

Diagrams in mathematics: To draw or not to draw?

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This paper describes the use of diagrams as self-explanatory tools. It considers the use of diagrams, in general, and more specifically, examines research that is currently being undertaken in the broad field of visualisation. The research participants referred to in this article were Advanced Certificate of Education students and the paper attempts to analyse their responses to questions based on simple area problems in mathematics. The outcome of this research underscores the strategic use of diagrams when dealing with problem solving. While this is an ongoing research project, the paper attempts to capture the current status of research on the use of diagrams.

Keywords: diagrams, representations, visualisation, spatial, self-explanation, problem-solving, experiential learning, mathematical symbols

Introduction

It is necessary to constantly revisit old ideas in order to improve the way we teach. A key strategy is to look at how cognition is influenced by ideas that are often taken for granted. There are researchers, for example, who argue that teaching and learning is “best conceived as a process, not in terms of outcomes” (Kolb & Kolb, 2005:2). One of the implications of this is that teaching and learning should arise out of the experiences of the learners themselves. As Dewey (1897:79) stated, “...education must be conceived as a continuing reconstruction of experience: ... the process and goal of education are one and the same thing”. In an attempt to find better approaches to teaching and learning, this research taps into the interest that has been shown in the efficacy of diagram usage when solving mathematical problems. This exploration is necessary despite the fact that some researchers are divided on the effectiveness of the diagram as a problem-solving tool (Simon, 1986).

According to Winn (1987), cited in Diezmann (1995:223), a “diagram is defined as an abstract visual representation that exploits spatial layout in a meaningful way, enabling complex processes and structures to be represented holistically”. This would imply that diagrams afford the viewer a physical form for a mental structure. It enables the problem solver, with the requisite prior knowledge, to find associations between different visual stimuli in the diagram with mental representations and understandings. This idea is firmly grounded in the belief that diagrams help to create representations of the problem, which mediate a solution (Goldman, 1989). More importantly though, diagrams should allow the viewer of the diagram to see a complete picture in the mind. The prior knowledge of the viewer will determine the understanding that is derived. Correct interpretation of the symbols in the diagram depends on the meanings that exist or on the simplicity of the diagram.

There are many questions associated with the use of diagrams. Are diagrams simply a heuristic (or method that encourages the learners to discover solutions for themselves) that assists in understanding and solving problems? Are diagrams a means to attaining higher levels of conviction and hence contributing to proof construction? Can learners’ reasoning skills be improved through the use of diagrams?

Theoretical perspectives

This research is framed within the constructivist paradigm and is underpinned by Kolb’s Experiential Learning Theory. Using diagrams when solving problems engages the learners actively in constructing meaning for themselves. The constructivist paradigm allows learners to use previously acquired representations and knowledge to develop new meaning from that which is currently being experienced. Much is known about the use of constructivist methods in teaching and learning (Sherman, 2000; Hua Liu

& Mathews, 2005; Mills, Bonner & Francis, 2006), based on the idea that teaching should be organised around allowing the learners to construct their own meaning from past and present experiences. While not much more needs to be said about the constructivist paradigm, Kolb's Experiential Learning Theory requires further unpacking.

Experiential Learning Theory (ELT)

ELT, according to Kolb (1984), places emphasis on experiences during the process of learning. This theory is different from cognitive and behavioural theories. In affording a central role to experience, Kolb postulated that the inevitable results for the learner would be empirical evidence, observation and reflection on the observed phenomena. Clearly, these experiences are effective only if the “here and now concrete experiences” and the “feedback processes” (Kolb, 1984:21) are genuine and real. According to Borzak (1981:9), experiential learning involves a “direct encounter with the phenomena being studied rather than merely thinking about the encounter, or only considering the possibility of doing something about it”. The direct engagement with the experience could possibly entail the drawing or the viewing of a pre-determined diagram. ELT defines learning as “the process whereby knowledge is created through the transformation of experience. Knowledge results from the combination of grasping and transforming experience” (Kolb, 1984:41). In claiming that knowledge “is a transformation process, being continuously created and recreated”, Kolb (1984:38) alludes to the fact that acquisition of knowledge is often a result of an iterative process. This research postulates that learning using mathematical diagrams are experiences that may arise out of many different types of mathematical actions. It focuses specifically on the experience with diagrams. The Kolb model (Figure 1) portrays what Kolb, Boyatzis and Mainemelis (1999:2-3) refer to as:

... two dialectically related modes of grasping experience -- Concrete Experience (CE) and Abstract Conceptualization (AC) -- and two dialectically related modes of transforming experience -- Reflective Observation (RO) and Active Experimentation (AE). According to the four-stage learning cycle depicted in Figure 1, immediate or concrete experiences are the basis for observations and reflections. These reflections are assimilated and distilled into abstract concepts from which new implications for action can be drawn. These implications can be actively tested and serve as guides in creating new experiences.

In grasping knowledge emanating from the experience of viewing or drawing diagrams, learners may not be able to physically feel the picture; however, their sense of vision will play an important role. Often, this experience is construed by the learner as a concrete one, and indeed these are concrete experiences because the learner uses particular skills to draw or may recall particular skills when interpreting an existing diagram. Through the interaction of the viewed stimulus (what the learner ‘sees’) and his/her prior knowledge, the learner is able to reflect on the symbols inherent in the diagram. New knowledge arises out of a particular sequence in which these visual symbols are interpreted. The reflection induced may create new knowledge in the form of a new concept or some generalisation. These concepts and generalisations can be tested with different applications and examples. These ideas are reflected in Figure 1.

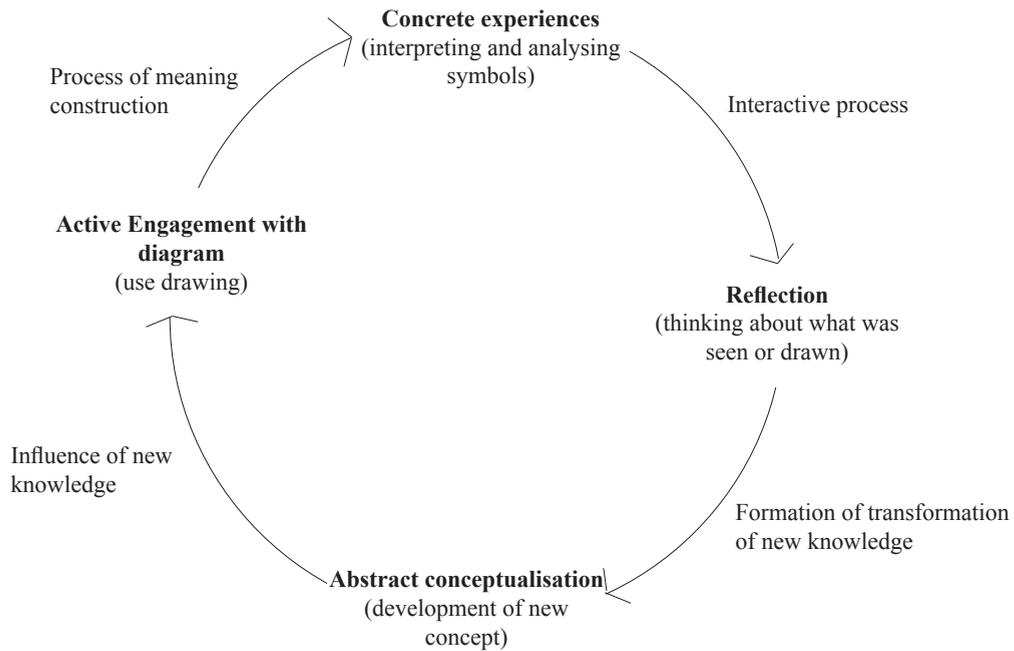


Figure 1: An adaptation of the Kolb model for experiential learning

The adaptation of the Kolb model illustrates the process that leads to either the acquisition of new knowledge or the transformation of old knowledge. Kolb's model focusses on the experiences related to learning in general. For the purposes of this study, the model was adapted to relate specifically to experiences involving the drawing of diagrams. The process begins with seeing a physical diagram or actually constructing one (not necessarily to scale). In interpreting or analysing the symbols inherent in the diagram, the learner begins to engage in a process of meaning construction. Meaning construction here is dependent on the learners' prior knowledge. Through the process of reflection and interaction with the new stimuli, meaning is enhanced. In reflecting and interacting with known stimuli in a diagram, the process of internalisation and externalisation takes place. This implies that, on seeing an external stimulus, the learner acknowledges the prior knowledge inherent in the diagram (whether mental or physical). In reflecting and interacting on the *a priori* knowledge, new knowledge is constructed. This becomes internalised as new knowledge. These new insights are again used to influence what is seen or added to the existing diagram. Hence an externalisation process ensues. In essence, there may be a process of iteration between the internalisation and externalisation of knowledge. Bertel's (2005) emphasis on this aspect is evident. He argues that diagrams are an essential part of a learner's mental processing in which mental constructions are externalized, internalized again, externalized, and so on. This type of analytical thinking results in new information being created or the transformation of old knowledge.

Current research

This paper describes a short, small scale research project that was conducted with the Advanced Certificate of Education (ACE) students at the University of KwaZulu-Natal. Eighty eight (88) students from three classes were asked to complete a three-question test. They were informed that the results were not going to be used for their year mark. However, the information obtained from analysing their responses would be important for future development of the mathematics modules. Half of all the questionnaires had diagrammatic representations with word problems and the remaining questionnaires had only the word

problems. The questionnaires were randomly distributed, with half of the students referred to as Group A students in each class receiving the questionnaires with diagrams. The remaining students, referred to as Group B students, received the questionnaires without diagrams. These questionnaires were administered by a contract staff member during the course of the semester. The ACE students are those who have had very little mathematical experience and this test was administered after they had completed an algebra module and were in the midst of the module on space, shape and measurement.

The first question (Figure 2) was a relatively familiar one and most of the students in Groups A and B answered it correctly.

A metal sheet of dimensions 20 cm by 5 cm was purchased to cut out metal disks with diameter 5 cm. How many disks can be obtained from the sheet?

Figure 2: Question as it appeared on the questionnaire

Some students (Group A) received the diagram (Figure 3) as well.



Figure 3: Diagram received by some students

Seventy percent of the Group A students who had the diagrams already drawn answered the question correctly, while 80% of those in Group B, who had to draw the diagrams themselves, correctly answered the question. The anomaly in the percentages is not necessarily significant due to the small difference. The problem was relatively simple and it set the scene for the next question.

The second question (Figure 4) was similar, but was deliberately manipulated to see whether students read, interpreted and understood the question itself. Although there was no solution to the problem, a suggested diagram was given.

A metal sheet of dimensions 20 cm by 5 cm was purchased to cut out metal triangles having dimensions base 10 cm and height 10 cm. How many triangles can be obtained from the sheet?



Figure 4: Question with diagram

The results here were far more anomalous than in the first question. Only 14% of those students from Group A (who received pre-drawn diagrams) answered the question correctly (that is, that it was not possible to cut out triangles). In contrast, 55% of the students from Group B (who received no diagrams) answered the question correctly. These results seem to contradict the idea that pre-drawn diagrams are tools that mediate understanding and solution of problems. It is significant and must be noted that all (55%) of the students in Group B who had answered the question, had correctly drawn a diagram. Of the remaining 45% of respondents in Group B, a few who had drawn the diagram arrived at incorrect solutions, but the majority of these respondents did not draw a diagram at all. Some may argue that the question itself was misleading, but it was expected that students should have been able to see that there was no solution possible. This could have been determined by using simple logic. No high level mathematics was required. The researcher was, in fact, attempting to show that by drawing diagrams students attain some level of understanding of the problem itself. The students who drew the diagram themselves stated with much conviction that "it was impossible to find a solution". It seems that in physically drawing the diagram the students were able to see for themselves the impossibility of finding a solution. This illustrates the idea that by actively engaging with the diagram students are, in fact, engaging in the process of meaning construction.

This perspective is strongly grounded in the belief that generating a diagram facilitates the conceptualisation of the problem structure (Van Essen & Hamaker, 1990). Cox (1999), Cox and Brna (1995), and Brna, Cox and Good (2001) articulated the idea that diagrams facilitate the self-explanation effect. They argue that, as graphical representations act to constrain the interpretation of a situation by limiting abstraction, they provide learners with more salient and vivid feedback to compare against their explanations. This contention is supported by Ainsworth and Loizou (2001). It seems that students who drew diagrams correctly, as self-explanatory artefacts, engaged with the text more than the students from Group A. In engaging with the text, the students then generated diagrams that made sense to them. In any case, by drawing the diagrams these students decreased the amount of information that they needed to remember as compared to those who did not draw diagrams. However, it is fallacious to assume that diagrams are spontaneously effective tools for students (see Dufour-Janvier, Bednarz & Belanger, 1987). Inadequate diagrammatic representations limit students' problem-solving capabilities (Klahr, 1978) because the visual stimuli inherent in the diagram is often at odds with the student's *a priori* knowledge. It is therefore important to investigate factors that influence problem representation (Goldman, 1986).

The argument goes further: representing word problem information using a diagram involves the decoding of linguistic information and the encoding of visual information (Diezmann, 2000). In this process of attempting to draw a diagram from the word problem, there are strong possibilities for the learner to acquire new knowledge. In creating a correct diagram from the verbal information given in the word problem, the learner has to carefully consider the different bits of information inherent in the problem. Herein lies the potential for the learner to be able to successfully make immediate inferences via this engagement with the information through internalisation and externalisation. This translation process carries the potential for knowledge acquisition (Karmiloff-Smith, 1990) through the re-organisation of information (Weinstein & Mayer, 1986) and subsequent inference making (Lindsay, 1995). This is strongly supported by the theory postulated in Figure 1. It may be that, on reading a word problem, the student creates mental images because of the symbolic nature of words. For example, if the word 'bird' is read, there may be an inclination for the reader to imagine the shape of a bird in flight. Different readers of the same word may imagine different birds, but in the case of mathematical words, there ought to be a tendency to create similar images. In contrast, it is difficult to imagine too many variations in the images formed when the word 'circle' is read. On reading such words and then constructing mental images, it is more than likely that the student would externalise these by physically drawing parts of the diagram. Then, by drawing the diagram, the student also relates the symbolic structure to prior knowledge (internalisation). All this contributes to meaning making, even before the actual physical act of solving the problem begins. It is this visual-analytic thinking that leads to transformation of existing knowledge or the creation of new knowledge. However, this only comes with the student actually working through

the words of the problem. Regarding diagrams, it seems that unless the diagrams are simple to interpret, students struggle to make meaning. This is possibly due to the non-engagement with the diagram creation or inadequate *a priori* knowledge.

Although the diagram (Figure 4) that accompanied the question for the first group could be construed as misleading, these students did not attempt to make sense of the information inherent in the question. A diagram on its own does not establish better understanding. It is the diagram generated as a self-explanatory tool that supports better solutions. Drawing these self-explanatory diagrams is, moreover, not easy. It takes much effort and requires considerable prior knowledge. Diagrams on their own have only symbolic value (students can identify aspects of the diagram), but when used in conjunction with text, the possibilities for meaning creation increases. In the case of the example in Figure 4, those students who attempted to engage with the diagram visualised the impossibility of finding a solution. It is perhaps significant that more students who had to draw the diagram discovered the impracticality of finding a reasonable solution, while those who had the diagrams drawn attempted to show that a solution was possible.

In order to further illustrate this argument, I draw on two different groups of people who answered similar questions. Firstly, as part of a master's research project, Budram (2010) asked a group of primary school learners the following question: **A farmer wants to fence off a square piece of land and he insists on using 12 poles on each side of the square. How many poles will he need?** Those learners who did not use a diagram generally gave the following response (Figure 5):

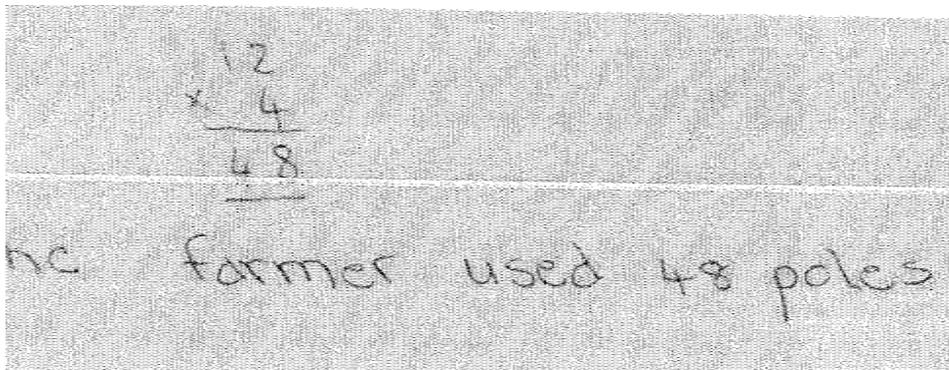


Figure 5: Learner response

The learners who arrived at the correct solution drew diagrams. This is one example (Figure 6):

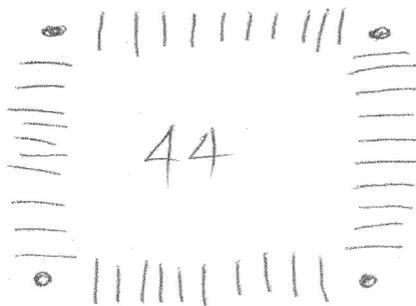


Figure 6: Learner drawing

The significance of these responses can only be grasped if compared to the responses of a group of trainee teachers in the Bachelor of Education programme at the University of KwaZulu-Natal. Sixty-nine

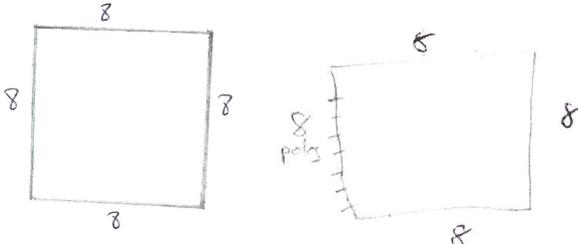
mathematics students in their third year of teacher training were asked a similar question: **A farmer wants to fence off a square piece of land and he insists on using 8 poles on each side of the square. How many poles will he need?** Those who did not draw diagrams generally responded as follows (Figure 7):

$$\begin{aligned} \text{Perimeter of square} &= \overset{\text{side} \times 4}{s \times 4} \\ &= 8 \times 4 \\ &= 32. \end{aligned}$$

each side will have 8 poles and a square piece of land has 4 sides \therefore the farmer will use ~~have~~ $8 \times 4 = 32$ poles.

Figure 7: Student solution

Some trainee teachers who drew diagrams still had the answer incorrect. This is illustrated in Figure 8.

$$\begin{aligned} 8 \times 4 \\ = 32 \end{aligned}$$


$$\begin{aligned} 8 + 8 + 8 + 8 \\ = 32 \end{aligned}$$

\therefore he will use 32 poles

Figure 8: Sample of student incorrect answers

Of the 69 students, fourteen did not draw the diagram, and of these 14 only one student had the correct solution. The remaining 55 students drew a diagram. Those who drew the diagram (23 students), as illustrated in Figure 9, had the solution incorrect. The remaining 32 students who drew the diagram, as shown in Figure 9, had the solution correct.

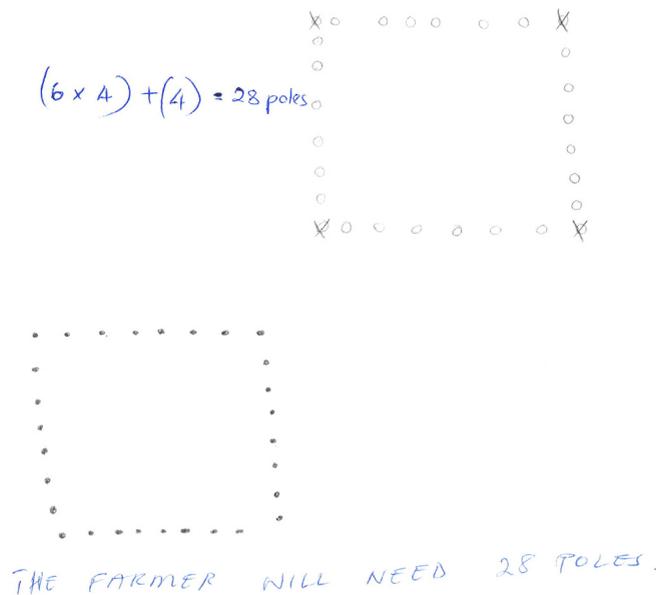


Figure 9: Sample of student correct answers

Again, when the correct and incorrect answers were examined, for both the trainee teachers and the primary school learners, it seems that diagrams drawn with understanding yielded better results. More importantly, the way the diagrams were drawn seemed to have had a significant effect on whether they solved the problem or not. When the diagram more closely resembled reality, the greater the possibility of obtaining a correct solution.

The third question that the ACE students attempted to answer demonstrated how, without sufficient knowledge, the students could not draw a reasonable self-explanatory diagram. It seems that the information contained in the text could not be interpreted, even by those who had the diagram drawn for them. This could not be directly attributed to language disadvantages because in most cases the students were very proficient in English.

The third question (Figure 11) yielded poor results from both sets of students.

An equilateral triangle has sides of length $2r$. Three identical circles, having radius r are constructed such that the vertices of the triangle are the centres of the circles. Calculate the area of the region between the circles.

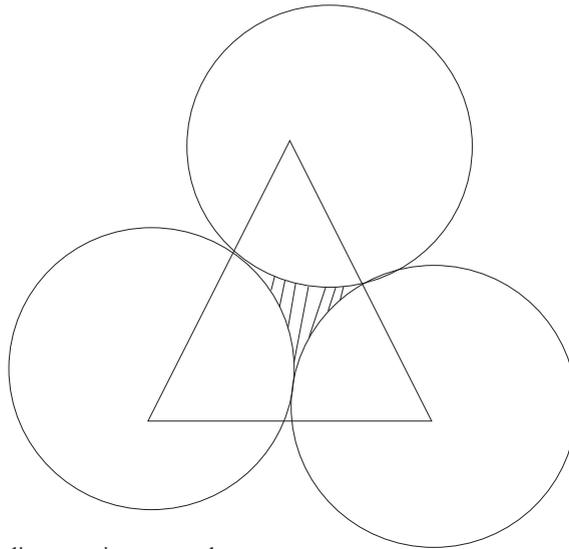


Figure 11: Question with diagram given to students

Eleven percent of Group A respondents and 7 percent of Group B respondents answered the question correctly. The third question showed that, if the question is difficult and beyond the experience of the student, the presence or absence of a diagram is immaterial.

Conclusion

Diagrams can be effective tools for sense making and should be used wisely when presenting word problems to students. Self-explanatory diagrams are true mediating artefacts that help learners develop better understanding of the mathematical problem; hence, constituting a possible means to solve the problem. Diagrams often make sense to the extent that learners can interpret the symbols inherent in the diagram. This depends on the prior learning and knowledge of the learner. Meaning is extracted from diagrams both visually and spatially. The act of drawing induces in the drawer the need to understand inherent ideas and concepts. This externalisation process of drawing reflects the person's understanding of these ideas and concepts. To conclude, learning through the drawing of diagrams is an area that is worthy of further research and should be pursued vigorously.

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