OPTIMAL PORTFOLIO SELECTION WITH UNCERTAIN IMPLICIT TRANSACTION COSTS IN A DYNAMIC STOCHASTIC FRAMEWORK

by

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Declaration

I, Sabastine Mushori, declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Signature..................................

Date: July 2019

Disclaimer

This document describes work undertaken as a PhD programme of study at the University of the Free State. All views and opinions expressed therein remain the sole responsibility of the author, and do not necessarily represent those of the institution.

Dedication

This study is dedicated to my wife Emily, my sons Roosevelt Tafadzwa, Providence Takudzwa and Sebastian (junior), and my daughter Sharon.

Abstract

This thesis proposes scenario-based approaches and decision models for some problem contexts in investment decision-making which include (i) optimal portfolio investment in periods of economic booms and recessions, (ii) the incorporation of uncertainty in implicit transaction costs incurred in initial trading and in subsequent rebalancing of portfolios, and (iii) the development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios. The methodological advances of the thesis offer several novel insights into the above decision problems. Firstly, the mean absolute deviation model is developed and extended into a stochastic multi-stage model that incorporates uncertainty of implicit transaction costs, asset returns and risk in optimal portfolio selection. This methodology allows investors and investment managers to choose optimal portfolios realising the impact of associated uncertain implicit transaction costs.

Secondly, a stochastic multi-stage trading cost model is developed that also takes into account uncertainty of implicit transaction costs, assets' returns and portfolio risk. This strategy generates optimal portfolios by minimising total implicit transaction costs incurred.

Thirdly, a multi-stage stochastic optimal portfolio policy that minimises maximum downside risk in the presence of uncertain implicit transaction costs is proposed.

This strategy is appropriate for a risk-averse and conservative investor who is highly concerned about the performance of his portfolio in an economic recession environment.

Fourthly, a dynamic stochastic methodology in optimal portfolio selection that maximises upside deviations (investment opportunities) and minimises maximum downside risk while taking into account uncertain implicit transaction costs incurred in initial trading and recourse times is developed.

Lastly, the mean-variance model is extended to become multi-period and to incorporate uncertainty in implicit transaction costs, asset returns and portfolio risk. All the proposed models capture uncertainty in implicit transaction costs, portfolio return and risk by way of scenarios.

Key words: stochastic modeling, uncertain implicit transaction costs, downside risk, uncertainty, investment opportunity.

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Abbreviations

MAD	Mean absolute deviation
MV	Mean-variance
MM	Minimax
MND	Maximum negative deviation
LSAD	Lower semi-absolute deviation
SMAD	Stochastic mean absolute deviation
SMADTC	Stochastic mean absolute deviation with transaction costs
SMUDTC	Stochastic maximum upside - downside deviation with transaction costs
SMNDTC	Stochastic maximum negative deviation with transaction costs
SMVTC	Stochastic mean-variance with transaction costs
VaR	Value-at-Risk
CVaR	Conditioanal Value-at-Risk
GAMS	General algebraic modeling system

Research Output

The following is a list of publications from this thesis.

Peer-rewiewed Journal Publications

1. Mushori, S. and Chikobvu, D. (2016) *Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg Stock Market*, International Journal of Operational Research, Vol. 26(2), 127-152.

2. Mushori, S. and Chikobvu, D. (2016) *A Stochastic multi-stage trading cost model in optimal portfolio selection*, Journal of Economics and Econometrics, Vol. 59(3), 32-66.

3. Mushori, S. and Chikobvu, D. (2017) *Optimal portfolio selection with stochastic maximum downside risk and uncertain implicit transaction costs*, Journal of Economic and Financial Sciences, Vol. 10(3), 411-423.

4. Mushori, S. and Chikobvu, D. (2018) *Optimal portfolio selection with uncertain implicit transaction costs in a dynamic stochastic mean-variance framework*, Studies in Nonlinear Dynamics and Econometrics (Under Review).

Chapter 1

Introduction

1.1 Background of the study

Financial markets are inherently volatile and are characterised by shifting values. There are financial risks and opportunities. The prices of individual securities are frequently changing for numerous reasons that include shifts in perceived value, localised supply and demand imbalances, and price changes in other investment sectors or the market as a whole. Reduced liquidity results in price volatility and market risk to any contemplated transaction. As a result of this volatility, transaction cost analysis (TCA) has become increasingly important in helping firms measure how effectively both perceived and actual portfolio orders are executed. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making processes [52].

As a result, banks, fund-management firms, financial consulting institutions and large institutional investors are faced with the challenges of managing their funds, assets and stocks towards selecting, creating, balancing and evaluating optimal portfolios on a continual basis. Financial crisis, economic imbalances, algorithmic trading and highly volatile movements of asset prices in recent times have raised alarms on the management of financial risks [12]. Extreme event risk is present in all areas of risk management. Whether we are concerned with market, credit, operational or insurance risk, one of the biggest challenges to the risk manager is to implement risk management models which allow for rare but high impact events, that also permit the measurement of their consequences.

In financial markets, the stability and sustainability of future pay-offs of an investment are largely determined by extreme changes in financial conditions rather than typical movements. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred in setting up a new portfolio or rebalancing an existing one must be included in any realistic portfolio selection analysis. Investment portfolios should be rebalanced to take account of changing market conditions and changes in funding.

1.2 Research problem, aim and objectives

Constructing a portfolio of investments is one of the most significant financial decisions facing individual investors, financial managers and institutions. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. This brings with it some trading costs, which can be either direct or indirect. Direct or explicit trading costs are observable and they include brokerage commissions, market fees, clearing and settlement costs, taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of trade. On the other hand, indirect or implicit costs are invisible, and these can broadly be put into three categories, namely market impact, opportunity costs and bid-ask spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments [35].

To provide investors with competitive portfolio returns, investment managers must manage transaction costs pro-actively. Perhaps the reason why managers find themselves in a difficult situation is that these implicit costs are ‘hidden’ in the stock price. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and, as variable costs, provide opportunities to improve the quality of execution. Some investors do not like too high costs as these are known to erode the profits of investment [35].

In this study, stochastic multi-stage mean-risk models with uncertain implicit trading costs in optimal portfolio selection are proposed. These models capture assets’ returns, implicit transaction costs and risk due to uncertainty. The models allow the investor to choose his or her desired implicit transaction cost value, and portfolio mean rate of return or risk level, where the risk is defined by the mean absolute deviation, maximum negative deviation and variance of assets’ returns from expected portfolio return.

1.2.1 Aim of the study

The aim of this study is to construct stochastic multi-stage mean-risk portfolio optimisation models with random transaction costs that capture assets’ returns and risk due to uncertainty. We consider uncertain implicit transaction costs since they are known to erode the benefit of investment if ignored. For the total performance of a portfolio, the quality of the implementation is as important as the decision itself. Implementation costs usually reduce portfolio returns with limited potential to generate upside potential [35].

1.2.2 Objectives of the study

This study explores the feasibility of improving existing and developing new optimisation techniques in solving optimisation problems in financial decision making. Models in the literature to date, to the best of the author's knowledge, do not incorporate implicit transaction costs by taking into account the fact that they are uncertain and that they are not always proportional to the assets' prices. Those models that consider implicit transaction costs do so by regarding implicit transaction costs as proportional to the assets' prices. Thus, the objectives of this study are:

(a) To develop a stochastic multi-stage mean-absolute deviation model that incorporates uncertainty of implicit trading costs, uncertainty of asset returns and uncertainty of risk in optimal portfolio selection;

(b) To develop a stochastic multi-stage trading cost model that generates optimal portfolios while minimising uncertain implicit transaction costs incurred by an investor during initial trading and in subsequent re-balancing of portfolios of investment;

(c) To construct a multi-stage stochastic maximum negative deviation model that optimises portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in subsequent re-balancing of portfolios of investment;

(d) To develop a multi-stage stochastic model that maximises portfolio gains and minimises maximum downside risk in the presence of uncertain implicit transaction costs incurred during initial trading and in subsequent rebalancing of portfolios;

(e) To improve and extend the mean-variance model by incorporating the uncertainty nature of implicit transaction costs, asset returns and risks in stochastic multi-stage financial optimisation of investment portfolios with recourse;

(f) To develop a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios.

1.3 Significance of the study

The study provides models that investment managers and individual investors can rely on as these models take into account the impact of uncertain implicit transaction costs in optimal portfolio selection. This has been a grey area for financial institutions, investment managers and individual investors. Portfolio selection methods that incorporate uncertain implicit transaction costs in the literature are mostly devoted to proportional transaction costs ([6], [9], [48], [49], [59]). Some models are best suited for conservative investors while others become very useful in periods of economic recession. In such economic uncertainties, the multi-stage stochastic maximum downside risk model that incorporates uncertainty of asset returns, risk and implicit transaction costs becomes appropriate. It is well documented in the literature that investors generally shun positions in which they would be subjected to catastrophic losses no matter how small the probability these losses carry [62]. Other models developed are appropriate for the more conservative investors.

The results and recommendations of this research will be of interest to individual investors, investment managers, fund managers, researchers, and other interested stakeholders in the financial industry.

1.4 Scientific contributions of the study

The main contributions of this study include:

(i) the development of a stochastic multi-stage mean-absolute deviation model that incorporates uncertainty of implicit trading costs, asset returns and risk in optimal portfolio selection;

(ii) the development of a stochastic multi-stage trading cost model that generates optimal portfolios while minimising uncertain implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of portfolios;

(iii) the construction of a multi-stage stochastic maximum negative deviation model that optimises portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in subsequent rebalancing of portfolios, applicable particularly in periods of economic recession;

(iv) the development of a multi-stage stochastic model that maximises portfolio gains and minimises maximum downside risk in the presence of uncertain implicit transaction costs incurred during initial trading and in subsequent rebalancing of portfolios;

(v) the development of a multi-stage stochastic mean-variance model that optimises portfolios in the presence of random implicit transaction costs incurred during initial trading and in subsequent re-balancing of portfolios;

(vi) the development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios.

Empirical comparative analysis with mean-absolute deviation, mean-variance and minimax models reveals that the stochastic multi-stage models generate superior optimal portfolios.

1.5 Thesis structure

This thesis comprises six chapters. In Chapter 2, literature on the development of three optimization models to be used for validation of proposed models is provided, showing how each strategy has evolved over the years. In Chapter 3[1], the stochas-

[1] 'Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg Stock Market, International Journal of Operational Research, Vol. 26, No. 2, pp127-152

tic programming theoretical framework applied in the development of the proposed stochastic financial optimisation models is introduced and formalised. The method of scenario generation is clearly explained. The general constraints applicable to all the proposed models are discussed explicitly. Transaction costs and balance constraints are developed. The procedure for measuring uncertain implicit transaction costs is explained.

In Chapter 4[2], the five stochastic multi-stage financial optimization models are logically developed, with specific constraints provided (where possible). In Chapter 5[3], each of the developed financial models is applied to historical data of securities on the Johannesburg Stock Market from January 2008 to September 2012. These historical data have been obtained courtesy of INet Bridge. The performance of the proposed models is evaluated by comparing portfolios developed from them and the mean-variance (MV), Mean absolute deviation (MAD) and minimax (MM) models.

The study is concluded by giving a summary of the research findings in Chapter 6. Direction for future research is also suggested.

[2]'Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg Stock Market, International Journal of Operational Research, Vol. 26, No. 2, pp127-152; 'A Stochastic multi-stage trading cost model in optimal portfolio selection, Journal of Economics and Econometrics, Vol. 59, No. 3, pp32-66; 'Optimal portfolio selection with stochastic maximum downside risk and uncertain implicit transaction costs, Journal of Economic and Financial Sciences, Vol. 10, No. 3, pp411-423; 'Investment opportunities, uncertain implicit transaction costs and maximum downside risk in dynamic stochastic financial optimization, International Journal of Economic and Financial Issues, Vol. 8, No. 4, pp256-264; 'Optimal portfolio selection with uncertain implicit transaction costs in a dynamic stochastic mean-variance framework', Studies in Nonlinear Dynamics and Econometrics (Under Review)

[3]'Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg Stock Market, International Journal of Operational Research, Vol. 26, No. 2, pp127-152; 'A Stochastic multi-stage trading cost model in optimal portfolio selection, Journal of Economics and Econometrics, Vol. 59, No. 3, pp32-66; 'Optimal portfolio selection with stochastic maximum downside risk and uncertain implicit transaction costs, Journal of Economic and Financial Sciences, Vol. 10, No. 3, pp411-423; 'Investment opportunities, uncertain implicit transaction costs and maximum downside risk in dynamic stochastic financial optimization, International Journal of Economic and Financial Issues, Vol. 8, No. 4, pp256-264; 'Optimal portfolio selection with uncertain implicit transaction costs in a dynamic stochastic mean-variance framework', Studies in Nonlinear Dynamics and Econometrics (Under Review)

Chapter 2

Literature Review

2.1 Introduction

Financial portfolio optimisation is a widely studied problem in mathematics, statistics, finance and computational studies. It involves determining an optimal combination of weights that are associated with financial assets held in a portfolio [12]. In practice, portfolio optimisation faces challenges by virtue of varying mathematical formulations, parameter estimation, business constraints and complex financial instruments. A formal approach towards making investment decisions for obtaining an optimal portfolio with a specific objective requires a mathematical formulation for the problem. Validation of the models from a mathematical perspective is challenging and requires rigorous calculations. The theoretical advances and computational techniques that appear in literature on financial portfolio optimisation reflect the on-going and progressive work in this field.

2.2 Constructing Portfolio Optimisation Models

A portfolio is the total collection of all investments held by an individual or institution, which includes stocks, bonds, real estates, options, futures, and alternative investments such as gold or limited partnerships [81]. An efficient portfolio is one that offers the highest return for a given level of risk. The set of efficient portfolios over different levels of risk constitute the efficient frontier. A portfolio consisting of a riskless asset and a risky asset has an efficient frontier which is a straight line. This happens since the riskless asset has no variance and the risk of the portfolio increases proportionally to the weighting of the risky asset. This is the capital allocation line [81].

Capital allocation is the allocation of funds between risky assets and riskless ones. The portfolio risk of risky assets is lowered by diversifying where assets with different and offsetting coefficients of correlation comprise the portfolio. However, there exist other risks that cannot be mitigated by diversification, for example, a rise in interest rates would affect all businesses as they all save or spend money. One can conceptually categorize all risks into diversifiable and non-diversifiable risk. Non-diversifiable risk is also called systematic or market risk [65].

Portfolio risk is the chance that the combination of assets or units, within the investments that one owns, fails to meet portfolio objectives [40]. As provided by Kapoor [50], the first important objective of a portfolio is to ensure that the investment is absolutely safe. Once investment safety is guaranteed, the portfolio should yield a steady current income. The current returns should at least match the opportunity cost of the funds of the investor. The third objective ensures that the portfolio appreciates in value in order to protect the investor from any erosion in purchasing power due to inflation. A good portfolio must consist of assets which can be marketed without difficulty. It is desirable to invest in companies listed on major stock exchanges which are actively traded. Since taxation is an important variable in total planning, a good portfolio should enable its owner to enjoy a favourable tax regime. As Kapoor [50] puts it, the portfolio should be developed considering not only income tax, but capital gains tax and gift tax as well.

Securities selection in an investment is made with a view to provide the investor with the maximum possible yield for a given level of risk or ensure minimum possible risk for a given level of return. However, portfolio selection is dependent on the attitude of the investor towards risk and other parameters. Investors can generally be classified into three categories namely risk-averse, risk-seeking and risk-neutral investors [50]. A risk-averse investor is an investor who prefers lower returns with known risks rather than higher returns with unknown risks. In other words, among various investments giving the same return with different levels of risk, the risk-averse investor always prefers the alternative with least risk. Risk-neutral is a term that is used to describe investors who are insensitive to risk. The risk-neutral investor effectively ignores the risk completely when making an investment decision. For example, when presented with two possible investments that carry different levels of risk, the risk-neutral investor considers just the expected return from each investment - their risks are irrelevant to him / her. Risk-seeking is defined as the acceptance of greater risk and uncertainty in investments or trading in exchange for anticipated higher returns [39]. Risk-seekers are more interested in capital gains from speculative assets than capital preservation from lower risk assets. Generally, higher risk implies higher return potential, although the quality of the asset in question must be considered beforehand to ascertain whether there is sufficient return potential to justify the risk involved. Some types of assets that risk-seeking investors would be attracted to are: small-cap equities, arbitrage investments, emerging market equities and debt, currencies of developing countries, junk bonds and commodity futures, to name just a few [39].

It can be said that central to the decision by the investor is portfolio risk. Portfolio risk can be defined as the chance that a combination of assets or units within the investment that one owns, fails to meet the investor's financial objectives [40]. There are two main forms of risk associated with trading, namely market risk and liquidity risk. Market risk can be defined as the capacity of an investor's trades to result in losses due to unfavourable price movements that affect the market as a whole. There are several factors that can cause market risk, but movement in any of the following can exert major pressure: stock prices, interest rates, foreign exchange rates and commodity prices [40]. The second type of risk, liquidity risk, can be explained as the possibility that an investor may be forced to trade an asset at a worse price than he / she anticipated [40]. For example, when trying to sell an illiquid stock, an investor may struggle to find a buyer resulting in the investor selling the stock for less than the current value. In some markets, liquidity risk can even mean that the trade affects negatively the price of the asset being sold or bought. This is generally more of an issue in emerging or low-volume markets, where there may not be enough investors in the market to trade with. A market's liquidity is the ease with which an asset or stock can be bought or sold without affecting its price. Opening and closing a position on a highly liquid market is easier and generally less risky than in an illiquid one.

Although financial risk has increased significantly in recent years, risk and risk management are not contemporary issues. With increasing global markets, risk management becomes critical as the risk may originate even thousands of miles away from the domestic market [36]. Financial markets react very quickly to any unfolding information. The economic climate and financial markets can be affected very quickly by changes in exchange rates, interest rates and commodity prices. As a result, it becomes imperative for investment firms, fund managers and banks to ensure financial risks are identified and managed properly.

Financial risk arises through countless transactions of a financial nature which include sales and purchases, investments and loans, and various other business activities. When financial prices change drastically, it can increase costs, reduce revenues, or otherwise adversely impact the profitability of an organisation. Financial risk management is a process that deals with the uncertainties resulting from financial markets. This process necessitates making organisational decisions about risks that are acceptable versus those that are not. Measuring or quantifying risk is important in understanding the potential features of risk that a financial institution faces. It helps to analyse the efficiency of risk control measures which is significant in the process of decision-making [36].

2.2.1 Types of portfolios

Generally, portfolios can be classified into three broad types which are considered in investment and these are: the aggressive portfolio, the balanced investment strategy and the defensive portfolio [50]. An aggressive investment strategy is a portfolio that attempts to maximise returns by taking a relatively higher degree of risk. It emphasizes capital appreciation as a primary investment objective, rather than income or safety of principal. Such a strategy would therefore have an asset allocation with a substantial weighting in stocks and a much smaller allocation to fixed income and cash [39]. Aggressive investment strategies are especially suitable for young adults, because a lengthy investment horizon enables them to ride out market fluctuations. Regardless of the investor's age, however, a high tolerance for risk is an absolute pre-requisite for an aggressive investment strategy. The aggressiveness of an investment strategy depends on the relative weight of high-reward, high-risk asset classes such as equities and commodities within the portfolio. For example, if portfolio A has an asset allocation of 75% equities, 15% fixed income and 10% commodities, it would be considered aggressive since 85% of the portfolio is weighted to equities and commodities. However, portfolio B with an asset allocation of 85% equities and 15% commodities would be considered more aggressive than portfolio B. It should be taken into consideration that an aggressive strategy demands more active management than a conservative "buy-and-hold" or index tracking strategy [39]. This is a result of the fact that an aggressive strategy is much more volatile than a conservative one, and could require frequent adjustments, depending on market conditions. More rebalancing would also be required to bring portfolio allocations back to their target levels. Volatility of the assets could lead allocations to deviate significantly from their original weights. Hence, risk management becomes very important when building and maintaining an aggressive portfolio. Keeping losses to a minimum and making profit are key to success in this type of portfolio [50].

The second strategy, the balanced investment strategy, is a method of portfolio allocation and management aimed at balancing portfolio risk and return. Such portfolios are generally diveded equally between equities and fixed-income securities.

The last strategy, a defensive investment strategy, is a conservative method of portfolio selection and management aimed at minimizing the risk of losing the principal (initial capital). This strategy entails regular portfolio rebalancing to maintain the investor's intended asset allocation; buying high-quality, short-maturity bonds and blue-chip stocks; diversifying across both sectors and countries; placing stop-loss orders; and holding cash and cash equivalents in down markets. Such a strategy is meant to protect investors against significant losses from major market downturns. Many portfolio managers adopt defensive investment strategies for less risk-averse clients such as retirees without steady salaries. In such a case, the objective is to protect existing capital and keep pace with inflation through modest growth [39].

2.3 Risk measures

Investors are constantly faced with a trade-off between adjusting potential returns for higher risk. However, the events of the global financial crisis of 2008 and others in the past, have demonstrated the necessity for adequate risk management [65]. Poor risk management can result in bankruptcies and can threaten collapse of an entire financial sector [43]. Many risk measures have been presented over the years, intended to be applicable to as many different risk sources that have so far been identified: market risk, credit risk, liquidity risk, operational risk to name just a few.

2.3.1 Some Basic Properties of Risk Measures

The following is a mathematical definition of a risk measure adopted from Artzner et al. [2]:

Definition

Let X be a random variable such that the risk measure of X, $\rho(X)$, is a functional with $\rho : X \to (-\infty, \infty)$ and $\rho(\ell^\infty) \subset \mathbf{R}$. We define a risk measure $\rho : X \to (-\infty, \infty)$ as a mapping of a random variable from the probability space to the set of real numbers, $\mathbf{R}$.

Coherent Risk Measures

Artzner et al. [2] defines a risk measure as coherent if it satisfies the following four properties: monotonicity, positive homogeneity, sub-additivity and translation invariance. Given that X and Y denote portfolio returns, $\rho(X)$ and $\rho(Y)$ are their risk measures respectively, with c as an arbitrary constant, the definitions of the axioms are as follows:

(i) Monotonicity: $\rho(X) \leq \rho(Y), \text{ if } X \leq Y$

(ii) Translation invariance: $\rho(X + c) = \rho(X) - c, \;\; c \in \mathbf{R}$

(iii) Sub-additivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$

(iv) Positive Homogeneity: $\rho(\lambda X) = \lambda \rho(X)$

It should be noted that X is considered riskless if and only if X is a constant with a probability of one, that is, there exists a constant k such that $P(X = k) = 1 \cdot \rho(X)$ denotes the risk value for the asset outcomes, X.

The axiom of *monotonicity* states that if, in each state of the world, return Y is always better than return X, then the risk associated with Y should be higher than that related to X.

The axiom of *Translation invariance* illustrates that adding (or deducting) a risk-free amount to (from) a portfolio and investing it in the reference instrument, results in a decrease (increase) of the risk of the portfolio by exactly the same amount. From this property, we have the following well-known fact in finance:

$$\text{If } c = \rho(X), \;\; \rho(X + c) = \rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$$

Hence, it is possible to hedge an underwritten risky position by simply adding a certain amount of risk-free instruments in the portfolio.

The axiom of *Sub-additivity* tends to relate to the concept of portfolio diversification. It states that a portfolio consisting of several assets is less risky compared to a portfolio made up of a single security, provided that the correlation among the assets is not equal to one.

In this sense, we can say that the *sub-additivity* property is setting an upper bound to possible portfolio risk, and hence to the amount of capital we need to allocate to the portfolio. Only when there is a well-founded possibility that the sources of the risks may act together, the global portfolio risk will equal the sum of the components' risks.

The *Positive Homogeneity* axiom means that, if (for instance) the exposure to a specific position doubles, then the risk measure related to that position doubles as well. However, in the case that the position size directly influences risk (consider liquidity risk), we should account for any possible outcome (e.g., difficulty in liquidating the position), and we might expect the risk to go beyond doubling.
In such a case we can have:

$$\rho(k \cdot X) \geq k \cdot \rho(X)$$

However, from the *Sub-additivity* axiom we have

$$\rho(k \cdot X) \leq k \cdot \rho(X)$$

Thus

$$\rho(k \cdot X) = \underbrace{\rho(X + X + \cdots + X)}_{\text{k-times}} \leq \underbrace{\rho(X) + \rho(X) + \cdots + \rho(X)}_{\text{k-times}} = k \cdot \rho(X)$$

Accordingly, the result gives us equality as given in the *Positive Homogeneity* axiom.

Convex Risk Measures

Convex risk measures are used as a way of introducing a better diversification benefit as compared to coherent risk measures. The idea of convexity is that it takes the properties of positive homogeneity and sub-additivity, and combines them to better potray the liquidity risk of a portfolio. The *convexity* axiom is given as follows:

(v) Convexity: For $\lambda \in [0, 1]$, and X, Y as defined under coherence risk measures, we have

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$$

We now have the following definitions as given by Zhou et al. [90].

Definitions:

(1) A risk measure $\rho(X)$ is called a monetary risk measure if $\rho(0)$ is finite, and if $\rho(X)$ satisfies the axioms of Monotonicity and Translation Invariance.

(2) A risk measure $\rho(X)$ satisfying the axioms of Positive Homogeneity, Consistency and Sub-additivity is called a deviation measure.

(3) A risk measure $\rho(X)$ satisfying the axioms of Translation Invariance, Sub-additivity, Positive Homogeneity and Monotonicity is called a coherent measure of risk.

(4) A risk measure $\rho(X)$ satisfying the axioms of Translation Invariance, Monotonicity and Convexity is called a convex measure of risk.

Additionally, a coherent risk measure can respect the following axioms:

(vi) Relevance: if $X \leq 0$ and $X \neq 0$, then $\rho(X) > 0$.

(vii) Strictness: $\rho(X) \geq -E_p[X]$.

(viii) Law Invariance: if $F_X = F_Y$, then $\rho(X) = \rho(Y)$.

The relevance axiom ensures that if a position always generates negative results (losses), then its risk is positive. The strictness axiom ensures that the measure is sufficiently conservative to exceed the common loss expectation. The law invariance axiom, which is presented for coherent risk measures by Kusuoka [53], ensures that two positions that have the same probability function have equal risks. This last characteristic is important for risk measurement in practice, when real data that are dependent on a law are employed.

Acceptance set

Given a coherent risk measure ρ, Artzner et al. [2] define the acceptance set as $A_p = \{X \in L^p : \rho(X) \leq 0\}$. The acceptance set is the set containing the positions that cause a situation with no loss. Let L^p_+ be the cone of non-negative elements of L^p and let L^p_- be its negative counterpart. Each coherent risk measure ρ has an acceptance set A_p that satisfies the following properties:

(a) A_p contains L^p_+,

(b) A_p has no intersection with L^p_-,

(c) A_p is a convex cone.

The risk measure associated with this set is

$$\rho(X) = \inf\{c : (X + c) \in A_p\}$$

This means the minimum capital required to be added to position X to make it acceptable. It is demonstrated by Artzner et al. [2] that if an acceptance set satisfies the previously defined properties, then the risk measure associated with this set is coherent. If a risk measure is coherent, then the acceptance set linked to this measure satisfies the required properties.

2.4 Deviation risk measures

In focusing on generalised deviations, Rockafeller et. al. [73] investigate functionals D on $L^2(\Omega)$ that obey some axioms from the properties of standard deviation. The following definition is provided.

Definition: (General deviation measures): Rockafeller et. al. [73]
A deviation measure is any functional $D : L^2(\Omega) \to [0, \infty]$ *satisfying*

(D1) $D(X + C) = D(X)$ *for all* X *and constants* C,

(D2) $D(0) = 0$, *and* $D(\lambda X) = \lambda D(X)$ *for all* X *and all* $\lambda > 0$,

(D3) $D(X + Y) \leq D(X) + D(Y)$ *for all* X *and* Y,

(D4) $D(X) \geq 0$ *for all* X, *with* $D(X) > 0$ *for non-constant* X.

Under these axioms, $D(X)$ depends only on $X - E(X)$, and it vanishes only if $X - E(X) = 0$ (as seen from $D4$ with $X - E(X)$ in place of X). According to Rockafeller et. al. [73], this property captures the idea that D measures the degree of *uncertainty* in X. They conclude that if D is a deviation measure, then so too are its *reflection* $\hat{D}$ and its symmetrization $\bar{D}$ given by

$$\hat{D}(X) = D(-X), \qquad \bar{D}(X) = \frac{1}{2}[D(X) + \hat{D}(X)]$$

Axiom $D2$ is positive homogeneity. Combining $D2$ with $D3$ gives linearity property, which implies D is a convex functional on $L^2(\Omega)$. The above properties of a deviation measure are also stated in [24], [71], and [51].

We now give descriptions of the most popular deviation risk measures in the literature.

2.4.1 Standard deviation

A measure of risk is a metric of high importance in the field of portfolio theory. The choice of risk measure tends to vary depending on the purpose. The standard deviation of asset returns is a common risk measure, which measures the dispersion of the data from its expected value. The mathematical definition of a portfolio standard deviation is:

$$\sigma_p = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[x_i - E(X)]^2} \tag{2.1}$$

where n is the number of assets in the portfolio, x_i is the sample outcome for each asset in the portfolio and $E(X)$ is the mean of the outcome of the assets in the portfolio. Alternatively, the variance is used as a risk measure instead of standard deviation. The variance is a symmetrical risk measure, and it was first used by Markowitz [60] in optimal portfolio selection.

2.4.2 Semi-variance

The semi-variance is the expected value of the squared negative deviations of possible outcomes from expected return. Markowitz [61] elucidates on risk quantification for optimal portfolio selection and recommends using the semi-variance instead of the variance. While the variance penalises both upside and downside deviations from expected return, the semi-variance only penalises dispersion below the expected return. The semi-variance is defined as:

$$\text{Semi-variance}(X) = E[\max(0, x_i - E(X))^2]$$

2.4.3 Mean Absolute Deviation

Konno and Yamazaki [46] used the mean absolute deviation (MAD) as a measure of risk in their linear optimisation model. This decision had the benefit of reducing the mathematical complexity inherent in the Mean-variance Markowitz model. The MAD is defined by

$$\text{MAD} = \frac{1}{n}\sum_{i=1}^{n}|x_i - E(X)| \tag{2.2}$$

where n is the number of assets in the portfolio, x_i is the sample outcome of asset i and $E(X)$ is the mean of the sample.

2.4.4 Maximum Negative Deviation

The maximum negative deviation (MND) or lower-semi-absolute deviation (LSAD) is a shortfall risk measure (downside risk) relative to the mean of the portfolio [38]. It is defined as:

$$MND = R(X) = E|\min[0, x_i - E(X)]| \tag{2.3}$$

2.4.5 Sharpe's risk measure

Sharpe's risk measure is also known as the Sharpe ratio or reward-to-volatility ratio [78]. The Sharpe's ratio is a measure of portfolio performance that gives the risk premium per unit of total risk. The total risk is measured by the portfolio's standard deviation of returns. The risk premium on a portfolio is the total portfolio return minus the risk-free return. The Sharpe ratio can be interpreted as the excess return above the risk-free rate per unit of risk, where portfolio standard deviation is the risk. It provides a portfolio risk measure in terms of determining the quality of the portfolio's return at a given level of risk. The Sharpe ratio is expressed as:

$$S_p = \frac{\mu_p - R_f}{\sigma_p} \tag{2.4}$$

where μ_p is the expected portfolio return, R_f is the risk-free rate of return, and σ_p is the portfolio standard deviation.

2.5 Safety measures

Safety measures of risk involve the probability of the portfolio return becoming worse than a certain level.

2.5.1 Value-at-Risk (VaR)

Value-at-Risk (VaR) is one of the very popular risk measures widely used in the financial industry. It describes the magnitude of likely losses a portfolio can be expected to suffer during "normal" market movements within a given period(Linsmeier and Pearson [58]). VaR is a number above which we have only $(1-\alpha)\%$ of losses in a given period and it represents what one can expect to lose with probability $\alpha\%$, where α is the significance level. VaR is given by

$$\mathrm{VaR}_\alpha(X) = \min\{z|F_X(z) \geq \alpha\} \tag{2.5}$$

Jorion [42] provides the following about VaR:

(1) To estimate the probability of the loss, with a confidence interval, we need to define the probability distributions of individual risks, the correlation across these risks and the effect of such risks on value. In fact, simulations are widely used to measure the VaR for an asset portfolio.

(2) The focus in VaR is clearly on downside risk and potential losses. It is used in banks and reflects their fear of a liquidity crisis, where a low-probability catastrophic occurrence creates a loss that wipes out the capital and creates a client exodus.

(3) There are three key elements of VaR (a specified level of loss in value, a fixed time period over which risk is assessed and a confidence interval). The VaR can be specified for an individual asset, a portfolio of assets or for an entire firm.

(4) While the VaR at investment banks is specified in terms of market risks (interest rate changes, equity market volatility and economic growth), there is no reason why the risks cannot be defined more broadly or narrowly in specific contexts. Thus, one could compute the VaR for a large investment project for a firm in terms of competitive and firm-specific risks, and the VaR for a gold mining company could be given in terms of gold price risk.

Three common statistical approaches to estimate VaR are non-parametric historical simulation methods, parametric methods based on econometric models with volatility dynamics and various extreme value theory (EVT) based methods, where commonly the generalised Pareto distribution (GPD) is assumed for the tail distribution.

Although VaR is a commonly used risk measure by banks and other financial institutions, it has received a great amount of criticism from academics stating its many short-comings:

1. An investor who subscribes to VaR is implicitly stating that he / she is indifferent between very small losses exceeding VaR and very high losses. This assumption is far from reality.

2. VaR is not sub-additive and hence not coherent. This means that the VaR of a portfolio with two instruments can be larger than the sum of VaR of these two instruments.

3. VaR is a non-smooth, non-convex and multi-extrema (many local minima) function that makes it difficult to use in portfolio optimization.

4. VaR further relies on a linear approximation of risk and often assumes a normal distribution (or t-distribution) of the underlying market data.

Due to the above-mentioned issues, Conditional Value-at-Risk was developed as an extension to VaR.

2.5.2 Conditional Value-at-Risk

For random variables with continuous distribution functions, $\mathrm{CVaR}_a(X)$ equals the conditional expectation of X subject to $X \geq \mathrm{VaR}_\alpha(X)$. This measure has a wide number of other names, including expected tail loss, tail Value-at-Risk and expected shortfall. The general definition of conditional value-at-risk (CVaR) for random variables with a possibly discontinuous distribution function is as follows (see Rockafellar and Uryasev [72]):

The CVaR of X with significance level $\alpha \in (0, 1)$ is the mean of the generalised α-tail distribution:

$$\mathrm{CVaR}_\alpha(X) = \int_{-\infty}^{\infty} z dF_X^\alpha(z) \tag{2.6}$$

where

$$\mathrm{CVaR}_\alpha(X) = \begin{cases} 0, & z < \mathrm{VaR}_\alpha(X) \\ \frac{F_X(z)-\alpha}{1-\alpha}, & z \geq \mathrm{VaR}_\alpha(X). \end{cases}$$

Pflug [68] follows a different approach and suggests to define CVaR via an optimization problem, which he borrowed from Rockafellar and Uryasev [72]:

$$\mathrm{CVaR}_\alpha(X) = \min_\beta\{\beta + \frac{1}{1-\alpha} E[X-\beta]^+\} \tag{2.7}$$

where $[t]^+ = \max\{0, t\}$. CVaR provides information that can be considered complementary to that given by VaR, as it measures the expected excess loss above the VaR, if a loss larger than VaR actually occurs. Thus, it calculates the average of the worst $(1-\alpha)$ losses.

CVaR is more sensitive to the shape of the loss distribution in the tail of the distribution whereas VaR often considers a normal or t- distribution.

2.6 Porfolio Optimization Models in the literature

2.6.1 The Markowitz Mean-Variance Model

Mean-variance (MV) analysis was introduced by Markowitz [60], who describes the basic formulations and the quadratic programming tools used to solve them. Mean-variance portfolio theory is based on the idea that the value of investment opportunities can be meaningfully measured in terms of mean return and variance of the return. This mean-variance analysis is based on the following assumptions:

(i) All investors are risk-averse; they prefer less risk to more for the same level of expected return,

(ii) expected returns for all assets are known,

(iii) the variances and covariances of all asset returns are known, and

(iv) there are no transaction costs or taxes. Furthermore, mean-variance analysis is based on single-period model of investment.

Markowitz [60] proposes the mean - variance (MV) optimisation model in which the portfolio that minimises the variance subject to the restriction of a given mean return is chosen as the optimum portfolio. The mathematical model proposed by Markowitz is as follows:

Minimise

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{2.8}$$

subject to

$$\rho \leq \sum_{i=1}^{n} r_i x_i,$$
$$1 = \sum_{i=1}^{n} x_i,$$
$$0 \leq x_i \leq u_i, i = 1, \cdots, n,$$

where σ_{ij} is the covariance between assets i and j, x_i is the proportion of wealth invested in asset i, r_i is the expected return of asset i in each period, ρ is the minimum rate of return desired by an investor, and u_i is the maximum proportion of wealth which can be invested in asset i.

A feasible solution x is called efficient if it has the maximal expected return among all portfolios with the same variance, or alternatively, if it has a minimum variance among all portfolios that have at least a certain expected return. The variance is used as a measure of risk. The collection of efficient portfolios forms the efficient frontier of the portfolio universe. Although financial analysts and economists were aware of risk prior to Markowitz's risk measure, their risk was more concerned with standard financial statement analysis, following a similar line of enquiry to that of Graham [23]. However, Markowitz [60], was the first to formalize portfolio risk, diversification and asset selection in a mathematically consistent framework. In this respect, Markowitz's Portfolio Theory was a significant innovation in risk measurement and optimal portfolio selection, for which he won the Nobel prize [44].

There is extensive literature that generalises Markowitz's work to the multi-period case. Mossin [67], Samuelson [76] and Merton [63], show how an investor optimally chooses his portfolio in a dynamic environment in the absence of transaction costs. This implementation of a dynamic policy brings with it rebalancing of portfolio weights and results in high transaction costs. A number of researchers have studied optimal portfolio selection in the presence of transaction costs. Constantinides [11] studies the multi-period mean-variance problem and shows that the optimal trading policy is characterised by a no-trade interval, such that if the risky-asset portfolio weight is inside the interval, then it is optimal not to trade, and if the portfolio weight is outside, then it is optimal to trade to the boundary of this interval. However, the study does not incorporate the impact of transaction costs in investment. Traditionally, researchers have assumed that market price impact is linear on the amount traded [54], and thus market impact costs are quadratic. Torre and Ferrari [83], Grinold and Kahn [22], and Almgren et.al. [1] show that the square root function is more appropriate for modeling market price impact. Thus suggesting that market impact costs grow at a rate slower than quadratic. Garleanu and Pedersen [20] consider a multi-stage setting that relies on modeling price changes and give closed-form expressions for the optimal dynamic portfolio optimisation in the presence of quadratic transaction costs. De Miguel et. al. [16] extends the analysis by Garleanu and Pedersen [20] to a case where they capture the distortions on market price through a power function representation of transaction costs with an exponent between numbers one and two. They show that there exists an analytical rebalancing region for every time period such that the optimal policy at each time period is to trade to the boundary of the corresponding rebalancing region. However, the use of a power function to model market impact does not give a good approximation since implicit transaction costs are uncertain.

Multi-stage decision problems under uncertainty are considered by Becker et. al. [4] and Darlington et al. [14] for non-linear problems. A mean-variance approach is adopted with a static transcription of the dynamic decision model. Gulpinar et. al. [25] incorporate proportional transaction costs, given stochastic data in the form of a scenario tree. Glen [21] considers a mean-variance portfolio rebalancing strategy with transaction costs comprising of fixed charges and variable costs that include market impact cost. These variable transaction costs are assumed to be non-linear functions of traded value. Approximating implicit transaction costs by a non-linear function seems inappropriate since implicit transaction costs are random.

In this thesis, the portfolio selection problem is addressed by applying stochastic programming. A multi-stage stochastic mean-variance model that captures asset returns, implicit transaction costs and risk due to uncertainty is constructed. Stochastic programming is adopted as it has a number of advantages over other techniques. Firstly, stochastic programming models can accommodate general distributions by means of scenarios. One does not have to explicitly assume a specific stochastic process for the securities' returns, but he or she can rely on the empirical distribution of these returns. Secondly, stochastic models can address practical issues such as transaction costs, turnover constraints, limits on securities and prohibition of short-selling. Regulatory and institutional or market-specific constraints can be accommodated. Thirdly, stochastic models can flexibly use different risk measures [37]. Hence, this study uses portfolio variance, mean absolute deviation and maximum negative deviation as risk measures.

The first distinguishing feature of our multi-stage stochastic mean-variance model with uncertain transaction costs (SMVTC) is that it takes into account the approximate nature of the set of discrete scenarios by considering the variance around each return scenario and the corresponding implicit cost incurred during trading. Secondly, the variance term allows for the variability of asset returns over the scenario tree. Hence, uncertainty on asset returns is represented by a discrete approximation of a multi-variate continuous distribution as well as the variability due to discrete approximation.

Models involving implicit transaction costs have also received a fair share in the literature. Xia and Tian [85] estimate implicit transaction cost in the Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. They use the Bayesian Gibbs sampling method proposed by Hasbrouck [29] to analyze implicit costs in the bull and bear markets. Hasbrouck [29] expands Roll's model [74] and suggests a Bayesian Gibbs sampling method based on the daily closing prices to estimate implicit costs, and shows that the correlation coefficient between the results of using the Bayesian method and those of using high-frequency data was 0.965. Kozmik [49] discusses asset allocation with transaction costs formulated as multi-stage stochastic programming model. He considers transaction costs as proportional to the value of the assets bought or sold, but does not take into account implicit costs in the model. He employs conditional-Value-at-Risk as a measure of risk. Brown and Smith [6] study the problem of dynamic portfolio optimization in a discrete-time finite-horizon setting, and again, consider proportional trading costs. Lynch and Tan [59] study portfolio selection problems with multiple risky assets. They develop analytic frameworks for the case with many assets taking proportional transaction costs. Korn [48] studies continuous-time portfolio optimization and takes into account proportional transaction costs. Cai et. al. [9] examine numerical solution of dynamic portfolio optimisation with transaction costs. While transaction costs are broad and include explicit as well as implicit costs, Cai et. al. [9] consider a case of proportional transaction costs which can be either implicit or explicit, whichever is greater. However, the study of trading costs requires the identification of the type of cost to be estimated in order to explore effective ways of having a good estimate. Thus in this study, we consider implicit transaction costs. These costs are invisible and difficult to measure. They can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments [35].

In this thesis, an optimal portfolio policy of a multi-period stochastic mean-variance investor subject to uncertain returns, risk and uncertain implicit transaction costs is proposed.

2.6.2 The Mean Absolute Deviation Model

The mean absolute deviation (MAD) model was first proposed by Konno and Yamazaki [46], in deterministic form, as an alternative to the famous and widely used mean-variance model by Markowitz [60]. The MAD is a dispersion-type risk linear programming (LP) computable measure that may be considered as an approximation of the variance when the absolute values replace the squares. Konno and Yamazaki [46] propose the mean absolute deviation model as a risk measure to overcome the weaknesses of the variance. This MAD model is equivalent to the mean-variance model by Markowitz [60] if the assets' returns are multivariate normally distributed. However, the MAD model is a special case of picewise linear risk model which is fast in optimising portfolios by means of linear programming unlike the mean-variance model that requires quadratic programming. Use of a linear model considerably reduces the time needed to reach a solution, thereby making it more appropriate for large-scale portfolio selection. It makes extensive calculations of the covariance matrix unnecessary, as opposed to the mean-variance model. The MAD model is also sensitive to outliers in historical data [8].
Konno and Yamazaki [46] propose the MAD model as given below:

Minimise

$$H(x) = \frac{1}{\tau}\sum_{t=1}^{\tau} |\sum_{i=1}^{n}(r_{it} - \bar{r}_{pt})x_{it}| \tag{2.9}$$

subject to

$$\rho \leq \sum_{i=1}^{n} r_{it} x_{it}, t = 1, \cdots, \tau,$$
$$1 = \sum_{i=1}^{n} x_{it}, t = 1, \cdots, \tau,$$
$$0 \leq x_{it} \leq u_i, i = 1, \cdots, n, t = 1, \cdots, \tau.$$

The modulus value in the objective function is defined as

$$|\sum_{i=1}^{n}(r_{it} - \bar{r}_{pt})x_i| = \begin{cases} \sum_{i=1}^{n}(r_{it} - \bar{r}_{pt})x_i, & \sum_{i=1}^{n}(r_{it} - \bar{r}_{pt})x_i \geq 0 \\ -\sum_{i=1}^{n}(r_{it} - \bar{r}_{pt})x_i, & \sum_{i=1}^{n}(r_{it} - \bar{r}_{pt})x_i < 0 \end{cases}$$

where $\bar{r}_{pt}$ is the portfolio return at end of period t, r_{it} is the return of asset i in period t, x_{it} is the weight of asset i in period t, ρ is the minimum expected portfolio return of period t and τ is the total number of periods considered in rebalancing the portfolio. u_i is an upperbound of each weight x_{it} of asset i and n is the number of assets in the portfolio.

The MAD model has its short comings. Ignoring the covariance matrix can cause great estimation risk [80]. The MAD also penalises not only the negative deviations, but also the positive deviations. Investors prefer higher positive deviations and avoid lower and negative deviations in the portfolio returns. Fama [17], explains that making a distinction between positive and negative returns is necessary if portfolio returns are asymmetrically distributed and stock returns are skewed. Fishburn [18], introduced downside-risk measures to deal with such problems. The advantage of downside-risk measures is that they only penalise returns below a given threshold level specified by the investor. Michalowski and Ogryczak [64] extend the MAD model to incorporate downside-risk aversion. Hoe et. al. [33] make an empirical comparison of the mean-variance, mean absolute deviation, minimax and mean-semi-variance models in portfolio optimization. They compare the portfolio compositions and performance of different optimal portfolios by using data of monthly returns from 54 stocks inluded in the Kuala Lumpur Composite index from January 2004 to December 2007. The most risk portfolio is found to be that of the mean-variance model while the MAD model generated portfolio had the least risky. The minimax model shows the highest performance (i.e., the model produces the best portfolio), followed by the MAD model, and the mean-variance gives the least performance. However, despite the good performance by the minimax model, it has its disadvantages. Because of its objective to minimise maximum loss, minimax is sensitive to outliers in historical data [87].

Most models presented in the literature are static models: A decision is made, then not further modified. They are essentially single-period models, since there is only one decision to be made, for the initial period. The purpose of this study is to construct a multi-stage stochastic mean absolute deviation portfolio optimisation model with random transaction costs that captures assets' returns and risk due to uncertainty. The model employs stochastic programming with recourse by taking into account rebalancing of portfolio composition as the uncertainty of returns gets realised. Mean absolute deviation models that are proposed in the literature do not take the uncertainty of the future into account. Most models that account for transaction costs in portfolio selection do not consider random transaction costs. Konno and Wijayanayake [45] proposed the deterministic mean absolute deviation model with transaction costs modeled by a concave function. They use a linear cost function as an approximation to the concave cost function. Gulpinar et. al. [25] propose a multi-stage mean- variance portfolio analysis with non-random transaction costs. Yu et. al. [89] propose a multi-period portfolio selection with l_∞ model (i.e., the maximum absolute deviation risk model). They employ the l_∞ function to control the risk in every period. However, no transaction costs are considered in the model. Again, the model does not account for uncertainty of future events.

2.6.3 The Minimax Model

It has been well documented in the literature that investors generally shun positions in which they would be subjected to catastrophic losses however small the probability these losses may carry. Such a “disaster avoidance motive” implies that investors care about extreme negative scenarios in investment and are averse to the risk of sharp price plunges [62]. Hence the potential loss from extreme undesirable returns should become a significant factor in asset pricing. Returns in economic recessions and booms are characterized by extreme movements [41]. The extreme movements of the market are not always reflected in all the individual stocks. Some individual stocks show an extreme reaction while others exhibit a milder reaction. It is in extreme cases that investors are highly concerned about the performance of their portfolios, particularly the downside movements.

The notion of tail risk or extreme downside risk has increasingly gained consideration in the asset pricing literature. In particular, contrary to assumptions of the standard Capital Asset Pricing Model (CAPM) of Sharpe [79] and Lintner [57], in which portfolio risk is fully captured by the variance of the portfolio return distribution, asset returns display significant negative skewness and excess kurtosis, both of which increase the likelihood of extreme negative returns [70]. On the studies that focus directly on the likelihood of extreme returns, Ruenzi and Weigert [75], use a copula-based approach to construct a systematic tail risk measure to show that stocks with high crash sensitivity, measured by lower tail dependence with the market, are associated with higher returns that can not be explained by traditional risk factors such as: the downside beta, co-skewness or co-kurtosis. These studies examine the variation in expected returns across individual stocks.

Young [87] introduces a linear programming model which maximises the minimum return or minimises the maximum loss (minimax) over time periods and applies it to stock indices from eight countries. The model developed by Young [87] is as follows:

Maximise

$$M_p \tag{2.10}$$

subject to

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i y_{it} - M_p, t = 1, \cdots, T, \\
G &\leq \sum_{i=1}^{n} w_i \bar{y}_i, \\
W &\geq \sum_{i=1}^{n} w_i, \\
0 &\leq \sum_{i=1}^{n} w_i,
\end{aligned}$$

where y_{it} is the return of one dollar invested in security i in time period t, $\bar{y}_i$ is the mean return of security i, w_i is the portfolio allocation to security i, M_p is the minimum return on the portfolio, G is the minimum level of return, and W is the total weight allocation.

The analysis shows that the model performs similarly to the classical mean-variance model of Markowitz [60]. Additionally, he argues that, when data is log-normally distributed or skewed, the minimax (MM) formulation might be a more appropriate method compared to the mean-variance formulation which is optimal for normally distributed data. Kamil et. al. [44] develop a single and two stage stochastic programming model with recourse for portfolio selection in which they minimise the maximum downside deviation of portfolio returns from the expected return. In this thesis, the formulation by Kamil et. al. [44] is extended and a multi-stage stochastic maximum negative deviation with transaction costs (SMNDTC) model which takes into account uncertainty of implicit transaction costs and asset returns in optimal portfolio selection is developed. The model being proposed considers downside risk and related implicit transaction costs in trading to give an investor or investment manager an option of selecting his or her portfolio knowing the implicit trading costs he or she is likely to incur. The uncertainty about future events is captured in stages and by means of scenarios.

2.6.4 Conditional Value-at-Risk model

The Conditional Value-at-Risk (CVaR) model is proposed by Rockafeller and Uryasev [72]. The CVaR is also known as mean excess loss, mean shortfall or tail Value-at-Risk. Value-at-Risk (VaR) measures the minimum loss corresponding to certain worst number of cases but it does not quantify how bad these worst losses are. CVaR quantifies this magnitude and is a measure of the expected loss corresponding to a number of worst cases, depending on the chosen confidence level. The CVaR can be defined as the conditional expectation of loss above the amount α at a specified probability level β over a given time horizon [56]. CVaR satisfies the four properties of coherence regarding risk which are:

(i) the wealth at risk declines when an amount of riskless wealth is added;

(ii) more wealth is preferred as compared to less wealth;

(iii) the aggregated risk of two investments is less than the sum of the two associated individual risks; and

(iv) risk must also grow with the same proportionality when the wealth at risk is multiplied by a positive factor [77].

This model is a linear programming model and is given as follows [72]:

Minimise

$$\alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} z_t \tag{2.11}$$

subject to

$$\begin{aligned}
&0 \leq z_t, t = 1, \cdots, T, \\
&\rho \leq \sum_{i=1}^{n} r_{it} x_i, \\
&1 = \sum_{i=1}^{n} x_i, \\
&0 \leq x_i,
\end{aligned}$$

where α is the lowest amount of loss, β is the probability that the loss will not exceed α, T is the number of periods, z_t is the variable, r_{it} is the realization of random variable R_i during period t, and x_i is the amount invested in asset i.

2.7 Stochastic programming theory

2.7.1 Introduction

This section gives the basic properties and theory on stochastic programming (SP). The main properties to be considered are formulations of deterministic equivalent programs to a stochastic program, the forms of the feasible region and objective function, and conditions for optimality. The focus will be on stochastic programs with recourse, and in particular, stochastic linear programs.
Methods of operations research, especially mathematical programming methods, are receiving broader acceptance in the economic and financial industry. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making processes [52]. Using the methodology, people observe and formulate real problems carefully, then model those problems in scientific way, by considering viewpoints as wide and reasonable as possible. A solution is then obtained to optimality. Those features of operational research characterises its applications in a variety of fields.

Probability theory and statistics help people better understand and recognise the rules by which systems operate. Generally, events will happen with probabilities. That different outcomes will occur under the same circumstances is explained by the fact that those outcomes all have positive probabilities. In many cases, it is more precise to predict what the future will more likely be rather than what the future will exactly be.

Stochastic programming deals with a class of optimisation models and algorithms in which some of the data may be subject to significant uncertainty. Such models are appropriate when data evolve over time and decisions need to be made prior to observing the entire data stream. For example, investment decisions in portfolio planning problems must be implemented before stock performance can be observed. Similarly, planning for power generation is done before the demand for electricity is realised. It is therefore important, under such circumstances, that models be developed in which plans are evaluated against a variety of future scenarios that represent alternative outcomes of data. Such models provides plans which are better able to hedge against future losses and catastrophic events [55].

Stochastic programming is mathematical programming with random parameters. In a real context of enterprise risk management, when future events need to be considered in business activity planning, uncertainty of parameters plays the key role. Stochastic programming provides a paradigm to include uncertainty into optimisation-based decision models. In particular, a multistage stochastic program with recourse is a multi-period mathematical program where parameters are assumed to be uncertain along the time path. The term recourse means that the decision variables adapt to the different outcomes of random parameters at each time period. The stochastic programming model allows one to handle several scenarios simultaneously while scenarios present different outcomes of random variables. It provides an adaptive policy that is close in spirit to the way decision-makers have to deal with uncertain futures in real life [19].

Stochastic programming makes modeling possible in the case where the parameters needed are random, i.e. the value could be from sets, continuous or discrete. It still has to be known what set it is and the corresponding behavior (probability distribution over this set). A big assumption of general stochastic programming is that the probability distributions of random parameters are known. In most cases, historical data or simulation with assumptions upon the statistical parameters is carried out. This historical market data is used in market risk management. However, this is rarely applicable in operational risk modeling, where historical data is limited and statistical parameters are complicated to evaluate [86].

Stochastic programming provides a general framework to model path dependence of the stochastic process within an optimization model. It allows for uncountably many states and actions which are combined with constraints and time-lags [15]. An important characteristic of stochastic programming is that it separates model formulation activities from solution activity, which is not the same with dynamic programming. This separation makes it not necessary for stochastic programming to obey all mathematical assumptions, unlike dynamic programming. This leads to a rich class of models for which a variety of algorithms can be developed [82].

2.7.2 Deterministic linear programs

Solutions for stochastic programs are generally achieved by first finding deterministic equivalent linear programs to stochastic programs. In this regard, a description of the development of a general deterministic linear program is given following closely a presentation by Birge and Louveaux [5].

A deterministic linear program consists of finding a solution to the following program:

Minimise

$$Z = c^T x \tag{2.12}$$

subject to

$$\begin{aligned} b &= Ax, \\ 0 &\leq x \end{aligned}$$

where x is an $(n \times 1)$ vector of decisions, and c, A and b are known vector and matrix of constants of sizes $(n \times 1)$, $(m \times n)$, and $(m \times 1)$, respectively. The value $Z = c^T x$ corresponds to the objective function, while $\{x|AX = b, x \geq 0\}$ defines the set of feasible solutions. An optimum solution x^* is a feasible solution such that $c^T x \geq c^T x^*$ for any feasible x if the objective is to minimise the function (Birge and

Louveaux [5]). Linear programs typically search for a minimal cost solution under some requirements (demand) to be met, or for a maximum profit solution under limited resources.

2.7.3 Decision and stages

Stochastic linear programs are linear programs in which some problem data may be considered uncertain. *Recourse programs* are programs where some decisions or recourse actions can be taken after uncertainty is disclosed. Data uncertainty can be represented as random variables, in which an accurate probabilistic description of the random variables is assumed available in the form of some probability distributions. It should be noted here that particular values of the random variables will only be known after the random experiment has occurred. The data vector $\xi = \xi(s)$ is only known after the random experiment has occurred. Thus, we have the following two sets of decisions to be made corresponding with the two sets of data, i.e., before and after the random experiment [5]. Decisions taken before the experiment are called first-stage decisions and those taken after the experiment are second-stage decisions. We represent first-stage decisions by the vector $\mathbf{x}$ and second-stage decisions by $\mathbf{y}(s)$ or $\mathbf{y}$. The sequence of decisions can then be described as

$$x \rightarrow \xi(s) \rightarrow y(s, x).$$

2.7.4 A Recourse Problem and its components

The term "recourse" refers to the opportunity to adapt a solution to the specific outcome observed. Thus, x stands for resource levels to be acquired, and s corresponds to a specific data scenario. The decision y adapts to the specific combination of x and s obtained. If the initial decision x is coupled with a bad outcome, the variable y offers an opportunity to recover to the fullest extent possible. In recourse problems, there are always two or more decision stages.

A resource problem is characterised by its scenario tree, scenario problems and nonanticipativity constraints. A *scenario* is a specific, complete realisation of the stochastic elements that might appear during the course of the problem. A *scenario tree* is defined as a structured distributional representation of the stochastic elements and the manner in which they may evolve over the period of time represented in the problem. A *scenario problem* is thus associated with a particular scenario and may be looked upon as a deterministic optimisation problem [31].

The nodes in a scenario tree are associated with a subset of decisions, with the root node corresponding to the initial decision stage, during which no specific information regarding the random variables is known. The leaf nodes correspond to the final decisions required, which are made after all available information has been obtained. Depending on the manner in which the problem is formulated, it may be necessary to include specific conditions to ensure that the decision sequence abides by the information associated with the scenario tree. Such conditions are *nonanticipativity constraints*, and they impose the condition that scenarios sharing the same information history up to a particular decision epoch must have the same decisions. These nonanticipativity constraints are a distinguishing feature of stochastic programs and methods of solution are designed to exploit their structure [31].

2.7.5 Two-Stage Stochastic Linear Programs with Fixed Recourse

Stochastic programming traces its roots to Dantzig [13], where the recourse model is first introduced, and Charnes and Cooper [10], where probabilistically constrained models are introduced. A stochastic program might be thought of as one of the ultimate operations research models. When some of the data elements in a linear program (LP) are most appropriately described using random variables, a stochastic linear program (SLP) results [31]. In this case, stochastic programs involve an artful blend of deterministic mathematical programs and stochastic models. In many cases, solution techniques for stochastic programs rely on statistical estimation and numerical approximation methods. In short, stochastic programming relies on a wide variety of operations research techniques.

Recourse models result when some of the decisions must be fixed before information relevant to the uncertainties is available, while some of them can be delayed until afterward. The two-stage stochastic linear program with fixed recourse has the following general form [5]:

Minimise

$$Z = c^T x + E_\xi[h(x, s)] \tag{2.13}$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

where

$$h(x, s) = \min \; q(s)^T y(s) \tag{2.14}$$

subject to

$$\begin{aligned} W(s)y(s) &= r(s) - T(s) \\ y(s) &\geq 0 \end{aligned}$$

where q is a second-stage objective vector and h is a second-stage value function with random argument. The above stochastic program can be written in compact form as presented by Birge and Louveaux [5] :

Minimise

$$Z = c^T x + E_\xi[\min\ q(s)^T y(s)] \tag{2.15}$$

subject to

$$\begin{aligned} Ax &= b \\ W(s) &= r(s) - T(s)x \\ x \geq 0,\ \ & y(s) \geq 0 \end{aligned}$$

where c is a known vector in $\mathbf{R}^{n_1}$, b is a known vector in $\mathbf{R}^{m_1}$, A and W are known matrices of size $m_1 \times n_1$ and $m_2 \times n_2$, respectively, and W is called the *recourse matrix*, which we assume to be fixed in this case. r is the right-hand side in second-stage, $T(s)$ is a technology matrix, and superscript T is the transpose of a matrix or vector. Fixing the recourse matrix enables characterisation of the feasibility region to be done in a convenient manner for computation purpose. Formulation of the stochastic linear program where the recourse matrix is not fixed is considered in the next section.

For each s, $T(s)$ is given as an $m_2 \times n_1$ matrix, with $q(s) \in \mathbf{R}^{n_1}$ and $r(s) \in \mathbf{R}^{m_2}$. The stochastic components of the problem produce the vector

$$\xi^T(s) = [q(s)^T, r(s)^T), T_1(s), \cdots, T_{m_2}(s)]$$

with $N = n_2 + m_2 + (m_2 \times n_1)$ components, where $T_i(s)$ is the i-th row of the technology matrix $T(s)$. E_ξ gives the mathematical expectation with respect to ξ. $\Theta \in \mathbf{R}^N$ is taken to be the smallest closed subset in $\mathbf{R}^N$ where $P(\xi \in \Theta) = 1$, and it is assumed that the constraints hold almost surely (a.s.).

The two-stage stochastic program (2.14) becomes equivalent to the deterministic linear program [5] :

Minimise

$$Z = c^T x + \Phi(x) \tag{2.16}$$

subject to

$$Ax = b$$
$$x \geq 0$$

where

$$\Phi(x) = E_\xi Q(x, \xi(s)) \tag{2.17}$$

and

$$Q(x, \xi(s)) = \min_y \{q(s)^T y | Wy = r(s) - T(s)x, y \geq 0\}. \tag{2.18}$$

Following the presentation by Birge and Louveaux [5], formulations (2.15) and (2.16)–(2.18) apply for both discrete and continuous random variables. This representation clearly illustrates the sequence of events in the recourse problem. First-stage decisions x are taken in the presence of uncertainty about future realisations of ξ. In the second stage, the actual value of ξ becomes known and some corrective actions or recourse decisions y can be taken. First-stage decisions are, however, chosen by taking their future effects into account. These future effects are measured by the value function or recourse function, $\Phi(x)$, which computes the expected value of taking decision x. $Q(x, \xi(s))$ is a second-stage value function with a random argument. q is a second-stage objective vector.

If T is not stochastic, the formulation (2.16) $-$ (2.18) can be replaced by [5]

Minimise

$$Z = c^T x + \psi(\chi) \tag{2.19}$$

subject to

$$Ax = b,$$
$$Tx - \chi = 0,$$
$$x \geq 0,$$

where

$$\psi(\chi) = E_{\xi}\psi(\chi, \xi(s))$$

and

$$\psi(\chi, \xi(s)) = \min\{q(s)^T y | Wy = r(s) - \chi, \ \ y \geq 0\}.$$

This formulation implies that x now corresponds to an m_2-dimensional *tender* $\chi = Tx$ to be bid against the outcomes $r(s)$ of the random events.

2.7.6 Simple Recourse

The presentation in this sub-section is adopted from Higle [31].

A special case of the recourse model known as the simple recourse model, arises when the constraint coefficients in the second stage problem, W, form an identity matrix I so that

$$h(x, s) = \min \ \ (q(s)^+ y(s)^+ + q(s)^- y(s)^-) \tag{2.20}$$

subject to

$$Iy(s)^+ - Iy^- = r(s) - T(s)x$$
$$y^+, y^- \geq 0$$

For any right-hand-side vector, $r(s) - T(s)x$, a feasible solution to (2.20) is easily determined by simply noting the sign of the right-hand side of each constraint and setting y^+ and y^- accordingly. Furthermore, provided that the i^{th} component of

$q(s)^+ - q(s)^-$ satisfies $q(s)^+ - q(s)^- > 0$ for each i, this feasible solution is also optimal [31].

It is also noted here that the sub-problem (2.20) can be equivalently represented as

$$h(x, s) = \sum_i h_i(x, s)$$

where

$$h_i(x, s) = \min \ (q(s)_i^+ y_i^+ + q(s)_i^- y_i^-) \tag{2.21}$$

subject to

$$y_i^+ - y_i^- = r(s)_i - \{T(s)x\}_i$$
$$y_i^+, y_i^- \geq 0.$$

Simple recourse situations arise, for example, when *target values* can be identified and the objective being to minimize deviations from these target values [31].

2.7.7 Multi-stage Recourse Problems

Following the above description, formulation of a multi-stage stochastic program with recourse can now be presented. A scenario formulation for the stochastic multi-stage program is introduced.

Most practical decision problems involve a sequence of decisions that react to outcomes that evolve over time. We start by considering the multi-stage stochastic linear program with fixed recourse, which takes the following form, as given by Birge and Louveaux [5]:

Minimise

$$Z = c^1 x_1 + E_{\xi^2}[\min \ c^2(s)x^2(s^2) + \cdots + E_{\xi^H}[\min \ c^H(s)x^H(s^H)] \cdots] \tag{2.22}$$

subject to

$$W_1 x^1 = r^1,$$
$$T^1(s^2)x^1 + W^2 x^2(s^2) = r^2(s^2),$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$T^{H-1}(s^H)x^{H-1}(s^{H-1}) + W^H x^H = r^H(s)$$
$$x^1 \geq 0;\ \ x^t(s^t) \geq 0,\ \ t = 2, \cdots, H;$$

where c^1 is a known vector in $\mathbf{R}^{n_1}$, r^1 is a known vector in $\mathbf{R}^{m_1}$, $\xi^1(s)^T = \{c^t(s), r^t(s)^t, T^{H-1}(s), \cdots, T_{\frac{t-1}{m_1}}\}$ is a random N_t - vector defined (Ω, Σ^t, P) (where $\Sigma^t \subset \Sigma^{t+1}$ for all $t = 2, \cdots, H$), and each W^t is a known $m_1 \times n_1$ matrix. (Ω, Σ^t, P) is a probability space where Ω is a set of all random events and P is the probability of events. The decisions x depend on the history up to time t. We also suppose that Θ^t is the support of ξ^t.

We now provide the deterministic equivalent form of this problem in terms of a dynamic program. Given that the stages are 1 to H, we define the states as $x^t(s^t)$. Since the only interaction between periods is through this realisation, we define a dynamic programming type of recursion. For terminal conditions, we have:

$$Q^H(x^{H-1}, \xi^H(s)) = \min\ c^H(s)x^H(s) \tag{2.23}$$

subject to

$$W^H x^H(s) = r^H(s) - T^{H-1}(s)x^{H-1},$$
$$x^H(s) \geq 0.$$

2.7.8 Discrete random variables

We now present some basic properties when ξ is a discrete random variable as adopted from Birge and Louveaux [5]. These properties are important in that they provide the existence of solutions of multi-stage stochastic programs, particularly solution of (2.16) – (2.18).

Let $K_1 = \{x|AX = b,\ x \geq 0\}$ be the set determined by the fixed constraints, that is, those that do not depend on the particular realisation of the random vector. For any given ξ, we define the "elementary feasibility set" as

$$K_2(\xi) = \{x|y \geq 0 \text{ exists such that } W(s)y = r(s) - T(s)x\}.$$

The following theorems in the literature are now presented which demonstrate that multi-stage stochastic programs have feasible optimal solutions. Some of these theorems will be presented without proofs (corresponding proofs are shown in the Appendix 1).

We begin by giving the following definition:

Definition (Birge and Louveaux [5])

Define pos$W = \{t|Wy = t, y \geq 0\}$, as the *positive hull* of W. This is the set of right-hand sides that can be obtained by a non-negative combination of the columns of W. The positive hull is a convex cone.

Theorem 1: (Birge and Louveaux [5])

(i) *For a given* ξ, *the elementary feasibility set* $K_2(\xi)$ *is a convex polyhedron,*

(ii) *When* ξ *is a finite discrete random variable,* K_2 *is a convex polyhedron.*

Proof

The proof is presented in Appendix 1. The proof states that the intersection of infinitely many convex polyhedra is a convex polyhedron.

For the second-stage value function, $Q(x, \xi)$, we have the following theorem:

Theorem 2: (Birge and Louveaux [5])

For any given ξ, the value function $Q(x, \xi)$ is

(i) *a piecewise linear convex function in (r, T);*

(ii) *a piecewise linear concave function in q;*

(iii) *a piecewise linear convex function in x for all $x \in K_2$.*

When ξ is a finite discrete random variable, $\Phi(x)$ is piecewise linear and convex on K_2.

Proof:

The proof is provided in Appendix 1. This proof provides for convexity of the second-stage value function $Q(x, \xi)$, which is a pre-requisite for an optimal solution. The following proposition shows that $\Phi(x)$ is finite.

2.7.9 General Cases

A description of the general cases is adopted from Birge and Louveaux [5].

For some fixed values of x and ξ, the value of the second-stage program, as given above, is

$$Q(x, \xi) = \min_y \{q(s)^T y | W(s)y = r(s) - T(s)x, y \geq 0\}.$$

We have previously defined the 'elementary feasibility set' as

$$K_2(\xi) = \{|y \geq 0 \text{ exists such that } W(s)y = r(s) - T(s)x\}$$

which can equivalently be written as

$$K_2(\xi) = \{x | Q(x, \xi) < \infty\}.$$

When ξ is not a discrete random variable, K_2 can be defined in two ways (Birge and Louveaux [5]):

$$K_2 = \{x | \Phi(x) < \infty\}$$

or

$$K_2^P = \cap_{\xi \in \Theta} K_2(\xi).$$

The set K_2^P defines the *possibility interpretation* of the second-stage feasibility set. A first-stage decision x belongs to K_2^P if, for all possible values of the random vector ξ, a feasible second-stage decision can be taken.

A problem that could arise and would cause the sets K_2^P and K_2 to be different, would be to have $Q(x, s)$ bounded above with probability one and yet having $\Phi(x)$, the expectation of $Q(x, \xi)$, unbounded.

Proposition 3 (Birge and Louveaux [5])
If ξ has finite second moments, then

$$P(\omega | Q(x, \xi) < \infty) = 1 \ \textit{implies} \ \Phi(x) < \infty.$$

Observe that the second-stage program is the linear program

$$Q(x, \xi) = \min\{q(s)^T y | Wy = r(s) - T(s)x, y \geq 0\}.$$

From Theorem 2, solving this linear program for x and ξ amounts to finding some optimal basis **B** for which

$$Q(x, \xi) = q_B(s)^T B^{-1}(r(s) - T(s)x).$$

Assuming $Q(x, \xi)$ is bounded above with probability one and that the same basis B would be optimal for all x and all ξ, it is sufficient condition for $\Phi(x)$ to be bounded because it implies $E_\xi(q_B^T B^{-1} r)$ and $E_\xi(q_B^T B^{-1} T x)$ are both bounded above.

Theorem 4: (Birge and Louveaux [5])
For a stochastic program with fixed recourse where ξ has finite second moments, the sets K_2 and K_2^P coincide.

Proof

We first consider $x \in K_2^P$. This implies $Q(x,\xi) < \infty$ with probability one, so that, by Proposition 3, $\Phi(x)$ is bounded above and $x \in K$.

We now consider $x \in K_2$. It follows that $\{\xi | Q(x,\xi) < \infty\}$ is a set of measure one. We observe that $Q(x,\xi) < \infty$ is equivalent to $\{r(s) - T(s)x\} \in \mathrm{pos}W$ and that $r(s) - T(s)x$ is a linear function of ξ, and $\{\xi \in \Sigma | Q(x,\xi) < \infty\}$ is a closed subset of Σ of measure one, for any set Σ of measure one. By definition of Θ, it follows that $\{\xi | Q(x,\xi) < \infty\} \subseteq \Theta$ and therefore $x \in K_2^P$. This completes the proof.

2.7.10 Special cases: Complete and simple recourse (Birge and Louveaux [5])

A complete recourse is also represented by $\mathrm{pos}W = \mathbf{R}^{m_2}$, which implies that W contains a positive linear basis of $\mathbf{R}^{m_2}$. Complete recourse is added to a model to ensure that results are always feasible.

A *simple recourse* offers additional computational advantages to stochastic programming solutions. For a simple recourse problem, $W = [I, -I]$, y is divided correspondingly as $(\mathbf{y}^+, \mathbf{y}^-)$, and $\mathbf{q} = (\mathbf{q}^+, \mathbf{q}^-)$. The optimal values of $y_i^+(s), y_i^-(s)$ are determined completely by the sign of $r(s) - T(s)x$ provided that $\mathbf{q}_i^+ + \mathbf{q}_i^- \geq 0$ with probability one. This finiteness result is in the following Theorem.

Theorem 5

Suppose the two-stage stochastic program in (2.14) *is feasible and has simple recourse, and that* ξ *has finite second moments. Then* $\Phi(x)$ *is finite if and only if* $\mathbf{q}_i^+ + \mathbf{q}_i^- \geq 0$ *with probability one.*

Proof

If $q_i^+(s) + q_i^-(s) < 0$ for $s \in \Omega_1$ where $P(\Omega_1) > 0$, then, for any feasible x in (2.14), for all $s \in \Omega_1$ where $r_i(s) - T_i(s)x > 0$, let $y_i^+(s) = r_i(s) - T_i(s)x + u$, $y_i^-(s) = u$. By letting $u \to \infty$, $Q(x, s) \to -\infty$. A similar argument applies if $r(s) - T_i(s)x \leq 0$. So $\Phi(x)$ is not finite.

If $q_i^+(s) + q_i^-() \geq 0$ with probability one, then
$Q(x, s) = \sum_{i=1}^{m_2}(q_i^+(s)(r_i(s) - T_i(s)x)^+ + q_i^-(-r_i(s) + T_i(s)x)^+)$,
which is finite for all s. Using proposition 3, we obtain the result.

2.7.11 Optimality conditions

We now consider optimality conditions for stochastic programs, in particular, the necessary and sufficient conditions for two-stage stochastic linear programs as adopted from Birge and Louveaux [5]. The following theorem gives some sufficient conditions to guarantee that a solution to the two-stage stochastic program (2.14) exists. In the theorem, we use the notation rc to denote *recession cone*, $\{v | u + \lambda v \in S, \text{ for all } \lambda \geq 0 \text{ and } u \in S\}$ when applied to a proper convex function, f.

Theorem 6: (Birge and Louveaux [5])
Suppose that the random elements ξ have finite second moments and one of the following:

(a) *the feasible region K is bounded; or*

(b) *the recourse function Φ is eventually linear in all recession directions of K, i.e., $\Phi(x + \lambda v) = \Phi(x + \bar{\lambda} v) + (\lambda - \bar{\lambda}) rc\Phi(v)$ for some $\bar{\lambda} \geq 0$ (dependent on x), all $\lambda \geq \bar{\lambda}$, and some constant recession value, $rc\Phi(v)$, for all v such that $x + \lambda v \in K$ for all $x \in K$ and $\lambda \geq 0$.*

Then, if problem (2.14) has a finite optimal value, it is attained for some $x \in \mathbf{R}^n$.

Proof

For (a) the proof follows immediately by noting that the objective is convex and finite on K, which is compact by assumption. The only possibility for not attaining an optimum is, therefore, when the optimal value is only attained asymptotically. By (b), along any recession direction v, we must have $rc\Phi(v) \geq 0$ for a finite value of $\phi(x + \lambda v)$. Hence, the optimal value must be attained. This completes the proof.

2.8 Summary

In this chapter, presentations on how some models used in optimal asset allocation have evolved with time have been given. These models were developed as essentially single-period models. Some researchers in the field extended the models in a number of ways as described in the preceding sections in an attempt to improve on efficiency of the models in modeling real-life situations. Some models were extended to become multi-period, but could not represent uncertainty in implicit transaction costs that are incurred by investors during initial trading and in rebalancing of their portfolios. Researchers who attempted to incorporate uncertain implicit transaction costs in optimal portfolio selection assumed that these costs are proportional to asset prices. Jansen and De Vries [41] found that in periods of economic recession and booms, extreme movements of the market are not always reflected in all stocks. Some stocks show an extreme reaction while others exhibit a milder reaction. Hence this study develops financial models that are stochastic and multi-period, incorporating uncertainty in implicit transaction costs, asset returns and portfolio risk by way of scenarios, the methodology of which is described in the chapters that follow.

Chapter 3

Methodology - Stochastic financial modeling with uncertain implicit transaction costs

3.1 Statement of the problem

A multi-period discrete-time optimal portfolio strategy over a given investment horizon is determined in this chapter. This strategy accounts for uncertainty in assets' returns and random transaction costs incurred during portfolio rebalancing. To define the problem, the entire planning horizon T is divided into two discrete time intervals T_1 and T_2, where $T_1 = 0, 1, \cdots, t, \cdots, \tau$ and $T_2 = \tau + 1, \cdots, T$. The portfolio is structured over this period in terms of both assets' returns and risk, where the risk is measured by the mean absolute deviation, the variance and the maximum negative deviation of assets' returns from expected portfolio return. Period T_1 defines the planning horizon, with the set of periods $I = \{[0, 1), [1, 2), \cdots, [\tau - 1, \tau)\}$. Thus, $t \in I$ refers to the t-th period, with $t = 0$ and $t = \tau$ giving the initial and final stages respectively. During T_1 an investor makes decisions and adjustments to his portfolio at each time-stage as some return realisations unfold. Thus, after the initial investment at $t = 0$, the portfolio may be restructured at discrete times $t = 1, 2, \cdots, \tau$, and after period τ, no further decisions are implemented until investment maturity at $t = T$. This restructuring of the portfolio brings with it transaction costs, as the investor sells or buys shares of some securities to balance his portfolio. A number of articles in the literature proposed models that do not account for these transaction costs. By ignoring such costs and considering only returns realised at time stages t, for $t = 1, 2, \cdots, \tau$, we are overvaluing the portfolio or the portfolio may not be optimal.

3.2 Scenario generation

A set of securities $L = \{i : i = 1, 2, \cdots, n\}$ is considered for an investment. Let $\{R_t\}_{t=1,2,\cdots,\tau} = \{R_1, \cdots, R_\tau\}$ be stochastic security returns at $t = 1, 2, \cdots, \tau$. The decision process is non-anticipative (i.e., a decision at a given stage does not depend on the future realisation of the random events). The decision at period t depends on the outcome at period $t - 1$. A scenario is defined as a possible realization of the stochastic variables $\{R_1, R_2, \cdots, R_\tau\}$. Hence the set of scenarios corresponds to the set of paths followed from the root to the leaves of a tree, S_τ, and nodes of the tree at level $t \geq 1$ corresponds to possible realisations of R_t. Each node at a level t corresponds to a decision which must be determined at time t, and depends in general on R_t, the initial wealth (W_0) of the portfolio and past decisions. Given the event history up to time t, R_t, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observations R_{t+1}. The branching process is represented by a scenario tree. An example of a scenario tree with 2 time periods and three-three branching structure is shown in Figure 3.1 below.

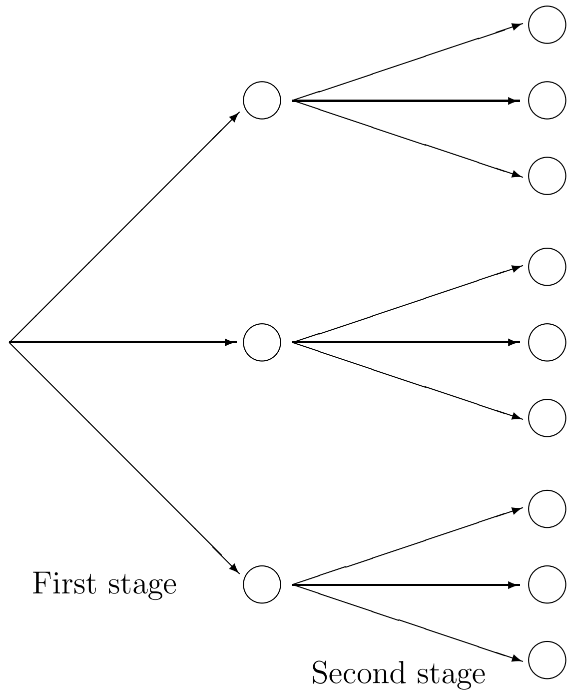

Figure 3.1: Scenarios

The uncertain return of the portfolio at the end of the period t is $R(x_{it}, r_{it})$, where x_{it} is the weight of asset i in period t and r_{it} is the realisation of the random variable R_i during period t. This is a random variable with a distribution function, say F, given by

$$F_{(R,\mu)} = P_r\{R(x_{it}, r_{it}) \leq \mu\}$$

It is assumed that the distribution F depends on the portfolio composition x. The expected return of the portfolio at the end of period t is

$$\bar{r}_{pt} = E[R(x_{it}, r_{it})].$$

In the above equation, p in $\bar{r}_{pt}$ denotes portfolio and t stands for period considered. Suppose the uncertain returns of the assets, R, in period t are represented by a finite set of discrete scenarios $\Omega(t) = \{s : s = 1, 2, \cdots, S(t)\}$, whereby the returns under a particular scenario $s \in \Omega$ take the values $R_s = (R_{1s}, R_{2s}, \cdots, R_{ns})^T$ with associated probability $p_s > 0$, where $\sum_{s\in\Omega} p_s = 1$. The expected portfolio return of period t is now given by

$$\bar{r}_{pt} = \sum_{s=1}^{S} p_s R_{ist} x_{ist}.$$

where R_{ist} is the return of asset i of scenario s in period t, and x_{ist} is the proportion of asset i of scenario s in period t.

3.3 Capital allocation

As the problem is dynamic in nature, the wealth to be invested varies with time. Since it is assumed that the investor joins the market at $t = 0$, with an initial wealth W_0, W_t becomes the wealth at period t. The wealth can be re-allocated among the n-assets at the beginning of each of the τ consecutive time periods for portfolio rebalancing. The rate of asset' return of security i of period t in scenario s within the planning horizon is denoted by $\{R_{ist}\}_{i\in L, s\in\Omega}$ for each stage t. The return R_{ist} has a mean defined by $r_{it} = E(R_{ist}), i \in L, s \in \Omega$, for each period $t \in I$.

Let x_{ist} be the proportion of the risky asset i of scenario s at the beginning of period t. An investor is seeking the best investment strategy, $x_t = [x_{1st}, x_{2st}, \cdots, x_{nst}]^T$ for each $t \in I$, such that

$$\textstyle\sum_{i=1}^{n} x_{it} = \sum_{i=1}^{n} \sum_{s=1}^{S} x_{ist} = 1,$$

where S is the total number of scenarios in period t. Let a_{ist} and v_{ist} be respectively the buying and selling proportions so that $x_{ist} + a_{ist} - v_{ist}$ is the proportion invested in the i^{th} asset of scenario s at the beginning of period t. Thus, we also have

$$\textstyle\sum_{s=1}^{S} (x_{ist} + a_{ist} - v_{ist}) = 1, i \in L, t \in I.$$

Let k_{ist} and l_{ist} be, respectively, the rate of buying and selling transaction costs of the proportion of asset i of scenario s bought or sold for portfolio rebalancing at the beginning of period t. It must be noted that the following are consequences of transaction costs Hakansson [27]:

(a) The fact that transactions have costs ensures that for the same asset, $a_{ist} \cdot v_{ist} = 0$; that is, the buy and sell variables corresponding to the same asset can never be nonzero simultaneously. For example, if an asset is being bought the investor incurs a cost only for the buying transaction and the cost for the selling transaction of the same asset is taken to be zero since the asset is not being sold.

(b) The incorporation of transaction costs in the model provides essential 'friction' which without it, the optimization process has complete freedom to re-allocate the portfolio every time period, which (if implemented) can result in significantly poorer realised performance than forecast, due to excessive transaction costs.

Hakansson [27] explains that in the absence of transaction costs, myopic policies are sufficient to achieve optimality. In models considered in this thesis, money can only be added at $t = 0$, and not in subsequent periods. Thus, only the initial wealth, W_0, is considered.

3.4 Transaction costs and balance constraints

In rebalancing the portfolio at any period $t > 0, t = 1, 2, \cdots, \tau$, buying and selling of securities take place. The buying and selling of securities during initial trading and rebalancing of the portfolio result in the investor incurring some explicit and implicit transaction costs. In this study, transaction costs are taken to mean implicit transaction costs, details of which are explained in section 3.7. The decision made at period t depends on x_{ist} and the yield of the investment in asset i of scenario s. Thus we have

$$r_{it} = R_{ist} \cdot p_s \cdot (x_{ist} + a_{ist} - v_{ist}), i \in L, s \in \Omega, t \in I, \tag{3.1}$$

and

$$0 \leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i} q_{jst}; i \in L, s \in \Omega \tag{3.2}$$

where $b_{ist} = b_{ist}(a_{ist})$ is the amount of money used for buying proportion a_{ist}, and $q_{ist} = q_{ist}(v_{ist})$ is the money obtained from selling proportion v_{ist} of asset i of scenario s in period t. Note here that A is the set containing all assets i for which volumes have been bought. Thus, constraint (3.2) justifies that the amount of money used to buy volumes of asset i should be at most the amount obtained from selling volumes of asset $j, (j \neq i)$ of period t. It is clear that

$$b_{ist} \geq 0, \;\; q_{ist} \geq 0, i \in L, s \in \Omega, t \in I$$

since one can either have money to buy an asset or not, and one can either get money from selling assets or not if there are no sales. Another constraint is given by

$$0 \leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I. \tag{3.3}$$

This constraint explains that the volume of asset i of scenario s in period t sold for portfolio rebalancing should not exceed the volume of the asset in the portfolio. The third constraint is the one discussed above, that is,

$$a_{ist} \cdot v_{ist} = 0, i \in L, s \in \Omega, t \in I. \tag{3.4}$$

It should be noted that the scenarios may reveal identical value for the uncertain quantities up to a certain period. These scenarios that share common information must yield the same decisions up to that period. Thus we have the constraint

$$x_{ist} = x_{is't},$$

for all scenarios s and s' with identical past up to time t.

3.5 Expected Wealth

Generally, the objective of an investor is to minimise portfolio risk while at the same time maximising expected portfolio return on investment, or achieving a prescribed expected return. Thus, the mean rate of return of the portfolio of period t is given by

$$\begin{aligned}\bar{r}_{pt} &= p_s \cdot R_{1st} \cdot (x_{1st} + a_{1st} - v_{1st}) + \cdots + p_s \cdot R_{nst} \cdot (x_{nst} + a_{nst} - v_{nst}) \\ &= \textstyle\sum_{i=1}^{n} p_s \cdot R_{ist} \cdot (x_{ist} + a_{ist} - v_{ist}), s \in \Omega, t \in I.\end{aligned}$$

The wealth of period t, without transaction costs, is given by

$$W_t = (1 + \bar{r}_{pt}) \cdot W_{t-1}, t \in I.$$

Clearly, it can be said that the expected rate of return is a linear function of $(x_{ist} + a_{ist} - v_{ist})$. Taking transaction costs into consideration results in the transaction cost of asset i in scenario s of period t as

$$k_{ist}a_{ist} + l_{ist}v_{ist}$$

where $k_{ist}a_{ist}$ is the cost for buying proportion a of asset i and $l_{ist}v_{ist}$ is the cost of selling proportion v of asset i in period t. Since a_{ist} and v_{ist} cannot be non-zero simultaneously for each asset i (as is shown later), this provides that either

$$k_{ist}a_{ist} = 0 \text{ or } l_{ist}v_{ist} = 0$$

or both are zero if there is no buying or selling of asset i in scenario s of period t. Allowing co-movement of assets' prices and corresponding implicit transaction costs during trading, each scenario asset price is associated with some corresponding implicit transaction cost. Thus the expected transaction cost of asset i in period t is given by

$$\textstyle\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}, i \in L, t \in I.$$

It is therefore observed that the total portfolio transaction cost of period t becomes

$$\textstyle\sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I.$$

Denoting the net expected portfolio return of period t by N_{pt}, we get

$$N_{pt} = \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}].$$

Thus, the wealth of period t taking transaction costs into account is given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1}, t \in I. \tag{3.5}$$

3.6 Lower and Upper Bounds of Variables

In portfolio optimisation, it is sometimes possible for an investor to sell an asset that one does not own. This is called short-selling or simply shorting. An investor borrows an asset which he then sells. Later, when the price of the asset falls on the market, the investor buys the asset and pays back to the lender. Short-selling is only profitable if the asset price declines. It is very risky for investors as potential for loss is great. Thus, bounds U_{ist} on decision variables are put to prevent short-selling and enforce further restrictions imposed by the investor. This yields the following constraint:

$$0 \leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I. \tag{3.6}$$

Constraint (3.6) ensures diversification of the portfolio. It restricts the investor from investing all his wealth in one security, which can be very risky.

3.7 Transaction cost measurement

An investor incurs transaction costs when buying or selling securities on the Stock Market. This takes place during initial trading and when rebalancing the portfolio in subsequent periods. Transaction costs are either explicit or implicit. Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. Implicit costs are embedded in the stock price, and hence are invisible. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of trade execution. These can broadly be put into three categories, namely market impact, opportunity costs and bid-ask spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments [35]. When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed' price that would have prevailed if the trade had not occurred [26].

Thus immediate trade execution is assumed and consider market impact to account for the total implicit costs. The calculation of implicit transaction cost follows that provided by Hau [30]. The spread mid-point benchmark is used and the transaction price is taken to be the last price of the month. The effective spread is calculated as twice the distance from the mid-price measured in basis points. Thus, for the mid-price P^M, mid-point of the bid-ask spread, and transaction price P^T, the effective spread (implicit transaction cost) is obtained as

$$SPREAD^{Trade} = 200 \times \frac{|P^T - P^M|}{P^M}.$$

We consider the transaction price to be the last price of the month for each security. These are prices of securities traded on the Johannesburg Stock Market from 1 January 2008 to 30 September 2012. The historical data is made available courtesy of I-Net Bridge through the University of the Free State's Department of Economics.

3.8 Summary

This chapter has described the general constraints that shall be considered in developing stochastic multi-stage optimal portfolio selection models in the next chapter.

These financial models shall all incorporate uncertainty in implicit trading costs, asset returns and portfolio risk. These models are developed to serve different contexts of situations that arise in real-life.

Chapter 4

Stochastic multi-stage financial optimisation models

Modern portfolio theory is about determining how to distribute capital among available securities such that, for a given level of risk, the expected return is maximised, or for a given level of return, the associated risk is minimised. Risk is any event or action that may adversely affect an organisation's or individual's ability to achieve objectives and execute strategies. Risks in this context, can be broadly grouped into two categories, namely business risks and financial risks. Business risks refer to risks that organisations take on willingly in order to add value to the organisation [39]. A good example is strategic risk. On the other hand, financial risks arise due to possible losses which are entirely market driven. Financial risks can be divided into market risk, credit risk and operational risk. In portfolio optimisation, the most common types of risks studied are the financial risks, and they are the main issue of concern in this work.

4.1 Stochastic Mean Absolute Deviation model with uncertain implicit transaction costs

4.1.1 Introduction

Methods of Operations Research, especially mathematical programming methods, are receiving broader acceptance in the economic and financial industry. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making processes [52]. This research intends to address the portfolio selection problem by applying stochastic programming. A multi-stage stochastic mean absolute deviation model that captures asset returns and risk due to uncertainty is constructed. The mean absolute deviation model was first proposed by Konno and Yamazaki [46], in deterministic form as an alternative to the famous and widely used mean-variance model by Markowitz [60]. Stochastic programming is adopted since it has a number of advantages over other techniques, as explained in section 2.7.

The MAD is a dispersion-type risk linear programming (LP) computable measure that may be considered as an approximation of the variance when the absolute values replace the squares. The use of a linear model considerably reduces the time needed to reach a solution, thereby making it more appropriate for large-scale portfolio selection. It makes extensive calculations of the covariance matrix unnecessary, as opposed to the mean-variance portfolio model. The MAD model is also sensitive to outliers in historical data [8].

4.1.2 Expected portfolio risk

The portfolio risk for any realisation of any period is measured by the mean absolute deviation of the realised returns relative to the expected portfolio return, $\bar{r}_{pt}$. Konno and Yamazaki [46] develop the deterministic mean absolute deviation model with portfolio risk expressed as

$$H_p = \frac{1}{\mathrm{T}} \sum_{t=1}^{T} | \sum_{i=1}^{n} (r_{it} - \bar{r}_{pt}) x_i |$$

where r_{it} is the realised return of asset i of period t, $\bar{r}_{pt}$ is the expected portfolio return per period, x_i is the amount of money invested in asset i, and H_p is the portfolio risk. This formulation of portfolio risk is extended by constructing a model that takes into account uncertainty of asset returns and randomness of transaction costs in portfolio rebalancing. Using the deterministic model as our basis, three key stochastic framework elements are incorporated to formulate the stochastic mean absolute deviation model. The first element considered is the concept of scenarios. Since the deterministic model represents one particular scenario, the inclusion of multiple scenarios in capturing uncertainty results in increase in the number of variable parameters. Thus uncertainty is represented by a set of distinct realisations $s \in \Omega$. This results in consideration of the scenario parameter, along with other parameters of security and time period. Secondly, the stochastic mean absolute deviation model allows for the implementation of recourse decisions as unfolding information on assets' returns gets realised. The third element is the probabilistic feature of the stochastic framework which assigns probabilities to scenarios. The parameter p_s represents scenario probability. Scenarios may reveal identical value for the uncertain quantities up to a certain period. Scenarios that share common information must yield the same decisions up to that period.

The stochastic mean absolute deviation also incorporates transaction costs incurred during portfolio rebalancing at each time period. Thus, the following portfolio risk is obtained:

$$H_p = \frac{1}{\tau} \sum_{t=1}^{\tau} | \sum_{s=1}^{S} p_s (R_{ist} - \bar{r}_{pt}) x_{ist} |, i \in L. \tag{4.1}$$

4.1.3 Multi-stage stochastic MAD model

The multi-stage portfolio selection problem[1] is expressed as a minimisation of portfolio risk subject to constraints describing the growth of the portfolio in all scenarios, a performance constraint, and bounds on the variables.

The final expected portfolio wealth is constrained to a particular value α and the optimization model is intended to find the least risky investment strategy to achieve this expected specified wealth. Alternatively, the same strategy can be achieved by constraining the net expected return to be at least λ, or the gross expected portfolio return to be at least h. Varying λ or h and re-optimising generates a set of optimal portfolios, forming the efficient frontier. The stochastic mean absolute deviation model, which is non-linear, is now stated as follows:

[1] 'Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg Stock Market, International Journal of Operational Research, Vol. 26, No. 2, pp127-152

Minimise

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau}|\sum_{s=1}^{S} p_s(R_{ist} - \bar{r}_{pt})x_{ist}| \tag{4.2}$$

subject to

$$\begin{aligned}
\lambda &\leq N_{p\tau} \\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I, \\
W_t &= (1 + N_{pt})W_{t-1}, t \in I, \\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j\neq i}^{n} q_{ist}, s \in \Omega, t \in I, \\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I, \\
0 &\leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &\leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I \\
0 &= a_{ist} \cdot v_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &\leq b_{ist}, 0 \leq q_{ist}, i \in L, s \in \Omega, t \in I, \\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I,
\end{aligned}$$

for scenarios s and s' with identical past up to period t. The last constraint imposes non-anticipativity to ensure that all scenarios with identical past up to a certain time period have the same decision at that time.The non-linear equation (4.2) shall be transformed into a linear equation at the end of this subsection.

In addition to constraining the final rate of return of the portfolio, constraints of the form

$$N_{pt} \geq \lambda_t, t \in I,$$

must be added to ensure any desired intermediate expected performance. It is assumed that no borrowing is done and the portfolio is self-financing. It deserves mention that the existence of a riskless asset among the securities is regarded as a special case in the stochastic mean absolute deviation formulation (4.2) above.

It has been noted earlier that for each asset i, a_{ist} and v_{ist} cannot simultaneously be non-zero. The following assertion can now be proved.

Theorem 7 (Adapted from [47])
Assume that there are transaction costs of model (4.2), the complementary constraint $a_{ist} \cdot v_{ist} = 0$ can be eliminated from the model.

Proof
Without loss of generality, consider $a_t \cdot v_t = 0$. Let $(x_1^*, \cdots, x_n^*; a_1^*, \cdots, a_n^*; v_1^*, \cdots, v_\tau^*)$ be an optimal solution of (4.2) without the complementary condition, and assume that $a_t \cdot v_t > 0, t \in M \subset I$.

For $t \in M$, let

$$\begin{pmatrix} a'_t \\ v'_t \end{pmatrix} = \begin{pmatrix} a_t^* - v_t^* \\ 0 \end{pmatrix}$$

if $a_t^* \geq v_t^* \geq 0$ and

$$\begin{pmatrix} a'_t \\ v'_t \end{pmatrix} = \begin{pmatrix} 0 \\ v_t^* - a_t^* \end{pmatrix}$$

if $v_t^* \geq a_t^* \geq 0$.
Then $(x_1^*, \cdots, x_n^*; a'_1, \cdots, a'_\tau; v'_1, \cdots, v'_\tau)$ satisfies all the constraints of (4.2). Also it has the same objective as $(x_1^*, \cdots, x_n^*; a_1^*, \cdots, a_\tau^*; v_1^*, \cdots, v_\tau^*)$. This completes the proof.

Thus, problem (4.2) can now be stated without the complementary constraint $a_{ist} \cdot v_{ist} = 0$. Let $y_{st} = (R_{ist} - r_{it})x_{ist}, i \in L, s \in \Omega, t \in I$. Since $R_{ist} = R_{ist}(x_{ist} + a_{ist} - v_{ist})$ and $r_{it} = r_{it}(x_{ist} + a_{ist} - v_{ist})$, it can be stated that $y_{st} = y_{st}(x_{ist} + a_{ist} - v_{ist})$ as well. Thus, formulation (4.2) leads to the following minimisation problem:

Minimise

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau}|\sum_{s=1}^{S} p_s y_{st}|$$

subject to

$$\begin{aligned}
\lambda &\le N_{pt}, t \in I,\\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I,\\
W_t &= (1 + N_{pt})W_{t-1}, t \in I,\\
0 &\le \sum_{i \in A} b_{ist} \le \sum_{j=1, j\neq i}^{n} q_{ist}, s \in \Omega, t \in I,\\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I,\\
0 &\le v_{ist} \le x_{ist}, i \in L, s \in \Omega, t \in I,\\
0 &\le x_{ist} \le U_{ist}, i \in L, s \in \Omega, t \in I,\\
0 &\le b_{ist}, 0 \le q_{ist}, i \in L, s \in \Omega, t \in I,\\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I,
\end{aligned}$$

for scenarios s and s' with identical past up to period t. The above non-linear program is equivalent to the following linear program

Minimise

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau} B_t \tag{4.3}$$

subject to

$$
\begin{aligned}
0 &\leq B_t + \sum_{s=1}^{S} p_s y_{st}, i \in L, t \in I, \\
0 &\leq B_t - \sum_{s=1}^{S} p_s y_{st}, i \in L, t \in I \\
\lambda &\leq N_{pt}, t \in I, \\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n} [\sum_{s=1}^{S} p_s R_{ist} \{k_{ist} a_{ist} + l_{ist} v_{ist}\}], t \in I, \\
W_t &= (1 + N_{pt}) W_{t-1}, t \in I, \\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s \in \Omega, t \in I, \\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I, \\
0 &\leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &\leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &\leq b_{ist}, 0 \leq q_{ist}, i \in L, s \in \Omega, t \in I, \\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I,
\end{aligned}
$$

for scenarios s and s' with identical past up to period t and where

$$B_t \geq |\sum_{s=1}^{S} p_s y_{st}|.$$

The first two constraints ensure that the deviation is absolute.

4.2 Stochastic Multi-stage trading cost model (SMADTC) with uncertain implicit transaction costs

4.2.1 Introduction

In this study, a stochastic multi-stage mean absolute deviation model[2] with trading cost (SMADTC) that minimises implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio is proposed. This is achieved by allowing the investor to choose his or her desired implicit transaction cost value, and portfolio mean rate of return or risk level, where the risk is defined by the mean absolute deviation of assets' returns from expected portfolio return. The multi-stage stochastic transaction cost model captures assets' returns, transaction costs and risk due to uncertainty.

4.2.2 The stochastic multi-stage trading cost (SMADTC) model portfolio risk

The portfolio risk considered in this model is the same as the one given in equation (4.1) which is

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau}|\sum_{s=1}^{S} p_s(R_{ist} - \bar{r}_{pt})x_{ist}|.$$

Let

$$y_{st} = (R_{ist} - \bar{r}_{pt})x_{ist}, i \in L, s \in \Omega, t \in I.$$

Since $R_{ist} = R_{ist}(x_{ist} + a_{ist} - v_{ist})$ and $\bar{r}_{pt} = \bar{r}_{pt}(x_{ist} + a_{ist} - v_{ist})$, the statement $y_{st} = y_{st}(x_{ist} + a_{ist} - v_{ist})$ results. Thus the portfolio risk becomes

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau}|\sum_{s=1}^{S} p_s y_{st}|.$$

[2]'A Stochastic multi-stage trading cost model in optimal portfolio selection, Journal of Economics and Econometrics, Vol. 59, No. 3, pp32-66

This risk function is non-linear, and since it is intended to have a linear programming model, the portfolio risk function is linearised and it becomes

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau} B_t$$

where $B_t \geq |\sum_{s=1}^{S} p_s y_{st}|$.

4.2.3 Multi-stage Stochastic Transaction Cost (SMADTC) Model

The multi-stage stochastic transaction cost model in optimal portfolio selection is presented as a minimisation of implicit trading costs incurred by an investor during initial trading and in subsequent re-balancing of the portfolio. The model is subjected to constraints describing the growth of the portfolio in all scenarios, some performance constraints and bounds on variables.

The final expected wealth is constrained to be at least a particular value, say α, desired by the investor. The portfolio risk, measured by the stochastic mean absolute deviation, is also constrained to be at most a value chosen by the investor, say d. The optimisation model provides an optimal investment strategy that minimises implicit transaction costs while achieving the specified expected wealth and desired risk level. Varying the expected return or risk and re-optimising generates a set of optimal portfolios, forming the efficient frontier. By letting the investor's desired minimum net portfolio return to be at least λ say, the stochastic multi-stage transaction cost model becomes

Minimise

$$C_p = \sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\} \tag{4.4}$$

subject to

$$\begin{aligned}
0 &\le B_t + \sum_{s=1}^{S} p_s y_{st}, i \in L, t \in I, \\
0 &\le B_t - \sum_{s=1}^{S} p_s y_{st}, i \in L, t \in I, \\
\lambda &\le N_{pt} \\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I, \\
W_t &= (1 + N_{pt})W_{t-1}, t \in I, \\
d &\ge \frac{1}{\tau}\sum_{t=1}^{\tau} B_t \\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I, \\
0 &\le \sum_{i \in A} b_{ist} \le \sum_{j=1, j \ne i}^{n} q_{ist}, s \in \Omega, t \in I, \\
0 &\le v_{ist} \le x_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &\le x_{ist} \le U_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &= a_{ist} \cdot v_{ist}, i \in L, s \in \Omega, t \in I, \\
b_{ist} &\ge 0, q_{ist} \ge 0, i \in L, s \in \Omega, t \in I, \\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I,
\end{aligned}$$

where C_p is the total portfolio transaction cost. The first two constraints ensure that the deviation is absolute. In addition to constraining the final rate of return of the portfolio, constraints of the form

$$N_{pt} \geq \lambda_t, t \in I$$

can be added to ensure any desired intermediate expected performance. Again, it is assumed that no borrowing is done and the portfolio is self-financing. It deserves mention that the existence of a riskless asset among the securities is regarded as a special case in the stochastic transaction cost model formulation (4.4) above.

It has been noted earlier that for each asset i, a_{ist} and v_{ist} cannot simultaneously be non-zero. Thus, model (4.4) can now be stated without the complementary constraint $a_{ist} \cdot v_{ist} = 0$ as the following linear program:

Minimise

$$C_p = \sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\} \tag{4.5}$$

subject to

$$
\begin{aligned}
0 &\leq B_t + \sum_{s=1}^{S} p_s y_{st}, i \in L, t \in I,\\
0 &\leq B_t - \sum_{s=1}^{S} p_s y_{st}, i \in L, t \in I,\\
N_{pt} &\geq \lambda\\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I,\\
W_t &= (1 + N_{pt})W_{t-1}, t \in I,\\
d &\geq \frac{1}{\tau}\sum_{t=1}^{\tau} B_t\\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I,\\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s \in \Omega, t \in I,\\
0 &\leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I,\\
0 &\leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I,\\
b_{ist} &\geq 0, q_{ist} \geq 0, i \in L, s \in \Omega, t \in I,\\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I.
\end{aligned}
$$

4.3 Stochastic Maximum Downside risk model with uncertain implicit transaction costs

4.3.1 Introduction

Banks, fund-management firms, financial consulting institutions and large institutional investors are faced with the challenges of managing their funds, assets and stocks towards selecting, creating, balancing and evaluating optimal portfolios on a continual basis. Financial crises, economic imbalances, algorithmic trading and highly volatile movements of asset prices in recent times have raised alarms on the management of financial risks [12]. Extreme event risk is present in all areas of risk management. One of the biggest challenges facing the risk manager is how to implement risk management models that allow for rare but high impact events, and permit the measurement of their consequences.

In financial markets, the stability and sustainability of future pay-offs of an investment are largely determined by extreme changes in financial conditions rather than typical movements. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred in setting up a new portfolio or rebalancing an existing one must be included in any realistic portfolio selection analysis. Investment portfolios should be rebalanced to account for changing market conditions and changes in funding.

In this section, a multi-stage stochastic maximum negative deviation (SMNDTC) model[3] with uncertain implicit transaction costs in optimal portfolio selection is proposed. Maximum negative deviation of asset returns from expected portfolio return is used as portfolio risk. The model takes into account downside risk and corresponding

[3] 'Optimal portfolio selection with stochastic maximum downside risk and uncertain implicit transaction costs, Journal of Economic and Financial Sciences, Vol. 10, No. 3, pp411-423

implicit transaction costs in trading in order to provide an investor or investment manager an option of selecting a portfolio knowing the implicit trading costs which are likely to be incurred. The study uses stochastic programming with recourse. The uncertainty about future asset returns and corresponding implicit transaction costs is captured in stages by means of scenarios. Implicit costs are taken to be random as it is in the buying or selling of assets (whose prices are stochastic) that these costs are incurred.

4.3.2 The SMNDTC model portfolio risk

During the period $[0;\tau]$, the downside risk of asset i of scenario s in period t is defined as

$$M_{ist} = |\min[0, R_{ist} - \bar{r}_{pt}]|.$$

Thus, the expected downside portfolio risk at any time period t becomes $\sum_{s=1}^{S} p_s M_{ist} x_{ist}$. This results in the expected downside portfolio risk H_p, for all time periods as

$$H_p = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s M_{ist} x_{ist}, i \in L.$$

If $\beta_t = \sum_{s=1}^{S} p_s M_{ist} x_{ist}$, the expected portfolio risk for the entire rebalancing phase becomes

$$H_p = \frac{1}{\tau} \sum_{t=1}^{\tau} \beta_t.$$

4.3.3 The multi-stage stochastic maximum negative deviation (SMNDTC) model

The objective in this problem is to obtain an optimal portfolio that minimises the expected portfolio risk subject to constraints describing the growth of the portfolio in all periods, a performance constraint and bounds on decision variables. Letting λ to be the investor's minimum desired expected portfolio mean return and θ to be the minimum acceptable transaction cost, the following optimisation model is obtained:

Minimise

$$H_p = \frac{1}{\tau}\sum_{t=1}^{\tau}\beta_t \tag{4.6}$$

subject to

$$
\begin{aligned}
W_t &= (1+N_{pt})W_{t-1}, t \in I,\\
\lambda &\leq N_{pt}, t \in I,\\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I,\\
0 &= \beta_t - \sum_{s=1}^{S} p_s M_{ist}x_{ist}, i \in L, t \in I,\\
\theta &\geq \sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}, i \in L, t \in I,\\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I,\\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j\neq i}^{n} q_{ist}, s \in \Omega, t \in I,\\
0 &\leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I,\\
0 &\leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I,\\
b_{ist} &\geq 0, q_{ist} \geq 0, i \in L, s \in \Omega, t \in I,\\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I.
\end{aligned}
$$

The model (4.6) has a non-linear objective function and the third constraint is also non-linear. The model is transformed into a linear stochastic programming model as follows. For each scenario s, let

$$K_{ist} \geq M_{ist} = |\min[0, R_{ist} - \bar{r}_{pt}]|, i \in L, t \in I.$$

Then the expected portfolio risk becomes

$$J_p = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s K_{ist} x_{ist}, i \in L,$$

where the expected portfolio risk at any time period t is given by $\sum_{s=1}^{S} p_s M_{ist} x_{ist}$. If $Z_t = \sum_{s=1}^{S} p_s K_{ist} x_{ist}$, then the expected portfolio risk for the period $[0, \tau]$ is

$$J_p = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t.$$

The programming model (4.6) above is then transformed into the following linear stochastic model:

Minimise

$$J_p = \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t \tag{4.7}$$

subject to

$$\begin{aligned}
W_t &= (1 + N_{pt})W_{t-1}, t \in I,\\
\lambda &\leq N_{pt}, t \in I,\\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I,\\
0 &= Z_t - \sum_{s=1}^{S} p_s K_{ist}x_{ist}, i \in L, t \in I,\\
K_{ist} &\geq M_{ist}, i \in L, s \in \Omega, t \in I,\\
\theta &\geq \sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}, i \in L, t \in I,\\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I,\\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j\neq i}^{n} q_{ist}, s \in \Omega, t \in I,\\
0 &\leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I,\\
0 &\leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I,\\
b_{ist} &\geq 0, q_{ist} \geq 0, i \in L, s \in \Omega, t \in I,\\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I.
\end{aligned}$$

The following theorem shows that models (4.6) and (4.7) yield the same optimal values.

Theorem 8

If x^* is an optimal solution to (4.6), then (x^*, J_p^*) is an optimal solution to (4.7). Conversely, if (x^*, J_p^*) is an optimal solution to (4.7), then x^* is an optimal solution to (4.6).

Proof

Without loss of generality, let $x^* = x_{ist}^*$, $J = J_p$ and $J^* = J_p{}^*$. If x^* is an optimal solution to (4.6), then (x^*, J^*) is a feasible solution to (4.7), where

$$J^* = \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist}^* \geq \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s M_{ist} x_{ist}^*.$$

If (x^*, J^*) is not an optimal solution to (4.7), then there exists a feasible solution (x, J) to (4.7) such that $J < J^*$, where

$$\begin{aligned} J &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist} \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - \bar{r}_{pt}] \cdot | x_{ist}. \end{aligned}$$

It is observed that

$$\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - \bar{r}_{pt}] \cdot | x_{ist} = J < J^*$$

and that

$$J < J^* = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - \bar{r}_{pt}] \cdot | x_{ist}^*.$$

This is a contradiction since x^* is an optimal solution to (4.6).

Conversely, if $(x^*, J^*)$is an optimal solution to (4.7), where

$$J^* = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| \cdot x_{ist}^*,$$

then x^* is an optimal solution of (4.6). Otherwise there exists a feasible solution x to (4.6) such that

$$\begin{aligned} J &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist} \\ &\leq \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| \cdot x^*_{ist} \\ &= J^* \end{aligned}$$

which contradicts that (x^*, J^*) is an optimal solution to (4.7). This completes the proof.

4.4 Stochastic modeling of Investment opportunities, Uncertain Implicit Transaction costs and Maximum Downside Risk: SMUDTC model

4.4.1 Introduction

Individual investors, investment managers and fund managers are all concerned with achieving optimal portfolios of a set of investment assets. Models for portfolio selection have evolved over the years starting with Markowitz [60] mean-variance formulation to more recent stochastic optimisation forms ([32]; [84]). Regardless of whether portfolios are selected for a bank's derivative mix, an investor's equity holdings or a firm's asset and liability management, the common objective in all models is the minimisation of some measure of risk while maximising some reward measure. The mean-variance (MV) model has enjoyed popularity over the years despite its criticisms. The mean-variance portfolio analysis has the following simplifying assumptions:

(i) the assets' returns are multivariate normally distributed,

(ii) the investor's utility function is quadratic, and

(iii) there are no transaction costs.

None of these is exactly true in actual markets. Many studies show that returns from hedge funds are not normally distributed [7]. Pratt [69] concludes that a quadratic utility function is very unlikely because it implies increasing absolute risk aversion. Volatility as a risk measure treats risks and opportunities equally yet investors are concerned about downside deviations (losses) and are happy of upside deviations (gains). Hakansson [27] explains that in the absence of transaction costs, myopic policies are sufficient to achieve optimality. The incorporation of transaction costs in any model provides essential friction which without it, the optimisation has complete freedom to reallocate the portfolio every time-period, which (if implemented) can result in significantly poorer realised performance than forecast, due to excessive transaction costs [27]. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments [35]. In this part of the study, a stochastic multi-stage upside-downside deviation (SMUDTC) model[4] with uncertain transaction costs is proposed that takes into account a risk-averse investors view of minimising maximum downside risk while maximising upside deviations (gains) in an uncertain environment. The model captures uncertainty in both portfolio risk and gain by way of scenarios, which is a representative and comprehensive set of possible realisations of the future. This is achieved by taking deviations around each return scenario. The SMUDTC model also takes into account uncertainty of implicit trading costs incurred by the investor during initial trading and in subsequent rebalancing of the portfolio.

4.4.2 SMUDTC model portfolio risk

The study assumes that the investor intends to choose a portfolio with minimum of the maximum downside deviations of asset returns relative to expected portfolio return. Let the downside risk of asset i of scenario s in period t be defined by

$$M_{ist} = |\min[0, R_{ist} - \bar{r}_{pt}]|.$$

This gives the expected portfolio risk in period t as $\sum_{s=1}^{S} p_s M_{ist} x_{ist}$. The expected portfolio risk for the period $[0, \tau]$ becomes

$$\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s M_{ist} x_{ist}, i \in L.$$

Letting $Z_t = \sum_{s=1}^{S} p_s M_{ist} x_{ist}$ results in the expected portfolio risk for the entire planning phase as

$$H_p = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t.$$

[4]‘Investment opportunities, uncertain implicit transaction costs and maximum downside risk in dynamic stochastic financial optimization, International Journal of Economic and Financial Issues, Vol. 8, No. 4, pp256-264

4.4.3 SMUDTC model portfolio gain

Portfolio gain is defined as an upside deviation of an asset return from expected portfolio return at any period t. The study assumes that the investor wants to maximise upside deviations of asset returns resulting in better investment opportunities. Let the upside deviation of asset i of scenario s of period t be defined by

$$\Upsilon_{ist} = \max[0, R_{ist} - \bar{r}_{pt}].$$

This gives the expected upside deviation of the portfolio of period t as $\sum_{s=1}^{S} p_s \Upsilon_{ist} x_{ist}$. Hence the expected upside deviation of the portfolio for all time-periods becomes

$$\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s \Upsilon_{ist} x_{ist}, i \in L.$$

If $Y_t = \sum_{s=1}^{S} p_s \Upsilon_{ist} x_{ist}$, then the expected portfolio gain for the entire rebalancing period is given by

$$\phi_p = \frac{1}{\tau} \sum_{t=1}^{\tau} Y_t.$$

4.4.4 The multi-stage stochastic optimization (SMUDTC) model

There is a bi-criteria objective to be satisfied in the study by maximising portfolio gain while minimising maximum downside risk. Since the intention of the investor is to minimise portfolio risk, it is achieved by maximising the negative of the downside risk, that is, maximising $-H_p = -\frac{1}{\tau} \sum_{t=1}^{\tau} Z_t$. This results in a single objective where the investor maximises the sum of the negative loss and gain of the portfolio of any period t. Thus, the objective function becomes

$$\text{Maximise} \qquad (\phi_p - H_p) = \frac{1}{\tau} \{ \sum_{t=1}^{\tau} Y_t - \sum_{t=1}^{\tau} Z_t \}.$$

Letting θ to be the minimum acceptable transaction cost and λ to be the minimum expected portfolio return, the following multi-stage stochastic model with uncertain implicit transaction costs is obtained:

Maximise

$$V_p = \frac{1}{\tau}\{\sum_{t=1}^{\tau} Y_t - \sum_{t=1}^{\tau} Z_t\} \tag{4.8}$$

subject to

$$\begin{aligned}
W_t &= (1 + N_{pt})W_{t-1}, t \in I, \\
\lambda &\leq N_{pt}, t \in I, \\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I, \\
0 &= Z_t - \sum_{s=1}^{S} p_s M_{ist} x_{ist}, i \in L, t \in I, \\
0 &= -Y_t - \sum_{s=1}^{S} p_s \Upsilon_{ist} x_{ist}, i \in L, t \in I, \\
\theta &\geq \sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}R_{ist}, i \in L, t \in I, \\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I, \\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s \in \Omega, t \in I, \\
0 &\leq v_{ist} \leq x_{ist}, i \in L, s \in \Omega, t \in I, \\
0 &\leq x_{ist} \leq U_{ist}, i \in L, s \in \Omega, t \in I, \\
b_{ist} &\geq 0, q_{ist} \geq 0, i \in L, s \in \Omega, t \in I, \\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I.
\end{aligned}$$

The model (4.8) has a non-linear objective function, and the third and fourth constraints are also non-linear. In order to transform the model into a stochastic linear programming model, the following changes are made.

For each scenario s, let

$$K_{ist} \geq M_{ist} = |\min[0, R_{ist} - \bar{r}_{pt}]|, i \in L, t \in I.$$

This gives the expected portfolio risk of period t as $\sum_{s=1}^{s} p_s K_{ist} x_{ist}$, with expected portfolio risk for the period $[0, \tau]$ becoming $\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{s} p_s K_{ist} x_{ist}$. Then letting $\Psi_t = \sum_{s=1}^{S} p_s K_{ist} x_{ist}, i \in L, t \in I$, gives the expected portfolio risk for the entire planning phase as

$$E = \frac{1}{\tau} \sum_{t=1}^{\tau} \Psi_t.$$

Similarly, for each scenario s, let

$$D_{ist} \geq \Upsilon_{ist} = \max[0, R_{ist} - \bar{r}_{pt}], i \in L, t \in I.$$

This results in $\sum_{s=1}^{S} p_s D_{ist} x_{ist}$ as the expected upside deviation of the portfolio of period t. Therefore, the expected upside deviation (gain) of the portfolio for the period $[0, \tau]$ becomes

$$\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^{S} p_s D_{ist} x_{ist}, i \in L.$$

By letting $Q_t = \sum_{s=1}^{S} p_s D_{ist} x_{ist}$, the expected portfolio gain for the entire planning phase becomes

$$F = \frac{1}{\tau} \sum_{t=1}^{\tau} Q_t.$$

The model (4.8) becomes equivalent to the following stochastic linear programming model.

Maximise

$$\Phi_p = \frac{1}{\tau} \{ \sum_{t=1}^{\tau} Q_t - \sum_{t=1}^{\tau} \Psi_t \} \tag{4.9}$$

subject to

$$
\begin{aligned}
W_t &= (1+N_{pt})W_{t-1}, t \in I,\\
\lambda &\le N_{pt}, t \in I,\\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I,\\
0 &= \Psi_t - \sum_{s=1}^{S} p_s K_{ist} x_{ist}, i \in L, t \in I,\\
0 &= -Q_t - \sum_{s=1}^{S} p_s D_{ist} x_{ist}, i \in L, t \in I,\\
K_{ist} &\ge M_{ist}, i \in L, s \in \Omega, t \in I,\\
D_{ist} &\ge \Upsilon_{ist}, i \in L, s \in \Omega, t \in I,\\
\theta &\ge \sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}R_{ist}, i \in L, t \in I,\\
1 &= \sum_{s=1}^{S} x_{ist}, i \in L, t \in I,\\
0 &\le \sum_{i \in A} b_{ist} \le \sum_{j=1, j\neq i}^{n} q_{ist}, s \in \Omega, t \in I,\\
0 &\le v_{ist} \le x_{ist}, i \in L, s \in \Omega, t \in I,\\
0 &\le x_{ist} \le U_{ist}, i \in L, s \in \Omega, t \in I,\\
b_{ist} &\ge 0, q_{ist} \ge 0, i \in L, s \in \Omega, t \in I,\\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I.
\end{aligned}
$$

The following Theorem shows that models (4.8) and (4.9) are equivalent and yield the same optimal values.

Theorem 9

If x^* is an optimal solution to (4.8), then (x^*, E^*, F^*) is an optimal solution to (4.9). Conversely, if (x^*, E^*, F^*) is an optimal solution to (4.9), then x^* is an optimal solution to (4.8).

Proof

Without loss of generality, let $x^* = x^*_{ist}$. If x^* is an optimal solution to (4.8), then (x^*, E^*, F^*) is a feasible solution to (4.9), where

$$\begin{aligned} E &= \frac{1}{\tau}\sum_{t=1}^{\tau}\Psi_t \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist} \\ &\geq \frac{1}{\tau}\sum_{t=1}\tau\sum_{s=1}^{S} p_s M_{ist} x_{ist} \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| \end{aligned}$$

and

$$\begin{aligned} F &= \frac{1}{\tau}\sum_{t=1}^{\tau} Q_t \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s D_{ist} x_{ist} \\ &\geq \frac{1}{\tau}\sum_{t=1}\tau\sum_{s=1}^{S} p_s \Upsilon_{ist} x_{ist} \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot (\max[0, R_{ist} - \bar{r}_{pt}]). \end{aligned}$$

If (x^*, E^*, F^*) is not an optimal solution to (4.8), then there exists a feasible solution (x, E, F) such that $E^* < E$ and $F^* < F$, where

$$\begin{aligned} E &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist} \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| \end{aligned}$$

and

$$\begin{aligned} F &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s D_{ist} x_{ist} \\ &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot (\max[0, R_{ist} - \bar{r}_{pt}]). \end{aligned}$$

It is observed that

$$\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x^*_{ist} \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| = E^* < E$$

and that

$$E^* < E = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - \bar{r}_{pt}]|.$$

Also

$$\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x^*_{ist} \cdot \max[0, R_{ist} - \bar{r}_{pt}] = F^* < F$$

and

$$F^* < F = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot \max[0, R_{ist} - \bar{r}_{pt}].$$

This is a contradiction since x^* is an optimal solution to (4.8).

Conversely, if (x^*, E^*, F^*) is an optimal solution to (4.9), then x^* is an optimal solution of (4.8). Otherwise, there exists a feasible solution x to (4.8) such that

$$\begin{aligned} E^* &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x^*_{ist} \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| \\ &< \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot |\min[0, R_{ist} - \bar{r}_{pt}]| \\ &= E \end{aligned}$$

and

$$
\begin{aligned}
F^* &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x^*_{ist} \cdot (\max[0, R_{ist} - \bar{r}_{pt}]) \\
&< \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot x_{ist} \cdot (\max[0, R_{ist} - \bar{r}_{pt}]) \\
&= F
\end{aligned}
$$

which contradicts that (x^*, E^*, F^*) is an optimal solution to (4.9). This completes the proof.

4.5 Stochastic mean-variance model (SMVTC) with uncertain implicit transaction costs

4.5.1 Introduction

Constructing a portfolio of investments is one of the most significant financial decisions facing individual investors, financial managers and institutions. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred in setting up a new portfolio or rebalancing an existing one must be included in any realistic portfolio selection analysis. Investment portfolios should be rebalanced to take account of changing market conditions and changes in funding. In this section, the famous mean-variance portfolio optimisation problem is extended to incorporate uncertainty of future asset returns and uncertainty in implicit transaction costs incurred during initial trading and when rebalancing the portfolio in subsequent times.

Mean-variance (MV) analysis was introduced by Harry Markowitz [60], who describe the basic formulations and the quadratic programming tools used to solve them. Mean-variance portfolio theory is based on the idea that the value of investment opportunities can be meaningfully measured in terms of mean return and variance of the return. This mean-variance analysis is based on the following assumptions:

(i) All investors are risk-averse; they prefer less risk to more for the same level of expected return,

(ii) expected returns for all assets are known,

(iii) the variances and covariances of all asset returns are known, and

(iv) there are no transaction costs or taxes.

Furthermore, mean-variance analysis is based on single-period model of investment. In this study, an optimal portfolio policy of a multi-period stochastic mean-variance investor subject to uncertain returns and uncertain implicit transaction costs is proposed.

4.5.2 Stochastic mean-variance (SMVTC) model expected portfolio risk

In the portfolio selection model proposed by Markowitz [60], the variance is used to represent the risk of the portfolio. Denoting x_i to be the proportion invested in stock i, $i \in L$, where $\sum_{i=1}^{n} x_i = 1, X = [x_1, x_2, \cdots, x_n]^T$ is called a portfolio. If we let R_i to be the return of asset i, the mean-variance model has portfolio risk (σ_p^2) expressed as

$$\sigma_p^2 = \frac{1}{n-1}\{\sum_{i=1}^{n}(R_i x_i - E[R_i x_i])^2\} = \frac{1}{n-1}\{\sum_{i=1}^{n}(R_i x_i - \alpha)^2\},$$

where $\alpha = E[R_i x_i]$. This formulation of risk is in deterministic form since it takes into account only one scenario of an asset's return. It is therefore extended by considering uncertainty of asset returns and uncertainty of implicit transaction costs in initial trading and in subsequent rebalancing of the portfolio. Thus incorporating into the deterministic formulation, three key stochastic framework elements of scenario s, time period t and probability p_s, as explained in earlier sections. Scenarios that share common information in a period must yield the same decisions up to that period. The stochastic variance also incorporates implicit transaction costs incurred during initial trading and in recourse periods. By considering the variance around each return scenario, the portfolio risk (variance) at each time period is obtained as

$$\sigma_p^2 = \frac{1}{n-1}\{\sum_{s=1}^{S}(p_s R_{ist} x_{ist} - E[R_{ist} x_{ist}])^2\} = \frac{1}{n-1}\sum_{s=1}^{S}\{p_s R_{ist} x_{ist} - \bar{r}_{pt}\}^2, i \in L, t \in I,$$

where $\bar{r}_{pt} = E[R_{ist}x_{ist}] = \sum_{s=1}^{S} p_s R_{ist} x_{ist}$. The portfolio risk over t- time periods becomes

$$\sigma_p^2 = \frac{1}{\tau(n-1)} \sum_{t=1}^{\tau} \sum_{s=1}^{S} \{p_s R_{ist} x_{ist} - \bar{r}_{pt}\}^2, i \in L,$$

where R_{ist}, $\bar{r}_{pt}$ and x_{ist} are as defined in earlier subsections. Let

$$G_t = \sum_{s=1}^{S} [p_s R_{ist} x_{ist} - \bar{r}_{pt}]^2, i \in L, t \in I.$$

The portfolio risk then becomes

$$H_p = \frac{1}{\tau(n-1)} \sum_{t=1}^{\tau} G_t.$$

4.5.3 The Multi-stage Stochastic Mean-Variance Model

The multi-stage stochastic portfolio selection model[5] is proposed as a minimisation of portfolio risk (variance) subject to constraints describing the growth of the portfolio in all scenarios, performance constraints and bounds on variables. The optimisation model provides an optimal investment strategy that minimises portfolio variance while achieving a prescribed portfolio mean return and the desired transaction cost level. By varying expected return or risk and re-optimizing yields a set of optimal portfolios comprising the efficient frontier. By letting the investor's desired minimum net portfolio mean return to be at least λ and the desired transaction cost to be at most θ, say, the following multi-stage stochastic mean-variance model with uncertain transaction costs is obtained:

Minimise

$$H_p = \frac{1}{\tau(n-1)} \sum_{t=1}^{\tau} G_t \tag{4.10}$$

[5]'Optimal portfolio selection with uncertain implicit transaction costs in a dynamic stochastic mean-variance framework', Studies in Nonlinear Dynamics and Econometrics (Under Review)

subject to

$$
\begin{aligned}
N_{pt} &\geq \lambda \\
N_{pt} &= \bar{r}_{pt} - \sum_{i=1}^{n}[\sum_{s=1}^{S} p_s R_{ist}\{k_{ist}a_{ist} + l_{ist}v_{ist}\}], t \in I, \\
W_t &= (1 + N_{pt})W_{t-1}, t \in I, \\
\theta &\geq \sum_{s=1}^{S} p_s\{k_{i,s,t}a_{i,s,t} + l_{i,s,t}v_{i,s,t}\}R_{ist}, i \in L, t \in I, \\
1 &= \sum_{s=1}^{S} x_{i,s,t}, i \in L, t \in I, \\
0 &\leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s \in \Omega, t \in I, \\
0 &\leq v_{i,s,t} \leq x_{i,s,t}, i \in L, s \in \Omega, t \in I, \\
0 &\leq x_{i,s,t} \leq U_{i,s,t}, i \in L, s \in \Omega, t \in I, \\
b_{ist} &\geq 0, q_{ist} \geq 0, i \in L, s \in \Omega, t \in I, \\
x_{ist} &= x_{is't}, i \in L, s \in \Omega, t \in I.
\end{aligned}
$$

It is assumed that at $t > 0$, no borrowing is done and the portfolio is self-financing. It also deserves mention here that the existence of a riskless asset among the securities is a special case in the stochastic multi-stage model. It has been noted earlier that for each asset i, a_{ist} and v_{ist} can not be non-zero simultaneously. Thus the model above can now be stated without the complementary constraint $a_{ist} \cdot v_{ist} = 0$.

4.6 Summary

The chapter has demonstrated how each proposed financial model is developed. Proofs to some concepts have been provided as expected in any development of mathematical theory. The next chapter validates the proposed financial models by using real market data from the Johannesburg Stock Market. Simulation of historical data is done.

Chapter 5

Empirical analysis of the stochastic financial optimization models

5.1 Introduction

Monthly historical securities' data from the Johannesburg Stock Market from January 2008 to September 2012 are collected. These historical data have been obtained courtesy of INet Bridge through University of the Free State's Department of Economics. The data are shown in tables in the Appendix and they comprise mean asset returns (Table A1) and asset transaction cost rates (Table A2). The performance of each of the proposed models is evaluated by comparing portfolios developed from them and the MV, MAD and MM models. The following criteria are used in the selection of stocks to comprise our initial portfolio:

(a) Stocks with negative mean returns for the entire period considered are excluded from the sample,

(b) Companies which were not on the list by January 2008 and only entered the JSE afterwards are excluded,

(c) Those assets having the highest positive mean returns are taken to become our initial portfolio.

Empirical distributions of asset returns in the study are computed from past monthly returns and are taken as equi-probable scenarios. A scenario, R_{ist}, for the return of asset i of period t is obtained as

$$R_{ist} = \frac{P_{i,t}}{P_{i,t-1}} - 1$$

where $P_{i,t}$ is the historical monthly price of asset i.

In the first stage of the analysis, 13 securities are chosen to comprise the initial portfolio. The mean returns, total implicit transaction costs and total wealth of each of the four optimal portfolios developed according to models 2.8, 2.9, 2.10 (i.e., MV, MAD, MM) and the respective proposed model are calculated and then compared.

In using the proposed models, five scenarios are considered for each asset return and the corresponding transaction cost, and each model is applied over one and / or two stages. However, comparison with other models is restricted to the first stage. Empirical distributions of the 13 securities are taken to comprise the initial portfolio. Since for each security there are 54 monthly returns, the months are numbered from 1 to 54, and random numbers used to select asset returns and associated transaction costs corresponding to scenarios of a security. Implicit transaction costs calculated from the effective bid-ask spread corresponding to each selected asset return are considered. It is assumed that these transaction costs are random since they are randomly selected together with corresponding asset' returns. Thus, a scenario comprises an asset return and the associated transaction cost. The implicit cost for each of the 13 assets is taken into account as it is in the buying or selling of each asset that the cost is incurred. The transaction cost is given as a rate. Each scenario is taken to be equally likely to occur.

It should be noted that scenarios are only considered for the proposed stochastic models. Other models use mean returns and risks calculated over the whole period under examination. First-stage optimal portfolios of all models are compared. For the proposed stochastic models, five scenarios are considered for each asset return and its corresponding implicit transaction cost at each time period, each with a probability of occurring of $\frac{1}{5}$. This consideration is taken noting that in stochastic programming, the scenario tree grows exponentially. At the end of the first stage, an investor decides on his first-stage optimal portfolio as given by the investor's chosen portfolio transaction cost, the diversification limit, the gross portfolio mean return or the net portfolio mean return as the case may be, and the associated risk as specified in the respective stochastic model under consideration. General Algebraic Modeling System (GAMS) software is used in the analysis to obtain optimal portfolios.

5.2 SMAD model application and analysis of results

The stochastic mean absolute deviation (SMAD) model is used to obtain optimal portfolios which are then compared with portfolios from the mean-variance (MV), mean absolute deviation (MAD) and minimax (MM) models. The analysis is presented in the following sections.

5.2.1 Stage 1

An investor who has R10000 to spend on his initial portfolio is considered. It has emerged that diversifying the investor's portfolio by decreasing the diversification limit from 0.4 to 0.1 results in the gross mean portfolio return remaining fixed at 0.008 and the risk becoming very small in each case. However, the net mean portfolio return fluctuates between 0.008 and 0.011, reflecting fluctuating transaction costs. Table 5.1 below has this information where the abbreviation 'D. L' means 'diversification limit' and the expression 'assets' stands for 'number of assets'.

Table 5.1: First Stage Optimal Portfolios

D.L	Gross mean	Assets	Net mean	MAD	Wealth
0.100	0.013	11	0.011	2.1684×10^{-19}	10114.320
0.125	0.013	12	0.011	2.40551×10^{-18}	10112.736
0.150	0.013	10	0.011	1.5992×10^{-18}	10111.780
0.175	0.013	10	0.011	2.6766×10^{-19}	10107.344
0.200	0.013	6	0.008	4.3368×10^{-19}	10081.526
0.225	0.013	6	0.008	1.0842×10^{-19}	10081.891
0.250	0.013	8	0.009	2.3039×10^{-19}	10085.628
0.275	0.013	10	0.008	0	10081.730
0.300	0.013	7	0.011	2.50722×10^{-19}	10105.436
0.325	0.013	6	0.010	7.2844×10^{-20}	10098.718
0.350	0.013	7	0.011	1.4×10^{-18}	10106.596
0.375	0.013	4	0.011	2.710×10^{-19}	10108.360
0.400	0.013	4	0.011	2.168×10^{-19}	10108.360

Each of the diversification limits is then fixed and the gross mean portfolio return allowed to increase from 0.013 to values when the portfolio becomes 'saturated', i.e., no more buying and selling of assets. Thus creating a set of optimal portfolios for each diversification limit considered. It is observed that, regardless of the diversification limit chosen, the net mean portfolio return for each chosen gross mean portfolio return is the same for all diversification limits considered. Thus transaction costs are the same for each gross mean portfolio return selected. Table $A2$ in Appendix 2 has this information. It is also evident that even the risk assumes the same values for each gross mean portfolio return considered for all diversification limits, except at the portfolio saturation point. Thus the efficient frontiers at the various diversification limits do not differ much. However, it emerges that the maximum wealth is achieved at the optimal portfolio saturation point for each diversification limit. This is also where the risk is highest. Efficient frontiers of net mean portfolio returns and gross mean portfolio returns reveal the impact of neglecting transaction costs in portfolio selection. The gap between the two frontiers shows the amount of transaction costs incurred by the investor when portfolio selection is done without taking implicit transaction costs into account. It is therefore up to each investor to choose the wealth-risk-diversification limit combination he or she desires.

5.2.2 Stage 2

As in stage 1, the diversification limit is allowed to vary from 0.2 to 0.4 as shown in Table 5.2 below. A first-stage optimal portfolio with five securities is considered. The results show that optimal portfolios in stage 2 have the same gross mean portfolio return and the same net mean portfolio return, regardless of the diversification limit. Here, it is noted that transaction costs for the optimal portfolios are the same. It is also observed that the risk assumes the same value at each diversification limit. Small variations in the expected wealth of optimal portfolios is due to rounding error of the net mean portfolio return during execution of the program. Similar results are observed with all Stage 1 optimal portfolios.

It is evident from the Table 5.2 findings that each optimal portfolio at each diversification limit considered, incurs an overall implicit transaction cost of $\frac{1}{7}$ of the returns on investment. Thus, implicit transaction costs amount to approximately 14.3% of the fund invested during initial trading.

Optimal portfolios at each diversification limit are also evaluated, the results of which are shown in Table $A4$ in Appendix 2. For each gross mean portfolio return, the net mean portfolio returns are the same and also equal are the associated risks. Unlike in stage 1, expected portfolio wealth is almost invariant for each pair of gross mean portfolio return and risk, and for each diversification limit. Thus efficient frontiers are almost the same regardless of diversification limit.

Table 5.2: Second Stage Optimal Portfolios

D.L	Gross mean	No. of Assets	Net mean	MAD(risk)	Wealth
0.200	0.007	5	0.006	0.013	10423.815
0.225	0.007	5	0.006	0.013	10424.277
0.250	0.007	5	0.006	0.013	10425.228
0.275	0.007	5	0.006	0.013	10424.843
0.300	0.007	5	0.006	0.013	10423.679
0.325	0.007	5	0.006	0.013	10424.327
0.350	0.007	5	0.006	0.013	10424.321
0.375	0.007	5	0.006	0.013	10424.290
0.400	0.007	5	0.006	0.013	10424.259

5.2.3 Comparison of Stage 1 and Stage 2 optimal portfolios

The in-sample analysis explains the advantages of portfolio rebalancing as depicted from the efficient frontiers of two sets of optimal portfolios in which the diversification limit is the same. Stage 2 portfolios are superior, hence the need for portfolio rebalancing. It is again clear that the implicit transaction costs incurred when buying initial portfolio and during portfolio rebalancing have a bearable impact on portfolio returns. Hence, ignoring transaction costs results in exagerated optimal portfolios.

5.2.4 Summary

In this part of the study, a multi-stage stochastic mean absolute deviation model with random transaction costs in optimal portfolio selection is proposed. Contributions of the study are viewed to include

(i) the development of a stochastic MAD model that optimises portfolios in the presence of random implicit transaction costs, and captures risk due to uncertainty,

(ii) the development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios.

It is found, from the data sample considered, that implicit transaction costs for investing in a fund at the Johannesburg Stock Market amounts to approximately 14.3% of returns on investment. The methodology allows investors and investment managers to choose optimal portfolios realising the impact of associated transaction costs. It is a linear programming (LP) model, and hence reduces considerably the time needed to reach a solution. It is therefore feasible for large-scale portfolio selection.

5.3 SMADTC model application and analysis of results

In this section, the stochastic mean absolute deviation with transaction costs (SMADTC) model is applied to data on securities from the Johannesburg Stock Market to obtain optimal portfolios. The results of the stochastic multi-stage trading cost model are given over two stages as outlined below. Again, as in the previous section, the performance of the SMADTC model is compared with the MV, MAD and MM models' performances.

5.3.1 Stage 1

As in above section, an investor who has R10000 to spend on his initial portfolio is considered. The optimal portfolios generated by the four models are shown in Table 5.3 below. The abbreviation 'D.L' stands for 'diversification limit'.
The information in Table 5.3 reveals that the portfolios generated by the MM model have the highest gross mean returns for each diversification limit considered, followed by the MV model. The SMADTC model generates optimal portfolios with least gross mean returns, up to when the diversification limit is 0.25. The portfolio gross mean returns from the MAD and the MV models show a decreasing trend as the portfolios become less diversified. The MM and SMADTC generated portfolios have gross mean returns that increase as portfolios become less diversified.

Although the MM model appears to generate optimal portfolios with highest gross wealth, it is evident that these portfolios also have the highest implicit transaction costs, resulting in the least net portfolio wealth. The SMADTC model generates optimal portfolios with the least transaction costs per each diversification limit. The MAD model portfolios have transaction costs that start being greater than those of MV portfolios, but becoming less as portfolios become less diversified.

The greatest advantage of the SMADTC lies in its ability to generate an optimal portfolio whose net wealth is always more than amount invested. The other models generate portfolios which are not optimal investments as intended because of the eroding effect of implicit transaction costs. The information also reveals that implicit transaction costs in SMADTC portfolios decline as the portfolio becomes less diversified. However, portfolio risk increases instead. It is also evident that portfolio wealth increases with decrease in portfolio diversification. The risk used in the analysis is given in subsection 4.2.2.

Table 5.3: Summary Statistics of Optimal Portfolios

D.L	Model	Mean return	Risk	Gross Wealth	Cost	Net Wealth
0.10	MAD	0.027	0.002	10269.80	432.00	9837.80
	MV	0.029	0.002	10289.02	426.34	9862.68
	MM	0.031	0.031	10314.80	502.10	9812.70
	SMADTC	0.014	0.001	10138.09	3.71	10134.38
0.15	MAD	0.025	0.022	10251.75	479.10	9772.65
	MV	0.028	0.002	10277.97	397.32	9880.65
	MM	0.033	0.033	10332.30	411.15	9921.15
	SMADTC	0.018	0.005	10174.99	2.81	10172.18
0.20	MAD	0.024	0.023	10240.20	223.20	10017.00
	MV	0.027	0.002	10268.25	378.64	9889.61
	MM	0.035	0.035	10346.80	503.60	9843.20
	SMADTC	0.019	0.006	10187.44	2.16	10185.28
0.25	MAD	0.023	0.025	10229.25	248.50	9980.75
	MV	0.026	0.002	10259.68	356.71	9902.97
	MM	0.036	0.036	10356.25	591.00	9765.25
	SMADTC	0.020	0.007	10194.80	1.70	10193.10

Table 5.3 Continued

0.30	MAD	0.023	0.025	10221.70	272.80	9948.90
	MV	0.026	0.002	10255.02	340.06	9914.96
	MM	0.036	0.036	10362.50	550.00	9812.50
	SMADTC	0.023	0.010	10226.24	1.56	10224.68
0.35	MAD	0.021	0.026	10214.90	270.55	9944.35
	MV	0.025	0.002	10250.44	318.87	9931.57
	MM	0.037	0.037	10368.30	502.10	9866.20
	SMADTC	0.025	0.012	10249.58	1.42	10248.16
0.40	MAD	0.021	0.027	10209.60	215.20	9994.40
	MV	0.025	0.002	10247.81	306.70	9941.12
	MM	0.037	0.037	10373.20	440.40	9932.80
	SMADTC	0.026	0.013	10256.72	1.28	10255.44

The wealth from a portfolio generated by SMADTC or MV model grows in value as the portfolio becomes less diversified. This is not the case with MAD and MM optimally generated portfolios whose wealths decline with decrease in diversification. The assets' weights of portfolios generated by the four models are compared for each diversification level. This analysis is shown in Table 5.4. It is evident that as the portfolios become less diversified, the assets in optimal portfolios become different, with the exception of MAD and MV generated portfolios that seem to maintain the same assets. While the SMADTC, MAD and MM models diversify portfolios as per the diversification level, there is little influence of such a restriction on assets in optimal portfolios generated by MV model. It is noted that the MV optimal portfolios comprise more assets in relatively smaller quantities than any portfolio generated by other models. In Table 5.4, the abbreviation 'STC' is used instead of 'SMADTC' to ensure that the table fits on the page.

Table 5.4: Assets' Percentage Compositions in Optimal Portfolios at different diversification levels

D.L	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10			10	10	10		10	10
	MV	10	7.8	10	10	10	4.5	3.9	3.7	10	9.2	10	10	10
	MM	10	10	10		10	10	10	10		10	10		10
	STC	10	10	10	10		10	10		10	10		10	10
0.15	MAD	15	15	15	15					15	10		15	
	MV	15	6.1	13.7	15	6.4			2.9	15	7.3	14.7	0.6	3.2
	MM					15	15	15	15		10	15		15
	STC		15	15				15		15	15		10	15
0.20	MAD	20		20	20					20			20	
	MV	20	4.0	10.8	20	3.1			3.4	20	4.4	13.1		1.3
	MM						20	20	20			20		20
	STC		20	20				20		20	20			
0.25	MAD	25			25					25			25	
	MV	24.5	2.3	8.7	22.3	0.3			3.9	25	1.1	11.8		
	MM						25	25	25			25		
	STC		25	25						25	25			
0.30	MAD	10			30					30			30	
	MV	22.8	1.8	8.6	21				3.9	30		11.9		
	MM						30	30	10			30		
	STC		30	30						10	30			
0.35	MAD				30					35			35	
	MV	20.5	1.2	8.1	19.5				3.8	35		11.8		
	MM						30	35				35		
	STC		35	30							35			

5.3.2 Stage 2

The MV, MM and MAD models are not considered in this stage. It is only the SMADTC model that is examined to assess its multi-stage potential in generating optimal portfolios. A first-stage optimal portfolio with six securities is considered. The second-stage optimal portfolios generated by the model reveal that as the investor diversifies his or her portfolio, implicit transaction costs increase while the portfolio wealth decreases, as shown in Table 5.5 below. This can be attributed to an increasing number of securities being included in the portfolio as the diversification limit decreases.

Table 5.5: Second-stage Optimal Portfolios

D.L	Gross mean	Assets	Net mean	Risk(MAD)	Cost	Wealth
0.175	0.005	6	0.005	0.003	5.402	10231.432
0.200	0.006	5	0.005	0.004	4.222	10240.097
0.225	0.007	5	0.006	0.005	3.683	10248.122
0.250	0.007	4	0.007	0.006	3.143	10256.147
0.275	0.008	4	0.008	0.006	2.936	10263.840
0.300	0.009	4	0.009	0.007	2.728	10271.534
0.325	0.010	4	0.009	0.008	2.520	10279.227
0.350	0.010	3	0.010	0.009	2.318	10286.915

A further analysis is done by holding the diversification limit constant and varying the portfolio gross mean return as shown in Table 5.6. Here, ‘assets’ imply ‘number of assets’. The results show that for the same portfolio gross mean return that can be achieved at different diversification limits, implicit transaction costs decrease with increasing diversification limit. The portfolio wealth also increases with increasing diversification limit. For the same risk, it is observed that the greater the diversification limit the lower the portfolio implicit transaction cost. It is also noted that for each diversification limit considered, an increase in portfolio gross mean return causes implicit transaction costs to rise.

Table 5.6: Stage 2 Optimal Portfolios: Portfolio rate of return constrained

D.L	Gross mean	Assets	Net mean	Risk(MAD)	Cost	Wealth
0.175	0.005	6	0.005	0.003	5.402	10231.432
	0.006	6	0.005	0.004	5.479	10240.062
	0.007	6	0.006	0.005	5.623	10250.103
	0.008	6	0.007	0.006	5.814	10260.096
	0.009	6	0.008	0.007	6.010	10270.085
	0.010	6	0.009	0.008	6.210	10290.069
	0.011	6	0.010	0.009	6.611	10289.853
	0.012	6	0.011	0.010	7.308	10299.341
	0.013	6	0.012	0.011	8.036	10308.797
	0.014	6	0.013	0.012	8.837	10318.180
	0.015	6	0.014	0.013	9.644	10327.557
	0.016	6	0.015	0.014	11.432	10335.954
	0.017	6	0.016	0.015	14.179	10343.392
0.200	0.006	5	0.005	0.004	4.222	10240.097
	0.007	5	0.007	0.005	4.323	10251.403
	0.008	5	0.008	0.006	4.474	10261.436
	0.009	5	0.009	0.007	4.666	10271.429
	0.010	6	0.010	0.008	4.954	10281.325
	0.011	6	0.010	0.009	5.491	10290.973
	0.012	6	0.011	0.010	6.028	10300.620
	0.013	5	0.012	0.011	6.630	10310.202

Table 5.6 Continued

D.L	Gross mean	Assets	Net mean	Risk(MAD)	Cost	Wealth
0.200	0.014	5	0.013	0.012	7.438	10319.579
	0.015	5	0.014	0.013	8.246	10328.956
	0.016	6	0.015	0.014	9.944	10337.442
	0.017	5	0.016	0.015	11.814	10345.756
	0.018	5	0.017	0.016	14.410	10353.348
0.250	0.007	4	0.007	0.006	3.143	10256.147
	0.008	4	0.008	0.006	3.202	10262.708
	0.009	4	0.009	0.007	3.292	10272.803
	0.010	4	0.010	0.008	3.458	10282.821
	0.011	4	0.011	0.009	3.650	10292.814
	0.012	4	0.012	0.010	3.841	10302.807

As in first stage, the performance of portfolios generated by the model during the second stage is analysed by considering the portfolio net return. Although it is obvious that portfolio performance calculated using the equation (5.1) will differ according to the risk measure chosen (MAD in this case), some interesting results emerged as shown in Table 5.7. Portfolio performance is measured as follows:

$$\text{Portfolio performance} = \frac{\text{Mean return}}{\text{risk}}. \tag{5.1}$$

It is evident that as the portfolio gross mean return increasingly vary for each diversification limit, the portfolio performance is on a decreasing trend.

Table 5.7: Stage 2 Optimal Portfolios' Performances

D.L	Net mean	Risk(MAD)	Performance	Cost
0.175	0.005	0.003	1.67	5.402
	0.005	0.004	1.25	5.479
	0.006	0.005	1.20	5.623
	0.007	0.006	1.17	5.814
	0.008	0.007	1.14	6.010
	0.009	0.008	1.13	6.210
	0.010	0.009	1.11	6.611
	0.011	0.010	1.10	7.308
	0.012	0.011	1.09	8.036
	0.013	0.012	1.08	8.837
	0.014	0.013	1.08	9.644
	0.015	0.014	1.07	11.432
	0.016	0.015	1.07	14.179
0.200	0.005	0.004	1.25	4.222
	0.007	0.005	1.40	4.323
	0.008	0.006	1.33	4.474
	0.009	0.007	1.29	4.666
	0.010	0.008	1.25	4.954
	0.010	0.009	1.11	5.491
	0.011	0.010	1.10	6.028
	0.012	0.011	1.09	6.630
	0.013	0.012	1.08	7.438
	0.014	0.013	1.08	8.246
	0.015	0.014	1.07	9.944
	0.016	0.015	1.07	11.81
	0.017	0.016	1.06	14.41

Table 5.7 Continued

D.L	Net mean	Risk(MAD)	Performance	Cost
0.250	0.007	0.006	1.17	3.143
	0.008	0.006	1.33	3.202
	0.009	0.007	1.29	3.292
	0.010	0.008	1.25	3.458
	0.011	0.009	1.22	3.650
	0.012	0.010	1.20	3.841
	0.013	0.011	1.18	4.763
	0.013	0.012	1.08	5.813
	0.014	0.013	1.08	6.863
	0.015	0.014	1.07	7.914
	0.016	0.015	1.07	9.013
	0.017	0.016	1.06	11.424
	0.018	0.017	1.06	14.709
	0.018	0.018	1.00	19.961
0.300	0.009	0.007	1.29	2.728
	0.010	0.008	1.25	2.840
	0.011	0.009	1.22	2.945
	0.012	0.010	1.20	3.128
	0.013	0.011	1.18	3.441
	0.014	0.012	1.17	4.450
	0.014	0.013	1.08	5.504
	0.015	0.014	1.07	6.737
	0.016	0.015	1.07	7.970
	0.017	0.016	1.06	9.202
	0.018	0.017	1.06	12.072
	0.018	0.018	1.00	16.109

5.3.3 Sensitivity Analysis

Sensitivity analysis of the SMADTC model is performed by calculating the sensitivity index (SI) for each parameter. This is achieved by finding the output percentage difference when varying one input parameter from its minimum value to its maximum value ([3], [34]). The following formula is used:

$$\text{Sensitivity Index} = \frac{D_{\max} - D_{\min}}{D_{\max}} \tag{5.2}$$

where $D_{\min}$ and $D_{\max}$ are the minimum and maximum output values respectively, resulting from varying the input parameter over its entire range. This figure-of-merit provides a good indication of parameter and model variability [28]. The minimum input value is taken to be zero and allow the maximum input value to vary from 0.1 to 0.3, as shown in Table 5.8. These maximum input values are bounds that given to the asset weights to ensure portfolio diversification. The phrase 'max value' refers to 'maximum input parameter value' and it gives the maximum input of the parameter considered. The sensitivity index corresponding to each asset weight is then calculated.The results in Table 5.8 reflect small percentages of model variability due to each input change of each parameter. This shows that the output values are not strongly reliant on certain individual parameters. Hence, the SMADTC seems to be a reliable portfolio optimisation model. x_i represents the weight of asset i in the portfolio.

However, when doing sensitivity analysis, the following is noted: The SMADTC model is stochastic and works by replacing one parameter by a less profitable one when a weight of an asset is taken to be zero, thereby achieving a less 'optimal portfolio'. Here, a less preferable asset is included in the 'less optimal portfolio'. This is so because of a condition in the model that ensures that the total weight of assets in a portfolio be unit. This, however, leads to relative sensitivity analysis. Thus, the $D_{\min}$ of model output is a relative value. Therefore, the relative sensitivity analysis of the model is performed by finding the difference between the optimal portfolio value when the asset under consideration attains maximum value and when it gets a value of zero (*i.e.*, excluded).

Table 5.8: SMADTC model Sensitivity Analysis

Max value	Parameter	Cost SI	Wealth SI
0.1	x_1	0.1618	0.0019
	x_2	-0.0607	0.0017
	x_3	0.0547	0.0016
	x_4	0.1534	-0.00006
	x_6	0.1348	-0.00005
	x_7	0.1078	0.0016
	x_9	0.0647	0.0004
	x_{10}	0.0377	0.0012
	x_{12}	0.1348	0.0010
	x_{13}	0.1240	0.0016
0.15	x_2	-0.0125	0.0052
	x_3	0.1281	0.0035
	x_7	0.2135	0.0028
	x_9	0.1281	0.0006
	x_{10}	0.0747	0.0017
	x_{12}	0.2598	0.0018
	x_{13}	0.2456	0.0029
0.2	x_2	-0.1065	0.0064
	x_3	0.2222	0.0039
	x_7	0.3704	0.0031
	x_9	0.2222	0.0074
	x_{10}	0.1296	0.0023

Table 5.8 Continued

Max value	Parameter	Cost SI	Wealth SI
0.25	x_2	-0.2153	0.0071
	x_3	0.3529	0.0049
	x_9	0.3529	0.0009
	x_{10}	0.2059	0.0029
0.3	x_2	-0.1859	0.0101
	x_3	0.4615	0.0059
	x_9	0.4615	0.0043
	x_{10}	0.2692	0.0034

Sensitivity analysis of the model is also carried out in which the gross portfolio mean return is allowed to vary by gradually increasing it and analysing changes in optimal portfolio asset weights. The results are shown in Table 5.9. It is observed that the model identifies assets with better returns but which have more implicit transaction costs whenever there is a change in the initial asset's weight. However, such changes are minimal. The abbreviation 'A.W.' stands for 'Asset weight' in the following table.

Table 5.9: Percentage portfolio compositions: Portfolio mean return constrained.

D.L	A.W.	0.014	0.016	0.018	0.02	0.022	0.024	0.026	0.028	0.030
0.1	x_1	10	10	10	10	10	10	10	10	10
	x_2	10	10	10	10	10	10	10	10	10
	x_3	10	10	10	10	10	10	10	10	10
	x_4	10	2.3	4.9	4.0					
	x_5		7.7	10	10	10	10	10	10	10
	x_6	10	10	5.1						
	x_7	10	10	10	10	10	10	10	10	10
	x_8								1.5	10
	x_9	10	10	10	10	10	10	10		
	x_{10}	10	10	10	10	10	10	10	8.5	
	x_{11}				6.0	10	10	10	10	10
	x_{12}	10	10	10	10	10	10	10	10	10
	x_{13}	10	10	10	10	10	10	10	10	10
D.L	A.W.	0.019	0.020	0.022	0.024	0.026	0.028	0.030	0.032	0.034
0.2	x_1					20	14.1	20	20	20
	x_2	20	20	20	20	20	20	20	20	20
	x_3	20	20	20	20	20	20	20	20	20
	x_5								8.5	15.2
	x_7	20	20	20	20		5.9			
	x_9	20	11.9							
	x_{10}	20	20	20	20	20	20	20	11.5	
	x_{11}			2.6	13.3	20	20	20	20	20
	x_{12}									4.8
	x_{13}		8.1	17.4	6.7					

Table 5.9 Continued

D.L	A.W.	0.023	0.024	0.026	0.028	0.030	0.032	0.034	0.036	0.038
0.3	x_1						4.1	10.2	18.2	26.3
	x_2	30	30	30	30	30	30	30	30	30
	x_3	30	30	30	30	30	30	29.8	21.8	13.7
	x_7		10							
	x_9	10								
	x_{10}	30	30	29.7	21.2	12.7	5.9			
	x_{11}				10.3	18.8	27.3	30	30	30

5.3.4 Summary

In this part of the study, a multi-stage stochastic transaction cost model is proposed which uses mean absolute deviation as its risk and takes into account uncertainty of assets' returns and of implicit transaction costs. The main contributions of this part of the study includes: (i) the development of a stochastic trading cost model that generates optimal portfolios while minimising implicit transaction costs incurred during initial trading and in subsequent re-balancing of the portfolios, (ii) the development of a strategy that captures uncertainty in stock prices and in corresponding implicit transaction costs by way of scenarios. To provide investors with competitive portfolio returns, investment managers must manage transaction costs pro-actively, because lower transaction costs mean higher portfolio returns. Ineffective transaction cost management may be very damaging in today's competitive environment, in which the difference between success and failure may be no more than a few basis points. The model has shown that it generates optimal portfolios whose net wealths are better than those generated by MV, MM and MAD models. It also has the advantage of generating a least-cost optimal portfolio as compared with the other three models.

The methodology allows investors or investment managers to choose optimal portfolios with acceptable implicit cost levels to be incurred while executing their trading. It is a suitable strategy for conservative investors. The methodology is a linear programming model and hence reduces considerably the time needed to reach a solution. It is feasible for large-scale portfolio selection. It is left, however, for future research to develop a model that accounts for both implicit and explicit trading costs while considering uncertainty of both asset prices and transaction costs. The study, like any other, has its own limitations. The use of an asset's closing price for each month in the calculation of an asset's return rate may not be the most accurate measure. However, since all assets' rates of returns have been obtained in the same way, this does not prejudice the findings.

5.4 SMNDTC model results

This section analyses optimal portfolios generated by the stochastic maximum negative deviation with uncertain transaction costs (SMNDTC) model discussed in section 4.3. The portfolio risk used is described in subsection 4.3.2. An investor who has $R10000$ to spend on the initial portfolio is again considered. The optimal portfolios describing the first-stage efficient frontier are shown in Table 10 below. As above, the abbreviation 'D. L' stands for 'diversification limit'.

Table 5.10: Stage 1 optimal portfolios: risk, cost and expected return unconstrained

D. L	Gross mean	Net mean	Risk	Cost	% cost	Netwealth
0.100	0.031	0.027	0.034	44.82	12.9	10265.78
0.125	0.034	0.029	0.031	55.15	14.7	10286.85
0.150	0.036	0.030	0.029	63.49	16.7	10297.01
0.175	0.038	0.032	0.027	60.04	15.8	10317.61
0.200	0.039	0.036	0.025	36.48	7.7	10357.12
0.225	0.040	0.036	0.025	39.47	10.0	10359.09
0.250	0.040	0.036	0.024	42.45	10.0	10361.05
0.275	0.041	0.035	0.024	44.28	12.1	10362.78
0.300	0.041	0.036	0.024	46.10	12.1	10364.50
0.325	0.041	0.037	0.023	47.93	9.8	10366.23
0.350	0.042	0.037	0.023	44.71	11.9	10372.89
0.375	0.042	0.038	0.023	38.98	9.5	10382.03
0.400	0.042	0.039	0.022	33.24	7.1	10391.16

As portfolios become less diversified, there is an increase in both the gross portfolio mean return and the net portfolio mean return. The risk declines and the net wealth increases with increasing diversification limit. However, the total implicit transaction cost rises from $R44.82$ to $R63.49$ as diversification limit increases from 0.1 to 0.15

respectively. Thereafter, the transaction cost declines in a fluctuating pattern to a lowest value of $R33.24$ obtained when the diversification limit is 0.4. Efficient frontiers of net mean portfolio returns and gross mean portfolio returns reveal the impact of neglecting implicit transaction costs in portfolio selection. The total implicit transaction costs incurred to achieve each optimal portfolio vary from 7.1% to 16.7% of returns on investment.

Analysis of portfolio composition of assets in optimal portfolios is carried out and the information is shown in Table 5.11 below. It is evident from the table results that the SMNDTC model allocates the maximum weight possible to each of the selected assets except when an even distribution is impossible. The model reflects consistency in selection of assets as portfolios become less diversified.

Table 5.11: Assets' percentage composition: Stage 1 optimal portfolios

D. L	AVI	ASR	APN	CSB	CLS	CML	PNC	CPI	IPL
0.100	10	10	10	10	10	10	10	10	10
0.125	12.5	12.5		12.5	12.5	12.5	12.5	12.5	12.5
0.150	15	15		15		10	15	15	15
0.175	17.5	17.5		12.5			17.5	17.5	17.5
0.200	20	20					20	20	20
0.225	22.5	22.5					22.5	10	22.5
0.250	25	25					25		25
0.275	27.5	27.5					27.5		17.5
0.300	30	30					30		10
0.325	32.5	32.5					32.5		2.5
0.350	35	35					30		
0.375	37.5	37.5					25		
0.400	40	40					20		

5.4.1 Sensitivity Analysis

Sensitivity analysis of the SMNDTC model is done by calculating the sensitivity index (SI) for each parameter. As in the section above, the methodology by Hoffman and Gardner [34], and Bauer and Hamby [3] is employed. Output percentage difference is calculated by varying one input parameter, $x_i, i = 1, \cdots, 13$ at a time, from its minimum value (zero in this case) to its maximum value (parameter value in optimal portfolio). Sensitivity indices are obtained using the same formula as in equation (5.2). Maximum output values of 0.1, 0.15, 0.2 up to 0.3 are considered which are the imposed diversification limits. The model being proposed is stochastic and works by replacing one parameter by a ‘less profitable one when a weight of zero is assigned to the asset of the optimal portfolio. Applying the above method yields a relative sensitivity analysis of the model, the results of which are shown in Table 5.12 below.

Table 5.12: SMNDTC model sensitivity analysis

D. L	Parameter	Proxy	Max value	Cost SI	Risk SI	Wealth SI
0.10	x_1	$x_7(10)$	0.10	0.0357	0.3235	0.0027
	x_2			-0.0076	0.0000	0.0026
	x_3			-0.0036	0.0588	0.0004
	x_4			0.4931	0.0882	-0.0011
	x_5			0.0134	0.0294	0.0008
	x_6			0.0442	0.1176	0.0007
	x_8			0.2700	-0.0294	0.0008
	x_{11}			0.0192	0.2353	0.0018
	x_{12}			0.0451	0.1765	0.0018
	x_{13}			0.0013	0.2353	0.0001

Table 5.12 Continued

D. L	Parameter	Proxy	Max value	Cost SI	Risk SI	Wealth SI
0.15	x_1		0.15	0.0128	0.3793	0.0019
	x_2			-0.0331	0.0000	0.0018
	x_4			0.4971	0.1379	-0.0038
	x_6	$x_5(10)$		0.0217	0.1724	-0.0011
	x_8			0.2608	0.0000	-0.0009
	x_{11}			-0.0047	0.2759	0.0006
	x_{12}			0.0227	0.2414	0.0006
0.20	x_1		0.20	-1.1239	0.3600	0.0075
	x_2			-1.2303	-0.0400	0.0074
	x_8	$x_4(20)$		-0.5482	-0.0400	0.0039
	x_{11}			-1.1645	0.2800	0.0058
	x_{12}			-1.1009	0.2400	0.0058
0.25	x_1		0.25	0.0436	0.4583	0.0021
	x_2	$x_{11}(25)$		-0.0707	0.0000	0.0020
	x_8			0.6620	0.0000	-0.0023
	x_{12}			0.0683	0.2917	0.0001
0.30	x_1		0.30	0.4583	0.4583	0.0025
	x_2	$x_{11}(10)$		0.0417	0.0417	0.0024
	x_8			0.0417	0.0417	-0.0028
	x_{12}			0.3333	0.3333	0.0000

It is observed that, for each diversification limit considered, the same asset is being chosen as the proxy. However, these proxies vary from one diversification limit to the other. The results show small percentages of model variability of optimal wealth due to changes in input parameter value. The wealth sensitivity index ranges from -0.38% to 0.75%. This is a good indication that the model output values are not significantly influenced by specific input values. Some wealth sensitivity indices are negative implying that the respective proxies result in better wealth. However, the cost sensitivity indices vary from -1.2303 to 0.6620. The negative indices show that the proxies result in higher implicit transaction costs compared to assets in the optimal portfolios. All such proxies cause a decline in portfolio wealth since all corresponding wealth S.I. values are positive.

5.4.2 Summary

In this part of the study, a multi-stage stochastic maximum downside risk model that incorporates uncertainty of asset returns and implicit transaction costs is proposed. The model best applies to periods of economic recessions which are characterised by extreme movements in asset prices. In such times, investors are highly concerned about the performance of their portfolios, particularly the downside movements. This part of the study extends the horizon of knowledge by:

(i) developing a multi-stage stochastic maximum negative deviation model that optimises portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in subsequent rebalancing of portfolios, and

(i) developing a strategy that captures uncertainty in stock returns and in corresponding implicit trading costs in extreme downside movements of stock prices by way of scenarios.

The methodology allows investors and investment managers to decide on optimal portfolios realising the associated implicit transaction costs. It is a linear programming model and hence it is feasible for large-scale portfolio selection as it reduces considerably the time needed to reach a solution.

5.5 SMUDTC model results

A stochastic multi-stage upside-downside deviation (SMUDTC) model with uncertain transaction costs described in section 4.4 is applied to historical data on securities from the Johannesburg Stock Market. The portfolio risk used in the analysis is given in subsection 4.4.2. The period under study is divided into two, giving in-sample and out-of-sample data. The in-sample period starts from January 2008 and ends March 2010. The second period starts from April 2010 to September 2012. In each period, the performance of each model is analysed. First-stage optimal portfolios generated by the SMUDTC model are compared with optimal portfolios from the MAD, MV and MM models. In a real-life environment, comparison of models is usually done by means of ex-post analysis. It is also noted that portfolio performances are usually affected by market trend, hence the need to subject the proposed SMUDTC model to these two periods and evaluate its performance. As in above proposed stochastic models, an investor who has R10 000 to spend on his initial portfolio is considered.

5.5.1 In-sample Analysis

In analysing the performance of each model, the gross portfolio mean return, portfolio risk, total implicit transaction costs incurred and the gross and net portfolio wealth are evaluated. Neither portfolio mean return nor portfolio risk is constrained, and diversification limits from 0.1 to 0.4 are used. The optimal portfolios generated by the four models are shown in Table 5.13 below. As above, the abbreviation 'D. L' stands for 'diversification limit'.

Table 5.13: Summary statistics of in-sample optimal portfolios

Div. Lim	Measure	MAD	MV	MM	SMUDTC
0.10	Mean return	0.027	0.030	0.034	0.003
	Risk	0.024	0.001	0.034	0.001
	Gross wealth	10265.76	10303.55	10339.30	10066.72
	Implicit cost	710.10	835.42	901.70	40.22
	Net wealth	9555.66	9468.12	9437.60	10026.50
0.15	Mean return	0.024	0.032	0.038	0.004
	Risk	0.027	0.001	0.038	0.001
	Gross wealth	10236.14	10316.92	10380.50	10065.00
	Implicit cost	855.25	929.73	721.45	29.33
	Net wealth	9380.89	9387.18	9659.05	10035.67
0.20	Mean return	0.023	0.033	0.041	0.003
	Risk	0.028	0.001	0.041	0.001
	Gross wealth	10225.12	10330.33	10409.00	10056.44
	Implicit cost	1092.00	818.23	746.40	25.36
	Net wealth	9133.12	9512.10	9662.60	10031.08
0.25	Mean return	0.022	0.033	0.043	0.003
	Risk	0.029	0.001	0.043	0.001
	Gross wealth	10216.15	10330.92	10428.00	10050.86
	Implicit cost	396.25	760.18	883.25	23.20
	Net wealth	9819.90	9570.74	9544.75	10027.66
0.30	Mean return	0.021	0.033	0.044	0.003
	Risk	0.029	0.001	0.044	0.001
	Gross wealth	10211.78	10331.02	10443.60	10045.83
	Implicit cost	445.30	757.81	1023.30	53.84
	Net wealth	9766.48	9573.21	9420.30	10028.63

Table 5.13 Continued

Div. Lim	Measure	MAD	MV	MM	SMUDTC
0.35	Mean return	0.021	0.033	0.046	0.003
	Risk	0.029	0.001	0.046	0.001
	Gross wealth	10207.83	10331.02	10457.80	10051.63
	Implicit cost	444.45	757.80	1083.60	18.79
	Net wealth	9763.38	9573.22	9374.20	10032.84
0.40	Mean return	0.020	0.033	0.047	0.004
	Risk	0.030	0.001	0.047	0.001
	Gross wealth	10204.72	10331.02	10469.20	10047.63
	Implicit cost	343.80	757.79	984.40	5.52
	Net wealth	9860.92	9573.23	9484.80	10042.11

It is observed that, for each given diversification limit, the SMUDTC optimal portfolios have the least gross mean portfolio return and the investor incurs the least implicit transaction costs. The MM optimal portfolios have the highest gross expected portfolio returns followed by the MV-generated optimal portfolios. However, the MM-investor incurs the greatest implicit trading cost followed by the MV-investor. It is evident, that despite having the least expected portfolio returns, SMUDTC optimal portfolios have the greatest portfolio wealth for every diversification limit considered. The MAD-generated optimal portfolios have greater wealth than those of MV and MM optimal portfolios. The trend remains the same as portfolios become less diversified.

5.5.2 Out-of-sample analysis

In a real-life environment, validation of models is usually done by ex-post analysis. This is achieved by making a comparative evaluation of model performance on in-sample and out-of-sample data sets. The period from April 2010 to September 2012 provides the out-of-sample data set for the SMUDTC model. Table 5.14 below shows the performance of the SMUDTC model together with MV, MAD and minimax models. The information in Table 5.14 shows that MM optimal portfolios have the highest gross expected portfolio returns followed by the MV and the MAD optimal portfolios respectively.

Table 5.14: Summary statistics of out-of-sample optimal portfolios

Div. Lim	Measure	MAD	MV	MM	SMUDTC
0.10	Mean return	0.027	0.029	0.031	0.005
	Risk	0.002	0.002	0.1	0.0009
	Gross wealth	10269.80	10289.02	10314.80	10064.57
	Implicit cost	432.00	426.34	502.10	15.36
	Net wealth	9837.80	9862.68	9812.70	10049.21
0.15	Mean return	0.025	0.028	0.033	0.006
	Risk	0.022	0.002	0.143	0.0009
	Gross wealth	10251.75	10277.97	10332.30	10070.17
	Implicit cost	479.10	397.32	411.15	17.27
	Net wealth	9772.65	9880.65	9921.15	10052.90
0.20	Mean return	0.024	0.027	0.035	0.007
	Risk	0.023	0.002	0.20	0.0007
	Gross wealth	10240.20	10268.25	10346.80	10073.55
	Implicit cost	223.20	378.64	503.60	8.00
	Net wealth	10017.00	9889.61	9843.20	10065.55

Table 5.14 Continued

Div. Lim	Measure	MAD	MV	MM	SMUDTC
0.25	Mean return	0.023	0.026	0.036	0.007
	Risk	0.025	0.002	0.250	0.0006
	Gross wealth	10229.25	10259.68	10356.25	10080.73
	Implicit cost	248.50	356.71	591.00	7.40
	Net wealth	9980.75	9902.97	9765.25	10073.33
0.30	Mean return	0.023	0.026	0.036	0.008
	Risk	0.025	0.002	0.250	0.0006
	Gross wealth	10221.70	10255.02	10362.50	10081.56
	Implicit cost	272.80	340.06	550.00	6.64
	Net wealth	9948.90	9914.96	9812.50	10074.92
0.35	Mean return	0.021	0.025	0.037	0.008
	Risk	0.026	0.002	0.333	0.0006
	Gross wealth	10214.90	10250.44	10368.30	10082.44
	Implicit cost	270.55	318.87	502.10	6.36
	Net wealth	9944.35	9931.57	9866.20	10076.08
0.40	Mean return	0.021	0.025	0.037	0.007
	Risk	0.027	0.002	0.333	0.0007
	Gross wealth	10209.60	10247.81	10373.20	10075.40
	Implicit cost	215.20	306.70	440.40	6.80
	Net wealth	9994.40	9941.12	9932.80	10068.60

It is also evident that the MM optimal portfolios have the highest portfolio risk with the SMUDTC optimal portfolio risks being the least for all diversification levels considered in the study. The SMUDTC optimal portfolios bear the least implicit transaction costs with the MM optimal portfolios having the highest transaction costs. It is also clear that the SMUDTC optimal portfolio has the greatest wealth for every diversification limit considered. All these features are evident in the in-sample analysis above. Thus, the SMUDTC investor has advantages of having high net expected portfolio returns and incurring the least implicit transaction costs during trading over his MM-, MV- and MAD- counterparts.

5.5.3 Sensitivity analysis

Model sensitivity analysis is carried out using both in-sample and out-of-sample data. This is done by finding the sensitivity index (SI) for each parameter, $x_i, i = 1, \cdots, 13$, where the output percentage difference is calculated by varying one input parameter from its minimum value (zero in this case) to its maximum value. As in the previous section, the sensitivity index is calculated using the formula in equation (5.2). Zero is taken to be the minimum input value for each parameter with the maximum value constrained by the diversification limit considered and is given as the best weight of the chosen asset in the optimal portfolio. Diversification limits of 0.1, 0.2 and 0.3 are considered. Table 5.15 shows in-sample model sensitivity analysis where there are very small variations in portfolio wealth resulting from each asset being removed from the optimal portfolio. The proxy optimal portfolio has wealth increase of at most 0.05% and decrease of at most 0.049%. This shows that the SMUDTC model is not significantly influenced by individual parameter choices. The risk sensitivity index varies from 0 to 0.2, with zero being the modal sensitivity index. This again may imply that model output values are not strongly dependent on certain individual parameters. The cost sensitivity indices range from -78.4% to 32.6%. These large values (numerically) are a result of very small implicit transaction costs incurred which have been used in the calculation. Table 5.13 shows the implicit transaction costs associated with SMUDTC optimal portfolios.

Table 5.15: In-sample SMUDTC model sensitivity analysis

D.L	Parameter	Max. Value	Cost SI	Risk SI	Wealth SI
0.1	x_1	0.1	0.0000	0.2297	-0.00045
	x_2	0.1	0.1000	0.1228	-0.00024
	x_4	0.1	0.0000	0.0397	0.00002
	x_5	0.1	0.0000	0.1228	-0.00024
	x_6	0.1	0.0000	0.1561	-0.00017
	x_7	0.1	0.1000	0.1367	-0.00034
	x_8	0.1	0.0000	0.3262	-0.00033
	x_9	0.1	0.0000	0.0552	-0.00012
	x_{10}	0.1	0.0000	0.0776	-0.00032
	x_{11}	0.1	0.0000	0.0756	-0.00014
0.2	x_1	0.2	0.0000	0.1356	-0.000073
	x_2	0.2	0.1000	-0.0410	-0.000047
	x_4	0.2	0.0000	-0.1798	0.000495
	x_6	0.2	0.0000	0.1893	0.00012
	x_9	0.2	0.0000	-01309	0.00022
0.3	x_1	0.3	0.1000	-0.7836	-0.000509
	x_2	0.3	0.2000	-0.5172	-0.000867
	x_3	0.3	0.0000	0.0164	-0.000662
	x_4	0.3	0.0000	-0.4888	-0.000078

5.5.4 Summary

Portfolio selection that incorporates transaction costs in the literature is mostly devoted to proportional transaction costs. There is extensive use of the mean-variance and MAD models in financial optimisation yet both models penalise upside deviations (gains) and downside deviations (losses) in the same way. It is important to differentiate upside deviations from downside deviations since positive deviation is desirable to any investor while negative deviation is undesirable. Hence in this part of the study, a new model that accounts for better investment opportunities (upside deviations), risk (downside deviations) and implicit transaction costs in a dynamic uncertain environment is proposed. The study considers uncertainty in asset returns, portfolio risk and implicit trading costs incurred during initial trading and in subsequent rebalancing of the portfolio. The proposed model is tested using real data from an emerging market and its performance is compared with those of mean-variance, MAD and minimax models. The results show that the proposed model generates optimal portfolios with least risk, highest portfolio wealth and minimum implicit transaction costs. The model is suitable for a risk-averse and conservative investor.

5.6 Stochastic Mean-Variance with transaction costs (SMVTC) model application and analysis of results

The portfolio risk in the SMVTC model is measured by the variance of asset returns from expected portfolio return as described in section 4.5.2. The period of study is again divided into two. The first period, which starts from January 2008 to March 2010, is where the four models are applied to validate the SMVTC model. The second period, from April 2010 to September 2012, is used to find out if the SMVTC model results in the first period are consistent. An investor who has R10000 to spend on his initial portfolio is again considered. The models are applied over the first stage only.

5.6.1 In-sample analysis

The three LP problems MV, MAD and MM defined above together with the SMVTC model are solved for each data set and for each diversification level. In analysing the performance of each model, the gross portfolio mean return, the risk, total implicit transaction costs incurred, and the gross and net portfolio wealths are considered. For each model, the diversification limits are taken from 0.1 to 0.4, and neither the mean portfolio return nor the portfolio risk is constrained. The optimal portfolios generated by the four models are shown in Table 5.16 below.

It is evident from the Table 5.16 results that, for the same diversification limit, MM-generated portfolios have the highest expected gross portfolio returns, followed by the MV, with the SMVTC-generated portfolios having the least expected gross returns. The MM optimal portfolios have the highest risk values followed by the MAD-generated portfolios. The SMVTC optimal portfolios have the least risk values. The table also shows that the MM optimal portfolios have the highest expected gross wealth followed by the MV optimal portfolios. The SMVTC portfolios have the least expected gross portfolio wealth for every diversification limit considered.

However, the total implicit transaction costs incurred for each optimal portfolio is least in SMVTC portfolios followed by MAD portfolios, with the greatest implicit costs being incurred by either an MV or MM investor. It is noted that the total expected net wealth is greatest in SMVTC-generated portfolios. The MAD optimal portfolios improve more on net expected wealth as portfolios become less diversified, compared to MV and MM portfolios. Thus, the SMVTC optimal portfolios have the least risk values, the least implicit transaction costs incurred, and they have the greatest expected net wealth.

Table 5.16: Summary statistics of in-sample optimal portfolios

D. Lim		MAD	MV	MM	SMVTC
0.10	Mean return	0.027	0.030	0.034	0.002
	Risk	0.024	0.001	0.034	2.0054E-8
	Gross wealth	10265.76	10303.55	10339.30	10011.32
	Cost	710.10	835.42	901.70	44.39
	Net wealth	9555.66	9468.12	9437.60	9966.94
0.15	Mean return	0.024	0.032	0.038	0.0009
	Risk	0.027	0.001	0.038	6.0619E-9
	Gross wealth	10236.14	10316.92	10380.50	10001.79
	Cost	855.25	929.73	721.45	56.61
	Net wealth	9380.89	9387.18	9659.05	9945.19
0.20	Mean return	0.023	0.033	0.041	0.0006
	Risk	0.028	0.001	0.041	8.3476E-9
	Gross wealth	10225.12	10330.33	10409.00	10001.53
	Cost	1092.00	818.23	746.40	37.13
	Net wealth	9133.12	9512.10	9662.60	9964.40

Table 5.16 Continued

D. Lim		MAD	MV	MM	SMVTC
0.25	Mean return	0.022	0.033	0.043	0.001
	Risk	0.029	0.001	0.043	1.98468E-8
	Gross wealth	10216.15	10330.92	10428.00	10009.96
	Cost	396.25	760.18	883.25	28.89
	Net wealth	9819.90	9570.74	9544.75	9981.07
0.30	Mean return	0.021	0.033	0.044	0.001
	Risk	0.029	0.001	0.044	2.18275E-8
	Gross wealth	10211.78	10331.02	10443.60	10007.14
	Cost	445.30	757.81	1023.30	53.84
	Net wealth	9766.48	9573.21	9420.30	9953.30
0.35	Mean return	0.021	0.033	0.046	0.002
	Risk	0.029	0.001	0.046	2.5204E-8
	Gross wealth	10207.83	10331.02	10457.80	10011.90
	Cost	444.45	757.80	1083.60	37.62
	Net wealth	9763.38	9573.22	9374.20	9974.29
0.40	Mean return	0.020	0.033	0.047	0.0005
	Risk	0.030	0.001	0.047	8.12541E-9
	Gross wealth	10204.72	10331.02	10469.20	10003.25
	Cost	343.80	757.79	984.40	10.60
	Net wealth	9860.92	9573.23	9484.80	9992.65

Table 5.17 below presents assets' percentage compositions of In-Sample optimal portfolios at increasing diversification levels. The assets in Table $A1$ in Appendix 2 are coded A_1, A_2, $\cdots$, A_{13} to stand in for AVI, ASR, $\cdots$, WHL, respectively, following the order in which they are given in Table A_1. This is done to ensure that Table 5.17 fits on the page. Selected common securities with high percentage compositions in

MV, MM and SMVTC optimal portfolios include A_1, A_8 and A_{13}, when portfolios are more diversified. It is also observed that SMVTC portfolios comprise assets in small quantities and being many, a characteristic which is less shared by even MV portfolios. Generally, assets common to both MV and SMVTC portfolios tend to be in larger quantities in MV than SMVTC optimal portfolios, for example, assets A_1, A_8 and A_9. The abbreviation 'SMC' stands for 'SMVTC' and is adopted to ensure that the table fits well on the page.

Table 5.17: Assets' percentage compositions of in-sample optimal portfolios at different diversification levels: mean return and risk unconstrained

D.L	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10				10	10	10	10	10
	MV	10	10	10	10	10		2	10	10		10	8	10
	MM	10	10			10	10	10	10		10	10	10	10
	SMC	9	9	9	1	8	9	8	7	7	7	8	9	10
0.15	MAD		15	15	15	15				15	10		15	
	MV	15	10	5	10	10			15	15		15		6
	MM	15					15	15	15		10	15		15
	SMC	4	15	15	7	3	2	7	9	8		9	8	15
0.20	MAD		20	20	20	20				20				
	MV	20	5		8	9			20	20		16		2
	MM	20					20	20	20					20
	SMC	20			1			10	9	10		17	20	13
0.25	MAD			25	25	25				25				
	MV	25	4		7	8			19	20		17		
	MM	25					25	25	25					
	SMC	4	7	5	2	7	7	7	6	10	9	4	7	25

In Table 5.18 below, a summary of descriptive statistics of sample data of securities from which each SMVTC in-sample optimal portfolio is generated is given. After identifying the securities in each optimal portfolio, regardless of the asset weights obtained, the mean return of all assets in the 'portfolio' sample for the entire in-sample period is calculated. The sample standard deviation, coefficient of variation, excess skewness and excess kurtosis are calculated to provide some ideas on the data on which the SMVTC model is applied to yield each optimal portfolio. In the table, 'port1' stands for 'portfolio 1', 'St.Dev' stands for 'standard deviation' and 'CV' stands for 'covariance'. Table 5.18 shows varying means and standard deviations with the smallest being 0.03343 and 0.00907, and the highest being 0.04298 and 0.102152 respectively. All sample skewness values are between -1 and -0.5, showing that the sample distributions are moderately skewed to the left. The sample excess kurtosis values are all positive, implying that the sample distributions are leptokurtic.

Table 5.18: Statistics of sample data from which SMVTC-optimal portfolios are constructed: portfolio risk and mean return unconstrained

Portfolio	Mean	St.Dev	CV	Sample skeness	Sample kurtosis
1	0.04288	0.09113	0.47059	-0.87485	1.74633
2	0.03343	0.09278	0.36029	-0.47931	1.58773
3	0.03842	0.00907	4.23745	-0.92248	1.31044
4	0.04138	0.09144	0.45257	-0.82627	1.63654
5	0.04298	0.10054	0.42745	-0.84668	1.22052
6	0.0385	0.102152	0.37689	-0.88476	1.12443
7	0.04271	0.07811	0.54682	-0.4426	1.62229

5.6.2 Out-of-sample analysis

There are many approaches that can be used to compare models, and one of the most commonly applied methods involves representing the ex-post asset returns of optimal portfolios over a given period. Generally, portfolio performances are usually affected by the market trend which makes it difficult to draw some uniform conclusions. However, the current case produces portfolio performances that have some common features. Comparison of out-of-sample optimal portfolios is shown in Table 5.19 below.

Table 5.19: Summary statistics of out-of-sample optimal portfolios

D. L		MAD	MV	MM	SMVTC
0.10	Mean return	0.027	0.029	0.031	0.002
	Risk	0.002	0.002	0.1	1.846E-8
	Gross wealth	10269.80	10289.02	10314.80	10011.09
	Cost	432.00	426.34	502.10	37.86
	Net wealth	9837.80	9862.68	9812.70	9973.23
0.15	Mean return	0.025	0.028	0.033	0.001
	Risk	0.022	0.002	0.143	1.9904E-8
	Gross wealth	10251.75	10277.97	10332.30	10009.32
	Cost	479.10	397.32	411.15	32.49
	Net wealth	9772.65	9880.65	9921.15	9976.84
0.20	Mean return	0.024	0.027	0.035	0.001
	Risk	0.023	0.002	0.2	1.8693E-8
	Gross wealth	10240.20	10268.25	10346.80	10007.71
	Cost	223.20	378.64	503.60	40.56
	Net wealth	10017.00	9889.61	9843.20	9967.15

Table 5.19 Continued

D. L		MAD	MV	MM	SMVTC
0.25	Mean return	0.023	0.026	0.036	0.001
	Risk	0.025	0.002	0.25	1.7098E-8
	Gross wealth	10229.25	10259.68	10356.25	10008.13
	Cost	248.50	356.71	591.00	37.20
	Net wealth	9980.75	9902.97	9765.25	9970.94
0.30	Mean return	0.023	0.026	0.036	0.001
	Risk	0.025	0.002	0.25	1.2561E-8
	Gross wealth	10221.70	10255.02	10362.50	10006.53
	Cost	272.80	340.06	550.00	27.84
	Net wealth	9948.90	9914.96	9812.50	9978.69
0.35	Mean return	0.021	0.025	0.037	0.002
	Risk	0.026	0.002	0.333	3.0025E-8
	Gross wealth	10214.90	10250.44	10368.30	10013.26
	Cost	270.55	318.87	502.10	26.95
	Net wealth	9944.35	9931.57	9866.20	9986.31
0.40	Mean return	0.021	0.025	0.037	0.001
	Risk	0.027	0.002	0.333	1.8485E-8
	Gross wealth	10209.60	10247.81	10373.20	10011.33
	Cost	215.20	306.70	440.40	21.26
	Net wealth	9994.40	9941.12	9932.80	9990.07

It is evident that MM optimal portfolios have the highest expected gross returns, incur the greatest implicit transaction costs and have the least expected portfolio wealths for every diversification limit considered except 0.15. As in in-sample analysis, the SMVTC optimal portfolios have smallest mean returns and risks, and with superior expected wealths. It is also evident that the SMVTC investor incurs the smallest implicit transaction costs.

A comparison of out-of-sample optimal portfolios assets' percentage compositions is shown in Table 5.20. As portfolios become less diversified, the MAD and MV optimal portfolios have common securities with high percentage compositions, for example, securities A_1, A_4 and A_9. The choice of assets in MM portfolios does not match any of the other models' optimal portfolios. Portfolios generated by the SMVTC and MV models have some common securities, but with reverse percentage compositions, for example, A_1, A_3, A_4 and A_5. Securities with high percentage compositions in SMVTC optimal portfolios do not comprise the MV optimal portfolios, for example, A_6, A_7 and A_{12}. The SMVTC portfolios tend to have the greatest number of assets, though in smaller quantities. Again 'SMC' is used to represent 'SMVTC' to ensure that the table fits well on the page.

Table 5.20: Assets' percentage compositions in out-of-sample optimal portfolios at different diversification levels: mean return and risk unconstrained

D.L	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10			10	10	10		10	10
	MV	10	8	10	10	10	5	4	4	10	9	10	1	10
	MM	10	10	10		10	10	10	10		10	10		10
	SMC	1	10	10	10	10	10	8	10		2	10	10	8
0.15	MAD	15	15	15	15					15	10		15	
	MV	15	6	14	15	6			3	15	7	15	1	3
	MM					15	15	15	15		10	15		15
	SMC	11	12		9		11	12	9	14		4	9	9
0.20	MAD	20		20	20					20			20	
	MV	20	4	11	20	3			3	20	4	13		1
	MM						20	20	20			20		20
	SMC		10	8	10	17	6	10	16			10	7	5
0.25	MAD	25			25					25			25	
	MV	25	2	9	22				4	25	1	12		
	MM						25	25	25			25		
	SMC	6	5	6	10	5	18	16		17		10	6	1
0.30	MAD	10			30					30			30	
	MV	23	2	9	21				4	30		12		
	MM						30	30	10			30		
	SMC	2	10	2	12	30		16	11			8	10	
0.35	MAD				30					35			35	
	MV	21	1	8	20				4	35		12		
	MM						30	35				35		
	SMC	17	6	33	4	3	2		4					31

5.6.3 Sensitivity analysis

Sensitivity analysis of the SMVTC model is done by finding sensitivity index (SI) for each parameter. Again, this is achieved by finding the output percentage difference when varying one input parameter, $x_i, i = 1, \cdots, 13$ from its minimum value (zero in this case) to its maximum value. The formula in equation (5.2) above is again used. Zero is taken as the minimum input value for each parameter with the maximum value being restricted by the diversification limit considered and is given as the best weight of the chosen asset in the optimal portfolio. The diversification limits of 0.1, 0.2 and 0.3 are considered.

Table 5.21: In-sample SMVTC model sensitivity analysis

D. L	Parameter	Max Value	Cost SI	Risk SI	Wealth SI
0.1	x_1	0.087	-0.52219	0.3537	0.0011
	x_2	0.086	-0.22235	0.4284	0.0009
	x_3	0.087	-0.123	0.5494	0.0009
	x_4	0.005	-0.03604	0.2563	0.0004
	x_5	0.079	-0.38049	0.5035	0.0010
	x_6	0.085	-0.23654	0.4579	0.0009
	x_7	0.081	-0.13111	-0.0277	0.0005
	x_8	0.071	-0.43636	0.5404	0.0010
	x_9	0.074	-0.35684	0.4354	0.0009
	x_{10}	0.072	-0.14823	0.2684	0.0004
	x_{11}	0.054	-0.21356	0.4858	0.0010
	x_{12}	0.087	-0.18901	0.3962	0.0009
	x_{13}	0.100	-0.11737	0.3432	0.0005

Table 5.21 Continued

D. L	Parameter	Max Value	Cost SI	Risk SI	Wealth SI
0.2	x_1	0.200	-0.41826	-1.1223	-0.00014
	x_3	0.002	0.01212	-1.7424	-0.00078
	x_4	0.005	-0.78858	0.1828	0.00029
	x_7	0.095	-0.39564	-0.5357	0.00010
	x_8	0.094	0.00539	-1.0943	-0.00051
	x_9	0.095	-0.34931	-1.6064	-0.00045
	x_{11}	0.177	-0.16079	-1.2772	-0.00052
	x_{12}	0.200	-0.13439	-0.3762	-0.00025
	x_{13}	0.134	-0.49475	-1.0061	-0.00004
0.3	x_1	0.141	0.29309	0.3428	0.00034
	x_2	0.126	-0.13336	0.8277	0.00107
	x_3	0.091	0.36906	0.3206	-0.00007
	x_4	0.011	-0.43091	0.2635	0.00061
	x_6	0.258	-0.27675	0.2767	0.00066
	x_7	0.032	-0.42106	0.0495	0.00007
	x_8	0.160	0.24424	0.5956	0.00023
	x_9	0.074	0.26226	-0.1817	-0.00006
	x_{11}	0.069	0.39636	0.6684	0.00040
	x_{12}	0.039	-0.52990	-0.8630	-0.00006

It should be noted that the SMVTC model is stochastic and works, as explained in above models, by replacing one parameter by a 'less' profitable one when the weight of the chosen asset is zero. The result in the table reflect small percentages of model variability due to each input change of each parameter. This shows that the output values are not strongly reliant on certain individual parameters. The cost sensitivity analysis of the in-sample reveals that most S.I. values are negative implying that the less preferred asset replacing the optimal-portfolio selected asset causes the investor to incur more costs. However, most of such assets result in reduced portfolio risk and a decline in portfolio wealth as most wealth S.I. values are positive.

5.6.4 Summary

A stochastic multi-stage mean-variance optimal portfolio strategy that takes into account uncertainty of asset returns and associated implicit transaction costs is proposed. The methodology allows investors or investment managers to choose optimal portfolios knowing implicit transaction cost levels to be incurred when executing the trading. It has emerged that the SMVTC-generated optimal portfolios have the least total implicit transaction costs, least risk and greatest net wealths compared to optimal portfolios generated by the mean-variance, mean absolute deviation and minimax models. However, common assets in optimal portfolios generated by both the MV and SMVTC models tend to be in larger quantities in MV than SMVTC portfolios. The strategy is suitable for conservative investors.

Chapter 6

Conclusion

In this Chapter, a summary of the work and results presented so far is given. The prospective research directions are discussed as well.

6.1 Summary

In this research, stochastic programming has been applied to find optimal portfolio strategies under market uncertainty. Five stochastic multi-stage portfolio optimisation models have been developed, taking into consideration uncertainty in implicit transaction costs. Models in the literature to date, to the best of the author's knowledge, do not incorporate implicit transaction costs by taking into account the fact that they are uncertain and that they are not always proportional to the assets' prices. Thus, implicit transaction costs have only been modeled as being proportional to asset prices. In this study, it has been developed and shown how uncertain implicit transaction costs can be embedded in financial optimisation models in order to have a more accurate representation of market uncertainty. The study considers scenario-based approaches and uncertain implicit transaction costs are taken as not proportional to asset prices.

Firstly, an extension of the Mean Absolute Deviation model to incorporate uncertainty in implicit transaction costs, and uncertainty in asset prices and portfolio risk is developed. This SMAD model shows that it generates optimal portfolios which are better than those generated by MV, MAD and MM models. It is suggested that implicit transaction costs incurred when buying initial portfolio and during portfolio rebalancing have a negative impact on portfolio returns. The analysis shows that implicit transaction costs account for approximately 14.3% of returns on investment.

Secondly, the SMADTC model is proposed whose objective function is to minimise total implicit costs incurred in initial trading and recourse periods, taking into account the investor's preferred portfolio risk. The model is shown to generate least-cost optimal portfolios with net wealth better than the MV, MAD and MM-generated optimal portfolios. In order to provide investors with competitive portfolio returns, investment managers must manage transaction costs pro-actively, because lower transaction costs mean higher portfolio returns. In this regard, the SMADTC model becomes a suitable tool. It is suitable for the conservative investors.

Consideration of a risk-averse and conservative investor who is concerned about the performance of his portfolio in an economic recession environment is made. A multistage stochastic optimal portfolio policy (SMNDTC) that minimises maximum downside risk in the presence of uncertainty in implicit transaction costs, asset returns and portfolio risk is developed. The model shows that it generates optimal portfolios with profitable returns despite extreme movements in asset prices. It is revealed that although the MM model appears to generate optimal portfolios with highest gross wealth, the portfolios also have the highest implicit transaction costs, resulting in having the least net portfolio wealth. The total implicit transaction costs incurred in initial trading of optimal portfolio vary from 7.1% to 16.7% of returns on investment.

The Mean-Variance and Mean Absolute Deviation models both penalise gains (upside deviations) and losses (downside deviations) in the same way, yet investors are concerned about downside deviations and are happy of upside deviations. They are also concerned about minimising uncertain implicit transaction costs. To overcome these difficulties, a dynamic stochastic multi-stage methodology (SMUDTC) in optimal portfolio selection that maximises upside deviations and minimises maximum downside risk while taking into account uncertain implicit transaction costs incurred in initial trading and in subsequent rebalancing of the portfolio is developed. The results show that the model generates optimal portfolios with least risk, highest portfolio wealth and minimum implicit transaction costs incurred when compared with MV, MAD and MM-generated portfolios.

Lastly, an extension of the Mean-Variance model is done to make it multi-period and incorporate uncertainty in implicit transaction costs, and uncertainty in asset returns and portfolio risk. Again results show that SMVTC-generated optimal portfolios have least total implicit transaction costs, least risk and greatest net wealth compared to optimal portfolios generated by MV, MAD and MM models. However, common assets in optimal portfolios generated by both MV and SMVTC models tend to be in larger quantities in MV than SMVTC optimal portfolios. This strategy is suitable for conservative investors.

6.2 Future work

To further address the issue of transaction costs, there is need to develop financial optimisation models that incorporate both explicit and implicit costs in an uncertain environment.

Chapter 7

Appendix 1

7.1 Stochastic linear programming with recourse

Some proofs of Theorems that are cited in the thesis and not proven are now given below.

Theorem 1 (Birge and Louveaux [5])

(a) *For a given ξ, the elementary feasibility set $K_2(\xi)$ is a convex polyhedron.*

(b) *When ξ is a finite discrete random variable, K_2 is a convex polyhedron.*

Proof

(a). Consider some x and ξ such that no $y \geq 0$ exists such that $W(s)y = r(s) - T(s)x$. Using the notation pos$W$, it is the same to say that we consider some x and ξ such that $r(s) - T(s)x \notin \text{pos}W(s)$. Thus, we have a point, $r(s) - T(s)x$, which does not belong to a convex set, pos$W(s)$. Then, there must exist some hyperplane, say $\{x|\sigma^T x = 0\}$, that separates $r(s) - T(s)x$ from pos$W$. This hyperplane satisfies $\sigma^T t < 0$ for $t \in \text{pos}W(s)$ and $\sigma^T(r(s) - T(s)x) > 0$. For one particular ξ, $W(s)$ is fixed and there can be only finitely many different such hyperplanes. This completes the proof.

(b). The intersection of finitely many convex polyhedra is a convex polyhedra. This completes the proof.

We state here that, for some fixed values of x and ξ, the value $Q(x, \xi)$ of the second-stage program is given by

$$Q(x, \xi) = \min_y\{q(s)^T y | W(s)y = r(s) - T(s)x, \;\; y \geq 0\}. \tag{7.0}$$

We state that an ill-defined model can result in being unbounded below or infeasible. However, unboundedness can be avoided by adding upper bounds on y, and infeasibility is avoided by considering $x \in K_2$. Thus, for $x \in K_2$, $Q(x, \xi)$ is finite for all ξ and we define

$$\Phi(x) = E_\xi Q(x, \xi) = \sum_{k=1}^{K} p_k Q(x, \xi)$$

where $k = 1, \cdots, K$ represents the K realisations of ξ. The equivalent deterministic program can be stated as

Minimise

$$z(x) = c^T x + \Phi(x)$$

Subject to

$$x \in K_1 \cap K_2.$$

where

$$K_1 = \{x | AX = b, \;\; x \geq 0\}$$

and

$$K_2(\xi) = \{x | y \geq 0 \text{ exists such that} W(s)y = r(s) - T(s)x\}$$

as defined in section 2.7.

Theorem 2: (Birge and Louveaux [5])

For any given ξ, the value function $Q(x, \xi)$ is

(i) *a piecewise linear convex function in r, T);*

(ii) *a piecewise linear concave function in q;*

(iii) *a piecewise linear convex function in x for all $x \in \mathbf{K_2}$.*

When ξ is a finite discrete random variable, $\Phi(x)$ is piecewise linear and convex on K_2.

Proof:

To prove convexity in (a) and (c), we just need to prove that $f(b) = \min\{q^T y | Wy = b\}$ is a convex function in b. We consider two different vectors, say b_1 and b_2, and some convex combination $b_\lambda = \lambda b_1 + (1-\lambda) b_2,\ \ \lambda \in (0,1)$.

Let y_1^* and y_2^* be some optimal solution of $\min\{q^T y | Wy = b\}$ for $b = b_1$ and $b = b_2$, respectively. Then we have $\lambda y_1^* + (1-\lambda) y_2^*$ as a feasible solution of $\min\{q^T y | Wy = b_\lambda\}$. By letting y_λ^* be an optimal solution of this last problem, we have

$$
\begin{aligned}
f(b_\lambda) &= q^T y_\lambda^* \leq q^T(\lambda y_1^* + (1-\lambda) y_2^*) \\
&= \lambda q^T y_1^* + (1-\lambda) q^T y_2^* \\
&= \lambda F(b_1) + (1-\lambda) f(b_2),
\end{aligned}
$$

which proves the required proposition. A similar proof can be given to show convexity in q. To prove piecewise linearity, we observe that solving (42) for given x and ξ leads to finding some square submatrix $B(s)$ of $W(s)$, called a basis, such that $y_B = B(s)^{-1}(r(s) - T(s)x,\ \ y_N = 0$, where y_B is the subvector associated with the columns of B and y_N includes the remaining components of y. A basis is feasible if $y_B \geq 0$ and a feasible basis is optimal if $\alpha_B(s)^T B(s)^{-1} W(s) \leq q(s)^T$. As long as these definitions hold, we have

$$Q(x,\xi) = q_B(s)^T B(s)^{-1}(r(s) - T(s)x),$$

which is linear in q, r, T and x on a domain defined by the feasibility and optimality conditions. Piecewise linearity follows from the existence of finitely many different optimal bases for the second-stage program. This completes the proof.

Chapter 8

Appendix 2

Here, information in tables that has been included in the discussion of the findings of empirical evaluation of proposed models is provided.

Table A1: Mean asset returns

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
1	0.099	0.422	-0.028	0.067	0.124	0.127	0.045
2	-0.102	-0.036	0.036	0.041	-0.057	-0.074	-0.078
3	-0.051	0.297	-0.027	0.143	-0.017	0.056	-0.082
4	-0.027	0.038	0.057	-0.021	0.014	-0.152	-0.042
5	-0.112	-0.036	-0.031	-0.138	-0.122	-0.188	-0.063
6	0.093	-0.125	0.32	0.036	0.062	0.099	0.304
7	0.084	0.071	0.113	0	0.179	0.04	0.134
8	-0.026	-0.193	-0.091	0.1	0	0.106	0.003
9	0.029	-0.173	-0.2	-0.018	0.03	-0.13	0.079
10	0.225	-0.2	0.101	-0.074	-0.048	0	-0.003
11	0.116	0.213	-0.103	0.04	0.118	-0.076	0.034
12	-0.074	-0.186	0.26	0.096	-0.029	-0.026	0.051
13	0.026	-0.139	-0.02	-0.035	-0.106	-0.022	-0.085
14	-0.136	0.294	-0.099	0.236	0.017	0.114	0.019
15	-0.046	0.02	-0.053	-0.043	0.097	0.163	0.076
16	0.061	0.069	0.111	0.022	0.064	-0.075	0.04
17	-0.029	0	0.141	-0.038	0.049	0.159	0.035
18	0.117	0.104	0.087	0.078	0.116	0.113	0.087
19	0.001	0.102	-0.017	0.014	-0.009	0.118	0
20	0	0.079	0.057	0.081	0.128	0.013	0.114

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
21	0.097	0.029	0.07	-0.077	0.079	0.048	0.056
22	-0.074	0.08	0.027	0.017	0.018	0.035	-0.092
23	0.083	0.004	0.082	0.056	0.083	0.048	0.074
24	0.012	-0.011	-0.084	-0.027	-0.028	0.017	0.005
25	0.033	0.036	0.053	-0.014	0.042	-0.006	0.095
26	0.089	0.069	0.12	0.034	0.074	0.101	0.032
27	0.008	0.018	0.05	-0.011	0.044	0.176	0.053
28	-0.031	-0.133	-0.052	-0.05	0.051	-0.07	0.069
29	-0.049	0.015	-0.039	0.072	0.043	0.028	-0.001
30	0.131	0.112	0.071	-0.027	0.065	0.16	0.128
31	0	-0.091	0.032	-0.068	0.037	0.004	-0.022
32	0.088	0.117	0.118	0.147	0.179	0.16	0.113
33	0.029	0.077	-0.005	0.212	0.034	0.014	0.157
34	0	0	-0.005	-0.049	-0.049	0.125	0.009
35	0.076	0.125	-0.011	0.057	-0.003	0.111	0.036
36	-0.005	0.055	-0.074	-0.048	-0.085	-0.069	-0.145
37	0.01	0.028	-0.043	-0.074	-0.029	-0.052	0.054
38	-0.01	0.056	-0.031	0.116	0.105	0.076	0.022
39	0.027	-0.012	0.025	-0.005	0.031	0.067	0.097
40	-0.028	0.012	0.077	0.043	-0.025	0.034	-0.047
41	0.042	0.014	-0.037	-0.023	-0.013	-0.018	0.065
42	0.033	0.02	-0.008	-0.035	-0.053	0.037	0.081
43	0.014	-0.018	0.014	0.011	0.043	-0.006	0.001
44	-0.011	-0.105	0.08	0.113	-0.097	0.005	-0.088
45	0.101	0.099	0.044	0.001	0.107	0.128	0.135

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
46	0.049	0.012	0.023	0.086	0.068	0.007	0.047
47	0.063	-0.041	-0.008	0.055	0.039	0	-0.003
48	0.045	0.086	0.031	-0.005	-0.141	0.117	0.081
49	0.057	0.106	0.09	-0.053	0.106	0.083	0.034
50	0.057	-0.043	0.091	0.133	0.019	0.038	0.057
51	0.039	0.098	0.061	0.04	0.046	0.025	0.115
52	0.01	0.057	-0.065	-0.007	0	-0.099	-0.01
53	0.027	0.068	0.07	0.054	0.149	0.051	0.074
54	0.182	-0.01	0.153	0.102	0.016	0.064	0.101
55	-0.009	0.007	-0.008	0.053	-0.003	0.02	0.049

Table A1 : Mean asset returns (continued)

Month	PNC	SPP	TRU	CPI	IPL	WHL
1	0.136	-0.008	0.02	0.177	0.053	0.057
2	0	-0.025	-0.038	-0.064	-0.009	-0.025
3	-0.022	0.155	0.004	-0.096	-0.034	-0.013
4	-0.068	-0.071	-0.105	-0.076	-0.308	-0.054
5	-0.146	-0.048	0.002	-0.115	-0.005	-0.097
6	-0.057	0.051	0.267	0.107	-0.165	0.155
7	0.288	0.038	0.073	-0.013	0.226	0.067
8	-0.141	-0.065	-0.064	0.085	0.094	-0.069
9	-0.285	0.129	0.136	-0.156	-0.07	-0.034
10	-0.272	-0.056	-0.043	0.037	-0.084	0.084
11	-0.042	0.06	0.075	0.036	0.184	0.038
12	0.049	-0.035	0.033	0.034	-0.137	0.058
13	-0.063	-0.031	-0.117	0	-0.138	-0.115
14	0.039	-0.026	0.032	0.081	0.207	-0.042
15	0.022	0.026	0.058	0.202	0.02	0.053
16	0.089	0.07	0.068	0.051	0.101	0.042
17	0.097	-0.004	0.019	0.035	-0.028	0.036
18	0	0.054	0.081	0.131	0.167	0.202
19	-0.004	0.043	-0.016	0.146	0.09	0.012
20	0.456	0.037	0.08	0.109	0.077	0.025
21	-0.033	0.066	0.059	0.049	0.02	0.083
22	-0.041	-0.041	-0.075	0.109	-0.006	-0.048
23	-0.02	0.074	0.048	0.111	0.092	0.072
24	0.1	-0.001	-0.028	-0.024	-0.092	0.024
25	0.125	0.028	0.166	0.065	0.17	0.127

Month	PNC	SPP	TRU	CPI	IPL	WHL
26	0.078	0.021	0.058	0.153	0.07	0.093
27	-0.023	0.029	0.017	0.076	-0.018	0.041
28	0.103	0.019	0.022	-0.03	-0.025	-0.013
29	-0.002	0.022	-0.013	0.197	-0.107	0.035
30	0.131	0.071	0.087	0.093	0.124	0.085
31	-0.021	-0.017	0.004	0.035	0.059	-0.053
32	0.114	0.105	0.196	0.116	0.111	0.099
33	0.068	0.025	-0.014	-0.021	0.007	0.015
34	0.092	0.062	0.043	0.019	0.072	-0.038
35	0.107	-0.033	-0.005	0.156	0.041	0.02
36	-0.034	-0.071	-0.11	-0.109	-0.137	-0.128
37	0.03	0.046	0	0.039	0.021	0.136
38	0.029	0	0.106	0.057	0.016	0.047
39	0.014	0.026	0.078	0.037	0.033	0.072
40	0.099	-0.05	-0.047	0.085	-0.013	-0.007
41	0.154	-0.026	0.012	-0.042	0.041	0
42	0.022	0.02	-0.012	0.001	-0.052	0.061
43	-0.076	0.035	0.071	0.04	0.007	0.155
44	0.116	0.011	-0.091	0.019	-0.091	-0.04
45	0.138	0.007	0.138	-0.052	0.116	0.154
46	-0.029	0.16	-0.012	0.011	0.007	0.001
47	0.044	-0.035	-0.068	-0.033	0.044	-0.035
48	0.185	0.02	0.059	0.024	0.122	0.077
49	0.059	0.041	0.034	0.015	0.063	0.074
50	0.039	0.004	-0.001	0.108	0.052	0.067

Month	PNC	SPP	TRU	CPI	IPL	WHL
51	0.113	0.05	0.028	0.083	0.089	0.01
52	-0.037	-0.097	-0.009	0.017	-0.021	0.003
53	0.035	0.035	0.089	-0.058	0.04	0.031
54	0.002	0.054	0.154	0.026	0.1	0.071
55	0.061	0.011	0.003	-0.012	0.005	0.016

Table A2 :Asset Transaction Cost rates

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
1	0.0030	0.0023	0.0028	0.0036	0.0050	0.0119	0.0073
2	0.0004	0.0009	0.0028	0.0029	0.0023	0.0061	0.0051
3	0.0054	0.0008	0.0022	0.0052	0.0032	0.0425	0.0088
4	0.0087	0.0064	0.0121	0.0040	0.0107	0.0011	0.0019
5	0.0043	0.0062	0.0027	0.0080	0.0034	0.0004	0.0229
6	0.0000	0.0140	0.0072	0.0386	0.0098	0.0067	0.0012
7	0.0066	0.0048	0.0195	0.0214	0.0147	0.0018	0.0128
8	0.0048	0.0044	0.0306	0.0171	0.0548	0.0137	0.0024
9	0.0042	0.0061	0.0063	0.0078	0.0132	0.0018	0.0007
10	0.0100	0.0220	0.0027	0.0018	0.0213	0.0025	0.0060
11	0.0060	0.0003	0.0112	0.0258	0.0041	0.0085	0.0136
12	0.0143	0.0045	0.0170	0.0049	0.0015	0.0258	0.0019
13	0.0003	0.0126	0.0010	0.0055	0.0083	0.0135	0.0037
14	0.0033	0.0051	0.0029	0.0041	0.0019	0.0015	0.0020
15	0.0064	0.0023	0.0004	0.0001	0.0228	0.0026	0.0082
16	0.0057	0.0114	0.0042	0.0054	0.0092	0.0139	0.0140
17	0.0010	0.0087	0.0435	0.0165	0.0099	0.0073	0.0105
18	0.0537	0.0047	0.0067	0.0212	0.0045	0.0109	0.0028
19	0.0007	0.0052	0.0027	0.0474	0.0056	0.0053	0.0113
20	0.0060	0.0117	0.0126	0.0856	0.0046	0.0295	0.0033
21	0.0061	0.0303	0.0052	0.0532	0.0169	0.0033	0.0127
22	0.0070	0.0003	0.0087	0.0194	0.0268	0.0149	0.0049
23	0.0076	0.0029	0.0046	0.0074	0.0003	0.0008	0.0103
24	0.0056	0.0473	0.0025	0.0069	0.0127	0.0055	0.0026
25	0.0018	0.0058	0.0095	0.0070	0.0079	0.0046	0.0056

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
26	0.0149	0.0074	0.0327	0.0072	0.0009	0.0019	0.0045
27	0.0037	0.0208	0.0032	0.0233	0.0104	0.0035	0.0165
28	0.0202	0.0065	0.0256	0.0301	0.0051	0.0021	0.0073
29	0.0068	0.0225	0.0141	0.0061	0.0018	0.0056	0.0116
30	0.0298	0.0000	0.0343	0.0068	0.0695	0.0253	0.0094
31	0.0072	0.0197	0.0027	0.0408	0.0084	2.0000	0.0115
32	0.0157	0.0153	0.0066	0.0679	0.0040	0.0749	0.0058
33	0.0139	0.0238	0.0015	0.0019	0.0040	0.0286	0.0111
34	0.0142	0.0024	0.0079	0.0073	0.0013	0.0105	0.0024
35	0.0074	0.0186	0.0068	0.0326	0.0198	0.0066	0.0114
36	0.0316	2.0000	0.0082	0.0146	0.0487	0.0074	0.0133
37	0.0041	0.0083	0.0433	0.0155	0.0369	0.0082	0.0075
38	0.0459	2.0000	0.0058	0.0014	0.0029	0.0354	0.0033
39	0.0006	0.0386	0.0012	0.0046	0.0334	0.0177	0.0339
40	0.0743	0.0225	0.0097	0.0044	0.0148	0.0202	0.0543
41	0.0210	0.0290	0.0103	0.0952	0.0144	0.1277	0.0402
42	0.0208	0.0022	0.0179	0.0222	0.0308	0.1053	0.0089
43	0.0217	2.0000	0.0177	0.0396	0.0238	2.0000	0.0052
44	0.0053	2.0000	0.0151	0.0619	0.0033	0.0942	0.0112
45	0.0124	0.0942	0.0077	0.0759	0.0183	0.0408	0.0083
46	0.0086	0.0083	0.0800	0.0352	0.0215	0.0258	0.0014
47	0.0169	0.0645	0.0026	0.0829	0.0064	0.0077	0.0271
48	0.0060	0.0072	0.0185	0.0769	0.0068	0.0198	0.0020
49	0.0209	2.0000	0.0255	0.1452	0.0000	0.0217	0.0047
50	0.0330	0.0108	0.0132	0.1125	0.0043	0.0089	0.0277

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
51	0.0040	0.0189	0.0016	0.0894	0.0086	0.0308	0.0114
52	0.0032	0.0028	0.0110	2.0000	0.0078	0.0048	0.0164
53	0.0317	0.0237	0.0026	0.0734	0.0007	0.0543	0.0101
54	0.0180	0.0220	0.0085	0.0112	0.0104	0.0151	0.0228

Table A2 :Asset Transaction Cost rates

Month	PNC	SPP	TRU	CPI	IPL	WHL
1	0.0319	0.0019	0.0009	0.0063	0.0058	0.0062
2	0.0323	0.0072	0.0033	0.0105	0.0132	0.0010
3	0.0038	0.0044	0.0241	0.0061	0.0067	0.0018
4	0.0049	0.0035	0.0028	0.0411	0.0040	0.0029
5	0.0058	0.0021	0.0031	0.0031	0.0005	0.0013
6	0.0046	0.0024	0.0056	0.0031	0.0050	0.0010
7	0.0315	0.0029	0.0066	0.0056	0.0050	0.0008
8	0.0086	0.0946	0.0540	0.0121	0.0542	0.0069
9	0.0263	0.0009	0.0068	0.0003	0.0019	0.0027
10	0.0148	0.0030	0.0128	0.0108	0.0052	0.0023
11	0.0202	0.0019	0.0063	0.0459	0.0121	0.0689
12	0.0043	0.0109	0.0029	0.0071	0.0078	0.0099
13	0.0067	0.0007	0.0023	0.0028	0.0042	0.0044
14	0.0127	0.0017	0.0033	0.0032	0.0043	0.0057
15	0.0197	0.0041	0.0036	0.0191	0.0084	0.0020
16	0.0072	0.0012	0.0177	0.0047	0.0100	0.0050
17	0.0074	0.0001	0.0185	0.0050	0.0008	0.0004
18	0.0122	0.0085	0.0077	0.0005	0.0002	0.0098
19	0.0088	0.0007	0.0015	0.0054	0.0085	0.0015
20	0.0081	0.0078	0.0324	0.0030	0.0027	0.0046
21	0.0179	0.0085	0.0022	0.0031	0.0037	0.0271
22	0.0228	0.0022	0.0011	0.0101	0.0097	0.0234
23	0.0106	0.0305	0.0077	0.0143	0.0010	0.0220
24	0.0293	0.0053	0.0024	0.0039	0.0137	0.0073
25	0.0212	0.0050	0.0127	0.0125	0.0205	0.0421

Month	PNC	SPP	TRU	CPI	IPL	WHL
26	0.0452	0.0019	0.0145	0.0145	0.0034	0.0121
27	0.0178	0.0024	0.0137	0.0069	0.0047	0.0223
28	0.0203	0.0093	0.0027	0.0053	0.0064	0.0071
29	0.0526	0.0043	0.0026	0.0037	0.0040	0.0019
30	0.0595	0.0113	0.0095	0.0065	0.0133	0.0016
31	2.0000	0.0058	0.0069	0.0127	0.0096	0.0028
32	0.0470	0.0020	0.0021	0.0140	0.0086	0.0300
33	0.0287	0.0151	0.0018	0.0047	0.0047	0.0114
34	0.0030	0.0023	0.0068	0.0149	0.0068	0.0025
35	0.0545	0.0032	0.0261	0.0091	0.0125	0.0184
36	0.0590	0.0017	0.0119	0.0092	0.0103	0.0032
37	0.0874	0.0142	0.0117	0.0012	0.0022	0.0117
38	0.0597	0.0039	0.0055	0.0241	0.0013	0.0080
39	0.2246	0.0186	0.0064	0.0102	0.0215	0.0058
40	0.0488	0.0107	0.0041	0.0015	0.0079	0.0302
41	0.0845	0.0252	0.1433	0.0481	0.0299	0.1060
42	0.0207	0.0077	0.0088	0.0236	0.0164	0.0105
43	0.2118	0.0357	0.0239	2.0000	0.0077	0.0008
44	0.0267	0.0080	0.0066	0.0142	0.0078	0.0123
45	0.2449	0.0741	0.0075	0.0714	0.0082	0.0370
46	0.2290	0.0050	0.0187	0.0546	0.0328	0.0481
47	0.0047	0.0093	0.0123	0.0165	0.0110	0.0169
48	0.0625	0.0012	0.0007	0.0479	0.0099	0.0273
49	0.1374	0.0228	0.0110	0.0147	0.0042	0.0069
50	0.0247	0.0179	0.0013	0.0165	0.0024	0.0608

Month	PNC	SPP	TRU	CPI	IPL	WHL
51	0.0114	0.0011	0.0155	0.0469	0.0007	0.0109
52	0.0270	0.0062	0.0650	0.0408	0.0336	0.0198
53	0.0110	0.0082	0.0633	0.0524	0.0507	0.0112
54	0.0228	0.0445	0.0136	0.0202	0.0033	0.0347

Table A3 : Stage 1 Efficient Frontiers at various Diversification limits

Div. Lim	Gross mean	No. of Assets	Net mean	MAD(risk)	Wealth
0.1	0.013	11	0.011	2.1684×10^{-19}	10114.320
	0.02	12	0.018	0.007	10178.771
	0.03	11	0.025	0.017	10253.167
	0.04	10	0.036	0.018	10355.180
0.125	0.013	12	0.011	2.4×10^{-18}	10112.736
	0.02	10	0.018	0.007	10181.466
	0.03	10	0.028	0.017	10276.063
	0.04	8	0.034	0.021	10344.850
0.15	0.013	10	0.011	1.599×10^{-18}	10111.780
	0.02	11	0.018	0.007	10182.074
	0.03	11	0.025	0.017	10248.470
	0.04	7	0.034	0.023	10336.510
0.2	0.013	6	0.008	4.33×10^{-19}	10081.526
	0.02	6	0.015	0.007	10151.170
	0.03	7	0.026	0.017	10255.143
	0.04	5	0.036	0.026	10363.520
0.25	0.013	8	0.009	2.3×10^{-19}	10085.628
	0.02	10	0.016	0.007	10158.532
	0.03	7	0.026	0.017	10256.967
	0.04	6	0.037	0.027	10366.508
0.3	0.013	7	0.011	2.5×10^{-19}	10105.436
	0.02	12	0.018	0.007	10175.404
	0.03	7	0.028	0.017	10282.641
	0.04	6	0.037	0.027	10369.217

Table A4: Stage 2 Efficient frontiers at various diversification limits

Div. Lim	Gross mean	No. of Assets	Net mean	MAD(risk)	Wealth
0.2	0.007	5	0.006	0.013	10423.815
	0.010	5	0.009	0.015	10460.266
	0.015	5	0.014	0.017	10513.668
	0.020	5	0.019	0.020	10562.684
	0.025	5	0.024	0.022	10611.495
	0.030	5	0.029	0.025	10664.308
	0.035	5	0.034	0.027	10714.813
0.225	0.007	5	0.006	0.013	10424.277
	0.010	5	0.009	0.015	10459.822
	0.015	5	0.014	0.017	10513.566
	0.020	5	0.019	0.020	10562.732
	0.025	5	0.024	0.022	10611.730
	0.030	5	0.029	0.025	10662.818
	0.035	5	0.034	0.027	10715.476
0.25	0.007	5	0.006	0.013	10425.228
	0.010	5	0.009	0.015	10461.394
	0.015	5	0.014	0.017	10513.540
	0.020	5	0.019	0.020	10562.598
	0.025	5	0.024	0.022	10611.730
	0.030	5	0.029	0.025	10660.993
	0.035	5	0.034	0.027	10714.008

Table A4 Continued

Div. Lim	Gross mean	No. of Assets	Net mean	MAD(risk)	Wealth
0.275	0.007	5	0.006	0.013	10424.843
	0.010	5	0.009	0.015	10460.696
	0.015	5	0.014	0.017	10513.540
	0.020	5	0.019	0.020	10562.598
	0.025	5	0.024	0.022	10611.730
	0.030	5	0.029	0.025	10660.733
	0.035	5	0.034	0.027	10712.199

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Publications

Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg stock market

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Abstract: We propose a multi-stage stochastic mean absolute deviation model with random transaction costs in optimal portfolio selection. We take implicit costs incurred in trading as our transaction costs. The multi-stage stochastic model captures risk due to uncertainty, as well as implicit transaction costs incurred by an investor during initial trading and subsequent rebalancing of the portfolio. We apply the model to securities on the Johannesburg stock market and find out that implicit transaction costs are at least 14.3% of returns on investment.

Keywords: stochastic mean absolute deviation; random transaction costs; uncertainty; stochastic programming; portfolio rebalancing.

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1 Introduction

Methods of operations research, especially mathematical programming methods, are receiving broader acceptance in the economic and financial industry. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making process. This research intends to address the portfolio selection problem by applying stochastic programming. We construct a multi-stage stochastic mean absolute deviation (MAD) model that captures asset returns and risk due to uncertainty.

The MAD model was first proposed by Konno and Yamazaki (1991), in deterministic form, as an alternative to the famous and widely used mean-variance model by Markowitz (1952). We have adopted stochastic programming as it has a number of advantages over other techniques. Firstly, stochastic programming models can accommodate general distributions by means of scenarios. We do not have to explicitly assume a specific stochastic process for the securities' returns, but we can rely on the empirical distribution of these returns. Secondly, they can address practical issues such as transaction costs, turnover constraints, limits on securities and prohibition of short-selling. Regulatory and institutional or market-specific constraints can be accommodated. Thirdly, they can flexibly use different risk measures. This presents the choice to optimise the appropriate risk measure or utility function.

MAD is a dispersion-type risk linear programming (LP) computable measure that may be considered as an approximation of the variance when the absolute values replace the squares. Konno and Yamazaki (1991) propose the MAD model as a risk measure to overcome the weaknesses of the variance. This MAD model is equivalent to the mean-variance model by Markowitz (1952) if the assets' returns are multivariate normally distributed. However, the MAD model is a special case of piecewise linear risk model which is fast in optimising portfolios by means of LP unlike the mean-variance model that requires quadratic programming. The use of a linear model considerably reduces the time needed to reach a solution, thereby making it more appropriate for large-scale portfolio selection. It makes extensive calculations of the covariance matrix unnecessary, as opposed to the mean-variance model. The MAD model is also sensitive to outliers in historical data (Byrne and Lee, 2004).

The MAD model has its short comings. Ignoring the covariance matrix can cause great estimation risk (Simaan, 1997). MAD also penalises not only the negative deviations, but also the positive deviations. Nevertheless, investors prefer higher positive deviations and avoid lower negative deviations in portfolio returns. Fama (1965) explains that making a distinction between positive and negative returns is necessary if portfolio returns are asymmetrically distributed and stock returns are skewed. Fishburn (1977) introduced downside-risk measures to deal with such problems. The advantage of downside-risk measures is that they only penalise returns below a given threshold level specified by the investor. Michalowski and Ogryczak (2001) extend the MAD model to incorporate downside-risk aversion. Hoe et al. (2010) make an empirical comparison of the mean-variance, MAD, minimax and mean-semi-variance models in portfolio optimisation. They compare the portfolio compositions and performance of different optimal portfolios by using data of monthly returns of 54 stocks included in the Kuala Lumpur Composite index from January 2004 to December 2007. They find the most risk portfolio to be that of the mean-variance model while the MAD model generated portfolio is the least risky. The minimax model shows the highest performance, followed by the MAD model, and the mean-variance gives the least performance. However, despite the good performance by the minimax model, it has its disadvantages. Because of its objective to minimise maximum loss, minimax is sensitive to outliers in historical data (Young, 1998).

Xiao and Tian (2012) estimate implicit transaction cost in Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. They use the Bayesian Gibbs sampling method proposed by Hasbrouck (2009) to analyse implicit costs in the bull and bear markets. Hasbrouck (2009) incorporates the Gibbs estimates into asset pricing specifications over a historical sample and find that effective cost is positively related to stock returns. Kozmik (2012) discusses an asset allocation with transaction costs formulated as a multi-stage stochastic programming model. He considers transaction costs as proportional to the value of the assets sold or bought, but does not consider implicit trading costs in the model. He employs conditional-value-at-risk as a risk measure. Moallemi and Saglam (2011) study dynamic portfolio selection models with Gaussian uncertainty using linear decision models incorporating proportional transaction costs. They assume that trading costs such as bid-ask spread, broker commissions, and exchange fees are proportional to the trade size. However, as stock prices follow a random-walk process, this would result in trading costs fluctuating due to a number of factors that include liquidity of stocks, market impact and so on. Considering proportional transaction costs in an uncertain environment does not provide good estimate of trading costs, especially implicit transaction costs. Brown and Smith (2011) study the problem of dynamic portfolio optimisation in a discrete-time finite-horizon setting, and again, consider proportional trading costs. Lynch and Tan (2006) study portfolio selection problems with multiple risky assets. They develop analytic frameworks for the case with many assets taking proportional transaction costs. Korn (2011) studies continuous-time portfolio optimisation and considers proportional transaction costs. Cai et al. (2013) examine numerical solution of dynamic portfolio optimisation with transaction costs. While transaction costs are broad and includes explicit as well as implicit costs, they consider a case of proportional transaction costs. These proportional trading costs can be either explicit or implicit, whichever is greater. However, the study of transaction costs requires the identification of the type of cost to be estimated in order to explore effective ways of having a good estimate. Thus, in our study, we concentrate on implicit transaction costs as these are invisible and difficult to measure. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, 2008).

Most models presented in the literature are static models: a decision is made, then not further modified. They are essentially single-period models, since there is only one decision to be made, for the first period. Real-life decision processes are more complicated. Although we must make an initial decision now, there will be many opportunities to adjust down the road. We do not, today, have a complete decision basis for future decisions – the future is unknown. Stochastic programming models hence are dynamic, covering multiple time periods with associated separate decisions, and they account for the stochastic decision process. The main features of stochastic programming are scenarios and stages. The uncertainty about future events are captured by a set of scenarios, which is a representative and comprehensive set of possible realisations of the future. Stochastic programming recognises that future decisions happen in stages. A first decision now, then after a certain time period, a second decision is made which depends upon the first stage decision and the events that occurred during the time period.

The purpose of this study is to construct a multi-stage stochastic MAD portfolio optimisation model with random transaction costs that captures assets' returns and risk due to uncertainty. The model employs stochastic programming with recourse by taking into account rebalancing of portfolio composition as the uncertainty of returns get realised. MAD models that are proposed in the literature do not take the uncertainty of the future into account. Of the models that account for transaction costs in portfolio selection, they do not consider random transaction costs. Konno and Wijayanayake (2001) proposed the deterministic MAD model with transaction costs modelled by a concave function. They use a linear cost function as an approximation to the concave cost function. Gulpinar et al. (2004) propose a multi-stage mean-variance portfolio analysis with non-random transaction costs. Yu et al. (2006) propose a multi-period portfolio selection with l_∞ model. They employ the l_∞ function to control the risk in every period. However, no transaction costs are considered in the model. Again, the model does not account for uncertainty of future events.

This paper is organised as follows. In Section 2, we discuss the formulation of the stochastic MAD model with random transaction costs for multi-period optimal portfolio selection. Transaction costs and portfolio rebalancing constraints are explained. Section 3 is devoted to discussing implicit transaction costs during trading and how they affect portfolio returns. Explicit transaction costs are not considered in this study as these can be known before execution of the trade. In Section 4, we demonstrate the application of the model to securities taken from the Johannesburg stock market, courtesy of INet Bridge. The merits of the model are clearly revealed in the application. We conclude, in Section 5, by giving a summary of our findings.

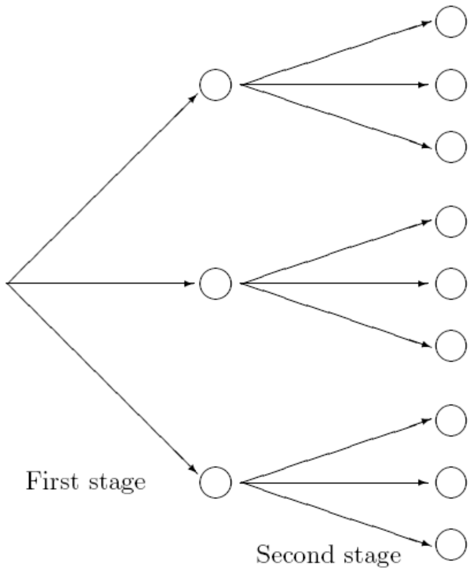
2 Problem statement

We determine a multi-period discrete-time optimal portfolio strategy over a given investment horizon. We shall achieve this by extending the MAD model to account for uncertainty in assets' returns and random transaction costs incurred during portfolio rebalancing. To define the problem, we divide the entire investment horizon T into two discrete time intervals T_1 and T_2, where $T_1 = 0, 1, \cdots, \tau$ and $T_2 = \tau + 1, \cdots, T$. The portfolio is structured over this period in terms of both assets' returns and risk, where the risk is measured by the MAD of assets' returns from expected portfolio return. Period τ defines the planning horizon. During T_1, an investor makes decisions and adjustments to his portfolio at each time-stage as some return realisations unfold. Thus, after the initial investment at $t = 0$, the portfolio may be restructured at discrete times $t = 1, 2, \cdots, \tau$; and after period τ, no further decisions are implemented until investment maturity at $t = T$. This restructuring of the portfolio brings with it transaction costs, as the investor sells or buys shares of some securities to balance his portfolio. A number of articles in the literature proposed models that do not account for these transaction costs. By ignoring such costs and considering only returns realised at time stages $t = 1, 2, \cdots, \tau$, we are overvaluing the portfolio or the portfolio may not be optimal.

2.1 Scenario generation

We consider a set of securities $I = \{i: i = 1, 2, \cdots, n\}$ for an investment. Let $R_t = \{R_1, \cdots, R_\tau\}$ be stochastic events at $t = 1, 2, \cdots, \tau$. The decision process is non-anticipative (i.e., a decision at a given stage does not depend on the future realisation of the random events). The decision at period t depends on the outcome at period $t - 1$. We define a scenario as a possible realisation of the stochastic variables $\{R_1, R_2, \cdots, R_\tau\}$. Hence, the set of scenarios corresponds to the set of paths followed from the root to the leaves of a tree, S_τ, and nodes of the tree at level $t \geq 1$ corresponds to possible realisations of R_t. Each node at a level t corresponds to a decision which must be determined at time t, and depends in general on R_t, the initial wealth of the portfolio and past decisions. Given the event history up to time t, R_t, the uncertainty in the next period is characterised by finitely many possible outcomes for the next observations R_{t+1}. The branching process is represented by a scenario tree. An example of a scenario tree with two time periods and three-three branching structure is shown in Figure 1.

Figure 1 Scenario tree



The uncertain return of the portfolio at the end of the period t is $R = R(x_t, r_t)$. This is a random variable with a distribution function, say F, given by

$$F(x, \mu) = p\{R(x, \tau) \leq \mu\}$$

We assume that the distribution F does not depend on the portfolio composition x. The expected return of the portfolio at the end of period t is

$$r_{pt} = E\left[R\left(x_t, R_t\right)\right] = r\left(x_t, R_t\right).$$

Suppose the uncertain returns of the assets, R_t, in period t are represented by a finite set of discrete scenarios $\Omega = \{s\colon s = 1, 2, \cdots, S\}$, whereby the returns under a particular scenario $s \in \Omega$ take the values $R_s = (R_{1s}, R_{2s}, \cdots, R_{ns})^T$ with associated probability $p_s > 0$, where $\sum_{s\in\Omega} p_s = 1.$ The mean return of assets in period t is

$$r_t = \sum\nolimits_{s\in\Omega} p_s R_{st}, t = 1, 2, \cdots, \tau.$$

The portfolio return under a particular realisation of asset return R_s of period t is denoted by $r_{st} = r(x_t, R_{st})$. The expected portfolio return of period t is now given by

$$r_{pt} = E\left[r\left(x_t, R_{st}\right)\right] = \sum\nolimits_{s\in\Omega} p_s r\left(x_t, R_{st}\right)$$

2.2 Capital allocation

As the problem is dynamic in nature, the wealth to be invested varies with time. Since we assume that the investor joins the market at $t = 0$, with an initial wealth W_0, W_t becomes the wealth at period t. The wealth can be re-allocated among the n-assets at the beginning of each of the τ consecutive time periods for portfolio rebalancing. The rate of asset' return of security i of period t in scenario s within the planning horizon is denoted by $R_i = (R_{1st}, R_{2st}, \cdots, R_{nst})$, where R_{ist} is the random return of security i of scenario s in period t. The return R_{st} has a mean defined by $r_{st} = E(R_{st})$, $t = 1, 2, \cdots, \tau$.

Let x_{ist} be the proportion of the risky asset i of scenario s at the beginning of period t. An investor is seeking the best investment strategy, $x_t = [x_{1st}, x_{2st}, \cdots, x_{nst}]$ for $t = 1, 2, \cdots, \tau$, such that

$$\sum\nolimits_{i=1}^{n} x_{it} = \sum\nolimits_{s=1}^{S} x_{ist} = 1, t = 1, 2, \cdots, \tau.$$

Let a_{ist} and v_{ist} be respectively the buying and selling volumes so that $x_{ist} + a_{ist} - v_{ist}$ is the volume invested in the i^{th} asset of scenario s at the beginning of period t. Thus, we also have

$$\sum\nolimits_{s=1}^{S} \left(x_{ist} + a_{ist} - v_{ist}\right) = 1, i = 1, 2, \cdots, n; t = 1, 2, \cdots, \tau.$$

Let k_{ist} and l_{ist} be, respectively, the rate of buying and selling transaction costs of the volume of asset i of scenario s bought or sold for portfolio rebalancing at the beginning of period t. It must be noted that the following are consequences of transaction costs:

a The fact that transactions have costs ensures that for the same asset, $a_{ist} \cdot v_{ist} = 0$; that is, the buy and sell variables corresponding to the same asset can never be non-zero simultaneously.

b The incorporation of transaction costs in the model provides essential 'friction' which without it the optimisation has complete freedom to re-allocate the portfolio every time period, which, (if implemented) can result in significantly poorer realised performance than forecast, due to excessive transaction costs.

Hakansson (1971) explains that in the absence of transaction costs, myopic policies are sufficient to achieve optimality. In our model, money can only be added at $t = 0$, and not in subsequent periods. Thus, only the initial wealth, W_0, is considered.

2.3 Transaction costs and balance constraints

In rebalancing the portfolio at any period $t > 0$, $t = 1, 2, \cdots, \tau$, buying and selling of securities take place. The buying and selling of securities during initial trading and rebalancing of the portfolio result in the investor incurring some explicit and implicit transaction costs. Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. On the other hand, implicit costs are invisible. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of execution. In this study, we take transaction costs to mean implicit transaction costs, details of which are explained in Section 3. The impact of transaction costs on the mean-variance model have been studied by Konno and Yamazaki (1991), and Gulpinar et al. (2004). The decision made at period t depends on x_{ist} and the yield of the investment in asset i of scenario s as

$$r_{it} = r_{ist} \cdot p_s \cdot (x_{ist} + a_{ist} - v_{ist}), i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \tag{2.1}$$

and

$$0 \leq \sum_{i \in A} b_{ist} = \sum_{j=1, j \neq i} q_{jst}; i = 1, \cdots, n; s = 1, \cdots, S \tag{2.2}$$

where $b_{ist} = b_{ist}(a_{ist})$ is the amount of money used for buying volume a_{ist}, and $q_{ist} = q_{ist}(v_{ist})$ is the money obtained from selling volume v_{ist} of asset i of scenario s in period t. Note here that A is the set containing all assets i for which volumes have been bought. Thus, constraint (2) justifies that the amount of money used to buy volumes of asset i should be the same as the amount obtained from selling volumes of asset j, ($j \neq i$) of period t. Another constraint is given by

$$0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \tag{2.3}$$

This constraint explains that the volume of asset i of scenario s in period t sold for portfolio rebalancing should not exceed the volume of the asset in the portfolio. The third constraint is the one discussed above, that is,

$$a_{ist} \cdot v_{ist} = 0 \tag{2.4}$$

It should be noted that the scenarios may reveal identical value for the uncertain quantities up to a certain period. These scenarios that share common information must yield the same decisions up to that period. Thus, we have the constraint

$$x_{ist} = x_{is't},$$

for all scenarios s and s' with identical past up to time t.

2.4 Expected wealth

Generally, the objective of an investor is to minimise portfolio risk while at the same time maximising expected portfolio return on investment, or achieving a prescribed expected return. Thus, the mean rate of return of the portfolio of period t is given by

$$r_{pt} = p_s \cdot r_{1st} \cdot (x_{1st} + a_{1st} - v_{1st}) + \cdots + p_s \cdot r_{nst} \cdot (x_{nst} + a_{nst} - v_{nst})$$
$$= \sum_{i=1}^{n} p_s \cdot r_{ist} \cdot (x_{ist} + a_{ist} - v_{ist}),\ s = 1, \cdots, S;\ t = 1, \cdots, \tau.$$

The wealth of period t, without transaction costs, is given by

$$W_t = (1 + r_{pt}) \cdot W_{t-1},\ t = 1, \cdots, \tau \tag{2.5}$$

Clearly, it can be said that the expected rate of return is a linear function of ($x_{ist} + a_{ist} - v_{ist}$). Taking transaction costs into consideration, we have the transaction cost of asset i in scenario s of period t to be

$$k_{ist} a_{ist} + l_{ist} v_{ist}$$

where $k_{ist}a_{ist}$ is the cost for buying volume a of asset i and $l_{ist}v_{ist}$ is the cost of selling volume v of asset i in period t. Since a_{ist} and v_{ist} cannot be non-zero simultaneously for each asset i (as is shown later in this section), we either have

$$k_{ist} a_{ist} = 0 \text{ or } l_{ist} v_{ist} = 0$$

or both being zero if there is no buying or selling of asset i in scenario s of period t. Allowing co-movement of assets' prices and corresponding implicit transaction costs during trading, we say that each scenario asset price is associated with it an implicit transaction cost. Thus, the expected transaction cost of asset i in period t is given by

$$\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\},\ i = 1, \cdots, n;\ t = 1, \cdots, \tau.$$

We therefore observe that the total portfolio transaction cost of period t becomes

$$\sum_{i=1}^{n} \left[\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \right],\ t = 1, 2, \cdots, \tau.$$

Denoting the net expected portfolio return of period t by N_{pt}, we get

$$N_{pt} = r_{pt} - \sum_{i=1}^{n} \left[\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \right].$$

Thus, the wealth of period t taking transaction costs into account is given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1},\ t = 1, \cdots, \tau. \tag{2.6}$$

2.5 Expected risk

The portfolio risk for any realisation of any period is measured by the MAD of the realised returns relative to the expected portfolio return, r_{pt}. Konno and Yamazaki (1991) develop the deterministic MAD model in an attempt to improve on the famous Markowitz (1952) mean-variance model. Their MAD model has portfolio risk expressed as

$$H(p) = \frac{1}{\mathrm{T}} \sum_{t=1}^{T} \left| \sum_{i=1}^{n} (r_{it} - r_i) x_i \right|$$

where *rit* is the realised return of asset *i* of period *t*, r_i is the expected return of asset *i* per period, and x_i is the proportion of wealth invested in asset *i*.

We therefore extend this formulation of portfolio risk by constructing a model that takes into account uncertainty of asset returns and randomness of transaction costs in portfolio rebalancing. Using the deterministic model as our basis, three key stochastic framework elements are incorporated to formulate the stochastic MAD model. The first element considered is the concept of scenarios. Since the deterministic model represents one particular scenario, the inclusion of multiple scenarios in capturing uncertainty result in the increase in the number of variable parameters. Thus, uncertainty is represented by a set of distinct realisations $s \in \Omega$. We now consider the scenario parameter, along with other parameters of security and time period. Secondly, the stochastic MAD model allows for the implementation of recourse decisions as unfolding information on assets' returns get realised. The third element is the probabilistic feature of the stochastic framework which assigns probabilities to scenarios. The parameter p_s represents scenario probability. Scenarios may reveal identical value for the uncertain quantities up to a certain period. Scenarios that share common information must yield the same decisions up to that period.

The stochastic MAD also incorporates transaction costs incurred during portfolio rebalancing at each time period. Thus, we obtain the following portfolio risk:

$$H\left(r_{pt}\right) = \frac{1}{\tau}\sum_{t=1}^{\tau}\left|\sum_{s=1}^{S} p_s\left(R_{st} - r_{st}\right)x_{st}\right| \tag{2.7}$$

where R_{st}, r_{st} and x_{st} are as defined earlier. It should be noted that information revealed in the literature show transaction costs being approximated by a linear function or a step function. In some cases, they are considered to be proportional to volume of asset bought or sold. In either case, we argue that since returns are random, transaction costs can as well be random. Hence, in our model, we consider random transaction costs. Yu et al. (2003) considered symmetric transaction costs for buying and selling during portfolio rebalancing in their general mean-risk model they proposed. This makes computation easier although real-life situations may be more complex than this. Depending on the demand of the asset, transaction cost for buying asset *i* may be different from transaction cost for selling asset *j*, if asset *i* is on great demand and asset *j* is not.

2.6 Lower and upper bounds of variables

In portfolio optimisation, it is sometimes possible for an investor to sell an asset that one does not own. This is called short-selling or simply shorting. An investor borrows an asset which he then sells. Later, when the price of the asset falls on the market, the investor buys the asset and pays back to the lender. However, short-selling is only profitable if the asset price declines. It is very risky for investors as potential for loss is great. Thus, bounds U_{ist} on decision variables are put to prevent short-selling and enforce further restrictions imposed by the investor. We have the following constraints:

$$0 \le x_{ist} \le U_{ist},\, i = 1, \cdots, n;\, s = 1, \cdots, S;\, t = 1, \cdots, \tau. \tag{2.8}$$

and

$$0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \tag{2.9}$$

Constraint (8) ensures diversification of the portfolio. It restricts the investor from investing all his wealth in one security, which can be very risky. Constraint (9) prohibits the selling of more of an asset i than his portfolio has.

2.7 Multi-stage stochastic MAD model

We express the multi-stage portfolio selection problem as a minimisation of portfolio risk subject to constraints describing the growth of the portfolio in all scenarios, a performance constraint, and bounds on the variables.

We constrain the final expected wealth to be a particular value α. The optimisation model intends to find the least risky investment strategy to achieve the expected specified wealth. Alternatively, we can achieve the same strategy by constraining the net expected return to be at least λ, or the gross expected portfolio return to be at least θ. Varying λ or θ and re-optimising generate a set of optimal portfolios, forming the efficient frontier. We now state the stochastic MAD model as follows:

Minimise

$$H(r_{pt}) = \frac{1}{\tau} \sum_{t=1}^{\tau} \left| \sum_{s=1}^{S} p_s (R_{ist} - r_t) x_{ist} \right| \tag{2.10}$$

subject to

$$N_{p\tau} \geq \lambda$$
$$W_t = (1 + N_{pt}) W_{t-1}, t = 1, \cdots, \tau,$$
$$0 \leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau$$
$$0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,$$
$$0 \leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,$$
$$0 = a_{ist} \cdot v_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,$$
$$b_{ist} \geq 0, q_{ist} \geq 0, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,$$
$$x_{ist} = x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,$$

In addition to constraining the final rate of return of the portfolio, constrains of the form

$$N_{pt} \geq \lambda_t, t = 1, \cdots, \tau,$$

can be added to ensure any desired intermediate expected performance. We assume that no borrowing is done and the portfolio is self-financing. It deserves mention that the existence of a riskless asset among the securities is regarded as a special case in the stochastic MAD formulation (10) above.

It has been noted earlier that for each asset i, a_{ist} and v_{ist} cannot simultaneously be non-zero. We now prove the assertion.

Theorem 1:

Assume that there are transaction costs of model (10), the complementary constraint $a_{ist} \cdot v_{ist} = 0$ can be eliminated from the model.

Proof:

Without loss of generality, let us consider $a_t \cdot v_t = 0$. Let $(x_1^*, \cdots x_n^*; a_1^*, \cdots, a_n^*; v_1^*, \cdots, v_\tau^*)$ be an optimal solution of (10) without the complementary condition, and let us assume that $a_t \cdot v_t > 0$, $t \in M \subset \{1, \cdots, \tau\}$:

For $t \in M$, let

$$\begin{pmatrix} a_t' \\ v_t' \end{pmatrix} = \begin{pmatrix} a_t^* - v_t^* \\ 0 \end{pmatrix}$$

if $a_t^* \geq v_t^* \geq 0$ and

$$\begin{pmatrix} a_t' \\ v_t' \end{pmatrix} = \begin{pmatrix} 0 \\ v_t^* - a_t^* \end{pmatrix}$$

if $v_t^* \geq a_t^* \geq 0$.

Then $(x_1^*, \cdots, x_n^*; a_1', \cdots, a_\tau'; v_1', \cdots, v_\tau')$ satisfies all the constraints of (10). Also, it has the same objective as $(x_1^*, \cdots, x_n^*; a_1^*, \cdots, a_\tau^*; v_1^*, \cdots, v_\tau^*)$. This completes the proof.

Thus, we can now state problem (10) without the complementary constraint $a_{ist} \cdot v_{ist} = 0$. Let us denote $y_{st} = (R_{ist} - r_{it})x_{ist}$, $i = 1, \cdots, n$; $s = 1, \cdots, S$; $t = 1, \cdots, \tau$. Since $R_{ist} = R_{ist}(x_{ist} + a_{ist} - v_{ist})$ and $r_{it} = r_{it}(x_{ist} + a_{ist} - v_{ist})$, we have $y_{st} = y_{st}(x_{ist} + a_{ist} - v_{ist})$ as well. Thus, formulation (10) leads to the following minimisation problem:

Minimise

$$H\left(r_{pt}\right) = \frac{1}{\tau}\sum_{t=1}^{\tau}\left|\sum_{s=1}^{S} p_s y_{st}\right| \tag{2.11}$$

subject to

$$\begin{aligned}
&N_{p\tau} \geq \lambda, \\
&W_t = \left(1 + N_{pt}\right)W_{t-1}, t = 1, \cdots, \tau, \\
&0 \leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&0 \leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&b_{ist} \geq 0, q_{ist} \geq 0, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&x_{ist} = x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau
\end{aligned}$$

which is equivalent to the linear programme:

Minimise

$$H\left(r_{pt}\right)=\frac{1}{\tau}\sum_{t=1}^{\tau}Z_t \tag{2.12}$$

subject to

$$0 \le Z_t + \sum_{s=1}^{S} p_s y_{st},\ i=1,\cdots,n;\ t=1,\cdots,\tau,$$
$$0 \le Z_t - \sum_{s=1}^{S} p_s y_{st},\ i=1,\cdots,n;\ t=1,\cdots,\tau,$$
$$N_{p\tau} \ge \lambda,$$
$$W_t = \left(1+N_{pt}\right)W_{t-1},\ t=1,\cdots,\tau,$$
$$0 \le \sum_{i\in A} b_{ist} \le \sum_{j=1, j\ne i}^{n} q_{ist},\ s=1,\cdots,S;\ t=1,\cdots,\tau,$$
$$0 \le v_{ist} \le x_{ist},\ i=1,\cdots,n;\ s=1,\cdots,S;\ t=1,\cdots,\tau,$$
$$0 \le x_{ist} \le U_{ist},\ i=1,\cdots,n;\ s=1,\cdots,S;\ t=1,\cdots,\tau,$$
$$b_{ist} \ge 0,\ q_{ist} \ge 0,\ i=1,\cdots,n;\ s=1,\cdots,S;\ t=1,\cdots\tau,$$
$$x_{ist} = x_{is't},\ i=1,\cdots,n;\ s=1,\cdots,S;\ t=1,\cdots,\tau,$$

where $Z_t \ge \left|\sum_{s=1}^{S} p_s y_{st}\right|$. The first two constraints ensure that the deviation is absolute.

3 Transaction cost measurement

From retail to more professional investors and practitioners, there is great concern on transaction costs as information gleaned from research reveal that lower transaction costs result in higher portfolio returns. Transaction costs include all costs associated with trading, which can be split into explicit and implicit costs.

Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. On the other hand, implicit costs are invisible. These can broadly be put into three categories, namely market impact, opportunity costs, and spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, 2008).

To provide investors with competitive portfolio returns, investment managers must manage transaction costs proactively. Perhaps the reason why managers find themselves in a difficult situation is that these implicit costs are 'hidden' in the stock price. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and, as variable costs, provide opportunities to improve the quality of execution. Missed trading opportunity costs are incurred when investors fail to fulfil their orders. Price movements or unavailability of the security (lack of liquidity) can be cause of such investor behaviour.

The contribution of opportunity costs to total implicit costs cannot be underestimated. The effects of market impact costs and opportunity costs are inversely proportional. Minimising market impact and reducing opportunity costs form a set of conflicting objectives. When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed price' that would have prevailed if the trade had not occurred. Thus, market impact can be explained as the additional costs incurred by the investor for immediate trade execution. This is the resultant change that occurs when the number of shares of stocks an investor wants to buy or sell exceeds the number other market participants are willing to buy or sell at that price. Although market impact cost tends to decrease with time, such a delay in execution would result in increasing opportunity costs. This gives rise to what is called the 'trader's dilemma'.

In this study, we assume immediate trade execution, thus taking market impact to account for the total implicit costs. We shall use the spread mid-point benchmark and the transaction price. The transaction price shall be the last price of the month. We follow the implicit transaction calculation as given by Hau (2006). We calculate the effective spread as twice the distance from the midprice measured in basis points. For a transaction price P^T and the mid-price P^M, mid-point of the bid-ask spread, we obtain the effective spread (implicit transaction cost) as

$$SPREAD^{Trade} = 200 \times \frac{|P^T - P^M|}{P^M}.$$

We take the last price as the transaction price for every stock considered, and for each month. These are prices of assets traded on the Johannesburg Stock Exchange from 1 January 2008 to 30 September 2012. These historical data have been obtained courtesy of INet Bridge.

4 Model application and analysis

In applying the model to analyse the historical data, we use GAMS software. The 13 assets used in the analysis are selected from securities in the Johannesburg stock market. Selection is on the basis of mean asset returns over the entire duration considered, and those assets having the highest mean returns are chosen to comprise the initial portfolio. Since we are using historical data, we consider empirical distributions computed from past returns as equiprobable scenarios. Observations of returns over, say N, overlapping periods of length δ_t are considered as the N_s possible outcomes (or scenarios) of the future returns and a probability of $\frac{1}{N_s}$ is assigned to each of them.

We assume that we have historical prices at each of the t periods, $t = 1, \cdots, \tau$, of stocks under consideration. For each period t, we compute the realised vector over the previous period, say 1 month, which is further considered as one of the Ns scenarios for the future returns on the assets. Thus, for example, a scenario R_{ist} for the return on asset i is obtained as

$$R_{i,s,t} = \frac{V_{i,t} - V_{i,t-1}}{V_{i,t-1}}$$

where

- R_{ist} is the rate of return of asset i of scenario s of period t
- $V_{i,t-1}$ is the closing price per share of asset i in period $t-1$
- $V_{i,t}$ is the closing price per share of asset i of period t.

On the basis of selecting assets with the best mean returns for the period 1 January 2008 to 30 September 2012, the 13 securities heading columns in Table 1 were chosen to comprise our initial portfolio. The assets mean returns are denoted by $R_1, R_2, \cdots, R_{13}$.

4.1 Scenario generation for stage 1

We consider five scenarios in our demonstration and apply the model over two stages. This consideration is taken noting that in stochastic programming the scenario tree grows exponentially. We take the empirical distributions of the 13 securities comprising our initial portfolio. Since for each security we have 54 monthly returns, we number the months from 1 to 54 and use random numbers to select asset returns corresponding to a scenario of a security. As explained above, we consider implicit transaction costs, and in particular, the market impact costs, obtained from the effective bid-ask spread, the calculation of which is also given in the above section. The effective bid-ask spread corresponding to each selected asset return for each scenario is used to calculate the market impact cost corresponding to the asset's return in that scenario. We consider these transaction costs as random since they are randomly selected together with corresponding returns.

4.2 End-of-first-stage portfolio selection

A GAMS software is used to execute the model. In identifying assets that should comprise the second stage portfolio, a condition is given in the model to show an expected value of each asset and assets with positive and higher expected values are chosen. It should be noted that each asset has five scenarios and each scenario is equally likely to occur and is given a probability of $\frac{1}{5}$. The portfolio return rate at this stage is obtained by dividing the sum of the assets' expected return rates by the number of assets in the optimal portfolio. The implicit cost for each of the 13 assets is taken into account as it is in the buying of each asset that the cost is incurred. The transaction cost is given as a rate.

4.3 End-of-first-stage portfolio rebalancing

At the end of the first stage, an investor decides on his first-stage optimal portfolio as given by the investor's chosen diversification limit, the gross portfolio mean return or the net portfolio mean return as the case may be, and the associated risk given by the MAD. It should be noted that the new monetary values of the assets become known after running the model for the second stage. As in the first stage, the securities from the first-stage optimal portfolio are each having five scenarios, each with a probability of occurring of $\frac{1}{5}$. Each scenario has, associated with it, a cost rate (buying or selling cost rate). Each cost rate also has a probability of $\frac{1}{5}$. After running the model, the new proportions of assets become known. This happens concurrently with the calculation of transaction cost of an asset when volume is bought or sold. These transaction costs are expected costs of the volume of an asset bought or sold. The product of asset proportion, asset cost rate and the associated probability gives the expected transaction cost of an asset during portfolio rebalancing. These transaction costs are known when the investor has decided on his or her optimal portfolio at the end of stage 2.

4.4 Analysis of results

4.4.1 Stage 1

We consider an investor who has R10000 to spend on his initial portfolio. We find that as we diversify the investor's portfolio by decreasing the diversification limit from 0.4 to 0.1, the gross mean portfolio return remains fixed at 0.013 and the risk is very small in each case. However, the net mean portfolio return fluctuates between 0.008 and 0.011, reflecting fluctuating transaction costs. Table 1 has this information where the phrase 'div. lim' means 'diversification limit'.

Table 1 First stage optimal portfolios

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD*	*Wealth*
0.100	0.013	11	0.011	2.1684×10^{-19}	10,114.320
0.125	0.013	12	0.011	2.40551×10^{-18}	10,112.736
0.150	0.013	10	0.011	1.5992×10^{-18}	10,111.780
0.175	0.013	10	0.011	2.6766×10^{-19}	10,107.344
0.200	0.013	6	0.008	4.3368×10^{-19}	10,081.526
0.225	0.013	6	0.008	1.0842×10^{-19}	10,081.891
0.250	0.013	8	0.009	2.3039×10^{-19}	10,085.628
0.275	0.013	10	0.008	0	10,081.730
0.300	0.013	7	0.011	2.50722×10^{-19}	10,105.436
0.325	0.013	6	0.010	7.2844×10^{-20}	10,098.718
0.350	0.013	7	0.011	1.4×10^{-18}	10,106.596
0.375	0.013	4	0.011	2.710×10^{-19}	10,108.360
0.400	0.013	4	0.011	2.168×10^{-19}	10,108.360

We then fix each of the diversification limits and let the gross mean portfolio return increase from 0.013 to values when the portfolio becomes 'saturated', i.e., no more buying and selling of assets. Thus, creating a set of optimal portfolios for each diversification limit considered. We note that, regardless of the diversification limit chosen, the net mean portfolio return for each chosen gross mean portfolio return is the same for all diversification limits considered. Thus, transaction costs are the same for each gross mean portfolio return selected. Table A3 in Appendix has this information. It is also evident that even the risk assumes the same values for each gross mean portfolio return considered for all diversification limits, except at the portfolio saturation point. Thus, the efficient frontiers at the various diversification limits do not differ much.

However, we observe that the maximum wealth is achieved at the optimal portfolio saturation point for each diversification limit. This is also where the risk is highest. Efficient frontiers of net mean portfolio returns and gross mean portfolio returns reveal the impact of neglecting transaction costs in portfolio selection. The gap between the two frontiers is the exaggeration that results. It is therefore up to each investor to choose the wealth-risk-diversification limit combination he or she desires.

4.4.2 Stage 2

As in stage 1, we allow the diversification limit to vary from 0.2 to 0.4 as shown in Table 2. We consider a first-stage optimal portfolio with five securities. We find that optimal portfolios in stage 2 have the same gross mean portfolio return and the same net mean portfolio return, regardless of the diversification limit. Here, we note that transaction costs for the optimal portfolios are the same. It is also observed that the risk assumes the same value at each diversification limit. Small variations in the expected wealth of optimal portfolios is due to rounding error of the net mean portfolio return during execution of the programme. Similar results are observed with all stage 1 optimal portfolios.

Table 2 Second stage optimal portfolios

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD(risk)*	*Wealth*
0.200	0.007	5	0.006	0.013	10,423.815
0.225	0.007	5	0.006	0.013	10,424.277
0.250	0.007	5	0.006	0.013	10,425.228
0.275	0.007	5	0.006	0.013	10,424.843
0.300	0.007	5	0.006	0.013	10,423.679
0.325	0.007	5	0.006	0.013	10,424.327
0.350	0.007	5	0.006	0.013	10,424.321
0.375	0.007	5	0.006	0.013	10,424.290
0.400	0.007	5	0.006	0.013	10,424.259

It is evident from the table findings that each optimal portfolio at each diversification limit considered, incurs an overall implicit transaction cost of $\frac{1}{7}$ of the returns on investment. Thus, implicit transaction costs amount to approximately 14.3% of the fund invested during initial trading.

We again find optimal portfolios at each diversification limit, the results of which are shown in Table A4 in Appendix. For each gross mean portfolio return, the net mean portfolio returns are the same and also equal are the associated risks. Unlike in stage 1, expected portfolio wealth is almost invariant for each pair of gross mean portfolio return and risk, and for each diversification limit. Thus, efficient frontiers are almost the same regardless of diversification limit.

4.4.3 Comparison of stage 1 and stage 2 optimal portfolios

The in-sample analysis explains the advantages of portfolio rebalancing as depicted from the efficient frontiers of two sets of optimal portfolios in which diversification limit is the same. Stage 2 portfolios are superior, hence the need for portfolio rebalancing. It is again clear that the transaction costs incurred when buying initial portfolio and during portfolio rebalancing have a bearable impact on portfolio returns. Hence, ignoring transaction costs results in exaggerated optimal portfolios.

4.5 Conclusions

In this study, we propose a multi-stage stochastic MAD model with random transaction costs in optimal portfolio selection. We view our contributions to include:

1 the development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios

2 the development of a stochastic MAD model that optimises portfolios in the presence of random implicit transaction costs, and which captures risk due to uncertainty.

The methodology allows investors and investment managers to choose optimal portfolios realising the impact of associated implicit transaction costs. It is a LP model, and hence reduces considerably the time needed to reach a solution. It is therefore feasible for large-scale portfolio selection. It is left for future research to have a model that takes into account uncertainty of both stock prices and implicit transaction costs as well as explicit trading costs. The study also has some limitations. The calculation of asset rate-of-return using the asset's closing price of the month may not be the most accurate measure of asset's monthly rate-of-return. However, since all assets' rates-of-returns are obtained in the same way, this does not prejudice the findings.

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Appendix

Table A1 (a) Mean asset returns

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
1	0.099	0.422	–0.028	0.067	0.124	0.127	0.045
2	–0.102	–0.036	0.036	0.041	–0.057	–0.074	–0.078
3	–0.051	0.297	–0.027	0.143	–0.017	0.056	–0.082
4	–0.027	0.038	0.057	–0.021	0.014	–0.152	–0.042
5	–0.112	–0.036	–0.031	–0.138	–0.122	–0.188	–0.063
6	0.093	–0.125	0.32	0.036	0.062	0.099	0.304
7	0.084	0.071	0.113	0	0.179	0.04	0.134
8	–0.026	–0.193	–0.091	0.1	0	0.106	0.003
9	0.029	–0.173	–0.2	–0.018	0.03	–0.13	0.079
10	0.225	–0.2	0.101	–0.074	–0.048	0	–0.003
11	0.116	0.213	–0.103	0.04	0.118	–0.076	0.034
12	–0.074	–0.186	0.26	0.096	–0.029	–0.026	0.051
13	0.026	–0.139	–0.02	–0.035	–0.106	–0.022	–0.085
14	–0.136	0.294	–0.099	0.236	0.017	0.114	0.019
15	–0.046	0.02	–0.053	–0.043	0.097	0.163	0.076
16	0.061	0.069	0.111	0.022	0.064	–0.075	0.04
17	–0.029	0	0.141	–0.038	0.049	0.159	0.035
18	0.117	0.104	0.087	0.078	0.116	0.113	0.087
19	0.001	0.102	–0.017	0.014	–0.009	0.118	0
20	0	0.079	0.057	0.081	0.128	0.013	0.114
21	0.097	0.029	0.07	-0.077	0.079	0.048	0.056
22	–0.074	0.08	0.027	0.017	0.018	0.035	–0.092
23	0.083	0.004	0.082	0.056	0.083	0.048	0.074
24	0.012	–0.011	–0.084	–0.027	–0.028	0.017	0.005
25	0.033	0.036	0.053	–0.014	0.042	–0.006	0.095
26	0.089	0.069	0.12	0.034	0.074	0.101	0.032
27	0.008	0.018	0.05	–0.011	0.044	0.176	0.053
28	–0.031	–0.133	–0.052	–0.05	0.051	–0.07	0.069
29	–0.049	0.015	–0.039	0.072	0.043	0.028	–0.001
30	0.131	0.112	0.071	–0.027	0.065	0.16	0.128
31	0	–0.091	0.032	–0.068	0.037	0.004	–0.022
32	0.088	0.117	0.118	0.147	0.179	0.16	0.113
33	0.029	0.077	–0.005	0.212	0.034	0.014	0.157
34	0	0	–0.005	–0.049	–0.049	0.125	0.009
35	0.076	0.125	–0.011	0.057	–0.003	0.111	0.036
36	–0.005	0.055	–0.074	–0.048	–0.085	–0.069	–0.145

Table A1 (a) Mean asset returns (continued)

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
37	0.01	0.028	–0.043	–0.074	–0.029	–0.052	0.054
38	–0.01	0.056	–0.031	0.116	0.105	0.076	0.022
39	0.027	–0.012	0.025	–0.005	0.031	0.067	0.097
40	–0.028	0.012	0.077	0.043	–0.025	0.034	–0.047
41	0.042	0.014	–0.037	–0.023	–0.013	–0.018	0.065
42	0.033	0.02	–0.008	–0.035	–0.053	0.037	0.081
43	0.014	–0.018	0.014	0.011	0.043	–0.006	0.001
44	–0.011	–0.105	0.08	0.113	–0.097	0.005	–0.088
45	0.101	0.099	0.044	0.001	0.107	0.128	0.135
46	0.049	0.012	0.023	0.086	0.068	0.007	0.047
47	0.063	–0.041	–0.008	0.055	0.039	0	–0.003
48	0.045	0.086	0.031	–0.005	–0.141	0.117	0.081
49	0.057	0.106	0.09	–0.053	0.106	0.083	0.034
50	0.057	–0.043	0.091	0.133	0.019	0.038	0.057
51	0.039	0.098	0.061	0.04	0.046	0.025	0.115
52	0.01	0.057	–0.065	–0.007	0	–0.099	–0.01
53	0.027	0.068	0.07	0.054	0.149	0.051	0.074
54	0.182	–0.01	0.153	0.102	0.016	0.064	0.101
55	–0.009	0.007	–0.008	0.053	–0.003	0.02	0.049

Table A1 (b) Mean asset returns

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
1	0.136	–0.008	0.02	0.177	0.053	0.057
2	0	–0.025	–0.038	–0.064	–0.009	–0.025
3	–0.022	0.155	0.004	–0.096	–0.034	–0.013
4	–0.068	–0.071	–0.105	–0.076	–0.308	–0.054
5	–0.146	–0.048	0.002	–0.115	–0.005	–0.097
6	–0.057	0.051	0.267	0.107	–0.165	0.155
7	0.288	0.038	0.073	–0.013	0.226	0.067
8	–0.141	–0.065	–0.064	0.085	0.094	–0.069
9	–0.285	0.129	0.136	–0.156	–0.07	–0.034
10	–0.272	–0.056	–0.043	0.037	–0.084	0.084
11	–0.042	0.06	0.075	0.036	0.184	0.038
12	0.049	–0.035	0.033	0.034	–0.137	0.058
13	–0.063	–0.031	–0.117	0	–0.138	–0.115
14	0.039	–0.026	0.032	0.081	0.207	–0.042
15	0.022	0.026	0.058	0.202	0.02	0.053
16	0.089	0.07	0.068	0.051	0.101	0.042

Table A1 (b) Mean asset returns (continued)

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
17	0.097	–0.004	0.019	0.035	–0.028	0.036
18	0	0.054	0.081	0.131	0.167	0.202
19	–0.004	0.043	–0.016	0.146	0.09	0.012
20	0.456	0.037	0.08	0.109	0.077	0.025
21	–0.033	0.066	0.059	0.049	0.02	0.083
22	–0.041	–0.041	–0.075	0.109	–0.006	–0.048
23	–0.02	0.074	0.048	0.111	0.092	0.072
24	0.1	–0.001	–0.028	–0.024	–0.092	0.024
25	0.125	0.028	0.166	0.065	0.17	0.127
26	0.078	0.021	0.058	0.153	0.07	0.093
27	–0.023	0.029	0.017	0.076	–0.018	0.041
28	0.103	0.019	0.022	–0.03	–0.025	–0.013
29	–0.002	0.022	–0.013	0.197	–0.107	0.035
30	0.131	0.071	0.087	0.093	0.124	0.085
31	–0.021	–0.017	0.004	0.035	0.059	–0.053
32	0.114	0.105	0.196	0.116	0.111	0.099
33	0.068	0.025	–0.014	–0.021	0.007	0.015
34	0.092	0.062	0.043	0.019	0.072	–0.038
35	0.107	–0.033	–0.005	0.156	0.041	0.02
36	–0.034	–0.071	–0.11	–0.109	–0.137	–0.128
37	0.03	0.046	0	0.039	0.021	0.136
38	0.029	0	0.106	0.057	0.016	0.047
39	0.014	0.026	0.078	0.037	0.033	0.072
40	0.099	–0.05	–0.047	0.085	–0.013	–0.007
41	0.154	–0.026	0.012	–0.042	0.041	0
42	0.022	0.02	–0.012	0.001	–0.052	0.061
43	–0.076	0.035	0.071	0.04	0.007	0.155
44	0.116	0.011	–0.091	0.019	–0.091	–0.04
45	0.138	0.007	0.138	–0.052	0.116	0.154
46	–0.029	0.16	–0.012	0.011	0.007	0.001
47	0.044	–0.035	–0.068	–0.033	0.044	–0.035
48	0.185	0.02	0.059	0.024	0.122	0.077
49	0.059	0.041	0.034	0.015	0.063	0.074
50	0.039	0.004	–0.001	0.108	0.052	0.067
51	0.113	0.05	0.028	0.083	0.089	0.01
52	–0.037	–0.097	–0.009	0.017	–0.021	0.003
53	0.035	0.035	0.089	–0.058	0.04	0.031
54	0.002	0.054	0.154	0.026	0.1	0.071
55	0.061	0.011	0.003	–0.012	0.005	0.016

Table A2 (a) Asset transaction cost rates

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
1	0.0030	0.0023	0.0028	0.0036	0.0050	0.0119	0.0073
2	0.0004	0.0009	0.0028	0.0029	0.0023	0.0061	0.0051
3	0.0054	0.0008	0.0022	0.0052	0.0032	0.0425	0.0088
4	0.0087	0.0064	0.0121	0.0040	0.0107	0.0011	0.0019
5	0.0043	0.0062	0.0027	0.0080	0.0034	0.0004	0.0229
6	0.0000	0.0140	0.0072	0.0386	0.0098	0.0067	0.0012
7	0.0066	0.0048	0.0195	0.0214	0.0147	0.0018	0.0128
8	0.0048	0.0044	0.0306	0.0171	0.0548	0.0137	0.0024
9	0.0042	0.0061	0.0063	0.0078	0.0132	0.0018	0.0007
10	0.0100	0.0220	0.0027	0.0018	0.0213	0.0025	0.0060
11	0.0060	0.0003	0.0112	0.0258	0.0041	0.0085	0.0136
12	0.0143	0.0045	0.0170	0.0049	0.0015	0.0258	0.0019
13	0.0003	0.0126	0.0010	0.0055	0.0083	0.0135	0.0037
14	0.0033	0.0051	0.0029	0.0041	0.0019	0.0015	0.0020
15	0.0064	0.0023	0.0004	0.0001	0.0228	0.0026	0.0082
16	0.0057	0.0114	0.0042	0.0054	0.0092	0.0139	0.0140
17	0.0010	0.0087	0.0435	0.0165	0.0099	0.0073	0.0105
18	0.0537	0.0047	0.0067	0.0212	0.0045	0.0109	0.0028
19	0.0007	0.0052	0.0027	0.0474	0.0056	0.0053	0.0113
20	0.0060	0.0117	0.0126	0.0856	0.0046	0.0295	0.0033
21	0.0061	0.0303	0.0052	0.0532	0.0169	0.0033	0.0127
22	0.0070	0.0003	0.0087	0.0194	0.0268	0.0149	0.0049
23	0.0076	0.0029	0.0046	0.0074	0.0003	0.0008	0.0103
24	0.0056	0.0473	0.0025	0.0069	0.0127	0.0055	0.0026
25	0.0018	0.0058	0.0095	0.0070	0.0079	0.0046	0.0056
26	0.0149	0.0074	0.0327	0.0072	0.0009	0.0019	0.0045
27	0.0037	0.0208	0.0032	0.0233	0.0104	0.0035	0.0165
28	0.0202	0.0065	0.0256	0.0301	0.0051	0.0021	0.0073
29	0.0068	0.0225	0.0141	0.0061	0.0018	0.0056	0.0116
30	0.0298	0.0000	0.0343	0.0068	0.0695	0.0253	0.0094
31	0.0072	0.0197	0.0027	0.0408	0.0084	2.0000	0.0115
32	0.0157	0.0153	0.0066	0.0679	0.0040	0.0749	0.0058
33	0.0139	0.0238	0.0015	0.0019	0.0040	0.0286	0.0111
34	0.0142	0.0024	0.0079	0.0073	0.0013	0.0105	0.0024
35	0.0074	0.0186	0.0068	0.0326	0.0198	0.0066	0.0114
36	0.0316	2.0000	0.0082	0.0146	0.0487	0.0074	0.0133
37	0.0041	0.0083	0.0433	0.0155	0.0369	0.0082	0.0075
38	0.0459	2.0000	0.0058	0.0014	0.0029	0.0354	0.0033

Table A2 (a) Asset transaction cost rates (continued)

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
39	0.0006	0.0386	0.0012	0.0046	0.0334	0.0177	0.0339
40	0.0743	0.0225	0.0097	0.0044	0.0148	0.0202	0.0543
41	0.0210	0.0290	0.0103	0.0952	0.0144	0.1277	0.0402
42	0.0208	0.0022	0.0179	0.0222	0.0308	0.1053	0.0089
43	0.0217	2.0000	0.0177	0.0396	0.0238	2.0000	0.0052
44	0.0053	2.0000	0.0151	0.0619	0.0033	0.0942	0.0112
45	0.0124	0.0942	0.0077	0.0759	0.0183	0.0408	0.0083
46	0.0086	0.0083	0.0800	0.0352	0.0215	0.0258	0.0014
47	0.0169	0.0645	0.0026	0.0829	0.0064	0.0077	0.0271
48	0.0060	0.0072	0.0185	0.0769	0.0068	0.0198	0.0020
49	0.0209	2.0000	0.0255	0.1452	0.0000	0.0217	0.0047
50	0.0330	0.0108	0.0132	0.1125	0.0043	0.0089	0.0277
51	0.0040	0.0189	0.0016	0.0894	0.0086	0.0308	0.0114
52	0.0032	0.0028	0.0110	2.0000	0.0078	0.0048	0.0164
53	0.0317	0.0237	0.0026	0.0734	0.0007	0.0543	0.0101
54	0.0180	0.0220	0.0085	0.0112	0.0104	0.0151	0.0228

Table A2 (b) Asset transaction cost rates

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
1	0.0319	0.0019	0.0009	0.0063	0.0058	0.0062
2	0.0323	0.0072	0.0033	0.0105	0.0132	0.0010
3	0.0038	0.0044	0.0241	0.0061	0.0067	0.0018
4	0.0049	0.0035	0.0028	0.0411	0.0040	0.0029
5	0.0058	0.0021	0.0031	0.0031	0.0005	0.0013
6	0.0046	0.0024	0.0056	0.0031	0.0050	0.0010
7	0.0315	0.0029	0.0066	0.0056	0.0050	0.0008
8	0.0086	0.0946	0.0540	0.0121	0.0542	0.0069
9	0.0263	0.0009	0.0068	0.0003	0.0019	0.0027
10	0.0148	0.0030	0.0128	0.0108	0.0052	0.0023
11	0.0202	0.0019	0.0063	0.0459	0.0121	0.0689
12	0.0043	0.0109	0.0029	0.0071	0.0078	0.0099
13	0.0067	0.0007	0.0023	0.0028	0.0042	0.0044
14	0.0127	0.0017	0.0033	0.0032	0.0043	0.0057
15	0.0197	0.0041	0.0036	0.0191	0.0084	0.0020
16	0.0072	0.0012	0.0177	0.0047	0.0100	0.0050
17	0.0074	0.0001	0.0185	0.0050	0.0008	0.0004
18	0.0122	0.0085	0.0077	0.0005	0.0002	0.0098
19	0.0088	0.0007	0.0015	0.0054	0.0085	0.0015

Table A2 (b) Asset transaction cost rates (continued)

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
20	0.0081	0.0078	0.0324	0.0030	0.0027	0.0046
21	0.0179	0.0085	0.0022	0.0031	0.0037	0.0271
22	0.0228	0.0022	0.0011	0.0101	0.0097	0.0234
23	0.0106	0.0305	0.0077	0.0143	0.0010	0.0220
24	0.0293	0.0053	0.0024	0.0039	0.0137	0.0073
25	0.0212	0.0050	0.0127	0.0125	0.0205	0.0421
26	0.0452	0.0019	0.0145	0.0145	0.0034	0.0121
27	0.0178	0.0024	0.0137	0.0069	0.0047	0.0223
28	0.0203	0.0093	0.0027	0.0053	0.0064	0.0071
29	0.0526	0.0043	0.0026	0.0037	0.0040	0.0019
30	0.0595	0.0113	0.0095	0.0065	0.0133	0.0016
31	2.0000	0.0058	0.0069	0.0127	0.0096	0.0028
32	0.0470	0.0020	0.0021	0.0140	0.0086	0.0300
33	0.0287	0.0151	0.0018	0.0047	0.0047	0.0114
34	0.0030	0.0023	0.0068	0.0149	0.0068	0.0025
35	0.0545	0.0032	0.0261	0.0091	0.0125	0.0184
36	0.0590	0.0017	0.0119	0.0092	0.0103	0.0032
37	0.0874	0.0142	0.0117	0.0012	0.0022	0.0117
38	0.0597	0.0039	0.0055	0.0241	0.0013	0.0080
39	0.2246	0.0186	0.0064	0.0102	0.0215	0.0058
40	0.0488	0.0107	0.0041	0.0015	0.0079	0.0302
41	0.0845	0.0252	0.1433	0.0481	0.0299	0.1060
42	0.0207	0.0077	0.0088	0.0236	0.0164	0.0105
43	0.2118	0.0357	0.0239	2.0000	0.0077	0.0008
44	0.0267	0.0080	0.0066	0.0142	0.0078	0.0123
45	0.2449	0.0741	0.0075	0.0714	0.0082	0.0370
46	0.2290	0.0050	0.0187	0.0546	0.0328	0.0481
47	0.0047	0.0093	0.0123	0.0165	0.0110	0.0169
48	0.0625	0.0012	0.0007	0.0479	0.0099	0.0273
49	0.1374	0.0228	0.0110	0.0147	0.0042	0.0069
50	0.0247	0.0179	0.0013	0.0165	0.0024	0.0608
51	0.0114	0.0011	0.0155	0.0469	0.0007	0.0109
52	0.0270	0.0062	0.0650	0.0408	0.0336	0.0198
53	0.0110	0.0082	0.0633	0.0524	0.0507	0.0112
54	0.0228	0.0445	0.0136	0.0202	0.0033	0.0347

Table A3 Stage 1 efficient frontiers at various diversification limits

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD (risk)*	*Wealth*
0.1	0.013	11	0.011	2.1684×10^{-19}	10,114.320
	0.02	12	0.018	0.007	10,178.771
	0.03	11	0.025	0.017	10,253.167
	0.04	10	0.036	0.018	10,355.180
0.125	0.013	12	0.011	2.4×10^{-18}	10,112.736
	0.02	10	0.018	0.007	10,181.466
	0.03	10	0.028	0.017	10,276.063
	0.04	8	0.034	0.021	10,344.850
0.15	0.013	10	0.011	1.599×10^{-18}	10,111.780
	0.02	11	0.018	0.007	10,182.074
	0.03	11	0.025	0.017	10,248.470
	0.04	7	0.034	0.023	10,336.510
0.2	0.013	6	0.008	4.33×10^{-19}	10,081.526
	0.02	6	0.015	0.007	10,151.170
	0.03	7	0.026	0.017	10,255.143
	0.04	5	0.036	0.026	10,363.520
0.25	0.013	8	0.009	2.3×10^{-19}	10,085.628
	0.02	10	0.016	0.007	10,158.532
	0.03	7	0.026	0.017	10,256.967
	0.04	6	0.037	0.027	10,366.508
0.3	0.013	7	0.011	2.5×10^{-19}	10,105.436
	0.02	12	0.018	0.007	10,175.404
	0.03	7	0.028	0.017	10,282.641
	0.04	6	0.037	0.027	10,369.217

Table A4 Stage 2 efficient frontiers at various diversification limits

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD (risk)*	*Wealth*
0.2	0.007	5	0.006	0.013	10,423.815
	0.010	5	0.009	0.015	10,460.266
	0.015	5	0.014	0.017	10,513.668
	0.020	5	0.019	0.020	10,562.684
	0.025	5	0.024	0.022	10,611.495
	0.030	5	0.029	0.025	10,664.308
	0.035	5	0.034	0.027	10,714.813
0.225	0.007	5	0.006	0.013	10,424.277
	0.010	5	0.009	0.015	10,459.822
	0.015	5	0.014	0.017	10,513.566
	0.020	5	0.019	0.020	10,562.732
	0.025	5	0.024	0.022	10,611.730
	0.030	5	0.029	0.025	10,662.818
	0.035	5	0.034	0.027	10,715.476

Table A4 Stage 2 efficient frontiers at various diversification limits (continued)

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD (risk)*	*Wealth*
0.25	0.007	5	0.006	0.013	10,425.228
	0.010	5	0.009	0.015	10,461.394
	0.015	5	0.014	0.017	10,513.540
	0.020	5	0.019	0.020	10,562.598
	0.025	5	0.024	0.022	10,611.730
	0.030	5	0.029	0.025	10,660.993
	0.035	5	0.034	0.027	10,714.008
0.275	0.007	5	0.006	0.013	10,424.843
	0.010	5	0.009	0.015	10,460.696
	0.015	5	0.014	0.017	10,513.540
	0.020	5	0.019	0.020	10,562.598
	0.025	5	0.024	0.022	10,611.730
	0.030	5	0.029	0.025	10,660.733
	0.035	5	0.034	0.027	10,712.199

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Brussels, 03 December 2016

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RE: Decision letter JEE-D-16-393

Dear Sabastine,

I have now heard from two referees who have read your manuscript entitled "*A Stochastic Multi-stage Trading Cost model in optimal portfolio selection*". Both referees are in favour of publication. Therefore, I am glad to let you know that your manuscript JEE-D-16-393 has been accepted for publication in the Journal of Economics and Econometrics.

Please fill in the enclosed JEE copyright form, scan it, and email it to back to the Editorial office.

Sincerely yours,

Professor Dr. Victor Ginzburg
Editor-in-Chief
Journal of Economics and Econometrics

A Stochastic Multi-stage Trading Cost model in optimal portfolio selection

Abstract

We propose a multi-stage stochastic trading cost model in optimal portfolio selection. This strategy captures uncertainty in implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio. We assume that implicit costs are stochastic as are asset returns. We use mean absolute deviation as our risk and apply the model to securities on the Johannesburg Stock Market. The model generates optimal portfolios by minimizing total implicit transaction costs incurred. It provides least-cost optimal portfolios whose net wealths are better than those generated by the mean-variance, minimax and mean absolute deviation models.

Key words: implicit transaction costs, stochastic programming.

1 Introduction

Financial markets are inherently volatile, characterized by shifting values, risks and opportunities. The prices of individual securities are frequently changing for numerous reasons that include shifts in perceived value, localized supply and demand imbalances, and price changes in other sector investments or the market as a whole. Reduced liquidity results in price volatility and market risk to any contemplated transaction. As a result of this volatility, transaction cost analysis (TCA) has become increasingly important in helping firms measure how effectively both perceived and actual portfolio orders are executed. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making processes. In this study, we propose a stochastic multi-stage mean absolute deviation model with trading costs (SMADTC) that minimizes implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio. This is achieved by allowing the investor to choose his or her desired implicit transaction cost value and portfolio mean rate of return or risk level, where the risk is defined by the mean absolute deviation of assets' returns from expected portfolio return. The multi-stage stochastic transaction cost model captures assets' returns, implicit transaction costs and risk due to uncertainty. We apply stochastic programming since it has a number of advantages over other techniques. Firstly, stochastic programming models can accommodate general distributions by means of scenarios. We do not have to explicitly assume a specific stochastic process for securities' returns, but we can rely on the empirical distribution of these returns. Secondly, they can address practical issues such as transaction costs, turnover constraints, limits on securities and prohibition of short-selling. Regulatory and institutional or market-specific constraints can be accommodated. Thirdly, they can flexibly use different risk measures.

Konno and Yamazaki [14] propose the mean absolute deviation (MAD) model, in deterministic form, as an alternative to the mean-variance (MV) model by Markowitz [17]. MAD is a dispersion-type risk linear programming (LP) computable measure that may be taken as an approximation of the variance when the absolute values replace the squares. It is equivalent to the mean-variance if the assets' returns are multivariate normally distributed. However, using a linear model considerably reduces the time needed to reach a solution, thus making the MAD model more appropriate for large-scale portfolio selection. It makes intensive calculations of the covariance matrix unnecessary as opposed to the mean-variance model. The MAD model is also sensitive to outliers in historical data (Byrne and Lee, [2]) Much financial research has been done regarding asset allocation, portfolio construction and performance attribution. However, for the total performance of a portfolio, the quality of the implementation is as important as the decision itself. Implementation costs usually reduce portfolio returns with limited potential to generate upside potential. A portfolio must strike what an investor believes to be an acceptable balance between risk and reward, having considered all costs incurred in the setting or rebalancing of the portfolio. Investment portfolios should be rebalanced to take account of changing market conditions and changes in funding. This brings with it some trading costs, which can be either direct or indirect. Direct trading costs are observable and they include brokerage commissions, market fees and taxes. Indirect costs are invisible and these include bid-ask spread, market impact and opportunity costs.

Of the literature that is devoted to modeling portfolio selection with transaction costs, the greater part concentrates on proportional transaction costs. Kozmik [15] discusses an asset allocation strategy with transaction costs formulated as a multi-stage stochastic programming model. He considers transaction costs as proportional to the value of the assets bought or sold, but does not consider implicit transaction costs in the model. He employs Conditional-Value-at-Risk as a risk measure. Moallemi and Saglam [18] study dynamic portfolio selection models with Gausian uncertainty using linear decision models incorporating proportional transaction costs. They assume that trading costs such as bid-ask spread, broker commissions and exchange fees are proportional to the trade size. However, as assets' prices follow a random-walk process, such price movement would result in randomly fluctuating transaction costs due to a number of factors that include asset liquidity, market impact and so on. In economic recessions and booms where asset returns are characterised by extreme movements, the extreme movements of the market are not always reflected in all individual stocks. Some individual stocks show an extreme reaction while others exhibit a milder reaction (Jansen and De Vries, [12]). Hence considering proportional transaction costs in an uncertain environment does not provide good estimate of trading costs, especially implicit transaction costs. Lynch and Tan [16] study portfolio selection problems with multiple risky assets. They develop analytic frameworks for the case with many assets taking proportional transaction costs. Xiao and Tian [20] estimate implicit transaction costs in Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. They use the Bayesian Gibbs sampling method proposed by Hasbrouck [10] to analyze implicit costs in the bull and bear markets. Hasbrouck [10] incorporates the Gibbs estimates into asset pricing specifications over a historical sample and find that effective cost is positively related to stock returns. Brown and Smith [3] study the problem of dynamic portfolio optimization in a discrete-time finite-horizon setting, and they also take into account proportional transaction costs. Cai, et al [4] examine numerical solutions of dynamic portfolio optimization with transaction costs. They consider proportional transaction costs which can be either explicit or implicit, whichever is greater. However, transaction cost analysis requires the identification of the type of cost to be estimated in order to explore effective ways of having a good estimate of it, hence enabling an investor to make an informed decision. Thus, in our study, we concentrate on implicit transaction costs as these are invisible and can easily erode the profits of an investment. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (Hondt and Giraud, [5]). Konno and Wajayanayake [13] propose the deterministic mean absolute deviation model with transaction costs modeled by a concave function. They use a linear cost function as an approximation to the concave cost function. Gulpinar, et al [6] propose a multi-stage mean-variance portfolio analysis with proportional transaction costs. Our study considers random transaction costs since asset prices are random.

Some investors do not like too high costs as these are known to erode the profits of investment. Most models in the literature are static models and they are essentially single-period models. There is only one decision to be made, for the first period. In real-life, decisions are made with possibilities of adjustment down the road, since the future is unknown. Stochastic programming models hence are dynamic, covering multiple-time periods with associated separate decisions, and they account for the stochastic decision process. The main features of stochastic programming are scenarios and stages. The uncertainty about future events is captured by a set of scenarios, which is a representative and comprehensive set of possible realizations of the future. Stochastic programming recognizes that future decisions happen in stages, incorporating earlier decisions and events that occurred during earlier time-periods.

The main contributions of this study include:

(i) The development of a stochastic multi-stage trading cost model that generates optimal portfolios while minimizing uncertain implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of portfolios;

(ii) The development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios.

This paper is organized as follows. In Section 2, we discuss the formulation of the stochastic trading cost model for multi-stage optimal portfolio selection. Transaction costs and portfolio rebalancing constraints are explained. Discussion on transaction cost calculation is given in this section. In Section 3, we demonstrate the application of the model to securities taken from the Johannesburg Stock Market courtesy of I-Net Bridge. We compare the model's performance with the performance of the mean-variance, minimax (MM) and mean absolute deviation models. We conclude, in Section 4, by giving a summary of our findings and the main contributions of this study, as well as directions on future research.

2 Materials and methods

We determine a multi-period discrete-time optimal portfolio strategy over a given investment horizon T. The period T is divided into two discrete intervals T_1 and T_2. T_1 defines the planning phase, where $T_1 = 0, 1, \cdots, \tau$. Period $T_2 = \tau + 1, \cdots, T$ is the period to investment maturity. During T_1, an investor makes decisions and adjustments to his portfolio at each of the τ periods as some assets' returns get realized. The initial investment takes place at $t = 0$, with portfolio restructuring at times $t = 1, 2, \cdots, \tau$. After period τ, no further decisions are implemented until investment maturity at $t = T$. A portfolio is restructured in terms of asset return and risk which is measured by the mean absolute deviation of assets' returns from expected portfolio return. This restructuring of the portfolio brings with it transaction costs, as the investor buys or sells shares of some securities to balance his portfolio. For a conservative investor, trade execution costs are paramount. Hence the need to minimize these costs. We thus consider an investor who is interested in getting an optimal portfolio while at the same time keeping transaction costs to the minimum.

2.1 Scenario Generation

Let $\mathbf{R} = \{R_1, \cdots, R_t\}$ be stochastic events at $t = 1, \cdots, T$, and consider $I = \{i : i = 1, 2, \cdots, n\}$ to be a set of securities for an investment. The decision process is non-anticipative (i.e., a decision at a given stage does not depend on the future realization of the random events). The decision at period t is dependent on R_{t-1}. We define a scenario as a possible realization of the stochastic variables $\{R_1, R_2, \cdots, R_t\}$. Hence the set of scenarios corresponds to the set of paths followed from the root to the leaves of a tree, S_τ, and nodes of the tree at level $t \geq 1$ corresponds to possible realizations of R_t. Each node at a level t corresponds to a decision which must be determined at time t, and depends in general on R_t, the initial wealth of the portfolio and past decisions. Given the event history up to time t, R_t, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observations R_{t+1}. The branching process is represented by a scenario tree. An example of a scenario tree with 2 time periods and three-three branching structure is shown in Figure 1 below.

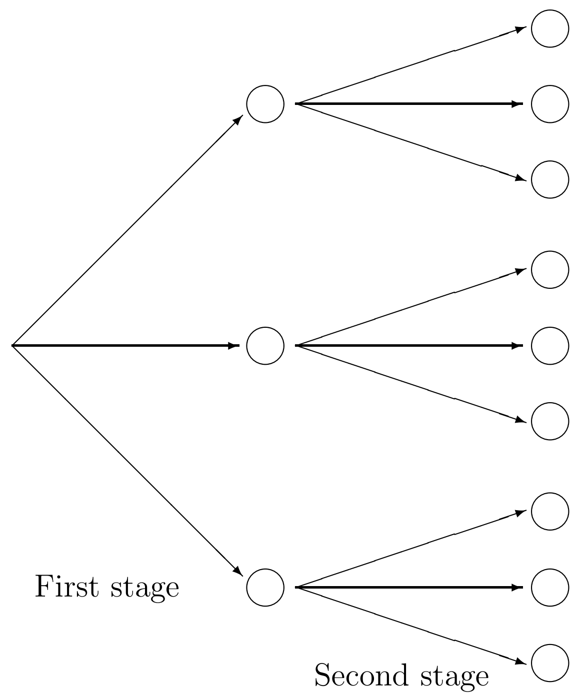

Figure 1: Scenario tree

The uncertain return of the portfolio at the end of the period t is $R = R(x_t, r_t)$. This is a random variable with a distribution function, say F, given by

$$F(x, \mu) = p\{R(x, r) \leq \mu\}$$

The expected return of the portfolio at the end of period t is

$$r_{pt} = E[R(x_t, R_t)] = r(x_t, R_t).$$

Suppose the uncertain returns of the assets, R_t, in period t are represented by a finite set of discrete scenarios $\Omega = \{s : s = 1, 2, \cdots, S\}$, whereby the returns under a particular scenario $s \in \Omega$ take the values $R_s = (R_{1s}, R_{2s}, \cdots, R_{ns})^T$ with associated probability $p_s > 0$, where $\sum_{s\in\Omega} p_s = 1$. The portfolio return under a particular realization of asset return R_s of period t is denoted by $r_{st} = r(x_t, R_{st})$. The expected portfolio return of period t is now given by

$$r_{pt} = E[r(x_t, R_{st})] = \sum_{s\in\Omega} p_s r(x_t, R_{st})$$

2.2 Model constraints

The dynamic portfolio selection in discrete time allows an investor to adjust dynamically his or her portfolio at successive stages. We assume that the investor joins the market at $t = 0$ with initial wealth W_0, and W_t being portfolio wealth at period t. The wealth can be partitioned among the n-assets at the beginning of each of the τ consecutive time periods to rebalance the portfolio. The investor is seeking an optimal investment strategy,

$x_t = [x_{1st}, x_{2st}, \cdots, x_{nst}]$ for $t = 1, 2, \cdots, \tau$, such that

$$\sum_{i=1}^{n} x_{it} = \sum_{s=1}^{S} x_{ist} = 1, t = 1, 2, \cdots, \tau.$$

where S is the total number of scenarios in period t. Let a_{ist} and v_{ist} be, respectively, the buying and selling proportions of asset i of scenario s of period t. Then we have $a_{ist} = \frac{A_{ist}}{W_t}$ and $v_{ist} = \frac{V_{ist}}{W_t}$ where A_{ist} is the amount of money used to buy new shares of asset i of scenario s in period t, and V_{ist} is the money obtained from selling shares of asset i of scenario s in period t. Thus, in portfolio rebalancing at the beginning of each time period t, we have

$$x_{i,s,t} = x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \quad (1)$$

We remark here that either a_{ist} or v_{ist} is zero at each rebalancing of the portfolio since we cannot buy and sell an asset at the same time. This results in the constraint

$$a_{ist} \cdot v_{ist} = 0 \quad (2)$$

We assume that the portfolio is self-financing and consider that money can only be added at $t = 0$ and not in subsequent periods.

The decision made by the investor at period t depends on $x_{i,s,t-1}$ and the yield of the investment of asset i of scenario s. Thus, the expected return of asset i in period t is given by

$$\begin{aligned} r_{it} &= \sum_{s \in Q} p_s \cdot R_{i,s,t} \cdot (x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}), i = 1, \cdots, n; t = 1, \cdots, \tau. \\ &= \sum_{s \in Q} p_s \cdot R_{ist} \cdot x_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau. \end{aligned} \quad (3)$$

for all scenarios s of asset i. Here, $Q \subset \Omega$ is a set of scenarios of asset i of period t. When rebalancing the portfolio, it is observed that

$$0 \leq \sum_{i} a_{ist} \leq \sum_{j,j\neq i} v_{ist}; i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \qquad (4)$$

This constraint ensures that the amount of money used to purchase shares of assets i should be at most equal to the amount of money obtained from selling shares of assets $j, (j \neq i)$ of period t during portfolio rebalancing.

The following constraint guarantees that the proportion of asset i of scenario s in period t sold for portfolio rebalancing should be at most the proportion of the asset in the portfolio,

$$0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \qquad (5)$$

The inequality (5) also ensures that no short-selling takes place. To provide for diversification of the portfolio, bounds U_{ist} on decision variables are considered resulting in the constraint

$$0 \leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \qquad (6)$$

It should be noted that scenarios may reveal identical value for the uncertain quantities up to a certain period. These scenarios that share common information must yield the same decisions up to that period. Thus we have the constraint

$$x_{ist} = x_{is't} \tag{7}$$

for all scenarios s and s' with identical past up to time t.

2.3 Expected portfolio Wealth

The investor aims to obtain an optimal strategy that minimizes transaction costs. Let k_{ist} and l_{ist} be, respectively, the rate of buying and selling transaction costs of the quantity of asset i of scenario s bought or sold for portfolio rebalancing at the beginning of period t. Thus, the transaction cost rate incurred by the investor for buying or selling proportions a_{ist} or v_{ist} respectively, of asset i of scenario s in period t is given by

$$k_{ist}a_{ist} + l_{ist}v_{ist}$$

We remark here that since $a_{ist} \cdot v_{ist} = 0$ (from (2)), we have either

$$k_{ist}a_{ist} = 0 \quad \text{or} \quad l_{ist}v_{ist} = 0 \tag{8}$$

or both being equal to zero. Thus, the expected transaction cost rate of the portfolio in period t becomes

$$\textstyle\sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}, i = 1, 2, \cdots, n; t = 1, 2, \cdots, \tau.$$

The gross mean rate of return of the portfolio of period t is given by

$$\begin{aligned} r_{pt} &= \sum_{s=1}^{S} p_s \cdot R_{i,s,t} \cdot (x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}), i = 1, \cdots, n; t = 1, \cdots, \tau. \\ &= \sum_{s=1}^{S} p_s \cdot R_{ist} \cdot x_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau. \end{aligned} \tag{9}$$

since we have a total of S scenarios in each period. The wealth of period t, without transaction costs, becomes

$$W_t = (1 + r_{pt}) \cdot W_{t-1}, t = 1, \cdots, \tau \tag{10}$$

Denoting the net expected portfolio return rate of period t by N_{pt}, we get

$$N_{pt} = r_{pt} - \textstyle\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}.$$

Thus, the portfolio wealth of period t taking transaction costs into account is given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1}, t = 1, \cdots, \tau. \tag{11}$$

2.4 Expected portfolio Risk

The portfolio risk for any realization of any period is measured by the mean absolute deviation of the realized returns relative to the expected portfolio return, r_{pt}. We consider deviation of a scenario return from expected portfolio return at any period t. Konno and Yamazaki [14] develop the deterministic mean absolute deviation model in an attempt to improve on the famous Markowitz [17] mean-variance model. Their mean absolute deviation model has portfolio risk expressed as

$$H(p) = \frac{1}{\mathrm{T}} \sum_{t=1}^{T} | \sum_{i=1}^{n} (r_{it} - r_i) x_i |$$

where r_{it} is the realized return of asset i of period t, r_i is the expected return of asset i per period, and x_i is the proportion of wealth invested in asset i. We therefore extend this formulation of portfolio risk by taking into account uncertainty of asset returns and transaction costs in portfolio rebalancing. Using the deterministic model as our basis, three key stochastic framework elements are incorporated to formulate the stochastic mean absolute deviation model. The first element considered is the concept of scenarios. Since the deterministic model represents one particular scenario, the inclusion of multiple scenarios in capturing uncertainty result in the increase in the number of variable parameters. Thus uncertainty is represented by a set of distinct realizations $s \in \Omega$. We now consider the scenario parameter, along with other parameters of security and time period. Secondly, the stochastic trading cost model allows for the implementation of recourse decisions as unfolding information on assets' returns get realized. The third element is the probabilistic feature of the stochastic framework which assigns probabilities to scenarios. The parameter p_s represents scenario probability. Scenarios may reveal identical value for the uncertain quantities up to a certain period. Scenarios that share common information must yield the same decisions up to that period. The stochastic mean absolute deviation also incorporates transaction costs incurred during portfolio rebalancing at each time period. Thus, we obtain the following portfolio risk:

$$H(r_{pt}) = \frac{1}{\tau}\sum_{t=1}^{\tau} |\sum_{s=1}^{S} p_s(R_{st} - r_{pt})x_{st}| \tag{12}$$

where R_{st}, r_{pt} and x_{st} are as defined earlier. Let us denote $y_{st} = (R_{ist} - r_{pt})x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau$. Since $R_{i,s,t} = R_{i,s,t}(x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t})$ and $r_{pt} = r_{pt}(x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t})$, we have $y_{s,t} = y_{s,t}(x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t})$ as well. Equivalently, we have $y_{st} = y_{st}(x_{ist})$. Thus the portfolio risk becomes

$$H(r_{pt}) = \frac{1}{\tau}\sum_{t=1}^{\tau} |\sum_{s=1}^{S} p_s y_{st}|$$

This risk function is non-linear, and since we intend to have a linear programming model we linearize the portfolio risk function and it becomes

$$H(r_{pt}) = \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t \tag{13}$$

where $Z_t \geq |\sum_{s=1}^{S} p_s y_{st}|$.

2.5 Multi-stage Stochastic Transaction Cost Model

We present the multi-stage stochastic transaction cost model in optimal portfolio selection as a minimization of implicit trading costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio. We subject the model to constraints describing the growth of the portfolio in all scenarios, some performance constraints and bounds on variables.

We constrain the final expected wealth to be at least a particular value, say α, desired by the investor. The portfolio risk, measured by the stochastic mean absolute deviation, is also constrained to be at most a value chosen by the investor, say θ_t, at any period t. The optimization model provides an optimal investment strategy that minimizes implicit transaction costs while achieving the specified expected wealth and desired risk level. Varying the expected return or risk and re-optimizing generate a set of optimal portfolios, forming the efficient frontier. If we let the investor's desired minimum net portfolio return to be at least λ say, we state the stochastic multi-stage transaction cost model as follows:

Minimize

$$\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \tag{14}$$

subject to

$$\begin{aligned}
0 &\leq Z_t + \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq Z_t - \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
N_{pt} &\geq \lambda_t \\
W_t &= (1 + N_{pt}) W_{t-1}, t = 1, \cdots, \tau, \\
\theta_t &\geq \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t \\
1 &= \sum_{s=1}^{S} x_{ist}, i = 1 \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq \sum_{i} a_{ist} \leq \sum_{j=1, j \neq i}^{n} v_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau \\
0 &\leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
0 &\leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
0 &= a_{ist} \cdot v_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
x_{ist} &= x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,
\end{aligned}$$

The first two constraints ensure that the deviation is absolute. In addition to constraining the final rate of return of the portfolio, constraints of the form

$$N_{pt} \geq \lambda_t, t = 1, \cdots, \tau,$$

can be added to ensure any desired intermediate expected performance. We assume that no borrowing is done and the portfolio is self-financing. It deserves mention that the existence of a riskless asset among the securities is regarded as a special case in the stochastic transaction cost model formulation (14) above.

It has been noted earlier that for each asset i, a_{ist} and v_{ist} cannot simultaneously be non-zero. Thus, we can now state problem (14) without the complementary constraint $a_{ist} \cdot v_{ist} = 0$ as the following linear program.

Minimize

$$C_{pt} = \sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \tag{15}$$

subject to

$$
\begin{aligned}
0 &\leq Z_t + \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq Z_t - \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
N_{pt} &\geq \lambda_t \\
W_t &= (1 + N_{pt}) W_{t-1}, t = 1, \cdots, \tau, \\
\theta_t &\geq \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t \\
1 &= \sum_{s=1}^{S} x_{ist}, i = 1 \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq \sum_{i} a_{ist} \leq \sum_{j=1, j \neq i}^{n} v_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau \\
0 &\leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
0 &\leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
x_{ist} &= x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,
\end{aligned}
$$

where C_{pt} is the total portfolio transaction cost rate at time t. Alternatively, the objective function can be expressed as minimizing the total portfolio transaction cost in period t as

$$C = [\textstyle\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}] \cdot W_{t-1}$$

2.6 Transaction cost measurement

An investor incurs transaction costs when buying or selling securities on the Stock Market. This takes place during initial trading and when rebalancing the portfolio in subsequent periods. Transaction costs are either explicit or implicit. Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. However, implicit costs are embedded in the stock price, and hence are invisible. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of trade execution.These can broadly be put into three categories, namely market impact, opportunity costs and spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (Hondt and Giraud, [5]). When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed' price that would have prevailed if the trade had not occurred.

Thus we assume immediate trade execution and regard market impact to account for the total implicit costs. We follow implicit transaction cost calculation as provided by Hau [9]. We use the spread mid-point benchmark and take the transaction price to be the last price of the month. We calculate the effective spread as twice the distance from the mid-price measured in basis points. Thus, for the mid-price P^M, mid-point of the bid-ask spread, and transaction price P^T, we obtain the effective spread (implicit transaction cost) as

$$SPREAD^{Trade} = 200 \times \frac{|P^T - P^M|}{P^M}$$

We consider the transaction price to be the last price of the month for each security. These are prices of securities traded on the Johannesburg Stock Market from 1 January 2008 to 30 September 2012. We got the historical data courtesy of I-Net Bridge.

3 Results and discussion

We use monthly historical data of securities on the Johannesburg Stock Market from January 2008 to September 2012. We evaluate the performance of the proposed SMADTC model by comparing portfolios developed from it and the mean-variance, mean absolute deviation and minimax (MM) models. These models are shown in appendix section. The following criteria are used in the selection of stocks to comprise our initial portfolio:

(a) Stocks with negative mean returns for the entire period considered are excluded from the sample,

(b) Companies which were not on the list by January 2008 and only entered the JSE afterwards are excluded.

(c) Those assets having the highest positive mean returns are taken to become our initial portfolio.

We take empirical distributions computed from past monthly returns as equiprobable scenarios. A scenario, R_{ist}, for the return of asset i of period t is obtained as

$$R_{ist} = \frac{P_{i,s,t}}{P_{i,s,t-1}} - 1$$

where $P_{i,s,t}$ is a historical monthly price of asset i.

In the first stage of analysis, 13 securities are chosen to comprise our initial portfolio. The mean returns, total transaction costs and total wealth of each of the four optimal portfolios developed according to models15, A_1, A_2 and A_3 are calculated and then compared.

In using the proposed model, we consider five scenarios for each asset return and the corresponding transaction cost, and apply the SMADTC model over two stages. However, comparison with other models is restricted to the first stage. We take empirical distributions of the 13 securities comprising our initial portfolio. Since for each security we have 54 monthly returns, we number the months from 1 to 54, and use random numbers to select asset returns and associated transaction costs corresponding to scenarios of a security. We take implicit transaction costs calculated from the effective bid-ask spread corresponding to each selected asset return. We assume that these transaction costs are random since they are randomly selected together with corresponding asset' returns. Thus, a scenario comprises an asset return and the associated transaction cost. The implicit cost for each of the 13 assets is taken into account as it is in the buying or selling of each asset that the cost is incurred. We give the transaction cost as a rate. We take each scenario to be equally likely to occur.

It should be noted that scenarios are only considered for the proposed SMADTC model. Other models use mean returns and risks calculated over the whole period under examination. First-stage optimal portfolios of all models are compared. For the SMADTC model, we consider five scenarios for each asset return in our demonstration, each with a probability of occurring of $\frac{1}{5}$, and apply the model over two stages. This consideration is taken noting that in stochastic programming the scenario tree grows exponentially. At the end of the first stage, an investor decides on his first-stage optimal portfolio as given by the investor's chosen portfolio transaction cost, the diversification limit, the gross portfolio mean return or the net portfolio mean return as the case may be, and the associated risk given by the mean absolute deviation.

3.1 Analysis of results

Results of the SMADTC model are given in stages.

Stage 1

We consider an investor who has R10000 to spend on his initial portfolio. The optimal portfolios generated by the four models are shown in Table 1 below. The phrase 'D.L' stands for 'diversification limit'.

The information in Table 1 reveals that the portfolios generated by the MM model have the highest gross mean returns for each diversification limit considered, followed by the MV model. The SMADTC model generates optimal portfolios with least gross mean returns, up to when the diversification limit is 0.25. The portfolio gross mean returns from the MAD and the MV models show a decreasing trend as the portfolios become less diversified. On the contrary, the MM and SMADTC generated portfolios have gross mean returns that increase as portfolios become less diversified.

Although the MM model appears to generate optimal portfolios with highest gross wealth, it is evident that these portfolios also have the highest implicit transaction costs, resulting in the least net portfolio wealth. The SMADTC model generates optimal portfolios with the least transaction costs per each diversification limit. The MAD model portfolios have transaction costs that start being greater than those of MV portfolios, but becoming less as portfolios become less diversified.

The greatest advantage of the SMADTC lies in its ability to generate an optimal portfolio whose net wealth is always more than amount invested. The other models generate portfolios which are not optimal investments as intended because of the eroding effect of implicit transaction costs. The information also reveals that implicit transaction costs in SMADTC portfolios decline as the portfolio becomes less diversified.

Table 1. Summary Statistics of Optimal Portfolios

D.L	Model	Mean return	Risk	Gross Wealth	Cost	Net Wealth
0.10	MAD	0.027	0.002	10269.80	432.00	9837.80
	MV	0.029	0.002	10289.02	426.34	9862.68
	MM	0.031	0.031	10314.80	502.10	9812.70
	SMADTC	0.014	0.001	10138.09	3.71	10134.38
0.15	MAD	0.025	0.022	10251.75	479.10	9772.65
	MV	0.028	0.002	10277.97	397.32	9880.65
	MM	0.033	0.033	10332.30	411.15	9921.15
	SMADTC	0.018	0.005	10174.99	2.81	10172.18
0.20	MAD	0.024	0.023	10240.20	223.20	10017.00
	MV	0.027	0.002	10268.25	378.64	9889.61
	MM	0.035	0.035	10346.80	503.60	9843.20
	SMADTC	0.019	0.006	10187.44	2.16	10185.28
0.25	MAD	0.023	0.025	10229.25	248.50	9980.75
	MV	0.026	0.002	10259.68	356.71	9902.97
	MM	0.036	0.036	10356.25	591.00	9765.25
	SMADTC	0.020	0.007	10194.80	1.70	10193.10

0.30	MAD	0.023	0.025	10221.70	272.80	9948.90
	MV	0.026	0.002	10255.02	340.06	9914.96
	MM	0.036	0.036	10362.50	550.00	9812.50
	SMADTC	0.023	0.010	10226.24	1.56	10224.68
0.35	MAD	0.021	0.026	10214.90	270.55	9944.35
	MV	0.025	0.002	10250.44	318.87	9931.57
	MM	0.037	0.037	10368.30	502.10	9866.20
	SMADTC	0.025	0.012	10249.58	1.42	10248.16
0.40	MAD	0.021	0.027	10209.60	215.20	9994.40
	MV	0.025	0.002	10247.81	306.70	9941.12
	MM	0.037	0.037	10373.20	440.40	9932.80
	SMADTC	0.026	0.013	10256.72	1.28	10255.44

However, portfolio risk increases instead. It is also evident that portfolio wealth increases with decrease in portfolio diversification.The wealth from a portfolio generated by SMADTC or MV model grows in value as the portfolio become less diversified. This is not the case with MAD and MM optimally generated portfolios whose wealths decline with decrease in diversification.

The assets' weights of portfolios generated by the four models are compared for each diversification level. Assets considered are numbered $A_1, \cdots, A_{13}$ and their descriptions are given in Appendix section. This analysis is shown in Table 2. It is evident that as the portfolios become less diversified, the assets in optimal portfolios become different, with the exception of MAD and MV generated portfolios that seem to maintain the same assets. While the SMADTC, MAD and MM models diversify portfolios as per the diversification level, there is little influence of such a restriction on assets in optimal portfolios generated by MV model. It is noted that the MV optimal portfolios comprise more assets in relatively smaller quantities than any portfolio generated by other models.

Table 2.Assets' Percentage Compositions in Optimal Portfolios.

D.L	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10			10	10	10		10	10
	MV	10	7.8	10	10	10	4.5	3.9	3.7	10	9.2	10	10	10
	MM	10	10	10		10	10	10	10		10	10		10
	SMADTC	10	10	10	10		10	10		10	10		10	10
0.15	MAD	15	15	15	15					15	10		15	
	MV	15	6.1	13.7	15	6.4			2.9	15	7.3	14.7	0.6	3.2
	MM					15	15	15	15		10	15		15
	SMADTC		15	15				15		15	15		10	15
0.20	MAD	20		20	20					20			20	
	MV	20	4.0	10.8	20	3.1			3.4	20	4.4	13.1		1.3
	MM						20	20	20			20		20
	SMADTC		20	20				20		20	20			
0.25	MAD	25			25					25			25	
	MV	24.5	2.3	8.7	22.3	0.3			3.9	25	1.1	11.8		
	MM						25	25	25			25		
	SMADTC		25	25						25	25			

0.30	MAD	10			30				30			30
	MV	22.8	1.8	8.6	21			3.9	30		11.9	
	MM					30	30	10			30	
	SMADTC		30	30					10	30		
0.35	MAD				30				35			35
	MV	20.5	1.2	8.1	19.5			3.8	35		11.8	
	MM					30	35				35	
	SMADTC		35	30						35		

Stage 2

The MV, MM and MAD models are not considered in this stage. It is only the SMADTC model that is examined to assess its multi-stage potential in generating competent optimal portfolios. We consider a first-stage optimal portfolio with six securities. The second-stage optimal portfolios generated by the model reveal that as the investor diversifies his or her portfolio, implicit transaction costs increase while the portfolio wealth decreases, as shown in Table 3 below. This can be attributed to an increasing number of securities being included in the portfolio as the diversification limit decreases.

Table 3. Second-stage Optimal Portfolios

D.L	Gross mean	Assets	Net mean	Risk(MAD)	Cost	Wealth
0.175	0.005	6	0.005	0.003	5.402	10231.432
0.200	0.006	5	0.005	0.004	4.222	10240.097
0.225	0.007	5	0.006	0.005	3.683	10248.122
0.250	0.007	4	0.007	0.006	3.143	10256.147
0.275	0.008	4	0.008	0.006	2.936	10263.840
0.300	0.009	4	0.009	0.007	2.728	10271.534
0.325	0.010	4	0.009	0.008	2.520	10279.227
0.350	0.010	3	0.010	0.009	2.318	10286.915

A further analysis is done by holding the diversification limit constant and varying the portfolio gross mean return as shown in Table 4. Here, 'assets' imply 'number of assets'. The results show that for the same portfolio gross mean return that can be achieved at different diversification limits, implicit transaction costs decrease with increasing diversification limit. The portfolio wealth also increases with increasing diversification limit. For the same risk, we observe that the greater the diversification limit the lower the portfolio implicit transaction cost. It is also noted that for each diversification limit considered, an increase in portfolio gross mean return causes implicit transaction costs to rise.

Table 4. Stage (2) Optimal Portfolios: Portfolio rate of return constrained

D.L	Gross mean	No. of Assets	Net mean	Risk(MAD)	Cost	Wealth
0.175	0.005	6	0.005	0.003	5.402	10231.432
	0.006	6	0.005	0.004	5.479	10240.062
	0.007	6	0.006	0.005	5.623	10250.103
	0.008	6	0.007	0.006	5.814	10260.096
	0.009	6	0.008	0.007	6.010	10270.085
	0.010	6	0.009	0.008	6.210	10290.069
	0.011	6	0.010	0.009	6.611	10289.853
	0.012	6	0.011	0.010	7.308	10299.341
	0.013	6	0.012	0.011	8.036	10308.797
	0.014	6	0.013	0.012	8.837	10318.180
	0.015	6	0.014	0.013	9.644	10327.557
	0.016	6	0.015	0.014	11.432	10335.954
	0.017	6	0.016	0.015	14.179	10343.392
0.200	0.006	5	0.005	0.004	4.222	10240.097
	0.007	5	0.007	0.005	4.323	10251.403

	0.008	5	0.008	0.006	4.474	10261.436
	0.009	5	0.009	0.007	4.666	10271.429
	0.010	6	0.010	0.008	4.954	10281.325
	0.011	6	0.010	0.009	5.491	10290.973
	0.012	6	0.011	0.010	6.028	10300.620
	0.013	5	0.012	0.011	6.630	10310.202
	0.014	5	0.013	0.012	7.438	10319.579
	0.015	5	0.014	0.013	8.246	10328.956
	0.016	6	0.015	0.014	9.944	10337.442
	0.017	5	0.016	0.015	11.814	10345.756
	0.018	5	0.017	0.016	14.410	10353.348
0.250	0.007	4	0.007	0.006	3.143	10256.147
	0.008	4	0.008	0.006	3.202	10262.708
	0.009	4	0.009	0.007	3.292	10272.803
	0.010	4	0.010	0.008	3.458	10282.821
	0.011	4	0.011	0.009	3.650	10292.814
	0.012	4	0.012	0.010	3.841	10302.807

As in first stage, we analyzed the performance of portfolios generated by the model during the second stage by considering the portfolio net return. Although it is obvious that portfolio performance calculated using the equation (16) will differ according to the risk measure chosen (MAD in this case), some interesting results emerged as shown in Table 5.

$$\text{Portfolio performance} = \frac{\text{Mean return}}{\text{risk}} \tag{16}$$

It is evident that as the portfolio gross mean return increasingly vary for each diversification limit, the portfolio performance is on a decreasing trend.

Table 5. Stage (2) Optimal Portfolios' Performances

D.L	Net mean	Risk(MAD)	Performance	Cost
0.175	0.005	0.003	1.67	5.402
	0.005	0.004	1.25	5.479
	0.006	0.005	1.20	5.623
	0.007	0.006	1.17	5.814
	0.008	0.007	1.14	6.010
	0.009	0.008	1.13	6.210

D.L	Net mean	Risk(MAD)	Performance	Cost
0.175	0.010	0.009	1.11	6.611
	0.011	0.010	1.10	7.308
	0.012	0.011	1.09	8.036
	0.013	0.012	1.08	8.837
	0.014	0.013	1.08	9.644
	0.015	0.014	1.07	11.432
	0.016	0.015	1.07	14.179
0.200	0.005	0.004	1.25	4.222
	0.007	0.005	1.40	4.323
	0.008	0.006	1.33	4.474
	0.009	0.007	1.29	4.666
	0.010	0.008	1.25	4.954
	0.010	0.009	1.11	5.491
	0.011	0.010	1.10	6.028
	0.012	0.011	1.09	6.630
	0.013	0.012	1.08	7.438

D.L	Net mean	Risk(MAD)	Performance	Cost
	0.014	0.013	1.08	8.246
	0.015	0.014	1.07	9.944
	0.016	0.015	1.07	11.81
	0.017	0.016	1.06	14.41
0.250	0.007	0.006	1.17	3.143
	0.008	0.006	1.33	3.202
	0.009	0.007	1.29	3.292
	0.010	0.008	1.25	3.458
	0.011	0.009	1.22	3.650
	0.012	0.010	1.20	3.841
	0.013	0.011	1.18	4.763
	0.013	0.012	1.08	5.813
	0.014	0.013	1.08	6.863
	0.015	0.014	1.07	7.914
	0.016	0.015	1.07	9.013
	0.017	0.016	1.06	11.424
	0.018	0.017	1.06	14.709
	0.018	0.018	1.00	19.961

D.L	Net mean	Risk(MAD)	Performance	Cost
0.300	0.009	0.007	1.29	2.728
	0.010	0.008	1.25	2.840
	0.011	0.009	1.22	2.945
	0.012	0.010	1.20	3.128
	0.013	0.011	1.18	3.441
	0.014	0.012	1.17	4.450
	0.014	0.013	1.08	5.504
	0.015	0.014	1.07	6.737
	0.016	0.015	1.07	7.970
	0.017	0.016	1.06	9.202
	0.018	0.017	1.06	12.072
	0.018	0.018	1.00	16.109

3.2 Sensitivity Analysis

We perform sensitivity analysis of the SMADTC model by calculating sensitivity index (SI) for each parameter. This is achieved by finding the output percentage difference when varying one input parameter from its minimum value to its maximum value (Hoffman and Gardner, [11]; Bauer and Hamby,

[1]). We use the formula

$$\text{Sensitivity Index} = \frac{D_{\max} - D_{\min}}{D_{\max}}$$

where $D_{\min}$ and $D_{\max}$ are the minimum and maximum output values respectively, resulting from varying the input parameter over its entire range. This figure-of-merit provides a good indication of parameter and model variability (Hamby D. M., [8]). We take the minimum input value to be zero and let the maximum input value vary from 0.1 to 0.35, as shown in Table 6. The phrase 'max value' refers to 'maximum input parameter value' and it gives the maximum input of the parameter considered. We calculate the sensitivity index corresponding to each asset weight.The result in the table reflect small percentages of model variability due to each input change of each parameter. This shows that the output values are not strongly reliant on certain individual parameters. Hence, the SMADTC model shows to be a reliable portfolio optimization model.

However, when doing sensitivity analysis, the following is noted. The SMADTC model is stochastic and works by replacing one parameter by a less profitable one when we take a weight of an asset to be zero, thereby achieving a less 'optimal portfolio'. Here, a less preferable asset is included in the 'less optimal portfolio'. This is so because of a condition in the model that ensures that the total weight of assets in a portfolio be unit. The weight x_i is for asset $A_i, i = 1, \cdots, 13$. This, however, leads to relative sensitivity analysis. Thus, the $D_{\min}$ of model output is a relative value. Therefore, we perform the relative sensitivity analysis of the model by finding the difference between the optimal portfolio value when the asset under consideration attains maximum value and when it gets a value of zero (*i.e.*, excluded).

Table 6.SMADTC model Sensitivity Analysis

Max value	Parameter	Cost SI	Wealth SI
0.1	x_1	0.1618	0.0019
	x_2	-0.0607	0.0017
	x_3	0.0547	0.0016
	x_4	0.1534	-0.00006
	x_6	0.1348	-0.00005
	x_7	0.1078	0.0016
	x_9	0.0647	0.0004
	x_{10}	0.0377	0.0012
	x_{12}	0.1348	0.0010
	x_{13}	0.1240	0.0016

Max value	Parameter	Cost SI	Wealth SI
0.15	x_2	-0.0125	0.0052
	x_3	0.1281	0.0035
	x_7	0.2135	0.0028
	x_9	0.1281	0.0006
	x_{10}	0.0747	0.0017
	x_{12}	0.2598	0.0018
	x_{13}	0.2456	0.0029
0.2	x_2	-0.1065	0.0064
	x_3	0.2222	0.0039
	x_7	0.3704	0.0031
	x_9	0.2222	0.0074
	x_{10}	0.1296	0.0023
0.25	x_2	-0.2153	0.0071
	x_3	0.3529	0.0049
	x_9	0.3529	0.0009
	x_{10}	0.2059	0.0029

Max value	Parameter	Cost SI	Wealth SI
0.3	x_2	-0.1859	0.0101
	x_3	0.4615	0.0059
	x_9	0.4615	0.0043
	x_{10}	0.2692	0.0034

We also carry out sensitivity analysis of the model in which we allow the gross portfolio mean return to vary by gradually increasing it and analysing changes in optimal portfolio asset weights. The results are shown in Table 7. The abbreviation 'A.weight' means 'Asset weight'. We observe that the model identifies assets with better returns but which have more implicit transaction costs whenever there is a change in the initial asset's weight. However, such changes are minimal.

Table 7.Percentage portfolio compositions: Portfolio mean return constrained.

D.Lim	A.weight	0.014	0.016	0.018	0.02	0.022	0.024	0.026	0.028	0.030
0.1	x_1	10	10	10	10	10	10	10	10	10
	x_2	10	10	10	10	10	10	10	10	10
	x_3	10	10	10	10	10	10	10	10	10
	x_4	10	2.3	4.9	4.0					
	x_5		7.7	10	10	10	10	10	10	10
	x_6	10	10	5.1						
	x_7	10	10	10	10	10	10	10	10	10
	x_8								1.5	10
	x_9	10	10	10	10	10	10	10		
	x_{10}	10	10	10	10	10	10	10	8.5	
	x_{11}				6.0	10	10	10	10	10
	x_{12}	10	10	10	10	10	10	10	10	10
	x_{13}	10	10	10	10	10	10	10	10	10

D.Lim	A.weight	0.019	0.020	0.022	0.024	0.026	0.028	0.030	0.032	0.034
0.2	x_1					20	14.1	20	20	20
	x_2	20	20	20	20	20	20	20	20	20
	x_3	20	20	20	20	20	20	20	20	20
	x_5								8.5	15.2
	x_7	20	20	20	20		5.9			
	x_9	20	11.9							
	x_{10}	20	20	20	20	20	20	20	11.5	
	x_{11}			2.6	13.3	20	20	20	20	20
	x_{12}									4.8
	x_{13}		8.1	17.4	6.7					
		0.023	0.024	0.026	0.028	0.030	0.032	0.034	0.036	0.038
0.3	x_1						4.1	10.2	18.2	26.3
	x_2	30	30	30	30	30	30	30	30	30
	x_3	30	30	30	30	30	30	29.8	21.8	13.7
	x_7		10							
	x_9	10								
	x_{10}	30	30	29.7	21.2	12.7	5.9			
	x_{11}				10.3	18.8	27.3	30	30	30

4 Conclusion

In this study, a multi-stage stochastic transaction cost model is proposed which uses mean absolute deviation as its risk and takes into account uncertainty of assets' returns and of implicit transaction costs. The main contributions of this study include (i) the development of a stochastic trading cost model that generates optimal portfolios while minimizing uncertain implicit transaction costs incurred during initial trading and in subsequent rebalancing of the portfolios, (ii) the development of a strategy that captures uncertainty in stock prices and in corresponding implicit transaction costs by way of scenarios. To provide investors with competitive portfolio returns, investment managers must manage transaction costs pro-actively, because lower transaction costs mean higher portfolio returns. Ineffective transaction cost management may be very damaging in today's competitive environment, in which the difference between success and failure may be no more than a few basis points. The model has shown that it generates optimal portfolios whose net wealths are better that those generated by MV, MM, and MAD models. It also has the advantage of generating a least cost optimal portfolio as compared to the other three models. The methodology allows investors or investment managers to choose optimal portfolios with acceptable implicit cost levels to be incurred while executing their trading. It is a suitable strategy for conservative investors. The methodology is a linear programming model and hence reduces considerably the time needed to reach a solution. It is feasible for large-scale portfolio selection. It is left, however, for future research to develop a model that accounts for both implicit and explicit trading costs while considering uncertainty of both asset prices and transaction costs. The study, like any other, has its own limitations. The use of an asset's closing price for each month in the calculation of an asset's return rate may not be the most accurate measure. However, since all assets' rates of returns have been obtained in the same way, this does not prejudice the findings.

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5 Appendix

We present models and data sets used in the validation of the SMADTC model.

5.1 Mean-variance model: A1

The mean-variance (MV) model is proposed by Harry Markowitz [17], in which the portfolio that minimizes the variance subject to the restriction of a given mean return is chosen as the optimum portfolio. The mathematical model proposed by Markowitz is as follows:

Minimize

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{1}$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} r_i x_i &\geq \rho, \\ \sum_{i=1}^{n} x_i &= 1, \\ 0 \leq x_i &\leq u_i, i = 1, \cdots, n. \end{aligned}$$

where σ_{ij} is the covariance between assets i and j, x_i is the proportion of wealth invested in asset i, r_i is the expected return of asset i in each period, ρ is the minimum rate of return desired by an investor, and u_i is the maximum proportion of wealth which can be invested in asset i.

5.2 Minimax model: A2

Young [21] propose the minimax (MM) model using minimum return as as measure of risk. This minimax model is equivalent to the mean-variance (MV) model if assets' returns are multivariate normally distributed. It is a linear programming model and is given as follows:

Maximize

$$M_p \tag{2}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{n} w_i y_{it} - M_p &\geq 0, t = 1, \cdots, T \\
\sum_{i=1}^{n} w_i \bar{y}_i &\geq G, \\
\sum_{i=1}^{n} w_i &\leq W, \\
w_i &\geq 0, i = 1, \cdots, n.
\end{aligned}$$

where y_{it} is the return of one dollar invested in security i in time period t, $\bar{y}_i$ is the mean return of security i, w_i is the portfolio allocation to security i, M_p is the minimum return on the portfolio, G is the minimum level of return, and W is the total allocation.

5.3 Mean absolute deviation model: A3

Konno and Yamazaki [14] propose the L_1 risk function (mean absolute deviation - MAD) and show that it behaves in the same manner as the mean - variance model of Harry Markowitz [17] when the assets' returns are multivariate normally distributed. They develop the MAD model as given below:

$$H(x) = \frac{1}{T}\sum_{t=1}^{T}|\sum_{i=1}^{n}(r_{it} - r_i)x_i| \tag{3}$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} r_i x_i &\geq \rho, \\ \sum_{i=1}^{n} x_i &= 1, \\ 0 \leq x_i &\leq u_i, i = 1, \cdots, n. \end{aligned}$$

where r_{it} is the realized return of asset i of period t, r_i is the expected return of asset i per period, and x_i is the proportion of wealth invested in asset i.

5.4 Data of assets

Table 8. Meaning of symbols

Symbol	A_1	A_2	A_3	A_4	A_5	A_6	A_7
Asset	AVI	ASR	APN	CSB	CLS	CML	MPC
Symbol	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	
Asset	PNC	SPP	TRU	CPI	IPL	WHL	

Table 9. Mean asset returns

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
1	0.099	0.422	-0.028	0.067	0.124	0.127	0.045
2	-0.102	-0.036	0.036	0.041	-0.057	-0.074	-0.078
3	-0.051	0.297	-0.027	0.143	-0.017	0.056	-0.082
4	-0.027	0.038	0.057	-0.021	0.014	-0.152	-0.042
5	-0.112	-0.036	-0.031	-0.138	-0.122	-0.188	-0.063
6	0.093	-0.125	0.32	0.036	0.062	0.099	0.304
7	0.084	0.071	0.113	0	0.179	0.04	0.134
8	-0.026	-0.193	-0.091	0.1	0	0.106	0.003
9	0.029	-0.173	-0.2	-0.018	0.03	-0.13	0.079
10	0.225	-0.2	0.101	-0.074	-0.048	0	-0.003
11	0.116	0.213	-0.103	0.04	0.118	-0.076	0.034
12	-0.074	-0.186	0.26	0.096	-0.029	-0.026	0.051
13	0.026	-0.139	-0.02	-0.035	-0.106	-0.022	-0.085
14	-0.136	0.294	-0.099	0.236	0.017	0.114	0.019
15	-0.046	0.02	-0.053	-0.043	0.097	0.163	0.076
16	0.061	0.069	0.111	0.022	0.064	-0.075	0.04
17	-0.029	0	0.141	-0.038	0.049	0.159	0.035

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
18	0.117	0.104	0.087	0.078	0.116	0.113	0.087
19	0.001	0.102	-0.017	0.014	-0.009	0.118	0
20	0	0.079	0.057	0.081	0.128	0.013	0.114
21	0.097	0.029	0.07	-0.077	0.079	0.048	0.056
22	-0.074	0.08	0.027	0.017	0.018	0.035	-0.092
23	0.083	0.004	0.082	0.056	0.083	0.048	0.074
24	0.012	-0.011	-0.084	-0.027	-0.028	0.017	0.005
25	0.033	0.036	0.053	-0.014	0.042	-0.006	0.095
26	0.089	0.069	0.12	0.034	0.074	0.101	0.032
27	0.008	0.018	0.05	-0.011	0.044	0.176	0.053
28	-0.031	-0.133	-0.052	-0.05	0.051	-0.07	0.069
29	-0.049	0.015	-0.039	0.072	0.043	0.028	-0.001
30	0.131	0.112	0.071	-0.027	0.065	0.16	0.128
31	0	-0.091	0.032	-0.068	0.037	0.004	-0.022
32	0.088	0.117	0.118	0.147	0.179	0.16	0.113

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
33	0.029	0.077	-0.005	0.212	0.034	0.014	0.157
34	0	0	-0.005	-0.049	-0.049	0.125	0.009
35	0.076	0.125	-0.011	0.057	-0.003	0.111	0.036
36	-0.005	0.055	-0.074	-0.048	-0.085	-0.069	-0.145
37	0.01	0.028	-0.043	-0.074	-0.029	-0.052	0.054
38	-0.01	0.056	-0.031	0.116	0.105	0.076	0.022
39	0.027	-0.012	0.025	-0.005	0.031	0.067	0.097
40	-0.028	0.012	0.077	0.043	-0.025	0.034	-0.047
41	0.042	0.014	-0.037	-0.023	-0.013	-0.018	0.065
42	0.033	0.02	-0.008	-0.035	-0.053	0.037	0.081
43	0.014	-0.018	0.014	0.011	0.043	-0.006	0.001
44	-0.011	-0.105	0.08	0.113	-0.097	0.005	-0.088
45	0.101	0.099	0.044	0.001	0.107	0.128	0.135

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
46	0.049	0.012	0.023	0.086	0.068	0.007	0.047
47	0.063	-0.041	-0.008	0.055	0.039	0	-0.003
48	0.045	0.086	0.031	-0.005	-0.141	0.117	0.081
49	0.057	0.106	0.09	-0.053	0.106	0.083	0.034
50	0.057	-0.043	0.091	0.133	0.019	0.038	0.057
51	0.039	0.098	0.061	0.04	0.046	0.025	0.115
52	0.01	0.057	-0.065	-0.007	0	-0.099	-0.01
53	0.027	0.068	0.07	0.054	0.149	0.051	0.074
54	0.182	-0.01	0.153	0.102	0.016	0.064	0.101
55	-0.009	0.007	-0.008	0.053	-0.003	0.02	0.049

Table 9. Mean asset returns (continued)

Month	PNC	SPP	TRU	CPI	IPL	WHL
1	0.136	-0.008	0.02	0.177	0.053	0.057
2	0	-0.025	-0.038	-0.064	-0.009	-0.025
3	-0.022	0.155	0.004	-0.096	-0.034	-0.013
4	-0.068	-0.071	-0.105	-0.076	-0.308	-0.054
5	-0.146	-0.048	0.002	-0.115	-0.005	-0.097
6	-0.057	0.051	0.267	0.107	-0.165	0.155
7	0.288	0.038	0.073	-0.013	0.226	0.067
8	-0.141	-0.065	-0.064	0.085	0.094	-0.069
9	-0.285	0.129	0.136	-0.156	-0.07	-0.034
10	-0.272	-0.056	-0.043	0.037	-0.084	0.084
11	-0.042	0.06	0.075	0.036	0.184	0.038
12	0.049	-0.035	0.033	0.034	-0.137	0.058
13	-0.063	-0.031	-0.117	0	-0.138	-0.115
14	0.039	-0.026	0.032	0.081	0.207	-0.042
15	0.022	0.026	0.058	0.202	0.02	0.053
16	0.089	0.07	0.068	0.051	0.101	0.042
17	0.097	-0.004	0.019	0.035	-0.028	0.036

Month	PNC	SPP	TRU	CPI	IPL	WHL
18	0	0.054	0.081	0.131	0.167	0.202
19	-0.004	0.043	-0.016	0.146	0.09	0.012
20	0.456	0.037	0.08	0.109	0.077	0.025
21	-0.033	0.066	0.059	0.049	0.02	0.083
22	-0.041	-0.041	-0.075	0.109	-0.006	-0.048
23	-0.02	0.074	0.048	0.111	0.092	0.072
24	0.1	-0.001	-0.028	-0.024	-0.092	0.024
25	0.125	0.028	0.166	0.065	0.17	0.127
26	0.078	0.021	0.058	0.153	0.07	0.093
27	-0.023	0.029	0.017	0.076	-0.018	0.041
28	0.103	0.019	0.022	-0.03	-0.025	-0.013
29	-0.002	0.022	-0.013	0.197	-0.107	0.035
30	0.131	0.071	0.087	0.093	0.124	0.085
31	-0.021	-0.017	0.004	0.035	0.059	-0.053
32	0.114	0.105	0.196	0.116	0.111	0.099
33	0.068	0.025	-0.014	-0.021	0.007	0.015
34	0.092	0.062	0.043	0.019	0.072	-0.038
35	0.107	-0.033	-0.005	0.156	0.041	0.02

Month	PNC	SPP	TRU	CPI	IPL	WHL
36	-0.034	-0.071	-0.11	-0.109	-0.137	-0.128
37	0.03	0.046	0	0.039	0.021	0.136
38	0.029	0	0.106	0.057	0.016	0.047
39	0.014	0.026	0.078	0.037	0.033	0.072
40	0.099	-0.05	-0.047	0.085	-0.013	-0.007
41	0.154	-0.026	0.012	-0.042	0.041	0
42	0.022	0.02	-0.012	0.001	-0.052	0.061
43	-0.076	0.035	0.071	0.04	0.007	0.155
44	0.116	0.011	-0.091	0.019	-0.091	-0.04
45	0.138	0.007	0.138	-0.052	0.116	0.154
46	-0.029	0.16	-0.012	0.011	0.007	0.001
47	0.044	-0.035	-0.068	-0.033	0.044	-0.035
48	0.185	0.02	0.059	0.024	0.122	0.077
49	0.059	0.041	0.034	0.015	0.063	0.074
50	0.039	0.004	-0.001	0.108	0.052	0.067
51	0.113	0.05	0.028	0.083	0.089	0.01
52	-0.037	-0.097	-0.009	0.017	-0.021	0.003
53	0.035	0.035	0.089	-0.058	0.04	0.031

Month	PNC	SPP	TRU	CPI	IPL	WHL
54	0.002	0.054	0.154	0.026	0.1	0.071
55	0.061	0.011	0.003	-0.012	0.005	0.016

Table 10.Asset Transaction Cost rates

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
1	0.0030	0.0023	0.0028	0.0036	0.0050	0.0119	0.0073
2	0.0004	0.0009	0.0028	0.0029	0.0023	0.0061	0.0051
3	0.0054	0.0008	0.0022	0.0052	0.0032	0.0425	0.0088
4	0.0087	0.0064	0.0121	0.0040	0.0107	0.0011	0.0019
5	0.0043	0.0062	0.0027	0.0080	0.0034	0.0004	0.0229
6	0.0000	0.0140	0.0072	0.0386	0.0098	0.0067	0.0012
7	0.0066	0.0048	0.0195	0.0214	0.0147	0.0018	0.0128
8	0.0048	0.0044	0.0306	0.0171	0.0548	0.0137	0.0024
9	0.0042	0.0061	0.0063	0.0078	0.0132	0.0018	0.0007
10	0.0100	0.0220	0.0027	0.0018	0.0213	0.0025	0.0060
11	0.0060	0.0003	0.0112	0.0258	0.0041	0.0085	0.0136
12	0.0143	0.0045	0.0170	0.0049	0.0015	0.0258	0.0019
13	0.0003	0.0126	0.0010	0.0055	0.0083	0.0135	0.0037
14	0.0033	0.0051	0.0029	0.0041	0.0019	0.0015	0.0020
15	0.0064	0.0023	0.0004	0.0001	0.0228	0.0026	0.0082
16	0.0057	0.0114	0.0042	0.0054	0.0092	0.0139	0.0140
17	0.0010	0.0087	0.0435	0.0165	0.0099	0.0073	0.0105

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
18	0.0537	0.0047	0.0067	0.0212	0.0045	0.0109	0.0028
19	0.0007	0.0052	0.0027	0.0474	0.0056	0.0053	0.0113
20	0.0060	0.0117	0.0126	0.0856	0.0046	0.0295	0.0033
21	0.0061	0.0303	0.0052	0.0532	0.0169	0.0033	0.0127
22	0.0070	0.0003	0.0087	0.0194	0.0268	0.0149	0.0049
23	0.0076	0.0029	0.0046	0.0074	0.0003	0.0008	0.0103
24	0.0056	0.0473	0.0025	0.0069	0.0127	0.0055	0.0026
25	0.0018	0.0058	0.0095	0.0070	0.0079	0.0046	0.0056
26	0.0149	0.0074	0.0327	0.0072	0.0009	0.0019	0.0045
27	0.0037	0.0208	0.0032	0.0233	0.0104	0.0035	0.0165
28	0.0202	0.0065	0.0256	0.0301	0.0051	0.0021	0.0073
29	0.0068	0.0225	0.0141	0.0061	0.0018	0.0056	0.0116
30	0.0298	0.0000	0.0343	0.0068	0.0695	0.0253	0.0094
31	0.0072	0.0197	0.0027	0.0408	0.0084	2.0000	0.0115
32	0.0157	0.0153	0.0066	0.0679	0.0040	0.0749	0.0058
33	0.0139	0.0238	0.0015	0.0019	0.0040	0.0286	0.0111
34	0.0142	0.0024	0.0079	0.0073	0.0013	0.0105	0.0024
35	0.0074	0.0186	0.0068	0.0326	0.0198	0.0066	0.0114

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
36	0.0316	2.0000	0.0082	0.0146	0.0487	0.0074	0.0133
37	0.0041	0.0083	0.0433	0.0155	0.0369	0.0082	0.0075
38	0.0459	2.0000	0.0058	0.0014	0.0029	0.0354	0.0033
39	0.0006	0.0386	0.0012	0.0046	0.0334	0.0177	0.0339
40	0.0743	0.0225	0.0097	0.0044	0.0148	0.0202	0.0543
41	0.0210	0.0290	0.0103	0.0952	0.0144	0.1277	0.0402
42	0.0208	0.0022	0.0179	0.0222	0.0308	0.1053	0.0089
43	0.0217	2.0000	0.0177	0.0396	0.0238	2.0000	0.0052
44	0.0053	2.0000	0.0151	0.0619	0.0033	0.0942	0.0112
45	0.0124	0.0942	0.0077	0.0759	0.0183	0.0408	0.0083
46	0.0086	0.0083	0.0800	0.0352	0.0215	0.0258	0.0014
47	0.0169	0.0645	0.0026	0.0829	0.0064	0.0077	0.0271
48	0.0060	0.0072	0.0185	0.0769	0.0068	0.0198	0.0020
49	0.0209	2.0000	0.0255	0.1452	0.0000	0.0217	0.0047
50	0.0330	0.0108	0.0132	0.1125	0.0043	0.0089	0.0277
51	0.0040	0.0189	0.0016	0.0894	0.0086	0.0308	0.0114
52	0.0032	0.0028	0.0110	2.0000	0.0078	0.0048	0.0164

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
53	0.0317	0.0237	0.0026	0.0734	0.0007	0.0543	0.0101
54	0.0180	0.0220	0.0085	0.0112	0.0104	0.0151	0.0228

Table 10. (continued)

Month	PNC	SPP	TRU	CPI	IPL	WHL
1	0.0319	0.0019	0.0009	0.0063	0.0058	0.0062
2	0.0323	0.0072	0.0033	0.0105	0.0132	0.0010
3	0.0038	0.0044	0.0241	0.0061	0.0067	0.0018
4	0.0049	0.0035	0.0028	0.0411	0.0040	0.0029
5	0.0058	0.0021	0.0031	0.0031	0.0005	0.0013
6	0.0046	0.0024	0.0056	0.0031	0.0050	0.0010
7	0.0315	0.0029	0.0066	0.0056	0.0050	0.0008
8	0.0086	0.0946	0.0540	0.0121	0.0542	0.0069
9	0.0263	0.0009	0.0068	0.0003	0.0019	0.0027
10	0.0148	0.0030	0.0128	0.0108	0.0052	0.0023
11	0.0202	0.0019	0.0063	0.0459	0.0121	0.0689
12	0.0043	0.0109	0.0029	0.0071	0.0078	0.0099
13	0.0067	0.0007	0.0023	0.0028	0.0042	0.0044
14	0.0127	0.0017	0.0033	0.0032	0.0043	0.0057

Month	PNC	SPP	TRU	CPI	IPL	WHL
15	0.0197	0.0041	0.0036	0.0191	0.0084	0.0020
16	0.0072	0.0012	0.0177	0.0047	0.0100	0.0050
17	0.0074	0.0001	0.0185	0.0050	0.0008	0.0004
18	0.0122	0.0085	0.0077	0.0005	0.0002	0.0098
19	0.0088	0.0007	0.0015	0.0054	0.0085	0.0015
20	0.0081	0.0078	0.0324	0.0030	0.0027	0.0046
21	0.0179	0.0085	0.0022	0.0031	0.0037	0.0271
22	0.0228	0.0022	0.0011	0.0101	0.0097	0.0234
23	0.0106	0.0305	0.0077	0.0143	0.0010	0.0220
24	0.0293	0.0053	0.0024	0.0039	0.0137	0.0073
25	0.0212	0.0050	0.0127	0.0125	0.0205	0.0421
26	0.0452	0.0019	0.0145	0.0145	0.0034	0.0121
27	0.0178	0.0024	0.0137	0.0069	0.0047	0.0223
28	0.0203	0.0093	0.0027	0.0053	0.0064	0.0071
29	0.0526	0.0043	0.0026	0.0037	0.0040	0.0019
30	0.0595	0.0113	0.0095	0.0065	0.0133	0.0016
31	2.0000	0.0058	0.0069	0.0127	0.0096	0.0028
32	0.0470	0.0020	0.0021	0.0140	0.0086	0.0300

Month	PNC	SPP	TRU	CPI	IPL	WHL
33	0.0287	0.0151	0.0018	0.0047	0.0047	0.0114
34	0.0030	0.0023	0.0068	0.0149	0.0068	0.0025
35	0.0545	0.0032	0.0261	0.0091	0.0125	0.0184
36	0.0590	0.0017	0.0119	0.0092	0.0103	0.0032
37	0.0874	0.0142	0.0117	0.0012	0.0022	0.0117
38	0.0597	0.0039	0.0055	0.0241	0.0013	0.0080
39	0.2246	0.0186	0.0064	0.0102	0.0215	0.0058
40	0.0488	0.0107	0.0041	0.0015	0.0079	0.0302
41	0.0845	0.0252	0.1433	0.0481	0.0299	0.1060
42	0.0207	0.0077	0.0088	0.0236	0.0164	0.0105
43	0.2118	0.0357	0.0239	2.0000	0.0077	0.0008
44	0.0267	0.0080	0.0066	0.0142	0.0078	0.0123
45	0.2449	0.0741	0.0075	0.0714	0.0082	0.0370
46	0.2290	0.0050	0.0187	0.0546	0.0328	0.0481
47	0.0047	0.0093	0.0123	0.0165	0.0110	0.0169
48	0.0625	0.0012	0.0007	0.0479	0.0099	0.0273
49	0.1374	0.0228	0.0110	0.0147	0.0042	0.0069
50	0.0247	0.0179	0.0013	0.0165	0.0024	0.0608

Month	PNC	SPP	TRU	CPI	IPL	WHL
51	0.0114	0.0011	0.0155	0.0469	0.0007	0.0109
52	0.0270	0.0062	0.0650	0.0408	0.0336	0.0198
53	0.0110	0.0082	0.0633	0.0524	0.0507	0.0112
54	0.0228	0.0445	0.0136	0.0202	0.0033	0.0347

OPTIMAL PORTFOLIO SELECTION WITH STOCHASTIC MAXIMUM DOWNSIDE RISK AND UNCERTAIN IMPLICIT TRANSACTION COSTS

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Abstract

A multi-stage stochastic optimal portfolio policy that minimizes downside risk in the presence of uncertain implicit transaction costs is proposed. As asset returns in economic recessions and booms are characterised by extreme movements, some individual stocks show an extreme reaction while others exhibit a milder reaction. The study therefore considers a risk-averse and conservative investor who is highly concerned about the performance of his portfolio in an economic recession environment. Maximum negative deviation is taken as the downside risk and stochastic programming is applied with stochastic data given in the form of a scenario tree. A set of discrete scenarios of asset returns is considered, taking the deviation around each return scenario. Thus uncertainties of asset returns and implicit transaction costs are represented by discrete approximations of a multi-variate continuous distribution. The portfolio is rebalanced at discrete time intervals as new information on returns get realised. First-stage optimal-portfolio results show that implicit transaction costs vary from 7.1% to 16.7% of returns on investment.

Keywords

Maximum downside risk, implicit transaction costs, discrete scenarios, uncertainty.

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1. INTRODUCTION

Banks, fund-management firms, financial consulting institutions and large institutional investors are faced with the challenges of managing their funds, assets and stocks towards selecting, creating, balancing and evaluating optimal portfolios on a continual basis. Financial crises, economic imbalances, algorithmic trading and highly volatile movements of asset prices in recent times have raised high alarms about the management of financial risks. Extreme event risk is present in all areas of risk management. Whether one is concerned with market, credit, operational or insurance risk, one of the biggest challenges facing the risk manager is how to implement risk management models that allow for rare but damaging events, and permit the measurement of their consequences.

In financial markets, the stability and sustainability of future pay-offs of an investment are largely determined by extreme changes in financial conditions rather than typical movements. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred in setting up a new portfolio or rebalancing an existing one must be included in any realistic portfolio selection analysis. Investment portfolios should be rebalanced to account for changing market conditions and changes in funding.

In this study, a multi-stage stochastic maximum negative deviation (SMNDTC) model with uncertain implicit transaction costs in optimal portfolio selection is proposed. Maximum negative deviation of asset returns from expected portfolio return is used as portfolio risk. The model takes into account downside risk and corresponding implicit transaction costs in trading in order to provide an investor or investment manager an option of selecting a portfolio knowing the implicit trading costs which are likely to be incurred. The study uses stochastic programming with recourse. The uncertainty about future asset returns and corresponding implicit transaction costs is captured in stages by means of scenarios. Implicit costs are taken to be random as it is in the buying or selling of assets (whose prices are stochastic) that these costs are incurred.

2. LITERATURE REVIEW

It is well documented in the literature that investors generally shun positions in which they would be subjected to catastrophic losses however small the probability these losses carry. Such a "disaster avoidance motive" (Menezes et al., 1980) implies that investors care about extreme negative scenarios in investment and are averse to the risk of sharp price plunges. Hence the potential loss from extreme undesirable returns should become a significant factor in asset pricing. Asset returns in economic recessions and booms are characterised by extreme movements (Jansen & De Vries, 1991). The extreme movements of the market are not always reflected in all the individual stocks. Some individual stocks show an extreme reaction while others exhibit a milder reaction. It is in extreme cases that investors are highly concerned about the performance of their portfolios, particularly the downside movements.

The notion of tail risk or extreme downside risk has increasingly gained consideration in the asset pricing literature. In particular, contrary to the assumptions of the standard Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965), in which portfolio risk is fully captured by the variance of the portfolio return distribution, asset returns display significant negative skewness and excess kurtosis, both of which increase the likelihood of extreme negative returns (Richard et al., 2015). In the studies that focus directly on the likelihood of extreme returns, Ruenzi and Weigert (2013) use a copula-based approach to construct a systematic tail risk measure and show that stocks with high crash sensitivity, measured by lower tail dependence with the market, are associated with higher returns that cannot be explained by traditional risk factors, downside beta, co-skewness or co-kurtosis. These studies examine the variation in expected returns across individual stocks.

Young (1998) introduces a linear programming model which maximizes the minimum return or minimizes the maximum loss (minimax) over time periods and applies this to the stock indices of eight countries. The analysis show that the model performs similarly to the classical mean-variance model of Markowitz (1991). Additionally, Young (1998) argues that when data is log-normally distributed or skewed, the minimax formulation might be a more appropriate method compared to the mean-variance formulation which is optimal for normally distributed data. Kamil et al. (2009) develop a single and two-stage stochastic programming model with recourse for portfolio selection in which the maximum downside deviation of asset returns from expected portfolio return is minimised. This study extends their formulation and develops a multistage stochastic maximum negative deviation (SMNDTC) model which takes into account uncertainty of implicit transaction costs and asset returns as well as recourse decisions in discrete time intervals in optimal portfolio selection.

Investment portfolios should be rebalanced to account for changing market conditions and changes in funding. The investor incurs transaction costs during initial trading and in subsequent rebalancing of the portfolio. Trading costs are either direct or indirect. Direct trading costs are observable and include brokerage commissions, market fees and taxes. Indirect costs are invisible and include bid-ask spread, market impact and opportunity costs. Some investors do not like overly high transaction costs, as these are known to erode the profits of investment. The model being proposed considers downside risk and corresponding implicit transaction costs in trading to give an investor or investment manager an option of selecting portfolios knowing the implicit trading costs which are likely to be incurred. The uncertainty about future asset returns and implicit costs is captured in stages and by means of scenarios. The main contributions of this study include:

- the development of a multi-stage stochastic maximum negative deviation model that optimizes portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in rebalancing of portfolios, and
- the development of a strategy that captures uncertainty in stock returns and in corresponding implicit trading costs in extreme downside movements of stock prices by way of scenarios.

3. PROBLEM STATEMENT

A set of securities $I = \{i: i = 1,2, \ldots, n\}$ is considered for an investment for the period $[0, T]$. The study seeks to determine a multi-period discrete-time optimal portfolio strategy subject to uncertain implicit transaction costs. The portfolio is structured in terms of asset return and downside risk measured as the maximum negative deviation of asset return from expected portfolio return. The strategy takes into account the approximate nature of a set of discrete scenarios by considering the negative deviation around each asset return scenario. Let $R^t =$

$\{R_1, \dots, R_t\}$ be stochastic events at time periods $t = 1,2, \dots, \tau$. The investment horizon T is divided into two discrete times T_1 and T_1 defined by $T_1 = [0, \tau]$ and $T_2 = (\tau, T]$. During T_1 an investor makes decisions and adjustments to his portfolio at each time-stage as new information on asset returns become available. The initial investment takes place at $t = 0$, with recourse decisions implemented at discrete times $t = 1,2, \dots, \tau$. After $t = \tau$, no further decisions are implemented until investment maturity at $t = T$.

Buying of the initial portfolio assets and implementation of recourse decisions result in the investor incurring some transaction costs, which can erode the value of the investment. The decision process is non-anticipative, that is, a decision at a particular stage does not depend on the future realization of the random events. The recourse decision at period t is dependent on the outcome at period $t-1$. Given the event history up to time t, R_t, the uncertainty in period $t+1$ is characterised by finitely many possible outcomes for the observations R_{t+1}. The branching process can be represented by a scenario tree. Below is an example of a scenario tree with two-time periods and a three-three branching structure.

It is considered that the uncertain asset returns, R_t, in period t are represented by a finite set of discrete scenarios $\Omega = \{s: s = 1,2, \dots, S\}$, where the returns under a particular scenario take values $R_s = \{R_{1,s}, R_{2,s}, \dots, R_{n,s}\}^T$ with associated probability $p_s > 0$, where $\sum_{s=1}^{S} p_s = 1$.

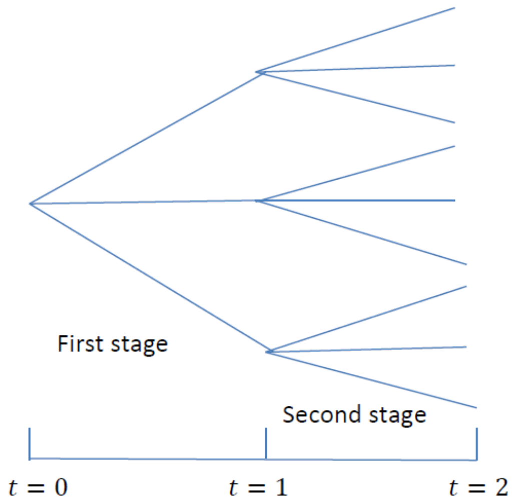

FIGURE 1: Scenario tree

3.1 Formulation of model constraints

In this problem, an investor dynamically adjusts a portfolio at successive discrete times as new information on asset returns arises. Initial portfolio selection takes place at $t = 0$ with wealth W_0 distributed among the n assets of the initial portfolio. The investor seeks to obtain an optimal strategy $x_t = [x_{1,1,t}, x_{1,2,t}, \dots, x_{2,1,t}, x_{2,2,t}, \dots, x_{1,s,t}]^T$, $t = 1,2, \dots, \tau$, at the end of the planning phase. It is noted that

$$\sum_{i=1}^{n} x_{i,t} = \sum_{s=1}^{S} x_{i,s,t} = 1, t = 1,2, \dots, \tau, \quad \textbf{(1)}$$

where S is the total number of scenarios in period t. Let $a_{ist} = \frac{A_{ist}}{W_t}$ and $v_{ist} = \frac{V_{ist}}{W_t}$, be, respectively, the buying and selling proportions of asset i of scenario s of period t, where A_{ist} is the amount of money used to buy new shares and V_{ist} is the money obtained from selling shares of asset i of scenario s of period t. It is derived that $x_{i,s,t} = x_{i,s,t-1} + a_{i,s,t} - v_{i.s,t}, i = 1,2,\dots,n; s = 1,\dots,S; t = 1,\dots,\tau,$ and also that $a_{i,s,t} \cdot v_{i,s,t} = 0$ since we cannot buy and sell the same asset at each time that recourse decisions are implemented.

The expected return of asset i of period t can now be stated as $r_{it} = \sum_{s\in Q} p_s \cdot R_{ist} \cdot x_{ist}$, $i = 1,\dots,n; t = 1,\dots,\tau$, where $Q \subset \Omega$ is a set of scenarios of asset i of period t. Thus, the gross expected return of the portfolio of period t becomes $r_{pt} = \sum_{s=1}^{S} p_s \cdot R_{ist} \cdot x_{ist}$, $i = 1,\dots,n; t = 1,\dots,\tau$. During portfolio rebalancing, it is ensured that

$$0 \leq v_{ist} \leq x_{ist;}\ i = 1,\dots,n; t = 1,\dots,\tau \tag{2}$$

since the portfolio is self-financing and there is no additional funding to the portfolio at $t > 0$. Thus the volume of asset i of scenario s in period t sold for portfolio rebalancing should not exceed the volume of the asset in the portfolio.

It is observed here that $v_{it} = \sum_{s\in Q} p_s \cdot v_{ist}$ and $x_{it} = \sum_{s\in Q} p_s \cdot x_{ist}$. In a self-financing portfolio being rebalanced, the amount of money gained from selling asset i of period t should be at most the amount of money used to buy asset j $(i \neq j)$ of the same period. This results in the constraint

$$0 \leq \sum_{i=A} a_{ist} = \sum_{j=1, j\neq i} v_{jst},\ i = 1,\dots,n; s = 1,\dots,S \tag{3}$$

where set A contains all assets for which volumes have been bought. To avoid short-selling, the constraint

$$0 \leq x_{ist} \leq U_{ist}, i = 1,\dots,n; s = 1,\dots,S; t = 1,\dots\tau, \tag{4}$$

is considered where U_{ist} is the maximum proportion allowed for scenario s of period t for each asset i. If k_{ist} and l_{ist} are the transaction cost rates for buying and selling, respectively, a unit volume of asset i in scenario s for portfolio rebalancing at the beginning of period t, then either $k_{ist} \cdot a_{ist} = 0$ or $l_{ist} \cdot v_{ist} = 0$ or both are zero. The transaction cost incurred by the investor for buying or selling asset i of scenario s in period t is given by $k_{ist} \cdot a_{ist} + l_{ist} \cdot v_{ist}$. Therefore, the expected transaction cost of the portfolio of period t is $\sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}$, $i = 1,\dots,n$; $t = 1,\dots,\tau$. This results in the net expected portfolio return, N_{pt}, of period t as $N_{pt} = r_{pt} - \sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}$, with the portfolio net wealth of period t given by

$$W_t = \left(1 + N_{pt}\right) \cdot W_{t-1}, t = 1,2,\dots,\tau. \tag{5}$$

Scenarios may reveal identical value for the uncertain quantities up to a certain period. Such scenarios must yield the same decisions up to that period. This results in the constraint

$$x_{ist} = x_{iht}, \tag{6}$$

for all scenarios s and h with identical past up to time t.

3.2 Portfolio risk

During the period $[0;\tau]$, the downside risk of asset i of scenario s in period t is defined as $K_{ist} = |\min[0, R_{ist} - r_{pt}]|$. Thus the expected downside portfolio risk at any time period t becomes $\sum_{s=1}^{S} p_s K_{ist} x_{ist}$.

This results in the expected downside portfolio risk, H, for all time periods as

$$H = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist} x_{ist}. \quad \textbf{(7)}$$

If $\beta_t = \sum_{s=1}^{S} p_s K_{ist} x_{ist}$, the expected portfolio risk for the entire rebalancing phase becomes $H = \frac{1}{\tau}\sum_{t=1}^{\tau} \beta_t$.

3.3 The multi-stage stochastic maximum negative deviation model

The objective in this problem is to obtain an optimal portfolio that minimizes the expected portfolio risk subject to constraints describing the growth of the portfolio in all periods, a performance constraint and bounds on decision variables. Letting θ to be the minimum desired expected portfolio mean return and λ to be the minimum acceptable transaction cost, the following optimization model is obtained:

Minimize
$$H = \frac{1}{\tau}\sum_{t=1}^{\tau} \beta_t \quad \textbf{(8)}$$

Subject to
$$\begin{aligned}
N_{pt} &\geq \theta \\
W_t &= (1 + N_{pt})W_{t-1}, t = 1, \dots, \tau, \\
0 &= \beta_t - \textstyle\sum_{s=1}^{S} p_s K_{ist} x_{ist}, \quad i = 1, \dots, n; t = 1, \dots, \tau, \\
0 &\leq \textstyle\sum_{i \in A} a_{ist} \leq \sum_{j=1, j \neq i}^{n} v_{ist}, \quad s = 1, \dots, S; t = 1, \dots, \tau, \\
\lambda &\geq \textstyle\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}, i = 1, \dots, n; t = 1, \dots, \tau, \\
1 &= \textstyle\sum_{s=1}^{S} x_{ist}, \quad i = 1, \dots, n; t = 1, \dots, \tau, \\
0 &\leq v_{ist} \leq x_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\
0 &\leq x_{ist} \leq U_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\
x_{ist} &= x_{iht}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots \tau.
\end{aligned}$$

The model **(8)** has a non-linear objective function and the third constraint is also non-linear. The model is transformed into a linear stochastic programming model as follows. For each scenario s, let$M_{ist} \geq K_{ist} = |\min[0, R_{ist} - r_{pt}]|, s = 1, \dots, S$. Then, the expected portfolio risk becomes $G = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s M_{ist} x_{ist}$, where the expected portfolio risk at any period t is given by $\sum_{s=1}^{S} p_s K_{ist} x_{ist}$. If $Z_t = \sum_{s=1}^{S} p_s M_{ist} x_{ist}$, then the expected portfolio risk for the period $[0, \tau]$ is $G = \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t$. The programming model **(8)** is transformed into the following linear stochastic model.

Minimize $$G = \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t \quad \textbf{(9)}$$

Subject to
$$
\begin{aligned}
N_{pt} &\geq \theta \\
W_t &= (1 + N_{pt})W_{t-1}, t = 1, \dots, \tau, \\
0 &= Z_t - \textstyle\sum_{s=1}^{S} p_s M_{ist} x_{ist}, \;\; i = 1, \dots, n; t = 1, \dots, \tau, \\
0 &\leq \textstyle\sum_{i\in A} a_{ist} \leq \sum_{j=1, j\neq i}^{n} v_{ist}, \;\; s = 1, \dots, S; t = 1, \dots, \tau, \\
\lambda &\geq \textstyle\sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}, i = 1, \dots, n; t = 1, \dots, \tau, \\
1 &= \textstyle\sum_{s=1}^{S} x_{ist}, \;\; i = 1, \dots, n; t = 1, \dots, \tau, \\
0 &\leq v_{ist} \leq x_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\
0 &\leq x_{ist} \leq U_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\
x_{ist} &= x_{iht}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots \tau.
\end{aligned}
$$

The following theorem shows that models **(8)** and **(9)** yield the same optimal values.

Theorem 1

If x^* is an optimal solution to **(8)**, then (x^*, G^*) is an optimal solution to **(9)**. Conversely, if (x^*, G^*) is an optimal solution to **(9)**, then x^* is an optimal solution to **(8)**.

Proof

Without loss of generality, let $x^* = x^*_{ist}$. If x^* is an optimal solution to **(8)**, then (x^*, G^*) is a feasible solution to **(9)**, where $G = \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s M_{ist}x_{ist} \geq \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist}x_{ist}$. If (x^*, G^*) is not an optimal solution to **(9)**, then there exists a feasible solution (x, G) to **(9)** such that $G < G^*$, where

$$
\begin{aligned}
G &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s K_{ist}x_{ist} \\
&= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist}
\end{aligned}
$$

It is observed that $\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist} = G < G^*$ and tha $G < G^* = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x^*_{ist}$t. This is a contradiction since x^* is an optimal solution of **(8)**.

Conversely, if (x^*, G^*) is an optimal solution of **(9)**, where

$$G = \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist},$$

then x^* is an optimal solution of **(8)**. Otherwise, there exists a feasible solution x to **(8)** such that

$$\begin{aligned} G &= \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist} \\ &< \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{s=1}^{S} p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist}^* \\ &= G^* \end{aligned}$$

which contradicts that (x^*, G^*) is an optimal solution to **(9)**. This completes the proof.

3.4 Measurement of transaction costs

Transaction costs incurred by an investor when buying or selling shares of securities at a stock market are broadly of two types, namely implicit and explicit costs. Explicit costs can easily be determined before execution of trade, as they do not rely on the trading strategy. These include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. Implicit costs, on the other hand, are invisible. They depend mainly on the trading characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of trade execution. They are of three categories, namely market impact, opportunity costs and spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt & Giraud, 2008). When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed' price that would have prevailed if the trade had not occurred. Thus, in this study, immediate execution of trade is assumed, and hence market impact accounts for the total implicit costs. Hau (2006) provides a methodology for calculating these implicit costs which this study adopts. The transaction price is taken to be the last price of the month and the spread mid-point is used as benchmark. The effective spread is then calculated as twice the distance from the mid-price measured in basis points. Thus obtaining the effective spread (implicit transaction cost) as

$$Spread^{Trade} = \frac{200 \times |P^T - P^M|}{P^M}$$

where P^T is the transaction price and P^M is the mid-point of the bid-ask spread.

4. DATA AND SAMPLE

The historical monthly data of securities on the Johannesburg Stock Market from January 2008 to September 2012 is considered, and the following criteria are used to select securities available for portfolio selection:

- stocks with negative mean returns for the entire period considered are excluded from the sample,
- companies which were not listed on the Johannesburg Stock Market by January 2008 and only entered afterwards are excluded, and
- securities having the highest positive mean returns for the entire period.

Empirical distributions computed from past monthly returns are taken as equi-probable scenarios. A scenario, R_{ist}, for the return of asset i of period t is calculated as $R_{ist} = \frac{P_{i,t}}{P_{i,t-1}} - 1$, where $P_{i,t}$ is the historical monthly price of asset i. Five scenarios are considered for each asset return and corresponding implicit cost at each time period and the model is applied over one stage. The initial portfolio is selected from 13 assets and empirical distributions of these securities are considered. Since for each security we have 54 monthly returns for the period under study, the months are numbered from 1 to 54 and random numbers used to select asset returns and associated transaction costs to get scenarios for each asset. It is assumed that transaction costs are random since they are randomly selected together with corresponding asset returns. Thus a scenario comprises an asset return and the associated transaction cost. The transaction cost is given as a rate and scenarios are taken to be equally likely to occur. Thus, each asset's return and transaction cost scenario has a probability of occurring of $\frac{1}{5n}$, where n is the number of assets in the portfolio. The number of scenarios is restricted to 5, since in stochastic programming the scenario tree grows exponentially. At the end of the first stage, the investor decides on the first-stage optimal portfolio as given by the investor's chosen portfolio risk, the gross portfolio mean return or net portfolio mean return as the case may be and the portfolio transaction cost.

4.1 Model application and results

The study considers an investor who has R10 000 to spend on the initial portfolio. The optimal portfolios describing the first-stage efficient frontier are shown in TABLE 1. The phrase 'D. Lim' stands for 'diversification limit'.

TABLE 1: Stage 1 optimal portfolios: risk, cost and expected return unconstrained

D. Lim	*Gross mean*	*Net mean*	*Risk*	*Cost*	*% Cost*	*Net wealth*
0.100	0.031	0.027	0.034	44.82	12.9	10265.78
0.125	0.034	0.029	0.031	55.15	14.7	10286.85
0.150	0.036	0.030	0.029	63.49	16.7	10297.01
0.175	0.038	0.032	0.027	60.04	15.8	10317.61
0.200	0.039	0.036	0.025	36.48	7.7	10357.12
0.225	0.040	0.036	0.025	39.47	10.0	10359.09
0.250	0.040	0.036	0.024	42.45	10.0	10361.05
0.275	0.041	0.035	0.024	44.28	12.1	10362.78
0.300	0.041	0.036	0.024	46.10	12.1	10364.50
0.325	0.041	0.037	0.023	47.93	9.8	10366.23
0.350	0.042	0.037	0.023	44.71	11.9	10372.89
0.375	0.042	0.038	0.023	38.98	9.5	10382.03
0.400	0.042	0.039	0.022	33.24	7.1	10391.16

Source: Authors' analysis

As portfolios become less diversified, there is an increase in both the gross portfolio mean return and the net portfolio mean return. The risk declines and the net wealth increases with increasing diversification limit. However, the total implicit transaction cost rises from R44.82 to R63.49 as the diversification limit increases from 0.1 to 0.15 respectively. Thereafter, the transaction cost declines in a fluctuating pattern to a lowest value of R33.24 obtained when the diversification limit is 0.4. Efficient frontiers of net mean portfolio returns and gross mean portfolio returns reveal the impact of neglecting implicit transaction costs in portfolio selection. The total implicit transaction costs incurred to achieve each optimal portfolio vary from 7.1% to 16.7% of returns on investment.

Analysis of portfolio composition of assets in optimal portfolios is carried out and the information is shown in TABLE 2. It is evident from the table results that the SMNDTC model allocates the maximum weight possible to each of the selected assets except when an even distribution is impossible. The model reflects consistency in selection of assets as portfolios become less diversified.

TABLE 2: Assets percentage composition: Stage 1 optimal portfolios

D. Lim	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*PNC*	*CPI*	*IPL*
0.100	10	10	10	10	10	10	10	10	10
0.125	12.5	12.5		12.5	12.5	12.5	12.5	12.5	12.5
0.150	15	15		15		10	15	15	15
0.175	17.5	17.5		12.5			17.5	17.5	17.5
0.200	20	20					20	20	20
0.225	22.5	22.5					22.5	10	22.5
0.250	25	25					25		25
0.275	27.5	27.5					27.5		17.5
0.300	30	30					30		10
0.325	32.5	32.5					32.5		2.5
0.350	35	35					30		
0.375	37.5	37.5					25		
0.400	40	40					20		

Source: Authors' analysis

4.1.1 Sensitivity analysis

Sensitivity analysis of the SMNDTC model is done by calculating the sensitivity index (SI) for each parameter. The methodology by Hoffman and Gardner (1983) and Bauer and Hamby (1991) is employed. Output percentage difference is calculated by varying one input parameter at a time, from its minimum value (zero in this case) to its maximum value (parameter value in optimal portfolio). Sensitivity indices are obtained as follows:

$$\text{Sensitivity Index (SI)} = \frac{D_{max} - D_{min}}{D_{max}}$$

where D_{min} and D_{max} are the minimum and maximum output values respectively. We consider maximum output values of 0.1, 0.15, 0.2 up to 0.3, which are the imposed diversification limits. The model being proposed is stochastic and works by replacing one parameter by a 'less profitable' one when we assign a weight of zero to the asset of the optimal portfolio. This happens because of the condition imposed by the model that the sum of assets' weights be unity.

This results in a relative sensitivity value. Hence, D_{min} value of the model output is a relative value. Thus, applying the above method yields a relative sensitivity analysis of the model, the results of which are shown in TABLE 3.

TABLE 3: SMNDTC model sensitivity analysis

D. Lim	*Parameter*	*Proxy*	*Max value*	*Cost SI*	*Risk SI*	*Wealth SI*
0.10	X1		0.10	0.0357	0.3235	0.0027
	X2			-0.0076	0.0000	0.0026
	X3			-0.0036	0.0588	0.0004
	X4			0.4931	0.0882	-0.0011
	X5	X7(10)		0.0134	0.0294	0.0008
	X6			0.0442	0.1176	0.0007
	X8			0.2700	-0.0294	0.0008
	X11			0.0192	0.2353	0.0018
	X12			0.0451	0.1765	0.0018
	X13			0.0013	0.2353	0.0001
0.15	X1		0.15	0.0128	0.3793	0.0019
	X2			-0.0331	0.0000	0.0018
	X4			0.4971	0.1379	-0.0038
	X6	X5(10)		0.0217	0.1724	-0.0011
	X8			0.2608	0.0000	-0.0009
	X11			-0.0047	0.2759	0.0006
	X12			0.0227	0.2414	0.0006
0.20	X1		0.20	-1.1239	0.3600	0.0075
	X2			-1.2303	-0.0400	0.0074
	X8	X4(20)		-0.5482	-0.0400	0.0039
	X11			-1.1645	0.2800	0.0058
	X12			-1.1009	0.2400	0.0058
0.25	X1		0.25	0.0436	0.4583	0.0021
	X2	X11(25)		-0.0707	0.0000	0.0020
	X8			0.6620	0.0000	-0.0023
	X12			0.0683	0.2917	0.0001

D. Lim	Parameter	Proxy	Max value	Cost SI	Risk SI	Wealth SI
0.30	X1		0.30	0.4583	0.4583	0.0025
	X2	X11(10)		0.0417	0.0417	0.0024
	X8			0.0417	0.0417	-0.0028
	X12			0.3333	0.3333	0.0000

Source: Authors' analysis

It is observed that, for each diversification limit considered, the same asset is being chosen as the proxy. However, these proxies vary from one diversification limit to the other. The results show small percentages of model variability of optimal wealth due to changes in input parameter value. The wealth sensitivity index ranges from -0.38% to 0.75%. This is a good indication that the model output values are not significantly influenced by specific input values. Some wealth sensitivity indices are negative, implying that the respective proxies result in better wealth. However, the cost sensitivity indices vary from -1.2303 to 0.6620. The negative indices show that the proxies result in higher implicit transaction costs compared to assets in optimal portfolios. All such proxies cause a decline in portfolio wealth, since all corresponding wealth S.I. values are positive.

5. CONCLUSION

In this study, a multi-stage stochastic maximum downside risk model that incorporates uncertainty of asset returns and implicit transaction costs is proposed. The model best applies to periods of economic recessions which are characterised by extreme movements in asset prices. In such times, investors are highly concerned about the performance of their portfolios, particularly the downside movements. The contribution of this study includes:

- the development of a multi-stage stochastic maximum negative deviation model that optimizes portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in subsequent rebalancing of portfolios, and
- the development of a strategy that captures uncertainty in stock returns and in corresponding implicit trading costs in extreme downside movements of stock prices by way of scenarios.

The methodology allows investors and investment managers to decide on optimal portfolios realizing the associated implicit transaction costs. It is a linear programming model and hence it is feasible for large-scale portfolio selection, as it reduces considerably the time needed to reach a solution. It is, however, left for further research to obtain a model that captures both implicit and explicit transaction costs in uncertain market environments.

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Optimal portfolio selection with uncertain implicit transaction costs in a dynamic stochastic Mean - Variance framework

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Optimal portfolio selection with uncertain implicit transaction costs in a dynamic stochastic Mean-Variance framework

Abstract

We study the optimal portfolio policy of a multi-period stochastic mean-variance investor subject to uncertain implicit transaction costs. Multi-stage stochastic programming is used to model this problem of financial optimization with stochastic data given in the form of a scenario tree. We extend Markowitz's mean-variance model and consider a set of discrete scenarios of asset returns, taking the variance around each return scenario. Thus uncertainties of asset returns and transaction costs of securities are represented by discrete approximations of a multi-variate continuous distribution. We apply the model to securities on the Johannesburg Stock Market. We find that the model generates optimal portfolios with least risk, least total implicit transaction costs and greatest net wealths compared to optimal portfolios generated by the mean-variance (MV), mean absolute deviation (MAD) and minimax (MM) models.

Key words: stochastic mean-variance model, implicit transaction costs, discrete scenarios, uncertainty.

JEL Classification: C01, C58, D81, G11

1 Introduction

Constructing a portfolio of investments is one of the most significant financial decisions facing individual investors, financial managers and institutions. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred in setting up a new portfolio or rebalancing an existing one must be included in any realistic portfolio selection analysis. Investment portfolios should be rebalanced to take account of changing market conditions and changes in funding. In this study, we extend the famous mean-variance portfolio optimization problem to incorporate uncertainty of future asset returns and uncertainty in implicit transaction costs incurred during initial trading and when rebalancing the portfolio in subsequent times.

Mean-variance (MV) analysis was introduced by Harry Markowitz (1952), who describe the basic formulations and the quadratic programming tools used to solve them. Mean-variance portfolio theory is based on the idea that the value of investment opportunities can be meaningfully measured in terms of mean return and variance of the return. This mean-variance analysis is based on the following assumptions: (i) All investors are risk-averse; they prefer less risk to more for the same level of expected return, (ii) expected returns for all assets are known, (iii) the variances and covariances of all asset returns are known, and iv there are no transaction costs or taxes. Furthermore, mean-variance analysis is based on single-period model of investment. In this study, we propose an optimal portfolio policy of a multi-period stochastic mean-variance investor subject to uncertain returns and uncertain implicit transaction costs.

2 Literature review

Multi-stage decision problems under uncertainty are considered by Becker et. al. (1994) and Darlington et al. (2008) for non-linear problems. A mean-variance approach is adopted with a static transcription of the dynamic decision model. Gulpinar et. al. (2004) incorporate proportional transaction costs, given stochastic data in the form of a scenario tree. Glen (2011) considers a mean-variance portfolio rebalancing strategy with transaction costs comprising of fixed charges and variable costs that include market impact cost. These variable transaction costs are assumed to be non-linear functions of traded value. Approximating implicit transaction costs by a non-linear function seems inappropriate since implicit transaction costs are random.

We address the portfolio selection problem by applying stochastic programming. We construct a multi-stage stochastic mean-variance model that captures asset returns, implicit transaction costs and risk due to uncertainty. We have adopted stochastic programming as it has a number of advantages over other techniques. Firstly, stochastic programming models can accommodate general distributions by means of scenarios. We do not have to explicitly assume a specific stochastic process for the securities' returns, but we can rely on the empirical distribution of these returns. Secondly, they can address practical issues such as transaction costs, turnover constraints, limits on securities and prohibition of short-selling. Regulatory and institutional or market-specific constraints can be accommodated. Thirdly, they can flexibly use different risk measures. Hence, this study uses portfolio variance as risk.

The first distinguishing feature of our multi-stage stochastic mean-variance (SMVTC) model is that it takes into account the approximate nature of the set of discrete scenarios by considering the variance around each return scenario and the corresponding implicit cost incurred during trading. Secondly, the variance term allows for the variability of asset returns over the scenario tree. Hence, uncertainty on asset returns is represented by a discrete approximation of a multi-variate continuous distribution as well as the variability due to discrete approximation.

Models involving implicit transaction costs have also received a fair share in the literature. Xia and Tian (2012) estimate implicit transaction cost in Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. They use the Bayesian Gibbs sampling method proposed by Hasbrouck (2009) to analyze implicit costs in the bull and bear markets. Hasbrouck (2009) expands Roll's model (1984) and suggests a Bayesian Gibbs sampling method based on the daily closing prices to estimate implicit cost, and proves that the correlation coefficient between the results of using the Bayesian method and those of using high-frequency data was 0.965. Kozmik (2012) discusses asset allocation with transaction costs formulated as multi-stage stochastic programming model. He considers transaction costs as proportional to the value of the assets bought or sold, but does not take into account implicit costs in the model. He employs conditional-Value-at-Risk as a measure of risk. Brown and Smith (2011) study the problem of dynamic portfolio optimization in a discrete-time finite-horizon setting, and again, consider proportional trading costs. Lynch and Tan (2010) study portfolio selection problems with multiple risky assets. They develop analytic frameworks for the case with many assets taking proportional transaction costs. Korn (2011) studies continuous-time portfolio optimization and takes into account proportional transaction costs. Cai et. al. (2013) examine numerical solution of dynamic portfolio optimization with transaction costs. While transaction costs are broad and include explicit as well as implicit costs, they consider a case of proportional transaction costs which can be either implicit or explicit, whichever is greater. However, the study of trading costs requires the identification of the type of cost to be estimated in order to explore effective ways of having a good estimate. Thus in our study, we consider implicit transaction costs. These costs are invisible and difficult to measure. They can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, (2008)).

We present a multi-period stochastic mean-variance (SMVTC) model with random implicit transaction costs in optimal portfolio selection. We achieve this by employing stochastic programming. Real-life decision processes are complicated. Although we must make an initial decision now, there will be many opportunities to adjust with time. Presently, we do not have a complete decision basis for future decisions - the future is unknown. Stochastic programming models hence are dynamic, covering multiple time periods with associated separate decisions, and they account for the stochastic decision process. Stochastic programming takes into account scenarios and stages. The uncertainties about future events are captured by a set of scenarios, which is a representative and comprehensive set of possible realizations of the future. The model employs stochastic programming with recourse by considering re-balancing of portfolio composition as the uncertainty of assets' returns get realized. We take implicit costs to be random as these are dependent on stock prices which follow a random-walk process.

The main contributions of this paper include:

(i) The development of a multi-stage stochastic mean-variance model that optimizes portfolios in the presence of random implicit transaction costs incurred during initial trading and in subsequent re-balancing of portfolios,

(ii) The development of a strategy that captures uncertainty in both stock prices and in corresponding implicit trading costs by way of scenarios.

3 Problem statement

The problem considered involves the determination of a multi-period discrete-time optimal portfolio strategy over an investment horizon $[0, T]$. We consider n risky assets and construct an optimal portfolio over the investment horizon. This portfolio is structured in terms of asset return and risk which is measured by the variance of the asset return from expected portfolio return. This strategy takes into account the approximate nature of a set of discrete scenarios by considering the variance around each asset return scenario.

3.1 Scenario generation

We consider a stochastic data process $\mathbf{R^t} = \{R_1, \cdots, R_t\}$ over an investment horizon T. The period T is divided into two discrete intervals, T_1 and T_2. $T_1 = \{t : t = 1, 2, \cdots, \tau\}$ defines the planning phase, where the investor makes decisions and adjustments to his portfolio at each of the τ periods as new information on assets' returns become available. During period $T_2 = \{t : t = \tau + 1, \cdots, T\}$ no re-adjustment of the portfolio takes place until investment maturity at $t = T$. We consider $I = \{i : i = 1, 2, \cdots, n\}$ to be a set of securities for an investment. The initial investment takes place at $t = 0$, with recourse decisions implemented at times $t = 1, 2, \cdots, \tau$. The decision process is non-anticipative (i.e., a decision at a given stage does not depend on the future realization of the random events). The decision at period t is dependent on R_{t-1}. A portfolio is structured in terms of asset return and risk which is measured by the variance of the asset return from expected portfolio return. At each of the re-structuring times, an investor buys or sells shares of some securities in order to optimize on his portfolio returns. The buying and selling of shares of securities makes the investor to incur some transaction costs. Thus, we consider a risk-averse and conservative investor who seeks to optimize his portfolio while keeping transaction costs to the minimum or at a prescribed level. We take the vector of the decision process to be $\mathbf{x} = \{x_1, x_2, \cdots, x_\tau\}^T$. We represent uncertainty of each asset return and corresponding transaction cost by a set of discrete scenarios. A scenario, $s \in \Omega$ is a possible realization of the stochastic variables $\{R_1, R_2, \cdots, R_\tau\}$. The set of scenarios corresponds to the set of paths followed from the root to the leaves of a tree, s_τ, and nodes of the tree at level $t > 1$ corresponds to possible realizations of R_t. Given the event history up to time t, R_t, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observations, R_{t+1}. An example of a scenario tree with 2 time periods and a three - three branching structure is shown in Figure 1 below.

[Figure 1 is here]

The sequence of decisions and realizations is given by $(x_1, R_1), x_2(x_1, R_1), R_2, \cdots, x_\tau(x_1, x_2, \cdots, x_{\tau-1}, R_1, R_2, \cdots, R_{\tau-1})$. We suppose the returns under a particular scenario $s \in \Omega = \{s : s = 1, 2, \cdots, S\}$ takes the values $R_s = (R_{is}, R_{2s}, \cdots, R_{ns})^T$ with associated probability $p_s > 0$, where $\sum_{s=1}^{S} p_s = 1$.

3.2 Constraints

The problem being studied is a dynamic one where an investor is allowed to adjust dynamically his or her portfolio at successive discrete times. The investor joins the market at $t = 0$ with an initial wealth W_0. This wealth is distributed among the n-assets at the beginning of each of the τ-consecutive time periods to allow re-balancing of the portfolio. The investor seeks to obtain an optimal strategy $x_t = [x_{1,1,t}, x_{1,2,t}, \cdots, x_{2,1,t}, x_{2,2,t}, \cdots, x_{i,s,t}]^T$, $t = 1, 2, \cdots, \tau$. We observe that $\sum_{i=1}^{n} x_{i,t} = \sum_{s=1}^{S} x_{i,s,t} = 1, t = 1, 2, \cdots, \tau$. Given that a_{ist} and v_{ist} are respectively the buying and selling proportions at time t, we have $x_{i,s,t} = x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau$. being the proportion of wealth invested in the i^{th} asset of scenario s at the beginning of period t. However, we state that $a_{i,s,t} \cdot v_{i,s,t} = 0$ at each re-balancing since we cannot buy and sell an asset at the same time. We assume that the portfolio is self-financing. Thus, money can only be added at $t = 0$. The expected return of asset i in period t is

$$\begin{aligned} r_{it} &= \sum_{s\in Q} p_s \cdot R_{ist} \cdot (x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}), i = 1, \cdots, n; t = 1, \cdots, \tau, \\ &= \sum_{s\in Q} p_s \cdot R_{ist} \cdot x_{ist}. \end{aligned}$$

where $Q \subset \Omega$ is a set of scenarios of asset i of period t. Similarly, the gross expected return of the portfolio of period t becomes

$$r_{pt} = \sum_{s=1}^{S} p_s \cdot R_{ist} \cdot x_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau.$$

since we have a total of S scenarios for all assets in each period. It should be noted that the volume of asset i in period t sold for portfolio re-balancing should not exceed the volume of the asset in the portfolio. Hence,

$$0 \leq \sum_{s\in Q} v_{ist} \leq \sum_{s\in Q} x_{ist}; i = 1, 2, \cdots, n; t = 1, 2, \cdots, \tau. \tag{1}$$

where v_{ist} is the volume of asset i of scenario s in period t sold for portfolio re-balancing. We observe that $v_{it} = \sum_{s\in Q} p_s \cdot v_{ist}$ and $x_{it} = \sum_{s\in Q} p_s \cdot x_{ist}$ Thus the above constraint becomes

$$0 \leq v_{it} \leq x_{it}, i = 1, 2, \cdots, n; t = 1, 2, \cdots, \tau. \tag{2}$$

We also state that the amount of money used to buy volumes of asset i should not exceed the amount got from selling volumes of asset $j (i \neq j)$ of the same period when re-balancing the portfolio. Therefore we have

$$0 \leq a_{it} \leq v_{jt}, j \neq i; i = 1, \cdots, n; t = 1, \cdots, \tau \tag{3}$$

We also state the constraint

$$0 \leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau \tag{4}$$

The left-hand inequality of the constraint guarantees that no short-selling occurs. The right inequality provides for the diversification of the portfolio, where U_{ist} are bounds on decision variables x_{ist}. The total transaction cost incurred by the investor for buying and selling proportions a_{ist} and v_{ist} respectively of asset i of scenario s in period t is given by $(k_{ist}a_{ist} + l_{ist}v_{ist})R_{ist}$ where it follows from above that either $k_{ist}a_{ist} = 0$ or $l_{ist}v_{ist} = 0$ or both being equal to zero. k_{ist} and l_{ist} are transaction cost rates for buying and selling shares of asset i of scenario s in period t during portfolio re-balancing. It should be noted that no re-balancing takes place if the new optimal portfolio is not better than the previous one. Thus the expected transaction cost of the portfolio of period t becomes

$$\sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}R_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau.$$

Denoting the net expected portfolio return of period t by N_{pt}, we obtain

$$N_{pt} = r_{pt} - \sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}R_{ist}.$$

This results in the net wealth of period t as

$$W_t = (1 + N_{pt}) \cdot W_{t-1}, t = 1, \cdots, \tau. \tag{5}$$

taking transaction costs into account.

3.3 Expected Risk

In the portfolio selection model proposed by Markowitz [18], the variance is used to represent the risk of the portfolio. Denoting x_i to be the proportion invested in stock i, $(i = 1, \cdots, n)$, where $\sum_{i=1}^{n} x_i = 1, M = (x_1, x_2, \cdots, x_n)$ is called a portfolio. If we let R_i to be the return of asset i, the mean-variance model has portfolio risk (σ_p^2) expressed as
$\sigma_p^2 = \frac{1}{n-1}\{\sum_{i=1}^{n}(R_ix_i - E[R_ix_i])^2\} = \frac{1}{n-1}\{\sum_{i=1}^{n}(R_ix_i - \mu)^2\}$
where $\mu = E[R_ix_i] = \frac{1}{n}\sum_{I=1}^{n} R_ix_i$. This formulation of risk is in deterministic form since it takes into account only one scenario of an asset's return. We therefore extend this formulation of portfolio risk by considering uncertainty of asset returns and uncertainty of implicit transaction costs in initial trading and in subsequent rebalancing of the portfolio. We incorporate into the deterministic formulation, three key stochastic framework elements. Firstly, we

accommodate multiple scenarios as a way of capturing uncertainty of asset returns and implicit transaction costs, and this results in an increase in the number of parameters. Thus, uncertainty is represented by a set of distinct realizations $s \in \Omega$. We now consider the scenario parameter along with other parameters of security and time period. Secondly, the stochastic multi-stage mean-variance model allows for the implementation of recourse decisions as unfolding information on assets' returns get realized. The third element is the probabilistic feature of the stochastic framework which assigns probabilities to scenarios. The parameter p_s denotes scenario probability. Scenarios that share common information in a period must yield the same decisions up to that period. The stochastic variance also incorporates implicit transaction costs incurred during initial trading and in recourse periods. We consider the variance around each return scenario and obtain the portfolio risk (variance) at each time period as
$\sigma_p^2 = \frac{1}{n-1}\{\sum_{s=1}^{S}(p_s R_{is} x_{is} - E[R_{is} x_{is}])^2\} = \frac{1}{n-1}\sum_{s=1}^{S}\{p_s R_{is} x_{is} - r_p\}^2$
where $r_p = E[R_{is} x_{is}] = \sum_{s=1}^{S} p_s R_{is} x_{is}$. The portfolio risk over t- time periods becomes

$$\sigma_p^2 = \frac{1}{\tau(n-1)}\sum_{t=1}^{\tau}\sum_{s=1}^{S}\{p_s R_{ist} x_{ist} - r_{pt}\}^2$$

where R_{ist}, r_{pt} and x_{ist} are as defined in earlier subsections. Let $Z_t = \sum_{s=1}^{S}[p_s R_{ist} x_{ist} - r_{pt}]^2, t = 1, \cdots, \tau; i = 1, \cdots, n$. We now have portfolio risk as

$$H = \frac{1}{\tau(n-1)}\sum_{t=1}^{\tau} Z_t$$

3.4 The Multi-stage Stochastic Mean-Variance Model

We propose the multi-stage stochastic portfolio selection model as a minimization of portfolio risk (variance) subject to constraints describing the growth of the portfolio in all scenarios, performance constraints and bounds on variables. The optimization model provides an optimal investment strategy that minimizes portfolio variance while achieving a prescribed portfolio mean return and the desired transaction cost level. By varying expected return or risk and re-optimizing yields a set of optimal portfolios comprising the efficient frontier. If we let the investor's desired minimum net portfolio

mean return to be at least λ and the desired transaction cost to be at most θ, say, we obtain the following multi-stage stochastic mean-variance model with uncertain transaction costs:
Minimize

$$H = \frac{1}{\tau(n-1)} \sum_{t=1}^{\tau} Z_t \tag{6}$$

subject to

$$\begin{aligned}
N_{p\tau} &\geq \lambda \\
W_t &= (1 + N_{pt})W_{t-1}, t = 1, \cdots, \tau, \\
\theta &\geq \sum_{s=1}^{S} p_s\{k_{i,s,t}a_{i,s,t} + l_{i,s,t}v_{i,s,t}\}R_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
1 &= \sum_{s=1}^{S} x_{i,s,t}, i = 1 \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq a_{it} \leq v_{jt}, j \neq i; i = 1, \cdots, n; t = 1, \cdots, \tau \\
0 &\leq v_{i,t} \leq x_{i,t}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq x_{i,s,t} \leq U_{i,s,t}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
x_{i,s,t} &= x_{i,s,'t}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,
\end{aligned}$$

We assume that at $t > 0$, no borrowing is done and the portfolio is self-financing. We mention here that the existence of a riskless asset among the securities is a special case in the stochastic multi-stage model. It has been noted earlier that for each asset i, a_{ist} and v_{ist} can not be non-zero simultaneously. Thus we state the model above without the complementary constraint $a_{ist} \cdot v_{ist} = 0$.

3.5 Transaction cost measurement

Transaction costs are costs incurred by an investor when buying or selling securities on the Stock Market. These costs are broadly of two types, namely explicit and implicit costs. Market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties are explicit costs. They can easily be determined before the execution of trade. Implicit costs, on the other hand, are strongly related to the trading strategy. They are market impact, opportunity costs and spread. These costs provide opportunities to improve the quality of trade execution. They can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, 2008).

In this study, we consider immediate trade execution. Thus market impact will account for the total implicit costs. We apply Hau (2006)'s approach to calculation of implicit transaction costs. We take the transaction price to be the security's last price of the month. The effective spread is taken to be twice the distance from the spread mid-price measured in basis points. Taking P^T as the transaction price and P^M as the mid-price, the effective spread (implicit transaction cost) is calculated as

$$SPREAD^{Trade} = 200 \times \frac{|P^T - P^M|}{P^M}$$

4 Data and Sample

We use monthly historical data of securities on the Johannesburg Stock Market from January 2008 to September 2012. We evaluate the performance of the proposed SMVTC model by comparing portfolios developed from it and the MV, MAD, and MM models. The following criteria are used in the selection of stocks to comprise our initial portfolio:

(a) Stocks with negative mean returns for the entire period considered are excluded from the sample,

(b) Companies which were not on the list by January 2008 and only entered the JSE afterwards are excluded.

(c) Those assets having the highest positive mean returns are taken to become our initial portfolio.

We take empirical distributions computed from past monthly returns as equiprobable scenarios. A scenario, R_{ist}, for the return of asset i of period t is obtained as $R_{ist} = \frac{P_{i,t}}{P_{i,t-1}} - 1$, where $P_{i,t}$ is the historical monthly price of asset i. The initial portfolio comprises 13 securities. The mean returns, total implicit transaction costs and total wealth of each of the four optimal portfolios developed according to models 6, 7, 8 and 9 are calculated and then compared. In using the proposed model, we consider five scenarios for each asset return and the associated transaction cost in each time period, and apply the SMVTC model over the first stage only. This is done since comparison with other models is restricted to the first stage. We take empirical distributions of the 13 securities comprising our initial portfolio. Since for each security we have 54 monthly returns, we number the months from 1 to 54, and use random numbers to select asset returns and associated transaction costs corresponding to scenarios of a security. We take implicit transaction costs calculated from the effective bid-ask spread corresponding to each selected asset return. We assume that these transaction costs are random since they are randomly selected together with associated asset' returns. Thus, a scenario comprises an asset return and the corresponding implicit transaction cost. The implicit cost for each of the 13 assets is taken into account as it is in the buying or selling of each asset that the cost is incurred. We give the transaction cost as a rate. We take each scenario to be equally likely to occur. It should be noted that scenarios are only considered for the proposed SMVTC model. Other models use mean returns and risks calculated over the whole period under examination. First-stage optimal portfolios of all models are compared. At the end of the first stage, an investor decides on his first-stage optimal portfolio as given by the investor's chosen portfolio transaction cost, the diversification limit, the gross portfolio mean return or the net portfolio mean return as the case may be, and the associated risk given by the variance. Basic statistics of sample data from which the SMVTC first-stage optimal portfolios are constructed are shown in Table 1 in the following section. We show sample statistics on portfolios generated at each of the diversification limits, $0.1, 0.15, \cdots, 0.4$. This helps to show if assumptions on the mean - variance model are met.

5 Results

5.1 In-sample analysis

We divide the period of study into two. The first period, which starts from January 2008 to March 2010, is where the four models are applied to validate the SMVTC model. The second period, from April 2010 to September 2012, is used to find out if the SMVTC model results in the first period are consistent.We consider an investor who has R10000 to spend on his initial portfolio. We only apply the models over the first stage.

For each dataset and for each diversification level, we solve all the three LP problems defined in Section 7 together with the SMVTC model. In analyzing the performance of each model we consider the gross portfolio mean return, the risk, total implicit transaction costs incurred, the gross and net portfolio wealths. We consider the diversification limits from 0.1 to 0.4 for each model, and neither the mean portfolio return nor the portfolio risk is constrained. The optimal portfolios generated by the four models are shown in Table 1 below. The phrase 'D. Lim' stands for 'diversification limit'.
It is evident from the Table 1 results that, for the same diversification limit, MM-generated portfolios have the highest expected gross portfolio returns, followed by the MV, with the SMVTC-generated portfolios having the least expected gross returns. The MM optimal portfolios have the highest risks followed by the MAD-generated portfolios. The SMVTC optimal portfolios have the least risks. The table also shows that the MM optimal portfolios have the highest expected gross wealth followed by the MV optimal portfolios. The SMVTC portfolios have the least expected gross portfolio wealth for every diversification limit considered.

However, the total implicit transaction costs incurred for each optimal portfolio is least in SMVTC portfolios followed by MAD portfolios, with the greatest implicit costs being incurred by either an MV or MM investor. We note that the total expected net wealth is greatest in SMVTC-generated portfolios. The MAD optimal portfolios improve more on net expected wealth as portfolios become less diversified,compared to MV and MM portfolios. Thus, the SMVTC optimal portfolios have the least risk, the least implicit transaction costs incurred, and they have the greatest expected net wealth.

Table 1: Summary statistics of in-sample optimal portfolios

D. Lim		MAD	MV	MM	SMVTC
0.10	Mean return	0.027	0.030	0.034	0.002
	Risk	0.024	0.001	0.034	2.0054E-8
	Gross wealth	10265.76	10303.55	10339.30	10011.32
	Cost	710.10	835.42	901.70	44.39
	Net wealth	9555.66	9468.12	9437.60	9966.94
0.15	Mean return	0.024	0.032	0.038	0.0009
	Risk	0.027	0.001	0.038	6.0619E-9
	Gross wealth	10236.14	10316.92	10380.50	10001.79
	Cost	855.25	929.73	721.45	56.61
	Net wealth	9380.89	9387.18	9659.05	9945.19
0.20	Mean return	0.023	0.033	0.041	0.0006
	Risk	0.028	0.001	0.041	8.3476E-9
	Gross wealth	10225.12	10330.33	10409.00	10001.53
	Cost	1092.00	818.23	746.40	37.13
	Net wealth	9133.12	9512.10	9662.60	9964.40
0.25	Mean return	0.022	0.033	0.043	0.001
	Risk	0.029	0.001	0.043	1.98468E-8
	Gross wealth	10216.15	10330.92	10428.00	10009.96
	Cost	396.25	760.18	883.25	28.89
	Net wealth	9819.90	9570.74	9544.75	9981.07
0.30	Mean return	0.021	0.033	0.044	0.001
	Risk	0.029	0.001	0.044	2.18275E-8
	Gross wealth	10211.78	10331.02	10443.60	10007.14
	Cost	445.30	757.81	1023.30	53.84
	Net wealth	9766.48	9573.21	9420.30	9953.30
0.35	Mean return	0.021	0.033	0.046	0.002
	Risk	0.029	0.001	0.046	2.5204E-8
	Gross wealth	10207.83	10331.02	10457.80	10011.90
	Cost	444.45	757.80	1083.60	37.62
	Net wealth	9763.38	9573.22	9374.20	9974.29
0.40	Mean return	0.020	0.033	0.047	0.0005
	Risk	0.030	0.001	0.047	8.12541E-9
	Gross wealth	10204.72	10331.02	10469.20	10003.25
	Cost	343.80	757.79	984.40	10.60
	Net wealth	9860.92	9573.23	9484.80	9992.65

Table 2 presents assets' percentage compositions of In-Sample optimal portfolios at increasing diversification levels. Highly selected common securities in MV, MM and SMVTC optimal portfolios included A_1, A_8 and A_{13}, when portfolios are more diversified. We observe that SMVTC portfolios comprise assets in small quantities and being many, a characteristic which is less shared by even MV portfolios. Generally, assets common to both MV and SMVTC portfolios tend to be in larger quantities in MV than SMVTC optimal portfolios, for example, assets A_1, A_8 and A_9.

Table 2:Assets' percentage compositions of in-sample optimal portfolios at different diversification levels: mean return and risk unconstrained.

D.L	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10				10	10	10	10	10
	MV	10	10	10	10	10		2	10	10		10	8	10
	MM	10	10			10	10	10	10		10	10	10	10
	SMVTC	9	9	9	1	8	9	8	7	7	7	8	9	10
0.15	MAD		15	15	15	15				15	10		15	
	MV	15	10	5	10	10			15	15		15		6
	MM	15					15	15	15		10	15		15
	SMVTC	4	15	15	7	3	2	7	9	8		9	8	15
0.20	MAD		20	20	20	20				20				
	MV	20	5		8	9			20	20		16		2
	MM	20					20	20	20					20
	SMVTC	20			1			10	9	10		17	20	13
0.25	MAD			25	25	25				25				
	MV	25	4		7	8			19	20		17		
	MM	25					25	25	25					
	SMVTC	4	7	5	2	7	7	7	6	10	9	4	7	25

In Table 3 below, we summarize descriptive statistics of sample data of securities from which each SMVTC in-sample optimal portfolio is generated. After identifying the securities in each optimal portfolio, regardless of the asset weights obtained, we calculate the mean return of all assets in the 'portfolio' sample for the entire in-sample period. The sample standard deviation, coefficient of variation, excess skewness and excess kurtosis are calculated to provide some ideas on the data on which the SMVTC model is applied to yield each optimal portfolio. In the table, 'port1' stands for 'portfolio 1', 'St.Dev' stands for 'standard deviation' and 'CV' stands for 'covariance'. Table 2 shows varying means and standard deviations with the smallest being 0.03343 and 0.00907, and the highest being 0.04298 and 0.102152 respectively. All sample skewness values are between -1 and -0.5, showing that the sample distributions are moderately skewed to the left. The sample excess kurtosis values are all positive, implying that the sample distributions are leptokurtic.

Table 3: Statistics of sample data from which SMVTC-optimal portfolios are constructed: portfolio risk and mean return unconstrained

Portfolio	Mean	St.Dev	CV	Sample skeness	Sample kurtosis
1	0.04288	0.09113	0.47059	-0.87485	1.74633
2	0.03343	0.09278	0.36029	-0.47931	1.58773
3	0.03842	0.00907	4.23745	-0.92248	1.31044
4	0.04138	0.09144	0.45257	-0.82627	1.63654
5	0.04298	0.10054	0.42745	-0.84668	1.22052
6	0.0385	0.102152	0.37689	-0.88476	1.12443
7	0.04271	0.07811	0.54682	-0.4426	1.62229

5.2 Out-of-sample analysis

In a real-life environment, comparison of models is usually done by means of ex-post analysis. There are many approaches that can be used to compare models, and one of the most commonly applied methods involves representing the ex-post asset returns of optimal portfolios over a given period. Generally, portfolio performances are usually affected by the market trend which makes it difficult to draw some uniform conclusions. However, our case produces portfolio performances that have some common features. Comparison of out-of-sample optimal portfolios is shown in Table 4 below.

Table 4: Summary statistics of out-of-sample optimal portfolios

D. Lim		MAD	MV	MM	SMVTC
0.10	Mean return	0.027	0.029	0.031	0.002
	Risk	0.002	0.002	0.1	1.846E-8
	Gross wealth	10269.80	10289.02	10314.80	10011.09
	Cost	432.00	426.34	502.10	37.86
	Net wealth	9837.80	9862.68	9812.70	9973.23
0.15	Mean return	0.025	0.028	0.033	0.001
	Risk	0.022	0.002	0.143	1.9904E-8
	Gross wealth	10251.75	10277.97	10332.30	10009.32
	Cost	479.10	397.32	411.15	32.49
	Net wealth	9772.65	9880.65	9921.15	9976.84
0.20	Mean return	0.024	0.027	0.035	0.001
	Risk	0.023	0.002	0.2	1.8693E-8
	Gross wealth	10240.20	10268.25	10346.80	10007.71
	Cost	223.20	378.64	503.60	40.56
	Net wealth	10017.00	9889.61	9843.20	9967.15
0.25	Mean return	0.023	0.026	0.036	0.001
	Risk	0.025	0.002	0.25	1.7098E-8
	Gross wealth	10229.25	10259.68	10356.25	10008.13
	Cost	248.50	356.71	591.00	37.20
	Net wealth	9980.75	9902.97	9765.25	9970.94
0.30	Mean return	0.023	0.026	0.036	0.001
	Risk	0.025	0.002	0.25	1.2561E-8
	Gross wealth	10221.70	10255.02	10362.50	10006.53
	Cost	272.80	340.06	550.00	27.84
	Net wealth	9948.90	9914.96	9812.50	9978.69
0.35	Mean return	0.021	0.025	0.037	0.002
	Risk	0.026	0.002	0.333	3.0025E-8
	Gross wealth	10214.90	10250.44	10368.30	10013.26
	Cost	270.55	318.87	502.10	26.95
	Net wealth	9944.35	9931.57	9866.20	9986.31
0.40	Mean return	0.021	0.025	0.037	0.001
	Risk	0.027	0.002	0.333	1.8485E-8
	Gross wealth	10209.60	10247.81	10373.20	10011.33
	Cost	215.20	306.70	440.40	21.26
	Net wealth	9994.40	9941.12	9932.80	9990.07

We observe that MM optimal portfolios have the highest expected gross returns, incur the greatest implicit transaction costs and have the least expected portfolio wealths for every diversification limit considered except 0.15. As in in-sample analysis, the SMVTC optimal portfolios have smallest mean returns and risks, and with superior expected wealths. It is also evident that the SMVTC investor incurs the smallest implicit transaction costs.

A comparison of out-of-sample optimal portfolios assets' percentage compositions is shown in Table 5. As portfolios become less diversified, the MAD and MV optimal portfolios have common securities with high percentage compositions, for example, securities A_1, A_4 and A_9. The choice of assets in MM portfolios does not match any of the other models' optimal portfolios. Portfolios generated by the SMVTC and MV models have some common securities, but with reverse percentage compositions, for example, A_1, A_3, A_4 and A_5. Securities with high percentage compositions in SMVTC optimal portfolios do not comprise the MV optimal portfolios, for example, A_6, A_7 and A_{12}. The SMVTC portfolios tend to have the greatest number of assets, though in smaller quantities.

Table 5:Assets' percentage compositions in out-of-sample optimal portfolios at different diversification levels: mean return and risk unconstrained.

D.Lim	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10			10	10	10		10	10
	MV	10	8	10	10	10	5	4	4	10	9	10	1	10
	MM	10	10	10		10	10	10	10		10	10		10
	SMVTC	1	10	10	10	10	10	8	10		2	10	10	8
0.15	MAD	15	15	15	15					15	10		15	
	MV	15	6	14	15	6			3	15	7	15	1	3
	MM					15	15	15	15		10	15		15
	SMVTC	11	12		9		11	12	9	14		4	9	9
0.20	MAD	20		20	20					20			20	
	MV	20	4	11	20	3			3	20	4	13		1
	MM						20	20	20			20		20
	SMVTC		10	8	10	17	6	10	16			10	7	5

D.Lim	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.25	MAD	25			25					25			25	
	MV	25	2	9	22				4	25	1	12		
	MM						25	25	25			25		
	SMVTC	6	5	6	10	5	18	16		17		10	6	1
0.30	MAD	10			30					30			30	
	MV	23	2	9	21				4	30		12		
	MM						30	30	10			30		
	SMVTC	2	10	2	12	30		16	11			8	10	
0.35	MAD				30					35			35	
	MV	21	1	8	20				4	35		12		
	MM						30	35				35		
	SMVTC	17	6	33	4	3	2		4					31

5.3 Sensitivity analysis

We perform sensitivity analysis of the SMVTC model by finding sensitivity index (SI) for each parameter. We do so by finding the output percentage difference when varying one input parameter from its minimum value (zero in this case) to its maximum value (Hoffman and Gardner, 1983; Bauer and Hamby, 1991). We use the formula, Sensitivity Index $= \frac{D_{\max}-D_{\min}}{D_{\max}}$, where $D_{\min}$ and $D_{\max}$ are the minimum and maximum output values respectively, resulting from varying the input parameter over its entire range. This figure-of-merit provides a good indication of parameter and model variability (Hamby, 1994). Zero is taken as the minimum input value for each parameter with the maximum value being restricted by the diversification limit considered and is given as the best weight of the chosen asset in the optimal portfolio. We consider diversification limits of 0.1, 0.2 and 0.3.

Table 6:In-sample SMVTC model sensitivity analysis

Div. Lim	Parameter	Max Value	Cost SI	Risk SI	Wealth SI
0.1	x_1	0.087	-0.52219	0.3537	0.0011
	x_2	0.086	-0.22235	0.4284	0.0009
	x_3	0.087	-0.123	0.5494	0.0009
	x_4	0.005	-0.03604	0.2563	0.0004
	x_5	0.079	-0.38049	0.5035	0.0010
	x_6	0.085	-0.23654	0.4579	0.0009
	x_7	0.081	-0.13111	-0.0277	0.0005
	x_8	0.071	-0.43636	0.5404	0.0010
	x_9	0.074	-0.35684	0.4354	0.0009
	x_{10}	0.072	-0.14823	0.2684	0.0004
	x_{11}	0.054	-0.21356	0.4858	0.0010
	x_{12}	0.087	-0.18901	0.3962	0.0009
	x_{13}	0.100	-0.11737	0.3432	0.0005
0.2	x_1	0.200	-0.41826	-1.1223	-0.00014
	x_3	0.002	0.01212	-1.7424	-0.00078
	x_4	0.005	-0.78858	0.1828	0.00029
	x_7	0.095	-0.39564	-0.5357	0.00010
	x_8	0.094	0.00539	-1.0943	-0.00051
Div. Lim	Parameter	Max Value	Cost SI	Risk SI	Wealth SI
	x_9	0.095	-0.34931	-1.6064	-0.00045
	x_{11}	0.177	-0.16079	-1.2772	-0.00052
	x_{12}	0.200	-0.13439	-0.3762	-0.00025
	x_{13}	0.134	-0.49475	-1.0061	-0.00004
0.3	x_1	0.141	0.29309	0.3428	0.00034
	x_2	0.126	-0.13336	0.8277	0.00107
	x_3	0.091	0.36906	0.3206	-0.00007
	x_4	0.011	-0.43091	0.2635	0.00061
	x_6	0.258	-0.27675	0.2767	0.00066
	x_7	0.032	-0.42106	0.0495	0.00007
	x_8	0.160	0.24424	0.5956	0.00023
	x_9	0.074	0.26226	-0.1817	-0.00006
	x_{11}	0.069	0.39636	0.6684	0.00040
	x_{12}	0.039	-0.52990	-0.8630	-0.00006

It should be noted that the SMVTC model is stochastic and works by replacing one parameter by a 'less' profitable one when the weight of the chosen asset is zero. Here, a less preferred asset is included in the 'new' optimal portfolio due to a condition in the model that ensures that the total weight of assets in the optimal portfolio be unit. Thus resulting in a relative sensitivity value. Hence, the D_{min} value of model output is a relative value. Therefore, by applying the method above we are finding relative sensitivity analysis of the model. The result in the table reflect small percentages of model variability due to each input change of each parameter. This shows that the output values are not strongly reliant on certain individual parameters. The cost sensitivity analysis of the in-sample reveals that most S.I. values are negative implying that the less preferred asset replacing the optimal-portfolio selected asset causes the investor to incur more costs. However,most of such assets cause a decline in portfolio risk and a decline in portfolio wealth as most wealth S.I. values are positive.

6 Conclusion

We propose a stochastic multi-stage mean-variance optimal portfolio strategy that takes into account uncertainty of asset returns and associated implicit transaction costs. The methodology allows investors or investment managers to choose optimal portfolios knowing implicit transaction cost levels to be incurred when executing the trading. It has emerged that the SMVTC-generated optimal portfolios have the least total implicit transaction costs, least risk and greatest net wealths compared to optimal portfolios generated by the mean-variance, mean absolute deviation and minimax models. However, common assets in optimal portfolios generated by both the MV and SMVTC models tend to be in larger quantities in MV than SMVTC portfolios. The strategy is suitable for conservative investors. It is left, however, for future research to develop a model that accounts for both implicit and explicit trading costs while considering uncertainty of both asset prices and transaction costs. The study, like any other, has its own limitations. The use of an asset's closing price for each month in the calculation of an asset's return rate may not be the most accurate measure. However, since all assets' rates of returns have been obtained in the same way, this does not prejudice the findings.

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7 Appendix

7.1 Validation models

We validate the SMVTC model by comparing its optimal portfolios with those generated by the deterministic mean - variance (MV) model, the MAD and the Minimax (MM) models. We therefore give a review of these models.

7.1.1 Mean-variance model

Harry Markowitz (1952) proposes the mean - variance optimization model in which the portfolio that minimizes the variance subject to the restriction of a given mean return is chosen as the optimum portfolio. The mathematical model proposed by Markowitz is as follows:

Minimize

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{7}$$

subject to $\Phi = \{\sum_{i=1}^{n} r_i x_i \geq \rho, \sum_{i=1}^{n} x_i = 1, 0 \leq x_i \leq u_i, i = 1, \cdots, n\}$ where σ_{ij} is the covariance between assets i and j, x_i is the proportion of wealth invested in asset i, r_i is the expected return of asset i in each period, ρ is the minimum rate of return desired by an investor, and u_i is the maximum proportion of wealth which can be invested in asset i.

7.1.2 Minimax model

Young (1998) proposes the minimax (MM) model using minimum return as as measure of risk. This minimax model is equivalent to the mean-variance (MV) model if assets' returns are multivariate normally distributed. It is a linear programming model and is given as follows:
Maximize

$$M_p \tag{8}$$

subject to $\Phi = \{\sum_{i=1}^{n} w_i y_{it} - M_p \geq 0, t = 1, \cdots, T; \sum_{i=1}^{n} w_i \bar{y}_i \geq G;$ $\sum_{i=1}^{n} w_i \leq W; w_i \geq 0, i = 1, \cdots, n\}$,
where y_{it} is the return of one dollar invested in security i in time period t, $\bar{y}_i$ is the mean return of security i, w_i is the portfolio allocation to

security i, M_p is the minimum return on the portfolio, G is the minimum level of return, and W is the total allocation.

7.1.3 Mean absolute deviation model

Konno and Yamazaki (1991) propose the L_1 risk function (mean absolute deviation - MAD) and show that it behaves in the same manner as the mean - variance model of Harry Markowitz (1952) when the assets' returns are multivariate normally distributed. They develop the MAD model as given below:

Minimize

$$H(x) = \frac{1}{T}\sum_{t=1}^{T}|\sum_{i=1}^{n}(r_{it} - r_i)x_i| \qquad (9)$$

subject to $\Phi = \{\sum_{i=1}^{n} r_i x_i \geq \rho; \sum_{i=1}^{n} x_i = 1; 0 \leq x_i \leq u_i, i = 1, \cdots, n.\}$.

Figures

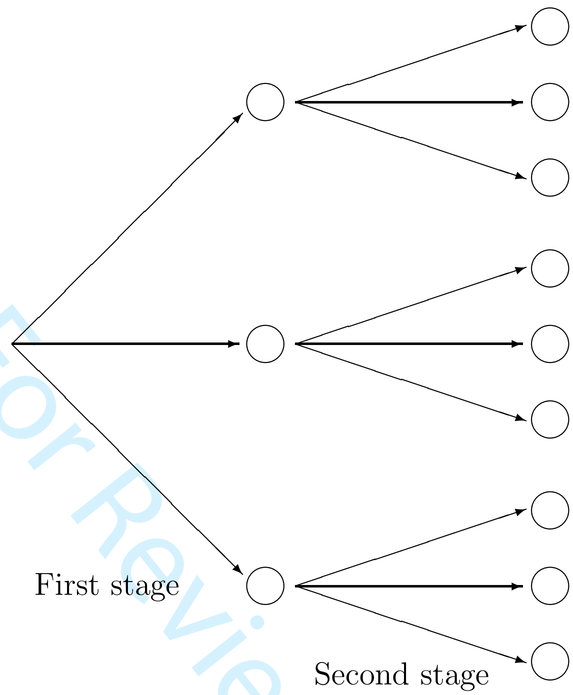

Figure 1: Scenario tree