) e^{if l h} \delta_0^{n-j-1},
\end{aligned}$$

simplifying similar exponential term from both side of the equation we have

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 \right) \hat{C}_n - \left(\frac{r}{2h} S_l + \frac{a}{h^2} S_l^2 \right) \hat{C}_n e^{ifh} \\
&+ \left(\frac{r}{2h} S_l - \frac{a}{h^2} S_l^2 \right) \hat{C}_n e^{-ifh} - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) \delta_0^{n-j-1},
\end{aligned}$$

factorizing out the iteration a time t_n

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 + \left(\frac{r}{2h} S_l - \frac{a}{h^2} S_l^2 \right) e^{-ifh} \right. \\
&\quad \left. - \left(\frac{r}{2h} S_l + \frac{a}{h^2} S_l^2 \right) e^{ifh} \right) \hat{C}_n - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) \delta_0^{n-j-1}.
\end{aligned}$$

This means that

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 - \frac{r}{2h} S_l (e^{ifh} - e^{-ifh}) - \frac{a}{h^2} S_l^2 (e^{ifh} \right. \\
&\quad \left. + e^{-ifh}) \right) \hat{C}_n - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) \delta_0^{n-j-1}, \\
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 - \frac{r S_l}{2h} (2i \sin(fh)) - \frac{a}{h^2} S_l^2 (2 \cos(fl)) \right) \hat{C}_n \\
&- (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) \delta_0^{n-j-1},
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{2a}{h^2} S_l^2 (1 - \cos(fh)) - i \frac{r S_l}{h} \sin(fh) \right) \hat{C}_n \\
&\quad - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) \delta_0^{n-j-1}, \\
& \left(\frac{1}{\delta_0} - 1\right) L_\alpha \hat{C}_{n+1} \\
&= \left(\left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{4a}{h^2} S_l^2 \left(\sin^2 \frac{fh}{2}\right) - i \frac{r S_l}{h} \sin(fh) \right) \\
&\quad - (1 - \delta_0) L_\alpha \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) \delta_0^{n-j-1}.
\end{aligned}$$

In fact

$$\delta_0 = \exp\left(\frac{\alpha \Delta t}{\alpha - 1}\right), \text{ with } 0 < \alpha < 1, \text{ we have } 0 < \delta_0 < 1$$

$$\sum_{j=0}^{n-1} \|\hat{C}_{j+1} - \hat{C}_j\| \delta_0^{n-j} < \sum_{j=0}^{n-1} \|\hat{C}_{j+1} - \hat{C}_j\|,$$

$$\text{Let } h = \min\left(\frac{r}{(1 - \delta_0) L_\alpha}, \Delta t\right),$$

$$\text{if } \|\hat{C}_{j+1} - \hat{C}_j\| < \delta_0 \text{ then } \sum_{j=0}^{n-1} \|\hat{C}_{j+1} - \hat{C}_j\| \delta_0^{n-j} \rightarrow 0,$$

and

$$\left\| \left(\frac{1}{\delta_0} - 1\right) L_\alpha \frac{\hat{C}_{n+1}}{\hat{C}_n} \right\| = \left\| \left(\frac{1}{\delta_0} - 1\right) L_\alpha + r + \frac{4a}{h^2} S_l^2 \left(\sin^2 \frac{fh}{2}\right) - i \frac{r S_l}{h} \sin(fh) \right\|.$$

This is

$$\left\| \frac{\hat{C}_{n+1}}{\hat{C}_n} \right\| = \left\| 1 + \frac{r \delta_0}{1 - \delta_0} + \frac{4a}{h^2} S_l^2 \left(\sin^2 \frac{fh}{2}\right) \frac{\delta_0}{1 - \delta_0} - i \frac{r S_l}{h} \sin(fh) \frac{\delta_0}{1 - \delta_0} \right\|.$$

An the conditional stability will be obtain if

$$\left\| 1 + \frac{r\delta_0}{1-\delta_0} + \frac{4a}{h^2} S_l^2 \left(\sin^2 \frac{fh}{2} \right) \frac{\delta_0}{1-\delta_0} - i \frac{rS_l}{h} \sin(fh) \frac{\delta_0}{1-\delta_0} \right\| < 1.$$

This will be ensured if we take $\Delta t \ll 1$, then $h = \Delta x = \min \left(\frac{r}{(1-\delta_0)L_\alpha}, \Delta t \right)$.

7.3.2. Numerical Simulations

We give here graphical solutions with different alpha values for some call and put options.

We consider the time fractional Black Scholes equation (5.6) with no dividend.

All Figures are illustrating a double barrier knockout option prices with strike price $K = 10$,

Knockout lower barrier price $DO = 3$ and knockout upper barrier $UO = 15$. The volatility is $\sigma = 0.45$ and the risk- free rate $r = 0.03$.

Figure 5, shows the classic Black Scholes prices which are obtained when equation (5.6) is reduced to the standard case $\alpha = 1$.

Figure 6, shows prices for values of alpha varying from $\alpha = 0.5$; $\alpha = 0.6$ and $\alpha = 0.67$. From figure 2 one has the feeling that the TFBSE with Caputo-Fabrizio fractional derivative (5.6), seems to show consistent memory property capable of capturing large jumps over time interval. The prices variations within less than a decile α value is remarkable, and suggests a sensitivity analysis over a range of α -values need to be conducted.

Figure 7, which illustrates prices for the standard case together with $\alpha 0.5$; $\alpha = 0.6$ and $\alpha = 0.9$, does reinforce that the sentiment that our solution seems to be consistent or very sensitive within specific ranges of α values. This gives us the feeling that the time fractional Black Scholes equation with Caputo-Fabrizio derivative could indeed successfully capture both high jumps over small interval of times, as well as it could satisfactorily depict the long range memory property.

Another noticeable feature from both figure 6 and figure 7 is that the pricing with TFBSE with Caputo-Fabrizio does not exhibit the same consistent property of under pricing the option -in comparison to the standard Black Scholes [26, 75]-when the strike price is close to the lower barrier, and overpricing it when the strike price is closer to the upper barrier. This can only comes to reinforce the already observed crossover behaviour of some fractional operator [56]. The Caputo-Fabrizio fractional derivative seems to show crossover in distribution from a certain range of values of α .

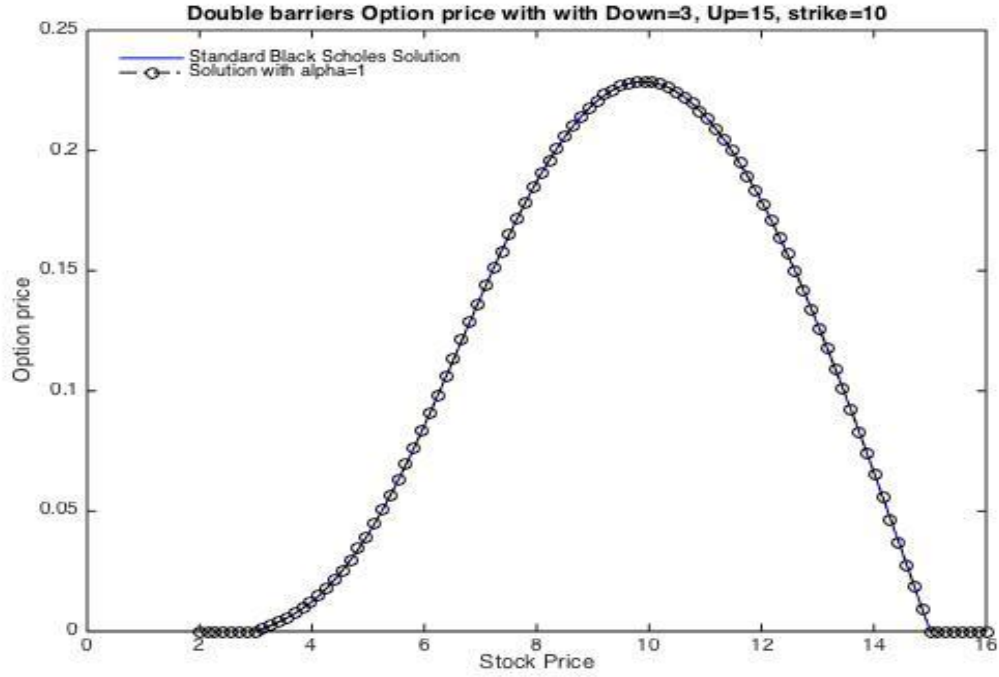


Figure 5: Double barrier option Prices-Parameters $\sigma = 0.45, r = 0.03, T = 1, K = 10, DO = 3, UO = 15$

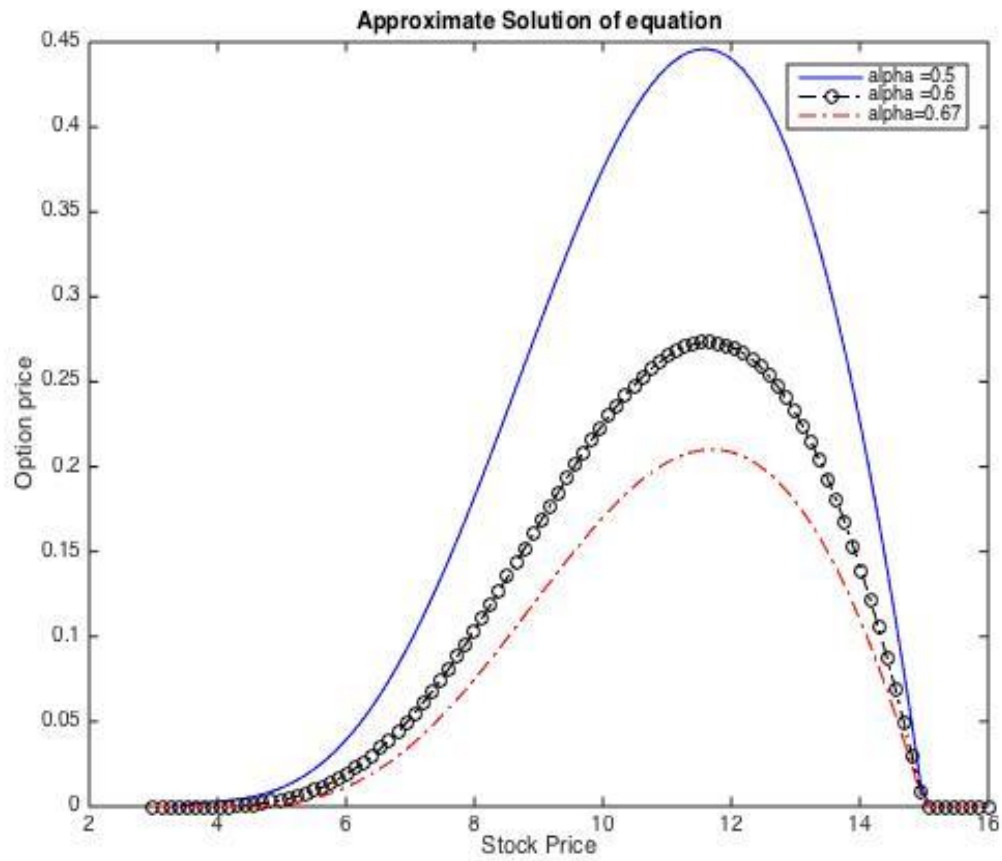


Figure 6: Double barrier option prices-Parameters $\sigma = 0.45, r = 0.03, K = 10, T = 1, D0 = 3, U0 = 15$

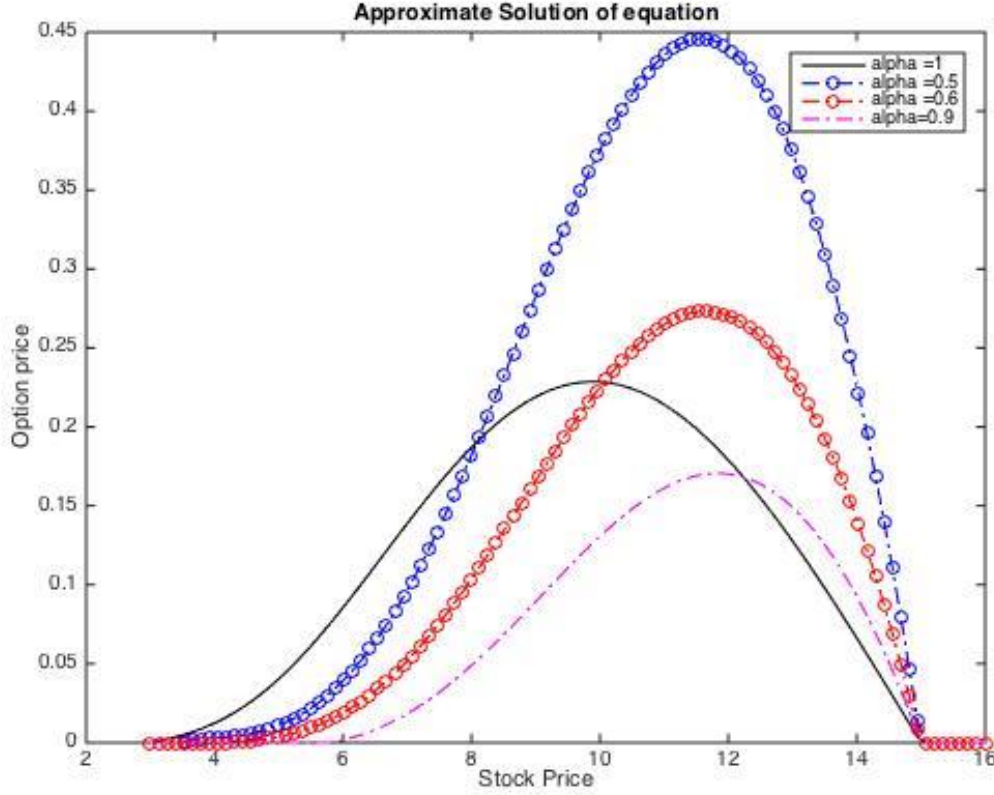


Figure 7: Double barrier option prices for α - values: $\sigma = 0.45, r = 0.03, K = 10, T = 1, DO = 3, UO = 15$

The new Caputo-Fabrizio TFBSE shows superior memory property and could satisfactorily capture both high jumps in short interval of times, as well as portray satisfactorily the long range memory property. Further investigations could focus on the sensitivity analysis of the solution to the time fractional Black Scholes equation with Caputo-Fabrizio derivative, and look in details how prices variations respond to α -values.

7.4 Numerical Solution of TFBSE with ABR Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given by the Atangana-Baleanu time derivative definition in the Riemann Liouville sense equation (2.10).

$$\begin{cases} {}^{ABR}_0 D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.9)$$

with

$${}^{ABR}_0\mathcal{D}_t^\alpha (C(S, t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t C(S, \tau) E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] d\tau, \alpha \in [0, 1].$$

Let

$$G(S, t) = \frac{B(\alpha)}{1-\alpha} \int_0^t C(S, \tau) E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] d\tau.$$

$${}^{ABR}_0\mathcal{D}_t^\alpha (C(S, t)) = \frac{\partial G(S, t)}{\partial t} = \frac{G(S, t + \Delta t) - G(S, t)}{\Delta t},$$

$${}^{ABR}_0\mathcal{D}_t^\alpha (C(S, t_j)) = \frac{G(S, t_{j+1}) - G(S, t_j)}{\Delta t} + O(\Delta t),$$

$$G(S, t_{j+1}) = \frac{B(\alpha)}{1-\alpha} \int_0^{t_{j+1}} C(S, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau,$$

$$G(S, t_j) = \frac{B(\alpha)}{1-\alpha} \int_0^{t_j} C(S, \tau) E_\alpha \left[-\alpha \frac{(t_j-\tau)^\alpha}{1-\alpha} \right] d\tau.$$

We then discretize the expressions further as:

$$\begin{aligned} G(S, t_{j+1}) &= \frac{B(\alpha)}{1-\alpha} \int_0^{t_{j+1}} C(S, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{C(S, t_{k+1}) + C(S, t_k)}{2} E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau + \mathcal{H}_{\alpha, j, k}^1, \end{aligned}$$

$$G(S, t_{j+1}) = \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \frac{C(S, t_{k+1}) + C(S, t_k)}{2} \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau + \mathcal{H}_{\alpha, j, k}^1. \quad (7.10)$$

Let

$$F_j^{\alpha, 1} = \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau.$$

$$F_j^{\alpha, 1} = (t_{j+1} - t_{k+1}) E_{\alpha, 2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] + (t_{j+1} - t_k) E_{\alpha, 2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right], \quad (7.11)$$

$$G(S, t_{j+1}) = \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \frac{C(S, t_{k+1}) + C(S, t_k)}{2} F_j^{\alpha,1} + \mathcal{H}_{\alpha,j,k}^1.$$

The additional term in equation (7.10) is expressed as:

$$\begin{aligned} \mathcal{H}_{\alpha,j,k}^1 &= \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} (C(S, \tau) - C(S, t_{k+1})) E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{(C(S, \tau) - C(S, t_{k+1}))(\tau - t_{k+1})}{\tau - t_{k+1}} E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{\partial C(S, t)}{\partial t} (\tau - t_{k+1}) E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau, \quad \tau < t < t_{j+1}. \\ \mathcal{H}_{\alpha,j,k}^1 &\leq \frac{B(\alpha)\Delta t}{1-\alpha} \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau, \quad (7.12) \\ &\leq \frac{B(\alpha)\Delta t}{1-\alpha} \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j F_j^{\alpha,1}, \\ \mathcal{H}_{\alpha,j,k}^1 &\leq \frac{B(\alpha)\Delta t}{1-\alpha} \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j \left((t_{j+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \\ &\quad \left. + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right] \right), \\ \mathcal{H}_{\alpha,j,k}^1 &\leq \frac{B(\alpha)\Delta t}{1-\alpha} \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) k^1. \end{aligned}$$

In a similar way,

$$\begin{aligned} G(S, t_j) &= \frac{B(\alpha)}{1-\alpha} \int_0^{t_j} C(S, \tau) E_\alpha \left[-\alpha \frac{(t_j - \tau)^\alpha}{1-\alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{k=1}^j \frac{C(S, t_{k+1}) + C(S, t_k)}{2} F_j^{\alpha,2} + \mathcal{H}_{\alpha,j,k}^2, \end{aligned}$$

where

$$F_j^{\alpha,2} = \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_j - \tau)^\alpha}{1 - \alpha} \right] d\tau,$$

$$F_j^{\alpha,2} = (t_j - t_{k+1})E_{\alpha,2} \left[-\alpha \frac{(t_j - t_{k+1})^\alpha}{1 - \alpha} \right] + (t_j - t_k)E_{\alpha,2} \left[-\alpha \frac{(t_j - t_k)^\alpha}{1 - \alpha} \right], \quad (7.13)$$

and

$$\begin{aligned} \mathcal{H}_{\alpha,j,k}^2 &= \frac{B(\alpha)}{1 - \alpha} \sum_{k=1}^j \int_{t_k}^{t_{k+1}} (C(S, \tau) - C(S, t_{k+1})) E_\alpha \left[-\alpha \frac{(t_j - \tau)^\alpha}{1 - \alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1 - \alpha} \sum_{k=1}^j \int_{t_k}^{t_{k+1}} \frac{(C(S, \tau) - C(S, t_{k+1}))(\tau - t_{k+1})}{\tau - t_{k+1}} E_\alpha \left[-\alpha \frac{(t_j - \tau)^\alpha}{1 - \alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1 - \alpha} \sum_{k=1}^j \int_{t_k}^{t_{k+1}} \frac{\partial C(S, t)}{\partial t} (\tau - t_{k+1}) E_\alpha \left[-\alpha \frac{(t_j - \tau)^\alpha}{1 - \alpha} \right] d\tau, \tau < t < t_j, \\ \mathcal{H}_{\alpha,j,k}^2 &\leq \frac{B(\alpha)\Delta t}{1 - \alpha} \max_{0 \leq t \leq t_j} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_j - \tau)^\alpha}{1 - \alpha} \right] d\tau, \\ &\leq \frac{B(\alpha)\Delta t}{1 - \alpha} \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=1}^j F_j^{\alpha,2}, \\ \mathcal{H}_{\alpha,j,k}^2 &\leq \frac{B(\alpha)\Delta t}{1 - \alpha} \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j \left((t_j - t_{k+1})E_{\alpha,2} \left[-\alpha \frac{(t_j - t_{k+1})^\alpha}{1 - \alpha} \right] \right. \\ &\quad \left. + (t_j - t_k)E_{\alpha,2} \left[-\alpha \frac{(t_j - t_k)^\alpha}{1 - \alpha} \right] \right), \\ \mathcal{H}_{\alpha,j,k}^2 &\leq \frac{B(\alpha)\Delta t}{1 - \alpha} \max_{0 \leq t \leq t_j} \left(\frac{\partial C(S, t)}{\partial t} \right) k^2. \end{aligned}$$

While discretizing the fractional derivative, we showed that the error term is bounded.

Providing for the error term we have that

$$\begin{aligned}
& {}^{ABR}_0\mathcal{D}_t^\alpha \left(C(S, t_j) \right) \\
&= \frac{B(\alpha)}{(1-\alpha)\Delta t} \left\{ \sum_{k=0}^j \frac{C(S, t_{k+1}) + C(S, t_k)}{2} \left((t_{j+1} \right. \right. \\
&\quad \left. \left. - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right] \right) \right. \\
&\quad \left. - \sum_{k=1}^j \frac{C(S, t_{k+1}) + C(S, t_k)}{2} F_j^{\alpha,2} \right\} + \frac{1}{\Delta t} (\mathcal{H}_{\alpha,j,k}^1 - \mathcal{H}_{\alpha,j,k}^2),
\end{aligned}$$

and

$$\left| \frac{1}{\Delta t} (\mathcal{H}_{\alpha,j,k}^1 - \mathcal{H}_{\alpha,j,k}^2) \right| = |\mathcal{H}^{\alpha,j}| \leq \frac{B(\alpha)}{1-\alpha} \max_{0 \leq t \leq t_j} \left(\frac{\partial C(S, t)}{\partial t} \right) k.$$

To derive the full numerical scheme of equation (5.9) we have that

$$\begin{aligned}
& {}^{ABR}_0\mathcal{D}_t^\alpha C(S, t) = rC(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S} \Leftrightarrow \\
& \frac{B(\alpha)}{(1-\alpha)\Delta t} \left\{ \sum_{k=0}^j \frac{C(S_l, t_{k+1}) + C(S_l, t_k)}{2} F_j^{\alpha,1} - \sum_{k=1}^j \frac{C(S_l, t_{k+1}) + C(S_l, t_k)}{2} F_j^{\alpha,2} \right\} + \mathcal{H}^{\alpha,j}, \\
&= rC(S_l, t_j) - \frac{1}{2} \sigma^2 S_l^2 \frac{C(S_{l+1}, t_j) - 2C(S_l, t_j) + C(S_{l-1}, t_j)}{(\Delta x)^2} \\
&\quad - r S_l \frac{C(S_{l+1}, t_j) - C(S_{l-1}, t_j)}{2\Delta x} \\
& \frac{B(\alpha)}{(1-\alpha)\Delta t} \left\{ \sum_{k=0}^j \frac{C(S_l, t_{k+1}) + C(S_l, t_k)}{2} \left((t_{j+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \right. \\
&\quad \left. \left. + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right] \right) \right. \\
&\quad \left. - \sum_{k=1}^j \frac{C(S_l, t_{k+1}) + C(S_l, t_k)}{2} \left((t_j - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_j - t_{k+1})^\alpha}{1-\alpha} \right] \right. \right. \\
&\quad \left. \left. + (t_j - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_j - t_k)^\alpha}{1-\alpha} \right] \right) \right\} + \mathcal{H}^{\alpha,j}, \\
&= rC(S_l, t_j) - \frac{1}{2} \sigma^2 S_l^2 \frac{C(S_{l+1}, t_j) - 2C(S_l, t_j) + C(S_{l-1}, t_j)}{h^2} \\
&\quad - r S_l \frac{C(S_{l+1}, t_j) - C(S_{l-1}, t_j)}{2h},
\end{aligned}$$

$$\begin{aligned}
& \frac{B(\alpha)}{(1-\alpha)\Delta t} \left\{ \left(\frac{1}{2} C_l^{j+1} + \frac{1}{2} C_l^j \right) \left((t_{j+1} - t_{j+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{j+1})^\alpha}{1-\alpha} \right] \right. \right. \\
& \quad \left. \left. + (t_{j+1} - t_j) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_j)^\alpha}{1-\alpha} \right] \right) \right. \\
& \quad + \sum_{k=0}^{j-1} \left(\frac{1}{2} C_l^{k+1} + \frac{1}{2} C_l^k \right) \left((t_{j+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \\
& \quad \left. + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right] - (t_j - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_j - t_{k+1})^\alpha}{1-\alpha} \right] \right. \\
& \quad \left. \left. - (t_j - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_j - t_k)^\alpha}{1-\alpha} \right] \right) \right\} + \mathcal{H}^{\alpha,j}, \\
& = r C_l^j - \frac{\sigma^2}{2h^2} S_l^2 C_{l+1}^j + \frac{\sigma^2}{h^2} S_l^2 C_l^j - \frac{\sigma^2}{2h^2} S_l^2 C_{l-1}^j - \frac{r}{2h} S_l C_{l+1}^j + \frac{r}{2h} S_l C_{l-1}^j.
\end{aligned}$$

Having showed in the lines above, that the error term $\mathcal{H}^{\alpha,j}$ from the discretization of the fractional derivative is bounded, we can therefore consider the numerical solution as :

$$\begin{aligned}
& \frac{B(\alpha)}{(1-\alpha)\Delta t} \left\{ \left(\frac{1}{2} C_l^{j+1} + \frac{1}{2} C_l^j \right) \Delta t E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_j)^\alpha}{1-\alpha} \right] \right. \\
& \quad + \sum_{k=0}^{j-1} \left(\frac{1}{2} C_l^{k+1} + \frac{1}{2} C_l^k \right) \left((j-k)\Delta t E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \\
& \quad \left. + (j-k+1)\Delta t E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right] \right. \\
& \quad \left. - (j-k-1)\Delta t E_{\alpha,2} \left[-\alpha \frac{(t_j - t_{k+1})^\alpha}{1-\alpha} \right] \right. \\
& \quad \left. \left. - (j-k)\Delta t E_{\alpha,2} \left[-\alpha \frac{(t_j - t_k)^\alpha}{1-\alpha} \right] \right) \right\}, \\
& = r C_l^j - \frac{\sigma^2}{2h^2} S_l^2 C_{l+1}^j + \frac{\sigma^2}{h^2} S_l^2 C_l^j - \frac{\sigma^2}{2h^2} S_l^2 C_{l-1}^j - \frac{r}{2h} S_l C_{l+1}^j + \frac{r}{2h} S_l C_{l-1}^j,
\end{aligned}$$

simplifying by Δt and solving for the next iteration we have:

$$\begin{aligned}
& \left(\frac{1}{2} C_l^{n+1} + \frac{1}{2} C_l^n \right) E_{\alpha,2} \left[-\alpha \frac{(\Delta t)^\alpha}{1-\alpha} \right] \\
& + \sum_{k=0}^{n-1} \left(\frac{1}{2} C_l^{k+1} + \frac{1}{2} C_l^k \right) \left((n-k) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k)^\alpha (\Delta t)^\alpha \right] \right. \\
& + (n-k+1) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k+1)^\alpha (\Delta t)^\alpha \right] \\
& - (n-k-1) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k-1)^\alpha (\Delta t)^\alpha \right] \\
& \left. - (n-k) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k)^\alpha (\Delta t)^\alpha \right] \right) \\
& = \frac{(1-\alpha)}{B(\alpha)} \left(r C_l^n - \frac{\sigma^2}{2h^2} S_l^2 C_{l+1}^n + \frac{\sigma^2}{h^2} S_l^2 C_l^n - \frac{\sigma^2}{2h^2} S_l^2 C_{l-1}^n - \frac{r}{2h} S_l C_{l+1}^n \right. \\
& \left. + \frac{r}{2h} S_l C_{l-1}^n \right).
\end{aligned}$$

Recall that $h = \Delta x$, and letting $d = \Delta t$, this becomes

$$\begin{aligned}
C_l^{n+1} &= \left(\frac{2(1-\alpha)}{B(\alpha) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} d^\alpha \right]} \left(r + \frac{\sigma^2}{h^2} S_l^2 \right) - 1 \right) C_l^n \\
&+ \frac{2(1-\alpha)}{B(\alpha) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} d^\alpha \right]} \left(\frac{r}{2h} S_l - \frac{\sigma^2}{2h^2} S_l^2 \right) C_{l-1}^n \\
&- \frac{2(1-\alpha)}{B(\alpha) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} d^\alpha \right]} \left(\frac{r}{2h} S_l + \frac{\sigma^2}{2h^2} S_l^2 \right) C_{l+1}^n \\
&+ \frac{1}{E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} d^\alpha \right]} \sum_{k=0}^{n-1} (C_l^{k+1} + C_l^k) \left((n-k+1) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k+1)^\alpha d^\alpha \right] \right. \\
&\left. - (n-k-1) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k-1)^\alpha d^\alpha \right] \right). \tag{7.14}
\end{aligned}$$

7.5 Numerical Solution of TFBSE with ABC Derivative

Here we consider the following time fractional Black-Scholes equation where the time fractional derivative is given by the Atangana-Baleanu time derivative definition in the Caputo-Fabrizio sense equation (2.11).

$$\begin{cases} {}^{ABC}_0D_t^\alpha C(S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - rC(S, t) = 0, (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.11)$$

with

$${}^{ABC}_0D_t^\alpha (C(S, t)) = \frac{B(\alpha)}{1-\alpha} \int_0^t \frac{\partial}{\partial x} C(S, \tau) E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] d\tau, \alpha \in [0, 1].$$

$$\begin{aligned} {}^{ABC}_0D_t^\alpha (C(S_l, t_{j+1})) &= \frac{B(\alpha)}{1-\alpha} \int_0^{t_{j+1}} \frac{C(S_{l+1}, \tau) - C(S_{l-1}, \tau)}{2\Delta x} E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{C(S_{l+1}, \tau) - C(S_{l-1}, \tau)}{2\Delta x} E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau, \\ &= \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^j \int_{t_k}^{t_{k+1}} C(S_{l+1}, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau \right. \\ &\quad \left. - \sum_{k=0}^j \int_{t_k}^{t_{k+1}} C(S_{l-1}, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau \right). \end{aligned}$$

Let

$$\begin{aligned} P_{l+1}(t_{j+1}) &= \sum_{k=0}^j \int_{t_k}^{t_{k+1}} C(S_{l+1}, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1}-\tau)^\alpha}{1-\alpha} \right] d\tau. \\ {}^{ABC}_0D_t^\alpha (C(S_l, t_{j+1})) &= \frac{B(\alpha)}{2(1-\alpha)h} [P_{l+1}(t_{j+1}) - P_{l-1}(t_{j+1})], \end{aligned}$$

$$\begin{aligned}
P_{l+1}(t_{j+1}) &= \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{C(S_{l+1}, t_{k+1}) - C(S_{l+1}, t_k)}{2} E_{\alpha} \left[-\alpha \frac{(t_{j+1} - \tau)^{\alpha}}{1 - \alpha} \right] d\tau + \mathcal{H}_{\alpha,j,k}^1, \\
&= \sum_{k=0}^j \frac{C(S_{l+1}, t_{k+1}) - C(S_{l+1}, t_k)}{2} \int_{t_k}^{t_{k+1}} E_{\alpha} \left[-\alpha \frac{(t_{j+1} - \tau)^{\alpha}}{1 - \alpha} \right] d\tau + \mathcal{H}_{\alpha,j,k}^1, \\
&= \sum_{k=0}^j \frac{C(S_{l+1}, t_{k+1}) - C(S_{l+1}, t_k)}{2} F_j^{\alpha,1} + \mathcal{H}_{\alpha,j,k}^1.
\end{aligned}$$

Similar to the ABR fractional operator type we have

$$\begin{aligned}
F_j^{\alpha,1} &= \int_{t_k}^{t_{k+1}} E_{\alpha} \left[-\alpha \frac{(t_{j+1} - \tau)^{\alpha}}{1 - \alpha} \right] d\tau, \\
&= (t_{j+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^{\alpha}}{1 - \alpha} \right] \\
&\quad + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^{\alpha}}{1 - \alpha} \right], \\
\mathcal{H}_{\alpha,j,k}^1 &= \sum_{k=0}^j \int_{t_k}^{t_{k+1}} C(S_{l+1}, \tau) - C(S_{l+1}, t_{k+1}) E_{\alpha} \left[-\alpha \frac{(t_{j+1} - \tau)^{\alpha}}{1 - \alpha} \right] d\tau, \\
&= \sum_{k=0}^j \int_{t_k}^{t_{k+1}} \frac{(C(S_{l+1}, \tau) - C(S_{l+1}, t_{k+1}))(\tau - t_{k+1})}{\tau - t_{k+1}} E_{\alpha} \left[-\alpha \frac{(t_{j+1} - \tau)^{\alpha}}{1 - \alpha} \right] d\tau, \\
&\leq \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j \int_{t_k}^{t_{k+1}} E_{\alpha} \left[-\alpha \frac{(t_{j+1} - \tau)^{\alpha}}{1 - \alpha} \right] d\tau, \leq \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=1}^j F_j^{\alpha,1}, \\
&\leq \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) \sum_{k=0}^j \left((t_{j+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^{\alpha}}{1 - \alpha} \right] \right. \\
&\quad \left. + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^{\alpha}}{1 - \alpha} \right] \right), \\
\mathcal{H}_{\alpha,j,k}^1 &\leq \max_{0 \leq t \leq t_{j+1}} \left(\frac{\partial C(S, t)}{\partial t} \right) k^2.
\end{aligned}$$

Proceeding in just a similar way as it is described in details above the numerical approximation of the fractional operator of the ABC type can be obtained as

$$\begin{aligned}
{}^{ABC}\mathcal{D}_t^\alpha \left(C(S_l, t_{j+1}) \right) &= \frac{B(\alpha)}{2(1-\alpha)h} [P_{l+1}(t_{j+1}) - P_{l-1}(t_{j+1})] \\
&= \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^j \int_{t_k}^{t_{k+1}} C(S_{l+1}, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau \right. \\
&\quad \left. - \sum_{k=0}^j \int_{t_k}^{t_{k+1}} C(S_{l-1}, \tau) E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau \right), \\
&= \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^j \frac{C(S_{l+1}, t_k) + C(S_{l+1}, t_{k+1})}{2} \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau \right. \\
&\quad \left. - \sum_{k=0}^j \frac{C(S_{l-1}, t_k) + C(S_{l-1}, t_{k+1})}{2} \int_{t_k}^{t_{k+1}} E_\alpha \left[-\alpha \frac{(t_{j+1} - \tau)^\alpha}{1-\alpha} \right] d\tau \right), \\
&= \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^j \frac{C(S_{l+1}, t_k) + C(S_{l+1}, t_{k+1})}{2} F_j^{\alpha,1} \right. \\
&\quad \left. - \sum_{k=0}^j \frac{C(S_{l-1}, t_k) + C(S_{l-1}, t_{k+1})}{2} F_j^{\alpha,1} \right) + \mathcal{H}^{\alpha,j},
\end{aligned}$$

where the error term $\mathcal{H}^{\alpha,j}$ as shown above is bounded .

$$\begin{aligned}
{}^{ABC}\mathcal{D}_t^\alpha C(S_l, t_{j+1}) &= \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^j \frac{C(S_{l+1}, t_k) + C(S_{l+1}, t_{k+1})}{2} F_j^{\alpha,1} \right. \\
&\quad \left. - \sum_{k=0}^j \frac{C(S_{l-1}, t_k) + C(S_{l-1}, t_{k+1})}{2} F_j^{\alpha,1} \right)
\end{aligned}$$

$$\begin{aligned}
& {}^{ABC}_0\mathcal{D}_t^\alpha C(S_l, t_{j+1}) \\
&= \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^j \left(\frac{C(S_{l+1}, t_k) + C(S_{l+1}, t_{k+1})}{2} \right. \right. \\
&\quad \left. \left. - \frac{C(S_{l-1}, t_k) + C(S_{l-1}, t_{k+1})}{2} \right) \left\{ (t_{j+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \right. \\
&\quad \left. \left. + (t_{j+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{j+1} - t_k)^\alpha}{1-\alpha} \right] \right\} \right),
\end{aligned}$$

With the above we can proceed to find a numerical scheme solution to the TFBSE (5.11) with ABC fractional operator. Having showed that the error term $\mathcal{H}^{\alpha,j}$ from the discretization of the fractional derivative is bounded, we can therefore consider the numerical solution as :

$${}^{ABC}_0\mathcal{D}_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0 \Leftrightarrow$$

$$\begin{aligned}
& \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^n \left(\frac{C(S_{l+1}, t_k) + C(S_{l+1}, t_{k+1})}{2} \right. \right. \\
&\quad \left. \left. - \frac{C(S_{l-1}, t_k) + C(S_{l-1}, t_{k+1})}{2} \right) \left\{ (t_{n+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{n+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \right. \\
&\quad \left. \left. + (t_{n+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{n+1} - t_k)^\alpha}{1-\alpha} \right] \right\} \right) \\
&= r C(S_l, t_n) - \frac{1}{2} \sigma^2 S_l^2 \frac{C(S_{l+1}, t_n) - 2C(S_l, t_n) + C(S_{l-1}, t_n)}{h^2} \\
&\quad - r S_l \frac{C(S_{l+1}, t_n) - C(S_{l-1}, t_n)}{2h},
\end{aligned}$$

this is

$$\begin{aligned}
& \frac{B(\alpha)}{2(1-\alpha)h} \left(\sum_{k=0}^n \left(\frac{C_{l+1}^k + C_{l+1}^{k+1}}{2} - \frac{C_{l-1}^k + C_{l-1}^{k+1}}{2} \right) \left\{ (t_{n+1} - t_{k+1}) E_{\alpha,2} \left[-\alpha \frac{(t_{n+1} - t_{k+1})^\alpha}{1-\alpha} \right] \right. \right. \\
& \quad \left. \left. + (t_{n+1} - t_k) E_{\alpha,2} \left[-\alpha \frac{(t_{n+1} - t_k)^\alpha}{1-\alpha} \right] \right\} \right) \\
& = r C_l^n - \frac{\sigma^2}{2h^2} S_l^2 C_{l+1}^n + \frac{\sigma^2}{h^2} S_l^2 C_l^n - \frac{\sigma^2}{2h^2} S_l^2 C_{l-1}^n - \frac{r}{2h} S_l C_{l+1}^n + \frac{r}{2h} S_l C_{l-1}^n,
\end{aligned}$$

letting $d = \Delta t$, and replacing $t_n = n \Delta t$ generally we have :

$$\begin{aligned}
& \frac{B(\alpha)}{2(1-\alpha)h} \left(\frac{1}{2} \sum_{k=0}^n \left(C_{l+1}^k + C_{l+1}^{k+1} - (C_{l-1}^k + C_{l-1}^{k+1}) \right) \left\{ (n-k)d E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k)^\alpha d^\alpha \right] \right. \right. \\
& \quad \left. \left. + (n-k+1)d E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k+1)^\alpha d^\alpha \right] \right\} \right) \\
& = r C_l^n - \frac{\sigma^2}{2h^2} S_l^2 C_{l+1}^n + \frac{\sigma^2}{h^2} S_l^2 C_l^n - \frac{\sigma^2}{2h^2} S_l^2 C_{l-1}^n - \frac{r}{2h} S_l C_{l+1}^n + \frac{r}{2h} S_l C_{l-1}^n,
\end{aligned}$$

this becomes

$$\begin{aligned}
C_{l+1}^{n+1} &= \frac{4(1-\alpha)h}{B(\alpha)dE_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} d^\alpha \right]} \left\{ \left(r + \frac{\sigma^2}{h^2} S_l^2 \right) C_l^n - \left(\frac{\sigma^2}{2h^2} S_l^2 + \frac{r}{2h} S_l \right) C_{l+1}^n \right. \\
& \quad \left. + \left(\frac{r}{2h} S_l - \frac{\sigma^2}{2h^2} S_l^2 \right) C_{l-1}^n \right\} + C_{l-1}^n + C_{l-1}^{n+1} - C_{l+1}^n \\
& \quad - \frac{1}{E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} d^\alpha \right]} \left(\sum_{k=0}^{n-1} \left(C_{l+1}^k + C_{l+1}^{k+1} \right. \right. \\
& \quad \left. \left. - (C_{l-1}^k + C_{l-1}^{k+1}) \right) \left\{ (n-k) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k)^\alpha d^\alpha \right] \right. \right. \\
& \quad \left. \left. + (n-k+1) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (n-k+1)^\alpha d^\alpha \right] \right\} \right). \tag{7.15}
\end{aligned}$$

REMARK

For almost all the numerical scheme solutions developed in this chapter, independently of the method used, the obtained numerical scheme, contains an additional cumbersome summation term, which renders stability analysis very challenging when it is not simply impossible to manage in some cases. Furthermore it also increases the difficulty in implementing the solution, resulting in bigger computational effort and times. This state of affairs motivated the next chapter. We developed a completely novel method of our own craft, capable of handling PDE and fractional PDE. We did so bearing in mind the double aspirations on one side of supporting higher order accuracy and improving the computational efficiency, as well as achieving formulas that are analytically easier to handle.

8. New Method for Integer and non Integer Order PDEs and Fractional PDEs.

We present in this chapter a novel method of our own craft that allows generalizing the use of the famous Adam-Bashforth method to Partial Differential Equations with local and non-local operator. The method derives a two steps Adam-Bashforth numerical scheme in Laplace space and the solution is taken back into the real space via inverse Laplace transform. The method yields a powerful numerical algorithm for fractional order derivative where the usually very difficult to manage summation in the numerical scheme disappears. We will also present the error analysis of the method, and discuss the stability of solutions that we obtained for applications on a wave equation like, on a fractional order diffusion equation and finally on our TFBSE with ABC fractional operator.

Adam-Bashforth method has been recognised as a powerful numerical tool to solve Partial Differential Equations (PDE) [82]. It is a numerical scheme that is used in many field of applied science, Epidemiology, engineering in dynamical systems, in chaotic problem [83-87]. The numerical scheme is good for both differential equations with classical derivatives, and differential equations with non-integer order derivatives. However the method is not fully applied to PDE with local and non-local operator as it was designed only for Ordinary Differential Equations (ODE) [82]. Various other methods are used instead for PDE with integer order differentiation, and those with real order derivatives [88-98]. Nonetheless, due to the accuracy and efficiency of Adam-Bashforth techniques, there is a need to extent the methods to PDE. By eliminating one variable and transforming the PDE to an ODE via Laplace Transform, the newly obtained ODE is then analysed in the Laplace space, and a further application of the inverse transform will return the solution to the real space. A new numerical scheme, which combines Adam-Bashforth Laplace transform and is capable of handling PDEs, is then achieved.

8.1 Numerical Method For PDE with Integer Order

Consider the following general PDE

$$\frac{\partial u(x, t)}{\partial t} = Lu(x, t) + Nu(x, t), \quad (8.1)$$

where L is a linear operator and N is a non-linear operator. We start by applying Laplace Transform on both sides of the equation, with respect to the x variable to obtain:

$$\begin{aligned} \mathcal{L}\left(\frac{\partial u(x, t)}{\partial t}\right) &= \mathcal{L}(Lu(x, t) + Nu(x, t)) \Rightarrow \frac{d}{dt} (u(p, t)) = \mathcal{L}(Lu(x, t) + Nu(x, t)) \\ \frac{d}{dt} (u(t)) &= F(u, t). \end{aligned} \quad (8.2)$$

where $u(t) = u(p, t)$ and $F(u, t) = \mathcal{L}(L u(x, t) + Nu(x, t))$.

Next we apply the fundamental theorem of calculus on equation (8.2) to obtain

$$u(t) = u(t_0) + \int_0^t F(u, \tau) d\tau.$$

This is also

$$u(t) = u_0 + \int_0^t F(u, \tau) d\tau,$$

when $t = t_{n+1}$ we have

$$u_{n+1} = u(t_{n+1}) = u_0 + \int_0^{t_{n+1}} F(u, \tau) d\tau,$$

when $t = t_n$ we have

$$u_n = u(t_n) = u_0 + \int_0^{t_n} F(u, \tau) d\tau,$$

It follows that

$$\begin{aligned} u_{n+1} - u_n &= \int_0^{t_{n+1}} F(u, \tau) d\tau - \int_0^{t_n} F(u, \tau) d\tau \\ u_{n+1} - u_n &= \int_{t_n}^{t_{n+1}} F(u, \tau) d\tau. \end{aligned}$$

If we approximate $F(u, t)$ with the Lagrange polynomial we have

$$F(u, t) \approx P(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F(u, t_n) + \frac{t - t_n}{t_{n-1} - t_n} F(u, t_{n-1}),$$

$$P(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1}.$$

We can therefore write

$$u_{n+1} - u_n = \int_{t_n}^{t_{n+1}} F(u, \tau) d\tau,$$

$$u_{n+1} - u_n = \int_{t_n}^{t_{n+1}} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1} \right) d\tau,$$

$$u_{n+1} - u_n = \frac{F_n}{t_n - t_{n-1}} \int_{t_n}^{t_{n+1}} (t - t_{n-1}) dt + \frac{F_{n-1}}{t_{n-1} - t_n} \int_{t_n}^{t_{n+1}} (t - t_n) dt,$$

$$u_{n+1} - u_n = \frac{F_n}{t_n - t_{n-1}} \left[\frac{1}{2} t^2 - t_{n-1} t \right]_{t_n}^{t_{n+1}} + \frac{F_{n-1}}{t_{n-1} - t_n} \left[\frac{1}{2} t^2 - t_n t \right]_{t_n}^{t_{n+1}}.$$

By letting

$$h = t_n - t_{n-1},$$

we have

$$\begin{aligned} u_{n+1} - u_n &= \frac{F_n}{h} \left(\frac{1}{2} t_{n+1}^2 - t_{n-1} t_{n+1} - \frac{1}{2} t_n^2 + t_{n-1} t_n \right) \\ &\quad - \frac{F_{n-1}}{h} \left(\frac{1}{2} t_{n+1}^2 - t_n t_{n+1} - \frac{1}{2} t_n^2 + t_n^2 \right), \\ &= \frac{F_n}{h} \left(\frac{1}{2} (t_{n+1} - t_n)(t_{n+1} + t_n) - t_{n-1}(t_{n+1} - t_n) \right) \\ &\quad - \frac{F_{n-1}}{h} \left(\frac{1}{2} (t_{n+1} - t_n)(t_{n+1} + t_n) - t_n(t_{n+1} - t_n) \right), \\ &= \frac{F_n}{h} \left(\frac{1}{2} h(t_{n+1} + t_n) - h t_{n-1} \right) - \frac{F_{n-1}}{h} \left(\frac{1}{2} h(t_{n+1} + t_n) - h t_n \right), \\ &= F_n \left(\frac{1}{2} (t_{n+1} + t_n) - t_{n-1} \right) - F_{n-1} \left(\frac{1}{2} (t_{n+1} + t_n) - t_n \right), \end{aligned}$$

$$\begin{aligned}
&= F_n \left(\frac{1}{2} ((n+1)h + nh) - (n-1)h \right) - F_{n-1} \left(\frac{1}{2} ((n+1)h + nh) - nh \right), \\
&= F_n \left(\frac{1}{2} nh + \frac{1}{2} h + \frac{1}{2} nh - nh + h \right) - F_{n-1} \left(\frac{1}{2} nh + \frac{1}{2} h + \frac{1}{2} nh - nh \right), \\
&= F_n \left(\frac{1}{2} h + h \right) - F_{n-1} \left(\frac{1}{2} h \right),
\end{aligned}$$

$$u_{n+1} = u_n + h \left(\frac{3}{2} F_n - \frac{1}{2} F_{n-1} \right). \quad (8.3)$$

Applying the inverse transform to return into the real space we have:

$$\mathcal{L}^{-1}(u_{n+1}) = \mathcal{L}^{-1} \left[u_n + h \left(\frac{3}{2} F_n - \frac{1}{2} F_{n-1} \right) \right],$$

$$u(x, t_{n+1}) = \mathcal{L}^{-1} \left[u_n + h \left(\frac{3}{2} F_n - \frac{1}{2} F_{n-1} \right) \right]. \quad (8.4)$$

We can then apply the forward or backward method in space to obtain

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \frac{3h}{2} F_i^n - \frac{h}{2} F_i^{n-1}. \quad (8.5)$$

8.2 New Numerical Method for PDE with non-Integer Order Derivative

To illustrate the method we consider the following general fractional PDE

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = L u(x, t) + N u(x, t), \quad (8.6)$$

where L is a linear operator and N is a non-linear operator. Applying Laplace transform on both sides of the equation (8.6) we have:

$$\mathcal{L} \left(\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \right) = \mathcal{L} (L u(x, t) + N u(x, t)).$$

For the Caputo type, fractional partial derivative this will be

$$\begin{aligned} {}^C D_t^\alpha u(p, t) &= \mathcal{L}(L u(x, t) + N u(x, t)), \\ {}^C D_t^\alpha u(p, t) &= F(u, t). \end{aligned}$$

This is

$${}^C D_t^\alpha u(t) = F(u, t), \quad (8.7)$$

where $u(t) = u(p, t)$ and $F(u, t) = \mathcal{L}(L u(x, t) + N u(x, t))$.

The next step is to apply the Caputo fractional Integral operator on equation (8.7). Doing so, we obtain

$$u(t) - u(t_0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} F(u, \tau) d\tau,$$

when $t = t_{n+1}$

$$u_{n+1} = u(t_{n+1}) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau,$$

when $t = t_n$

$$u_n = u(t_n) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau,$$

$$u_{n+1} - u_n = \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau - \int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau \right], \quad (8.8)$$

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau = \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau,$$

we approximate $F(u, t)$ with the following Lagrange polynomial

$$\begin{aligned} F(u, t) &\approx P(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F(u, t_n) + \frac{t - t_n}{t_{n-1} - t_n} F(u, t_{n-1}), \\ P(t) &= \frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1}. \end{aligned}$$

The first fractional integral in equation (8.8) can then be expressed as:

$$\begin{aligned}
\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau &= \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1} \right) dt, \\
&= \sum_{j=0}^n \left[\frac{F_n}{t_n - t_{n-1}} \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1}) dt \right. \\
&\quad \left. + \frac{F_{n-1}}{t_{n-1} - t_n} \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_n) dt \right].
\end{aligned}$$

Setting $h = t_{n+1} - t_n$ we have it is equal to

$$\sum_{j=0}^n \left[\frac{F_n}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1}) dt - \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_n) dt \right].$$

We continue by implementing the following change of variable.

We let $y = t_{n+1} - t, dt = -dy, t = t_{n+1} - y$

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1}) dt &= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} y^{\alpha-1} (-y + t_{n+1} - t_{n-1}) (-dy), \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} y^{\alpha-1} (-y + 2h) dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} y^{\alpha-1} (y - 2h) dy \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y^\alpha - 2hy^{\alpha-1}) dy, \\
&= \frac{1}{\alpha+1} [y^{\alpha+1}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} - \frac{2h}{\alpha} [y^\alpha]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}},
\end{aligned}$$

$$\begin{aligned}
& \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1}) dt \\
&= \frac{1}{\alpha + 1} \left((t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right) \\
&\quad - \frac{2h}{\alpha} \left((t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right).
\end{aligned}$$

In a similar way, by letting we have that

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_n) dt &= - \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} y^{\alpha-1} (-y + t_{n+1} - t_n) dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y^{\alpha} - hy^{\alpha-1}) dy, \\
&= \frac{1}{\alpha + 1} [y^{\alpha+1}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} - \frac{h}{\alpha} [y^{\alpha}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}}, \\
\int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_n) dt \\
&= \frac{1}{\alpha + 1} \left((t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right) \\
&\quad - \frac{h}{\alpha} \left((t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right).
\end{aligned}$$

Therefore it will follow that

$$\begin{aligned}
& \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau \\
&= \frac{F_n}{h} \left\{ \frac{1}{\alpha + 1} \sum_{j=0}^n \left[(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right] \right. \\
&\quad \left. - \frac{2h}{\alpha} \sum_{j=0}^n \left[(t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right] \right\} \\
&\quad - \frac{F_{n-1}}{h} \left\{ \frac{1}{\alpha + 1} \sum_{j=0}^n \left[(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right] \right. \\
&\quad \left. - \frac{h}{\alpha} \sum_{j=0}^n \left[(t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right] \right\}.
\end{aligned}$$

Let us evaluate the summations in details:

$$\begin{aligned}
\sum_{j=0}^n \left[(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right] &= (t_{n+1} - t_1)^{\alpha+1} - (t_{n+1} - t_0)^{\alpha+1} \\
&+ (t_{n+1} - t_2)^{\alpha+1} - (t_{n+1} - t_1)^{\alpha+1} \\
&+ (t_{n+1} - t_3)^{\alpha+1} - (t_{n+1} - t_2)^{\alpha+1} \\
&+ (t_{n+1} - t_4)^{\alpha+1} - (t_{n+1} - t_3)^{\alpha+1} \\
&+ (t_{n+1} - t_5)^{\alpha+1} - (t_{n+1} - t_4)^{\alpha+1} \\
&+ (t_{n+1} - t_6)^{\alpha+1} - (t_{n+1} - t_5)^{\alpha+1} \\
&+ \dots - \dots \\
&+ (t_{n+1} - t_{n-1})^{\alpha+1} - (t_{n+1} - t_{n-2})^{\alpha+1} \\
&+ (t_{n+1} - t_n)^{\alpha+1} - (t_{n+1} - t_{n-1})^{\alpha+1} \\
&+ (t_{n+1} - t_{n+1})^{\alpha+1} - (t_{n+1} - t_n)^{\alpha+1} \\
&= -(t_{n+1} - t_0)^{\alpha+1},
\end{aligned}$$

likewise,

$$\begin{aligned}
\sum_{j=0}^n \left[(t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right] &= (t_{n+1} - t_1)^{\alpha} - (t_{n+1} - t_0)^{\alpha} \\
&+ (t_{n+1} - t_2)^{\alpha} - (t_{n+1} - t_1)^{\alpha} \\
&+ (t_{n+1} - t_3)^{\alpha} - (t_{n+1} - t_2)^{\alpha} \\
&+ (t_{n+1} - t_4)^{\alpha} - (t_{n+1} - t_3)^{\alpha} \\
&+ (t_{n+1} - t_5)^{\alpha} - (t_{n+1} - t_4)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
& + (t_{n+1} - t_6)^\alpha - (t_{n+1} - t_5)^\alpha \\
& + \quad \dots \quad - \quad \dots \\
& + (t_{n+1} - t_{n-1})^\alpha - (t_{n+1} - t_{n-2})^\alpha \\
& + (t_{n+1} - t_n)^\alpha - (t_{n+1} - t_{n-1})^\alpha \\
& + (t_{n+1} - t_{n+1})^\alpha - (t_{n+1} - t_n)^\alpha \\
& = -(t_{n+1} - t_0)^\alpha,
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau \\
& = \frac{F_n}{h} \left[-\frac{1}{\alpha+1} (t_{n+1} - t_0)^{\alpha+1} + \frac{2h}{\alpha} (t_{n+1} - t_0)^\alpha \right] \\
& - \frac{F_{n-1}}{h} \left[-\frac{1}{\alpha+1} (t_{n+1} - t_0)^{\alpha+1} + \frac{h}{\alpha} (t_{n+1} - t_0)^\alpha \right], \\
& = \frac{F_n}{h} \left[-\frac{1}{\alpha+1} (n+1)^{\alpha+1} h^{\alpha+1} + \frac{2h}{\alpha} (n+1)^\alpha h^\alpha \right] \\
& - \frac{F_{n-1}}{h} \left[-\frac{1}{\alpha+1} (n+1)^{\alpha+1} h^{\alpha+1} + \frac{h}{\alpha} (n+1)^\alpha h^\alpha \right], \\
& = h^\alpha \left[\left(\frac{2(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) F_n - \left(\frac{(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) F_{n-1} \right].
\end{aligned}$$

The second fractional integral in (8.8) is evaluated similarly let us also give a detailed account of its evaluation.

$$\begin{aligned}
& \int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1} \right) dt, \\
& = \sum_{j=0}^{n-1} \left[\frac{F_n}{t_n - t_{n-1}} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} (t - t_{n-1}) dt \right. \\
& \quad \left. + \frac{F_{n-1}}{t_{n-1} - t_n} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} (t - t_n) dt \right].
\end{aligned}$$

Setting $h = t_{n+1} - t_n$ we have it is equal to

$$\frac{F_n}{h} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} (t - t_{n-1}) dt - \sum_{j=0}^{n-1} \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} (t - t_n) dt.$$

We also continue by implementing the following change of variable.

We let $y = t_n - t, dt = -dy, t = t_n - y$

$$\begin{aligned} & \int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau \\ &= \frac{F_n}{h} \sum_{j=0}^{n-1} \int_{t_n-t_j}^{t_n-t_{j+1}} -y^{\alpha-1} (-y + t_n - t_{n-1}) dy \\ & \quad - \frac{F_{n-1}}{h} \sum_{j=0}^{n-1} \int_{t_n-t_j}^{t_n-t_{j+1}} y^{\alpha-1} y dy, \\ &= \frac{F_n}{h} \sum_{j=0}^{n-1} \int_{t_n-t_j}^{t_n-t_{j+1}} (y^\alpha - hy^{\alpha-1}) dy - \frac{F_{n-1}}{h} \sum_{j=0}^{n-1} \int_{t_n-t_j}^{t_n-t_{j+1}} y^\alpha dy, \\ &= \frac{F_n}{h} \sum_{j=0}^{n-1} \left[\frac{1}{\alpha+1} y^{\alpha+1} - \frac{h}{\alpha} y^\alpha \right]_{t_n-t_j}^{t_n-t_{j+1}} - \frac{F_{n-1}}{h} \sum_{j=0}^{n-1} \left[\frac{1}{\alpha+1} y^{\alpha+1} \right]_{t_n-t_j}^{t_n-t_{j+1}}, \\ &= \frac{F_n}{h} \sum_{j=0}^{n-1} \left[\frac{(t_n - t_{j+1})^{\alpha+1}}{\alpha+1} - \frac{(t_n - t_j)^{\alpha+1}}{\alpha+1} - \frac{h}{\alpha} (t_n - t_{j+1})^\alpha + \frac{h}{\alpha} (t_n - t_j)^\alpha \right] \\ & \quad - \frac{F_{n-1}}{h} \sum_{j=0}^{n-1} \left[\frac{(t_n - t_{j+1})^{\alpha+1}}{\alpha+1} - \frac{(t_n - t_j)^{\alpha+1}}{\alpha+1} \right], \\ &= \frac{F_n}{h} \left\{ \frac{1}{\alpha+1} \sum_{j=0}^{n-1} [(t_n - t_{j+1})^{\alpha+1} - (t_n - t_j)^{\alpha+1}] \right. \\ & \quad \left. - \frac{h}{\alpha} \sum_{j=0}^{n-1} [(t_n - t_{j+1})^\alpha - (t_n - t_j)^\alpha] \right\} \\ & \quad - \frac{F_{n-1}}{h} \frac{1}{\alpha+1} \sum_{j=0}^{n-1} [(t_n - t_{j+1})^{\alpha+1} - (t_n - t_j)^{\alpha+1}]. \end{aligned}$$

As we proved it few steps above

$$\sum_{j=0}^{n-1} [(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1}] = -(t_n - t_0)^{\alpha+1},$$

and

$$\sum_{j=0}^{n-1} [(t_n - t_{j+1})^{\alpha} - (t_n - t_j)^{\alpha}] = -(t_n - t_0)^{\alpha}.$$

Therefore

$$\begin{aligned} \int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau &= \frac{F_n}{h} \left\{ \frac{-(t_n - t_0)^{\alpha+1}}{\alpha + 1} + \frac{h(t_n - t_0)^{\alpha}}{\alpha} \right\} + \frac{F_{n-1}}{h} \left\{ \frac{(t_n - t_0)^{\alpha+1}}{\alpha + 1} \right\}, \\ &= \frac{F_n}{h} \left(\frac{-n^{\alpha+1} h^{\alpha+1}}{\alpha + 1} + \frac{n^{\alpha} h^{\alpha+1}}{\alpha} \right) + \frac{F_{n-1}}{h} \left(\frac{n^{\alpha+1} h^{\alpha+1}}{\alpha + 1} \right). \end{aligned}$$

We can thus write

$$\int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau = h^{\alpha} \left[\left(\frac{n^{\alpha}}{\alpha} - \frac{n^{\alpha+1}}{\alpha + 1} \right) F_n + \frac{n^{\alpha+1}}{\alpha + 1} F_{n-1} \right].$$

We can now substitute back in equation (8.8) the newly obtained expressions of the two fractional integrals it then reads as follows

$$\begin{aligned} u_{n+1} - u_n &= \frac{h^{\alpha}}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^{\alpha}}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha + 1} \right) F_n - \left(\frac{(n+1)^{\alpha}}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha + 1} \right) F_{n-1} \right] \\ &\quad - \frac{h^{\alpha}}{\Gamma(\alpha)} \left[\left(\frac{n^{\alpha}}{\alpha} - \frac{n^{\alpha+1}}{\alpha + 1} \right) F_n + \frac{n^{\alpha+1}}{\alpha + 1} F_{n-1} \right]. \end{aligned}$$

This is

$$\begin{aligned} u_{n+1} - u_n &= \frac{h^{\alpha}}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^{\alpha}}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha + 1} - \frac{n^{\alpha}}{\alpha} + \frac{n^{\alpha+1}}{\alpha + 1} \right) F_n \right. \\ &\quad \left. - \left(\frac{(n+1)^{\alpha}}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha + 1} + \frac{n^{\alpha+1}}{\alpha + 1} \right) F_{n-1} \right]. \end{aligned}$$

The fractional numerical scheme in Laplace space can then be written as:

$$\begin{aligned}
u_{n+1} - u_n &= \frac{h^\alpha}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_n \right. \\
&\quad \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_{n-1} \right].
\end{aligned} \tag{8.9}$$

Remark 8.1 For $\alpha = 1$ we recover the classic Adam-Basforth Numerical scheme. In fact

$$\begin{aligned}
u_{n+1} - u_n &= \frac{h}{\Gamma(1)} \left[\left(2n + 2 - n + \frac{n^2 - (n+1)^2}{2} \right) F_n \right. \\
&\quad \left. - \left(n + 1 + \frac{n^2 - (n+1)^2}{2} \right) F_{n-1} \right], \\
&= h \left[\left(n + 2 - n - \frac{1}{2} \right) F_n - \left(n + 1 - n - \frac{1}{2} \right) F_{n-1} \right].
\end{aligned}$$

This is

$$u_{n+1} - u_n = h \left(\frac{3}{2} F_n - \frac{1}{2} F_{n-1} \right).$$

To obtain a numerical scheme in the real space we need to apply the inverse Laplace to equation (8.9). We obtain the following iterative scheme in the real space:

$$\begin{aligned}
u(x_i, t_{n+1}) &= \mathcal{L}^{-1} \left\{ u_n \right. \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_n \right. \\
&\quad \left. \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_{n-1} \right] \right\}.
\end{aligned} \tag{8.10}$$

The above equation can be discretized in x using any classical method, including but not limited to forward or backward difference, we can then achieve the following expression:

$$\begin{aligned}
u(x_i, t_{n+1}) &= u_n(x_i, t_n) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_i^n \right. \\
&\quad \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_i^{n-1} \right].
\end{aligned} \tag{8.11}$$

8.3 Error Analysis of the Method

Let

$${}_0^C D^\alpha u(x, t) = Lu(x, t) + Nu(x, t), \quad (8.12)$$

be a general fractional partial differential equation. As we established it earlier the numerical solutions using Laplace Adam-Bashforth method is given as:

$$\begin{aligned} u(x_i, t_{n+1}) = \mathcal{L}^{-1} \Big\{ & u_n \\ & + \frac{h^\alpha}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_n \right. \\ & \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_{n-1} \right] + R_n^\alpha \Big\}, \quad (8.13) \end{aligned}$$

where

$$R_n^\alpha < \infty.$$

Proof: Following the derivation presented earlier we have

$$u_{n+1} - u_n = \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} F(u, \tau) d\tau - \int_0^{t_n} (t_n - \tau)^{\alpha-1} F(u, \tau) d\tau \right],$$

where

$$F(u, \zeta) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1} + \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i),$$

$$\begin{aligned} u_{n+1} - u_n = \frac{1}{\Gamma(\alpha)} \Bigg[& \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1} \right) dt \\ & - \int_0^{t_n} (t_n - \tau)^{\alpha-1} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F_{n-1} \right) dt \\ & + \int_0^{t_{n+1}} \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} dt \\ & - \int_0^{t_n} \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i) (t_n - \tau)^{\alpha-1} dt \Bigg]. \end{aligned}$$

This is equal to

$$u_{n+1} - u_n = \frac{h^\alpha}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_n - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_{n-1} \right] + R_n^\alpha.$$

Therefore one can easily deduce that

$$R_n^\alpha = \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_{n+1}} \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} dt - \int_0^{t_n} \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i) (t_n - t)^{\alpha-1} dt \right),$$

$$|R_n^\alpha| \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_{n+1}} \left| \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} \right| dt + \int_0^{t_n} \left| \frac{F^{(2)}(u, \zeta)}{2!} \prod_{i=0}^1 (t - t_i) (t_n - t)^{\alpha-1} \right| dt \right),$$

$$|R_n^\alpha| \leq \frac{h^2}{8 \Gamma(\alpha)} \max_{\zeta \in (0, t_{n+1})} \{F^{(2)}(u, \zeta)\} \left(\int_0^{t_{n+1}} \left| \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} \right| dt + \int_0^{t_n} \left| \prod_{i=0}^1 (t - t_i) (t_n - t)^{\alpha-1} \right| dt \right),$$

$$|R_n^\alpha| \leq \frac{h^2}{8 \Gamma(\alpha)} \max_{\zeta \in (0, t_{n+1})} \{F^{(2)}(u, \zeta)\} \left(\int_0^{t_{n+1}} | (t_{n+1} - t)^{\alpha-1} | dt + \int_0^{t_n} | (t_n - t)^{\alpha-1} | dt \right)$$

$$|R_n^\alpha| \leq \frac{h^2}{8 \Gamma(\alpha)} \max_{\zeta \in (0, t_{n+1})} \{F^{(2)}(u, \zeta)\} \left(\frac{t_{n+1}^\alpha + t_n^\alpha}{\alpha} \right).$$

We ultimately can put an upper bound on the error term as:

$$|R_n^\alpha| \leq \frac{h^{\alpha+2}}{8 \Gamma(\alpha+1)} \max_{\zeta \in (0, t_{n+1})} \{F^{(2)}(u, \zeta)\} ((n+1)^\alpha + n^\alpha) < \infty.$$

8.4 Applications of the Method

Let us provide general applications of the method. We will illustrate the use of the method to solve a PDE with integer order derivative, on a general fractional operator P.D.E, and on the TFBSE with ABC fractional operator.

8.4.1. Integer order P.D.E

Consider the following wave like equation:

$$\frac{\partial u(x, t)}{\partial t} = c \frac{\partial u(x, t)}{\partial x}. \quad (8.14)$$

Applying Laplace transform on both sides we get the following

$$\frac{du(t)}{dt} = c(p u(p, t) - u(0, t)).$$

Silencing the variable p writing $u(p, t) = u(t)$ the later equation can be written as

$$\frac{du(t)}{dt} = F(t, u(t)).$$

It was proved earlier developing the method that the solution in the Laplace space of the later equation is given by:

$$\begin{aligned} u_{n+1} &= u_n + h \left(\frac{3}{2} F(t_n, u(t_n)) - \frac{1}{2} F(t_{n-1}, u(t_{n-1})) \right), \\ &= u_n + h \left(\frac{3}{2} F_n - \frac{1}{2} F_{n-1} \right). \end{aligned}$$

Applying the inverse Laplace transform we have

$$u(x, t_{n+1}) = u(x, t_n) + \frac{3hc}{2} \frac{\partial u(x, t_n)}{\partial x} - \frac{hc}{2} \frac{\partial u(x, t_{n-1})}{\partial x}.$$

Now we discretize in the space variable to obtain

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \frac{3hc}{2} \left[\frac{u(x_{i+1}, t_n) - u(x_i, t_n)}{\Delta x} \right] - \frac{hc}{2} \left[\frac{u(x_{i+1}, t_{n-1}) - u(x_i, t_{n-1})}{\Delta x} \right].$$

Letting $u(x_i, t_n) = u_i^n$, and $\Delta x = l$ it becomes

$$u_i^{n+1} = u_i^n + \frac{3hc}{2} \frac{u_{i+1}^n - u_i^n}{l} - \frac{hc}{2} \frac{u_{i+1}^{n-1} - u_i^{n-1}}{l}.$$

The iteration scheme can then be reorganised as:

$$u_i^{n+1} = \left(1 - \frac{3hc}{2l}\right) u_i^n + \frac{3hc}{2l} u_{i+1}^n - \frac{hc}{2l} u_{i+1}^{n-1} + \frac{hc}{2l} u_i^{n-1}. \quad (8.15)$$

(i) Stability Analysis

Assume we have a Fourier expansion in space

$$u(x, t) = \sum_f \hat{u} \exp(jfx).$$

Replacing in equation (8.15) it becomes

$$\hat{u}_{n+1} e^{jifl} = \left(1 - \frac{3hc}{2l}\right) \hat{u}_n e^{jifl} + \frac{3hc}{2l} \hat{u}_n e^{j(i+1)fl} - \frac{hc}{2l} \hat{u}_{n-1} e^{j(i+1)fl} + \frac{hc}{2l} \hat{u}_{n-1} e^{jifl}.$$

This is

$$\hat{u}_{n+1} = \left(1 - \frac{3hc}{2l}\right) \hat{u}_n + \frac{3hc}{2l} \hat{u}_n e^{jfl} - \frac{hc}{2l} \hat{u}_{n-1} e^{jfl} + \frac{hc}{2l} \hat{u}_{n-1},$$

$$\hat{u}_{n+1} = \left(1 - \frac{3hc}{2l} + \frac{3hc}{2l} e^{jfl}\right) \hat{u}_n + \left(\frac{hc}{2l} - \frac{hc}{2l} e^{jfl}\right) \hat{u}_{n-1},$$

$$\frac{\hat{u}_{n+1}}{\hat{u}_n} = \left(1 - \frac{3hc}{2l} + \frac{3hc}{2l} e^{jfl}\right) + \left(\frac{hc}{2l} - \frac{hc}{2l} e^{jfl}\right) \frac{\hat{u}_{n-1}}{\hat{u}_n}.$$

We can rearrange this as

$$\begin{aligned} \frac{\hat{u}_{n+1}}{\hat{u}_n} &= 1 - \frac{3hc}{2l} (1 - \cos(fl)) + j \frac{3hc}{2l} \sin(fl) + \frac{hc}{2l} (1 - \cos(jfl)) \frac{\hat{u}_{n-1}}{\hat{u}_n} \\ &\quad - j \frac{hc}{2l} \sin(fl) \frac{\hat{u}_{n-1}}{\hat{u}_n}, \end{aligned}$$

$$\frac{\hat{u}_{n+1}}{\hat{u}_n} = 1 - \frac{3hc}{l} \sin^2\left(\frac{fl}{2}\right) + \frac{hc}{l} \sin^2\left(\frac{fl}{2}\right) \frac{\hat{u}_{n-1}}{\hat{u}_n} + j \frac{hc}{2l} \left(3 - \frac{\hat{u}_{n-1}}{\hat{u}_n}\right).$$

Let us prove that we have $\forall n |u_n| < |u_0|$. For $n = 0$

$$\left|\frac{\hat{u}_1}{\hat{u}_0}\right| = \left|1 - \frac{3hc}{2l} \sin^2\left(\frac{fl}{2}\right)\right|.$$

This means that we have

$$\begin{aligned} \left|\frac{\hat{u}_1}{\hat{u}_0}\right| < 1 &\Leftrightarrow -1 < 1 - \frac{3hc}{2l} \sin^2\left(\frac{fl}{2}\right) < 1, \\ &\Rightarrow -2 < -\frac{3hc}{2l} \sin^2\left(\frac{fl}{2}\right) < 0, \\ &\Rightarrow 0 < \frac{3hc}{4l} \sin^2\left(\frac{fl}{2}\right) < 1. \end{aligned}$$

The previous condition will certainly be achieved if

$$0 < \frac{3hc}{4l} < 1,$$

as

$$\begin{aligned} 0 < \frac{3hc}{4l} \sin^2\left(\frac{fl}{2}\right) &< \frac{3hc}{4l}, \\ \frac{3hc}{4l} < 1 &\Rightarrow \frac{h}{l} < \frac{4}{3} c. \end{aligned}$$

This is

$$\frac{\Delta t}{\Delta x} < \frac{4}{3} c. \quad (8.16)$$

Now, let us assume $\forall j \leq n, |\hat{u}_n| < |\hat{u}_0|$ and prove that $|\hat{u}_{n+1}| < |\hat{u}_0|$.

The condition (8.16) ensures that the coefficients in the following equation are positive.

$$\hat{u}_{n+1} = \left(1 - \frac{3hc}{2l} + \frac{3hc}{2l} e^{jfl}\right) \hat{u}_n + \left(\frac{hc}{2l} - \frac{hc}{2l} e^{jfl}\right) \hat{u}_{n-1}.$$

The later then implies that

$$|\hat{u}_{n+1}| = \left(1 - \frac{3hc}{2l}\right) |\hat{u}_n| + \frac{3hc}{2l} |\hat{u}_n| |e^{jfl}| - \frac{hc}{2l} |\hat{u}_{n-1}| |e^{jfl}| + \frac{hc}{2l} |\hat{u}_{n-1}|.$$

Using the induction hypothesis this becomes

$$|\hat{u}_{n+1}| < \left(1 - \frac{3hc}{2l}\right) |\hat{u}_0| + \frac{3hc}{2l} |\hat{u}_0| - \frac{hc}{2l} |\hat{u}_0| + \frac{hc}{2l} |\hat{u}_0|,$$

and therefore, we obtain

$$|\hat{u}_{n+1}| < \left(1 - \frac{3hc}{2l} + \frac{3hc}{2l} - \frac{hc}{2l} + \frac{hc}{2l}\right) |\hat{u}_0|,$$

$$|\hat{u}_{n+1}| < |\hat{u}_0|.$$

We can therefore conclude that the numerical scheme solution presented above is conditionally stable.

8.4.2. Fractional order P.D.E

Consider the following fractional order P.D.E, where the fractional derivative is given in the Caputo sense.

$$\partial^\alpha u(x, t) = d \frac{\partial^2 u(x, t)}{\partial x^2}. \quad (8.17)$$

Applying Laplace transform on both sides, we get the following

$${}_0^C D_t^\alpha u(p, t) = d \mathcal{L} \left\{ \frac{\partial^2 u(x, t)}{\partial x^2} \right\} = d \mathcal{L} (p^2 u(p, t) - p u(0, t) - u(0, t)).$$

Silencing the variable p , writing $u(p, t) = u(t)$ the later equation can be rewritten as

$${}_0^C D_t^\alpha u(t) = d F(u(t), t).$$

We proved earlier its solution in the Laplace space is given by (eq 8.9)

$$u_{n+1} - u_n = \frac{h^\alpha}{\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_n - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) F_{n-1} \right].$$

Applying now the inverse transform we obtain:

$$u(x, t_{n+1}) = u(x, t_n) + \frac{h^\alpha}{\Gamma(\alpha)} \delta_n^\alpha d \frac{\partial^2 u(x, t_n)}{\partial x^2} - \frac{h^\alpha}{\Gamma(\alpha)} \delta_n^{\alpha,1} d \frac{\partial^2 u(x, t_{n-1})}{\partial x^2},$$

where

$$\delta_n^\alpha = \frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1},$$

and

$$\delta_n^{\alpha,1} = \frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}.$$

Discretizing in the space variable, we have

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \frac{h^\alpha}{\Gamma(\alpha)} \delta_n^\alpha d \frac{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n))}{(\Delta x)^2} - \frac{h^\alpha}{\Gamma(\alpha)} \delta_n^{\alpha,1} d \frac{u(x_{i+1}, t_{n-1}) - 2u(x_i, t_{n-1}) + u(x_{i-1}, t_{n-1}))}{(\Delta x)^2}.$$

Letting $u(x_i, t_n) = u_i^n$, and $\Delta x = l$ the equation becomes

$$u_i^{n+1} = u_i^n + \frac{h^\alpha}{\Gamma(\alpha)} \delta_n^\alpha d \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{l^2} - \frac{h^\alpha}{\Gamma(\alpha)} \delta_n^{\alpha,1} d \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{l^2}.$$

This is better rearranged as:

$$\begin{aligned} u_i^{n+1} = & \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \right) u_i^n + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha u_{i+1}^n + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha u_{i-1}^n \\ & - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} u_{i+1}^{n-1} + \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} u_i^{n-1} \\ & - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} u_{i-1}^{n-1}. \end{aligned} \quad (8.18)$$

(i) Stability Analysis of the Numerical Scheme solution to the Fractional Order PDE

Assume we have the following Fourier expansion in space

$$u(x, t) = \sum_f \hat{u}(t) \exp(jfx).$$

While letting

$$u(x_i, t_n) = u_i^n = \hat{u}_n \exp(jfi\Delta x) = \hat{u}_n e^{jfil}.$$

Equation (8.18) becomes

$$\begin{aligned} \hat{u}_{n+1} e^{jfil} = & \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \right) \hat{u}_n e^{jfil} + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \hat{u}_n e^{jf(i+1)l} \\ & + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \hat{u}_n e^{jf(i-1)l} - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} e^{jf(i+1)l} \\ & + \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} e^{jfil} - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} e^{jf(i-1)l}, \end{aligned}$$

this is

$$\begin{aligned}\hat{u}_{n+1} = & \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha\right) \hat{u}_n + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \hat{u}_n e^{jfl} + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \hat{u}_n e^{-jfl} \\ & - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} e^{jfl} + \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} e^{-jfl},\end{aligned}$$

which implies

$$\begin{aligned}\hat{u}_{n+1} = & \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha\right) \hat{u}_n + \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} \\ & + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \hat{u}_n (e^{jfl} + e^{-jfl}) - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} (e^{jfl} + e^{-jfl}),\end{aligned}$$

further we have

$$\begin{aligned}\hat{u}_{n+1} = & \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha\right) \hat{u}_n + \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} + \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \hat{u}_n (2 \cos(fl)) \\ & - \frac{h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \hat{u}_{n-1} (2 \cos(fl))\end{aligned}$$

and

$$\begin{aligned}\hat{u}_{n+1} = & \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha + \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \cos(fl)\right) \hat{u}_n \\ & + \left(\frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \cos(fl)\right) \hat{u}_{n-1}.\end{aligned}$$

Thus we can write

$$\hat{u}_{n+1} = \left(1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha (1 - \cos(fl))\right) \hat{u}_n + \left(\frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} (1 - \cos(fl))\right) \hat{u}_{n-1}. \quad (8.19)$$

From equation (8.19) we have

$$\begin{aligned}\frac{\hat{u}_{n+1}}{\hat{u}_n} = & 1 - \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha (1 - \cos(fl)) + \left(\frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} (1 - \cos(fl))\right) \frac{\hat{u}_{n-1}}{\hat{u}_n} \\ = & 1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) + \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \sin^2\left(\frac{fl}{2}\right) \frac{\hat{u}_{n-1}}{\hat{u}_n}.\end{aligned}$$

For $n = 0$

$$\frac{\hat{u}_1}{\hat{u}_0} = 1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right),$$

$$\left| \frac{\hat{u}_1}{\hat{u}_0} \right| < 1 \Leftrightarrow -1 < 1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) < 1 \Rightarrow 0 < \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) < 2,$$

$$0 < \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) < 2 \Rightarrow 0 < \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) < 1.$$

This will certainly be achieved if

$$0 < \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha < 1,$$

as

$$\frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) < \frac{2h^\alpha d}{l^2 \Gamma(\alpha)}.$$

Therefore,

$$0 < \frac{2h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha < 1 \Rightarrow \frac{h^\alpha}{l^2} < \frac{\Gamma(\alpha)}{2d\delta_n^\alpha}.$$

This is

$$\frac{(\Delta t)^\alpha}{(\Delta x)^2} < \frac{\Gamma(\alpha)}{2d\delta_n^\alpha}. \quad (8.20)$$

We will now prove by induction that $\forall n \ |\hat{u}_n| < |\hat{u}_0|$. We proved already that $|\hat{u}_1| < |\hat{u}_0|$, let us assume that $|\hat{u}_n| < |\hat{u}_0|$ and prove that $|\hat{u}_{n+1}| < |\hat{u}_0|$.

$$|\hat{u}_{n+1}| = \left| \left(1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right) \right) \hat{u}_n + \left(\frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \sin^2\left(\frac{fl}{2}\right) \right) \hat{u}_{n-1} \right|.$$

$$\text{Let } A_1 = 1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2\left(\frac{fl}{2}\right), \quad A_2 = \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \sin^2\left(\frac{fl}{2}\right)$$

Because of condition imposed by (8.20), we have both $A_1, A_2 > 0$.

$$\begin{aligned}
|\hat{u}_{n+1}| &= \left| \left(1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2 \left(\frac{fl}{2} \right) \right) \hat{u}_n + \left(\frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \sin^2 \left(\frac{fl}{2} \right) \right) \hat{u}_{n-1} \right|, \\
&= A_1 |\hat{u}_n| + A_2 |\hat{u}_{n-1}|.
\end{aligned}$$

By the induction hypothesis we have that

$$|\hat{u}_{n+1}| < A_1 |\hat{u}_0| + A_2 |\hat{u}_0|.$$

This means that

$$|\hat{u}_{n+1}| < (A_1 + A_2) |\hat{u}_0|.$$

But

$$A_1 + A_2 = 1 - \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^\alpha \sin^2 \left(\frac{fl}{2} \right) + \frac{4h^\alpha d}{l^2 \Gamma(\alpha)} \delta_n^{\alpha,1} \sin^2 \left(\frac{fl}{2} \right) = 1$$

Therefore

$$|\hat{u}_{n+1}| < |\hat{u}_0|.$$

This concludes the proof that the numerical scheme solution to the fractional order PDE is stable.

8.4.3. Illustrative Graphical Simulations of some Solutions

We justify further the applicability of our proposed method by considering a fractional reaction diffusion system [100].

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u \mu(u - b)(1 - u) - \kappa \frac{uv}{u + v}, \quad (8.21)$$

subject to zero flux boundary conditions and initial functions

$$\begin{aligned}
u_0(x, y, 0) &= 1 - e^{-10 \left(x - \frac{\beta}{2} \right)^2 + \left(y - \frac{\beta}{2} \right)^2}, \\
v_0(x, y, 0) &= e^{-10 \left(x - \frac{\beta}{2} \right)^2 + \left(y - \frac{\beta}{2} \right)^2}.
\end{aligned} \quad (8.22)$$

In the experiment, we observed that the distribution of the species is similar; hence we only report the spatial evolution of u –species.

Figure 8 is obtained with $\beta = 1, \delta = 0.7, \mu = 4.4, c = 1.92, b = 0.5$, and $\kappa = 5.33$

Figure 9 shows the exact solution of equation (8.14) with $c = 3, u(x, 0) = \exp(x), u(0, t) = \exp(ct)$. Figure 10 gives the method's solution of (8.14) with the same parameter conditions as in figure 9. Figure 11 also gives the exact solution of equation (8.14) with $c = 3, u(x, 0) = \cos(x), u(0, t) = \cos(ct)$. Figure 12 shows the method's solution with the same set of parameters as in figure 11.

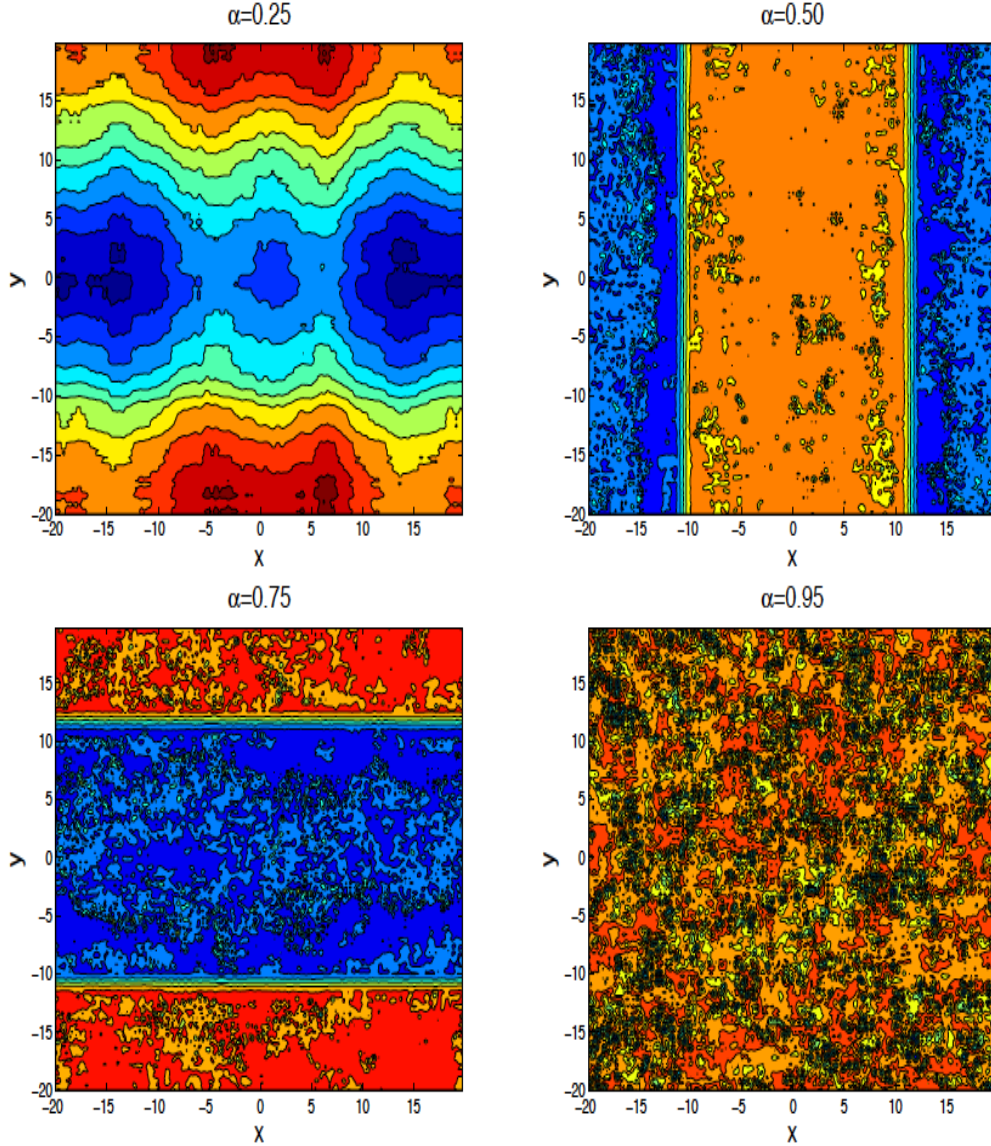


Figure 8: Dynamical behaviour of 2D reaction diffusion system (8.21) showing the distribution of species u at different values of α .

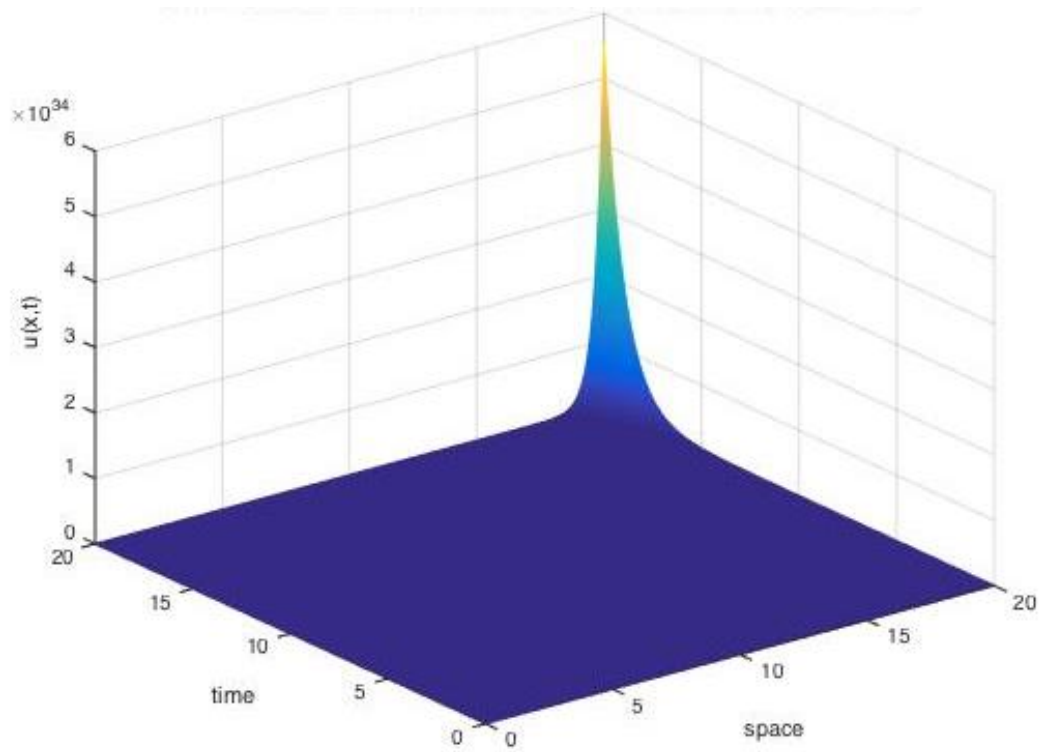


Figure 9: Exact solution of (8.14) with $c=3$, $u(x,0)=\exp(x)$, $u(0,t)=\exp(ct)$.

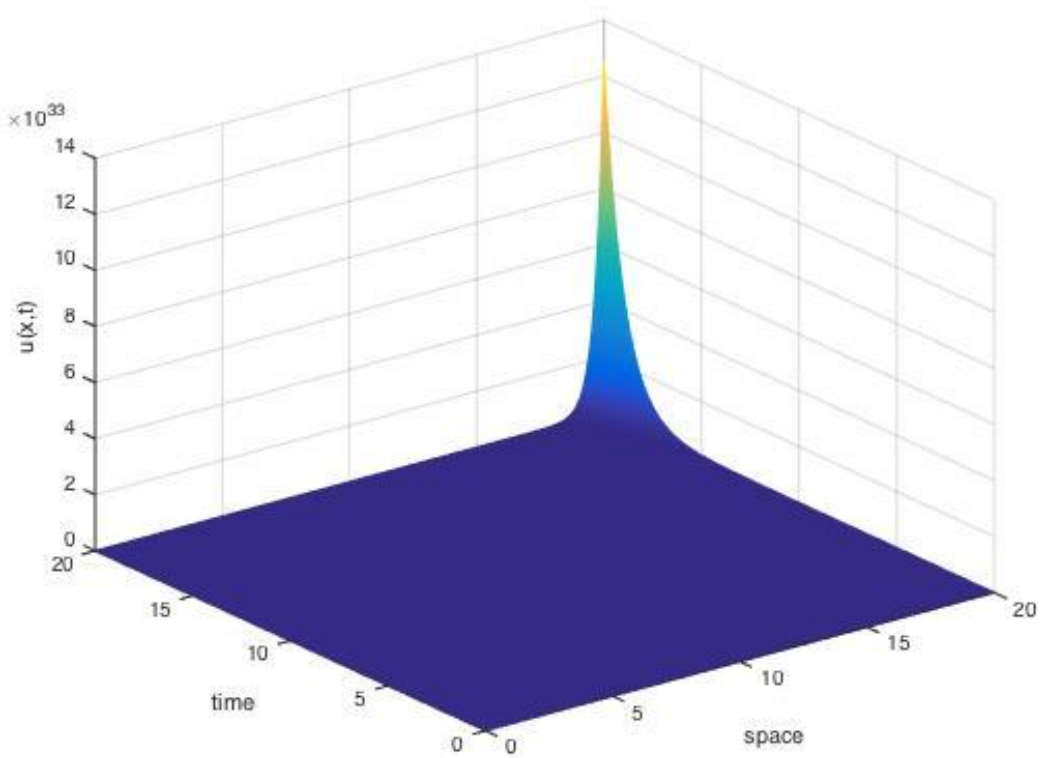


Figure 10: Our method solution of (8.14) same parameters as in figure 9.

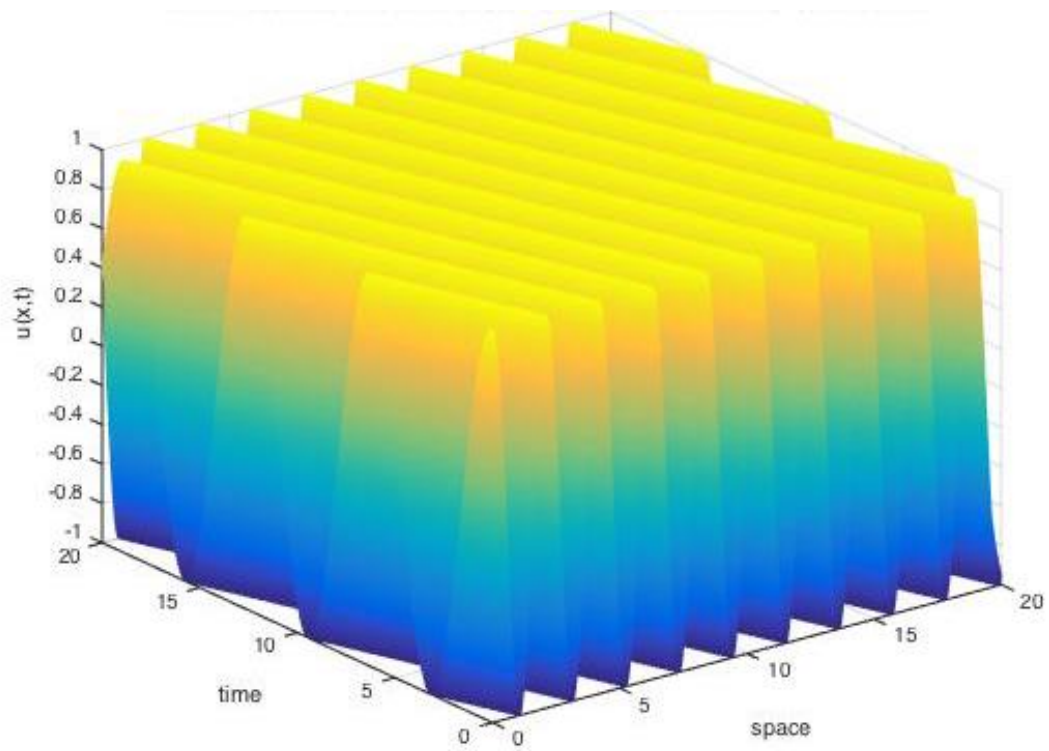


Figure 11: Exact solution of (8.14) $c=3$, $u(x,0)=\cos(x)$, $u(0,t)=\cos(ct)$.

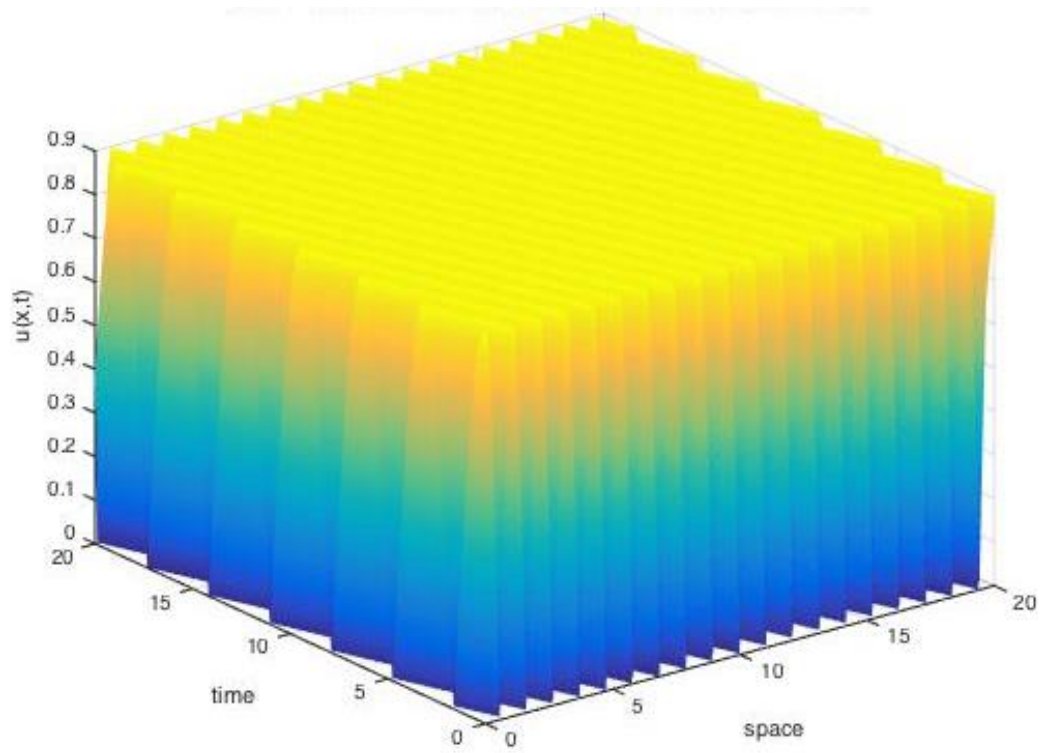


Figure 12: Method's solution of (8.14) same parameters as in figure 11.

To extend the well-known Adams-Bashforth numerical scheme to partial differential equations with integer and non-integer order derivatives, we introduced a new reliable and efficient numerical scheme. The method is a combination of Laplace transform, Adams-Bashforth and (forward or backward numerical scheme). We developed the method for general partial differential equations with local and nonlocal differentiation operator of the Caputo type. We presented in detail the error analysis and the convergence of the method. In the case of fractional partial differential equations, the method provides a numerical algorithm that is easier to implement. Unlike the conventional methods, forward, backward, Crank-Nicholson, the cumbersome summation that always appears in the additional term of their numerical algorithm for the case of fractional partial differential equations, does not exist with our method. This leads to an easier proof of stability and convergence. We illustrated the method by solving two partial differential studied the stability of each example. The proof shows without doubt that our method is very stable and also converges very quickly to the exact solution. We believe this method will turn out to be a very useful numerical scheme that will help solving nonlinear and linear partial differential equations with local and nonlocal operators equations including wave equation for the local case and a diffusion equation for fractional case. We studied the stability of each example. The proof shows without doubt that our method is very stable and also converges very quickly to the exact solution. We believe this method will turn out to be a very useful numerical scheme that will help solving nonlinear and linear partial differential equations with local and nonlocal operators.

In the following subsection we adapt the method to solve non-integer PDE with differentiation operator of the ABC type and use it to solve our ABC TFBSE (5. 11).

8.4.4. New Method for P.D.E with differentiation operator with non local and non singular kernel of the AB type

Let us consider the following general fractional P.D.E

$${}^{ABC}_0D_t^\alpha u(x, t) = L u(x, t) + N u(x, t), \quad (8.23)$$

where

$${}^{ABC}_0D_t^\alpha u(x, t) = \frac{B(\alpha)}{1 - \alpha} \int_0^t \frac{\partial u(x, \zeta)}{\partial \zeta} E_\alpha \left[\frac{-\alpha(t - \zeta)^\alpha}{1 - \alpha} \right] d\zeta ,$$

where $B(\alpha)$ is a normalization function such that , $B(0) = B(1) = 1$,

L and N represent respectively a linear and a nonlinear operator. Applying Laplace transform to (8.23) we have the following:

$$\begin{aligned}\mathcal{L}\left({}^{ABC}_0 D_t^\alpha u(x, t)\right) &= \mathcal{L}(L u(x, t) + N u(x, t)), \\ {}^{ABC}_0 D_t^\alpha u(p, t) &= \\ G(u, t),\end{aligned}\quad (8.24)$$

where $u(t) = u(p, t)$ and $G(u, t) = \mathcal{L}(L u(x, t) + N u(x, t))$.

Let us define and then apply the Atangna-Baleanu integral associated with the non-local and non-singular fractional derivative in (8.24).

Definition: The fractional integral associated with the AB fractional derivative is defined as:

$${}^{AB}_0 I_t^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(y)(t-y)^{\alpha-1} dy.$$

Applying the AB-Integral to (8.24) we can write

$$u(t) - u(t_0) = \frac{1-\alpha}{B(\alpha)} G(u, t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t G(u, \zeta)(t-\zeta)^{\alpha-1} d\zeta.$$

By denoting $u(t_n) = u_n$ the following arguments are straightforwardly made:

$$\begin{aligned}u_{n+1} &= u_0 + \frac{1-\alpha}{B(\alpha)} G(u, t_{n+1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} G(u, \zeta)(t_{n+1}-\zeta)^{\alpha-1} d\zeta, \\ u_n &= u_0 + \frac{1-\alpha}{B(\alpha)} G(u, t_n) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_n} G(u, \zeta)(t_n-\zeta)^{\alpha-1} d\zeta,\end{aligned}$$

and finally

$$\begin{aligned}
u_{n+1} - u_n &= \frac{1-\alpha}{B(\alpha)} (G(u, t_{n+1}) - G(u, t_n)) \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\int_0^{t_{n+1}} G(u, \zeta) (t_{n+1} - \zeta)^{\alpha-1} d\zeta \right. \\
&\left. - \int_0^{t_n} G(u, \zeta) (t_n - \zeta)^{\alpha-1} d\zeta \right]. \tag{8.25}
\end{aligned}$$

The next step is to define a Lagrangian polynomial approximation of the function $G(u, t)$.

$$G(u, t) \approx P(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} G_n + \frac{t - t_n}{t_{n-1} - t_n} G_{n-1}.$$

The first integral term in (8.25) can subsequently be expressed as

$$\begin{aligned}
\int_0^{t_{n+1}} G(u, \zeta) (t_{n+1} - \zeta)^{\alpha-1} d\zeta &= \sum_{j=0}^n \int_{t_j}^{t_{j+1}} G(u, \zeta) (t_{n+1} - \zeta)^{\alpha-1} d\zeta, \\
&= \sum_{j=0}^n \int_{t_j}^{t_{j+1}} P(t) (t_{n+1} - t)^{\alpha-1} dt, \\
&= \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} G_n + \frac{t - t_n}{t_{n-1} - t_n} G_{n-1} \right) (t_{n+1} - t)^{\alpha-1} dt, \\
&= \sum_{j=0}^n \left(\frac{G_n}{t_n - t_{n-1}} \int_{t_j}^{t_{j+1}} (t - t_{n-1}) (t_{n+1} - t)^{\alpha-1} dt \right. \\
&\quad \left. + \frac{G_{n-1}}{t_{n-1} - t_n} \int_{t_j}^{t_{j+1}} (t - t_n) (t_{n+1} - t)^{\alpha-1} dt \right), \\
&= \sum_{j=0}^n \left(\frac{G_n}{h} \int_{t_j}^{t_{j+1}} (t - t_{n-1}) (t_{n+1} - t)^{\alpha-1} dt \right. \\
&\quad \left. - \frac{G_{n-1}}{h} \int_{t_j}^{t_{j+1}} (t - t_n) (t_{n+1} - t)^{\alpha-1} dt \right),
\end{aligned}$$

where here we have $h = t_{j+1} - t_j$, for all $j = 0, 1, 2, \dots, n$. By making the following change of variable, $y = t_{n+1} - t$, and $dt = -dy$ we have.

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} (t - t_{n-1})(t_{n+1} - t)^{\alpha-1} dt &= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} -(t_{n+1} - t_{n-1} - y)y^{\alpha-1} dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y - (t_{n+1} - t_{n-1}))y^{\alpha-1} dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y - 2h)y^{\alpha-1} dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y^\alpha - 2hy^{\alpha-1}) dy, \\
&= \frac{1}{\alpha+1} [y^{\alpha+1}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} - \frac{2h}{\alpha} [y^\alpha]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}}, \\
&= \frac{1}{\alpha+1} \left((t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right) - \frac{2h}{\alpha} \left((t_{n+1} - t_{j+1})^\alpha - (t_{n+1} - t_j)^\alpha \right).
\end{aligned}$$

Likewise

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} (t - t_n)(t_{n+1} - t)^{\alpha-1} dt &= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} -(t_{n+1} - t_n - y)y^{\alpha-1} dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y - (t_{n+1} - t_n))y^{\alpha-1} dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y - h)y^{\alpha-1} dy, \\
&= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y^\alpha - hy^{\alpha-1}) dy, \\
&= \frac{1}{\alpha+1} [y^{\alpha+1}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} - \frac{h}{\alpha} [y^\alpha]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}}, \\
&= \frac{1}{\alpha+1} \left((t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right) - \frac{h}{\alpha} \left((t_{n+1} - t_{j+1})^\alpha - (t_{n+1} - t_j)^\alpha \right).
\end{aligned}$$

We can then write

$$\begin{aligned}
& \int_0^{t_{n+1}} G(u, \zeta)(t_{n+1} - \zeta)^{\alpha-1} d\zeta \\
&= \frac{G_n}{h} \left(\frac{1}{\alpha+1} \sum_{j=0}^n [(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1}] \right. \\
&\quad \left. - \frac{2h}{\alpha} \sum_{j=0}^n [(t_{n+1} - t_{j+1})^\alpha - (t_{n+1} - t_j)^\alpha] \right) \\
&\quad - \frac{G_{n-1}}{h} \left(\frac{1}{\alpha+1} \sum_{j=0}^n [(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1}] \right. \\
&\quad \left. - \frac{h}{\alpha} \sum_{j=0}^n [(t_{n+1} - t_{j+1})^\alpha - (t_{n+1} - t_j)^\alpha] \right),
\end{aligned}$$

We proved earlier above in subsection 8.2 that

$$\begin{aligned}
& \sum_{j=0}^n [(t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1}] = -(t_{n+1} - t_0)^{\alpha+1} \text{ and,} \\
& \sum_{j=0}^n [(t_{n+1} - t_{j+1})^\alpha - (t_{n+1} - t_j)^\alpha] = -(t_{n+1} - t_0)^\alpha;
\end{aligned}$$

therefore

$$\begin{aligned}
& \int_0^{t_{n+1}} G(u, \zeta)(t_{n+1} - \zeta)^{\alpha-1} d\zeta \\
&= \frac{G_n}{h} \left(-\frac{1}{\alpha+1} (t_{n+1} - t_0)^{\alpha+1} + \frac{2h}{\alpha} (t_{n+1} - t_0)^\alpha \right) \\
&\quad - \frac{G_{n-1}}{h} \left(-\frac{1}{\alpha+1} (t_{n+1} - t_0)^{\alpha+1} + \frac{h}{\alpha} (t_{n+1} - t_0)^\alpha \right), \\
&= \frac{G_n}{h} \left(-\frac{h^{\alpha+1}}{\alpha+1} (n+1)^{\alpha+1} + \frac{2h^{\alpha+1}}{\alpha} (n+1)^\alpha \right) \\
&\quad - \frac{G_{n-1}}{h} \left(-\frac{h^{\alpha+1}}{\alpha+1} (n+1)^{\alpha+1} + \frac{h^{\alpha+1}}{\alpha} (n+1)^\alpha \right), \\
&= h^\alpha \left(\frac{2(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} + \right) G_n \\
&\quad - h^\alpha \left(\frac{(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1}.
\end{aligned}$$

This is

$$\begin{aligned}
& \int_0^{t_{n+1}} G(u, \zeta)(t_{n+1} - \zeta)^{\alpha-1} d\zeta \\
&= h^\alpha \left(\frac{2(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) G_n \\
&- h^\alpha \left(\frac{(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1}.
\end{aligned} \tag{8.26}$$

By following the same steps we evaluate the second fractional integral in (8.25) as follows:

$$\begin{aligned}
\int_0^{t_n} G(u, \zeta)(t_n - \zeta)^{\alpha-1} d\zeta &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} G(u, \zeta)(t_n - \zeta)^{\alpha-1} d\zeta, \\
&= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} P(t)(t_n - t)^{\alpha-1} dt, \\
&= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} G_n + \frac{t - t_n}{t_{n-1} - t_n} G_{n-1} \right) (t_n - t)^{\alpha-1} dt, \\
&= \sum_{j=0}^{n-1} \left(\frac{G_n}{h} \int_{t_j}^{t_{j+1}} (t - t_{n-1})(t_n - t)^{\alpha-1} dt \right. \\
&\quad \left. - \frac{G_{n-1}}{h} \int_{t_j}^{t_{j+1}} (t - t_n)(t_n - t)^{\alpha-1} dt \right).
\end{aligned}$$

Applying the change of variable $y = t_n - t$, we have $t = t_n - y$ and $dt = -dy$

$$\begin{aligned}
& \int_0^{t_n} G(u, \zeta)(t_n - \zeta)^{\alpha-1} d\zeta \\
&= \sum_{j=0}^{n-1} \left(\frac{G_n}{h} \int_{t_n-t_j}^{t_n-t_{j+1}} -(h-y)y^{\alpha-1} dy - \frac{G_{n-1}}{h} \int_{t_n-t_j}^{t_n-t_{j+1}} yy^{\alpha-1} dt \right), \\
&= \sum_{j=0}^{n-1} \left(\frac{G_n}{h} \int_{t_n-t_j}^{t_n-t_{j+1}} (y^\alpha - hy^{\alpha-1}) dy - \frac{G_{n-1}}{h} \int_{t_n-t_j}^{t_n-t_{j+1}} y^\alpha dt \right), \\
&= \sum_{j=0}^{n-1} \left(\frac{G_n}{h} \left[\frac{1}{\alpha+1} y^{\alpha+1} - \frac{h}{\alpha} y^\alpha \right]_{t_n-t_j}^{t_n-t_{j+1}} - \frac{G_{n-1}}{h} \left[\frac{1}{\alpha+1} y^{\alpha+1} \right]_{t_n-t_j}^{t_n-t_{j+1}} \right), \\
&= \sum_{j=0}^{n-1} \left\{ \frac{G_n}{h} \left(\frac{1}{\alpha+1} ((t_n - t_{j+1})^{\alpha+1} - (t_n - t_j)^{\alpha+1}) \right. \right. \\
&\quad \left. \left. - \frac{h}{\alpha} ((t_n - t_{j+1})^\alpha - (t_n - t_j)^\alpha) \right) \right. \\
&\quad \left. - \frac{G_{n-1}}{h} \frac{1}{\alpha+1} ((t_n - t_{j+1})^{\alpha+1} - (t_n - t_j)^{\alpha+1}) \right\}, \\
&= \frac{G_n}{h} \frac{1}{\alpha+1} \sum_{j=0}^{n-1} ((t_n - t_{j+1})^{\alpha+1} - (t_n - t_j)^{\alpha+1}) \\
&\quad - \frac{G_n}{h} \frac{h}{\alpha} \sum_{j=0}^{n-1} ((t_n - t_{j+1})^\alpha - (t_n - t_j)^\alpha) \\
&\quad - \frac{G_{n-1}}{h} \frac{1}{\alpha+1} \sum_{j=0}^{n-1} ((t_n - t_{j+1})^{\alpha+1} - (t_n - t_j)^{\alpha+1}), \\
&= \frac{1}{h} \frac{G_n}{\alpha+1} (-(t_n - t_0)^{\alpha+1}) - \frac{G_n}{\alpha} (-(t_n - t_0)^\alpha) \\
&\quad - \frac{1}{h} \frac{G_{n-1}}{\alpha+1} (-(t_n - t_0)^{\alpha+1}), \\
&= \frac{1}{h} \frac{G_{n-1}}{\alpha+1} (t_n - t_0)^{\alpha+1} - \frac{1}{h} \frac{G_n}{\alpha+1} (t_n - t_0)^{\alpha+1} + \frac{G_n}{\alpha} (t_n - t_0)^\alpha, \\
&= \frac{1}{h} \frac{G_{n-1}}{\alpha+1} h^{\alpha+1} n^{\alpha+1} - \frac{1}{h} \frac{G_n}{\alpha+1} h^{\alpha+1} n^{\alpha+1} + \frac{G_n}{\alpha} h^\alpha n^\alpha, \\
&= h^\alpha \left(\left(\frac{n^\alpha}{\alpha} - \frac{n^{\alpha+1}}{\alpha+1} \right) G_n + \frac{n^{\alpha+1}}{\alpha+1} G_{n-1} \right).
\end{aligned}$$

This is

$$\int_0^{t_n} G(u, \zeta)(t_n - \zeta)^{\alpha-1} d\zeta = h^\alpha \left(\left(\frac{n^\alpha}{\alpha} - \frac{n^{\alpha+1}}{\alpha+1} \right) G_n + \frac{n^{\alpha+1}}{\alpha+1} G_{n-1} \right). \quad (8.27)$$

Combining (8.26), (8.27), into (8.25) we obtain:

$$\begin{aligned} u_{n+1} - u_n &= \frac{1-\alpha}{B(\alpha)} (G(u, t_{n+1}) - G(u, t_n)) \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[h^\alpha \left(\frac{2(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) G_n \right. \\ &\quad \left. - h^\alpha \left(\frac{(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1} \right. \\ &\quad \left. - h^\alpha \left(\left(\frac{n^\alpha}{\alpha} - \frac{n^{\alpha+1}}{\alpha+1} \right) G_n + \frac{n^{\alpha+1}}{\alpha+1} G_{n-1} \right) \right] \end{aligned}$$

This is then written as:

$$\begin{aligned} u_{n+1} - u_n &= \frac{1-\alpha}{B(\alpha)} (G(u, t_{n+1}) - G(u, t_n)) \\ &\quad + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_n \right. \\ &\quad \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1} \right]. \quad (8.28) \end{aligned}$$

Remark 8.2 For $\alpha = 1$ equation (8.28) becomes

$$\begin{aligned} u_{n+1} - u_n &= \frac{0}{B(1)} (G_{n+1} - G_n) \\ &\quad + \frac{h}{B(1)\Gamma(1)} \left[\left(2(n+1) - n + \frac{n^2 - (n+1)^2}{2} \right) G_n \right. \\ &\quad \left. - \left(n+1 + \frac{n^2 - (n+1)^2}{2} \right) G_{n-1} \right], \\ &= h \left[\left(2n+2 - n - n - \frac{1}{2} \right) G_n - \left(n+1 - n - \frac{1}{2} \right) G_{n-1} \right], \\ &= h \left(\frac{3}{2} G_n - \frac{1}{2} G_{n-1} \right). \end{aligned}$$

This is

$$u_{n+1} = u_n + h \left(\frac{3}{2} G_n - \frac{1}{2} G_{n-1} \right),$$

the standard Adam-Bashforth numerical scheme. We then apply the inverse Laplace transform to the equation (8.28) to define the numerical scheme solution in the real space.

$$\begin{aligned} u(x, t_{n+1}) = \mathcal{L}^{-1} \left\{ u(x, t_n) + \frac{1-\alpha}{B(\alpha)} (G_{n+1} - G_n) \right. \\ \left. + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_n \right. \right. \\ \left. \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1} \right] \right\}. \end{aligned} \quad (8.29)$$

Equation (8.29) can then be discretized in the space variable x using a classic method including but not limited to forward or backward numerical scheme.

$$\begin{aligned} u(x_i, t_{n+1}) = \mathcal{L}^{-1} \left\{ u(x_i, t_n) + \frac{1-\alpha}{B(\alpha)} (G_i^{n+1} - G_i^n) \right. \\ \left. + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_i^n \right. \right. \\ \left. \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_i^{n-1} \right] \right\}. \end{aligned} \quad (8.30)$$

8.4.5. Error Analysis

The solution to the general equation (8.23) is formulated by equation (8.29).

Proposition 8.1

The error R_n^α resulting in using the new numerical scheme (8.30) is bounded, that is:

$$\begin{aligned} u(x, t_{n+1}) = \mathcal{L}^{-1} \left\{ u_n + \frac{1-\alpha}{B(\alpha)} (G_{n+1} - G_n) \right. \\ \left. + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_n \right. \right. \\ \left. \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1} \right] \right\} + R_n^\alpha, \end{aligned} \quad (8.31)$$

where $R_n^\alpha < \infty$.

Proof

Earlier, we established that

$$\begin{aligned} u_{n+1} - u_n = \frac{1-\alpha}{B(\alpha)} (G_{n+1} - G_n) \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\int_0^{t_{n+1}} G(u, \zeta) (t_{n+1} - \zeta)^{\alpha-1} d\zeta - \int_0^{t_n} G(u, \zeta) (t_n - \zeta)^{\alpha-1} d\zeta \right], \end{aligned}$$

where

$$G(u, \zeta) = \frac{t - t_{n-1}}{t_n - t_{n-1}} G_n + \frac{t - t_n}{t_{n-1} - t_n} G_{n-1} + \frac{G^{(2)}(u, \xi)}{2!} \prod_{i=0}^1 (t - t_i),$$

$$\begin{aligned} u_{n+1} - u_n = \frac{1-\alpha}{B(\alpha)} (G_{n+1} - G_n) \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\int_0^{t_{n+1}} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} G_n + \frac{t - t_n}{t_{n-1} - t_n} G_{n-1} \right) (t_{n+1} - t)^{\alpha-1} dt \right. \\ \left. - \int_0^{t_n} \left(\frac{t - t_{n-1}}{t_n - t_{n-1}} G_n + \frac{t - t_n}{t_{n-1} - t_n} G_{n-1} \right) (t_n - t)^{\alpha-1} dt \right] \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\frac{G^{(2)}(u, \xi)}{2!} \int_0^{t_{n+1}} \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} dt \right. \\ \left. - \frac{G^{(2)}(u, \xi)}{2!} \int_0^{t_n} \prod_{i=0}^1 (t - t_i) (t_n - t)^{\alpha-1} dt \right] + R_n^\alpha. \end{aligned}$$

This should be equal to the numerical scheme we derived in (8.28) accounting for the error term:

$$\begin{aligned} u_{n+1} - u_n &= \frac{1-\alpha}{B(\alpha)} (G_{n+1} - G_n) \\ &\quad + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_n \right. \\ &\quad \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1} \right] + R_n^\alpha. \end{aligned}$$

By comparison, the following equality is directly made, and we have

$$\begin{aligned} R_n^\alpha &= \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\frac{G^{(2)}(u, \xi)}{2!} \int_0^{t_{n+1}} \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} dt \right. \\ &\quad \left. - \frac{G^{(2)}(u, \xi)}{2!} \int_0^{t_n} \prod_{i=0}^1 (t - t_i) (t_n - t)^{\alpha-1} dt \right], \\ |R_n^\alpha| &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\int_0^{t_{n+1}} \left| \frac{G^{(2)}(u, \xi)}{2!} \prod_{i=0}^1 (t - t_i) (t_{n+1} - t)^{\alpha-1} \right| dt \right. \\ &\quad \left. + \int_0^{t_n} \left| \frac{G^{(2)}(u, \xi)}{2!} \prod_{i=0}^1 (t - t_i) (t_n - t)^{\alpha-1} \right| dt \right], \\ &\leq \frac{\alpha h^2}{8 B(\alpha)\Gamma(\alpha)} \max_{\xi \in (0, t_{n+1})} G^{(2)}(u, \xi) \left(\int_0^{t_{n+1}} |t_{n+1} - t|^{\alpha-1} dt \right. \\ &\quad \left. + \int_0^{t_n} |t_n - t|^{\alpha-1} dt \right), \\ &\leq \frac{\alpha h^2}{8 B(\alpha)\Gamma(\alpha)} \max_{\xi \in (0, t_{n+1})} G^{(2)}(u, \xi) \left(\frac{t_{n+1}^\alpha + t_n^\alpha}{\alpha} \right), \\ |R_n^\alpha| &\leq \frac{\alpha h^{\alpha+2}}{8 B(\alpha)\Gamma(\alpha)} \max_{\xi \in (0, t_{n+1})} G^{(2)}(u, \xi) ((n+1)^\alpha + n^\alpha) < \infty. \end{aligned}$$

8.4.6. Application of the Method on the ABC TFBSE

We presented in subsections 8.4.1 and 8.4.2, which were the core work the published research paper [99], applications of this method on a standard classic integer PDE and on a fractional PDE respectively. We will present now an application on our Time fractional Black Scholes Equation with the fractional differentiation operator of the ABC type (5.11).

$$\begin{cases} {}^{ABC}_0D_t^\alpha C(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + r S \frac{\partial C(S, t)}{\partial S} - r C(S, t) = 0, & (t, S) \in (0, +\infty) \times (0, T), \\ C(0, t) = C_0 = p(t), \quad C(\infty, t) = q(t), \quad C(S, T) = v(S), \end{cases} \quad (5.11)$$

with ${}^{ABC}_0D_t^\alpha (C(S, t)) = \frac{B(\alpha)}{1-\alpha} \int_0^t \frac{\partial C(S, \tau)}{\partial \tau} E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] d\tau.$

$B(\alpha)$ is a normalisation constant function such that $B(0) = B(1) = 1$.

Following the guidelines presented above, we start by applying the Laplace transform on (5.11) and obtain

$${}^{ABC}_0D_t^\alpha C(p, t) = \mathcal{L} \left(r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S} \right).$$

By silencing the variable p and denoting $C(p, t) = G(t)$, we can rewrite the later equation as:

$${}^{ABC}_0D_t^\alpha C(t) = G(C(t), t), \quad (8.32)$$

where

$$G(C(t), t) = \mathcal{L} \left(r C(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} - r S \frac{\partial C(S, t)}{\partial S} \right).$$

The method establishes that the solution to solution to (8.32) in the Laplace space is given by equation (8.28):

$$\begin{aligned}
C_{n+1} - C_n &= \frac{1-\alpha}{B(\alpha)} (G_{n+1} - G_n) \\
&+ \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_n \right. \\
&\quad \left. - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1} \right) G_{n-1} \right].
\end{aligned}$$

To which we then apply the inverse Laplace transform to obtain

$$\begin{aligned}
C(S, t_{n+1}) &= C(S, t_n) \\
&+ \frac{1-\alpha}{B(\alpha)} \left(rC(S, t_{n+1}) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_{n+1})}{\partial S^2} - r S \frac{\partial C(S, t_{n+1})}{\partial S} - rC(S, t_n) \right. \\
&\quad \left. + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_n)}{\partial S^2} + r S \frac{\partial C(S, t_n)}{\partial S} \right) \\
&+ \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\delta_n^\alpha \left(rC(S, t_n) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_n)}{\partial S^2} - r S \frac{\partial C(S, t_n)}{\partial S} \right) \right. \\
&\quad \left. - \delta_n^{\alpha,1} \left(rC(S, t_{n-1}) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t_{n-1})}{\partial S^2} - r S \frac{\partial C(S, t_{n-1})}{\partial S} \right) \right], \quad (8.33)
\end{aligned}$$

where

$$\delta_n^\alpha = \frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1},$$

and

$$\delta_n^{\alpha,1} = \frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}.$$

Next we discretize equation (8.33) in its space variable. It becomes

$$\begin{aligned}
C(S_i, t_{n+1}) = & C(S_i, t_n) \\
& + \frac{1-\alpha}{B(\alpha)} \left(rC(S_i, t_{n+1}) \right. \\
& - \frac{1}{2} \sigma^2 S_i^2 \frac{C(S_{i+1}, t_{n+1}) - 2C(S_i, t_{n+1}) + C(S_{i-1}, t_{n+1})}{(\Delta x)^2} \\
& - r S_i \frac{C(S_{i+1}, t_{n+1}) - C(S_i, t_{n+1})}{\Delta x} - rC(S_i, t_n) \\
& + \frac{1}{2} \sigma^2 S_i^2 \frac{C(S_{i+1}, t_n) - 2C(S_i, t_n) + C(S_{i-1}, t_n)}{(\Delta x)^2} \\
& \left. + r S_i \frac{C(S_{i+1}, t_n) - C(S_i, t_n)}{\Delta x} \right) \\
& + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left[\delta_n^\alpha \left(rC(S_i, t_n) \right. \right. \\
& - \frac{1}{2} \sigma^2 S_i^2 \frac{C(S_{i+1}, t_n) - 2C(S_i, t_n) + C(S_{i-1}, t_n)}{(\Delta x)^2} \\
& - r S_i \frac{C(S_{i+1}, t_n) - C(S_i, t_n)}{\Delta x} \left. \right) \\
& - \delta_n^{\alpha,1} \left(rC(S_i, t_{n-1}) \right. \\
& - \frac{1}{2} \sigma^2 S_i^2 \frac{C(S_{i+1}, t_{n-1}) - 2C(S_i, t_{n-1}) + C(S_{i-1}, t_{n-1})}{(\Delta x)^2} \\
& \left. \left. - r S_i \frac{C(S_{i+1}, t_{n-1}) - C(S_i, t_{n-1})}{\Delta x} \right) \right].
\end{aligned}$$

We simplify the notation further by letting $C(S_i, t_n) = C_i^n$, and $\Delta x = l$ the equation becomes then

$$\begin{aligned}
C_i^{n+1} = & C_i^n \\
& + \frac{1-\alpha}{B(\alpha)} \left\{ rC_i^{n+1} - \frac{1}{2} \sigma^2 S_i^2 \frac{C_{i+1}^{n+1} - 2C_i^{n+1} + C_{i-1}^{n+1}}{l^2} - r S_i \frac{C_{i+1}^{n+1} - C_i^{n+1}}{l} \right. \\
& - rC_i^n + \frac{1}{2} \sigma^2 S_i^2 \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{l^2} + r S_i \frac{C_{i+1}^n - C_i^n}{l} \left. \right\} \\
& + \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left\{ \delta_n^\alpha \left(rC_i^n - \frac{1}{2} \sigma^2 S_i^2 \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{l^2} - r S_i \frac{C_{i+1}^n - C_i^n}{l} \right) \right. \\
& \left. - \delta_n^{\alpha,1} \left(rC_i^{n-1} - \frac{1}{2} \sigma^2 S_i^2 \frac{C_{i+1}^{n-1} - 2C_i^{n-1} + C_{i-1}^{n-1}}{l^2} - r S_i \frac{C_{i+1}^{n-1} - C_i^{n-1}}{l} \right) \right\},
\end{aligned}$$

$$\begin{aligned}
C_i^{n+1} = C_i^n &+ \frac{1-\alpha}{B(\alpha)} \left\{ \left[r + \frac{\sigma^2 S_i^2}{l^2} + \frac{r S_i}{l} \right] C_i^{n+1} - \left[r + \frac{\sigma^2 S_i^2}{l^2} + \frac{r S_i}{l} \right] C_i^n - \left[\frac{\sigma^2 S_i^2}{2l^2} + \frac{r S_i}{l} \right] C_{i+1}^{n+1} \right. \\
&+ \left. \left[\frac{\sigma^2 S_i^2}{2l^2} + \frac{r S_i}{l} \right] C_{i+1}^n - \frac{\sigma^2 S_i^2}{2l^2} C_{i-1}^{n+1} + \frac{\sigma^2 S_i^2}{2l^2} C_{i-1}^n \right\} \\
&+ \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \left\{ \delta_n^\alpha \left(\left[r + \frac{\sigma^2 S_i^2}{l^2} + \frac{r S_i}{l} \right] C_i^n - \left[\frac{\sigma^2 S_i^2}{2l^2} + \frac{r S_i}{l} \right] C_{i+1}^n - \frac{\sigma^2 S_i^2}{2l^2} C_{i-1}^n \right) \right. \\
&- \left. \delta_n^{\alpha,1} \left(\left[r + \frac{\sigma^2 S_i^2}{l^2} + \frac{r S_i}{l} \right] C_i^{n-1} - \left[\frac{\sigma^2 S_i^2}{2l^2} + \frac{r S_i}{l} \right] C_{i+1}^{n-1} - \frac{\sigma^2 S_i^2}{2l^2} C_{i-1}^{n-1} \right) \right\}.
\end{aligned}$$

Now by letting

$$A_i = r + \frac{\sigma^2 S_i^2}{l^2} + \frac{r S_i}{l}; \quad B_i = \frac{\sigma^2 S_i^2}{2l^2} + \frac{r S_i}{l} \text{ and } D_i = \frac{\sigma^2 S_i^2}{2l^2},$$

the last equation can also be expressed as:

$$\begin{aligned}
C_i^{n+1} = C_i^n &+ \frac{1-\alpha}{B(\alpha)} \{ A_i C_i^{n+1} - A_i C_i^n - B_i C_{i+1}^{n+1} + B_i C_{i+1}^n - D_i C_{i-1}^{n+1} + D_i C_{i-1}^n \} \\
&+ \frac{\alpha h^\alpha}{B(\alpha)\Gamma(\alpha)} \{ \delta_n^\alpha (A_i C_i^n - B_i C_{i+1}^n - D_i C_{i-1}^n) \\
&- \delta_n^{\alpha,1} (A_i C_i^{n-1} - B_i C_{i+1}^{n-1} - D_i C_{i-1}^{n-1}) \},
\end{aligned}$$

grouping likewise terms we can then rewrite the numerical scheme as:

$$\begin{aligned}
&\left(1 - \frac{1-\alpha}{B(\alpha)} A_i \right) C_i^{n+1} \\
&= \left\{ 1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} A_i \right\} C_i^n - \frac{1-\alpha}{B(\alpha)} B_i C_{i+1}^{n+1} \\
&+ \left\{ \frac{1-\alpha}{B(\alpha)} B_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} B_i \right\} C_{i+1}^n - \frac{1-\alpha}{B(\alpha)} D_i C_{i-1}^{n+1} \\
&+ \left\{ \frac{1-\alpha}{B(\alpha)} D_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} D_i \right\} C_{i-1}^n - \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha)\Gamma(\alpha)} A_i C_i^{n-1} \\
&+ \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha)\Gamma(\alpha)} B_i C_{i+1}^{n-1} \\
&+ \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha)\Gamma(\alpha)} D_i C_{i-1}^{n-1}. \tag{8.34}
\end{aligned}$$

8.4.7. Stability Analysis of the Numerical Scheme

To analyse the stability of the numerical solution (8.34) assume that we have a Fourier expansion of the form

$$C(S, t) = \sum_f \hat{C}(t) e^{jfs} \text{ while denoting } C(S_i, t_n) = C_i^n,$$

$$C_i^n = \hat{C}(t_n) e^{jfi\Delta x} \text{ written alternatively as } C_i^n = \hat{C}_n e^{jfil},$$

equation (8.34) is then

$$\begin{aligned} & \left(1 - \frac{1-\alpha}{B(\alpha)} A_i\right) \hat{C}_{n+1} e^{jfil} \\ &= \left\{1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} A_i\right\} \hat{C}_n e^{jfil} - \frac{1-\alpha}{B(\alpha)} B_i \hat{C}_{n+1} e^{jf(i+1)l} \\ &+ \left\{\frac{1-\alpha}{B(\alpha)} B_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} B_i\right\} \hat{C}_n e^{jf(i+1)l} - \frac{1-\alpha}{B(\alpha)} D_i \hat{C}_{n+1} e^{jf(i-1)l} \\ &+ \left\{\frac{1-\alpha}{B(\alpha)} D_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} D_i\right\} \hat{C}_n e^{jf(i-1)l} - \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} A_i \hat{C}_{n-1} e^{jfil} \\ &+ \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} B_i \hat{C}_{n-1} e^{jf(i+1)l} + \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} D_i \hat{C}_{n-1} e^{jf(i-1)l}. \end{aligned}$$

Simplifying similar terms, we have

$$\begin{aligned} & \left(1 - \frac{1-\alpha}{B(\alpha)} A_i\right) \hat{C}_{n+1} \\ &= \left\{1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} A_i\right\} \hat{C}_n - \frac{1-\alpha}{B(\alpha)} B_i \hat{C}_{n+1} e^{jfl} \\ &+ \left\{\frac{1-\alpha}{B(\alpha)} B_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} B_i\right\} \hat{C}_n e^{jfl} - \frac{1-\alpha}{B(\alpha)} D_i \hat{C}_{n+1} e^{-jfl} \\ &+ \left\{\frac{1-\alpha}{B(\alpha)} D_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} D_i\right\} \hat{C}_n e^{-jfl} - \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} A_i \hat{C}_{n-1} \\ &+ \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} B_i \hat{C}_{n-1} e^{jfl} + \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} D_i \hat{C}_{n-1} e^{-jfl}. \end{aligned}$$

Rearranging by grouping similar terms together we have,

$$\begin{aligned}
& \left(1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{1-\alpha}{B(\alpha)} B_i e^{jfl} + \frac{1-\alpha}{B(\alpha)} D_i e^{-jfl}\right) \hat{C}_{n+1} \\
&= \left\{1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} A_i + \left(\frac{1-\alpha}{B(\alpha)} B_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} B_i\right) e^{jfl} \right. \\
&\quad \left. + \left(\frac{1-\alpha}{B(\alpha)} D_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} D_i\right) e^{-jfl}\right\} \hat{C}_n \\
&\quad - \left(\frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} A_i - \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} B_i e^{jfl} - \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha) \Gamma(\alpha)} D_i e^{-jfl}\right) \hat{C}_{n-1}.
\end{aligned}$$

For $n = 0$

$$\begin{aligned}
& \left(1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{1-\alpha}{B(\alpha)} B_i e^{jfl} + \frac{1-\alpha}{B(\alpha)} D_i e^{-jfl}\right) \hat{C}_1 \\
&= \left\{1 - \frac{1-\alpha}{B(\alpha)} A_i + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} A_i + \left(\frac{1-\alpha}{B(\alpha)} B_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} B_i\right) e^{jfl} \right. \\
&\quad \left. + \left(\frac{1-\alpha}{B(\alpha)} D_i - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} D_i\right) e^{-jfl}\right\} \hat{C}_0.
\end{aligned}$$

This is also

$$\begin{aligned}
& \left(1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl})\right) \hat{C}_1 \\
&= \left\{1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right. \\
&\quad \left. + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha) \Gamma(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl})\right\} \hat{C}_0,
\end{aligned}$$

$$\begin{aligned}
& \left(1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right) \hat{C}_1 \\
& = \left\{ 1 - \left(\frac{1-\alpha}{B(\alpha)} - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} \right) (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\} \hat{C}_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| 1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_1\| \\
& = \left\| 1 - \left(\frac{1-\alpha}{B(\alpha)} - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} \right) (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_0\| \\
& < \left\| 1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_0\|,
\end{aligned}$$

$$if \ 0 < \frac{1-\alpha}{B(\alpha)} - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} < 1.$$

Let us examine now the inequalities.

$$if \ 0 < \frac{1-\alpha}{B(\alpha)} - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} < 1 \Rightarrow$$

$$\alpha h^\alpha \delta_o^\alpha < (1-\alpha)\Gamma(\alpha) \text{ and } (1-\alpha)\Gamma(\alpha) < \alpha h^\alpha \delta_o^\alpha + B(\alpha)\Gamma(\alpha).$$

Since

$$\delta_o^\alpha = \frac{2}{\alpha} - \frac{1}{\alpha+1} = \frac{\alpha+2}{\alpha(\alpha+1)}.$$

The first condition inequality $\alpha h^\alpha \delta_o^\alpha < (1-\alpha)\Gamma(\alpha)$ is

$$h^\alpha < \frac{1-\alpha^2}{\alpha+2} \Gamma(\alpha) < \Gamma(\alpha) \text{ since } 0 < \alpha < 1.$$

The upper inequality,

$$(1 - \alpha)\Gamma(\alpha) < \alpha h^\alpha \delta_o^\alpha + B(\alpha)\Gamma(\alpha) \Rightarrow (1 - \alpha)\Gamma(\alpha) < \left(\frac{\alpha + 2}{\alpha + 1} + B(\alpha)\right)\Gamma(\alpha),$$

(since $h^\alpha < \Gamma(\alpha)$)

$$\Rightarrow 1 - \alpha < \frac{\alpha + 2}{\alpha + 1} + B(\alpha),$$

$$\Rightarrow \alpha^2 + (1 + B(\alpha))(\alpha + 1) > 0.$$

This is always surely achieved whenever $0 < \alpha < 1$. The two inequalities bold down to the first one.

$$\begin{cases} h^\alpha < \Gamma(\alpha) \\ \alpha^2 + (1 + B(\alpha))(\alpha + 1) > 0 \end{cases} \Leftrightarrow h^\alpha < \Gamma(\alpha). \quad (8.35)$$

Thus for $h^\alpha < \Gamma(\alpha)$.

$$\begin{aligned} & \left\| 1 - \frac{1 - \alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_1\| \\ & < \left\| 1 - \frac{1 - \alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_0\| \Rightarrow \|\hat{C}_1\| < \|\hat{C}_0\|. \end{aligned}$$

Let us consider now the induction hypothesis $\|\hat{C}_n\| < \|\hat{C}_0\|$, and prove that $\|\hat{C}_{n+1}\| < \|\hat{C}_0\|$.

$$\begin{aligned} & \left\| 1 - \frac{1 - \alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_{n+1}\| \\ & = \left\| \left\{ 1 - \frac{1 - \alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right. \right. \\ & \quad + \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \Big\} \hat{C}_n \\ & \quad \left. - \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha)\Gamma(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \hat{C}_{n-1} \right\|, \end{aligned}$$

$$\begin{aligned}
& \left\| 1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_{n+1}\| \\
& < \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \left(\frac{1-\alpha}{B(\alpha)} - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} \right) \right\| \|\hat{C}_n\| \\
& + \left\| \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha)\Gamma(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_{n-1}\|.
\end{aligned}$$

By induction hypothesis we have

$$\begin{aligned}
& \left\| 1 - \frac{1-\alpha}{B(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_{n+1}\| \\
& < \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \left\{ \frac{1-\alpha}{B(\alpha)} - \frac{\alpha h^\alpha \delta_n^\alpha}{B(\alpha)\Gamma(\alpha)} \right\} \right\| \|\hat{C}_0\| \\
& + \left\| \frac{\alpha h^\alpha \delta_n^{\alpha,1}}{B(\alpha)\Gamma(\alpha)} (A_i - B_i e^{jfl} - D_i e^{-jfl}) \right\| \|\hat{C}_0\|.
\end{aligned}$$

With the condition (8.35) we can clearly have

$$\begin{aligned}
& \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_{n+1}\| \\
& < \frac{1}{2} \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_0\| \\
& + \frac{1}{2} \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_0\|.
\end{aligned}$$

$$\begin{aligned}
& \text{If } \max \left\{ \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\|, \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \right\} \\
& = \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\|,
\end{aligned}$$

then it is straightforward that

$$\begin{aligned}
& \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_{n+1}\| \\
& < \frac{1}{2} \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_0\| \\
& + \frac{1}{2} \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_0\|.
\end{aligned}$$

$$\|\hat{C}_{n+1}\| < \frac{1}{2} \|\hat{C}_0\| + \frac{1}{2} \|\hat{C}_0\| = \|\hat{C}_0\|, \text{ this is } \|\hat{C}_{n+1}\| < \|\hat{C}_0\|.$$

$$\begin{aligned}
\text{If } \max \left\{ \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\|, \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \right\} \\
= \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\|,
\end{aligned}$$

then we will have that

$$\begin{aligned}
& \left\| 1 - (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_{n+1}\| \\
& < \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_{n+1}\|,
\end{aligned}$$

and

$$\begin{aligned}
& \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_{n+1}\| \\
& < \frac{1}{2} \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_0\| \\
& + \frac{1}{2} \left\| (A_i - B_i e^{jfl} - D_i e^{-jfl}) \frac{1-\alpha}{B(\alpha)} \right\| \|\hat{C}_0\|
\end{aligned}$$

By the induction hypothesis and thus

$\|\hat{C}_{n+1}\| < \|\hat{C}_0\|$. As a result the numerical scheme solution equation (8.34) is conditionally stable whenever (8.35) is satisfied that is whenever $h^\alpha < \Gamma(\alpha)$.

Conclusion and outlook

Chapter 1 offered an overview of the global analysis we undertook. Its rationale founded in the growing use of differentiation operators of fractional calculus known for their suitable properties, to attempt to address deficiencies of the standard Black-Scholes models. We gave Chapter 2 a potted description of those tools of fractional calculus, their properties and their application in various disciplines of applied science. We presented the definitions of the most predominantly used fractional differentiation operators.

In Chapter 3 and Chapter 4 respectively, we made an extensive review of financial securities pricing, focusing on two main axes: 1. non-simulation based pricing approaches around the central Black-Scholes-Merton model, 2. applications on European type of options and extension to fractional versions of the classic Black Scholes models. Then we explored technical elements of definition and the characteristics of financial derivatives, their types and payoff functions, impacting their valuation and trading. We proceeded by looking at the derivation of the standard Black Scholes model, and how from it, how a Time Fractional Black Scholes Equations (TFBSE) can be obtained.

Being fully cognisant of the state of research on the question, we formulated our new Time Fractional Black Scholes Equations (TFBSEs) in Chapter 5, incorporating the most used and interesting fractional derivative based on known properties. For the specific exotic type of option we chose to work on, the literature on its corresponding fractional version is relatively meagre. A lot still needs to be done. Three of our five equations namely with Caputo-Fabrizio fractional derivative, Atangana-Baleanu both in the Riemann and Caputo sense, were then being formulated for the first time in the literature. Using different methods, we proved the existence and uniqueness of solutions to our equations.

In Chapter 6 and 7, respectively using Laplace and Sumudu transforms we developed five new semi analytical solutions to our TFBSEs we proved their stability. We derived 6 numerical scheme solutions of the TFBSEs. We tabulated the convergence rate and, margins of Riemann Liouville and Caputo numerical scheme solutions to compare them with available results in the literature. Both numerical schemes obtained showed an improvement of the existing solutions in the literature, with faster converging schemes. For the new TFBSE with the time fractional derivative defined in the sense of Caputo-Fabrizio.

We discussed the stability of our solution and illustrated by pricing a double barrier knock out Call option for various values of the order of differentiation α . We also compared to the standard case prices obtained when $\alpha = 1$. Unlike available results in the literature, who consistently underpriced-in comparison to the standard Black Scholes-when the strike price is closer to the down-and-out barrier, and overpriced when the strike price is closer to the up-and-out barrier, the new CF-TFBSE shows scale fluctuating behaviors. We concluded that indeed the new TFBSE has superior memory property and could satisfactorily capture both high jumps in short interval of times, as well as portray satisfactorily the long range memory property. Further investigations could focus on the sensitivity analysis of the solution to the time fractional Black Scholes equation with Caputo-Fabrizio derivative, and look in details how price variations respond to *alpha* values.

So far in the literature on TFBSEs it is always a daunting task to analyze the stability of the numerical scheme solutions, mainly because of an additional summation consubstantial to the nature of fractional operators. The numerical schemes we derived in Chapter 7 were no exception to that. This motivated the body of work of Chapter 8. We extended the well-known Adams-Bashforth numerical scheme to partial differential equations with integer and non-integer order derivatives. We constructed a new reliable and efficient method. The method is a combination of Laplace transform, Adams-Bashforth and (forward or backward numerical scheme). We conceived the method for general PDEs with local and nonlocal differentiation. We presented in detail the error analysis and the convergence of the method. In the case of fractional partial differential equations, the method provides a numerical algorithm that is easier to implement. Unlike the conventional methods, forward, backward, Crank-Nicholson, the cumbersome summation that always appears in the additional term of their numerical algorithm, for the case of fractional partial differential equations, does not exist with our method. This led to an easier proof of stability and convergence. We illustrated the method by solving two partial differential equations including wave equation for the local case and a diffusion equation for fractional case. We studied the stability of each example. The proof shows without doubt that our method is very stable and also converges very quickly to the exact solution. Finally we presented the error analysis of the method, and discuss the stability of the method solution of our TFBSE with ABC fractional operator.

Overall we strongly believe fractional differentiation operators' hold the keys to shortcomings of TFBSEs. Changes in the fractional diffusion equation when the operators are of Caputo-Fabrizio and Atangana-Baleanu revealed different diffusive regimes from Gaussian to non-Gaussian distributions. Crossover between diffusive regimes for small times or sub-diffusive for long times, are a feature observed in several empirical systems.

The method we introduced will turn out to be a very useful numerical scheme that will help solving nonlinear and linear partial differential equations with local and nonlocal operators. It simplifies the analysis, has no projection error, and the numerical implementation of solutions will be concerned only with truncation errors that can be assessed depending on the regularity of the solutions. The potential of the method for anomalous diffusion problems is immense.

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