Conceptualising the function concept: An image functions intervention

by

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Declaration

I, Christiaan Venter, declare that the thesis, Conceptualising the function concept: An image functions intervention, submitted for the qualification Doctor of Philosophy with specialisation in Higher Education Studies at the University of the Free State is my own independent work.

All the references that I have used have been indicated and acknowledged by means of complete references.

I further declare that this work, in part or as a whole, has not previously been submitted by me at another university or faculty for the purpose of obtaining a qualification.

6 March 2020

C. Venter **Date**

Ethics Statement

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13-May-2019

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We are pleased to inform you that your application for ethical clearance has been approved. Your ethical clearance is valid for twelve (12) months from the date of issue. We request that any changes that may take place during the course of your study/research project be submitted to the ethics office to ensure ethical transparency. furthermore, you are requested to submit the final report of your study/research project to the ethics office. Should you require more time to complete this research, please apply for an extension. Thank you for submitting your proposal for ethical clearance; we wish you the best of luck and success with your research.

Yours sincerely

Dr. Petrus Nel

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Language editing

14 January 2020

TO WHOM IT MAY CONCERN

I, Beverley Wilcock, hereby confirm that I have copy- and structurally edited the thesis entitled: *"Conceptualising the function concept: An image functions intervention"* for Mr C Venter.

Please note that I returned the edited document with recommended changes to the client and I have thus not reviewed the final document with the accepted/rejected changes. It therefore remains the client's responsibility to effect the suggested changes.

I am a registered language practitioner with the South African Translators' Institute with many years' experience in editing and translation in the academic and higher education sector.

Yours sincerely,

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Abstract

This thesis comprises three articles submitted for publication, sandwiched between an orientation chapter and a final reflection. The thread that runs through the thesis and connects the three articles is the mathematical concept of function. Students studying Mathematics are expected to have a good understanding of the function concept and its many subtleties and nuanced representations. However, the contrary has been shown to be true by ample research over the last 50 years in countries all over the world. The first article gave details on how photographs or images could be considered as representations of functions. This article used APOS (Action-Process-Object-Schema) theory to determine a genetic decomposition (GD) for the function concept. From this GD, activities were designed for an intervention based on defining and working with image functions, that is working with images/photographs as functions. The Image Functions Intervention (IFI) was then analysed from the APOS theoretical perspective and shown to adhere to the mental structures determined in the GD of the function concept. A first use of the IFI was evaluated by means of a questionnaire and qualitative analysis. The conclusion was that the IFI led to a broadened concept image, specifically regarding what can constitute as a function. The second article analysed the effectiveness of the IFI by means of quantitative analyses. In a randomised control design, an experimental and a control group both completed the Function Concept Inventory (FCI) as a pre-test. The experimental group completed the IFI and then both groups completed the FCI again as post-test. The experimental group showed a significant increase in their scores after the intervention. However, as there was no significant difference between the post-test scores of the experimental and the control groups, it could not be concluded unequivocally that the IFI caused the observed improvements. The last article used qualitative analysis with three data instruments to again gauge the possible effects of the IFI. Specifically, the main objective was to investigate to what extent the IFI could assist participants to develop an object conception of functions. Although multiple participants showed improvement on their understanding, only one participant managed to display a transitioning into the object level of understanding. Overall, the IFI showed merit and it was concluded that the IFI should be adapted and expanded based on the results and conclusions from the three articles. Further research should be undertaken to evaluate and explore the use of the IFI regarding the improvement of function concept understanding.

Keywords: function concept; APOS theory; intervention; image functions

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Orientation

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List of Acronyms

Acronym	Meaning
APOS	Action-Process-Object-Schema
FCI	Function Concept Inventory
GD	Genetic Decomposition
IFI	Image Functions Intervention
LMS	Learning Management System

ORIENTATION

1. Introduction

This first chapter serves to show how the three articles forming the core of this thesis are part of one coherent whole, and to provide the necessary background to the study.

As this thesis follows the format of three interconnected articles, the typical full literature review is not presented as a stand-alone chapter, but is rather incorporated into the individual articles. As all three articles dealt with aspects of the same topic, it was sometimes unavoidable to have certain information repeated in the articles. It was inevitable; however, it should be possible to read each article as an independent work.

2. Purpose and necessity of the research

This study explored the use of the Image Functions Intervention (IFI) in improving the comprehension of the function concept.

That the function concept is fundamentally important to mathematics can be accepted as a commonly shared opinion. As stated by Selden and Selden in Harel and Dubinsky (1992:1) "…the function concept, having evolved with mathematics, now plays a central and unifying role". And more recently, "[t]he concept of function is central to students' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyse graphs" (Son & Hu, 2015:4). O'Shea, Breen and Jaworski (2016:279) reiterate with "[f]unctions are central to present day mathematics" and elaborate "...going beyond calculus, functions are widely used in the comparison of abstract mathematical structures".

Despite the high value attached to an adequate understanding of functions and the function concept, a full and nuanced comprehension is not common among students (Carlson *et al.*, 2002:353). Doorman *et al.* (2012:1243) confirm the difficulty in learning the function concept and in particular state, "[f]unctions have different faces, and to make students perceive these as faces of the same mathematical concept is a pedagogical challenge".

What it boils down to is that students sit with an inadequate and/or erroneous function concept image. According to Tall and Vinner (1981:151), the concept image constitutes the "total cognitive structure that is associated with the concept". This entails all definitions, properties, ideas, theorems and examples that a student has grouped under the heading of function over his or her mathematical career so to speak. Although a student may know the formal definition of a function, when exposed to a problem, the full concept image will be utilised to solve the problem. Doorman *et al.* (2012:1245) also consider the concept image very important and have as one of their specific goals, the overcoming of a "too-limited" function concept image.

As the function concept is fundamental, yet misunderstood, the suggestion is that students should be introduced to the idea in such a manner that the resulting concept image will be as rich and accurate as possible. It is in these respects that the exploration of image functions is potentially very useful.

Much research has been done in confirming the difficulty with the function concept and/or trying to address the problem. Recent research includes that of Chimhande, Naidoo and Stols (2017), which confirmed that the difficulty is prevalent at school level with the mental construction at an action level of understanding, the lowest level according to the Action-Process-Object-Schema (APOS) theory (Arnon *et al.*, 2014). Doorman *et al.* (2012) explored the use of computer tools in aiding the transition to a structural view, the object level of APOS (section 4.1 discusses APOS theory in more detail), of functions. Makonye (2014) provided a theoretical analysis focusing on the use of multiple representations to foster a nuanced concept image through approaches where the function concept is kept embedded in students' reality as far as possible.

This study differed with respect to two aspects. Firstly, its pedagogy was new in that it used a new design based on image functions. Secondly, it was in contrast with prior research in that the intervention, the IFI, directly studied the structural view/object level of the concept whereas previous research (mentioned at the beginning of this paragraph) aimed to produce such a view as a result of activities and reflections on the more procedural level.

From the arguments above and further informed by 17 years of personal experience as a mathematics lecturer, it is known that the concept of function as well as its applications is important in mathematics, yet students' struggle with the concept is ongoing. The question arises if a specific pedagogy, that is "mathematics for teaching" (Hoover, Mosvold, Ball & Lai, 2016:4), based on students' active learning through an image functions intervention, can improve their conceptual understanding.

3. Focus of the research

Primary research question:

To what extent can the study of image functions improve function conceptualisation?

To answer the primary research question, the following **secondary questions** were investigated:

1. Within the APOS theoretical framework, how appropriate is an intervention based on image functions, to bring about an improved function conceptualisation?
2. To what extent can the average achievement on a function concept inventory be improved by completing an intervention programme based on image functions?
3. In what way can the said intervention programme assist in the development of an object conception of function over the short and long term?

Primary research aim:

Explore the effect of studying the function concept through image functions on students' function comprehension.

Objectives:

1. Within the APOS framework:
    a. Determine a genetic decomposition of the function concept.
    b. Design and then preliminary validate an image functions intervention.
2. Determine if and to what extent an image functions intervention can contribute to improved performance on a function concept inventory.
3.
    a. Determine if a shift can be observed in the number of students with an action or process conception of function to an object conception of function.
    b. Determine if the intervention can have a lasting positive effect on function comprehension, thus maintaining the object level of conceptualisation.

4. Research design

4.1. Theoretical framework for the study

Working with constructivist ideas, Dubinsky and McDonald (2001) as well as others before them such as Breidenbach, Dubinsky, Hawks and Nichols (1992), formulated the APOS framework (Arnon *et al.*, 2014) for modelling the learning of mathematical concepts.

Using the APOS framework, the development of the function concept can be modelled where the conceptualisation passes through levels in a non-linear way, starting with actions (A), then processes (P), objects (O) and finally mental schemas (S). APOS theory can help us understand how the learning takes place by explaining what we see when participants are trying to "construct their understanding of a mathematical concept" (Dubinsky & McDonald, 2001:1).

Using APOS theory, researchers such as Carlson, Oehrtman and Engelke (2010) and more recently, O'Shea *et al.* (2016) developed tests and specifically, a function concept inventory (FCI) in the case of O'Shea, to measure the understanding of the function concept. The FCI was used in this study as a quantitative measure of improvement in the conceptualisation.

The use of APOS as a theoretical framework has recently been used successfully in function concept studies; for example, Chimhande, Naidoo and Stols (2017) and Maharaj (2010). This study also used APOS theory as the framework, wherein the pre- and post-intervention levels of function conceptualisation were cast. The intervention, the IFI, was based on image functions and was designed to let participants enact directly with functions at the object level.

4.2. Design and methodology

A mixed-methods approach was used in this study, as the study not only endeavoured to quantitatively investigate if an image functions intervention can boost function concept comprehension but also tried to elucidate students' thinking. Students might get "the right answer" more often, yet their underlying thinking might not have improved or was not corrected. Creswell and Creswell (2018:32) emphasise that it is a core assumption in mixed-methods research that combining the quantitative and qualitative approaches lead to a more complete understanding of the research problem. This combination process occurs at the design, sampling, data collection and analysis levels of the research (Ary *et al.*, 2018:518).

The mixed-methods research – quantitative and qualitative – was conducted within the postpositivist paradigm. This paradigm enables one to engage the causal aspects and the probing for meaning that are fundamental to this study. According to Maree (2016:59), the postpositivist approach makes room for the "multifaceted reality” and the mental construction of this reality that is core to the APOS framework. APOS was developed within a constructivist paradigm emanating from the reflective abstraction of Piaget (1971). In constructivism, “human beings construct meaning as they engage with the world they are interpreting” (Creswell & Creswell, 2018:38). This is what the researcher thought a student should do in working with functions and as such, construct their concept image as defined by Tall and Vinner (1981:151).

In the **first leg of this study**, an intervention based on studying images as functions was designed, implemented and critically evaluated. Following APOS theory (Arnon *et al.*, 2014), a genetic decomposition (GD) was firstly determined. This GD presented the mental structures at the action, process, object and schema levels that were deemed necessary for the learning of the function concept. With this GD in mind, activities were designed for the Image Functions Intervention (IFI). To validate the IFI, firstly a theoretical analysis was used to verify if the IFI adhered to the GD. Secondly, a questionnaire (see Appendix A) gathering qualitative data was given to participants that had gone through the intervention. This questionnaire’s data was qualitatively analysed to establish proof of principle concerning the usefulness of the IFI.

The **second leg of this study** followed the classical pre-test-intervention-post-test model and gathered quantitative and qualitative data concurrently in keeping with a convergent parallel mixed-methods design (Maree, 2016:318). This is also referred to as a “triangulation design” as the purpose is to have the quantitative and qualitative data converge/triangulate. This was appropriate for this study as it can aid in validating the results.

All participants started by completing the function concept inventory (FCI) designed by O'Shea *et al.* (2016). The FCI was used as is (see Appendix B). This formed the pre-test and provided mostly quantitative data on the students' level of function comprehension, but a few questions in the inventory invited explanations and thus provided qualitative data on function comprehension. After the pre-test was completed, *simple random selection* was used to split the participants into two roughly equally sized groups, the experimental and the control groups. The defining characteristic of simple random selection is that each individual has an equal and independent chance of being selected for either group (Ary *et al.*, 2018:50). Immediately following the pre-test, the participants in the experimental group followed the intervention for a period of two weeks. The intervention was followed in a self-directed online manner through the learning management system (LMS) used by the local higher education institute. In the week after the intervention was completed, the same FCI was given to all participants (experimental and control groups) and as such used as a post-test for further quantitative and qualitative data gathering. This is typical of a convergent parallel mixed-methods approach (Maree, 2016:318). The participants of the experimental group also completed a questionnaire (see Appendix C) designed to gather further qualitative data in order to perform more in-depth analysis on their experiences and probe their thought processes regarding the function concept. As seen in Chimhande, Naidoo and Stols (2017:3), qualitative data can be used successfully to “elicit and analyse” the participants’ levels of function conceptualisation. At the end of the semester, qualitative data regarding the function concept were gathered from the experimental group’s participants’ examination scripts to investigate if any previous improvements in function conceptualisation had remained.

The quantitative aspects of the study were used to investigate if the IFI, within the specific context, could cause improved performance on the FCI. This would indicate improved function conceptualisation (O’Shea *et al.*, 2016). True to the postpositivist approach, the aim was not to generalise the result, but rather to provide evidence that is valid and reliable in terms of the existing phenomena (Maree, 2016:60). The qualitative aspects were used to investigate if participants could move towards an object level understanding of the function concept.

The flow of the second leg of the study can be seen in Figure 1.

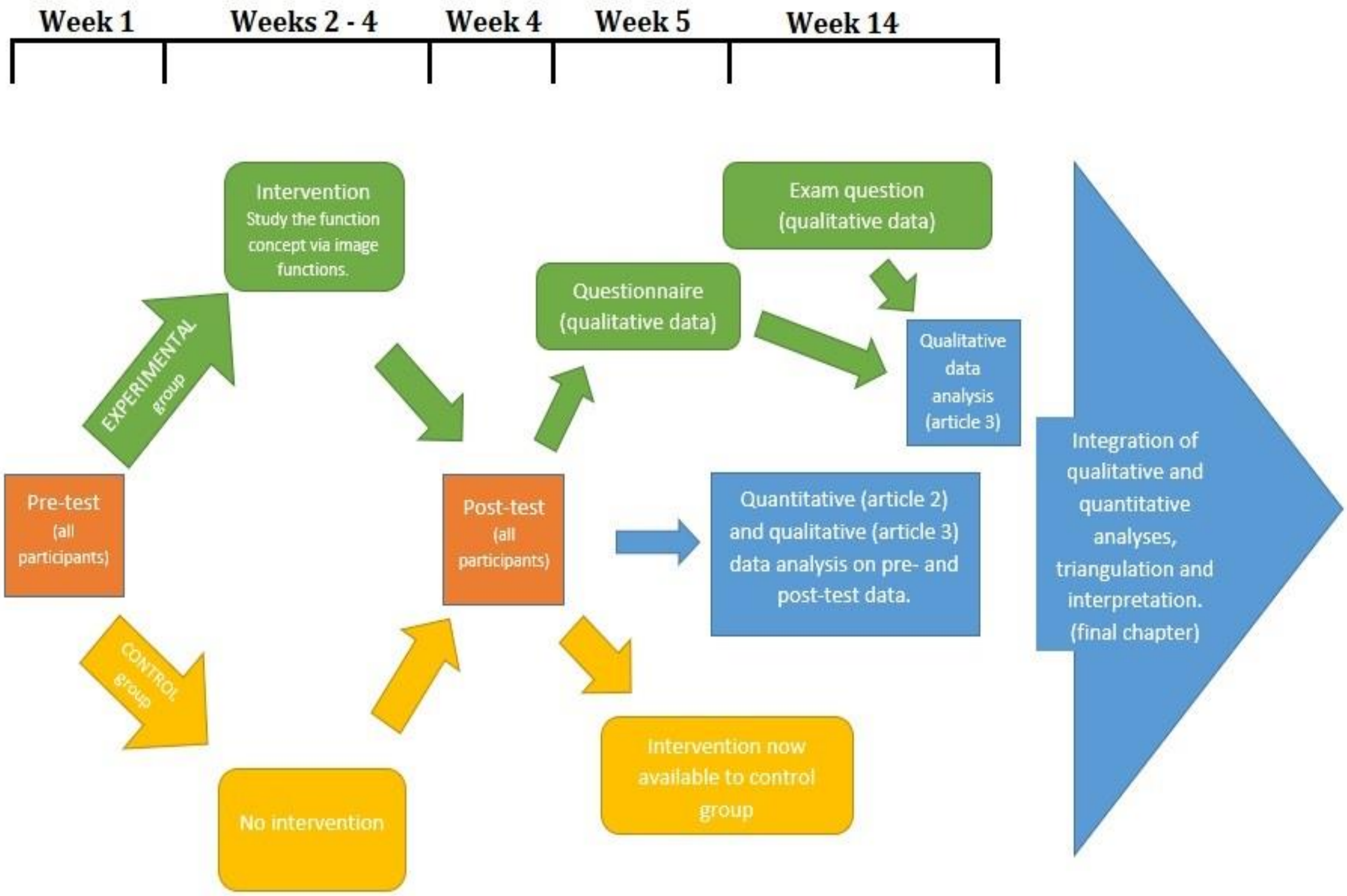

Figure 1. Research flow for the 2nd leg of the study.

4.3. Data collection

For the **first leg** of the study, which is reported in the first article, only qualitative data were gathered. This was done by means of a short questionnaire with three closed-ended questions (see Appendix A).

The **second leg** of the study used three data collection instruments. The first was the Function Concept Inventory (FCI) developed by O'Shea *et al.* (2016) and was used as is (see Appendix B). An inventory is different from a typical test used for assessment in that it is intended and thus designed in such a way as to specifically measure conceptual understanding (O'Shea *et al.*, 2016:281). The FCI consists of thirteen questions. Marks are allocated for ten of these questions, which thus gathers quantitative data. The three remaining questions of the FCI are open-ended and used to gather qualitative data on the participants' function concept understanding.

The second data instrument was a questionnaire consisting of three open-ended questions (see Appendix C) gathering qualitative data on participants' function concept understanding.

The third data instrument consisted of a single open-ended question (see Article 3) from the students' Calculus examination. The question pertained to the function concept. This examination formed the final summative assessment of their Calculus module.

The FCI was already validated (O'Shea *et al.*, 2016) as an appropriate test to determine the level of understanding a participant has with respect to the function concept. In O'Shea *et al.* (2016) the validity and reliability of the FCI was first determined first by pilot studies and subjectively by subject experts' consensus. Then Rasch Analysis (Bond & Fox 2007) was used to validate the test in terms of the test items combining to test a single construct, namely the trait of conceptual understanding of functions.

Qualitative data gathering in the form of questionnaires are quite common in education research and at least in part, have been used successfully in peer-reviewed research such as Doorman *et al.* (2012), O'Shea *et al.* (2016) and Epstein (2013). The aim is to bore down into the types of understanding required and the reasoning behind the students' choices.

4.4. Selection of research participants

For this study, convenience sampling (Maree, 2016:197) was used in order to gather accessible populations of students.

For the **first leg** of the study, the sample used was from a class of 1st year Calculus students at the local higher education institution. This sample was used for a preliminary validation of the usefulness of the IFI and as such, was connected to the second part of the first objective of this study, given in section three of this chapter.

For the **second leg** of the study, the sample used was from a class of 2nd year Calculus students. This sample was used for in-depth research on the effectiveness of the IFI via quantitative and qualitative analysis, reported on in articles two and three respectively. In turn, these two articles addressed the second and third objectives of the study. Using accessible populations was necessary, as it was not practically feasible to implement the intervention with groups at other institutions. Using the group of second year students was appropriate since the particular group's students were either mathematics majors or studying towards qualifications requiring a high level of mathematics. The function concept was therefore fundamental towards the further learning of this particular group of prospective participants.

First leg of the study: Proof of principle

A group of twenty-seven students in a first year Calculus class of 2019 was available for sampling. This group received the intervention in a classroom setting and then completed a questionnaire with three closed-ended questions. Details of the analysis are conveyed in Article one.

Second leg of the study: In-depth research

Approximately 91 students in the 2nd year group were available for sampling. All participants could write the pre-test. After the pre-test was completed, simple random sampling was used to form two groups (Maree, 2016:192), the experimental and the control. The experimental group participated in the intervention, whereas the control group did not. The experimental group's participants also completed a questionnaire, gathering qualitative data that was used to investigate if and how the participants' function conceptualisation had matured.

Randomisation is the best way to control extraneous variables and consequently to build confidence in possible inferences about the effectiveness of the intervention programme. As Ary *et al.* (2018:271) state, randomisation as part of an experimental study is the "gold standard for determining 'what works' in educational research".

4.5. Presentation of research findings

The research findings were presented in the format of three articles:

Article 1: An APOS Design of an Image Functions Intervention: A Qualitative Study

The first article followed the APOS methodology and started by determining a genetic decomposition for the function concept. Using this genetic decomposition, activities were designed for an intervention based on studying photographs/images as functions. The activities led the participants through the APOS levels and as such started by reinforcing the function as an action and then as a process. Lastly, the Image Function Intervention (IFI) aimed to get the participant to the desirable object level of understanding.

The article then moved to give a preliminary validation of the usefulness of the IFI in two parts. Part one dealt with a theoretical analysis of the IFI to ensure that it is adhering to the genetic decomposition and also that it was forming an exploratory base wherein some of the fundamental difficulties associated with the function concept could be addressed. In part two of the validation, the IFI was implemented in a classroom setting and a questionnaire gathering qualitative data was given to the participants. The data from this questionnaire was then analysed and the results reported.

The results from the theoretical analysis and the results from the qualitative data analysis were then considered collectively. This was done in an effort to establish if there was any indication that the IFI could be beneficial in at least some settings for some of the participants. Thus, we were trying to establish proof of principle. Proof of principle would give us reason to investigate the effects of the IFI further.

Article 2: Quantitative Analysis of the Effectiveness of an Image Functions Intervention

The second article reported on the use of the Image Functions Intervention (IFI) with a group of second year Calculus students. This article aimed to see if the IFI could have a significant effect on the performance of participants on the Function Concept Inventory (FCI) designed by O'Shea *et al.* (2016).

The article followed the pre-test-intervention-post-test model, with the FCI used as the pre-test and the post-test. All participants took the pre-test after which the group was split into an experimental and a control group of equal size using random selection. The experimental group completed the IFI after which all participants, therefore from both groups, wrote the post-test.

The FCI provided mostly quantitative data and some qualitative data. In this article, only the quantitative data were analysed. The qualitative data of the FCI were analysed in the third article.

The article also endeavoured to look for specific patterns of improvement, if any, and looked to identify areas or aspects where understanding was (still) lacking.

If a significant difference between the experimental and the control groups could be observed, this would help to further strengthen the case for the usefulness of the IFI.

Article 3: Developing an Object Conception of Function: An Intervention Study with Qualitative Analysis

The third article again reported on the use of the Image Functions Intervention (IFI) with the group of second year Calculus students reported on in the second article. However, in this article the aim was to establish if the IFI could assist participants to develop an object conception of the function concept. This was done by analysing qualitative data obtained from three different data instruments. The first instrument consisted of the three open-ended questions that formed part of the FCI that was given to all participants as both the pre-test and the post-test. All participants thus had the opportunity to complete the FCI twice, but only the experimental group completed the IFI after the pre-test. As stated previously, after the pre-test, random selection was used to divide the participants into two groups, the experimental and the control.

The second source of qualitative data was the questionnaire (see Appendix C) given to participants of the experimental group after completion of the IFI. This questionnaire used open-ended questions to gather data on participants' understanding of the function concept. The questions in this questionnaire were such that they were only relevant to participants that had completed the intervention. Therefore this questionnaire was not given to participants of the control group.

The third data instrument was one question from the students' Calculus examination. This examination formed the final summative assessment of their Calculus module.

Through qualitative data analysis, this article looked for shifts towards an object conception of functions in terms of APOS theory, and looked to see if the IFI could have a lasting effect on exhibiting an object conception of functions.

The qualitative validation of the IFI would again, similar to the quantitative validation in article two, strengthen the case for the usefulness of the IFI.

5. Value of the research

This study specifically aimed to expand the pedagogical content knowledge (PCK) of mathematics and in particular, with respect to the teaching and learning of functions in the higher education setting. See for example Ball, Thames & Phelps (2008) and Hoover *et al.* (2016) for more on the PCK of mathematics. The function concept is such a fundamental concept; it weaves through mathematics but also through applications in all of the natural sciences and increasingly also the other sciences. Therefore, a positive effect can be propagated through science in general. This study would also have a direct value for the students who would be learning about functions in the researcher's future classes and hopefully students in similar settings across South Africa.

6. Presentation of the thesis

Following this orientation chapter, the three articles forming the core of the thesis will each be presented as an individual chapter. As the requirement is that the articles should be considered publishable, the layout used will be that associated with a typical journal article. After the three articles, a final chapter follows that summarises the findings of the three articles and considers the triangulation of the different research methods applied in the study.

The following appendices are included:

Appendix A: The questionnaire used in article 1.

Appendix B: The Function Concept Inventory (FCI).

Appendix C: The questionnaire used in article 3.

Appendix D: Examples of interactive tasks in the IFI.

7. Conclusion

In this chapter, it was shown that the function concept is fundamentally important. It was further shown that the function concept is quite often understood at an unsatisfactory level. The function concept has behind it a large amount of published theoretical analysis (Dubinsky & Wilson, 2013), yet not enough headway has been made into addressing the problem of inadequate conceptualisation. The study reported on in this thesis is an attempt to get sufficiently practical in addressing the problem. It proposes that a specially designed intervention, the Image Functions Intervention (IFI), can be used to improve the understanding of the function concept.

Article one reports on the design and theoretical analysis of the IFI, which explores photographs/images as functions. Article one also provides an initial validation of the use of the IFI – a proof of principle.

Article two reports on the quantitative analysis that was used to determine if the IFI could cause an improvement on the scores of participants on the Function Concept Inventory (FCI). The FCI (O'Shea *et al.* 2016) was designed specifically to test the conceptual understanding of functions. Article two furthermore considers and analyses the results of specific questions from the FCI and compares these to other published results.

Article three reports on the qualitative analysis that was used to determine if the IFI could assist participants to reach an object level of understanding of the function concept. When using Action-Process-Object-Schema (APOS) theory as a model of how the understanding of a mathematical concept evolves through levels, the object level of conceptualisation is highly desirable but very infrequently observed.

The final chapter aims to consolidate the findings of the three articles and give overarching conclusions with respect to the primary and secondary research questions of this study.

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ARTICLE 1 – An APOS Design of an Image Functions Intervention: A Qualitative Study

Abstract

Despite the function concept being fundamental to mathematics, an adequate understanding of this concept is often lacking. This problem is prevalent at all levels of education and is reported in many countries. Could an intervention that explores photographs/images as functions help? The first objective was to determine a genetic decomposition (GD) of the function concept. This decomposition follows the APOS theory to identify the mental structures, at the action, process, object and schema levels, which are needed to learn the concept. Subsequently, activities were designed for the image functions intervention (IFI). The second objective was to validate this intervention. Keeping to APOS methodology, a literature review and personal experience were used to determine the GD. As an initial validation of the intervention, the first step was a theoretical analysis to ensure the intervention adhered to the GD previously determined. The second part of the validation was by means of the qualitative analysis of questionnaire data gathered after the first implementation of the intervention. The GD was successfully adapted from literature. It portrayed the mental structures and the mechanisms needed to move between the APOS levels. The theoretical analysis validated the intervention as adhering to the GD and providing opportunities for addressing common conceptual difficulties. The qualitative analysis provided evidence of the participants' expanded concept images but was inconclusive with respect to enhanced function comprehension. It was concluded that the designed intervention is theoretically sound, has merit and shows promise with respect to increased understanding of the function concept.

Keywords: image functions, APOS theory, genetic decomposition, function concept

1. Introduction

That the function concept is fundamentally important to mathematics can be accepted as a commonly shared opinion. As stated by Selden and Selden in Harel and Dubinsky (1992:1) '…the function concept, having evolved with mathematics, now plays a central and unifying role'. And more recently, '[t]he concept of function is central to students' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyze graphs' (Son & Hu, 2015: 4). O'Shea, Breen and Jaworski (2016:279) reiterate with '[f]unctions are central to present day mathematics' and elaborate '...going beyond calculus, functions are widely used in the comparison of abstract mathematical structures'.

Despite the high value attached to an adequate understanding of functions and the function concept, a full and nuanced comprehension is not common among students (Carlson *et al.*, 2002:353; Sajka, 2003:229). Doorman *et al.* (2012:1243) confirm the difficulty in learning the function concept and in particular state, '[f]unctions have different faces, and to make students perceive these as faces of the same mathematical concept is a pedagogical challenge'. This challenge is ongoing despite more than 50 years of research, producing 'a vast literature on teaching and learning the function concept' (Dubinsky & Wilson, 2013:84). That it remains such a challenge can partly be understood in the light of the difficulties evident in the history of the development of the function concept. The concept is said to be an epistemological obstacle (Sierpinska, 1992:28) as it is and has been so prevalent and persistent over a long time. The other reason could be attributed to what Dubinsky and Wilson (2013:86) highlight as the little attention that has been paid to research that applies the theoretical analyses (which is plentiful) to develop 'pedagogical strategies for helping students overcome these difficulties'. Simply put: (1) the concept of function is a difficult concept and (2) we have not been getting sufficiently practical in designing appropriate interventions/instructional treatments/didactical designs.

Some excellent work has been done in getting practical, but seemingly, more is needed. Ayers *et al*. (1988) and Breidenbach *et al*. (1992) considered the use of simple programming environments to provide practical activities in creating and using functions. Tall *et al*. (2000) considered the use of the 'function box/machine' as a strong cognitive root to anchor the different ideas connected with the function concept. Reed (2007) researched the effect of having students actively engage with the history of the concept of function. Salgado and Trigueros (2015) based their design and activities on models and modelling. In this paper, an intervention based on image functions is designed and analysed. One aspect where this intervention will differ from the implementations mentioned earlier, is in that participants are meant to use the intervention in a self-directed manner. Therefore, there is no teacher or lecturer involved. The need for such a self-directed intervention arises firstly from time-constraints with respect to direct contact time with students and secondly from the advantage of not needing teachers/lecturers to first become acquainted with the underlying ideas and content of the intervention.

The mathematics class is often filled with good intentions. Good intentions unfortunately do not guarantee good comprehension. With the function concept in mind, Akkoç and Tall (2005:7) points out that even in the face of a specific design, the outcome might not be achieved. They discuss a course that was designed to make the function concept foundational and an organising principle, but instead 'many students focus on the individual properties of each representation without connecting them together'. In order to increase the probability of a design being successful, it should be based on research and theory. Salgado and Trigueros (2015) provide a good example of such a design informed by Action-Process-Object-Schema (APOS) theory. Their design uses models and modelling. They first motivated their use of modelling by referring to research showing how modelling can raise motivation and interest, assist in identifying specific learning difficulties and facilitate learning and concept construction. Thereafter a genetic decomposition (defined in the literature review section) was constructed from which activities could be designed.

In light of the understanding of what reasonable design implies, this article will use an APOS theoretical analysis, supplemented by the literature on the learning of the function concept, to determine if it would be reasonable to expect the image functions intervention to improve understanding of the function concept. The following **research questions** were formulated:

1. Using the APOS theoretical framework, what can be regarded as appropriate mental structures for the learning of the function concept?
2. How valid is the design of the image functions intervention?

To answer the first question while keeping in line with the methodology of APOS theory, the current literature as well the researcher's own experience will be incorporated to create a genetic decomposition of the concept of function (Dubinsky, 2000:2; Maharaj, 2010:42). In this genetic decomposition, the appropriate mental structures at the action, process and object levels will be identified.

The second question will be answered in two parts: (1) The designed image functions intervention will be portrayed and analysed to validate if it adheres to the genetic decomposition determined earlier. (2) The intervention will be implemented within a classroom setting. Afterwards, the intervention's effects will be investigated through a questionnaire collecting qualitative data. The analysis of this data will look for indications of improved or broadened understanding of the function concept.

2. Literature review

2.1. Conceptual difficulties of the function concept

The concept or notion of a function is in its essence quite abstract but is often understood at a level where much of the abstract nature is not truly comprehended or might even be entirely lost. A student might for example directly equate the function concept to the existence of a formula (Dubinsky & Wilson, 2013; Sierpinska, 1992; Vinner & Dreyfus, 1989). One of the prominent indications of a lack of depth in the understanding of the function concept is the restrictiveness applied to what constitutes a function. If a student starts to fixate on particular types or certain representations, s/he loses much of the richness of the function concept.

Being able to recognise a certain formula or graph as (representing) a function is of course a necessary skill, but not sufficient in providing the student with the correct concept aspects and cognitive reasoning to be able to grasp and utilise higher mathematical concepts. For example, something as immediate as the inverse of a function, concepts such as limits, derivatives and not forgetting ideas that are even more abstract such as topological homeomorphism and category theory, remain out of reach. Thompson (1994:39) argues that a fundamental difficulty is students' lack of connections between the various representations of the same function. What is it that is being represented? Thompson (1994:39) names this 'something', the 'core concept of function', that which is left unchanged when moving between the different representations.

What it boils down to is that students sit with an inadequate and/or erroneous function concept image. According to Tall and Vinner (1981:151), the concept image constitutes the 'total cognitive structure that is associated with the concept'. This entails all definitions, properties, ideas, theorems and examples that a student has grouped under the heading of function over his or her mathematical career so to speak. Although a student may know the formal definition of a function, when exposed to a problem, the full concept image will be utilised to solve the problem. Doorman *et al.* (2012:1245) also consider the concept image very important and have as one of their specific goals, the overcoming of a 'too-limited' function concept image.

As the function concept is fundamental, yet misunderstood, the suggestion is that students should be introduced to the idea in such a manner that the resulting concept image will be as rich and accurate as possible. It is in these respects that the exploration of image functions is potentially very useful.

Much research has been done in confirming the difficulty with the function concept and/or trying to address the problem. Recent research includes that of Chimhande, Naidoo and Stols (2017), which confirmed that the difficulty is prevalent at school level. They showed that the mental constructions were typically at the action level of understanding, which is the lowest level according to Action-Process-Object-Schema (APOS) theory (Arnon *et al.*, 2014). Doorman *et al.* (2012) explored the use of computer tools in aiding the transition to a structural view of function; that is the object level of understanding. Makonye (2014) also provided a theoretical analysis focusing on the use of multiple representations to foster a nuanced concept image through approaches where the function concept is kept embedded in students' reality as far as possible.

Where this article will differ is firstly in its pedagogy. It will use a new design based on image functions. Secondly, it will be in contrast with prior research in that the intervention directly studies the structural view/object level of the concept whereas previous research aimed to produce such a view because of activities and reflections on the more procedural level.

2.2. Background on APOS theory

Working with constructivist ideas, Dubinsky and McDonald (2001) as well as others before them such as Breidenbach, Dubinsky, Hawks and Nichols (1992), formulated the APOS framework (Arnon *et al.*, 2014) for modelling the learning of mathematical concepts. Using the APOS framework, the development of the function concept can be modelled where the conceptualisation passes through stages in a non-linear way, starting with actions (A), then processes (P), objects

(O) and finally mental schemas (S). APOS theory can help us understand how the learning takes place by explaining what we see when participants are trying to ‘construct their understanding of a mathematical concept’ (Dubinsky & McDonald, 2001:1).

2.3. The genetic decomposition of the function concept

Genes are the building blocks of life and so to determine a genetic decomposition of a mathematical concept is to break down the learning of the concept into its imagined building blocks. The word imagined is used here as in following APOS theory, the breakdown is, among other things, dependent on the researcher’s own knowledge (Dubinsky, 2000:2; Maharaj, 2010:42). The researcher would use personal experience, completed research and observations to imagine and create a set of necessary mental structures and mechanisms at the action, process and object level. These structures and mechanisms are what someone who is learning the concept could need and use along the path of conceptual understanding (Arnon *et al*., 2014). Having the mental structures available makes it possible to judge at which level of conceptualisation a particular person is at, with respect to a specific mathematical concept.

Keeping to the analogy of building, if the genetic decomposition describes the progressive structures of the mathematical concept (the building), then the support needed to reach these structures would be described as the scaffolding. Part of the second objective of this article, namely the validation of the intervention, is to ensure that the intervention is appropriate. It must be appropriate on two fronts: (1) addressing the mathematical content in line with the genetic decomposition and (2) as scaffolding to support the student’s ‘construction of knowledge and skill’ (Bakker, Smit & Wegerif, 2015:1048).

From Arnon *et al*. (2014:27) we get the formal definition:

> A genetic decomposition is a hypothetical model that describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept.

Necessarily we then need to define what a mental structure is. Again, from Arnon *et al*. (2014:26):

> A mental structure is any relatively stable structure (something constructed in one's mind) that an individual uses to make sense of mathematical situations.

The genetic decomposition given in Table 1 is based on the decomposition given in Arnon *et al*. (2014:29). Extensions and/or expansions are based on the researcher's own experience complemented by current literature on the topic. The genetic decomposition given in Table 1 conveys the mental structures of the function concept at the action, process, object and schema levels. Furthermore, it also describes the mechanisms of progression, namely Interiorization, Encapsulation and Activity. In the APOS theory, these mechanisms are the means by which one can transition from one level to the next level of conceptualisation.

Table 1: Genetic decomposition of the function concept.

Action	Take an element of one set and apply an explicit rule, typically an (algebraic) expression, to determine a unique value belonging to another set.
From Action to Process: Interiorization Repeating this action and especially with sets with different kinds of elements, starts the interiorization by helping the student to reflect on the action and to see the pattern of *choosing* from one set, the domain, then *doing* something and then *obtaining* something else. Special emphasis must be placed on getting the student to consciously think about the chosen and the determined 'somethings' as belonging to specific sets. This is necessary, as from the researcher's own experience, students at the Action level will be satisfied once they 'get the answer', and not reflect further on the situation.	

<table>
<tr><td>Process</td><td>A dynamic transformation of inputs in the domain to outputs in the range without any explicit calculations needed.</td></tr>
<tr><td colspan="2">From Process to Object: Encapsulation

When it becomes necessary to think about applying an action or a process to the function (as a process), the dynamic process needs to be made static. The process needs to be captured and seen in its totality. Doing this encapsulates the function as a process to become the function as an object (Asiala et al., 1996, p.8; Arnon et al., 2014, p.30).</td></tr>
<tr><td>Object</td><td>Identify the word function as a noun. A noun has properties that can be listed. The noun is described by adjectives. A function could be for example, rapidly changing, smooth, constant etc.</td></tr>
<tr><td colspan="2">From Object to Schema: Activity

‘A schema is only constructed when it is functioning, and it only functions through experience: then that which is essential is not the schema as structure in itself but the structuring activity that gives rise to schemas.’ Piaget 1975/1985 as quoted in Arnon et al. (2014, p.110).</td></tr>
<tr><td>Schema</td><td>A dynamic mental framework, which a person might not be consciously aware of, that describes the function concept as simultaneously existing as an action, a process and an object and that links and relates these different underlying mental structures. A person evokes his or her schema when confronted with a problem involving the topic of functions. Specific examples of functions such as rational or trigonometric functions along with their properties and relations will also be included in the schema.</td></tr>
</table>

3. Theory of Image Functions

Consider the photographs or digital images in Figure 1. In Figure 1(a), the photograph of the horse consists of a finite number of pixels, or picture elements. This is easy to see in the zoomed image in Figure 1(b) where we can distinguish individual elements of the eye of the horse. To each position in the image, a unique colour is assigned. We therefore have a function.

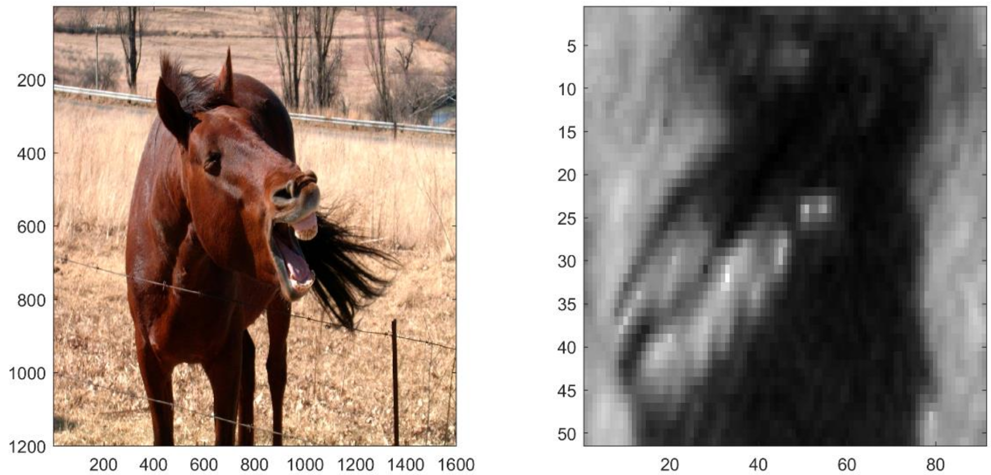

Figure 1. (a) Photograph of a horse. (b) Zooming in on the horse's eye.

Defining the function

A digital image, $f(x, y)$, is a function with both x and y being positive integers. Any combination of such an x and y will form an ordered pair that will denote the position of a particular pixel in the image. Corresponding to each ordered pair is a unique colour. Typically, the different colours are represented using the RGB (red, green, blue) colour space. Any specific output of an image function is then an ordered triple providing the specific combination of red, blue and green. Typically, a scale of 256 different shades of red are used and the same for green and blue (Gonzalez & Woods, 2017). If we then let the first shade be represented by 0, the last shade would then be represented by 255. Using these typical values, 256^3 potential combinations of red, green and blue are possible. For example, the triple (255, 0, 0) will be bright red as it contains the full complement of red and zero contributions of green and blue. (255, 255, 0) is bright yellow, (0, 255, 0) is bright green and (57, 229, 212) would be called turquoise by some.

If we only consider the possible outputs where the three components of each triple are equal, we end up with what is commonly referred to as a greyscale image, where outputs are shades of grey. For example, (0, 0, 0) is black, (255, 255, 255) would be white and (30, 30, 30) would be a dark grey. The image in Figure 1(b) is an example of a greyscale image. As the three values in each triple will be equal, the outputs for greyscale images each consist of a single number that represents the light intensity at a particular pixel.

We now consider all the usual aspects of functions in the light of the greyscale image of the horse's eye – that is Figure 1(b).

Domain

In the case of the eye of the horse, the image has exactly 51 rows and 91 columns. The domain of this image function is the set of ordered pairs:

$$\{(x, y)|1 \leq x \leq 51, 1 \leq y \leq 91, x \in Z^+, y \in Z^+\}$$

Z^+ is the set of positive integers.

Range

The word range can refer to two different concepts, namely the codomain and the image of the function, so care should be taken in using it. The codomain for a greyscale image is easily specified as the set $\{s \in Z^+|0 \leq s \leq 255\}$. This is then the set of shades of grey from which any greyscale image could be 'choosing'.
When the term range is referring to the *image* of the function, it will consist of all shades of grey actually present in the particular 'picture'. Here then the image of the function and the picture-image of the function are the same set. The picture set would normally have repeated values/colours and would thus be a different multiset from the function image.

Surjectivity and Injectivity of Image Functions

An image function would seldom be surjective. With colour images using the RGB colour space, we have 256^3 = 16777216 unique elements in the codomain and most often, many of these colours would not be present in the image. Being closer to surjective is normally desirable when it comes to images, as this would generally mean the image has higher contrast. Greyscale images typically have (only) 256 unique elements in the codomain; thus, being surjective has a much higher probability than in the case of colour images. It is clear that most images would not be injective either because it is highly probable that different pixels have exactly the same colour or shade of grey.

Existence of the Inverse Function

As for all functions, the inverse will exist if the function is injective. In the previous paragraph, we saw that it is highly improbable for an image function to be injective and consequently it is highly unlikely for the inverse to exist. With the high resolution of modern cameras, it is quite common for digital images to consist of millions of pixels. For greyscale images of such high resolution, it would then be impossible to have an inverse, as greyscale images only have 256 output options available. Even for colour images with 16777216 possible output options, it will still happen often that at least two pixels will have the same colour. Therefore, the probability of the inverse existing is small.

Continuity

Consider any point (x_0, y_0) in the domain of our image $f(x, y)$. Then we can show that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

and therefore, that the image is continuous on its domain.

Proof: Let $\epsilon > 0$. Let $0 < \delta < 1$. If $\left|\sqrt{(x - x_0)^2 + (y - y_0)^2}\right| < \delta < 1$ then $f(x, y) = f(x_0, y_0)$ because the domain of f is a subset of $Z \times Z$. Therefore, $|f(x, y) - f(x_0, y_0)| = 0 < \epsilon$. As the limit exists at any point in the domain and the limit is equal to the function value at that point, the function is continuous at any point in its domain.

Differentiability

A digital image is not differentiable at any point, yet a discrete derivative in the form of a difference quotient plays an important role in image processing. Applications where sudden changes such as steps, ramps, edges, lines or isolated dots need to be identified and/or accentuated, often rely in part on some discrete implementation of a derivative. From Calculus, we know that the derivative of a constant is zero, which translates to the important requirement of derivative-based filters to give back a small or even zero response in a homogeneous region of an image. See for example Gonzalez and Woods (2017) for more on the implementation of derivative filters and for example, Shrivakshan and Chandrasekar (2012) for more on edge detection techniques through the use of derivative filters.

4. Methods

4.1. Study design

This paper firstly uses an APOS theoretical analysis of the Image Functions Intervention (IFI) and secondly a qualitative approach to establish an initial proof of principle in the IFI having a positive effect on the understanding of the function concept.

Consequently, this section consists of two main parts. In the first part, the IFI will be portrayed and the criteria given whereby it was analysed. In the second part, a questionnaire will be discussed that was used to gather qualitative data on the initial effects of the IFI. A qualitative method was used here to allow the exploration of participants’ perceptions and allow for unexpected feedback on the intervention.

4.2. Part 1: The Image Functions Intervention

As a general context, the intervention deals with finding a missing student of which one recent photograph was available on the student's Facebook page. This photograph, however, was taken in low light conditions and as a result needs some processing before it will be helpful in finding the missing student.

This theme runs like a story throughout the intervention. This theme was chosen as participants are familiar with the context; they can easily understand the contingency relationships involving the variables that are present; and they are generally interested in the type of context (Donovan & Bradsford, 2005:359; Eggleton, 1992). Besides this story, the theory concerning image functions is conveyed and interwoven with reflective questions and specific activities. These activities are specifically designed along the principles below to keep in line with the genetic decomposition obtained in the literature review section:

1. Activities directly link with the mental structures determined in the genetic decomposition (Salgado & Trigueros 2015:107).
2. Activities address the categories of conceptual understanding (Dubinsky & Wilson, 2013:85–86).
3. Activities form an experiential base for the aspects of the function concept to be studied (Dubinsky & Wilson, 2013:90).

The principles given above will be used as the criteria for analysing the intervention.

Activity 1: Ask the participant to draw an 8x8 grid on paper as in Figure 2(a).

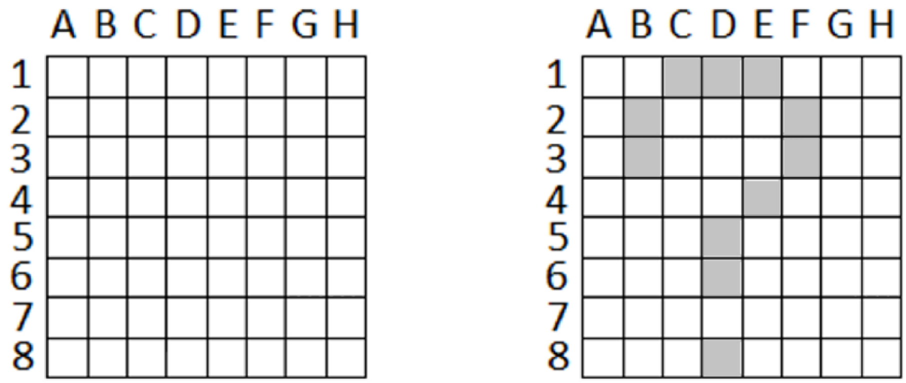

Figure 2. (a) An empty 8x8 grid. (b) Result obtained at the successful completion of Activity 1.

The participant must then take a pencil and shade the blocks at positions B2, B3, C1, D1, D5, D6, D8, E1, E4, F2, and F3. Accurate shading leads to the result given in Figure 2(b). This activity aims to connect the repeated action of assigning a shade of grey to a specific position in the grid, to a function value that is assigned to specific input. This activity is therefore aimed at helping students to construct the function concept as an action, which is the first stage of understanding according to APOS theory.

Students are asked to reflect on the activity by letting them provide answers to questions pertaining to uniqueness aspects of functions and asking questions to let them think about the choices that can be made with respect to input and output.

From this activity, there is a natural flow in letting the participant discover that an image is a function.

Activity 2: Let participants visualise some function outputs by using the commonly available software, Paint's, colour editor. Different combinations of either red, green and blue or cyan, magenta, yellow and black can be set up. This is an easy to use experiential playground where participants are asked to try out for example negative values or non-integer values or values larger than the maximum used in the colour scheme. This allows participants to engage with the idea of the range of an image function practically. At a later stage, participants must also think about the domain by realising that the image only has a finite number of rows and columns. The concept of the domain of a function is thus a quite practical ‘thing’ with respect to image functions.

Activity 3: In this activity, the aim is to assist participants in constructing the function concept as a process.

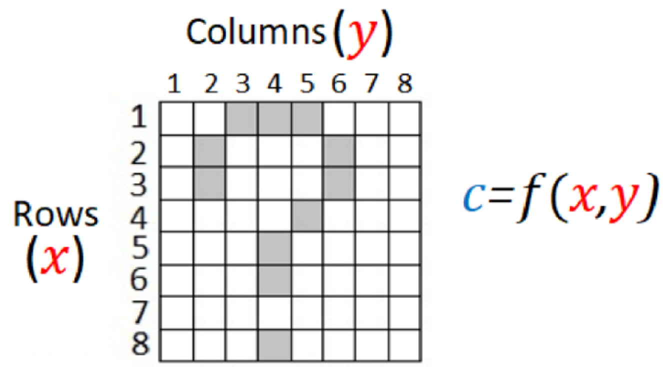

Figure 3. The function regarded as a process.

Participants are brought back to the image function formed in Activity 1, but now both the rows and columns are depicted by positive integers as in Figure 3. Participants are then led to compare the point-by-point, successive way in which they shaded each individual block, to what will happen inside a camera. Inside the camera all the ‘blocks’ get shaded simultaneously. The function is then a process of taking the entire domain at once and filling it with the range. Participants are again led to think about the specifics of the domain and the range, the uniqueness property and the aspects of being injective and surjective. Thinking about the inverse process is also introduced here, for example questions such as ‘Yellow is a colour that is present in this image (Figure 4). If you now make yellow the input, what will be the output?’, ‘Do you think the image function shown will have an inverse?’

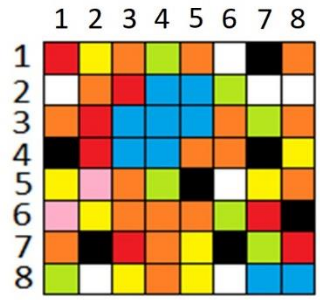

Figure 4. Will this function be injective?

Activity 4: With this activity, the aim is to assist participants in constructing the function concept as an object. The story in which the intervention is set is brought to a peak here. The participant sees how the knowledge of image functions, together with simple contrast stretching, is used to enhance the photograph discussed at the start of the intervention. It is enhanced to such a degree that sufficient information can be gathered from the image to assist in identifying the place where the photograph was taken. The contrast stretching seen in this activity is achieved through function composition.

As an introduction, the activity lets the participant explore contrast stretching with

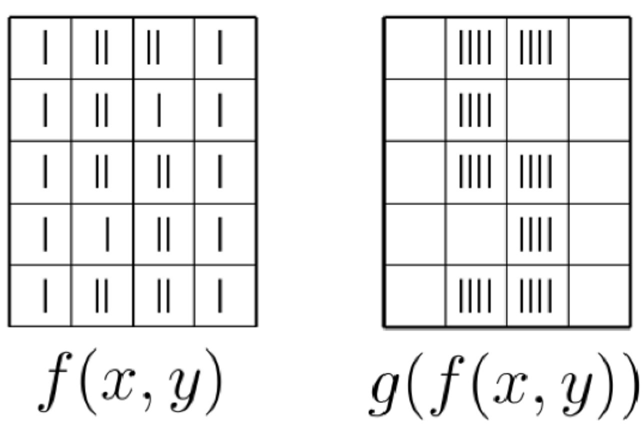

Figure 5. (a) Low contrast image. (b) Increased contrast after function composition.

pen and paper. The participant is asked to draw the 5x4 grid as pictured in Figure 5(a). The number of vertical lines, v, in each cell (pixel) can be considered the colour of that cell. Figure 5(a) is then an image function, say $f(x, y)$. The participant is then asked to draw a second 5x4 empty grid and then fill in its values by applying the function $g(v) = 4v - 4$ on the original image of Figure 5(a). A new image function $h(x, y)$ is thus created through function composition. We obtain Figure 5(b) through the function composition $h(x, y) = g\big(f(x, y)\big)$. This composition stretches the contrast to such an extent that we can now see the number 5 or the letters present in the image. The 5 (or s) was of course already present in the original image, but it was difficult to distinguish it from its background. It was difficult to distinguish due to the low contrast of the original image. Once the introductory contrast stretching is completed, the participant is brought back to the photograph associated with the missing person's case. This photograph is seen here in Figure 6(a).

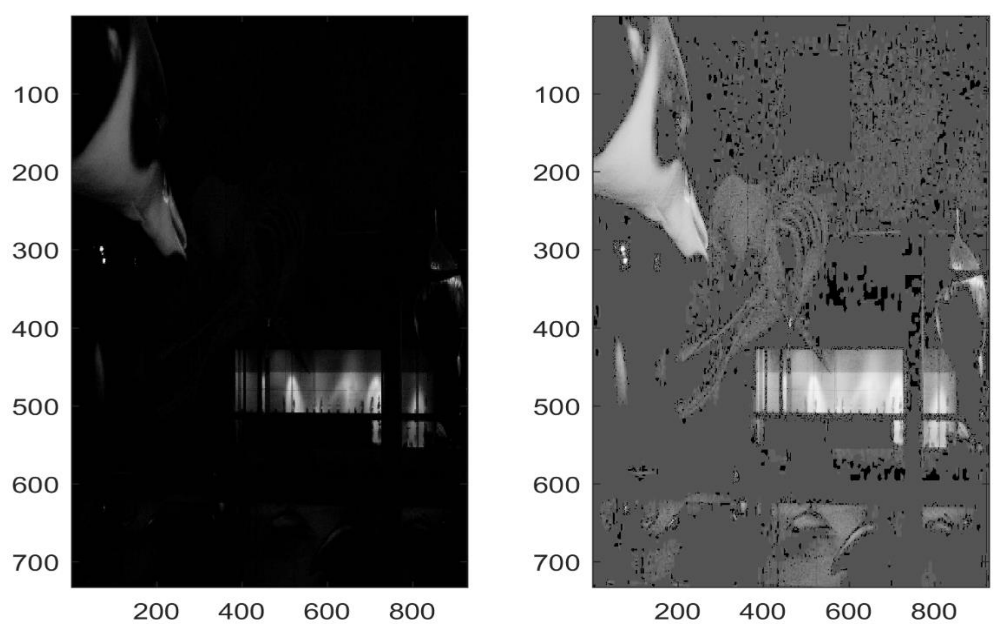

Figure 6. (a) Low contrast image. (b) Increased contrast after function composition.

Participants are initially asked to describe in their own words what is wrong with the photograph, or consequently the (image) function in Figure 6(a). How can a function be 'wrong', or for that matter be described? As an object, the function acquires global properties such as having low contrast; therefore, a small average difference between adjacent pixels across its domain.

This activity lets a participant realise that a low-contrast image can be improved by regarding it as one single thing – an object – that can be transformed by another function. Function composition is used to transform the original image function, $f(x, y)$, into a new and improved image function $g(f(x, y))$.

$$new_{object} = g\left(old_{object}\right)$$

If we carefully design the transformation function, g, we can obtain the desired results and again describe the new function as a whole. A function then becomes a noun and a noun can be described by adjectives. We might say that the new function is beautiful, it is clear, it has high dynamic range, it is smooth or was planned for the image of the intervention, it has improved contrast as can be seen in Figure 6(b).

4.3. Part 2: Proof of principle in using the IFI

This first practical implementation of the intervention had one objective that tied in with answering the second research question as was given at the end of the introduction of this article. The objective was to show that the intervention could contribute, however small, to an enhanced understanding of the function concept.

Achieving this objective would give us proof of principle; therefore, determining if the IFI is sensible and worthwhile to investigate further. Proof of principle, together with qualitative analyses indicating what the content of an intervention should entail, assists in determining the need and validation for further testing (Pressley, Graham & Harris, 2006:7).

4.4. Sampling and data collection

The intervention was implemented in a classroom setting with a group of twenty-seven students in a first-year Calculus course. This is not the intended method by which the intervention will be implemented, as it was designed to be a self-directed mini module where a participant follows his/her own pace and can actively engage with the various activities of the intervention. However, to gage the initial reaction of participants to the material and activities, it was decided that a classroom setting, together with a questionnaire at the end, would be sufficient and still enable us to achieve the objective of a proof of principle.

This group was chosen for convenience but fulfilled the minimum criteria of having prior knowledge of the function concept. From casual observation, the group had male and female members and these members were from at least three different ethnic backgrounds.

The questionnaire was handed out to all twenty-seven participants and it was made clear that participation was voluntary and would be anonymous. No personally identifying information was asked, as this was not deemed necessary for a proof of principle endeavour. The questionnaire consisted of three grammatically closed but conceptually open questions (Worley, 2015:19) as this still allowed any elaboration the participant might wish to provide.

1. Having completed the image functions intervention, have you realised/learnt something in particular of the function concept?
2. Is there an aspect of the function concept that is now clearer to you?
3. Is there some aspect of the function concept that you might have thought about in some way before, but now realise that you were wrong in some sense?

4.5. Data analysis

The method of Content Analysis with emergent coding was used (Maree, 2016:111). The responses to the three questions were searched for any indications relating to the objective of the questionnaire. After reading the responses a few times, four themes were identified:

(1) What constitutes as a function?

Students commonly struggle with misconceptions with regard to what and what cannot be regarded as a function (Dubinsky & Wilson, 2013:85–86). More discussion on this topic is given as part of the analysis of the Image Functions Intervention under the Results section part 1.

(2) Functions are connected to real life.

From reading through the participant responses, it seems that many were almost surprised to find functions being used in such an everyday type of topic as photographs. This theme does not represent an improved understanding of the function concept but can possibly assist in making the topic interesting to participants. This interest can increase their motivation, which is key to effective learning (Eggleton, 1992:1). Connecting the function concept knowledge to everyday experience also assists later retrieval and application (Donovan & Bradsford, 2005:364).

(3) Domain and range.

This theme relates to the understanding that a function requires a set of allowed inputs, and associated to each input, a unique output. The outputs form a set as well. From the genetic decomposition, we saw that clear understanding of domain and range are required to construct the function concept at the action level and to interiorise the actions to start constructing at the process level.

(4) Function inverses.

At the process level, being able to reverse the function and form the inverse function is required (Breidenbach *et al.*, 1992).

Each response given to the three questions was read and reread in order to judge if it contained any indication that the participant experienced an improved or broadened understanding of the function concept. The indication was then classified as belonging to one of the four themes.

4.6. Validity

It was not the intention of the questionnaire to deliver a generalizable result. The questionnaire was part of the effort to establish a proof of principle. This can be interpreted as looking for proof that the Image Functions Intervention can have value, at least in some settings with some participants.

4.7. Ethical considerations

Ethical clearance for the research conveyed in this paper was obtained through the university's General Human Research Ethics Committee (GHREC), reference number UFS-HSD2019/0006/1505.

5. Results

Following the methods section, this section will also consist of two main parts. The first part will convey the results of analysing the Image Functions Intervention (IFI). The second part will convey the results of the data analysis on the questionnaire.

5.1. Part 1: Analysis of the IFI

The intervention follows the APOS general trajectory in the sense that information and in particular the activities are ordered to first let participants retouch on the function concept as an **A**ction, then moving on to a **P**rocess and then finally to the function as an **O**bject. In the discussion of the activities above, it was already indicated how the activities aim to guide the participant to construct the desired mental constructs that emerged in the genetic decomposition. To expand on this and so validating our first design principle, we track the genetic decomposition:

1. At the **action level,** a student is expected to take an element from the domain and find its corresponding value from the range. This is what is required in Activity 1. Repeating the actions (Asiala *et al.*, 1996:7; Dubinksy & McDonald, 2001:3) together with encouraged reflection about the actions and the involved sets (expanded on in Activity 2), triggers the necessary **interiorization** mechanism to lead the participant to a process understanding of function.
2. At the **process level,** a student must now be able to capture the creation of the image of the function mentally (in the sense of the range) as a whole. Possibly infinitely many function evaluations can be imagined taking place simultaneously in the mind. Dubinsky and McDonald (2001:3) describe that at the process conception, the individual can think about infinitely repeating the same kind of action, as no external stimuli – such as following the steps of a formula – are still needed. Activity 3 assists here by juxtaposing the point-by-point creation of a photograph by individual actions with the actual chemical/electric process that is going on inside a camera to form the photograph all at once. Inside the camera, the entire film or sensor is illuminated and so all pixels get their values at the same time. The action of one 'element of light' reaching one point on the film/sensor to make one specific colour at that point/pixel, can be imagined to be repeated simultaneously for all the millions of pixels in the eventual photograph.
3. At the **object level,** a student must now be able to grasp a function as one static entity. Activity 4 aides the **encapsulation** of the former dynamic transformation process to a 'thing', to become a noun with associated adjectives. The encapsulation of functions at the process level by performing actions or other processes on these functions is reported to be key to transitioning to the object level of understanding (Asiala, *et al.* 1996). Yet, this type of encapsulation is missing from experience when it comes to functions (Asiala *et al.*, 1996:8). In the IFI, this is exactly the type of encapsulation emphasised by means of the function composition in Activity 4. Actions and processes are applied to the objects in order to achieve the specified goals.

4. As the intention with the intervention was only to look to improve understanding up until the object level, the **schema level** is not discussed in this analysis of the intervention.

The strategy in analysing the IFI is to consider the prerequisite design principles and showing if and to what extent the intervention manages to adhere to these principles. The first principle of adhering to the genetic decomposition was discussed above. In keeping to the second and third design principles, the intervention has to form a basin wherein the conceptual difficulties and other required concept aspects associated with the function concept, can be explored in a familiar context, namely that of photographs. To see if the intervention is compliant, we will look in turn at the most commonly occurring difficulties (Dubinsky & Wilson, 2013:85–86) and other function aspects:

1. What constitutes as a function?
    a. Using image functions encourages the realisation that a formula is not necessary to have a function. Vinner and Dreyfus (1989) categorised students' definitions of a function into six categories: Correspondence, Dependence Relation, Rule, Operation, Formula and Representation. Ultimately all six of these categories can be valuable viewpoints when dealing with various functions, function properties and applications. However, if a student focuses too much on the idea of a formula or an equation for a function, the construct could easily be cemented in his or her thinking that only formulas, or rules/correspondences having known formulas/equations, can be considered as functions. This kind of restrictiveness is quite common (Vinner & Dreyfus, 1989; Sierpinska, 1992; Breidenbach *et al.,* 1992; Dubinsky & Wilson, 2013).
    From the start of the intervention, the participant is put in a state of disequilibrium by introducing an everyday thing such as a photograph as a function. This state of disequilibrium, in the style of Piaget (Wadsworth, 1978:80), is necessary here as the participant's concept image of functions has been formed and *re-enforced* over a number of years already.

Therefore, by throwing the participant off balance, room is created for the restructuring of the concept image.

It is the suggestion of this paper that the IFI's reintroduction to functions via image functions can challenge the fixation and restricted thinking in terms of formulas and equations. This is necessary to develop beyond the Action level.

b. Learners also often focus on the symbols representing the variables, instead of the quantities they are representing. In the IFI, working with the image functions is done without the need for symbols. Familiar or intuitively understood terminology such as row, column and grey level are used.

c. Vinner and Dreyfus (1989:361) also showed that any seemingly irregular behaviour such as discontinuity, a split domain or 'The idea that the graph of a function has to have a stable character…'are erroneous ideas that learners often use to disqualify some rules/graphs as functions. The IFI, by working with image functions, again has the advantage above using linear functions for example, in that images change character easily across the two-dimensional domain. This is even more apparent when the object conception of function has been reached after Activity 4. Therefore, students will not be left with the idea that a function needs to act 'nicely' in any way to be considered a function. In contrast, Carlson and Oehrtmann (2005:2) mention the case of students thinking that constant functions are not functions 'because they do not vary'. Also see Bakar and Tall (1991). Confrey and Smith (1991) refer to the constant function as an example of a 'monster' function and confirm that learners exclude it as a function because they expect a function to 'covary'. In working with the IFI's activities, it occurs naturally that portions of the image will have the colour or grey level stay constant over smaller or even large areas. A participant can thus discover intuitively that an image function and by extension all other functions, are allowed to display seemingly non-regular behaviour over certain portions of their domains.

2. Univalence and Injectivity.

In working with the image functions of the IFI, the often-problematic univalence property is made practical and simultaneously important. In addition, the confusion that is often seen (Harel & Dubinsky, 1992) between the univalence property and the function being injective is addressed in a tangible way. In terms of image functions, it is evident that at any specific input (pixel), the photo has only one output (colour) and as such exhibits the univalence property. Furthermore, if we determine that for the particular photo, the colour at any pixel does not occur at any other pixel, we have determined that the photo/image function is one-to-one, thus injective. Reflections to evoke these realisations with participants are delivered through Activity 3 of the IFI.

3. Multiple representations.

   The activities of the IFI provide opportunities to ask new questions about multiple representations of functions. The multiple representations idea garners plenty of attention in textbooks as well as research (Stewart, 2015:10; Confrey & Smith, 1991; Carlson & Oehrtman, 2005). However, according to Thompson (1994:39), students still miss that which stays unchanged between these representations. Thompson (1994:39) referred to this as the 'core concept'. How would one represent an image function differently than by means of the photograph? Setting up a table that states explicitly the colour that belongs at each position, can illuminate the fact that the representation itself is *not* the function. Students can be led to realise that similar to an equation or a graph, the purpose of a table or the photograph is to tell us the output that belongs to any specific input. Why do we then have or use multiple representations? We only use them if they are helping us to understand and/or analyse the function. A blind person could get no value from a graph of a function, but could there maybe be a way to listen to a function? Could an audio representation be created and would it be helpful?

In validating the design from a theoretical viewpoint, we are moving the intervention towards a correct organisation of the knowledge to optimise deep understanding. This is key in developing sufficient expertise to solve problems that flow directly from the topic(s) involved as well as related problems (Donovan & Bransford, 2005:16). Naturally, the theoretical validation will need to be followed by practical validation, which is the content of research currently in progress. In

APOS theory, the genetic decomposition and the interventions based on it are validated by testing if the mental structures formed by participants are indeed those that were predicted (Arnon *et al*., 2014; Dubinsky, 1991). This leads to a feedback loop where the interpretation of the results will lead the researcher to make changes to the genetic decomposition and/or the intervention and then repeat the implementation and testing. These steps can be repeated in the loop until a type of convergence is achieved. For this article, qualitative data were also gathered by means of a questionnaire, as a way of externally validating the intervention. This is done by studying the improvement of participants in understanding the topic of interest.

5.2. Part 2: Analysis of the questionnaire data

In the Methods section, four themes were discussed that emerged from repeated reading of the participants' responses. The responses were then analysed individually within the structure provided by the four themes. This is done while keeping in mind that we are looking for indications that the participant experienced an improved or broadened understanding of the function concept. In the analysis to follow, {x,y} will mean that participant x gave the response while answering question y. For ease of reference, the three questions are repeated:

(1) Having completed the IFI, have you realised or learnt something in particular of the function concept?
(2) Is there an aspect of the function concept that is now clearer to you?
(3) Is there some aspect of the function concept that you might have thought about in some way before, but now realise that you were wrong in some sense?

Theme 1: What constitutes as a function?

The analysis illustrates that participants are showing an expansion of their 'concept image' (Tall & Vinner, 1981:151) related to what can also be considered a function. Consider the following responses:

{1,1} 'Yes, I realized functions have a broader meaning and that it forms a big part of our technological lives.'

{13,1} 'Yes, that all images are also functions.'

{21,1} 'Yes a function can be determined in different ways.'

{18,3} 'Yes, that not all functions are graphed on cartesian planes.'

Theme 2: Functions are connected to real life.

Again, we see participants' concept images expanded. Here it relates to a realisation that functions can be useful and specifically useful outside of mathematics itself. Some participants realise that functions can be part of their lived experiences. Consider the following representative responses:

{2,1} 'Yes, it can be used for various purposes.'

{5,1} 'functions can be used for alot of puposes out side of maths.'

{6,1} 'Yes, I understand that math is used everywhere.'

{22,1} 'Yes, how unclear images been processed to have clear pictures about something and I did not know that how functions are used in life.'

{19,3} 'Yes. I didn't know functions can relate to real life experiences and applies to images and is involved in biometric scanners.'

{25,3} 'Yes, I have learned that functions can be used for more than calculating or predicting change.'

Theme 3: Domain and range

A few participants gave some indication of increased understanding of the aspects of input and output. Clarity on input and output, as relating to domain and range, would assist in transitioning from an action level to a process level of function conceptualisation. Consider the following responses:

{4,1} 'Yes there is a input and a output'

{9,1} 'Yes, there is an input and output'

{3,2} 'Inputs and outputs of a function'

Theme 4: Function inverses

Some participants reported to have gained increased clarity on the aspect of function inverses. To understand the function concept at the process level, one needs to be able to reverse the actions of the function and then progress to formulating the inverse function and/or deciding if the inverse function will exist. The following responses are representative:

{13,2} 'Yes, the differentiation between inverse functions'

{15,2} 'The derivatives or inverse of functions.'

{3,3} 'Yes, inverse of a function.'

6. Discussion and conclusions

In this article, our first objective was to present a genetic decomposition of the function concept. Not only does the genetic decomposition serve as the basis for designing a learning intervention, it also serves to make the consequent analysis of data more reliable (Arnon *et al*., 2014:38). Through empirical study, the success of APOS theory has been shown through using the genetic decomposition not only to describe the mental constructions of participants but also to design 'effective instruction' (Weller *et al*., 2003). The genetic decomposition was determined, showing the necessary mental structures at the action, process, object and schema levels and so answering the first research question.

The presented genetic decomposition was used to design an intervention specifically using image functions. As part one of the validation of the Image Functions Intervention (IFI), the activities of the IFI were shown to keep true to the genetic decomposition. Important to note is that the IFI provides opportunities for encapsulation of processes through the function composition used in the contrast stretching activities. In APOS theory, encapsulation is the mechanism by which conceptualisation evolves from the Process to the Object level (Arnon *et al.*, 2014). Asiala *et al.* (1996) reported that this type of encapsulation, namely the encapsulation of functions considered at the Process level, is necessary to transition to the Object level of function conception. Asiala *et al.* (1996) further reported that this type of encapsulation is mostly lacking in our experience with functions. Besides adhering to the genetic composition and providing encapsulation opportunities, the IFI was also shown to create opportunities to address some prominent conceptual difficulties associated with the function concept.

Part two of the validation of the intervention entailed the implementation of the intervention in a classroom setting, with subsequent qualitative analysis of a questionnaire given to participants. This is a first iteration in the loop of implementation → results/data → analysis and updating of the intervention that is standard in the APOS research methodology (Arnon *et al.*, 2014). The intervention was put to the test in a classroom setting with twenty-seven first year Calculus students. From the results of the analysis, we saw that at least some participants reported gaining new insight on inputs and outputs of functions and the inverses of functions. Furthermore, it seems safe to conclude that for at least *some* participants, their ideas concerning what constitutes a function have been broadened. However, the improved understanding of the function concept we were looking to observe here, cannot be independently verified as it can be based only on the participants' own reporting. This is partly the fault of the questions of the questionnaire. They were intended to be sufficiently inviting to lead participants to provide rich responses from which true improved understanding could be judged. This did not happen to a sufficient measure. Seemingly, the questions were formulated too closed-ended. On a positive note, we can conclude from the results that the IFI has the potential to enrich the concept image (Tall & Vinner, 1981:151). The concept image is closely related to the Schema level of APOS theory (Dubinsky & McDonald, 2001:3) in the sense that it is the construct one will utilise when confronted with solving an actual problem. Concluding that the concept image of participants can be enriched is based on the many instances of participants making a new connection between functions and everyday/real life and losing some of the restrictiveness regarding what constitutes as a function. This enriched concept image, together with the positive results from the theoretical analysis of the IFI, gives us sufficient reason to claim proof of principle.

Therefore, the conclusion is that there is sufficient indication that the use of the IFI has merit and is therefore worthwhile to explore further. Subsequent qualitative and quantitative research will aim to verify, not only if the genetic decomposition is a true predictor of the mental constructions of participants, but also if the IFI can actually manage to improve participants' understanding of the function concept.

Competing interests

The author declares that no financial or personal relationship(s) exist that may have exerted inappropriate influence with regard to the writing of this article.

Disclaimer

The views and opinions expressed in this article are those of the authors and do not necessarily reflect the official policy or position of any affiliated agency of the authors.

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ARTICLE 2 – Quantitative Analysis of the Effectiveness of an Image Functions Intervention

Abstract

The concept of function is fundamental to Mathematics. Furthermore, research concerning the learning and teaching of the concept has received ample attention over the last fifty years. Nonetheless, the adequate understanding of the concept remains uncommon amongst high school learners and university students. This article reports on the use of an intervention based on the study of photographs as functions. Quantitative data analysis was used to investigate the effectiveness of this intervention. The study followed an experimental design with an experimental group (n = 10) and a control group (n = 7). Data were collected from participants that completed a function concept inventory as the pre- and post-test. The function concept inventory was determined to be a valuable diagnostic tool as it successfully indicated the elements of understanding with which participants struggled. The experimental group showed a significant increase in performance on the inventory, whereas the control group did not. However, the intervention could not unequivocally be concluded as the cause of the improved performance, as there was no significant difference between the post-test scores of these two groups.

1. Introduction

By means of quantitative data analysis, this paper reports on the effectiveness of an intervention based on image functions in improving participants' understanding of the concept of function.

The mathematical concept of function remains problematic and the problem is quite prevalent, from high school learners and university students to teachers. It is also reported to occur in many countries all over the world. This is evident from the many research articles conveying this message over the last thirty years and up until the present day (Carlson *et al.*, 2002; Chimhande *et al.*, 2017; Doorman *et al.,* 2012; Dubinsky & Wilson, 2013; Harel & Dubinsky, 1992; Son & Hu, 2015). The researcher's own experience of more than seventeen years of university teaching also confirms what the literature reports on students often lacking a comprehensive understanding of the function concept. They are missing what Thompson (1994:39) refers to as the 'core concept', namely that which is consistent/equivalent/unchanged between the various possible representations of any particular function.

The rationale for this study is entrenched in the fundamental importance of the function concept. Modern mathematics and its application have become ever more dependent on the thorough understanding of the many aspects of functions (O'Shea *et al.*, 2016:279; Selden & Selden in Harel & Dubinsky, 1992:1). According to Dubinsky and Wilson (2013:86) too little has been done in implementing the abundant theoretical analyses of the difficulties with the function concept. This intervention study is one attempt in developing a workable pedagogy for assisting with the deep understanding of the function concept. Two examples of prior attempts that showed promise include the use of programming environments (Breindenbach *et al*., 1992) and the study by Reed (2007) where participants were led to engage with the history of the function concept and its development. Further research is still needed as the diversity of settings and of participants is so substantial that it cannot be expected that a blanket approach can be sufficient.

Significant to this research is establishing if a quantitatively measured improvement in function concept understanding can be observed as the result of an intervention based on getting to know and working with image functions. If this improvement can be observed, the study will expand the pedagogical content knowledge of mathematics (Ball *et al.*, 2008; Hoover *et al.*, 2016) and in particular, with respect to the teaching and learning of functions in the higher education setting. The Image Functions Intervention (IFI) used in this research was developed as the first part of a larger study. The IFI was firstly analysed and showed to be valid from a theoretical perspective. Secondly, the intervention was administered in a classroom setting followed by a questionnaire collecting qualitative data. The analysis of this data showed that in principle at least, the intervention could have a positive effect on participants' function comprehension and in particular, the intervention broadened the participants' perception of what constitutes a function. This is relevant as realising what and what not constitutes a function, is seen as one of the common conceptual difficulties associated with the function concept (Dubinsky & Wilson, 2013; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). The theoretical analysis and the positive results obtained after the first implementation of the IFI provided the motivation for further investigation into the possible effects of the IFI.

The objective of this paper is to determine if and to what extent the IFI can contribute to improved performance on the Function Concept Inventory (FCI) developed by O'Shea *et al.* (2016). As the FCI was specifically developed to measure conceptual understanding of the function concept, an increase on the inventory scores will be considered as indicative of an enhanced understanding of the function concept.

2. Methodology

This paper formed part of a larger intervention study using a mixed-methods approach, with this paper being the quantitative leg of the study. A mixed-methods approach was used, as the larger study not only endeavoured to quantitatively investigate if an image functions intervention could boost function concept comprehension, but also tried to elucidate students' thinking. Combining the quantitative and qualitative approaches leads to a more complete understanding of the research problem (Creswell & Creswell, 2018:32).

The research for this paper followed a randomised control design with the classical pre-test-intervention-post-test model. The intervention was implemented in a self-directed online manner through the learning management system (LMS) used by the local higher education institution. In this paper, the influence of the intervention was evaluated quantitatively using the pre- and post-test scores on the Function Concept Inventory (FCI).

Convenience sampling (Maree, 2016:197) was used in order to gather an accessible population, namely the group of 91 second-year Calculus students at the local higher education institution. Using an accessible population was necessary, as it was not practically feasible to implement the intervention with groups at other institutions.

Initially all students in the class were invited to participate in the study. All who accepted the invitation then completed the FCI as the pre-test. Simple random sampling was then used with the aim to form two equally sized groups, the experimental and the control. Due to attrition, the final groups were not of equal size with the experimental group having ten participants and the control group only seven. Table 2 shows that the randomisation was still effective as there was no statistical difference between the two groups on the pre-test scores.

Immediately following the pre-test, the intervention programme followed for a period of three weeks, involving only the participants of the experimental group. In the week after the intervention was completed, the FCI was given to all participants again and as such, used as the post-test. Giving the same test reduces the instrumentation threat to internal validity. However, the possibility of participants using their knowledge of the test during the re-testing increases. For this reason, it was important to have some time pass between the pre-test and the post-test (Creswell & Creswell, 2018:223).

2.1. The intervention

A full exposition and analysis of the Image Functions Intervention (IFI) is given in Venter (2019). As such, only a summary of the IFI and the theoretical framework on which it depends is given here.

The theoretical framework used is that of APOS (action-process-object-schema) theory (Arnon *et al*., 2014). According to APOS theory, the understanding of a mathematical concept passes through stages or levels, but not necessarily in a linear fashion or along the same steps. A person's understanding typically must first reach the Action level. At this level, a function is only seen in terms of explicit steps, such as substituting one value at a time into a formula. Secondly, one can reach the Process level where it becomes possible to view the function mentally as producing a set of outputs from a defined set of inputs. This level also requires one to think about how the steps might be (inter)changeable and thinking about the inverse process. At the Object level, it becomes possible to consider the function as a single construct having global properties and one is able to comprehend and apply other actions and/or processes to the function. When the Schema level is reached, a coherent ability is present to ascertain the applicability of the concept to solve problems. In particular, one must be able to determine which stage or stages of conceptualisation are practically important in solving the problem. At the Schema level, links formed between the Action, Process and Object levels can be recalled and applied as and when necessary.

A key aspect of the APOS theory is the genetic decomposition (GD) of a mathematical concept. Arnon *et al.* (2014) describe the GD as a model set up to show the mental structures and mechanisms that someone might use while learning a mathematical concept. These structures are set up at the Action, Process, Object and Schema levels of understanding. The mechanisms of the GD are those that enable a person's understanding to transition between the different levels of conceptualisation postulated by APOS theory.

In Venter (2019), a GD for the function concept was determined. Then the IFI was analysed from an APOS theoretical perspective and showed to be linked to the mental structures determined in the GD. The IFI was also showed to be addressing common conceptual difficulties of the function concept and forming an experiential base for the aspects of the function concept to be studied (Dubinsky & Wilson, 2013:85-86; Salgado & Trigueros 2015:107).

The IFI is based on the use of image functions put in the context of a dilemma relating to the disappearance of a student. This context of a missing person's case was chosen to help in getting the participant interested and motivated to learn.

Participants are firstly led to realise that a photograph or image can be considered as representing a function. An image function is seen to have as inputs the positions of the individual picture elements or pixels that make up the image. The output at any position is the unique colour of the pixel at that position. The IFI then proceeds, through various activities, to explore the concept of function and to address the conceptual difficulties associated with the function concept. The activities are designed to let the participant firstly establish confidence at the Action level of understanding. Subsequent activities lead the participant to discover the function concept at the Process level and the Object level. The Object level of conceptualisation is the highest goal of the IFI. In APOS theory, the mechanism for moving from the Process level to the Object level is called encapsulation. This is achieved by letting actions or processes transform the function, while the function is considered as a process. The IFI implements the encapsulation by means of function composition activities. The function compositions stretch the contrast of an important photograph associated with the missing person. In doing so, sufficient information could be retrieved from the photo to solve the case. Asiala *et al.* (1996) reported that actions and processes that transform functions as processes are mostly missing from the experience of learners of the function concept. The conclusion drawn (Asiala *et al.*, 1996:8) was that the difficulty in reaching the Object level, is attributable to this missing experience. The IFI, through its function composition activities, tries to fill this experience gap to enable the necessary encapsulation.

2.2. The Function Concept Inventory

O'Shea *et al.* (2016) developed the Function Concept Inventory (FCI) (see Appendix B). Their aim with the FCI was firstly to provide teachers and lecturers an instrument by which they could gauge their learners/students' conceptual understanding of the function concept. The second aim was to provide researchers a way of evaluating the effects of different pedagogies or interventions on the conceptual understanding of participants. An inventory is different from a typical test used for assessment in that it is intended and thus designed in such a way as to specifically measure conceptual understanding (O'Shea *et al.*, 2016:281). The FCI was validated (O'Shea *et al.*, 2016) as an appropriate instrument to determine the level of understanding a student has with respect to the function concept. The validity and reliability of the FCI was determined first by pilot studies and subjectively by subject experts' consensus. Furthermore, Rasch Analysis (Bond & Fox 2007) was used by the FCI's creators to validate the inventory in terms of the test items combining to test a single construct, namely the trait of conceptual understanding of functions.

Before using the FCI, its appropriateness needed to be verified. Firstly, as the FCI was not adapted for this paper, the FCI was accepted as still internally valid. Secondly, this paper was not aiming to gauge the participants' actual conceptual understanding, but only look for increased performance on the FCI after the intervention. Therefore, the possibility of having different settings and/or populations would have minimal risk to the appropriateness of the FCI as data collection instrument.

2.3. Data collection and analysis

The FCI was completed online, as the pre-test and again as the post-test, through the learning management system (LMS) used by the institution. Participants answered ten multiple-choice questions with a maximum total score of twenty.

In representing and analysing the quantitative data, both descriptive and inferential statistics were used, with the aid of IBM SPSS Statistics for Windows, version 25.0 (2017 IBM Corp., Armonk, NY). With less than twenty students in both the control and experimental groups, the Mann-Whitney U-test was used to test for significant differences between the groups. The Wilcoxon signed rank test with the exact method was used to test for significant in-group changes. These two non-parametric tests are applicable here as the groups are small and the dependent variables in question, namely the FCI score and the gain score, cannot be assumed to have normal distributions (Kvam & Vidakovic, 2007:131). To qualify as small groups, it is noted from literature that 30 participants are often considered the cut-off (Dzikiti & Girdler-Brown, 2017:41; Harris *et al.*, 2008:1491; Maree, 2016:251). With the small groups and the non-parametric tests used in this paper, the disadvantage is with respect to the power of the test. The non-parametric tests are less efficient (Kvam & Vidakovic, 2007:3) than their parametric counterparts. As such, the probability is higher that the null hypothesis (no difference between the groups) might not be rejected although being false.

Effect sizes were also calculated to assist in drawing conclusions on the practical significance of any possibly statistically significant differences observed (Maree, 2016:233). Effect sizes were calculated as Pearson's correlation coefficient (r) using the formula (Mononen & Aunio, 2014:462; Rosenthal, 1994),

$$r = z/\sqrt{N},$$

where z is the z-score obtained from the test and N is the total number of observations. As a general rule of thumb, Cohen (1988) gives us that an r = 0.1 would be interpreted as a small effect size, r = 0.3 as medium and r = 0.5 as large.

This study was given ethical clearance by the General Human Research Ethics Committee at the institution where the research was conducted.

3. Results and discussion

Thirty-five participants completed the FCI as the pre-test. Of the initial 35 participants, only 17 completed the pre-test and the post-test. As such, only the data of those 17 participants were used for analysis. Table 1 shows the demographics of the two groups, as well as combined.

Table 1: Demographics of participants

	Experimental (n = 10)	**Control (n = 7)**	**Combined (N = 17)**
Mean age	20.51	20.14	20.36
Gender			
Male	9	5	14
Female	1	2	3
Race			
Black	7	4	11
White	3	3	6

Table 2 shows firstly the comparative results in the pre-test between the experimental and control groups. It is important that these two groups are as similar as possible before the intervention. The Mann-Whitney U test showed no statistically significant ($p < 0.05$) differences between the two groups on the pre-test FCI scores. Secondly, Table 2 shows the post-test and gain score means and standard deviations. Gain scores were calculated as the difference between the post-test and pre-test scores for each individual participant.

From table 2 we see that there are no significant differences across the two groups between the post-test scores or with respect to the gain scores.

Table 2: Means, standard deviations and effect sizes for the FCI.

	Intervention (I)	**Control (C)**					
	M (SD) n = 10	**M (SD)** n = 7	**U**	**z**	**Asymp. Sig. (2-tailed)**	**Between group compari-sons**	**Effect size (r)**
Pre-test	13.734 (3.097)	14.143 (2.129)	31.50	-0.342	0.732	I = C	-0.08
Post-test	14.967 (3.750)	15.193 (2.785)	34.50	-0.49	0.961	I = C	-0.12
Gain scores	1.233 (1.572)	1.050 (2.398)	29.5	-0.541	0.588	I = C	-0.13

Comparisons on the FCI (max 20 marks) pre- and post-test by group, using the Mann-Whitney test with $\alpha = 0.05$.

To consider the growth of participants within a group, the Wilcoxon signed rank test with the exact method (Maree, 2016) was used. Table 3 shows the results of this test on the data of the intervention group and the control group.

Table 3: In-group pairwise comparisons.

	Pre-test M (SD)	**Post-test M (SD)**	**z**	**Asymp. Sig. (2-tailed)**	**Pairwise Comparison**	**Effect size (r)**
Intervention group (n = 10)	13.734 (3.097)	14.967 (3.760)	-1.863	0.078	Post > Pre	-0.42
Control group (n = 7)	14.143 (2.129)	15.193 (2.785)	-0.734	0.563	Post = Pre	-0.20

In-group comparisons between pre-test and post-test results for the FCI (max 20 marks) using the Wilcoxon signed rank test with $\alpha = 0.05$.

For the intervention group, if we use the directional alternative hypothesis that the post-test mean is larger than the pre-test mean, we obtain a p-value of 0.078/2 =

0.039 < 0.05. Thus, there is a significant difference between the pre- and the post-test results for the intervention group. For the control group we find no significant difference between pre- and post-test results. From the intervention group (n=10), six participants improved their FCI score after the intervention, three obtained the same score pre- and post-intervention and one participant scored slightly less post-intervention. From the control group (n=7), four participants scored higher on the post-test than the pre-test, one obtained the same score and two participants scored less.

In the follow-up research conducted by O’Shea *et al.* (2016), 217 participants from various backgrounds completed the FCI. The results obtained in that study is insightful when compared to what was observed in this paper. Firstly, they reported difficulty measures for each of the multiple-choice questions from the inventory. The three questions they found to be most difficult (questions 5, 8 and 9 in Appendix B), were all taking spots in the top four most difficult questions in the results of this paper. Most difficult here corresponds to lowest average mark across all participants. The same was observed if participants from the intervention and the control groups were considered separately. In the results of this paper, the single question (question 12 in Figure 1) that obtained the lowest average mark was however not picked up as particularly difficult by the research of O’Shea *et al.* (2016). The question that participants from both the intervention and the control groups scored the lowest on was the very last question of the inventory, shown in Figure 1.

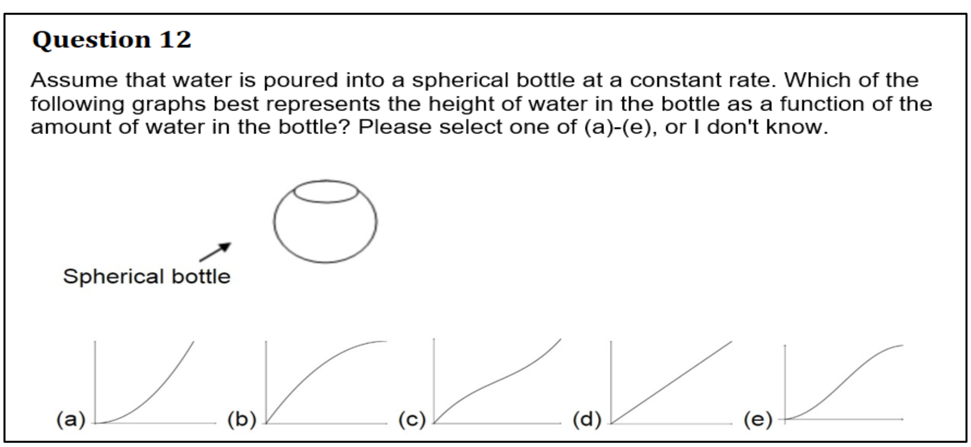

Figure 1: Question from the FCI that achieved the lowest average mark.

After the intervention, the situation was unchanged with the question in Figure 1 still seemingly being the most difficult. There was no increase in the average mark obtained for this question. This question deals with co-variational reasoning, which O'Shea *et al.* (2016) regarded as one of six elements of understanding associated with the function concept. Carlson *et al.* (2002:353) reported that co-variational reasoning was an aspect that 'even academically talented undergraduate students have difficulty with'. This difficulty is clear among the participants in this paper's research. Further research is needed to investigate if this difficulty is prevalent among learners and students in South Africa. This would be in contrast with the groups of students from Ireland and the UK where the O'Shea *et al.* (2016) study was conducted. If this aspect of function understanding is found to be particularly prevalent for some populations, it is an aspect that should be addressed in future versions of the IFI.

The easiest question of the inventory, as identified by O'Shea *et al.* (2016), was question 3. This question deals with an increasing function staying increasing under certain operations. This question was also determined to be the easiest for participants from the intervention and control groups of this paper. Furthermore, no change was observed with respect to the average mark for this question from before to after the intervention.

Special mention must be made of questions one, ten and eleven (Figure 2) from the inventory. These questions were the only ones for which the average marks increased post-intervention. However, this increase was only observed with the intervention group's participants. For questions ten and eleven, the average marks of the participants of the control group were unchanged while for question one, the average mark decreased. Care should be taken not to conclude too much from these improved average marks, as considering only these positive results would be biased. However, if we consider the elements of understanding that these questions deal with, namely distinguishing between functions and equations and handling the formal definition of function, it is encouraging to see the participants of the experimental group's improvement. In terms of APOS theory, these aspects concern the transition of function understanding from the action level to the process level. The IFI actively encourages this transition through specific activities.

Q1. Let f(x) =3x+5. Given the equation 3a+5=2, which of the following are true? (There may be more than one true statement.)

(a) f(a)=3a+5,	True	☐	False	☐
(b) f(a) = 2,	True	☐	False	☐
(c) f(x)=2 for all x,	True	☐	False	☐
(d) f(x)=2 for some value of x.	True	☐	False	☐

Q10. Suppose f(x) is a function defined for all real values of x. For each of the following statements decide if the statement is always, sometimes or never true.

i. There are two different real numbers a and b such that f(a) = f(b).

Always ☐ Sometimes ☐ Never ☐

ii. There are three different real numbers a, b, c such that f(a) = b and f(a) = c.

Always ☐ Sometimes ☐ Never ☐

Q11. Suppose f(x) is a function such that f(3) = 5 and f(4) = 12. Which of the following is true? (Please circle one of (a)-(d), or (e).)

(a) f(7) = 5+12 = 17, (b) f(7) = f(3) + 4 = 9, (c) f(7) = 5(12) = 60,

(d) It is not possible to find the value of f(7) from the information given.

(e) I don't know.

Figure 2: Questions from the FCI on which participants showed improvement.

4. Conclusions

This paper reports on the very first attempt of using the Image Functions Intervention (IFI) in its intended form to effect participants understanding of the function concept. Participants used the IFI in a self-directed manner through an interactive online package delivered through a learning management system. Participants' understanding was measured as a quantitative result on the Function

Concept Inventory (FCI) that was developed by O'Shea *et al.* (2016). The FCI was in part developed to be used in studies such as this one where there is the need to test conceptual understanding of functions.

In using the FCI, high attrition of participants was experienced. This was problematic as it meant final sample sizes were small. From a sampling frame of 91 possible participants, 51 started the FCI as pre-test, but only 35 completed it. Furthermore, from the 35 that completed the FCI as pre-test, only seventeen eventually completed the FCI a second time as the post-test. Possible explanations for the high attrition are that participants were experiencing time-pressure, as the semester tends to get busier towards the end, which coincided with the post-test. Another possibility is that participants might have experienced the FCI as hard work and therefore they might need more motivation to complete it. The fact that four participants (two each from the intervention and control groups) scored lower the second time they attempted the FCI, might corroborate the idea that participants lacked motivation while attempting the FCI. For the efficient use of the FCI, these and other possibilities need investigation. Using a much larger sampling frame could help to deliver larger numbers of completed pre- and post-tests. Some form of incentive might be needed to motivate participants to complete the FCI to the best of their ability.

The use of the FCI should however be encouraged as it showed potential in assisting a teacher/lecturer to identify specific elements of function understanding that a group of learners/students might be needing help with. Therefore, the FCI can be used as a valuable diagnostic tool.

With regard to the objective of this article, a statistically significant improvement (pre- to post-intervention) was observed on the FCI scores of the intervention group's participants. Thus, the intervention demonstrated a tendency to effect function understanding positively. However, this improvement cannot be unequivocally established as caused by the intervention, as no statistically significant difference was observed between the scores of the intervention and the control groups post-intervention.

Although the intervention group showed a statistically significant improvement, with a medium effect size of $|r| = 0.42$, the effectiveness of the IFI could be masked in this study by the small number of participants. In follow-up research, substantial effort should be made to increase the sampling size. This would make the statistical tests more sensitive to the effects of the intervention (Kvam & Vidakovic, 2007). It would then also be apt to investigate participants' experience with regard to the FCI. Also needed in follow-up research is to introduce the actual time spent by participants on the intervention, as a second independent variable.

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ARTICLE 3 – Developing an Object Conception of Function: An Intervention Study with Qualitative Analysis

Abstract

This paper attempted to contribute to the knowledgebase dealing with the problematic learning and understanding of the mathematical concept of a function. The function concept has been described as an epistemological obstacle. The objective was to determine the effects of an intervention based on considering images as functions. Specifically, this paper investigated if the intervention could assist participants to reach an object level of conceptualisation. Qualitative analysis was used to determine participants' (n = 10) levels of conceptualisation before and after the intervention. The results showed that specifically the definition of a function was problematic for participants. Furthermore, improved conceptualisation was observed with some participants. Importantly, the formal definition of the function concept was identified as a possible obstacle towards higher levels of conceptualisation.

Keywords: function concept, APOS theory, intervention, qualitative analysis

1. Introduction

> "The keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts – it is rather a complete emancipation from such notions." (Schaaf, 1930:500)

The quote above was published approximately *nine* decades ago in 1930. For a very long time then, it has been known that the function concept is of great importance to mathematics. Yet it has not been long enough to circumvent the high prevalence of inadequate understanding of functions. Sierpinska (1992) described the function concept as an epistemological obstacle. The problem with learning and fully understanding the concept has been persistent over a long time and widely occurring.

This paper investigated the effects of exploring and working with image functions, that is considering images or photographs as representing functions. Specifically, the research was aimed at exploring the effects of an intervention based on image functions and on the understanding of the mathematical concept of a function. The effects of the intervention were investigated by means of qualitative data analysis. The intervention was developed (Venter, 2019) to adhere to the APOS (action-process-object-schema) theoretical framework (Arnon *et al.*, 2014) of how the learning of mathematical concepts takes place. In Venter (2019), the Image Functions Intervention (IFI) was firstly analysed from a theoretical perspective and found to adhere to the genetic decomposition (defined in the Theoretical framework section) of the function concept. Secondly, an initial analysis of the effects of the intervention showed that participants might obtain a broadened concept image, specifically pertaining to what constitutes as a function. This aspect is one of the core conceptual difficulties reported by Dubinsky and Wilson (2013:86). The concept image is defined (Tall & Vinner, 1981:151) as the total of all aspects, definitions, anecdotes, problem situations and more that someone might associate with the concept. Therefore, when confronted with a problem involving functions, it is of key importance what the person has incorporated into his or her concept image.

Although the initial analysis (Venter, 2019) showed a broadening of the concept image, it could not independently provide evidence that the IFI improves function comprehension from an APOS perspective. This paper continues and expands the investigation of the effectiveness of the IFI.

The Research Question

To what extent can the Image Functions Intervention (IFI) assist in the development of an object conception of function over the short and long term?

To answer this question, this paper has two objectives:

1. Determine if a shift can be observed in the number of students with an action or process conception of function to an object conception of function.
2. Determine if the intervention can have a lasting positive effect on function comprehension, thus maintaining the object level of conceptualisation.

2. Theoretical framework

2.1. APOS theory

Action-Process-Object-Schema theory is a theoretical framework and a research methodology (Arnon *et al.*, 2014). It has its origins in the reflexive abstraction of Piaget (1971), but its current form can be traced back to research and papers such as Dubinsky (1991), Breidenbach *et al.* (1992), Asiala *et al.* (1996) as well as Dubinsky and McDonald (2001).

In this paper, APOS theory is used as a theoretical framework to provide a way of modelling the learning of a mathematical concept. Using this framework, the learning of a concept is said to evolve through stages or levels (Arnon *et al.*, 2014). The first is the action level. A student at this level needs external stimuli and cannot yet construct or work with the concept entirely in the mind. Something external such as an expression or equation is required. At the second level, the concept is understood as a process. At the process level, it becomes possible to construct the concept in the mind and thinking of the underlying actions that make up the process without actually performing any of these actions. Next, the learner must reach the object level. Now the process is said to have been encapsulated and it becomes possible to think about other actions and processes that might be performed on this object. The last level is the schema level. At this level, a student will be able to move freely between considering the concept as an action, process or object and will be led by the requirements of the particular problem at hand.

As an analogy, we can consider the concept of cake. If you are attempting to create the cake and simply read off each instruction, and follow it as best you can, you are at the action level. If in your mind you keep track of the end product and how each step is contributing to the end product, you are at the process level. Especially if you know which steps can be swapped without the result differing or you know the effect of omitting a specific step. Once you are able to think about the finished cake and its possible attributes, you are entering the object level. You would then be able to think about other actions and processes that could transform the cake. For example, cutting the cake in pieces or freezing the cake. At the schema level, you can start solving problems by thinking about the cake at different levels. If you find that the cake is too dry, you might need to think about the cake at the action and process levels to find the error in the recipe or procedures. If you need to classify different cakes based on appearance or you have to transport an enormous cake for a birthday, you will need the object level. To say you truly understand the concept of cake, one would expect you to be comfortable in thinking and working with cakes at the action, process and object levels, and thus be operating at the schema level.

2.2. The function concept: APOS conception levels

A key aspect of the APOS theoretical framework is the genetic decomposition (GD) of a concept. The GD conveys the mental structures, at the different APOS levels, that are needed to learn the concept. Furthermore, the GD provides the mechanisms that will enable the transitioning between the levels (Arnon *et al.*, 2014).

Using the GD of the function concept (Venter, 2019) and other research literature, the indicators in Table 1 were set up to enable the data analysis of this paper and the consequent placement of individual participants. The focus of this paper is only up until the object level. Furthermore, the detection of the schema level seems overly ambitious and therefore the indicators in Table 1 and the instrument specific indicators given later only go up until the object level.

Table 1. Indicators and counter-indicators of APOS level attainment.

General definition	Indicators of the level	Counter-indicators of the level
Action A student at the Action level is restricted to transforming mathematical objects by using external prompts such as formulas or expressions. Steps or instructions are needed (Asiala *et al*.,1996; Dubinsky & Wilson, 2013).	• Ability to substitute numbers into an expression and calculate (Breidenbach *et al.*, 1992). • Able to compose functions given by simple formulas (Breidenbach *et al.*, 1992). • Can recall the definition of a function (Breidenbach *et al.*, 1992).	Before the action stage, a student is said to exhibit a pre-function response to questions such as: "What is a function?" (Breidenbach *et al.*, 1992). No useful conveying of function ideas is present.
Process A student at the Process level constructs the function mentally and can realise the complete transformation of elements from the domain, to elements of the range. No external prompts are needed (Dubinsky & Wilson, 2013; Arnon *et al.*, 2014).	• Understand general composition of functions and reversal of functions (Dubinsky & Wilson, 2013). • Provides a definition of a function that includes mention of the inputs, outputs and a rule (Arnon *et al.*, 2014). • Can realise which steps of the process can be swapped or even left out for particular cases. • Ability to determine whether a function has an inverse as the reversal of a process (Arnon *et al.*, 2014).	• Difficulty interpreting a situation as a function unless a formula is given (Asiala *et al.*,1996). • Function composition is too difficult in atypical situations. For example, in the absence of formulas or with piece-wise defined functions (Breidenbach *et al.*, 1992). • Commonly not using the definition when confronted with problems (Tall & Vinner, 1981).
Object Through transforming the process by actions and/or other processes, the process is encapsulated to become an object. The dynamic process becomes a static entity (Asiala *et al.,* 1996; Dubinsky & Wilson, 2013; Arnon *et al.*, 2014).	• Able to think about and convey the global properties of a particular function or a type of function, for example periodic, smooth or constant, monotonic (Venter, 2019). • Able to form sets of functions, perform operations on functions and even 'construct a function that is a limit of a sequence of functions' (Arnon *et al.*, 2014).	• Cannot easily move between different representations of the function. • Considers a piece-wise defined function as consisting of multiple functions. • Struggles to create a function example that is connected with some real-life situation (Chimhande, Naidoo & Stols, 2017:5).

2.3. The function concept at the object level

In the cake analogy considered earlier in the theoretical framework, the object level is probably most well known to everyone and it would probably be safe to say that everyone understands cakes at the object level. The action and process levels would be considered the more difficult levels.

When it comes to the function concept, the opposite is often true. Chimhande, Naidoo and Stols (2017) found five or six learners to be operating at the action level and the sixth at the process level. Breidenbach *et al.* (1992) reported that of 62 students in their study, approximately 40% were performing at the action level and the other 60% were actually still at a pre-function level. A pre-function level implies no indication of any function concept understanding whatsoever. Dubinsky and Wilson (2013:86) reports on more research that confirms that the action level is unfortunately quite dominant.

So why is the object level so elusive when it comes to the function concept? Asiala *et al.* (1996) reports that going from a process understanding to an object understanding is achieved by encapsulating the process. This is achieved by reflecting on situations where actions or other processes are performed on the original process. This is 'considered to be extremely difficult'. The explanation that Asiala *et al.* (1996:8) offers for this difficulty is that we do not have much experience of actions being performed on what are interpreted as processes. For the function concept, Arnon *et al.* (2014:21) reports that 'encapsulation allows one to apply transformations of functions such as forming a set of functions, defining arithmetic operations on such a set and equipping it with a topology'. These actions are performed, not on a process, but directly on the objects. Furthermore, a learner of the function concept would only encounter these topics quite late along the learning path and only in more advanced mathematics courses. This confirms the problem mentioned by Asiala *et al.* (1996:8) that the learners of the function concept are not gaining experience in performing actions on functions as processes and are thus denied the necessary encapsulation.

The intervention used in this study (the IFI) has as its primary objective the development of an object understanding of the function concept. The necessary encapsulation is addressed by the IFI by using function composition. Function composition is considered as a process acting on another process (Asiala *et al.*, 1996; Breidenbach *et al.*, 1992) and is thus exactly what is prescribed for the encapsulation to take place.

3. Research design

A qualitative design based on textual analysis is used for this paper to enable the detection of the effects of the intervention on the true understanding of the function concept within the APOS framework. It is by means of qualitative study that the *nature* of the improved understanding can be investigated (Hesse-Biber & Leavy, 2011:151). Dubinsky and Wilson (2013), Chimhande, Naidoo and Stols (2017) and recently, Borji and Martinez-Planell (2019) showed the successful use of qualitative data to analyse participants' level of conceptualisation.

Within the APOS framework, improved understanding means that we should look for movement along the action-process-object-schema line. Care should however be taken when considering the APOS levels, in the sense that an understanding at all APOS levels is required. Therefore, an understanding of the function concept at the object level for example, does not eliminate the need for a participant to be able to understand functions at the action and process levels.

Textual analysis was used on three instruments to detect the respective APOS levels of comprehension presented by participants. The first instrument was the open-ended questions on the Function Concept Inventory (O'Shea *et al.,* 2016). Participants completed the FCI before and after completing the Image Functions Intervention (IFI) based on the study of images/photographs as functions. The second instrument is an open-ended questionnaire given to participants who had completed the IFI. The third instrument was a question taken from the final summative assessment of the participants in the Calculus module they were following.

3.1. The intervention

The intervention used in this paper is based on the introduction and transformation of images or photographs as functions. The Image Functions Intervention (IFI) was designed and analysed from a theoretical perspective to see if it adhered to the learning envisioned by the APOS framework (Arnon *et al.*, 2014). It was determined that the IFI follows APOS principles and could be considered theoretically sound (Venter, 2019).

The IFI is cast in the context of a missing person's case. This context then provides the motivation for learning how image functions work and how they can be transformed in order to play a part in solving the case.

At first, the participant is led to discover that an image or photograph can be considered as representing a function. The input to such a function is the position of a pixel and the corresponding output is then the colour of that pixel. The position is typically given as a two-element vector <x,y> with x and y denoting the row and column numbers respectively. The colour of the pixel is typically portrayed by a three-element vector <r,g,b> denoting the specific combination of red (r), green (g) and blue (b).

Thereafter the participant is exposed to specifically designed activities. They are designed to adhere to the genetic decomposition of the function concept and to provide an exploratory base for addressing the conceptual difficulties associated with the function concept. The IFI builds toward activities dealing with transformations of images using function composition. These function composition activities allow processes to act on the image function, with the image function itself considered a process. Through these activities, the intervention attempts to provide the encapsulation mechanism whereby the participant's function comprehension can evolve. In terms of APOS terminology, the participant is therefore led to an object conception of function, which is the primary objective of this paper.

3.2. Selection of participants

For this study, convenience sampling (Maree, 2016:197) was used to recruit participants from a group of 91 second-year students registered for a second year Calculus module at a local university in the Free State, South Africa. This group was chosen as all possible participants would have prior experience with the function concept and all participants could be given access to the intervention, which must be delivered online, in a controlled fashion. The group to which the participants belonged was not studying the function concept per se in any of their modules. This group consisted of students that were either studying Mathematics as a major or were registered for qualifications where Mathematics is required to a high level, for example Actuarial Science, Engineering and Computer Science. The function concept was therefore fundamental towards the further learning of the particular prospective participants.

3.3. Data collection methods and instruments

Three different instruments were used for data collection. From the potential 91 participants in the sampling frame, only ten participants completed the intervention. This paper will report only on the data from these ten participants.

The first instrument used was the Function Concept Inventory (FCI) developed by O'Shea *et al.* (2016). The FCI was developed with the purpose of evaluating participants' conceptual understanding of the function concept. Mostly it consists of closed-ended questions meant for quantitative analysis, which was conducted and submitted for publication. For this paper however, the three open-ended questions from the FCI were used as they give the opportunity for qualitative analysis. After completing the FCI, the participants completed the intervention – that is the IFI. Following the IFI, they completed the FCI a second time.

The second instrument used was a short questionnaire with three open-ended questions pertaining to the function concept. This questionnaire was set up to gather further qualitative data to be used to determine the APOS comprehension levels of the participants.

The third instrument was completed nine weeks after the completion of the questionnaire (data instrument 2). Participants wrote their final summative assessment examination pertaining to the Calculus module the students were following. One question from this examination was considered directly applicable to the function concept and was therefore chosen for qualitative data collection. The purpose of this last data instrument is to determine if participants could retain, after a longer period has passed, a higher level of function conceptualisation.

Data instrument 1 (administered week 1 and week 4)

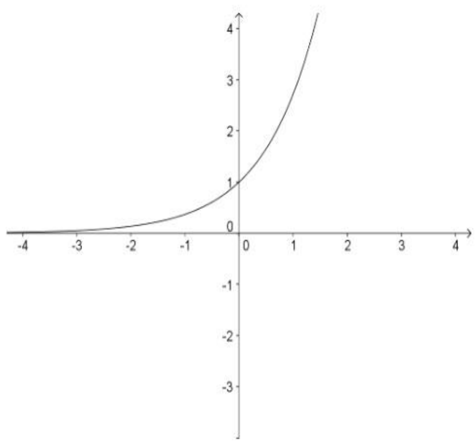
The three questions from the FCI:

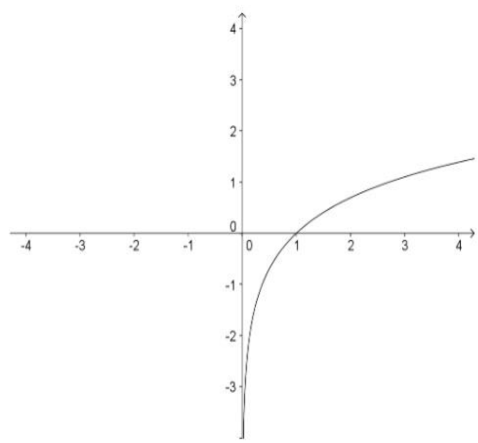
Question 1: *Look at what you see in a), b) and c) below and write in words a description of each of them separately using any terminology that you know. [You might use some or all of the words function, equation, solution, graph].*

a) $y = x^2-5x+6$
b) $(x-2)(x-3) = 0$
c) $(x-2)(x-3) = -1$

Write down as many ways as you can in which they are related to each other.

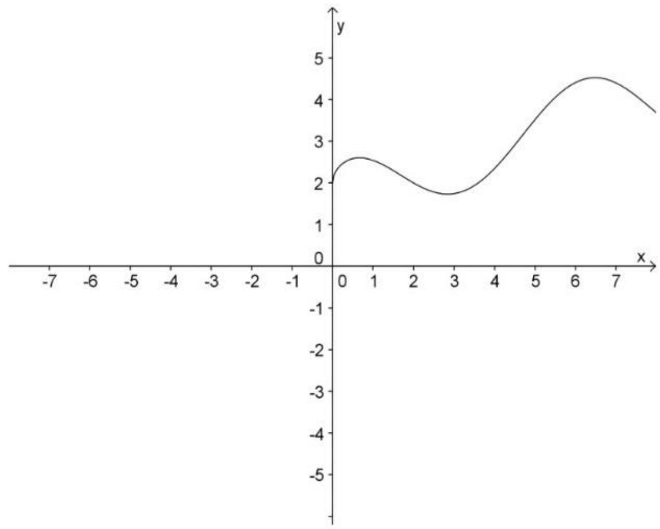
Question 2: *Look at the graphs in (i) and (ii) below.*

(i)


(ii)


The claim is that Graph (ii) represents a function which is the inverse of the function represented by graph (i). Explain what this means and state whether you think the claim is true.

Question 3: *Look at the graph given below and imagine, on the same axes, a graph to represent its inverse if you think there is an inverse. Explain clearly why you think there is an inverse or not an inverse.*



Data instrument 2 (administered week 5)

The three questions from the questionnaire given after the intervention:

Question 1: In your own words, give a definition of what a function is.

Question 2: Write down any example of a function.

Question 3: Give an example of a relation that would not qualify as a function.

Data instrument 3 (administered week 14)

The single question chosen from the examination:

Question: *(part A) Explain what a piecewise continuous function is and (part B) also explain how the convergence of such a function's Fourier series is defined at a jump discontinuity of the function.*

3.4. Data analysis

In the analysis of the data, we are looking for any indications that a participant has understanding at the action, process or object level. None of the data instruments used in this paper was deemed applicable to the schema level and as such we are not looking for indications with respect to the schema level. Furthermore, in the literature consulted, the schema level is not reported as being reached by any participants and it is therefore regarded as overly ambitious to look for such indications here.

From the general indicators given in Table 1 in the Theoretical Framework section, indicators that are more specific were set up for each question of each data collection instrument. The purpose is to provide a way to use each question to classify the participant's comprehension as belonging to a specific APOS level.

Data instrument 1

Question 1

Action: *Mentions that a value must be substituted to get an output value. Mentions typical textbook ideas such that 'it is a parabola' or 'a quadratic function'.*

Process: *Makes the connection regarding domain and range and can see that "b" and "c" are special cases of "a", with a specific output value chosen.*

Object: *Mentions that "a" is an equation that we can, if needed or required, interpret as representing a function. The equation itself is not the function. It is merely one way of representing some function that we still would need to describe or define.*

Question 2

Action: *Only notices that the two graphs seem to be reflections of one another about the line $y = x$. No attempt or mention is made of a possible verification of this suspicion.*

Process: *Can realise that for any point (x,y) on graph (i), the point (y,x) seems to be on graph (ii).*

Object: *Mentions that graphs (i) and (ii) seem to be reflections about y = x. Sees that (i) and (ii) has swopped domains and ranges. Conveys a sense of visual verification of the reflection about y = x.*

Question 3

Action: *Only mentions the use of the horizontal line test, without explanation. Mentions a point-wise construction of the potential inverse and possible verification of the result.*

Process: *Can explain that the inverse relationship would be a one-to-many correspondence and thus not a function.*

Object: *By visualising the reflection of the curve about the line y = x, one could see that the reflection could not be the graph of a function with x as input variable and y still as output variable.* (Note: A curve that fails the vertical line test can possibly still be representing a function. For example, $x = y^2$ can be the equation of a function, if one considers values of y as the input.)

Data instrument 2

Question 1

Action: *Responds with some version of a memorised definition of the function concept, but there is a tendency to omit key aspects or using inappropriate terminology.*

Process: *Responds by considering the inputs, outputs and mentions a rule, relationship or correspondence and emphasises the uniqueness aspect of function values.*

Object: *Note: The typical textbook definition of a function is at the process level, so going beyond that would be unexpected.*

Question 2

Action: *Responds with only a formula or an expression with no regard to domain and range.*

Process: *Conveys the rule or correspondence with an indication of the domain and the range.*

Object: *Elaborates on the example by giving applications of the function and/or classifying it.*

Question 3

Action: *Provides a typical textbook example without any qualification.*

Process: *Gives an example and shows how the univalence requirement is not satisfied.*

Object: *Elaborating on the example by explaining how the failing of the univalence requirement can be understood within the context of some application or real-life situation.*

Data instrument 3

Part A

Action: *To understand the concepts in this question and explain them, your understanding of the function concept needs to be at the process level at least. Any responses that seem out of place will be classified as at the Action level.*

Process: *Mentions the type of discontinuities that could be present and the one-sided limits that would exist at the positions of the different discontinuities.*

Object: *Responds by at least mentioning the domain being split and the different relationships or rules that could possibly be present on the different subdomains.*

Part B

Action: *Gives the formula for the sum of the Fourier series at the jump discontinuities without explaining anything.*

Process: *Gives the formula for the sum of the Fourier series at the jump discontinuities and explain, as a general construct, how the values are determined.*

Object: *Can relate how the sum of the Fourier series would be equal to the function, except at any jump discontinuity. At the jump discontinuity, the sum of the Fourier series will be equal to the average of the two one-sided limits about the discontinuity.*

3.5. Procedure

For each participant, his/her responses, before and after the intervention, to the open-ended questions of the FCI (data instrument 1) were carefully read and reread to look for indications of exhibiting understanding at any particular APOS level. The indications were along the guidelines given for the particular data collection instrument determined earlier. A tally of all such indications (Dubinsky & Wilson, 2013:96) was kept and is reported in Table 2 with pre- and post-intervention results.

The same procedure for data instrument 1 was also followed with the questionnaire (data instrument 2). This was done with the purpose of seeing if the same conclusion would be reached regarding each participant's APOS level, compared to the conclusion from the post-intervention analysis. Therefore, the questionnaire tries to confirm the post-intervention conclusions.

For the third data instrument, each participant's answer to the question was analysed and a decision made for each with respect to the APOS level attained. If a participant gave indications that matched with more than one APOS level, the highest level presented was assigned to the participant. The purpose of this third instrument was to investigate if there were any long-term effects on participants' understanding of the function concept.

4. Results and discussion

Table 2. Tally of indications at different APOS levels.

	Week 1					**Week 4**					**Week 5**					**Week 14**
	Pre-intervention					**Post-intervention**					**Questionnaire**					**Exam Question**
Parti-cipant	**Pre**	**A**	**P**	**O**	**Conclusion**	**Pre**	**A**	**P**	**O**	**Conclusion**	**Pre**	**A**	**P**	**O**	**Conclusion**	**Conclusion**
1		7	2		A		5	4		A/P		2	1		A/P	A
2		5	4		A/P		9			A					N	A
3		5	4		A/P		8	1		A		3			A	A
4		9			A		9			A	1	2			A	A
5		6	3		A/P		6		3	A/P/O		2	1		A/P	P
6	5	4			Pre/A	3	6			A	2	1			Pre/A	A
7			6	3	P/O			6	3	P/O		2	1		A/P	A
8		8	1		A		6	3		A/P					N	A
9		3	2	4	P/O		2	7		P					N	N
10		3	6		P		6	3		A/P					N	A

Pre – pre-function, A – action, P – process, O – Object, N – no participation

Prior to the intervention, seven participants (participants 1 to 6 and 8) were determined to be predominantly at the action level with three of these (participants 2, 3 and 5) said to be transitioning to the process level. One of the seven (participant 6 in Table 2) seems to still be in transition from a pre-function level to an action level. Taking into consideration that the participants are all in a second year Calculus module (Sequences and Series), this relatively high ratio of 70% at the action level or transitioning to a process level, is disappointing, but not entirely unexpected. As was mentioned in the Theoretical Framework, other research studies *also* found the action level to be most prevalent.

Post-intervention, four participants (participants 1, 5, 6 and 8) have improved slightly by moving into a transition phase to their next level. Important to notice is that only one participant, participant 5, has improved to a position that includes transitioning to the object level. Furthermore, it is noted that four participants (participants 2, 3, 9 and 10) seem to have regressed slightly. This regress is unexpected on a repeat on the same instrument and needs investigation. A possible explanation is that participants might have felt less motivated to invest the same amount of attention the second time.

Considering the questionnaire (data instrument 2), we note from Week 5 of Table 2 that of the six participants that completed it, 50% retained their post intervention levels (participants 1, 3 and 4) and the other 50% regressed slightly (participants 5, 6 and 7). In this case, the slight regress could be due to the difficulty of the questions themselves. At their essence, these questions are lying at least at the process level.

On the examination question (data instrument 3), Week 14 in Table 2 indicates that eight of the nine participants who completed the question, were judged to be at the action level. For five participants (participants 1, 5, 7, 8 and 10) this is a regress from their post intervention levels (Week 4 in Table 2). For participants 2, 3, 7, and 10 it is a regress from their pre-intervention levels (Week 1 in Table 2), which could imply that the question (data instrument 3) itself is at a higher difficulty level than the questions of data instrument 1. Considering again that the participants are second year Calculus students, the ratio of 89% operating at the action level is very high and confirms that the function concept remains an epistemological obstacle (Sierpinska, 1992:28).

When considering only the four participants (participants 1, 5, 6 and 8) that showed improvement from before to after the intervention (compare Weeks 1 and 4 in Table 2), it is notable that for three of these participants (participants 1, 5 and 8), the improvement was specifically for the second question of data instrument 1. Participants 1 and 8 improved from an action level to a process level on this question, while participant 5 improved from a process level to an object level. Consider participant 1's answers before and after the intervention:

<u>Before</u> the intervention:

An inverse of a function is a mirror of the function about the line y = x. (i) is a function and (ii) is the image of (i) about the line y = x. (ii) is the inverse of (i).

After the intervention:

An inverse of a function is a reflection of a function by the line y = x, f' is an inverse of function f if for every (x, y) element of D, $f(x_i) = y_i$ --> $f'(y_i) = x_i$. It is true that Graph (ii) is an inverse of Graph (i).

Although the after answer still has some issues, such as referring to an element (x,y) from an undefined set D, growth has certainly taken place here. The before answer only conveys a standard property of inverses, which is typical of the action level. In addition, no attempt at verification is made. In the after answer, we see the participant incorporating the necessary and sufficient requirements that the original function must be injective and surjective. What is still missing here is an attempt at verification. For participant 8, we saw similar growth taking place:

Before the intervention:

Graph in (ii) is the reflection of graph in (i) about the line y=x, which is the inverse of graph in (i)

After the intervention:

Graph (i) is reflected about the line y=x to give us the graph (ii), and this means that the coordinates of graph (i) being (x,y) in graph (ii) are (y,x), I think this claim is true because by taking point (1,0) in graph one you can see in graph (ii) is now (0,1).

In the before answer, participant 8, similar to participant 1, only conveys a standard property of inverses and without any attempt at verification. In the after answer there is an indication that the participant realises that every point can be considered as reflected individually but simultaneously, that is as a process. The participant attempts to verify this by using a specific point.

Now consider participant 5's answers before and after the intervention:

<u>Before</u> the intervention:

yes this is true both functions has an asymptote at the line y=x and both functions never cross into the third quardrant and they cancel each other out (i) is the exponential function that has an asymptote at x=0 (ii) is the logarithmic function that has an asymptote at y=0

<u>After</u> the intervention:

the first is a exponential and the second one its a natural logarithm functions they are inverses of each other for the exponential function it has an asymptote at the line y = 0 and the log function also has an asymptote at x = 0

In the before answer, the participant incorrectly states that the curves have the line $y = x$ as asymptote. Furthermore, stating that both *functions* 'never cross into the third quardrant [sic]', is confusing the *representation* of the function with the function itself. This disregards the fact that axes and quadrants are aspects specific to this particular type of representation and not general to the function concept. In the after answer, the participant classifies both graphs correctly as belonging to certain classes of functions and then uses knowledge of the correspondence between these two classes of functions to deduce that the second graph represents the inverse function of the function represented by the first graph. S/he also proceeds to identify the asymptotes of the two graphs correctly and seemingly notices that asymptotes have switched axes, as would be the case for inverses. This is because the domain and range of the function become the range and domain of the inverse function respectively.

Another notable occurrence to consider is the growth shown by participant 6 from before to after the intervention. Before the intervention, participant 6 was judged to be operating at predominantly the pre-function level and then made a large improvement after the intervention to be judged at predominantly the action level. Let us consider participant 6's answers to especially question 1 of data instrument 1.

Before the intervention:

a) is the parabola facing up b) factor of the parabola when you multiply it gives you a parabola c) this is the parabola plus one it has been factored.

After the intervention:

a) dependent variable of y is equal to independent x squared subtract independent x variable multiple it by five add six b) bracket inside independent variable x subtract two multiple by brackets inside a variable x subtract three it equal to zero, is a factorization c) product of inside the bracket independent variable x subtract two and inside bracket independent variable x subtract tree [three] it is equal to negative one

In the before-answer, stating that it is a 'parabola facing up', is considered to be at the action level as it is conveying some form of typical textbook knowledge. Parts b) and c) are nonsensical and convey no function comprehension. In the after answer, the participant is able to notice that the equation given in part a) can constitute a function by correctly identifying a dependent and an independent variable. The responses to parts b) and c) of the question are showing actions performed on the independent variable, but still cannot correctly relate these equations to the function identified in part a). There is also no reference to domain and/or range and therefore the participant is judged to operate at the action level after the intervention.

A theme that emerged as very prominent from the analysis concerns the inadequacy of participants dealing with the function concept in terms of the definition and examples adhering to the definition. Participants generally do not display an adequate understanding of the definition of a function and are displaying the typical restrictiveness of thinking only in terms of formulas and/or equations. This is a common conceptual difficulty described by Dubinsky and Wilson (2013:85) and is reported across the literature (Breidenbach *et al.,* 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989). In data instrument 2, all of the examples of supposed functions and non-functions that the participants gave, were simple formulas with no qualification in terms of domain and range. Some examples were even clearly incorrect. Participant 6 was collectively judged to be at the transition phase between the pre-function level and the action level. This participant gave $x^2 + 5x - 6 = 0$ as an example of a function and y = cos(x) as an example of a non-function, clearly showing a pre-function level of understanding. All other participants were judged to be at the action level with respect to the definition of a function.

At the action level of understanding, the participant stays fixed on the representation itself. It could be an equation, a graph, a table or some other representation, but the problem is that if one cannot start seeing past the representation, one cannot start to evolve to the process and object levels of understanding. Seeing past the representation and mentally grasping that which stays unchanged between different representations, is what Thompson (1994:39) referred to as the 'core concept'.

Not understanding that the domain and range are inseparable from defining your function, also keeps one at the action level. If, for example, it is asked to determine the domain of $f(x) = \sqrt{x-4}$, this question is lacking if the codomain of the function has not been stated already. Without stating the target set, that is the codomain, at the outset, we are letting the *formula* dictate to us by depending on the individual's personal image concept of natural domain. This again enforces the false notion that the formula *is* the function and not just a possible representation of the function.

5. Conclusions

This paper was set on using Action-Process-Object-Schema (APOS) theory as a model of how the learning of the function concept can take place. The two objectives for this paper concerned the development of an object level of understanding of the function concept over the short and the long term. The question was if the Image Functions Intervention (IFI) could bring about the desired object conceptualisation. The analysis of qualitative data from three different data instruments showed that we could not conclude that the IFI could deliver the object conceptualisation on any regular basis. However, the analysis showed that at least four (40%) of the participants experienced some growth in their understanding, with one participant reaching a transition stage to the object level. The analysis also provided valuable insight into the difficulty participants experienced concerning the definition of the function concept. This difficulty seems to be an obstacle on the path to conceptualisation at the object level as it hinders participants from forming their own examples of functions and non-functions, which adheres to the definition. Something to note is that the typical definition of a function as a relation between two sets, might itself be part of the problem. This is due to this type of definition being at the process level of understanding. That the definition is acting as an obstacle is speculation at this stage. Research on the truth of this speculation seems warranted judging from the heavy presence of function definition problems we saw in this paper.

The IFI still has a strong case for further use and investigation based firstly on its theoretical analysis (Venter, 2019) and secondly on the encouraging findings in this paper with respect to the observed growth of participants. Most importantly, the theoretical analysis showed that the IFI, through activities involving function composition, provides the necessary encapsulation of processes (Asiala *et al.*, 1996) to form functions as objects.

The APOS research methodology prescribes that the intervention is now adapted to incorporate what was learnt from the analysis after implementation. Two main changes to the IFI will involve firstly the creation of activities that deal specifically with the definition of the function concept. The definition needs to be addressed as the relation between two sets (process level) *and* as a set of ordered pairs (object level). The second change to the IFI will involve expanding the activities involving function composition to allow more and varied practice with encapsulating image functions as processes.

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FINAL REFLECTION

1. Research questions and related findings of the study

The **primary research question** was: To what extent can the study of image functions improve function comprehension?

To answer the primary research question, the study considered various **secondary questions:**

Research question 1: How valid and appropriate is the use of an image functions intervention when considering its theoretical analysis within the APOS framework?

In article one, the study of image functions was incorporated into a specially designed intervention, the Image Functions Intervention (IFI). The IFI was then analysed from an APOS theoretical perspective to see if it adhered to the genetic decomposition of the function concept. It was determined that the IFI could theoretically lead a participant through the conceptualisation levels by assisting the participant to establish the necessary mental structures prescribed by the genetic decomposition. Furthermore, it was shown that the IFI provided an exploratory base for the usual aspects of functions that are dealt with by students. General aspects of functions such as the definition, domain and range and injectivity can be dealt with, and specifically made applicable with respect to image functions. Image functions can provide a mathematical playground to consider function composition, continuity and differentiability. In particular, function composition was shown to be of key importance as it provides the participant with the encapsulation experience that is necessary to evolve from the process level of conception to the object level (Asiala *et al.*, 1996). The IFI uses function composition as a method to increase the contrast of a photograph that forms part of the context of the IFI. The context being a missing person’s case in which the said photograph plays a central role.

As the second part in the validation of the use of the IFI, article 1 reported on the implementation of the IFI in a classroom and the subsequent collection of qualitative data through a questionnaire. The analysis of this data showed that the IFI was particularly useful to broaden participants' concept images of functions, with particular emphasis on what can *also* be considered as representing functions.

Research question 2: To what extent can the average achievement on the Function Concept Inventory (FCI) be improved by completing an intervention programme based on image functions?

Article 2 reported that the experimental group of participants, therefore the group that completed the IFI, showed a statistically significant improvement on their scores on the FCI, with a medium effect size of 0.42. Furthermore, the control group did *not* show a significant improvement. However, no significant difference was detected between the post-intervention results of the experimental group compared to the control group. Therefore, it could not be concluded that the IFI must unequivocally be considered the cause of the observed improvements.

Research question 3: In what way can the said intervention programme assist in the development of an object conception of function over the short and long term?

In article 1, it was shown that the IFI uses function composition. The function composition is used in such a way that it enacts a process on the original image function while the original image function itself is being regarded as a process. This necessary encapsulation is the APOS theory mechanism that brings about an object conception of function. Thus, from a theoretical perspective it seems the IFI has at least, in part, the required toolkit.

In article 3, the qualitative analysis showed that four (in a group of ten) participants *improved* their conceptualisation over the short term. However, only one participant was judged to have improved enough to be considered transitioning into the object level of understanding.

1.1. Summary

To summarise and to answer the primary research question: The study of image functions, via a specially designed intervention within the APOS theoretical framework, *can* improve function understanding. In particular:

a) Participants can experience a broadening of their concept images by expanding their notions of what can constitute a function and how functions *are* a part of real-life. Take for example the following extractions from responses reported on in article 1: "I realized functions have a broader meaning", "a function can be determined in different ways", "functions can relate to real life experiences and applies to images and is involved in biometric scanners", "learned that functions can be used for more than calculating or predicting change".

b) Article 1 gave account of some participants' self-reported improved understanding of function inverses. The analysis of article 3 showed that some participants did in fact improve their understanding of inverses. For example, participant 8 improved from an action level to a process level with respect to function inverses while participant 5 improved from a process level to an object level. An indication that the improvement was probably caused by the intervention, was the participants' responses including ideas relating to injectivity and surjectivity. These aspects were directly dealt with in the intervention, yet *not* dealt with in any modules/classes the participants were enrolled for.

c) Article 2 reported a statistically significant improvement on the FCI scores of the experimental group (the group that completed the intervention) from before to after completing the intervention. Scoring better on the FCI implies an improved function conceptualisation.

d) Article 3 reported on four participants that showed improved overall conceptualisation levels within the APOS framework.

1.2. Triangulation

In the Orientation chapter, section 4.2, it was explained that the study consisted of two legs. The first leg consisted of the design and theoretical analysis of the intervention as well as an initial qualitative analysis to provide proof of principle with respect to the positive effects of the intervention. In the second leg of the study, a triangulation design was followed, that is a convergent parallel mixed-methods design (Maree, 2016:318). Quantitative and qualitative data collection methods were simultaneously applied using the same participants. The findings of the quantitative (article 2) and qualitative (article 3) methods are now compared along the areas of the triangulation protocol (O'Cathain *et al.*, 2010).

Agreement: Article 2 and article 3 showed that participants struggled with the formal definition of function and with the distinction between formulas, equations and functions. Both articles also showed that some participants improved their understanding with respect to these aspects. Both articles showed that at least some participants improved their understanding of functions in general after the intervention.

Partial agreement and dissonance: Considering only the group (n = 10) of participants that received the intervention, we can make the following comparisons.

The quantitative results of article 2 reported that six participants (participants 2, 3, 5, 7, 8 and 10) showed improvement directly after the intervention, three showed no change (participants 1, 6 and 9) and one participant (participant 4) regressed slightly.

The qualitative results of article 3 reported that four participants (participants 1, 5, 6 and 8) showed improvement, two participants (participants 4 and 7) showed no net change and four participants (participants 2, 3, 9, and 10) slightly regressed.

Therefore, the qualitative and quantitative results only agree with respect to participants 5 and 8. This seemingly low level of agreement needs further discussion. A reason that the qualitative results seemed less positive in comparison with the quantitative, might be connected to participants' levels of motivation while completing the same data instrument, the FCI, for a second time. In article 2, it was speculated that the high attrition rate observed with respect to the FCI, could be caused by participants' motivation levels. With the quantitative results based on multiple-choice questions and the qualitative results on open-ended questions, it is plausible that participants having low motivation would find it easier to complete the multiple-choice questions than the open-ended questions. This however needs confirmation in follow-up research, but it could explain why participants 2, 3 and 10 showed improvement on the quantitative test but regressed according to the qualitative analysis.

Silence: Article 2 showed that participants struggled with co-variational thinking, whereas article 3 gave no indication either way with respect to co-variational thinking. Article 3 showed that participants seemingly did not receive a long-term benefit with respect to function understanding. Article 2 however, could not give an indication with respect to long-term effects. It could be valuable in future quantitative research involving the IFI, as in article 2, to include a delayed test to measure long-term effects.

2. Limitations of the study

2.1. Sample selection and size

Throughout the study, only convenience sampling was used. This fact, together with the small sample size of article 2 (N =17), means that the findings of this study is not generalisable. This is not a problem for this study itself as from the postpositivist paradigm adopted in the Orientation chapter, generalisation is not a primary goal. However, this aspect should be noted when planning future research and in comparison with other studies' results.

Other settings could have an influence on results. Different populations might experience the Image Functions Intervention (IFI) differently than the two groups used in this study. Together with possible variations in educational background, the alternate experience could influence the results and findings.

In follow-up research, more settings should be used and with larger samples.

2.2. Addressing of the Schema level

In APOS theory, the schema level is the highest level. This level is however not reported to be occurring with any participants in any of the literature consulted for this study. In fact, the action level is quite dominant (Dubinsky & Wilson, 2013), with the process level occurring in some instances and the object level being rare. For this reason, the schema level was not planned for and ultimately not addressed in article three. The analysis of article three confirmed that none of the participants was close to the schema level. In article one, the schema level was included in the genetic decomposition of the function concept. However, no activities were designed for the intervention that was specifically aimed at the schema level. Future versions of the intervention will hopefully need to include the schema level.

2.3. Prior research

Concerning the primary research question of this thesis, namely investigating the effects of studying image functions on the understanding of the function concept, there are no prior research results. In digital image processing, it is common to model photographs/images as functions but using this in the pedagogy relating to functions is absent in the literature. Therefore, this study is limited by the lack of comparisons that can be made.

2.4. Collection of data

The questionnaire used in article one had three questions that were intended to evoke sufficient responses from the participants to allow independent judgement on any improvements with respect to function understanding. However, the questions ended up too closed-ended and often resulted in participants only responding with a simple ‘yes’ or ‘no’. For future use, these questions should be revised and/or the questionnaire redesigned to ensure sufficient responses are gathered from participants.

The function concept inventory (FCI) that was used in article two, had a high attrition rate. Of the 51 participants that started the FCI as the pre-test, only 17 participants fully completed the FCI as both pre-test and post-test. This had the influence of reducing the sample size substantially. The reason for the attrition should be investigated in order to avoid it in future research. If unavoidable, it should be considered at the planning stage.

In article three, the difficulty level of the questions of the second and third data instruments turned out to be higher than those of data instrument one. This could have had the effect of making the comparisons between results of the different instruments invalid. Therefore, it is possible that some participants that actually improved their understanding of the function concept were judged to have stayed constant or even to have regressed. Future research should endeavour to ensure that instruments that will be used for comparisons over time should be of the same difficulty.

3. Recommendations

In the previous section on limitations, some recommendations with respect to future research on the use of the IFI and the FCI were already discussed. This section will discuss some more recommendations that are based on the conclusions from the three articles of this thesis:

1. Expand the Image Functions Intervention (IFI) to include more, and a greater variety of activities concerning the composition of functions. The composition of functions was already considered a key part of the IFI as the composition provided the encapsulation mechanism to evolve from a process to an object conceptualisation of functions. However, judging from the results of article three in particular, the conclusion is that much more is needed with respect to function composition. Future research can then investigate if the new version of the IFI can assist participants in reaching the object level of understanding.

2. Article two reported that on analysing the results from the FCI, participants in this study scored the lowest on a question concerning co-variational reasoning. The experimental and the control groups answered the same question the worst. Furthermore, after the intervention was completed, there was no change in the average scores of the experimental group with respect to this question. However, in the results reported by O'Shea *et al.* (2016), this question was found to be only at medium difficulty. Research is needed to investigate if reasons can be found for this contrast. One possible reason is the different educational backgrounds of participants from South Africa (this study) in comparison to the participants in the UK and Ireland (O'Shea *et al.'s* study).

3. In article two, it was concluded that the actual time that a participant spends on the IFI could be introduced as an extra independent variable. It is reasonable to suspect that the amount of time spent will influence how participants fare on the FCI as post-test. If a positive correlation can be verified, it would add weight to the conclusion that the IFI is the cause of observed improvements.

4. In article three, we saw that participants struggled with the definition of the function concept. The typical definition of the function concept that students are exposed to, is at the process level. Consider the example given below. This definition is from Stewart (2015:11) but can be considered as representative of what first year Calculus students are exposed to.

   > A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

   This definition seems to naturally correspond to or evoke the function concept at the process level. Firstly, this could already be problematic for a student whose understanding of the function concept is only at the action level. Secondly, this process level type of definition might be an obstacle for students on their path towards understanding at the object level. Research is therefore needed to investigate how students are relating to a definition at the process level (a rule or correspondence between two sets), compared to a definition at the more abstract object level (a function being a set of ordered pairs).

4. Autobiographical reflection

From the get-go, I felt like a fish out of water. With my background in Mathematics and Applied Mathematics, the aspects of educational research I had to engage with in this study, initially felt quite foreign to me. That was despite almost two decades of teaching experience at the higher education level. Over the years, I had attended some in-service courses on teaching, but always felt that they were not quite adapted to the special needs of a mathematics classroom. Three years after starting this journey, I realise that I am still a bit of an outsider with respect to educational research, but I am very thankful for the learning and growth that was afforded me along this path.

To see a student's eyes suddenly light up after I have explained a concept to him or her has always brought me great joy. On the other hand, seeing a student get frustrated by 'not getting it', is quite disheartening. For this reason, I experienced the topic of this study at a personal level. *My* students struggle with the function concept.

In undertaking this study, I experienced aspects that were brand-new to me. For example, applying for ethical clearance and learning about paradigms, methodologies and sampling are aspects that research in Mathematics itself never prepared me for. Now that the study is at its end, I can conclude that I really enjoyed the qualitative analysis. I am no expert regarding qualitative analysis, but I can see the powerful role that this type of research methodology could play in unlocking the mysteries surrounding the learning and teaching of some mathematical concepts and aspects. I look forward to diving into this field.

The results from this study did not prove that the Image Functions Intervention (IFI) would cause improved understanding of the function concept. However, I am convinced that the IFI, after some tweaks and expansions, can be used to bring about positive change. I am convinced that the study of image functions can provide a mathematical playground wherein the conceptual difficulties associated with functions can be dealt with.

References

Dubinsky, E. & Wilson, R.T. 2013. High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, 32, 83–101

Maree, K. 2016. *First steps in research*, 2nd ed. Pretoria: Van Schaik Publishers.

O'Cathain, A., Murphy, E., Nicholl, J. 2010. Three techniques for integrating data in mixed methods studies. *BMJ*, 341:1147—1150

O'Shea, A., Breen, S. & Jaworski, B. 2016. The development of a function concept inventory. *International Journal of Research in Undergraduate Mathematics Education*, 2, 279–296

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Appendix A.

QUESTIONNAIRE 1

This questionnaire was used in Article 1. It was given to participants after they were presented the Image Functions Intervention (IFI) material in a classroom setting.

Instructions: Take your time and please complete each question freely and honestly.

1. Having completed the image functions intervention, have you realized/learned something in particular of the concept of function?

2. Is there an aspect of the function concept that is now 'more clear' to you?

3. Is there some aspect of the concept of function that you might have thought about in some way before, but now realize that you were wrong in some sense?

Appendix B.

FUNCTION CONCEPT INVENTORY

The Function Concept Inventory (FCI) was developed by O'Shea *et al.* (2016). The FCI was used as pre-test and post-test in the quantitative data collection of Article 2. Questions 2, 7a and 7b were also used for qualitative data collection for Article 3.

[O'Shea, A., Breen, S. & Jaworski, B. 2016. The development of a function concept inventory. *International Journal of Research in Undergraduate Mathematics Education*, 2: 279–296]

FUNCTION CONCEPT INVENTORY

Notes:

1. This inventory forms part of a PhD study by Christiaan Venter.
2. This inventory will be completed by all participants. It will be completed online through the learning management system, namely Blackboard.
3. Completion of this inventory is voluntary.
4. Through using the Blackboard learning management system to complete this inventory, a student gives permission for the gathering of the following data:
   (1) student number (2) surname (3) full names (4) gender
5. All information gathered will be kept confidential.
6. Thank you for being willing to complete this inventory.

Instructions:

1. Answer each question by choosing the correct answer(s).
2. Some questions have more than one correct answer. Read each question carefully to ensure you answer it in the correct fashion.
3. If you don't know the answer and the question has the option of choosing 'I don't know', choose that option freely. You cannot get penalized in any way by completing this inventory.

Q1. Let $f(x) = 3x+5$. Given the equation $3a+5 = 2$, which of the following are true? (There may be more than one true statement.)

(a) $f(a) = 3a+5$,	True ☐	False ☐
(b) $f(a) = 2$,	True ☐	False ☐
(c) $f(x) = 2$ for all x,	True ☐	False ☐
(d) $f(x) = 2$ for some value of x.	True ☐	False ☐

Q2. *Look at* what you see in a), b) and c) below and *write in words a description of each of them separately* using any terminology that you know. [You might use some or all of the words function, equation, solution, graph]

a) $y = x^2-5x+6$

b) $(x-2)(x-3) = 0$

c) $(x-2)(x-3) = -1$

Write down as many ways as you can in which they are related to each other.

Q3. Suppose f(x) is an increasing function.

a) Is it true that 3f(x) must be an increasing function?

Yes ☐ No ☐

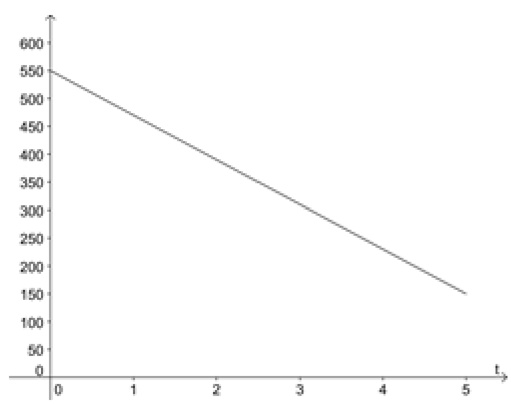
b) Is it true that f(x)+3 must be an increasing function?

Yes ☐ No ☐

Q4. A small business purchases a piece of equipment for R550. After 5 years the equipment will be outdated and will be sold for R150 as scrap metal. Each of the equation, the graph and the table of values shown below claims to represent the relationship between the value of the piece of equipment (in euros) and time (in years), assuming that the piece of equipment depreciates by the same amount each year. Please circle all options for which you think this claim is correct.

(a) $f(t) = 80t-550$,

(b)



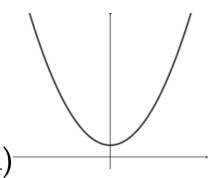
(c)

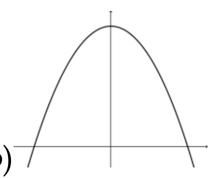
Time (in years)	Value (in rand)
0	550
1	430
2	350

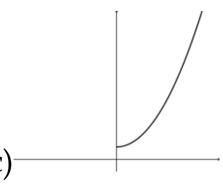
3	270
4	230
5	150

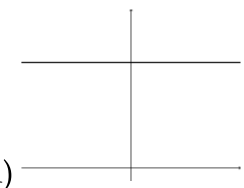
(d) I don’t know

Q5. Suppose f(x) is a function defined for all real values of x. Which of the following are possible graphs of $g(x) = [f(x)]^2$? (Please circle all of the options you think are possibilities.)

(a) 

(b) 

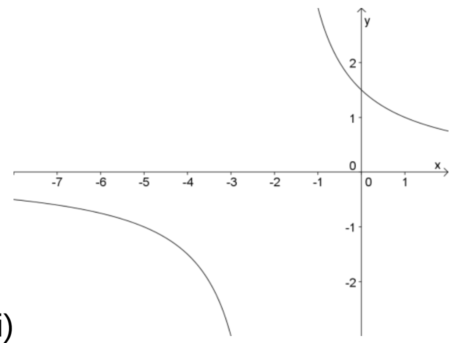
(c) 

(d) 

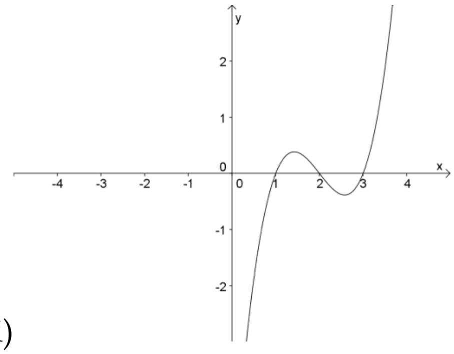
(e) I don’t know

Q6. In the table given, *match* each of the functions in (a), (b), (c) below with *one* of the graphs in (i), (ii), (iii) and (iv). *Mark* any key points on each of the graphs to help show how the graph and function are related.

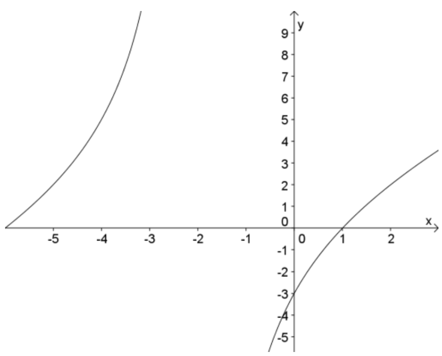
(a) $y = 9 - x^2$, (b) $y = 3/(x+2)$, (c) $y = (x-1)(x^2-5x+6)$



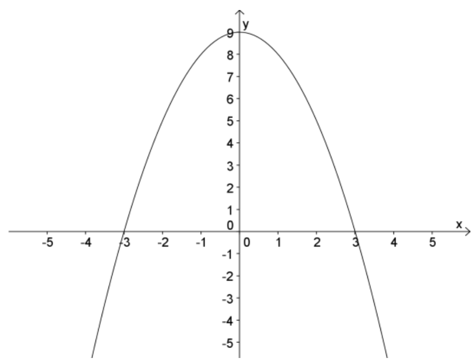
(i)



(ii)



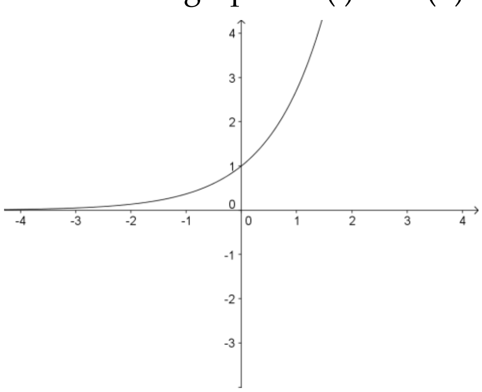
(iii)



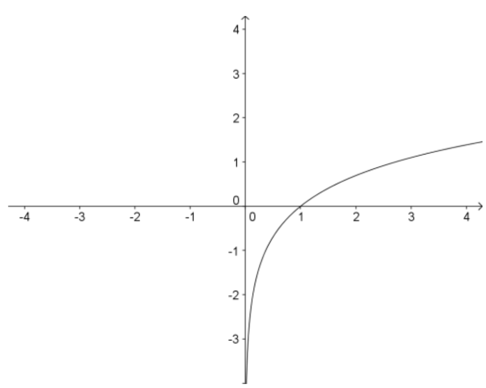
(iv)

	(a)	(b)	(c)	Explanation
(i)				
(ii)				
(iii)				
(iv)				

Q7. Look at the graphs in (i) and (ii) below.



(i)

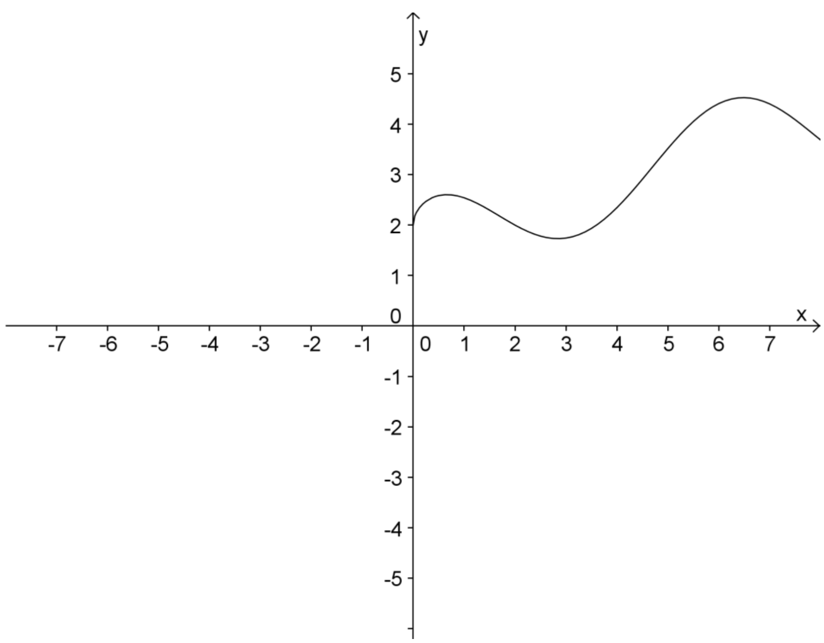


(ii)

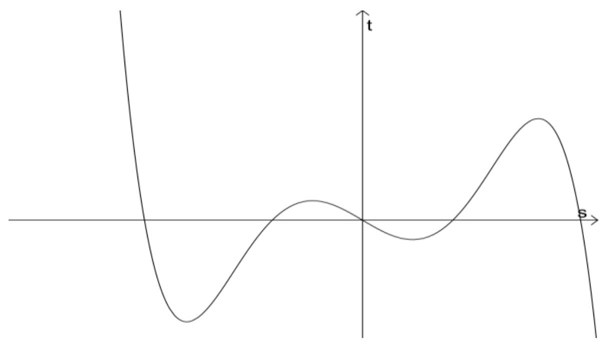
a) The *claim* is that Graph (ii) represents a function which is the *inverse* of the function represented by graph (i).

Explain what this means and *state* whether you think the claim is true.

b) Look at the graph given below and imagine, on the same axes, a graph to represent its inverse *if* you think there *is* an inverse. Explain clearly why you think there *is* an inverse or *not* an inverse.



Q8. Consider the curve below which describes a relationship between t and s.



Which of the following is true? (Please circle one of (a)-(d), or (e).)

a) s is a function of t but t is not a function of s.
b) t is a function of s but s is not a function of t.
c) s is a function of t and t is a function of s.
d) s is not a function of t and t is not a function of s.
e) I don't know.

Q9. Let s denote the size of a house and p denote its selling price. The tables below show the sizes and prices of houses sold each month by a suitable estate agent.

May

s	p
$100m^2$	R300,000
$120m^2$	R375,000
$75m^2$	R257,000
$90m^2$	R300,000
$110m^2$	R350,000

June

s	p
$100m^2$	R302,000
$120m^2$	R370,000
$70m^2$	R200,000
$131m^2$	R400,000
$120m^2$	R350,000

July

s	p
$110m^2$	R330,000
$90m^2$	R270,000
$80m^2$	R240,000
$50m^2$	R125,000
$120m^2$	R360,000

For which months is it true that p could be a *function* of s? (Please circle all of the options you think could be functions.)

(a) May (b) June (c) July (d) I don't know

Q10. Suppose f(x) is a function defined for all real values of x. For each of the following statements decide if the statement is always, sometimes or never true.

i. There are two different real numbers a and b such that f(a) = f(b).

Always ☐ Sometimes ☐ Never ☐

ii. There are three different real numbers a, b, c such that f(a) = b and f(a) = c.

Always ☐ Sometimes ☐ Never ☐

Q11. Suppose f(x) is a function such that f(3) = 5 and f(4) = 12. Which of the following is true? (Please circle one of (a)-(d), or (e).)

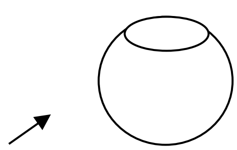
(a) f(7) = 5+12 = 17, (b) f(7) = f(3) + 4 = 9, (c) f(7) = 5(12) = 60,

(d) It is not possible to find the value of f(7) from the information given.

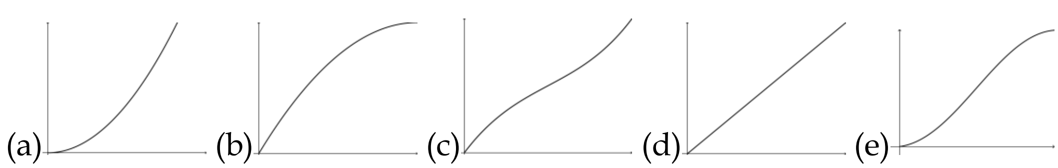
(e) I don't know.

Q12. Assume that water is poured into a spherical bottle at a constant rate. Which of the following graphs best represents the height of water in the bottle as a function of the amount of water in the bottle?

(Please circle one of (a)-(e), of (f).)



Spherical bottle



(f) I don't know.

Appendix C.

QUESTIONNAIRE 2

This questionnaire was used in Article 3. It was given to participants that completed the Image Functions Intervention (IFI).

Instructions: Take your time and please complete each question freely and honestly.

1. In your own words, give a definition of what a function is.

2. Write down any two examples of functions.

3. Can you give an example of a relation that would NOT qualify as a function?

Appendix D.

ASPECTS OF THE IMAGE FUNCTIONS INTERVENTION

The Image Functions Intervention (IFI) was set up with the use of the commercially available software *Articulate 360 Storyline* by Articulate Global Inc. The software is used as an e-learning authoring tool and can be used to create interactive learning material. For this study, the software was used to create a SCORM package that could be loaded into the LMS that is the learning management system, used by the local higher education institute. SCORM is a technical standard that makes possible the correct communication between various outputs from authoring software, and environments like an LMS.

In Article 1, the IFI's various activities were portrayed and analysed. Following these activities, the IFI, when implemented in the LMS, uses various kinds of questions which participants must answer to be able to proceed with the IFI. Three examples are given below of such questions to illustrate some of the interactivity of the IFI. Figures 1, 2 and 3 are from screenshots taken while running the IFI in a browser.

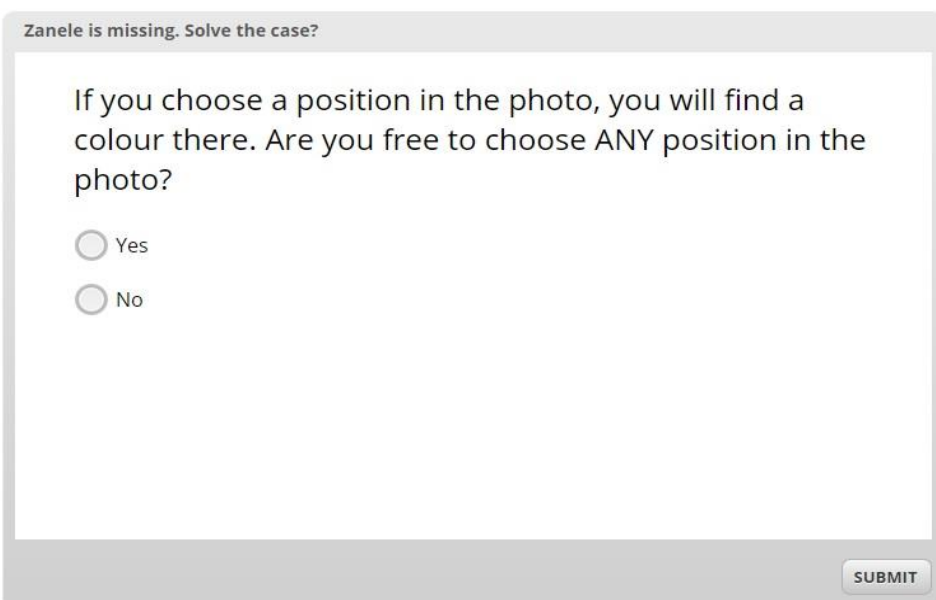


Figure 1. Dichotomous question type.

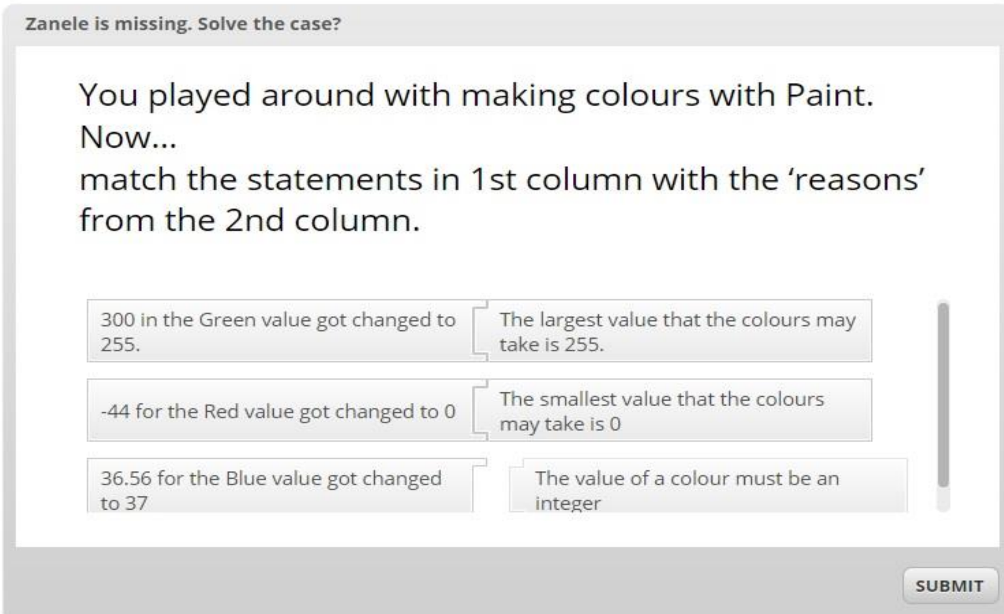


Figure 2. Puzzle pieces question type.

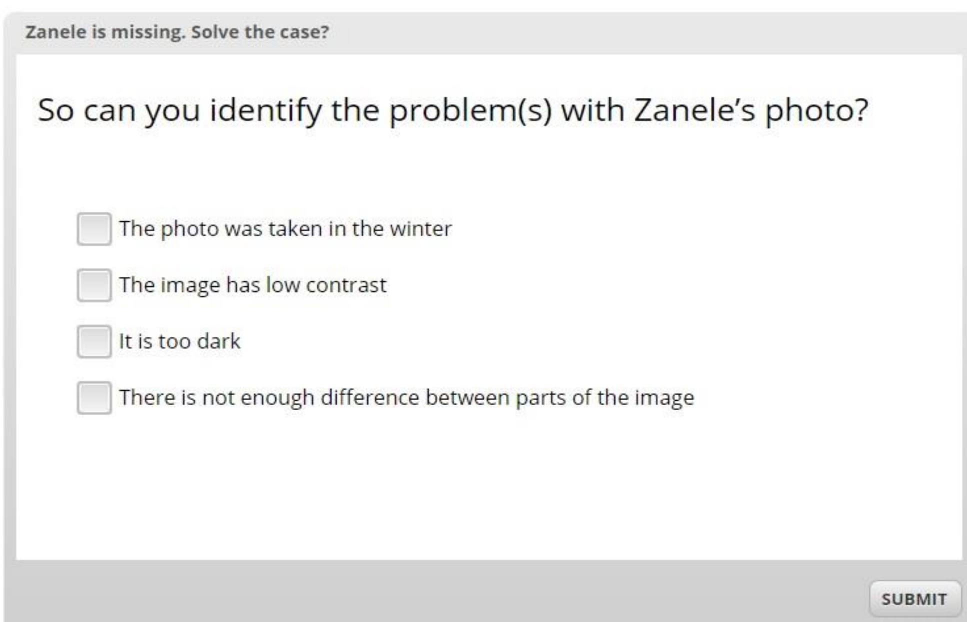


Figure 3. Multiple correct answers.