GROUNDWATER FLOW: NEW MODELS FOR LEAKY AND SELF-SIMILAR LEAKY AQUIFERS WITH NONLOCAL DIFFERENTIAL OPERATORS

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DECLARATION

"I, Ramotsho Mmanthupi Amanda, declare that the Doctorate Degree research dissertation, I herewith submit for the Doctorate Degree qualification in Geohydrology at the University of the Free State is my independent work, and that I have not previously submitted it for a qualification at another institution of higher education".

In addition, the following four publishable articles will be submitted for review

1. **Ramotsho, A.M and Atangana, A. (2020). Analyzing the new generalized equation of groundwater flowing within a leaky aquifer using Power law, Exponential decay law and Mittag-Leffler law**
2. **Ramotsho, A.M and Atangana, A. (2020). Application of the new numerical method with Caputo fractal-fractional derivative on the self-similar leaky aquifer equations**
3. **Ramotsho, A.M and Atangana, A. (2020). Application of the new numerical method with Caputo-Fabrizio fractal-fractional derivative on the self-similar leaky aquifer equations**
4. **Ramotsho, A.M and Atangana, A. (2020). Application of the new numerical method with Atangana-Baleanu fractal-fractional derivative on the self-similar leaky aquifer equations**

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"The Lord is my shepherd i shall not want" Psalm: 23.

Madiba said "What count in life is not the mere fact that we have lived. It is what difference we have made to the lives of others that will determine the significance of the life we lead". I owe my sincere appreciation to my supervisor Prof. Atangana and his wife Dr. Atangana for their guidance and support.

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''DEVELOP A PASSION FOR LEARNING. IF YOU DO, YOU WILL NEVER CEASE TO GROW '' ANTHONY J.N ANGELO

ABSTRACT

During 2018 Ramotsho and Atangana published the article with the following title: "Derivation of a groundwater flow model within leaky and self-similar aquifers: Beyond Hantush model". The scope of the work covered the explanation and background of leaky aquifers, existing equations, failure to apply the existing equations, derivation of the new generalized equation which represents the flow of groundwater in a leaky aquifer, introduction of self-similar leaky aquifers and the two derived equations. The conception of heterogeneity was covered in the article; however, the concept of memory effect and long term interaction was omitted. The proposed scope of this thesis is to introduce the three concepts of which are exponential law, Mittag Leffler law and power law into the new generalized equation, which represents the flow of water in a leaky aquifer.

It has been proven in literature that the concept of fractional differentiation is well accounted for in nature and in the field. In chapter six, fractal-fractional operators are introduced and furthermore the different new numerical operators with different fractional operators are presented. The concept of fractal-fractional operators is introduced into the two equations for water flowing within a self-similar leaky aquifer. The results are presented in terms of simulations, where cone of depression figures against time or space are given. It has been depicted that as the order is increased the cone of depression decreases. This represents the real-life scenario or concept of permeability. The figures have been fully analyzed in chapter ten.

LIST OF GREEK NOTATIONS

α	Alpha
β	Beta
$\int$	Definite Integral
Δ	Delta
$\mathbf{e}$	Exponential function
Γ	Gamma function
∞	Infinity
λ	Lambda
$\mathcal{L}$	Laplace Transform operator
$\lim$	Limits
ω	Omega
∂	Partial Derivative operator
$\emptyset$	Phi
π	Pi
ρ	Rho
$(\sum \quad)$	Summation
τ	Tau

LIST OF ABBREVIATIONS AND NOTATIONS

ABBREVIATION	DESCRIPTION
AB	Atangana Baleanu
ABMs	Adam-Bashforth Methods
AMMs	Adam-Moulton Methods
BDFs	Backward Differential Formulas
BEM	Backward Euler Method
C	Caputo
CF	Caputo-Fabrizio
CNM	Crank Nicholson Method
FDMs	Finite Differential Methods
FEM	Forward Euler Method
FEMs	Finite Element Methods
t^{α}	Fractal
H	Hydraulic head
LPD	Lagrange Polynomial Method
LMMs	Linear Multistep Methods
Q	Pumping Rate/Discharge rate
R	Radial distance
RKMs	Runge Kutta Methods
TM	Trapezoidal Method
K	Vertical Hydraulic Conductivity
W	Well function
S	Storativity
b/B	Thickness
∂_t	Time derivative
T	Transmissivity

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CHAPTER 1

1 BACKGROUND

Water plays a significant role in our lives; it is needed for production, consumption and also for economic growth. South Africa is one of the countries that experience a growing or looming water crisis. Groundwater is now recognized as the main source of water for bulk supply. Groundwater is stored in an aquifer; an aquifer is classified as a rock unit that has a potential to store and allow movement of water**.** There are four types of aquifers defined in literature namely confined, unconfined, perched and leaky aquifer (Harter, 2003). However the focus of this research is the leaky aquifer, it is been defined in literature as a fully saturated aquifer, it is bounded below by an impermeable layer and above by an aquitard (Kruseman and de Ridder, 1994).

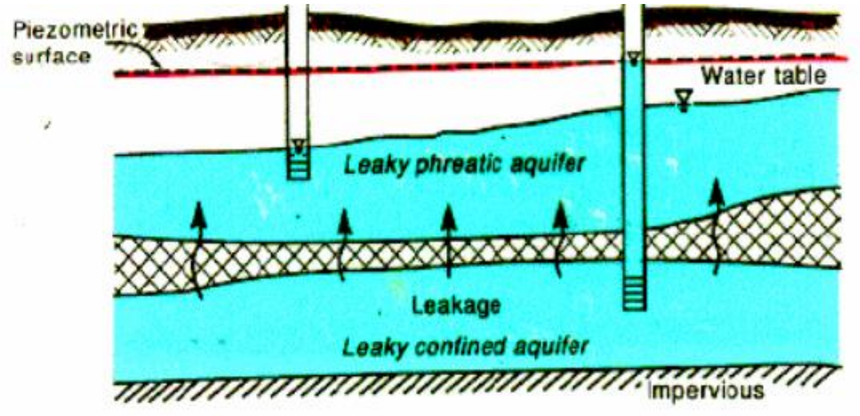


Figure 1: Simplified diagram indicating a phreatic and confined leaky aquifer (Bear, 2000)

Hantush and Jacob previously came up with a mathematical solution, which is capable of producing hydraulic properties of semi-confined aquifers; however they developed a model for homogeneous and isotropic aquifers with assumptions and limitations (Hantush and Jacob, 1955). The model that was developed by Hantush and Jacob cannot be used to investigate or study heterogeneous aquifers. These models are compiled using classical physical laws, with no memory.

1.1 EQUATION REPRESENTING THE FLOW OF GROUNDWATER: DERIVED BY THEIS AND LATER SIMPLIFIED BY HANTUSH AND ATANGANA

The equation for groundwater flow within an aquifer was derived by Theis using the known heat model assuming the medium transfers heat within a homogeneous system. However, the mathematical equation derived by Theis uses the local derivative (Atangana and Unlu, 2015). There is a relationship between the properties of Darcy's Law and the flow of heat, whereby the hydraulic heads is proportional to temperature, thermal gradient to pressure-gradient, conductivity is proportional to permeability and specific heat is analogous to specific yield (Theis, 1935). Therefore, the equation below represented the flow of groundwater within an aquifer.

$$\left(\emptyset\beta\rho_w g\frac{\partial h}{\partial t}+\alpha\rho_w g\frac{\partial h}{\partial t}\right)=\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z} \qquad (1)$$

1.2 NEW GENERALIZED GROUNDWATER FLOW EQUATION FOR LEAKY AQUIFERS

The new generalized equation was derived using the continuity equation. The inflow, outflow of water within an aquifer and storage are related by continuity equation given as,

$$\frac{dS}{dt}=inflow-outflow \qquad (2)$$

When leakage factor is taken into account, the storage equation can be given as follows,

$$\frac{\partial v}{\partial t}=Q_1+Q_3-Q_2 \qquad (3)$$

Taking into account that water flows within an aquifer through advection and dispersion, $\frac{\partial v}{\partial t}$ is given as,

$$\frac{\partial v}{\partial t}=S(2\pi r)dr\frac{\partial h}{\partial t} \qquad (4)$$

Q_1, Q_2 and Q_3 can be given as,

$$Q_1 = 2\pi(r + dr)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right]K \tag{5}$$

$$Q_2 = 2\pi rb\left[\frac{\partial h}{\partial r}\right]K \tag{6}$$

$$Q_3 = 2\pi rdr\frac{h(r,t)}{\lambda^2}Kb \tag{7}$$

Substituting equation (4), (5), (6) and (7) into equation (3),

$$S(2\pi r)dr\frac{\partial h}{\partial t} = 2\pi(r + dr)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right]K + 2\pi rdr\frac{h(r,t)}{\lambda^2}Kb - 2\pi rb\left[\frac{\partial h}{\partial r}\right]K \tag{8}$$

$$\frac{S(2\pi r)dr\frac{\partial h}{\partial t}}{2\pi rdr} = \frac{2\pi(r + dr)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right]K}{2\pi rdr} + \frac{2\pi rdr\frac{h(r,t)}{\lambda^2}Kb}{2\pi rdr} - \frac{2\pi rb\left[\frac{\partial h}{\partial r}\right]K}{2\pi rdr} \tag{9}$$

$$S\frac{\partial h}{\partial t} = \frac{(r + dr)b\left[\frac{\partial}{r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right]K}{rdr} + \frac{h(r,t)}{\lambda^2}Kb - \frac{b\left[\frac{\partial h}{\partial r}\right]K}{dr} \tag{10}$$

Transmissivity equals to conductivity and thickness, let (T) be represented by (Kb),

$$T = Kb \tag{11}$$

$$S\frac{\partial h}{\partial t} = T\frac{(r + dr)\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right]}{rdr} + T\frac{h(r,t)}{\lambda^2} - T\frac{\left[\frac{\partial h}{\partial r}\right]}{dr} \tag{12}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{(r + dr)\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right]}{rdr} + \frac{h(r,t)}{\lambda^2} - \frac{\left[\frac{\partial h}{\partial r}\right]}{dr} \tag{13}$$

Therefore,

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{1}{dr} + \frac{1}{r}\right)\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{\partial h}{\partial r}\right] + \frac{h(r,t)}{\lambda^2} - \frac{\left[\frac{\partial h}{\partial r}\right]}{dr} \tag{14}$$

The above version can be simplified in terms of factorization and canceling of like terms,

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{1}{dr}\frac{\partial h}{\partial r} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{h(r,t)}{\lambda^2} - \frac{1}{dr}\left[\frac{\partial h}{\partial r}\right] \tag{15}$$

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{h(r,t)}{\lambda^2} \tag{16}$$

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\left(1+\frac{\Delta r}{r}\right)+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{h(r,t)}{\lambda^2} \tag{17}$$

The above is therefore the exact equation that represents the movement of groundwater in a leaky aquifer.

1.3 PROBLEM STATEMENT

Groundwater flow equations are developed based on classical equations. Leaky aquifers disobey the fundamental rules or assumptions that come with these equations. Existing numerical solutions of flow within a leaky aquifer are based on differential equations which also incorporates a leakage effect additional to the original groundwater flow equation (Hantush, 1960).

These solutions are limited and do not incorporate the flow of water in a heterogeneous media. Generalized groundwater flow equation for leaky aquifer and the two equations for self-similar leaky aquifer have been developed to take into account the flow and heterogeneity of the media. There is a gap in the equations; the equations do not take into account the memory effect or interaction, they were developed using the local differential operators.

1.4 AIM AND OBJECTIVES

The aim of this research is to derive groundwater movement equations for leaky aquifer and a self-similar leaky aquifer using non-local differential operators.

Objectives:

1.1.1 Literature review of a leaky aquifer and the existing models

1.1.2 Background of Theis model and the upgraded version by Hantush and Atangana

1.1.3 Introduction of a new generalized groundwater movement equation of a leaky aquifer

1.1.4 To review the existence of local operators, non-local operators and different numerical schemes

1.1.5 Apply different non-local operators on the generalized groundwater flow equation of a leaky aquifer and

1.1.6 Model self-similar leaky aquifer equations with non-local operators

1.5 RESEARCH FRAMEWORK/ METHODOLOGY

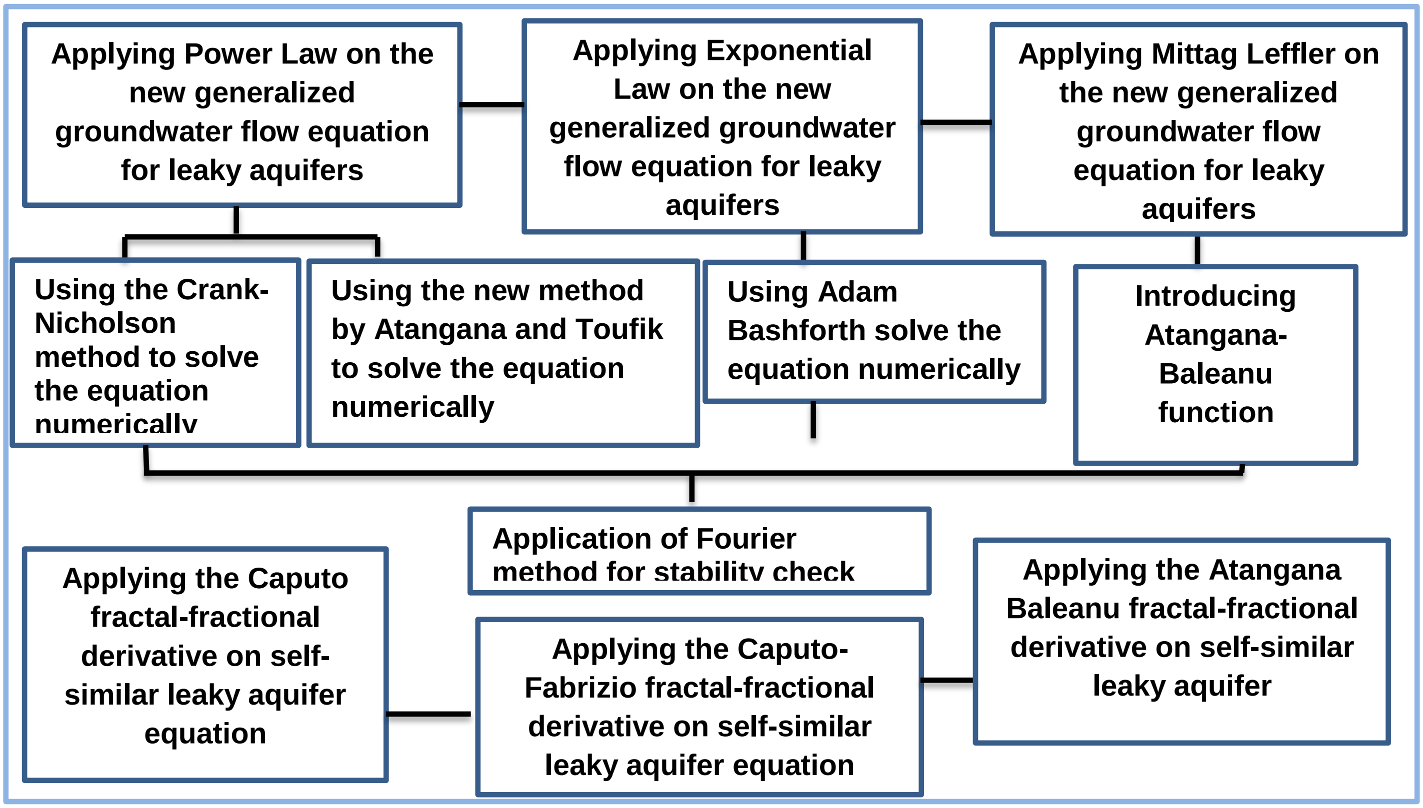


Figure 2: Research Framework outlining the main chapters of the thesis

1.6 APPROACH- STRUCTURE OF DISSERTATION

The dissertation is structured into eleven chapters which are consolidated from different sub-headings. Chapter one covers the background of leaky aquifers in general and also introduces the new generalized groundwater equation for leaky aquifers. Chapter two provides literature review of local operators, non-local operators and different numerical schemes that are used. Chapter three is application of power law on the new generalized equation. Chapter four covers application of exponential decay law on the new generalized equation. Chapter five is basically application of Mittag-Leffler law new generalized equation. Chapter six introduces the fractal-fractional derivatives and self-similar leaky aquifers. Chapter seven, eight and nine covers application of the new fractal-fractional derivatives on the self-similar leaky aquifer equations. In chapter ten, figures for Caputo and Caputo-Fabrizio are presented and interpreted. Chapter eleven concludes the aim, objectives and analysis of this thesis.

CHAPTER 2

2 LITERATURE REVIEW OF LOCAL OPERATORS, NON-LOCAL OPERATORS AND DIFFERENT NUMERICAL SCHEMES

2.1 LOCAL OPERATORS

Fractional calculus is considered as bible of mathematics, but recently it is widely used in many fields to model real-world problem. It has been recently applied to model complex biological systems with non-linear behavior and long-term memory. Fractional calculus is also applied in physical components and motions (Tarasov, 2016). Fractional derivative is used to model real problems in ecology, diffusion processes, viscoelastic system, control processing, fractions stochastic system, signal processing and in biology (Changpin et al., 2011).

Fractional derivative is defined by Marchaud, Riemann, Grünwald, Letnikov, Liouville, Riesz and Caputo-Fabrizio. Scaling variables such as (x) according to x^{α}, it is been described as fractal derivative in mathematics and applied mathematics. Porous media, aquifer and turbulence usually exhibit fractal properties. Local fractional derivative has been proposed in a sense that it can assist in describing or rather understanding and studying properties of fractal objects and processes in relation to them (Tarasov, 2016).

Local operators are classified or defined as local product of fields and their space-time derivative, in an interacting quantum field of theory (Zimmermann, 1970). The classical physical laws such as Darcy's Law, Fick's Law of diffusion and Fourier's Law cannot be used to assess heterogeneous media. Recently fractal derivative has been proposed to assess aquifers with such properties. Differentiation equation is delineated as indicated below,

$$k'(\alpha) = \lim_{v \to 0} \frac{k(v+\alpha) - k(\alpha)}{v} \tag{18}$$

From the classical term explanation, k is assumed to be a function defined at least on an interval containing the number a in its interior. If the assumption is correct and the limit is valid for$k'(a)$, then the change in k at a is called the classical derivative. The classical derivative of a linear function has a constant value equal to its classical slope

(Grossman and Katz, 1972). Fractional operators generalize classical operators. If the fractional operator is given in certain limits one can obtain the identity by using the limits, but however if the limits are ignored one will obtain a classical operator. For example if the fractional operator is given by the parameter s $\in(0, 1)$, then letting s$\to 0^+$ one obtains the identity and letting s$\to 1^-$ one gets the classical operator (Bucur, 2017).

2.2 NONLOCAL OPERATORS

Nonlocal derivatives have been identified as useful tool in many branches. Nonlocal operators are applied or rather used in modeling. Several models are used to describe or assist in anomalous diffusion processes, viscoelasticity, signal processing, geomorphology, materials sciences, fractals and so forth. These derivatives are capable of capturing long-term interactions. Fractional differentiation has the ability to describe memory effect. It has been captured in literature that the reverse memory is described by the integral as the inverse operator of the derivatives if and only the fractional derivative is capable of describing the memory (Atangana and Gomez, 2018).

Nonlocal behavior, introduced by the following integral operators: the fractional Laplacian, the Caputo and Marchaud fractional derivatives has been investigated by Bucur and proved that the nonlocal character given by Caputo and Marchaud induces some properties similar to those of the fractional Laplacian (Bucur, 2017). It has been proven that fractional Laplacian well describes non-local diffusion phenomena.

The Caputo and Marchaud local operators can be applied to describe casual system, also called a non-anticipative system. Their current models given at a specific time depends on the past. Bucur indicated or proved that Caputo stationary functions are locally dense in the space of smooth functions. The Caputo fractional derivative can be used to model long-range interaction. Furthermore, the extension operator of the Marchaud derivative is a local operator defined in one dimension whose trace is the original nonlocal operator itself (Bucur, 2017).

Nonlocal operators have been identified to replace the Brownian motion with other Lévy processes. Non-local operators are linked to stochastic processes, for example the pure jump Lévy processes whose infinitesimal generators are non-local operators. The pure jump processes was confirmed to be more appropriate to model financial markets

(Voigt, 2017). Non-local operators have been used to derive equations for certain flows and functional for manipulation of digitized images. This simply implies that the points in the same image domain can interact directly.

The Partial Differential Equations and variation techniques have been applied to non-local framework, as many of them are based processes, minimization and compaction methods (Gilboa and Osher, 2008). Non-local operator $(\mathcal{L}_k)$ with Dirichlet function boundary conditions that are applicable to homogeneous media were used to drive equations for non-trivial solutions (Servadei and Valdinoci, 2013). In 2016 a distinctive explicable was derived with integral conditions that accommodate the time-fractional type equation using the Caputo derivative which represents power law type. The research was based on a rectangular domain and the non-locality of the applied series expansion method the value problem was used (Karimov et al., 2016).

The non-local operators were initially considered following a semi-group; however, this group is linked to Markov process. This process is attained only when the prospect of each step is controlled by the antecedent steps that were acquired before. Manuscript is available that clearly indicate or rather suggest a new derivative that accommodates both non-singular kernel and non-locality. This new derivative can be applied in modeling the flow of heat in material with different scale and those with heterogeneous media. Previously heterogeneity was not taken into account and which made it difficult to analyse heterogeneous media. The new derivative is generalization of Mittag-Leffler function, Caputo, Riemann-Liouville approach and Laplace transform operator (Atangana and Baleanu 2016). Atangana indicated that the approach of using Riemann-Liouville and Caputo concept do not satisfy the basic properties of the Newtonian concept.

2.3 NUMERICAL SOLUTIONS

The characteristics and physical properties of water flowing within an aquifer are determined using the well-known groundwater flow equations, which are derived from classical equations. This results in Mathematical models for example partial differential equations with the first partition conditions, which are solved using analytical and numerical solutions. Analytical and numerical solutions can be seen complementary rather than competing, as numerical methods approximate solutions of mathematical equations. This can only be done when the algebraic methods cannot determine the exact solutions (Remani, 2013).

Analytical solutions are exact whereas numerical solutions are only approximations. Analytical solutions provide a reality check on the performance of numerical solutions. Analytical solutions are easily applied whereas numerical solutions are demanding a use of computer system. Analytical solutions can be applied to simple model where classical equations are used and numerical solutions can be applied on heterogeneous models with complex boundary conditions (Butcher, 2008).

2.3.1 NUMERICAL TECHNIQUES FOR ORDINARY DIFFERENTIAL EQUATIONS

Ordinary differential equations (ODE’s) have been escalated to assist as mathematical models in many fields and are considered as initial value problem. However it is less possible to get solutions that can be expressed in closed form for this type of mathematical equations, therefore numerical methods are been utilized to determine approximate solutions of ODE’s. Numerical methods for solving such equations are classified into two categories namely linear multistep methods and one-step methods.

2.3.1.1ONE STEP METHODS

2.3.1.1.1 FORWARD EULER’S METHOD (FEM)

The FEM method is used to approximate the numerical solution for first ordinary differential operators.

The solution was given as,

$$s_{w+1} = s_w + hf(j_w, s_w) \tag{19}$$

The solution increases from j_w to j_{w+1}=j_{w+h}. The method is simple to implement but there are errors in the solution which is indicated as the step error O (h^2) by (press et al., 1992) as indicated by Weisstein. For the solution to be adhered the so-called Courant-Friedrichs-Lewy condition must be fulfilled which states that when a function has a space discretization step that is greater than the modeling number should be omitted. Hence, Amen outlined that the local error in FEM decreases only when the square of the step dimension and the global error decreases linearly with the step dimension (Amen et al., 2004).

The difference was given as,

$$E_w = s(j_w) - j_w \tag{20}$$

2.3.1.1.2 BACKWARD EULER'S METHOD (BEM)

The BEM on the other hand is an implicit method as it solves the contemporary phase and the later phase time of a given equation. Weisstein outlined that the method is used widely to solve ODE that uses the following equation as opposed to the forward Euler's method.

The solution was represented as follows,

$$y_{s+1} - y_s = hf(t_{s+1,} y_{s+1}) \tag{21}$$

The BEM requires rearrangement of equation as opposed to the FE method, however they both have the same errors of which local and the global errors (Amen et al., 2004).

2.3.1.1.3 TRAPEZOIDAL METHOD (TM)

TM method is defined as a simple average of the implicit and the explicit Euler's methods. The method estimates the integrals or the area beneath the graph.

The solution was represented as,

$$2w(s+e) = 2w(s) + ew^1(s) + w^1(s+e) \tag{22}$$

However, the errors will be different from both the Euler's methods. The local error decreases as the cube of the step dimension. The global error goes down as the square of the step dimension also decreases. The method is taken to be more accurate than the Euler's methods (Amen et al., 2004). The derivation of the three equations cannot be discussed in this research, as they will not be used to come up with the analytical solution for the derived groundwater flow equation, as it will require a two-step method to be solved.

2.3.1.1.4 RUNGE KUTTA METHODS (RKMs)

RKMs fall under the implicit and explicit iterative methods that can be used for discretization for the approximate solutions of ODEs. Euler's method is only first order method which is simply different from the RKMs as the method aims to achieve higher accuracy. The method is performed or tested in different orders of accuracy (Kaw and Kalu, 2008).

2.3.1.2 LINEAR MULTISTEP METHODS (LMMs)

In the above discussion it was clear that only one point j_w was required to calculate the next approximation j_{w+1}, but for LMMs two steps are needed j_w and j_{w-1} to be able to approximate $j_{w+1.}$ The method only involves linear combinations for notation simplicity (Süli). Three commonly known and applied multistep expressions which are linear are backward differentiation formulas, Adams Moulton and Adams Bashforth methods.

2.3.1.3 BACKWARD DIFFERENTIAL FORMULAS (BDFs)

BDFs are generally used for the solution of stiff differential mathematical equations and differential algebraic mathematical equations. However, for differential algebraic equations the numerical method will only work if the equation has index no greater than one or has other special properties. For a given time t_n outlined as its function values $y(t)$ at t_n, this method or formulas can estimate the derivative of the specified function using the available data and increasing the accuracy of the estimation (Gear, 2007).

2.3.1.3.1 ADAMS-BASHFORTH METHODS (ABMs)

Adam methods estimate the integral with Lagrange polynomial method (LPM) within the interval $[t_n, t_{n+1}]$. They are classified into two known explicit and implicit methods, the Explicit-Adams Bashforth solutions and the Implicit-Adams Moulton solutions (Zeltkevic, 1998). The ABMs are classified as explicit multistep methods. They use transient and steady state information to derive a numerical solution of a complex function in a form of (t_{n+1}).

2.3.1.3.2 ADAMS-MOULTON METHODS (AMMs)

The numerical solution will not be discussed in this chapter. The method is not used for partial differential methods with both local and non-local operators; it was used to derive a numerical solution that can accommodate such operators with both localities (Atangana and Gnitchogna, 2017).

Single ODE is given as,

$$s(e_{w+2}) = s(e_{w+1}) + \frac{3h}{2} f\big(\mathrm{e}_{\mathrm{w}+1}, \mathrm{s}(\mathrm{e}_{\mathrm{w}+1})\big) - \frac{\mathrm{h}}{2} f(\mathrm{e}_{\mathrm{w}}, \mathrm{s}(\mathrm{e}_{\mathrm{w}})) \tag{23}$$

2.3.2 NUMERICAL TECHNIQUES FOR PARTIAL DIFFERENTIAL EQUATIONS

PDEs are currently used in technology, mathematics, natural science, engineering and other fields. PDE’s cannot assessed using the same methods as ODEs or neither the analytical methods due to their characteristics, boundaries, geometries and problems encountered. Most PDEs have no closed solution on complex domains; it is usually possible to derive solutions for some basic equations when simple domains are provided (Mathies, 2002).

Numerical methods for PDEs are categorized into Lagrangian and Euclidien algorithms. These two categories are differentiated in terms of the continuum nodes. It can further be divided to mesh free and mesh non-free. The mesh free is widely and usually used as they can accommodate any geometric changes of the domain of interest (Gonzalez-Casanova, 2005).

2.3.2.1 FINITE DIFFERENCE METHODS (FDMs)

This principle uses the differential quotients to approximate the differential derivatives. It is attributed in space and time or points. FDMs are capable of converting PDEs into discrete number of algebraic equations. The method is considered the oldest to solve differential equations, most popular and conceptually simple (Comsol, 2018).

The first order finite difference spatial approximation can be estimated in terms of backward, forward and central approximations, which was derived using Taylor series. However, the second order approximation differentiates the forward and backward finite difference approximations. First order 2D temporal finite difference approximations can be estimated using the explicit, implicit time approximations and Crank Nicolson.

2.3.2.1.1 CRANK NICOLSON METHOD (CNM)

Due to the unstable explicit schemes that are available and their limitations in terms of convergence, the Crank Nicolson scheme was developed and identified as the most superior because of stability, convergence and accuracy. The method is considered to have less truncation error.

The solution is defined in literature as,

$$s_w^{u+1} = s_w^u + \frac{\Delta t}{2\Delta x^2}\left[\left(e_{w-1}^u - 2e_w^u + e_{w+1}^u\right) + \left(e_{w-1}^{u+1} - 2e_w^{u+1} + e_{w+1}^{u+1}\right)\right] \tag{24}$$

Crank-Nicholson method is considered an implicit numerical solution scheme as the values to be computed are not just a function of values at the past time phase which are not obtainable (Mastorakis, 2006).

2.3.2.2 FINITE ELEMENT METHODS (FEMs)

The FEM is also considered to be popular, its applicability increases with availability of software programs and computer power. This method can select test and basic functions and it offers great freedom in the selection of discretization.

FEMs differ from the FDMs model, as they can approximate the flow equation by integrating rather than differentiating. As opposed to the rectangular shape of cell required by FDMs, FEMs cells shape can be triangular or almost any polygonal shape and the cell size can vary (Comsol, 2018).

CHAPTER 3

3 NUMERICAL SOLUTION OF THE NEW GENERALIZED MODEL OF GROUNDWATER MOVEMENT WITHIN A LEAKY AQUIFER USING POWER LAW

3.1 INTRODUCTION

The existence of power-law behaviour in nature grabbed attention and is applied in different practices or fields. The onset of power-law behaviour that is captured in earliest level can be analyzed directly in terms of the fundamental differential equations (Visser and Yunes, 2008). Power law has been recently utilized to model nature and its complexities. However the singularity associated with power law unfortunately disadvantages its use (Atangana and Gomez, 2018). The memory depicted with power law is usually not complete, fading and has no initial conditions.

Fractal derivatives are defined through a fractional integral in a sense that they elaborate the ratio of change between two quantities in fractal space. We acknowledge that some materials are heterogeneous and cannot be assessed using the classical Newtonian derivatives. Ordinary calculus and standard functions are not-applicable in assessing complex systems or heterogeneous and viscous media (Atangana and Gomez, 2018). Fractional differential operators are capable of dealing with boundary conditions in a simpler manner and can obtain exact expressions to illustrate composite structures with anomalous behaviours (Tateishi et al., 2017).

3.1.1 RIEMANN-LIOUVILLE OPERATOR

The Riemann-Liouville is applied in pure mathematics for example in finding or determining solution of integer-order differential equations. The Riemann-Liouville operator was the most fractional derivative to be used before the newly proposed different fractional-time operators (Atangana, 2017). This tool has been identified as the only mathematical tool to assist in assessing all physical problems.

The Riemann-Liouville derivative is given as,

$$ {}_aD_t^a h(t) = \frac{d^z}{dt^z}\, {}_aD_t^{-(z-a)} h(t) = \frac{d^z}{dt^z}\, {}_aI_t^{(z-a)} h(t) \tag{25} $$

The memory in the Riemann-Liouville was obtained by introducing the fractional integral and obtaining the fractional derivative. It has been stated that this operator has a precise average value over $x\, \epsilon[1, \infty]$ only if $x^{-k}, k > 2$ (Atangana and Gomez, 2018).

Atangana and Gomez indicated that the impulsive of the Riemann-Liouville fractional derivative indicates that the derivative does not have the required operators to pass from one scale to the other scale (Atangana and Gomez, 2018). It can only be useful to describe classical mechanics phenomena and it is scale invariant.

3.1.2 CAPUTO OPERATOR

The Caputo fractional derivative is defined via a modified Riemann-Liouville fractional integral; it is easy to use the Caputo fractional derivative as it does not depend on the initial conditions. Caputo made the first contribution into the Riemann-Liouville operators by transforming the convolution of fractional to power in order to get the initial conditions (Atangana, 2018).

The Caputo fractional time derivative is given as,

$$ D_t^a f(t) = \frac{1}{\Gamma(1-a)} \int_a^t \frac{d}{d\tau} \frac{h(\tau)}{(t-\tau)^a}\, d\tau \tag{26} $$

Caputo indicated that if a homogeneous media is given, the elastic properties of the media are simplified in terms of strain s and stresses. The strains and the stresses must be within a limited portion of field as they are linearly related by two parameters (Caputo, 1967). The Caputo derivative is introduced into equation (17),

$$ \frac{S}{T}\, {}_0^cD_t^\beta h(r,t) = \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\, {}_0^cD_r^\alpha h(r,t) + \frac{h(r,t)}{\lambda^2} \tag{27} $$

3.1.2.1 APPLYING THE CRANK NICHOLSON SCHEME INTO THE NEW CLASSICAL EQUATION THAT REPRESENTS THE MOVEMENT OF GROUNDWATER IN A LEAKY AQUIFER

The Crank Nicholson scheme is a second-order method given in time. The numerical scheme will only be applied to solve equation (17) as it cannot be applied into equation (27) due to the singularity. Discretizing equation (27),

$$D_t^a f(t) = \frac{1}{\Gamma(1-a)} \int_0^t \frac{d}{d\tau} h(0)(t-\tau)^{-a} d\tau \tag{28}$$

Therefore,

$$D_t^a f(t_{n+1}) = \frac{1}{\Gamma(1-a)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{h^{j+1} - h^j}{\Delta t} (t_{n+1} - \tau)^{-a} d\tau \tag{29}$$

$$D_t^a f(t_{n+1}) = \frac{1}{\Gamma(1-a)} \sum_{j=0}^{n} \frac{h^{j+1} - h^j}{\Delta t} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-a} d\tau \tag{30}$$

Let,

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-a} d\tau = \sigma_{n,j}^{\alpha} \tag{30.1}$$

and,

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-\beta} d\tau = \sigma_{n,j}^{\beta} \tag{30.2}$$

Therefore,

$$\frac{S}{T} \sum_{j=0}^{n} \left(h_i^{j+1} - h_i^j\right) \sigma_{n,j}^{\beta} = \frac{\partial^2 h}{\partial r^2} \left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r} \sum_{l=0}^{m} (h_{i+1}^{n+1} - h_{i-1}^{n+1}) \sigma_{n,j}^{\alpha} + \frac{h(r,t)}{\lambda^2} \tag{31}$$

$$\frac{S}{T} \sum_{j=0}^{n} \left(h_i^{j+1} - h_i^j\right) \sigma_{n,j}^{\beta} = \frac{1}{2} \left(\frac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{(\Delta r)^2} + \frac{h_{i+1}^{n} - 2h_i^{n} + h_{i-1}^{n}}{(\Delta r)^2} \right) \left(1 + \frac{\Delta r}{r_i}\right)$$

$$+\frac{1}{r_i}\sum_{l=0}^{m}(h_{i+1}^{n+1}-h_{i-1}^{n+1})\sigma_{n,j}^{\alpha}+\frac{h_i^{n+1}}{\lambda^2} \tag{32}$$

In order to simplify the notation let,

$\lambda_1=\frac{S}{T}\sigma_{n,j}^{\beta}$, $\lambda_2=\frac{1}{2(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)$, $\lambda_3=\frac{1}{r_i}\sigma_{n,j}^{\alpha}$, $\lambda_4=\frac{1}{\lambda^2}$

$$\sum_{j=0}^{n-1}\left(h_i^{j+1}-h_i^{j}\right)\lambda_1+\lambda_1(h_i^{n+1}-h_i^{n})=\lambda_2(h_{i+1}^{n+1}-2h_i^{n+1}+h_{i-1}^{n+1}+h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n})$$

$$+\lambda_3\sum_{l=0}^{m}(h_{i+1}^{n+1}-h_{i-1}^{n+1})+\lambda_4h_i^{n+1} \tag{33}$$

$$\sum_{j=0}^{n-1}\left(h_i^{j+1}-h_i^{j}\right)\lambda_1=\lambda_2(h_{i+1}^{n+1}-2h_i^{n+1}+h_{i-1}^{n+1}+h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n})-\lambda_1(h_i^{n+1}-h_i^{n})$$

$$+\lambda_3\sum_{l=0}^{m}(h_{i+1}^{n+1}-h_{i-1}^{n+1})+\lambda_4h_i^{n+1} \tag{34}$$

$$h_i^{n+1}(\lambda_1+2\lambda_2-\lambda_4)=h_i^{n}(\lambda_1-2\lambda_2)\;+\lambda_2(h_{i+1}^{n+1}+h_{i-1}^{n+1}+h_{i+1}^{n}+h_{i-1}^{n})$$

$$-\sum_{l=0}^{n-1}\left(h_i^{j+1}-h_i^{j}\right)\lambda_1+\;\lambda_3\sum_{l=0}^{m}(h_{i+1}^{n+1}-h_{i-1}^{n+1}) \tag{35}$$

The final numerical equation with the Crank Nicholson method is presented.

3.1.2.1.1 STABILITY ANALYSIS

The stability was verified using the Fourier series, the iteration method and recursion method. Fourier series is a very useful technique used to evaluate the stability of FDMs, Crank Nicholson method applied above is one of the FDMs. The other methods that were developed for the analysis of stability are mostly limited to linear problems.

$$h_i^{n+1}(\lambda_1+2\lambda_2-\lambda_4)=h_i^{n}(\lambda_1-2\lambda_2)\;\;+\lambda_2(h_{i+1}^{n+1}+h_{i-1}^{n+1}+h_{i+1}^{n}+h_{i-1}^{n})$$

$$-\sum_{j=0}^{n-1}\left(h_i^{j+1}-h_i^{j}\right)\lambda_1+\;\lambda_3\sum_{l=0}^{m-1}\left(h_{i+1}^{l+1}-h_{i-1}^{l+1}\right) \tag{36}$$

$$\delta_{n+1}e^{iklr}(\lambda_1 + 2\lambda_2 - \lambda_4) = \delta_n e^{iklr}(\lambda_1 - 2\lambda_2)$$

$$+\lambda_2\left(\delta_{n+1}e^{ikl(r+\Delta r)} + \delta_{n+1}e^{ikm(r-\Delta r)} + \delta_n e^{ikl(r+\Delta r)} + \delta_n e^{ikl(r-\Delta r)}\right)$$

$$-\sum_{j=0}^{n-1}\left(\delta_{j+1}e^{iklr} - \delta_j e^{iklr}\right)\lambda_1 + \lambda_3\sum_{l=0}^{m-1}\left(\delta_{l+1}e^{ikl(r+\Delta r)} - \delta_{l+1}e^{ikl(r-\Delta r)}\right) \quad (37)$$

Divide both sides by e^{iklr} and group the like terms,

$$\delta_{n+1}(\lambda_1 + 2\lambda_2 - \lambda_4) = \delta_n(\lambda_1 - 2\lambda_2) + \lambda_2\delta_{n+1}e^{ikl\Delta r} + \lambda_2\delta_{n+1}e^{-ikl\Delta r} + \lambda_2\delta_n e^{ikl\Delta r}$$

$$+\lambda_2\delta_n e^{-ikl\Delta r} - \sum_{j=0}^{n-1}\left(\delta_{j+1} - \delta_j\right)\lambda_1 + \lambda_3\sum_{l=0}^{m-1}\left(\delta_{l+1}e^{ikl\Delta r} - \delta_{l+1}e^{-ikl\Delta r}\right) \quad (38)$$

$$\delta_{n+1}\left(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2 e^{ikl\Delta r} - \lambda_2 e^{-ikl\Delta r}\right) = \delta_n\left(\lambda_1 - 2\lambda_2 + \lambda_2 e^{ikl\Delta r} + \lambda_2 e^{-ikl\Delta r}\right)$$

$$-\sum_{j=0}^{n-1}\left(\delta_{j+1} - \delta_j\right)\lambda_1 + \lambda_3\sum_{l=0}^{m-1}\delta_{l+1}\left(e^{ikl\Delta r} - e^{-ikl\Delta r}\right) \quad (39)$$

Therefore,

$$\delta_{n+1}(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])$$

$$= \delta_n(\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)]) - \sum_{j=0}^{n-1}\left(\delta_{n-j}\right)\lambda_1 + \lambda_3\sum_{l=0}^{m-1}(\delta_{l+1}[2i\sin(kl\Delta r)]) \quad (40)$$

If,

$\mathrm{n} = 0$

Therefore,

$$\delta_{n+1}(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)]) = \delta_n(\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)])$$

$$-\lambda_3\sum_{l=0}^{m-1}(\delta_{l+1}[2i\sin(kl\Delta r)]) \quad (41)$$

Applying the continuous recursion,

If,

$\mathrm{m} = 0$

Therefore,

$$\delta_{n+1}(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)]) = \delta_n(\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)]) \quad (42)$$

To check the stability of the above equation, iteration and recursion method will be applied.

To prove that,

$\left|\frac{\delta_1}{\delta_0}\right| < 1$

Let,

$\mathrm{n} = 0$

$$\left|\frac{(\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)])}{(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])}\right| < 1 \quad (43)$$

Assuming that the following different Conditions were applied:

Condition 1

If,

$$\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)] > 0 \quad (44)$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)] > 0 \quad (45)$$

Therefore

$$\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)] < \lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)] \quad (46)$$

$$\lambda_2[4\cos(kl\Delta r)] < 4\lambda_2 - \lambda_4 \quad (47)$$

$$\cos(kl\Delta r) < 1 - \frac{\lambda_4}{4\lambda_2} \quad (48)$$

Condition 2

If,

$$\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)] < 0 \quad (49)$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)] < \quad (50)$$

Therefore,

$$-\lambda_1 + 2\lambda_2 - \lambda_2[2\cos(kl\Delta r)] < -\lambda_1 - 2\lambda_2 + \lambda_4 + \lambda_2[2\cos(kl\Delta r)] \quad (51)$$

$$4\lambda_2 < \lambda_4 + 4\lambda_2[\cos(kl\Delta r)] \tag{52}$$

$$1 < \frac{\lambda_4}{4\lambda_2} + \cos(kl\Delta r) \tag{53}$$

Condition 3

If,

$$\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)] > 0 \tag{54}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)] < 0 \tag{55}$$

Therefore,

$$\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)] < -\lambda_1 - 2\lambda_2 + \lambda_4 + \lambda_2[2\cos(kl\Delta r)] \tag{56}$$

$$2\lambda_1 < \lambda_4 \tag{57}$$

Condition 4

If,

$$\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)] < 0 \tag{58}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)] > 0 \tag{59}$$

Therefore,

$$-\lambda_1 + 2\lambda_2 - \lambda_2[2\cos(kl\Delta r)] < \lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)] \tag{60}$$

$$-2\lambda_1 < -\lambda_4 \tag{61}$$

$$2\lambda_1 > \lambda_4 \tag{62}$$

We assume that when all integers are greater than 1,

$|\delta_{l+1}| < |\delta_0|$

When,
$\mathrm{n} = 0$, then at m+1

$$\delta_1(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)]) = \delta_0(\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)]) - \lambda_3 \sum_{l=0}^{m} (\delta_{l+1}[2i\sin(kl\Delta r)]) \tag{63}$$

$$|\delta_1(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)| = \left|\delta_0(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r)) - \lambda_3 \sum_{l=0}^{m} (\delta_{l+1}[2i\sin(kl\Delta r)])\right| \tag{64}$$

$$|\delta_1||(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)| \le |\delta_0||(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))|$$

$$+\left|\lambda_3 \sum_{l=0}^{m} (\delta_{l+1}[2i\sin(kl\Delta r)])\right| \tag{65}$$

$$|\delta_1||(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)| \leq |\delta_0||(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))|$$

$$+\sum_{l=0}^{m} (|\delta_{l+1}||\lambda_3||2i\sin(kl\Delta r)|) \tag{66}$$

Hypothetically the above can be written as follows,

$$|\delta_1||(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)|$$

$$< |\delta_0||(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))| + 2|\delta_0||\lambda_3|\sum_{l=0}^{m}(1) \tag{67}$$

$$|\delta_1||(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)| < |\delta_0||(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))|$$

$$+2|\delta_0||\lambda_3|(m+1) \tag{68}$$

$$\left|\frac{\delta_1}{\delta_0}\right| < \frac{|(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))| + 2|\lambda_3|(m+1)}{|(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)|} \tag{69}$$

$$\frac{|(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))| + 2|\lambda_3|(m+1)}{|(\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)|} < 1 \tag{70}$$

Considering different conditions of equation (70):

Condition 1

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) + 2\lambda_3(m+1) > 0 \tag{71}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) > 0 \tag{72}$$

Therefore,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) + 2\lambda_3(m+1) < \lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) \tag{73}$$

$$4\lambda_2\cos(kl\Delta r) + 2\lambda_3(m+1) < 4\lambda_2 - \lambda_4 \tag{74}$$

$$\cos(kl\Delta r) + \frac{\lambda_3(m+1)}{2\lambda_2} + \frac{\lambda_4}{4\lambda_2} < 1 \tag{75}$$

Condition 2

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2 \cos(kl\Delta r) + 2\lambda_3(m+1) < 0 \tag{76}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2 \cos(kl\Delta r) < 0 \tag{77}$$

Therefore,

$$-\lambda_1 + 2\lambda_2 - 2\lambda_2 \cos(kl\Delta r) - 2\lambda_3(m+1) < -\lambda_1 - 2\lambda_2 + \lambda_4 + 2\lambda_2 \cos(kl\Delta r) \tag{78}$$

$$4\lambda_2 - 2\lambda_3(m+1) < \lambda_4 + 4\lambda_2 \cos(kl\Delta r) \tag{79}$$

$$\cos(kl\Delta r) + \frac{\lambda_4}{4\lambda_2} + \frac{\lambda_3(m+1)}{2\lambda_2} > 1 \tag{80}$$

Condition 3

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2 \cos(kl\Delta r) + 2\lambda_3(m+1) < 0 \tag{81}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2 \cos(kl\Delta r) > 0 \tag{82}$$

Or

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2 \cos(kl\Delta r) + 2\lambda_3(m+1) > 0 \tag{83}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2 \cos(kl\Delta r) < 0 \tag{84}$$

The resulting solution for both conditions is the same.

$$-\lambda_1 + 2\lambda_2 - 2\lambda_2 \cos(kl\Delta r) - 2\lambda_3(m+1) < \lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2 \cos(kl\Delta r) \tag{85}$$

$$-2\lambda_3(m+1) < -\lambda_4 + 2\lambda_1 \tag{86}$$

$$\lambda_1 + \lambda_3(m+1) < \frac{\lambda_4}{2} \tag{87}$$

For all n>0,
Then,

$$\left|\sum_{l=0}^{m} \delta_{l+1}\right| < |\delta_0|(m+1) \tag{88}$$

We assume that,

$$|\delta_n| < |\delta_0| \tag{89}$$

$$\delta_{\mathrm{n+1}}(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])$$

$$= \delta_n(\lambda_1 - 2\lambda_2 + \lambda_2[2\cos(kl\Delta r)]) - \sum_{j=0}^{n-1}\left(\delta_{n-j}\right)\lambda_1 + \lambda_3 \sum_{l=0}^{m-1}(\delta_{l+1}[2i\sin(kl\Delta r)]) \tag{90}$$

$$|\delta_{\mathrm{n+1}}(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])|$$

$$= \left|\delta_n(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r)) - \sum_{j=0}^{n-1}\left(\delta_{n-j}\right)\lambda_1 + \lambda_3 \sum_{l=0}^{m-1}(\delta_{l+1}[2i\sin(kl\Delta r)])\right| \tag{91}$$

Considering the above assumptions:

$$|\delta_{\mathrm{n+1}}(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])|$$

$$< \left|\delta_0(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r)) - \sum_{j=0}^{n-1}(\delta_0)\lambda_1 + 2\lambda_3 \sum_{l=0}^{m-1}(\delta_0)\right| \tag{92}$$

$$|\delta_{\mathrm{n+1}}||(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])|$$

$$< |\delta_0||(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))| - |\delta_0||\lambda_1|(n+1)| + 2\lambda_3|\delta_0|(m+1)| \tag{93}$$

$$\left|\frac{\delta_{\mathrm{n+1}}}{\delta_0}\right| < \frac{|(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))| - |\lambda_1|(n+1) + |\,2\lambda_3|(m+1)}{|(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])|} \tag{94}$$

$$\frac{|(\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r))| - |\lambda_1|(n+1) + |\,2\lambda_3|(m+1)}{|(\lambda_1 + 2\lambda_2 - \lambda_4 - \lambda_2[2\cos(kl\Delta r)])|} < 1 \tag{95}$$

Different conditions of equation (95) were applied and the solutions below were obtained.

Condition 1

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) - \lambda_1(n+1) + 2\lambda_3(m+1) > 0 \tag{96}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) > 0 \tag{97}$$

Therefore,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) - \lambda_1(n+1) + 2\lambda_3(m+1) < \lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) \tag{98}$$

$$4\lambda_2\cos(kl\Delta r) - \lambda_1(n+1) + 2\lambda_3(m+1) < 4\lambda_2 - \lambda_4 \tag{99}$$

$$\cos(kl\Delta r) - \frac{\lambda_1(n+1)}{4\lambda_2} + \frac{\lambda_3(m+1)}{2\lambda_2} + \frac{\lambda_4}{4\lambda_2} < 1 \tag{100}$$

Condition 2

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) - \lambda_1(n+1) + 2\lambda_3(m+1) < 0 \tag{101}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) < 0 \tag{102}$$

Therefore,

$$-\lambda_1 + 2\lambda_2 - 2\lambda_2\cos(kl\Delta r) + \lambda_1(n+1) - 2\lambda_3(m+1) < -\lambda_1 - 2\lambda_2 + \lambda_4 + 2\lambda_2\cos(kl\Delta r) \tag{103}$$

$$4\lambda_2 + \lambda_1(n+1) - 2\lambda_3(m+1) < \lambda_4 + 4\lambda_2\cos(kl\Delta r) \tag{104}$$

$$1 < \cos(kl\Delta r) - \frac{\lambda_1(n+1)}{4\lambda_2} + \frac{\lambda_3(m+1)}{2\lambda_2} + \frac{\lambda_4}{4\lambda_2} \tag{105}$$

Condition 3

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) - \lambda_1(n+1) + 2\lambda_3(m+1) > 0 \tag{106}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) < 0 \tag{107}$$

or

If,

$$\lambda_1 - 2\lambda_2 + 2\lambda_2\cos(kl\Delta r) - \lambda_1(n+1) + 2\lambda_3(m+1) < 0 \tag{108}$$

and,

$$\lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r) > 0 \tag{109}$$

The resulting solution for both conditions is the same.

$$-\lambda_1 + 2\lambda_2 - 2\lambda_2\cos(kl\Delta r) + \lambda_1(n+1) - 2\lambda_3(m+1) < \lambda_1 + 2\lambda_2 - \lambda_4 - 2\lambda_2\cos(kl\Delta r)$$

$$\lambda_1(n+1) - 2\lambda_3(m+1) < 2\lambda_1 \tag{110}$$

$$\lambda_1(n+1) - 2\lambda_1 < 2\lambda_3(m+1) - \lambda_4 \tag{111}$$

$$\lambda_1(n+1-2) < 2\lambda_3(m+1) - \lambda_4 \tag{112}$$

3.1.2.2 APPLYING THE NEW NUMERICAL APPROXIMATION COMPILED BY ATANGANA AND TOUFIK

The concept of fractional derivative is widely used recently and several studies have been conducted. The concept of non-singular kernel and non-local was introduced in order to expand the limitations of the two known power law methods namely the Riemann-Liouville and Caputo fractional derivatives. However, it was recently identified that the current numerical schemes cannot be used to solve the PDEs. Equation (27) cannot be analysed using the current numerical schemes. Contemporary a new scheme was developed in order to analyze PDEs, which incorporates the two well-known concepts, the two-step Lagrange polynomial and the fundamental theorem of fractional calculus (Toufik and Atangana, (2017)).

Toufik and Atangana scheme derivation is given as,

$$\begin{cases} {}^{ABC}_{0}D_f(t) = h\big(t, f(t)\big) \\ f(0) = f_0 \end{cases} \tag{113}$$

$$f(t) - f(0) = \frac{1-\alpha}{ABC\ (\alpha)} h\big(t, f(t)\big) + \frac{\alpha}{\Gamma(\alpha) \times ABC(\alpha)} \int_0^t h\big(\tau, y(\tau)\big)(t-\tau)^{\alpha-1} d\tau \tag{114}$$

At t_{z+1},

$$f(t_{z+1}) - f(0) = \frac{1-\alpha}{ABC\ (\alpha)} h\big(t_z, f(t_z)\big) + \frac{\alpha}{\Gamma(\alpha) \times ABC(\alpha)} \int_0^{t_{z+1}} f\big(\tau, y(\tau)\big)(t_{z+1}-\tau)^{\alpha-1} d\tau$$

$$= \frac{1-\alpha}{ABC\ (\alpha)} h\big(t_z, f(t_z)\big) + \frac{\alpha}{\Gamma(\alpha) \times ABC(\alpha)} \sum_{w=0}^{z} \left(\int_w^{t_{w+1}} h\big(\tau, f(\tau)\big)(t_{z+1}-\tau)^{\alpha-1} d\tau \right) \tag{115}$$

Two-step Lagrange polynomial,

$$P_w(\tau) = \frac{\tau - t_{w-1}}{t_w - t_{w-1}} h\big(t_w, f(t_w)\big) - \frac{\tau - t_w}{t_w - t_{w-1}} f\big(t_{w-1}, y(t_{w-1})\big) \tag{116}$$

$$P_w(\tau) = \frac{h\big(t_w, f(t_w)\big)}{x}(\tau - t_{w-1}) - \frac{h\big(t_{w-1}, f(t_{w-1})\big)}{x}(\tau - t_w)$$

$$\simeq \frac{h(t_w, f_w)}{x}(\tau - t_{w-1}) - \frac{h(t_{w-1}, f_{w-1})}{x}(\tau - t_w) \tag{117}$$

Substituting equation (117) into equation (115),

$$
\begin{aligned}
f_{z+1} = f(0) + \frac{1-\alpha}{ABC\ (\alpha)} h\big(t_z, f(t_z)\big) \\
+ \frac{\alpha}{\Gamma(\alpha)\times ABC(\alpha)} \sum_{w=0}^{z} \left(\frac{h(t_w, f_w)}{x} \int_w^{t_{w+1}} (\tau - t_{w-1})(t_{z+1}-\tau)^{\alpha-1} d\tau \right) \\
- \frac{h(t_{w-1}, f_{w-1})}{x} \int_w^{t_{w+1}} (\tau - t_w)(t_{w+1}-\tau)^{\alpha-1} d\tau
\end{aligned}
\tag{118}
$$

To make the above equation in complex, the given elementary terms will be used. Let,

$$\mathrm{A}_{\alpha,\mathrm{w},1} = \int_w^{t_{w+1}} (\tau - t_w)(t_{z+1}-\tau)^{\alpha-1} d\tau \tag{119}$$

$$\mathrm{A}_{\alpha,\mathrm{w},1} = \mathrm{x}^{\alpha+1} \frac{(z+1-w)^{\alpha}(z-w+2+\alpha) - (z-w)^{\alpha}(z-w+2+2\alpha)}{\alpha(\alpha+1)} \tag{120}$$

$$\mathrm{A}_{\alpha,\mathrm{w},2} = \int_w^{t_{w+1}} (\tau - t_w)(t_{z+1}-\tau)^{\alpha-1} d\tau \tag{121}$$

$$\mathrm{A}_{\alpha,\mathrm{w},2} = \mathrm{x}^{\alpha+1} \frac{(z+1-w)^{\alpha+1} - \big((z-w)^{\alpha}(z-w+1+\alpha)\big)}{\alpha(\alpha+1)} \tag{122}$$

The following numerical scheme is obtained,

$$
\begin{aligned}
f_{z+1} = f(0) + \frac{1-\alpha}{ABC\ (\alpha)} h\big(z, f(t_z)\big) \\
+ \frac{\alpha}{ABC(\alpha)} \sum_{w=0}^{z} \left(\frac{x^{\alpha} f(t_w, y_w)}{\Gamma(\alpha+2)} \big((z+1-w)^{\alpha}(z-w+2+\alpha) - (z-w)^{\alpha}(z-w+2+2\alpha)\right) \\
- \frac{x^{\alpha} h(t_{w-1}, f_{w-1})}{\Gamma(\alpha+2)} \big((z+1-w)^{\alpha+1} - \big(z-w)^{\alpha}(z-w+1+\alpha)\big)
\end{aligned}
\tag{123}
$$

The above version of numerical scheme will be used to analyze equation below.

$${}_{0}^{c}D_t^{\beta} h(r,t) = \frac{T}{S}\left[\frac{\partial^2 h}{\partial r^2}\left(1+\frac{\Delta r}{r}\right) + \frac{1}{r}\, {}_{0}^{c}D_r^{\alpha} h(r,t) + \frac{h(r,t)}{\lambda^2}\right] \tag{124}$$

Let the function above be represented as follows,

$${}_{0}^{c}D_t^{\beta} h(r,t) = f(r,t,h(r,t)\ ) \tag{125}$$

at interval r_s, t_{v+1}, F$\left(r_s, t_{v,} h_s^v\right)$ will be given as follows,

$$F\left(r_s, t_{v,} h_s^v\right) = \frac{T}{S}\left[\frac{h_{s+1}^v - 2h_s^v + h_{s-1}^v}{(\Delta r)^2}\left(1 + \frac{\Delta r}{r_s}\right) + \frac{h_s^v}{\lambda^2} + \frac{1}{r_s}\ {}_0^cD_r^\alpha h(r_s, t_v)\right] \tag{126}$$

However ${}_0^cD_r^\alpha h(r_s, t_v)$ can be written as,

$$\begin{aligned}
{}_0^cD_r^\alpha h(r_s, t_v) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{r_i}\frac{\partial}{\partial\tau}h(\tau, t_v)(r_s - \tau)^{-\alpha}d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{w=0}^{i-1}\left(\int_{r_w}^{r_{w+1}}\frac{h(r_{s+1}, t_v) - h(r_{s-1}, t_v)}{\Delta r}(r_s - \tau)^{-\alpha}d\tau\right) \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\right)\int_{r_w}^{r_{w+1}}(r_s - \tau)^{-\alpha}d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\right)\int_{r_s - r_{w+1}}^{r_s - r_w} y^{-\alpha}dy \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\left[\frac{(r_s - r_w)^{1-\alpha}}{1-\alpha} - \frac{(r_s - r_{w+1})^{1-\alpha}}{1-\alpha}\right]\right) \\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\left[(r_s - r_w)^{1-\alpha} - (r_s - r_{w+1})^{1-\alpha}\right]\right)
\end{aligned} \tag{127}$$

Let,

$$r_s = \Delta r\ s, r_w = \Delta r w \tag{128}$$

and,

$$r_{w+1} = \Delta r(w+1) \tag{129}$$

Therefore,

$${}_0^cD_r^\alpha h(r_s, t_v) = \frac{(\Delta r)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\left[(s-w)^{1-\alpha} - (s-w-1)^{1-\alpha}\right]\right) \tag{130}$$

From the solution above, F$\left(r_s, t_{v,} h_s^v\right)$ and $F\left(r_s, t_{v-1,} h_s^{v-1}\right)$ can be written as follows,

$$F\left(r_s, t_{v,} h_s^v\right) = \frac{T}{S}\left[\frac{h_{s+1}^v - 2h_s^v + h_{s-1}^v}{(\Delta r)^2}\left(1 + \frac{\Delta r}{r_s}\right) + \frac{h_s^v}{\lambda^2}\right] + \frac{T}{S}\left[\frac{1}{r_s}\frac{(\Delta r)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\left[(s-w)^{1-\alpha} - (s-w-1)^{1-\alpha}\right]\right)\right] \quad (131)$$

and,

$$F\left(r_s, t_{v-1,} h_s^{v-1}\right) = \frac{T}{S}\left[\frac{h_{s+1}^{v-1} - 2h_s^{v-1} + h_{s-1}^{v-1}}{(\Delta r)^2}\left(1 + \frac{\Delta r}{r_s}\right) + \frac{h_s^{v-1}}{\lambda^2}\right] + \frac{T}{S}\left[\frac{1}{r_s}\frac{(\Delta r)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{w=0}^{s-1}\left(\frac{h_{s+1}^{v-1} - h_{s-1}^{v-1}}{\Delta r}\left[(s-w)^{1-\alpha} - (s-w-1)^{1-\alpha}\right]\right)\right] \quad (132)$$

Discretizing ${}_0^cD_t^\beta h(r,t)$ the following is obtained,

$$h(r,t) - h(r,0) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f\left(r, \tau, h(r,\tau)\right) d\tau \quad (133)$$

at interval r_s, t_{z+1},

$$h_s^{z+1} = h_s^0 + \sum_{v=0}^{z}\left[\frac{(\Delta t)^\beta}{\Gamma(\beta+2)} f\left(r_s, t_{v,} h_s^v\right)\{(z+1-v)^\beta(z-v+2+\beta) - (z-v)^\beta(z-v+2+2\beta)\}\right] - \sum_{v=o}^{z}\left[\frac{(\Delta t)^\beta}{\Gamma(\beta+2)} f\left(r_s, t_{v-1}, h_s^{v-1}\right)\{(z+1-v)^{\beta+1} - (z-v)^\beta(z-v+1+\beta)\}\right] \quad (134)$$

Therefore, the solution can be given as,

$$h_s^{z+1} = h_s^0$$

$$\sum_{v=0}^{z}\left[\frac{(\Delta t)^\beta}{\Gamma(\beta+2)}\frac{T}{S}\left\{\frac{h_{s+1}^v - 2h_s^v + h_{s-1}^v}{(\Delta r)^2}\left(1 + \frac{\Delta r}{r_s}\right) + \frac{h_s^v}{\lambda^2}\right\}\{(z+1-v)^\beta(z-v+2+\beta) - (z - v)^\beta(z-v+2+2\beta)\}\right]$$

$$+\sum_{v=0}^{z}\left[\frac{(\Delta t)^\beta}{\Gamma(\beta+2)}\frac{T}{S}\left\{\frac{1}{r_s}\frac{(\Delta r)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{w=0}^{s-1}\frac{h_{s+1}^v - h_{s-1}^v}{\Delta r}\left[(s-w)^{1-\alpha} - (s-w-1)^{1-\alpha}\right]\right\}\{(z+1 - v)^\beta(z-v+2+\beta) - (z-v)^\beta(z-v+2+2\beta)\}\right]$$

$$-\sum_{v=o}^{z}\left[\frac{(\Delta t)^{\beta}}{\Gamma(\beta+2)}\frac{T}{S}\left\{\frac{h_{s+1}^{v-1}-2h_{s}^{v-1}+h_{s-1}^{v-1}}{(\Delta r)^{2}}\left(1+\frac{\Delta r}{r_{s}}\right)+\frac{h_{s}^{v-1}}{\lambda^{2}}\right\}\{(z+1-v)^{\beta+1}-(z-v)^{\beta}(z-v+1+\beta)\}\right]$$

$$-\sum_{v=o}^{z}\left[\frac{(\Delta t)^{\beta}}{\Gamma(\beta+2)}\frac{T}{S}\left\{\frac{1}{r_{s}}\frac{(\Delta r)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{w=0}^{s-1}\frac{h_{s+1}^{v-1}-h_{s-1}^{v-1}}{\Delta r}[(s-w)^{1-\alpha}-(s-w-1)^{1-\alpha}]\right\}\{(z+1-v)^{\beta+1}-(z-v)^{\beta}(z-v+1+\beta)\}\right] \quad (135)$$

The above numerical scheme can be used to generate figures for different values of alpha and beta.

CHAPTER 4

4 NUMERICAL SOLUTION TO THE GENERALIZED EQUATION OF GROUNDWATER MOVEMENT IN A LEAKY AQUIFER WITH EXPONENTIAL DECAY LAW

Exponential decay law appears in many scenarios, it was applied back without proper definition and use. As opposed to the power law, it can capture the initial or beginning condition and it is complete.

4.1 CAPUTO FABRIZIO

Caputo and Fabrizio were the first researchers to indicate that the current fractional differential operators are not reliable for heterogeneous media and always produce misleading analyses (Atangana, 2018). New exact meaning of fractional derivative with a non-singular kernel for both temporal and spatial variables, whereby they used Laplace transform for temporal variables and Fourier transform for spatial variables was presented. The Caputo-Fabrizio fractional derivative it is represented as,

$$_{CF}D_{at}^{\beta}z(t) = \frac{1}{1-\alpha}\int_{\alpha}^{t} z'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds \tag{136}$$

This new proposed derivative can be used to study non-local system and capable of analyzing heterogeneous materials and their fluctuations (Caputo and Fabrizio, 2015). The law of exponential decay will be applied on the modified or new generalized equation which represents groundwater movement in a leaky aquifer. The equation can be written as follows,

$$^{CF}_{0}D_{t}^{\beta}h(r,t) = \frac{T}{S}\left[\frac{\partial^2 h}{\partial r^2}\left(1+\frac{\Delta r}{r}\right)+\frac{1}{r}\,{}^{CF}_{0}D_{r}^{\alpha}h(r,t)+\frac{h(r,t)}{\lambda^2}\right] \tag{137}$$

Where $^{CF}_{0}D_{t}^{\beta}$ is the Caputo-Fabrizio derivative defined as,

$$^{CF}_{0}D_{t}^{\beta}z(t) = \frac{M(\beta)}{1-\beta}\int_{0}^{t} z'(x)exp\left[-\frac{\beta(t-x)}{1-\beta}dx\right] \tag{138}$$

Where M (β) is the normalization function,

If w$\geq$ 1 than,

$$ {}_{0}^{CF}D_t^{\beta} z(t_w) = \frac{M(\beta)}{1-\beta}\int_0^{t_w} z'(\tau) exp\left[-\frac{\beta(t_w-\tau)}{1-\beta}d\tau\right] \tag{139} $$

$$ = \frac{M(\beta)}{1-\beta}\sum_{l=0}^{w}\left(\frac{f(t_{l+1})-f(t_l)}{\Delta t}\right)\int_{t_w-t_{l+1}}^{t_w-t_l} z'(\tau) exp\left[-\frac{\beta(t_w-\tau)}{1-\beta}d\tau\right] \tag{140} $$

$$ = \frac{M(\beta)}{1-\beta}\sum_{l=0}^{w}\left(\frac{z(t_{l+1})-z(t_l)}{\Delta t}\right)\frac{1-\beta}{\beta} exp\Big|_{t_l-t_{l+1}}^{t_w-t_l} \tag{141} $$

$$ = \frac{M(\beta)}{1-\beta}\sum_{l=0}^{w}\left(\frac{z(t_{l+1})-z(t_l)}{\Delta t}\right)\left\{exp\left[\frac{\beta}{1-\beta}(t_w-t_l)\right]-exp\left[\frac{\beta}{1-\beta}(w-t_{l+1})\right]\right\} \tag{142} $$

Let,

$$ exp\left[\frac{\beta}{1-\beta}(t_w-t_l)\right]-exp\left[\frac{\beta}{1-\beta}(t_n-t_{l+1})\right] = A_l^{\beta} \tag{143} $$

$$ {}_{0}^{CF}D_t^{\beta} z(t_w) = \frac{M(\beta)}{\beta}\sum_{j=0}^{n}\left(\frac{z(t_{l+1})-z(t_l)}{\Delta r}\right)A_l^{\alpha} \tag{144} $$

4.1.1 NUMERICAL APPROXIMATION USING THE ADAM-BASHFORTH METHOD

Adam Bashforth method will be used to numerically analyse the mathematical equation, the method is represented as follows,

$$ \frac{\partial x}{\partial t} = h\big(s,t,x(s,t)\big) \tag{145} $$

$$ x(s,t)-x(s,0) = \int_0^t h\big(s,t,x(s,\tau)\big)\,d\tau \tag{146} $$

The boundary conditions can be given as $t = (t_w)$ and $t = (t_{w+1})$,

At t_{w+1},

$$ x(s,t_{w+1})-u(s,0) = \int_0^{t_{w+1}} h\big(s,\tau,x(s,\tau)\big)\,d\tau \tag{147} $$

At t_w,

$$ x(s,t_w)-x(s,0) = \int_0^{t_w} h\big(s,\tau,x(s,\tau)\big)\,d\tau \tag{148} $$

$t_{w+1} - t_w$,

$$x(s, t_{w+1}) - x(s, t_w) = \int_0^{t_{w+1}} h\big(s, \tau, x(s, \tau)\big)\, d\tau - \int_0^{t_w} h\big(s, \tau, x(s, \tau)\big)\, d\tau \qquad (149)$$

Conventions: applying reversing limit integral,

$$x(s, t_{w+1}) - x(s, t_w) = \int_0^{t_{w+1}} h\big(s, \tau, x(s, \tau)\big)\, d\tau + \int_{t_w}^{0} h\big(s, \tau, x(s, \tau)\big)\, d\tau \qquad (150)$$

$$x(s, t_{w+1}) - x(s, t_w) = \int_{t_w}^{t_{w+1}} h\big(s, \tau, x(s, \tau)\big)\, d\tau \qquad (151)$$

Application of the LPD to approximate$\frac{\partial x}{\partial t}$,

$$p(t) = \frac{\tau - t_{w-1}}{t_w - t_{w-1}}\ h\big(s, t_w, x(s, t_{w-1})\big) + \frac{\tau - t_w}{t_{w-1} - t_w} h\big(s, t_{w-1}, x(s, t_{w-1})\big) \qquad (152)$$

$$x(s, t_{w+1}) - x(s, t_w) =$$

$$\int_{t_w}^{t_{w+1}} \left[\frac{\tau - t_{w-1}}{t_w - t_{w-1}}\ h\big(s, t_w, x(s, t_w)\big) + \frac{\tau - t_w}{t_{w-1} - t_n} h\big(s, t_{w-1}, x(s, t_{w-1})\big)\right] \qquad (153)$$

$$x(s, t_{w+1}) - x(s, t_w) = \frac{3\Delta t}{2} h\big(s, t_w, x(s, t_w)\big) - \frac{\Delta t}{2} h\big(s, t_{w-1}, x(s, t_{w-1})\big) \qquad (154)$$

At s_j,

$$x_j^{w+1} - x_j^{w} = \frac{3\Delta t}{2} h\big(s_j, t_w, x_j^{w}\big) - \frac{\Delta t}{2} h\big(s_j, t_{w-1}, x_j^{w-1}\big) \qquad (155)$$

Introducing Caputo Fabrizio Fractional Derivative,

$$ {}_{0}^{CF}D_t^{\beta} h(r, t) = \frac{T}{S}\left[\frac{\partial^2 h(r, t)}{\partial r^2}\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\frac{\partial h(r, t)}{\partial r} + \frac{h(r, t)}{\lambda^2}\right] \qquad (156)$$

$$x(s, t) - x(s, 0) = \frac{1 - \beta}{M(\beta)} h\big(s, t, x(s, t)\big) + \frac{\beta}{M(\beta)} \int_0^{t_{w+1}} h\big(s, \tau, x(s, \tau)\big)\, d\tau \qquad (157)$$

At t_{w+1},

$$x(s,t_{w+1})-x(s,0)=\frac{1-\beta}{M(\beta)}f\big(s,t_w,x(s,t_w)\big)+\frac{\beta}{M(\beta)}\int_0^{t_{w+1}}h\big(s,\tau,x(s,\tau)\big)\,d\tau \quad (158)$$

At t_w,

$$x(s,t_w)-x(s,0)=\frac{1-\beta}{M(\beta)}h\big(s,t_{w-1},x(s,t_{w-1})\big)+\frac{\beta}{M(\beta)}\int_0^{t_{w+1}}h\big(s,\tau,x(s,\tau)\big)\,d\tau \quad (159)$$

When $t_{w+1}-t_w$ we get the following,

$$\begin{aligned}x(s,t_{w+1})-x(s,t_w)&=\frac{1-\beta}{M(\beta)}\Big[h\big(s,t_w,x(s,t_w)\big)-h\big(s,t_{w-1},x(s,t_{w-1})\big)\Big]\\&+\frac{\beta}{M(\beta)}\int_0^{t_{w+1}}h(s,\tau,x(s,\tau))\,d\tau \qquad (160)\end{aligned}$$

At s_j,

$$\begin{aligned}x_j^{w+1}-x_j^w&=\frac{1-\beta}{M(\beta)}\left[h\left(s_j,t_w,x\big(s_{j,}x_j^w\big)\right)-h\left(s_j,t_{w-1},x\big(s_j,x_j^{w-1}\big)\right)\right]\\&+\frac{\beta}{M(\beta)}\left[\frac{3\Delta t}{2}h\big(s_j,t_w,x_j^w\big)-\frac{\Delta t}{2}h\big(s_j,t_{w-1},x_j^{w-1}\big)\right] \qquad (161)\end{aligned}$$

Therefore the function can be given as,

$$\begin{aligned}x_j^{w+1}-x_j^w&=\frac{1-\beta}{M(\beta)}\frac{T}{S}\left[\left(\frac{x_{j+1}^w-2x_j^w+x_{j-1}^w}{(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1}{r_j}\frac{x_{j+1}^w-x_j^w}{\Delta r}+\frac{x_j^w}{\lambda^2}\right]\\&-\frac{1-\beta}{M(\beta)}\frac{T}{S}\left[\left(\frac{x_{j+1}^{w-1}-2x_j^{w-1}+x_{j-1}^{w-1}}{(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1}{r_j}\frac{x_{j+1}^{w-1}-x_j^{w-1}}{\Delta r}+\frac{x_j^{w-1}}{\lambda^2}\right]\\&+\frac{\beta}{M(\beta)}\frac{3T\Delta t}{2S}\left[\left(\frac{x_{j+1}^w-2x_j^w+x_{j-1}^w}{(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1}{r_j}\frac{u_{j+1}^w-u_j^w}{\Delta r}+\frac{u_j^w}{\lambda^2}\right]\\&-\frac{\beta}{M(\beta)}\frac{T\Delta t}{2S}\left[\left(\frac{x_{j+1}^{w-1}-2x_j^{w-1}+x_{j-1}^{w-1}}{(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)++\frac{1}{r_j}\frac{x_{j+1}^{w-1}-u_j^{nw-1}}{\Delta r}+\frac{u_j^{w-1}}{\lambda^2}\right](162)\end{aligned}$$

Grouping the like terms for simplification,

$$x_j^{w+1} = x_j^{w}\left(\begin{array}{c} 1+\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S}-\frac{1-\beta}{M(\beta)}\frac{2T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2\lambda^2 S} \\ -\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2Sr_j\Delta r} \end{array}\right)$$

$$+x_{j+1}^{w}\left(\begin{array}{c} \frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right) \\ +\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2Sr_j\Delta r} \end{array}\right)$$

$$+x_{j-1}^{w}\left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)\right)$$

$$-x_{j+1}^{w-1}\left(\begin{array}{c} \left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\left(\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right) \\ +\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2Sr_j\Delta r} \end{array}\right)$$

$$-x_{j}^{w-1}\left(\begin{array}{c} \left(\frac{1-\beta}{M(\beta)}\frac{2T}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S} \\ -\left(\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2\lambda^2 S} \end{array}\right)$$

$$-x_{j-1}^{w-1}\left(\left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\left(\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)\right) \quad (163)$$

Let,

$$a_1 = \left(\begin{array}{c} 1+\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S}-\frac{1-\beta}{M(\beta)}\frac{2T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2\lambda^2 S} \\ -\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2Sr_j\Delta r} \end{array}\right) \quad (164)$$

$$a_2 = \left(\begin{array}{c} \frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right) \\ +\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2Sr_j\Delta r} \end{array}\right) \quad (165)$$

$$a_3 = \left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)\right) \quad (166)$$

$$a_4 = \begin{pmatrix} \left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r} \\ +\left(\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right) \\ +\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2Sr_j\Delta r} \end{pmatrix} \tag{167}$$

$$a_5 = \begin{pmatrix} \left(\frac{1-\beta}{M(\beta)}\frac{2T}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{T}{Sr_i\Delta r}+\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S} \\ -\left(\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)-\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2\lambda^2 S} \end{pmatrix} \tag{168}$$

And,

$$a_6 = \left(\left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\left(\frac{1-\beta}{M(\beta)}\frac{T\Delta t}{2S(\Delta r)^2}\right)\left(1+\frac{r_{j+1}-r_j}{r_j}\right)\right) \tag{169}$$

$$x_j^{w+1} = x_j^w a_1 + x_{j+1}^w a_2 + x_{j-1}^w a_3 - x_{j+1}^{w-1} a_4 - x_j^{w-1} a_5 - x_{j-1}^{w-1} a_6 \tag{170}$$

4.1.1.1 STABILITY ANALYSIS USING THE VON NEUMANN

The Von Neumann stability method incorporates the Fourier’s series and the Euler formulas. This method was developed in Los Alamos during the course of World War II by Von Neumann and was considered classified until its definition in Cranck and Nicholson (Charney et at., 1950). At present, this is the most widely applied technique for stability analysis.

$$\rho(x,t) = \sum_f \left(\hat{\rho}(t)e^{i\Delta x k_x}\right) \tag{171}$$

$$\begin{aligned} \hat{\rho}_{n+1}e^{i\Delta x k_x} &= a_1\hat{\rho}_n e^{i\Delta x k_x} + a_2\hat{\rho}_n e^{i(x+\Delta x)k_x} + a_3\hat{\rho}_n e^{i(\Delta x-x)k_x} \\ &\quad + a_4\hat{\rho}_{n-1}e^{i(x+\Delta x)k_x} + a_5\hat{\rho}_{n-1}e^{i\Delta x k_x} + a_6\hat{\rho}_{n-1}e^{i(\Delta x-x)k_x} \end{aligned} \tag{172}$$

Divide both sides by $e^{i\Delta x k_x}$ and grouping like terms,

$$\hat{\rho}_{n+1} = \hat{\rho}_n a_1 + \hat{\rho}_n a_2 e^{ixk_x} + \hat{\rho}_n a_3 e^{-ixk_x} + \hat{\rho}_{n-1} a_4 e^{ixk_x} + \hat{\rho}_{n-1} a_5 + \hat{\rho}_{n-1} a_6 e^{-ixk_x} \tag{173}$$

$$\hat{\rho}_{n+1} = \left(a_1 + a_2 e^{ixk_x} + a_3 e^{-ixk_x}\right)\hat{\rho}_n + \left(a_4 e^{ixk_x} + a_5 + a_6 e^{-ixk_x}\right)\hat{\rho}_{n-1} \tag{174}$$

When n=0 and $\hat{\rho}_{-1}$ will not be applicable,

$$\hat{\rho}_1 = \left(a_1 + a_2 e^{ixk_x} + a_3 e^{-xik_x}\right)\hat{\rho}_0 + \left(e^{ixk_x} a_4 + a_5 + e^{-xik_x} a_6\right)\hat{\rho}_{-1} \tag{175}$$

$$\hat{\rho}_1 = \left(a_1 + a_2 e^{ixk_x} + a_3 e^{-xik_x}\right)\hat{\rho}_0 \tag{176}$$

$$\frac{\hat{\rho}_1}{\hat{\rho}_0} = \left(a_1 + a_2 e^{ixk_x} + a_3 e^{-ixk_x}\right) \tag{177}$$

$$\left|\frac{\hat{\rho}_1}{\hat{\rho}_0}\right| \leq \left|\left(a_1 + a_2 e^{ixk_x} + a_3 e^{-ixk_x}\right)\right| < 1 \tag{178}$$

Applying Euler's formulas for simplification,

$$\left|\frac{\hat{\rho}_1}{\hat{\rho}_0}\right| \leq |(a_1 + a_2[cos(xk_x) + i\,sin(xk_x)] + a_3[cos(xk_x) - i\,sin(xk_x)])| < 1 \tag{179}$$

$$\left|\frac{\hat{\rho}_1}{\hat{\rho}_0}\right| \leq |[\mathrm{a}_1 + \cos(\mathrm{xk_x})\,(\mathrm{a}_2{+}\mathrm{a}_3) + \mathrm{i}\sin(\mathrm{xk_x})\,(\mathrm{a}_2{-}\mathrm{a}_3)]| < 1 \tag{180}$$

$$|a_1| + |(a_2{+}a_3)||cos(xk_x)| + |(a_2{-}a_3)||i\,sin(xk_x)| < 1 \tag{181}$$

Substituting $a_1, a_{2\ and}\ a_3$ in the equation above,

$$\begin{aligned}(a_2{+}a_3) = {} & \left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2Sr_j\Delta r}\right)\\ & +\left(\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{2S(\Delta r)^2}\left(1+\frac{r_{j+1}-r_j}{r_j}\right)\right)\end{aligned} \tag{182}$$

$$\begin{aligned}(a_2{+}a_3) &= \frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{2r_{j+1}}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)+\frac{1-\beta}{M(\beta)}\frac{3T\Delta t}{S(\Delta r)^2}\left(\frac{r_{j+1}}{r_j}\right)\\ &= \frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)\end{aligned} \tag{183}$$

$$(\mathrm{a}_2 - \mathrm{a}_3) = \left(\begin{array}{c} \frac{1-\beta}{M(\beta)} \frac{T}{S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) \\ + \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} \\ + \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) \\ + \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2S r_j \Delta r} \end{array} \right) - \left(\begin{array}{c} \frac{1-\beta}{M(\beta)} \frac{T}{S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) \\ + \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) \end{array} \right)$$

$$= \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} \left(\frac{2 + 3\Delta t}{2}\right) \tag{184}$$

Therefore,

$$\left| \left(\begin{array}{c} 1 + \frac{1-\beta}{M(\beta)} \frac{T}{\lambda^2 S} - \frac{1-\beta}{M(\beta)} \frac{2T}{S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) \\ - \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} + \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2\lambda^2 S} \\ - \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) - \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2S r_j \Delta r} \end{array} \right) \right|$$

$$+ \left| \frac{1-\beta}{M(\beta)} \frac{T}{S(\Delta r)^2} \left(\frac{r_{j+1}(2 + 3\Delta t)}{r_j}\right) + \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} \left(\frac{2 + 3\Delta t}{2}\right) \right| |cos(x k_x)|$$

$$+ \left| \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} \left(\frac{2 + 3\Delta t}{2}\right) \right| |j\, sin(x k_x)| < 1 \tag{185}$$

If $a_1 < 0$ *and* $(a_2 + a_3)$ *and* $(\mathrm{a}_2 - \mathrm{a}_3) > 0$,

$$-1 - \frac{1-\beta}{M(\beta)} \frac{T}{\lambda^2 S} + \frac{1-\beta}{M(\beta)} \frac{2T}{S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) + \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} - \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2\lambda^2 S}$$

$$+ \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{S(\Delta r)^2} \left(1 + \frac{r_{j+1} - r_j}{r_j}\right) + \frac{1-\beta}{M(\beta)} \frac{3T\Delta t}{2S r_j \Delta r}$$

$$+ \left[\frac{1-\beta}{M(\beta)} \frac{T}{S(\Delta r)^2} \left(\frac{r_{j+1}(2 + 3\Delta t)}{r_j}\right) + \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} \left(\frac{2 + 3\Delta t}{2}\right) \right] cos(x k_x)$$

$$+ \frac{1-\beta}{M(\beta)} \frac{T}{S r_j \Delta r} \left(\frac{2 + 3\Delta t}{2}\right) < 1 \tag{186}$$

$$-1-\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S}\left(1+\frac{3\Delta t}{2}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{4+6\Delta t}{2}\right)$$

$$+\left[\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)\right]cos(xk_x)<1 \tag{187}$$

Simplifying the above equation, the resulting equation is obtained,

$$-\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S}\left(1+\frac{3\Delta t}{2}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_i}\right)(cos(xk_x)+1)$$

$$+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{4+6\Delta t}{2}+\left(\frac{2+3\Delta t}{2}\right)cos(xk_x)\right)<2 \tag{188}$$

If $a_1, (a_2+a_3)\ and\ (\mathrm{a}_2-\mathrm{a}_3)\ >0$,

$$1+\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S}\left(1+\frac{3\Delta t}{2}\right)-\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_i}\right)-\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)$$

$$+\left[\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_j}\right)+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)\right]cos(xk_x)$$

$$+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)<1 \tag{189}$$

Therefore, the final stability analysis can be given as,

$$\frac{1-\beta}{M(\beta)}\frac{T}{\lambda^2 S}\left(1+\frac{3\Delta t}{2}\right)\ +\frac{1-\beta}{M(\beta)}\frac{T}{S(\Delta r)^2}\left(\frac{r_{j+1}(2+3\Delta t)}{r_j}\right)(cos(xk_x)-1)$$

$$+\frac{1-\beta}{M(\beta)}\frac{T}{Sr_j\Delta r}\left(\frac{2+3\Delta t}{2}\right)cos(xk_x)<0 \tag{190}$$

CHAPTER 5

5 GENERALIZED MODEL OF GROUNDWATER MOVEMENT IN A LEAKY AQUIFER WITH MITTAG-LEFFLER LAW

5.1 INTRODUCTION

The Mittag-Leffler function, which is greatly applied for its capability to compare different laws, was named after a great Swedish mathematician (1846-1927). The well-known Gosta Magnus Mittag-Leffler conducted a study to vary the interaction between dependent and independent variables in general theory of functions. The function has been applied in fractional order integral equations in the field of transport, complicated undefined systems and many more (Haubold et al., 2011).

The special function is given as follows in literature,

$$E_{\alpha,\beta}(\lambda z^{\alpha}) \tag{191}$$

and its general form is given as,

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\left(\frac{(\lambda z^{\alpha})^{k}}{\Gamma(\beta+\alpha k)'}\right) \quad \alpha>0,\beta>0,\lambda<\infty\ and\ \lambda=-\alpha(1-\alpha)^{-1} \tag{192}$$

Mittag-Leffler type/function has been potentially used by mathematicians, engineers, scientists and doctors (Haubold et al., 2011). The Mittag-Leffler function is capable of describing long term behaviour and can be utilized for different time distributions and renewal process (Gorenflo et al., 2002). The Mittag-Leffler function can non-locally describe all physical phenomena that were initially illustrated by exponential decay law (Atangana and Gomez, 2018).

5.2 APPLICATION OF MITTAG-LEFFLER LAW

The practical question one would ask is the following: Why do we need such differential operator to model the flow within a leaky aquifer? Well, one needs to understand that each differential operator depicts mean square displacement that helps to capture how the water maneuvers within a geological genesis. While this point have not been the main focus in this field and was not considered to be very critical, it is important account for every drop of water within a geological formation can be viewed as a particle moving within media, where the geometry of such media leads the movement of the particle. As the water moves within a geological formation, it encounters some challenges or resistance due to different properties of the media.

Due to heterogeneity the water can flow in one direction/portion and follow a specific natural law, for instance, exponential decay law which is more associated with the flow within the clay. The clay hinders the water to pass very quickly as opposed to sand or silt, thus the velocity of the water is decreasing until eventually the water become stagnant and such process can be depicted very well using the differential operator whose kernels are exponential decay law. Nevertheless, if instead the water goes through such geological formation and is able to keep constant velocity for longer time, one will be obliged to use the differential operator whose kernel is power law decay.

However there are more complex situations, where for instance the aquifer has dual or multi-layers horizontally, this could pose serious problem to model such situation, in this situation the flow can follow first the power law decay process and then later the exponential law. This scenario can practically be related to the situation where water is flowing slowly within a portion of a geological formation, then later become stagnant due to poor porosity and permeability. This situation cannot be depicted using the classical differentiation or fractional differentiation with power and exponential decay law kernels, as they do not accommodate cross over behaviour for mean square displacement. Therefore, a differential operator whose kernel has crossover behaviour can be used, which is the generalized Mittag-Leffler kernel. Applying the AB fractional derivative on equation (17),

$$ {}_{a}^{AB}D_{t}^{\beta} h(x,t) = \frac{T}{S}\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\left(1+\frac{\Delta r}{r}\right) + \frac{1}{r}\,{}_{a}^{AB}D_{r}^{\alpha} h(r,t) + \frac{h(r,t)}{\lambda^{2}}\right] \tag{193} $$

However ${}_{a}^{AB}D_{r}^{\alpha} h\left(r_i, t_j\right)$ can be written as,

$$ {}_{a}^{AB}D_{r}^{\alpha} h\left(r_i, t_j\right) = \frac{AB(\alpha)}{(1-\alpha)} \int_{0}^{r_i} \frac{\partial h(\tau, t_j)}{\partial \tau} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(r_i-\tau)^{\alpha}\right] d\tau \tag{194} $$

$$ {}_{a}^{AB}D_{r}^{\alpha} h\left(r_i, t_j\right) = \frac{AB(\alpha)}{(1-\alpha)} \sum_{k=0}^{i-1} \left(\int_{r_k}^{r_{k+1}} \frac{h\left(r_{i+1}, t_j\right) - h\left(r_{i-1}, t_j\right)}{\Delta r} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(r_i-\tau)^{\alpha}\right]\right) d\tau \tag{195} $$

$$ {}^{AB}_{\ a}D^{\alpha}_{r} h\left(r_i, t_j\right) = \frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{i-1}\left(\frac{h^{j}_{i+1}-h^{j}_{i-1}}{\Delta r}\right)\int_{r_k}^{r_{k+1}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(r_i-\tau\right)^{\alpha}\right]d\tau \tag{196} $$

We recall that the Mittag-Leffler in general form can be given as follows,

$$ E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(r_i-\tau\right)^{\alpha}\right] = \sum_{l=0}^{\infty}\left(\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}\left(t_n-\tau\right)^{\alpha l}}{\Gamma(\alpha l+1)}\right) \tag{197} $$

Therefore,

$$ \int_{r_k}^{r_{k+1}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(r_i-\tau\right)^{\alpha}\right]d\tau = \int_{r_k}^{r_{k+1}}\sum_{l=0}^{\infty}\left(\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}\left(t_n-\tau\right)^{\alpha l}}{\Gamma(\alpha l+1)}\right)\ d\tau \tag{198} $$

$$ \int_{r_k}^{r_{k+1}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(r_i-\tau\right)^{\alpha}\right]d\tau = \sum_{l=0}^{\infty}\left(\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}}{\Gamma(\alpha l+1)}\right)\int_{r_k}^{r_{k+1}}\left(t_n-\tau\right)^{\alpha l}\ d\tau \tag{199} $$

Let Y= $t_n-\tau$,

$$ \int_{r_k}^{r_{k+1}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(r_i-\tau\right)^{\alpha}\right]d\tau = \sum_{l=0}^{\infty}\left(\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}}{\Gamma(\alpha l+1)}\right)\int_{r_j-r_k}^{r_j-r_{k+1}} Y^{\alpha l}\left(-dy\right) $$

$$ = \sum_{l=0}^{\infty}\left(\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}}{\Gamma(\alpha l+1)}\right)\frac{Y^{\alpha l+1}}{\alpha l+1}\Bigg|\begin{matrix} r_j-r_k \\ r_j-r_{k+1}\end{matrix} $$

$$ = \sum_{l=0}^{\infty}\left(\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}}{\Gamma(\alpha l+1)}\right)\left[\frac{\left(r_j-r_k\right)^{\alpha l+1}}{\alpha l+1}-\frac{\left(r_j-r_{k+1}\right)^{\alpha l+1}}{\alpha l+1}\right] $$

$$ = \sum_{l=0}^{\infty}\left[\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}\left(r_j-r_k\right)^{\alpha l+1}}{\Gamma(\alpha l+1)(\alpha l+1)}-\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{l}\left(r_j-r_{k+1}\right)^{\alpha l+1}}{\Gamma(\alpha l+1)(\alpha l+1)}\right] $$

$$ = r_j-r_k\sum_{l=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\left(r_j-r_k\right)^{\alpha}\right)^{l}}{\Gamma(\alpha l+2)} - r_j-r_{k+1}\sum_{l=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\left(r_j-r_{k+1}\right)^{\alpha}\right)^{l}}{\Gamma(\alpha l+2)} $$

$$ = \left(r_j-r_k\right)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\left(r_j-r_k\right)^{\alpha}\right]-\left(r_j-r_{k+1}\right)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\left(r_j-r_{k+1}\right)^{\alpha}\right] \tag{200} $$

Let,

$$r_j = j\Delta r, r_k = k\Delta r \tag{201}$$

and,

$$r_{k+1} = (k+1)\Delta r \tag{202}$$

Therefore,

$$\int_{r_k}^{r_{k+1}} E_\alpha\left[-\frac{\alpha}{1-\alpha}(r_i-\tau)^\alpha\right]d\tau = (j\Delta r - k\Delta r)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(j\Delta r - k\Delta r)^\alpha\right]$$

$$-(j\Delta r-(k+1)\Delta r)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(j\Delta r-(k+1)\Delta r)^\alpha\right]$$

$$= \Delta r(j-k)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha (j-k)^\alpha\right]$$

$$-\Delta r\big(j-(k+1)\big)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha \big(j-(k+1)\big)^\alpha\right] \tag{203}$$

$${}_{a}^{AB}D_r^\alpha h(r_i,t_j) = \frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{i-1} h_{i+1}^j - h_{i-1}^j$$

$$\left\{(j-k)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha (j-k)^\alpha\right] - \big(j-(k+1)\big)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha\big(j-(k+1)\big)^\alpha\right]\right\} \tag{204}$$

We represent ${}_{a}^{AB}D_t^\beta h(x,t)$ as,

$${}_{a}^{AB}D_t^\beta h(x,t) = f\big(x,t,h(x,t)\big) \tag{205}$$

And the new fractional integral is given as,

$${}_{a}^{AB}I_t^\beta h(x,t) = \frac{1-\beta}{AB(\beta)}h(t) + \frac{\beta}{\mathrm{AB}(\beta)\Gamma(\beta)}\int_a^t h(\tau)(t-\tau)^{\beta-1}d\tau \tag{206}$$

Therefore ${}_{a}^{AB}D_t^\beta h(x,t)$ can be drafted as,

$${}_{a}^{AB}D_t^\beta h(x,t) = \frac{1-\beta}{AB(\beta)}f\big(x,t,h(x,t)\big) + \frac{\beta}{\mathrm{AB}(\beta)\Gamma(\beta)}\int_0^t f\big(x,\tau,h(x,\tau)\big)(t-\tau)^{\beta-1}d\tau \tag{207}$$

Discretizing the above equation, the following is obtained,

$$h(x,t) - h(x,0) = \frac{1-\beta}{AB(\beta)}f\big(x,t,h(x,t)\big) + \frac{\beta}{\mathrm{AB}(\beta)\Gamma(\beta)}\int_0^t f\big(x,\tau,h(x,\tau)\big)(t-\tau)^{\beta-1}d\tau \tag{208}$$

at interval x_i, t_{j+1},

$$h(x_i,t_{j+1})-h(x_i,0)=\frac{1-\beta}{AB(\beta)}f\left(x_i,t_{j+1},h(x_i,t_{j+1})\right)$$

$$+\frac{\beta}{\mathrm{AB}(\beta)\Gamma(\beta)}\sum_{l=0}^{j}\int_{t_j}^{t_{j+1}}f(x_i,\tau,h(x_i,\tau))(t_{j+1}-\tau)^{\beta-1}d\tau \tag{209}$$

We recall that,

$$\beta_j=\begin{cases}\dfrac{1}{\Gamma(\alpha+2)} & j=0\\ \dfrac{(j-1)^{\alpha+1}-2j^{\alpha+1}+(j+1)^{\alpha+1}}{\Gamma(\alpha+l)} & j=1,2,\dots.n-1\end{cases} \tag{210}$$

The final equation can be given as,

$$h_i^{j+1}=h_i^0+\frac{1-\beta}{AB(\beta)}\frac{T}{S}\left[\frac{h_{i+1}^j-2h_i^j+h_{i-1}^j}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^j}{\lambda^2}\right.$$

$$+\frac{1}{r_i}\frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{i-1}\left(h_{i+1}^j-h_{i-1}^j\right)\left\{(j-k)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha(j-k)^\alpha\right]\right.$$

$$\left.\left.-\left(j-(k+1)\right)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha\left(j-(k+1)\right)^\alpha\right]\right\}\right]$$

$$+\sum_{j=\alpha}^{n-1}(\beta_{n-j})\frac{T}{S}\left[\frac{h_{i+1}^j-2h_i^j+h_{i-1}^j}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^j}{\lambda^2}\right.$$

$$+\frac{1}{r_i}\frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{i-1}\left(h_{i+1}^j-h_{i-1}^j\right)\left\{(j-k)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha(j-k)^\alpha\right]\right.$$

$$\left.\left.-\left(j-(k+1)\right)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}\Delta r^\alpha\left(j-(k+1)\right)^\alpha\right]\right\}\right] \tag{211}$$

Let,

$$\alpha_1=\frac{1-\beta}{AB(\beta)}\frac{T}{S(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right) \tag{212}$$

$$\alpha_2=\frac{1-\beta}{AB(\beta)}\frac{T}{S\lambda^2} \tag{213}$$

$$\alpha_3 = \left\{\begin{matrix} \dfrac{1-\beta}{AB(\beta)}\dfrac{T}{S}\dfrac{1}{r_i}\dfrac{AB(\alpha)}{(1-\alpha)} \\ \left\{\begin{matrix} (j-k)E_{\alpha,2}\left[\dfrac{-\alpha}{1-\alpha}\Delta r^{\alpha}(j-k)^{\alpha}\right] \\ -\big(j-(k+1)\big)E_{\alpha,2}\left[\dfrac{-\alpha}{1-\alpha}\Delta r^{\alpha}\big(j-(k+1)\big)^{\alpha}\right] \end{matrix}\right\} \end{matrix}\right\} \tag{214}$$

$$\alpha_4 = \left(1+\frac{\Delta r}{r_i}\right)\frac{T}{S}\sum_{j=\alpha}^{n-1}\left(\beta_{n-j}\right) \tag{215}$$

$$\alpha_5 = \frac{T}{S\lambda^2}\sum_{j=\alpha}^{n-1}\left(\beta_{n-j}\right) \tag{216}$$

$$\alpha_6 = \left\{\begin{matrix} \left\{\begin{matrix} (j-k)E_{\alpha,2}\left[\dfrac{-\alpha}{1-\alpha}\Delta r^{\alpha}(j-k)^{\alpha}\right] \\ -\big(j-(k+1)\big)E_{\alpha,2}\left[\dfrac{-\alpha}{1-\alpha}\Delta r^{\alpha}\big(j-(k+1)\big)^{\alpha}\right] \end{matrix}\right\} \\ \displaystyle\sum_{j=\alpha}^{n-1}\left(\beta_{n-j}\right)\frac{T}{S}\frac{1}{r_i}\frac{AB(\alpha)}{(1-\alpha)} \end{matrix}\right\} \tag{217}$$

Therefore, the final numerical solution can be given as,

$$\begin{aligned} h_i^{j+1} = h_i^0 + \alpha_1\left(h_{i+1}^j - 2h_i^j + h_{i-1}^j\right) + \alpha_2 h_i^j + \alpha_3\sum_{k=0}^{i-1}\left(h_{i+1}^j - h_{i-1}^j\right) + \alpha_4 \\ + \left(h_{i+1}^j - 2h_i^j + h_{i-1}^j\right) + \alpha_5 h_i^j + \alpha_6\sum_{k=0}^{i-1}\left(h_{i+1}^j - h_{i-1}^j\right) \end{aligned} \tag{218}$$

5.3 STABILITY EVALUATION

To analyze the stability of the above equation, Von Neumann stability method is used. The method has been fully explained in the previous chapters.

$$\begin{aligned} h_i^{j+1} = h_i^0 + \alpha_1\left(h_{i+1}^j - 2h_i^j + h_{i-1}^j\right) + \alpha_2 h_i^j + \alpha_3\sum_{k=0}^{i-1}\left(h_{i+1}^j - h_{i-1}^j\right) + \alpha_4\left(h_{i+1}^j - 2h_i^j + h_{i-1}^j\right) \\ + \alpha_5 h_i^j + \alpha_6\sum_{k=0}^{i-1}\left(h_{i+1}^j - h_{i-1}^j\right) \end{aligned} \tag{219}$$

When,

$$h_i^j = \delta_j e^{ikmx} \tag{220}$$

Therefore,

$$\delta_{j+1} e^{ikmx} = \delta_0 e^{ikmx} + \alpha_1\left(\delta_j e^{ikm(x+\Delta x)} - 2\delta_j e^{ikmx} + \delta_j e^{ikm(x-\Delta x)}\right) + \alpha_2 \delta_j e^{ikmx} + \alpha_3 \sum_{k=0}^{i-1}\left(\delta_j e^{ikm(x+\Delta x)} - \delta_j e^{ikm(x-\Delta x)}\right) + \alpha_4\left(\delta_j e^{ikm(x+\Delta x)} - 2\delta_j e^{ikmx} + \delta_j e^{ikm(x-\Delta x)}\right) + \alpha_5 \delta_j e^{ikmx} + \alpha_6 \sum_{k=0}^{i-1}\left(\delta_j e^{ikm(x+\Delta x)} - \delta_j e^{ikm(x-\Delta x)}\right) \quad (221)$$

Dividing both sides by e^{ikmx},

$$\delta_{j+1} = \delta_0 + \alpha_1\left(\delta_j e^{ikm\Delta x} - 2\delta_j + \delta_j e^{-ikm\Delta x}\right) + \alpha_2 \delta_j + \alpha_3 \sum_{k=0}^{i-1}\left(\delta_j e^{ikm\Delta x} - \delta_j e^{-ikm\Delta x}\right) + \alpha_4\left(\delta_j e^{ikm\Delta x} - 2\delta_j + \delta_j e^{-ikm\Delta x}\right) + \alpha_5 \delta_j + \alpha_6 \sum_{k=0}^{i-1}\left(\delta_j e^{ikm\Delta x} - \delta_j e^{-ikm\Delta x}\right) \quad (222)$$

Taking out the common factors,

$$\delta_{j+1} = \delta_0 + \alpha_1 \delta_j\left(e^{ikm\Delta x} - 2 + e^{-ikm\Delta x}\right) + \alpha_2 \delta_j + \alpha_3 \delta_j \sum_{k=0}^{i-1}\left(e^{ikm\Delta x} - e^{-ikm\Delta x}\right) + \alpha_4 \delta_j\left(e^{ikm\Delta x} - 2 + e^{-ikm\Delta x}\right) + \alpha_5 \delta_j + \alpha_6 \delta_j \sum_{k=0}^{i-1}\left(e^{ikm\Delta x} - e^{-ikm\Delta x}\right) \quad (223)$$

We recall that,

$$\frac{e^{i\theta} + e^{i\theta}}{2i} = cos\theta \quad (224)$$

and,

$$\frac{e^{i\theta} - e^{i\theta}}{2i} = sin\theta \quad (225)$$

Therefore,

$$\delta_{j+1} = \delta_0 - \alpha_1 4 \sin\left(\frac{km\Delta x}{2}\right)^2 \delta_j + \alpha_2 \delta_j + 2i\alpha_3 \sum_{k=0}^{i-1}(\sin(km\Delta x))\delta_j - \alpha_4 4 \sin\left(\frac{km\Delta x}{2}\right)^2 \delta_j + \alpha_5 \delta_j + 2i\alpha_6 \sum_{k=0}^{i-1}(\sin(km\Delta x))\delta_j \quad (226)$$

We assume that for all j=0 $\left|\delta_j\right| < |\delta_0|$,

$$\delta_{j+1} = \delta_0 - \alpha_1 4 \sin\left(\frac{km\Delta x}{2}\right)^2 \delta_0 + \alpha_2\delta_0 + 2i\alpha_3 \sum_{k=0}^{i-1}(\sin(km\Delta x))\delta_0 - \alpha_4 4 \sin\left(\frac{km\Delta x}{2}\right)^2 \delta_0 + \alpha_5\delta_0 + 2i\alpha_6 \sum_{k=0}^{i-1}(\sin(km\Delta x))\delta_0 \quad (227)$$

Let,

$$1 - \alpha_1 4 \sin\left(\frac{km\Delta x}{2}\right)^2 + \alpha_2 - \alpha_4 4 \sin\left(\frac{km\Delta x}{2}\right)^2 + \alpha_5 = a \quad (228)$$

$$2i\alpha_3 \sum_{k=0}^{i-1}(\sin(km\Delta x)) + 2i\alpha_6 \sum_{k=0}^{i-1}(\sin(km\Delta x)) = ib \quad (229)$$

Therefore,

$$\delta_1 = \delta_0(a + ib) \quad (230)$$

$$\left|\frac{\delta_1}{\delta_0}\right| = |a + ib| = \sqrt{a^2 + b^2} \quad (231)$$

We know that,

$$\left|\frac{\delta_1}{\delta_0}\right| < 1 \quad (232)$$

Therefore,

$$\sqrt{a^2 + b^2} < 1 \quad (233)$$

We assume that for all j>0 $\left|\delta_j\right| < |\delta_0|$,

$$\delta_{j+1} = \delta_0 - \delta_j \left[\alpha_1 4 \sin\left(\frac{km\Delta x}{2}\right)^2 + \alpha_2 + 2i\alpha_3 \sum_{k=0}^{i-1}(\sin(km\Delta x)) - \alpha_4 4 \sin\left(\frac{km\Delta x}{2}\right)^2 + \alpha_5 + 2i\alpha_6 \sum_{k=0}^{i-1}(\sin(km\Delta x)) \right] \quad (234)$$

Let,

$$\alpha_1 4 \sin\left(\frac{km\Delta x}{2}\right)^2 + \alpha_2 - \alpha_4 4 \sin\left(\frac{km\Delta x}{2}\right)^2 + \alpha_5 = a \quad (235)$$

and,

$$2i\alpha_3 \sum_{k=0}^{i-1} (\sin(km\Delta x)) + \ 2i\alpha_6 \sum_{k=0}^{i-1} (\sin(km\Delta x)) = ib \tag{236}$$

Therefore,

$$\delta_{j+1} = \delta_0 - \delta_j (a + ib) \tag{237}$$

$$|\delta_{j+1}| \leq |\delta_0| + |\delta_j||a + ib| \tag{238}$$

and we recall that$|\delta_0| = |\delta_j|$, therefore the final solution can be given as

$$|\delta_{j+1}| < |\delta_0| + |\delta_0||a + ib| \tag{239}$$

$$|\delta_{j+1}| < |\delta_0|(1 + |a + ib|) \tag{240}$$

CHAPTER 6

6 SELF SIMILAR LEAKY AQUIFERS AND INTRODUCTION OF FRACTAL-FRACTIONAL DIFFERENTIATION

Atangana and Ramotsho (2018) introduced a new mathematical model that can be used to assess water flowing within a replicated aquifer, which is only scaled. It is given in literature that self-similar aquifers manifest resembling patterns at increasing small scales and they are known as enlarging evolving symmetry. Recently the concepts of fractal-fractional derivatives and integrals have been widely applied for different scenarios in applied mathematics and to predict chaotic behaviors. The figure below illustrates the concept of self-similarity in a geological formation.

Figure 3: Kink Banding- The picture shows self-similarity pattern in banding by Hunt (2000).

The Atangana-Baleanu, Riemann-Liouville and Caputo-Fabrizio operators have been used to denote the lack of the semi-group principle in modeling real-world scenarios. The operators were also used to capture non-local natural drawbacks that exhibit fractal behaviors. These operators have been proved to account for the fractal effect, capturing the memory (prehistory or original deformation is not forgotten), elasticity and non-locality (Atangana, 2017).

Atangana and Gomez published the article were the three operators were compared. The Riemann-Liouville indicates a constant scale while the Atangana-Baleanu and Caputo-Fabrizio showed interchange properties for the mean square displacement (Atangana and Gomez-Aguilar, 2018). Some articles compared the operators using different alpha values and showed the correlation of both at some point.

The three types of operators have been classified in terms of exponential law, Mittag-Leffler and power law. The operators that are classified as power law operators are Riemann-Liouville and Caputo operators. The Caputo-Fabrizio operator is classified as exponential decay law and the Atangana-Baleanu as a function of Mittag-Leffler function (Atangana and Qureshi, 2019).

The Caputo derivative was introduced in 1967 for analyzing the deformed solids with linear short time elastic responses. The operators were initially used to assess linear viscoelastic materials as opposed to the Riemann-Liouville operators that were used to analyse non-linear viscoelastic materials. It is documented in literature that the non-linear operators cannot account for linear operators (Hristov, 2019).

6.1 FRACTAL-FRACTIONAL OPERATORS AND THE NEW NUMERICAL METHOD FOR CLASSICAL AND FRACTIONAL DERIVATIVES

Fractal-fractional operators are classified or elaborated as characteristics which have been encountered in all fields of technology or science. The operators have non-integer features which are useful. The two terms are broad as compared to non-integers with singularity (Cattani, 2017). This chapter will only introduce the new numerical schemes for fractional derivatives. Chapter seven, eight and nine will cover the numerical schemes for fractal-fractional derivatives.

Different numerical schemes have been explained and some were used in the chapters above. The most applied methods are Adam-Bashforth methods. However the methods were derived from Lagrange Polynomial Method (LPM), of which are less accurate as compared to Newton polynomials. The new numerical scheme was derived using the well-known Newton’s interpolation polynomial.

6.1.1 THE NEW NUMERICAL SCHEME FOR ODE AND PDE WITH CLASSICAL DERIVATIVE

The method with classical derivative can be given as follows using the Cauchy problem,

$$\frac{dy(t)}{dt} = f\big(t, y(t)\big) \tag{241}$$

Where the function is non-linear indicating fractal-fractional, the numerical scheme that can be used to assess such derivative is converted and derived as given below.

$$y(t) - y(0) = \int_0^t f\big(\tau, y(\tau)\big) d\tau \tag{242}$$

At point,

$$t_{n+1} = (n+1)\Delta t \tag{243}$$

and,

$$y(t_{n+1}) - y(0) = \int_0^{t_{n+1}} f\big(\tau, y(\tau)\big) d\tau \tag{244}$$

At point,

$$t_n = n\Delta t \tag{245}$$

and,

$$y(t_n) - y(0) = \int_0^{t_n} f\big(\tau, y(\tau)\big) d\tau \tag{246}$$

Taking the difference,

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f\big(\tau, y(\tau)\big) d\tau \tag{247}$$

We can now consider the approximation of$f\big(\tau, y(t)\big)$ as a Newton polynomial,

$$P_n(\tau) = f\big(t_{n-2}, h(t_{n-2})\big) + \left[\frac{f\Big(t_{n-1,} h(t_{n-1})\Big) - f\Big(t_{n-2,} h(t_{n-2})\Big)}{\Delta t}\right](\tau - t_{n-2})$$

$$+\left[\frac{f\left(t_{n,}h(t_n)\right)-2f\left(t_{n-1,}h(t_{n-1})\right)+f\left(t_{n-2,}h(t_{n-2})\right)}{2(\Delta t)^2}\right](\tau-t_{n-2})(\tau-t_{n-1}) \qquad (248)$$

Replacing the polynomial into the original equation, the following is attributed,

$$y^{n+1}-y^n=f\left(t_{n-2,}y(t_{n-2})\right)\Delta t+\int_{t_n}^{t_{n+1}}\frac{f\left(t_{n-1,}y^{n-1}\right)-f(t_{n-2},y^{n-2})}{\Delta t}(\tau-t_{n-2})d\tau$$

$$+\int_{t_n}^{t_{n+1}}\frac{f\left(t_{n,}y^n\right)-2f(t_{n-1},y^{n-1})+f(t_{n-2},y^{n-2})}{2(\Delta t)^2}(\tau-t_{n-2})(\tau-t_{n-1})d\tau \quad (249)$$

By integrating we obtain the following,

$$y^{n+1}-y^n=f\left(t_{n-2,}y(t_{n-2})\right)\Delta t+\frac{f\left(t_{n-1,}y^{n-1}\right)-f(t_{n-2},y^{n-2})}{\Delta t}\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})d\tau$$

$$+\frac{f\left(t_{n,}y^n\right)-2f(t_{n-1},y^{n-1})+f(t_{n-2},y^{n-2})}{2(\Delta t)^2}\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})(\tau-t_{n-1})d\tau \quad (250)$$

The integrals can be calculated and given as follows,

$$\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})d\tau=\frac{5}{2}(\Delta t)^2 \qquad (251)$$

$$\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})(\tau-t_{n-1})d\tau=\frac{23}{6}(\Delta t)^2 \qquad (252)$$

Replacing the above into the scheme the following is obtained,

$$y^{n+1}-y^n=f\left(t_{n-2,}y(t_{n-2})\right)\Delta t+\frac{f\left(t_{n-1,}y^{n-1}\right)-f(t_{n-2},y^{n-2})}{\Delta t}\frac{5}{2}(\Delta t)^2$$

$$+\frac{f\left(t_{n,}y^n\right)-2f(t_{n-1},y^{n-1})+f(t_{n-2},y^{n-2})}{2(\Delta t)^2}\frac{23}{6}(\Delta t)^2 \qquad (253)$$

We can simplify the above version by cancelling out Δt and grouping the like terms.

$$y^{n+1}-y^n=f\left(t_{n-2,}y(t_{n-2})\right)\Delta t+\left[f\left(t_{n-1,}y^{n-1}\right)-f(t_{n-2},y^{n-2})\right]\frac{5}{2}\Delta t$$

$$+\left[f\left(t_{n,}y^n\right)-2f(t_{n-1},y^{n-1})+f(t_{n-2},y^{n-2})\right]\frac{23}{12}\Delta t \qquad (254)$$

$$y^{n+1}=y^{n}+\frac{5}{12}f(t_{n-2},y^{n-2})\Delta t-\frac{4}{3}f\left(t_{n-1,}y^{n-1}\right)\Delta t\ \frac{23}{12}f\left(t_{n,}y^{n}\right)\Delta t \qquad (255)$$

6.1.2 THE NEW NUMERICAL SCHEME FOR ODE AND PDE WITH CAPUTO FRACTIONAL DERIVATIVE

The method which includes the Caputo fractional derivative can be given as follows using the Cauchy problem,

$${}_{0}^{C}D_{t}^{\alpha}=f\left(t,y(t)\right) \qquad (256)$$

The above equation is transformed where,

$$y(t)-y(0)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}f\left(\tau,y(\tau)\right)(t-\tau)^{\alpha-1}d\tau \qquad (257)$$

At point,

$$t_{n+1}=(n+1)\Delta t \qquad (258)$$

and,

$$y(t_{n+1})-y(0)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t_{n+1}}f\left(\tau,y(\tau)\right)(t_{n+1}-\tau)^{\alpha-1}d\tau \qquad (259)$$

$$y(t_{n+1})-y(0)=\frac{1}{\Gamma(\alpha)}\sum_{j=2}^{n}\int_{t_j}^{t_{j+1}}f\left(\tau,y(\tau)\right)(t_{n+1}-\tau)^{\alpha-1}d\tau \qquad (260)$$

Therefore replacing the Newton polynomial into the above equation,

$$y^{n+1}=y^{0}+\frac{1}{\Gamma(\alpha)}\sum_{j=2}^{n}\int_{t_j}^{t_{j+1}}\left\{\begin{array}{c} f\left(t_{j-2},y^{j-2}\right) \\ +\dfrac{f\left(t_{j-1,}y^{j-1}\right)+f\left(t_{j-2},y^{j-2}\right)}{\Delta t} \\ \times\left(\tau-t_{j-2}\right) \\ +\dfrac{f\left(t_{j,}y^{j}\right)-2f\left(t_{j-1},y^{j-1}\right)+f\left(t_{j-2},y^{j-2}\right)}{2(\Delta t)^{2}} \\ \times\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right) \end{array}\right\}(t_{n+1}-\tau)^{\alpha-1}d\tau$$

(261)

This can be written as,

$$y^{n+1} = y^0 + \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} f\left(t_{j-2}, y^{j-2}\right) \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} \frac{f\left(t_{j-1,} y^{j-1}\right) + f\left(t_{j-2}, y^{j-2}\right)}{\Delta t} \int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} \frac{f\left(t_{j,} y^{j}\right) - 2f\left(t_{j-1}, y^{j-1}\right) + f\left(t_{j-2}, y^{j-2}\right)}{2(\Delta t)^2}$$

$$\times \int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)\left(\tau - t_{j-1}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau \qquad (262)$$

The above integrals can be calculated and grouped as,

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha}}{\alpha} \left[(n - j + 1)^{\alpha} - (n - j)^{\alpha}\right] \qquad (263)$$

$$\int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)} \begin{bmatrix} (n - j + 1)^{\alpha}(n - j + 3 + 2\alpha) \\ -(n - j)^{\alpha} n - j + 3 + 3\alpha) \end{bmatrix} \qquad (264)$$

$$\int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)\left(\tau - t_{j-1}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}$$

$$+ \left[\begin{array}{c} (n - j + 1)^{\alpha} \begin{bmatrix} 2(n - j)^2 \\ +(3\alpha + 10)(n - j) \\ +(2\alpha^2 + 9\alpha + 12) \end{bmatrix} \\ -(n - j)^{\alpha} \begin{bmatrix} 2(n - j)^2 \\ +(5\alpha + 10)(n - j) \\ +(6\alpha^2 + 18\alpha + 12) \end{bmatrix} \end{array} \right] \qquad (265)$$

Therefore substituting the above integrals in the equality the following is obtained,

$$h^{n+1} = h^0 + \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=2}^{n} F\left(t_{j-2}, h^{j-2}\right)\left[(n - j + 1)^{\alpha} - (n - j)^{\alpha}\right]$$

$$+ \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=2}^{n} \left(F\left(t_{j-1,} h^{j-1}\right) - F\left(t_{J-2,} h^{j-2}\right)\right) \begin{bmatrix} (n - j + 1)^{\alpha}(n - j + 3 + 2\alpha) \\ -(n - j)^{\alpha} n - j + 3 + 3\alpha) \end{bmatrix}$$

$$+\frac{(\Delta t)^{\alpha}}{2\Gamma(\alpha+3)}\sum_{j=2}^{n}\Big(F\big(t_{j,}h^{j}\big)-2F\big(t_{j-1,}h^{j-1}\big)+F\big(t_{j-2,}h^{j-2}\big)\Big)$$

$$\times\begin{bmatrix}(n-j+1)^{\alpha}[2(n-j)^{2}+(3\alpha+10)(n-j)+(2\alpha^{2}+9\alpha+12)]\\-(n-j)^{\alpha}[-2(n-j)^{2}+(5\alpha+10)(n-j)+(6\alpha^{2}+18\alpha+12)]\end{bmatrix}\tag{266}$$

Therefore,

$$h^{n+1}=h^{0}+\frac{\beta(\Delta t)^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=2}^{n}t_{j-2}^{\beta-1}f\big(t_{j-2},h^{j-2}\big)[(n-j+1)^{\alpha}-(n-j)^{\alpha}]$$

$$+\frac{\beta(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=2}^{n}t_{j-1}^{\beta-1}f\big(t_{j-1,}h^{j-1}\big)-t_{j-2}^{\beta-1}f\big(t_{j-2,}h^{j-2}\big)\begin{bmatrix}(n-j+1)^{\alpha}(n-j+3+2\alpha)\\-(n-j)^{\alpha}n-j+3+3\alpha)\end{bmatrix}$$

$$+\frac{\beta(\Delta t)^{\alpha}}{2\Gamma(\alpha+3)}\sum_{j=2}^{n}t_{j}^{\beta-1}f\big(t_{j,}h^{j}\big)-2t_{j-1}^{\beta-1}f\big(t_{j-1,}h^{j-1}\big)+t_{j-2}^{\beta-1}f\big(t_{j-2,}h^{j-2}\big)$$

$$\times\begin{bmatrix}(n-j+1)^{\alpha}[2(n-j)^{2}+(3\alpha+10)(n-j)+(2\alpha^{2}+9\alpha+12)]\\-(n-j)^{\alpha}[-2(n-j)^{2}+(5\alpha+10)(n-j)+(6\alpha^{2}+18\alpha+12)]\end{bmatrix}\tag{267}$$

6.1.3 THE NEW NUMERICAL SCHEME FOR ODE AND PDE WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

The method was developed using the Cauchy problem and implementing or introducing the Caputo-Fabrizio fractional derivative

Let,

$${}_{0}^{CF}D_{t}^{\alpha}y(t)=f\big(t,y(t)\big)\tag{268}$$

The equation is reformulated as follows,

$$y(t)-y(0)=\frac{1-\alpha}{\mathrm{M}(\alpha)}f\big(t,y(t)\big)+\frac{\alpha}{\mathrm{M}(\alpha)}\int_{0}^{t}f\big(\tau,y(\tau)\big)d\tau\tag{269}$$

At point,

$$t_{n+1}=(n+1)\Delta t\tag{270}$$

and,

$$y(t_{n+1}) - y(0) = \frac{1-\alpha}{\mathrm{M}(\alpha)} f\big(t_n, y(t_n)\big) + \frac{\alpha}{\mathrm{M}(\alpha)} \int_0^{t_{n+1}} f\big(\tau, y(\tau)\big) d\tau \quad (271)$$

At point,

$$t_n = n\Delta t \quad (272)$$

and,

$$y(t_n) - y(0) = \frac{1-\alpha}{\mathrm{M}(\alpha)} f\big(t_{n-1}, y(t_{n-1})\big) + \frac{\alpha}{\mathrm{M}(\alpha)} \int_0^{t_n} f\big(\tau, y(\tau)\big) d\tau \quad (273)$$

At point,

$$t_{n+1} - t_n \quad (274)$$

and,

$$y(t_{n+1}) - y(t_n) = \frac{1-\alpha}{\mathrm{M}(\alpha)} \big[f\big(t_n, y(t_n)\big) - f\big(t_{n-1}, y(t_{n-1})\big)\big] + \frac{\alpha}{\mathrm{M}(\alpha)} \int_{t_n}^{t_{n+1}} f\big(\tau, y(\tau)\big) d\tau \,(275)$$

$$y(t_{n+1}) - y(t_n) = \frac{1-\alpha}{\mathrm{M}(\alpha)} \big[f\big(t_n, y(t_n)\big) - f\big(t_{n-1}, y(t_{n-1})\big)\big] + \frac{\alpha}{\mathrm{M}(\alpha)} \int_{t_n}^{t_{n+1}} f\big(\tau, y(\tau)\big) d\tau \,(276)$$

$f(t, y(t))$ was represented using the Newton Polynomial in equation (248).

$$y^{n+1} - y^n = \frac{1-\alpha}{\mathrm{M}(\alpha)} \big[f\big(t_n, y(t_n)\big) - f\big(t_{n-1}, y(t_{n-1})\big)\big]$$
$$+ \frac{\alpha}{\mathrm{M}(\alpha)} \int_j^{t_{j+1}} \left\{ \begin{array}{c} f\big(t_{j-2}, y^{j-2}\big) \\ + \dfrac{f\big(t_{j-1,} y^{j-1}\big) - f\big(t_{j-2,} y^{j-2}\big)}{\Delta t} \big(\tau - t_{j-2}\big) \\ + \dfrac{f\big(t_{j,} y^{j}\big) - 2f\big(t_{j-1,} h^{j-1}\big) + f\big(t_{j-2,} h^{j-2}\big)}{2(\Delta t)^2} \\ \times (\tau - t_{-2})\big(\tau - t_{j-1}\big) \end{array} \right\} d\tau \quad (277)$$

The following is obtained,

$$h^{n+1}-h^{n}=\frac{1-\alpha}{\mathrm{M}(\alpha)}\left[f\left(t_{n},y(t_{n})\right)-f\left(t_{n-1},y(t_{n-1})\right)\right]+\frac{\alpha}{\mathrm{M}(\alpha)}\left\{\begin{array}{c}f\left(t_{j-2},y^{j-2}\right)\Delta t\\+\dfrac{f\left(t_{j-1,}y^{j-1}\right)-f\left(t_{j-2,}y^{j-2}\right)}{\Delta t}\displaystyle\int_{t_{n}}^{t_{n+1}}\left(\tau-t_{j-2}\right)d\tau\\+\dfrac{f\left(t_{j,}y^{j}\right)-2f\left(t_{j-1,}y^{j-1}\right)+f\left(t_{j-2,}y^{j-2}\right)}{2(\Delta t)^{2}}\\\times\displaystyle\int_{t_{j}}^{t_{n+1}}\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)d\tau\end{array}\right\}\tag{278}$$

Therefore the following equalities can be written as,

$$\int_{t_{j}}^{t_{j+1}}\left(\tau-t_{j-2}\right)d\tau=\frac{5}{2}(\Delta t)^{2}\tag{279}$$

$$\int_{t_{j}}^{t_{j+1}}\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)d\tau=\frac{23}{6}(\Delta t)^{3}\tag{280}$$

The above scheme can be given as,

$$y^{n+1}=y^{n}+\frac{1-\alpha}{\mathrm{M}(\alpha)}\left[f(t_{n},y^{n})-f\left(t_{n-1},y(t_{n-1})\right)\right]+\frac{\alpha}{\mathrm{M}(\alpha)}\sum_{j=2}^{n}\left\{\begin{array}{c}f\left(t_{j-2},y^{j-2}\right)\Delta t+\left[f\left(t_{j-1,}y^{j-1}\right)-f\left(t_{j-2,}y^{j-2}\right)\right]\dfrac{5}{2}\Delta t\\+\left[f\left(t_{j,}y^{j}\right)-2f\left(t_{j-1,}y^{j-1}\right)+f\left(t_{j-2,}y^{j-2}\right)\right]\dfrac{23\Delta t}{12}\end{array}\right\}\tag{281}$$

Therefore the final numerical scheme is denoted as,

$$y^{n+1}=y^{n}+\frac{(1-\alpha)\beta}{\mathrm{M}(\alpha)}\left[f(t_{n},y^{n})-f\left(t_{n-1},y(t_{n-1})\right)\right]$$

$$+\frac{\alpha}{\mathrm{M}(\alpha)}\sum_{j=2}^{n}\left\{-\frac{4}{3}f\left(t_{j-1,}y^{j-1}\right)\Delta t+\frac{5}{12}f\left(t_{j-2},y^{j-2}\right)\Delta t+\frac{23}{12}f\left(t_{j,}y^{j}\right)\Delta t\right\}\tag{282}$$

6.1.4 THE NEW NUMERICAL SCHEME FOR ODE AND PDE WITH ATANGANA BALEANU FRACTIONAL DERIVATIVE

The Cauchy problem was analyzed,

$${}^{ABC}_{\ \ \ 0}D_{t}^{\alpha,}y(t)=f(t,y(t))\tag{283}$$

Applying the Atangana-Baleanu fractional derivative on the equation above and integrating the equation,

$$y(t) - y(0) = \frac{1-\alpha}{AB(\alpha)} f(t, y(t)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t f(\tau, y(\tau))(t-\tau)^{\alpha-1} d\tau \quad (284)$$

At point

$$t_{n+1} = (n+1)\Delta t \quad (285)$$

and,

$$y(t_{n+1}) - y(0) = \frac{1-\alpha}{AB(\alpha)} f\big(t_n, y(t_n)\big) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \quad (286)$$

This can also be given as,

$$h(t_{n+1}) = h(0) + \frac{1-\alpha}{AB(\alpha)} f\big(t_n, y(t_n)\big) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} f(\tau, y(\tau)) d\tau \quad (287)$$

The Newton polynomial was used to approximate the function$f\big(t, h(t)\big)$ in equation (248), therefore the equation above can be written as,

$$y^{n+1} = y^0 + \frac{1-\alpha}{\mathrm{AB}(\alpha)} f\big(t_n, y(t_n)\big)$$
$$+\frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} \left\{ \begin{array}{c} f\big(t_{j-2}, y^{j-2}\big) \\ + \dfrac{f\big(t_{j-1,} y^{j-1}\big) - f\big(t_{J-2,} y^{j-2}\big)}{\Delta t} \big(\tau - t_{j-2}\big) \\ + \dfrac{f\big(t_{j,} y\big) - 2f\big(t_{j-1,} y^{j-1}\big) + F\big(t_{j-2,} y^{j-2}\big)}{2(\Delta t)^2} \\ \times \big(\tau - t_{j-2}\big)\big(\tau - t_{j-1}\big) \end{array} \right\} (t_{n+1}-\tau)^{\alpha-1} d\tau \quad (288)$$

This can be represented or simplified as follows,

$$y^{n+1} = y^0 + \frac{1-\alpha}{\mathrm{AB}(\alpha)} f\big(t_n, y(t_n)\big)$$

$$+\frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)}\sum_{j=2}^{n}\left\{\begin{array}{c}\int_{t_j}^{t_{j+1}} f\left(t_{j-2}, y^{j-2}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau \\ +\int_{t_j}^{t_{j+1}}\frac{f\left(t_{j-1,}y^{j-1}\right)-f\left(t_{J-2,}y^{j-2}\right)}{\Delta t}\left(\tau-t_{j-2}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau \\ +\int_{t_j}^{t_{j+1}}\frac{f\left(t_{j,}y^{j}\right)-2f\left(t_{j-1,}y^{j-1}\right)+f\left(t_{j-2,}y^{j-2}\right)}{2(\Delta t)^2} \\ \times\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau\end{array}\right\} \tag{289}$$

Therefore, the summations can be given as,

$$\begin{aligned}y^{n+1} &= y^0+\frac{1-\alpha}{\mathrm{AB}(\alpha)}f\left(t_n, y(t_n)\right) \\ &+\frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)}\sum_{j=2}^{n} f\left(t_{j-2}, y^{j-2}\right)\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau \\ &+\frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)}\sum_{j=2}^{n}\frac{f\left(t_{j-1,}y^{j-1}\right)-f\left(t_{j-2,}y^{j-2}\right)}{\Delta t}\int_{t_j}^{t_{j+1}}\left(\tau-t_{j-2}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau \\ &+\frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)}\sum_{j=2}^{n}\frac{f\left(t_{j,}y^{j}\right)-2f\left(t_{j-1,}y^{j-1}\right)+f\left(t_{j-2,}y^{j-2}\right)}{2(\Delta t)^2} \\ &\times\int_{t_j}^{t_{j+1}}\times\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau\end{aligned} \tag{290}$$

The integrals can be calculated and given as,

$$\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau=\frac{(\Delta t)^{\alpha}}{\alpha}\left[(n-j+1)^{\alpha}-(n-j)^{\alpha}\right] \tag{291}$$

$$\int_{t_j}^{t_{j+1}}\left(\tau-t_{j-2}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau=\frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)}\begin{bmatrix}(n-j+1)^{\alpha}(n-j+3+2\alpha) \\ -(n-j+1)^{\alpha}(n-j+3+3\alpha)\end{bmatrix} \tag{292}$$

$$\begin{aligned}\int_{t_j}^{t_{j+1}}\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau &= \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} \\ &\times\begin{bmatrix}(n-j+1)^{\alpha}\begin{bmatrix}2(n-j)^2 \\ +(3\alpha+10)(n-j) \\ +(2\alpha^2+9\alpha+12)\end{bmatrix} \\ -(n-j)^{\alpha}\begin{bmatrix}2(n-j)^2 \\ +(5\alpha+10)(n-j) \\ +(6\alpha^2+18\alpha+12)\end{bmatrix}\end{bmatrix}\end{aligned} \tag{293}$$

Replacing the calculated integrals in the equality above, the final numerical scheme can be given as,

$$\begin{aligned} y^{n+1} = y^{0} &+ \frac{1-\alpha}{\mathrm{AB}(\alpha)} f\big(t_{n}, y(t_{n})\big) \\ &+ \frac{\alpha(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+1)} \sum_{j=2}^{n} f\big(t_{j-2}, y^{j-2}\big)\left[(n-j+1)^{\alpha}-(n-j)^{\alpha}\right] \\ &+ \frac{\alpha(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+2)} \sum_{j=2}^{n} \left[f\big(t_{j-1,} y^{j-1}\big) - f\big(t_{J-2,} y^{j-2}\big)\right] \times \begin{bmatrix} (n-j+1)^{\alpha}(n-j+3+2\alpha) \\ -(n-j)^{\alpha}(n-j+3+3\alpha) \end{bmatrix} \\ &+ \frac{\alpha(\Delta t)^{\alpha}}{2\mathrm{AB}(\alpha)\Gamma(\alpha+3)} \sum_{j=2}^{n} f\big(t_{j,} y^{j}\big) - 2f\big(t_{j-1,} f^{j-1}\big) + f\big(t_{j-2,} y^{j-2}\big) \\ &\times \begin{bmatrix} (n-j+1)^{\alpha}\left[2(n-j)^{2}+(3\alpha+10)(n-j)+(2\alpha^{2}+9\alpha+12)\right] \\ -(n-j)^{\alpha}\left[2(n-j)^{2}+(5\alpha+10)(n-j)+(6\alpha^{2}+18\alpha+12)\right] \end{bmatrix} \end{aligned} \tag{294}$$

CHAPTER 7

7 DEFINITIONS OF FRACTAL-FRACTIONAL OPERATORS IN TERMS OF POWER LAW: CAPUTO SENSE

Definition 1: Accede that $f(t)$ which is fractal differentiable on (a, b) with order β then the fractal-fractional derivative of $f(t)$with order α in the Caputo sense having power law type is been given as follows:

$$ {}^{FFP}_{\ 0}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{dt^{\beta}} (t-\tau)^{n-\alpha-1} d\tau \tag{295} $$

When,

$n-1 < \alpha \le n$, $0 < n-1 < \beta \le n$

$$ {}^{FFP}_{\ 0}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{df(\tau)}{dt^{\beta}} (t-\tau)^{-\alpha} d\tau \frac{t^{1-\alpha}}{\alpha} \tag{296} $$

The more generalized version is given as,

$$ {}^{FFP}_{\ 0}D_t^{\alpha,\beta\lambda,} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d^{\lambda}}{dt^{\beta}} (t-\tau)^{n-\alpha-1} d\tau \tag{297} $$

When,

$n-1 < \alpha \le n$, $0 < n-1 < \lambda, \beta \le n$,

$$ {}^{FFP}_{\ 0}D_t^{\alpha,\beta,\lambda} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d^{\lambda}}{dt} (t-\tau)^{-\alpha} d\tau \frac{t^{1-\alpha}}{\alpha} \tag{298} $$

Definition 2: Accede that $f(t)$ be continuous on (a, b) which is open variables then the fractal-fractional integral of $f(t)$ with order α having power law is given as follows:

$$ {}^{FFP}J_{0,t}^{\alpha,} f(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\beta-1} f(\tau) d\tau \tag{299} $$

7.1 NUMERICAL SCHEME: USING THE NEW CAPUTO FRACTAL-FRACTIONAL NUMERICAL METHOD

The method was initiated or developed using the Cauchy problem,

$$ {}^{FFP}_{\ 0}D_t^{\alpha,\beta} h(t) = f\big(t, h(t)\big) \tag{300} $$

Integrating the equation above,

$$h(t) - h(0) = \frac{\alpha\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} f(\tau, h(\tau))(t-\tau)^{\alpha-1} d\tau \tag{301}$$

$F\big(t, h(t)\big)$ can be represented as $\beta t^{\beta-1} f\big(t, h(t)\big)$ thus,

$$h(t) - h(0) = \frac{1}{\Gamma(\alpha)} \int_0^t F(\tau, h(\tau))(t-\tau)^{\alpha-1} d\tau \tag{302}$$

At the point $t_{n+1} = (n+1)\Delta t,$ where the function F is non-linear,

$$h(t_{n+1}) - h(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F(\tau, h(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \tag{303}$$

Therefore,

$$h(t_{n+1}) - h(0) = \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} F(\tau, h(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \tag{304}$$

The Newton polynomial is given as,

$$\begin{aligned} P_n(\tau) = f\big(t_{n-2}, h(t_{n-2})\big) &+ \left[\frac{f\left(t_{n-1,}h(t_{n-1})\right) - f\left(t_{n-2,}h(t_{n-2})\right)}{\Delta t}\right](\tau - t_{n-2}) \\ &+ \left[\frac{f\left(t_{n,}h(t_n)\right) - 2f\left(t_{n-1,}h(t_{n-1})\right) + f\left(t_{n-2,}h(t_{n-2})\right)}{2(\Delta t)^2}\right](\tau - t_{n-2})(\tau - t_{n-1}) \end{aligned} \tag{305}$$

By replacing the Newton polynomial as given in equation (248) the following is obtained,

$$h^{n+1} = h^0 + \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} \left\{ \begin{array}{c} F\big(t_{j-2}, h^{j-2}\big) \\ +\dfrac{F\big(t_{j-1,}h^{j-1}\big) - F\big(t_{J-2,}h^{j-2}\big)}{\Delta t}\big(\tau - t_{j-2}\big) \\ +\dfrac{F\big(t_{j,}h^{j}\big) - 2F\big(t_{j-1,}h^{j-1}\big) + F\big(t_{j-2,}h^{j-2}\big)}{2(\Delta t)^2} \\ \times \big(\tau - t_{j-2}\big)\big(\tau - t_{j-1}\big) \end{array} \right\} (t_{n+1} - \tau)^{\alpha-1} d\tau \tag{306}$$

The following equality is obtained,

$$
\begin{aligned}
h^{n+1} &= h^0 + \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} F\left(t_{j-2}, h^{j-2}\right) \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} \frac{F\left(t_{j-1,} h^{j-1}\right) - F\left(t_{J-2,} h^{j-2}\right)}{\Delta t} \int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{n} \frac{F\left(t_{j,} h^{j}\right) - 2F\left(t_{j-1,} h^{j-1}\right) + F\left(t_{j-2,} h^{j-2}\right)}{2(\Delta t)^2} \\
&\times \int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)\left(\tau - t_{j-1}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau
\end{aligned}
\tag{307}
$$

The above integrals can be calculated and grouped as,

$$
\int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha}}{\alpha} \left[(n - j + 1)^{\alpha} - (n - j)^{\alpha}\right]
\tag{308}
$$

$$
\int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau = \frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha + 1)} \begin{bmatrix} (n - j + 1)^{\alpha}(n - j + 3 + 2\alpha) \\ -(n - j)^{\alpha} n - j + 3 + 3\alpha) \end{bmatrix}
\tag{309}
$$

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} \left(\tau - t_{j-2}\right)\left(\tau - t_{j-1}\right)(t_{n+1} - \tau)^{\alpha-1} d\tau &= \frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha + 1)(\alpha + 2)} \\
&+ \begin{bmatrix} (n - j + 1)^{\alpha} \begin{bmatrix} 2(n - j)^2 + (3\alpha + 10)(n - j) \\ +(2\alpha^2 + 9\alpha + 12) \end{bmatrix} \\ -(n - j)^{\alpha} \begin{bmatrix} 2(n - j)^2 + (5\alpha + 10)(n - j) \\ +(6\alpha^2 + 18\alpha + 12) \end{bmatrix} \end{bmatrix}
\end{aligned}
\tag{310}
$$

Therefore substituting the above integrals in the equality, the following is obtained,

$$
\begin{aligned}
h^{n+1} &= h^0 + \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha + 1)} \sum_{j=2}^{n} F\left(t_{j-2}, h^{j-2}\right)\left[(n - j + 1)^{\alpha} - (n - j)^{\alpha}\right] \\
&+ \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha + 2)} \sum_{j=2}^{n} \left(F\left(t_{j-1,} h^{j-1}\right) - F\left(t_{J-2,} h^{j-2}\right)\right) \begin{bmatrix} (n - j + 1)^{\alpha}(n - j + 3 + 2\alpha) \\ -(n - j)^{\alpha} n - j + 3 + 3\alpha) \end{bmatrix} \\
&+ \frac{(\Delta t)^{\alpha}}{2\Gamma(\alpha + 3)} \sum_{j=2}^{n} \left(F\left(t_{j,} h^{j}\right) - 2F\left(t_{j-1,} h^{j-1}\right) + F\left(t_{j-2,} h^{j-2}\right)\right) \\
&\times \begin{bmatrix} (n - j + 1)^{\alpha}[2(n - j)^2 + (3\alpha + 10)(n - j) + (2\alpha^2 + 9\alpha + 12)] \\ -(n - j)^{\alpha}[-2(n - j)^2 + (5\alpha + 10)(n - j) + (6\alpha^2 + 18\alpha + 12)] \end{bmatrix}
\end{aligned}
\tag{311}
$$

Therefore,

$$h^{n+1}=h^{0}+\frac{\beta(\Delta t)^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=2}^{n}t_{j-2}^{\beta-1}f\left(t_{j-2},h^{j-2}\right)[(n-j+1)^{\alpha}-(n-j)^{\alpha}]$$

$$+\frac{\beta(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=2}^{n}t_{j-1}^{\beta-1}f\left(t_{j-1,}h^{j-1}\right)-t_{j-2}^{\beta-1}f\left(t_{j-2,}h^{j-2}\right)\begin{bmatrix}(n-j+1)^{\alpha}(n-j+3+2\alpha)\\-(n-j)^{\alpha}n-j+3+3\alpha)\end{bmatrix}$$

$$+\frac{\beta(\Delta t)^{\alpha}}{2\Gamma(\alpha+3)}\sum_{j=2}^{n}t_{j}^{\beta-1}f\left(t_{j,}h^{j}\right)-2t_{j-1}^{\beta-1}f\left(t_{j-1,}h^{j-1}\right)+t_{j-2}^{\beta-1}f\left(t_{j-2,}h^{j-2}\right)$$

$$\times\begin{bmatrix}(n-j+1)^{\alpha}[2(n-j)^{2}+(3\alpha+10)(n-j)+(2\alpha^{2}+9\alpha+12)]\\-(n-j)^{\alpha}[-2(n-j)^{2}+(5\alpha+10)(n-j)+(6\alpha^{2}+18\alpha+12)]\end{bmatrix}\qquad(312)$$

7.1.1 IMPLEMENATTION OF CAPUTO FRACTAL-FRACTIONAL DERIVATIVE ON THE SELF=SIMILAR LEAKY AQUIFER EQUATION: SCENARIO 1

The mathematical model below was incorporated to assess self-similar leaky aquifers, the flow out is also through a self-similar leaky aquifer.

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\left(1+\frac{dr}{r}\right)+\frac{1}{r}\frac{\partial h}{\partial r^{\alpha}}+\frac{h(r,t)}{\lambda^{2}}\qquad(313)$$

Applying the Caputo fractal-fractional derivative into the function,

$$\frac{S}{T}{}_{0}^{c}D_{t}^{\beta,\alpha}h(r,t)=\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h(r,t)}{\partial r^{\alpha}}\right)\left(1+\frac{dr}{r}\right)+\frac{1}{r}\frac{\partial h(r,t)}{\partial r^{\alpha}}+\frac{h(r,t)}{\lambda^{2}}\qquad(314)$$

Differentiating the single and double fractal derivative the following is obtained,

$$\frac{S}{T}{}_{0}^{c}D_{t}^{\beta,\alpha}h(r,t)=\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h(r,t)}{\partial r^{\alpha}}\right)\left(1+\frac{dr}{r}\right)+\frac{1}{r}\frac{\partial h(r,t)}{\partial r^{\alpha}}\frac{r^{1-\alpha}}{\alpha}+\frac{h(r,t)}{\lambda^{2}}\qquad(315)$$

$${}_{0}^{c}D_{t}^{\beta,\alpha}h(r,t)\ =\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^{2}h(r,t)}{\partial r^{2}}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\frac{\partial h(r,t)}{\partial r}\right]\left(1+\frac{dr}{r}\right)$$

$$+\frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r,t)}{\lambda^{2}}\qquad(316)$$

The equation can be simplified by writing the following expression,

$$f\big(\alpha,r,t,h(r,\tau)\big)=\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^{2}h(r,t)}{\partial r^{2}}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\frac{\partial h(r,t)}{\partial r}\right]\left(1+\frac{dr}{r}\right)$$

$$+\frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r,t)}{\lambda^{2}} \tag{317}$$

Such that,

$$ {}_{0}^{C}D_{t}^{\beta,\alpha}h(r,t)\ =\ f\big(\alpha,r,t,h(r,\tau)\big) \tag{318}$$

Discretizing at $\left(x_{i,}t_{n+1}\right)$ and the numerical scheme will be given as,

$$h(r_i,t_{n+1})-h(r_i,0)=\frac{\alpha}{\Gamma(\beta)}\int_{0}^{t_{n+1}}\tau^{\alpha-1}(t_{n+1}-\tau)^{\beta-1}f(\alpha,r_i,t_{n+1},h(r_i,\tau))\,d\tau \tag{319}$$

$$\begin{aligned}
h_i^{n+1}&=h_i^{0}+\frac{\alpha(\Delta t)^{\beta}}{\Gamma(\beta+1)}\sum_{j=2}^{n}\ t_{j-2}^{\alpha-1}\left\{f\left(\alpha,r_i,t_{j-2},h\big(r_i,t_{j-2}\big)\right)\right\}\\
&+\frac{\alpha(\Delta t)^{\beta}}{\Gamma(\beta+2)}\sum_{j=2}^{n}\left\{\begin{matrix}t_{j-1}^{\alpha-1}f\left(\alpha,r_i,t_{j-1},h\big(r_i,t_{j-1}\big)\right)\\-t_{j-2}^{\alpha-1}f\left(\alpha,r_i,t_{j-2},h\big(r_i,t_{j-2}\big)\right)\end{matrix}\right\}\left[\begin{matrix}(n-j+1)^{\beta}(n-j+3+2\beta)\\-(n-j)^{\beta}n-j+3+3\beta)\end{matrix}\right]\\
&+\frac{\alpha(\Delta t)^{\beta}}{2\Gamma(\beta+3)}\sum_{j=2}^{n}\left(\begin{matrix}t_{j}^{\alpha-1}f\left(\alpha,r_i,t_j,h\big(r_i,t_j\big)\right)\\-2t_{j-1}^{\alpha-1}f\left(\alpha,r_i,t_{j-1},h\big(r_i,t_{j-1}\big)\right)\\+t_{j-2}^{\alpha-1}f\left(\alpha,r_i,t_{j-2},h\big(r_i,t_{j-2}\big)\right)\end{matrix}\right)\\
&\times\left[\begin{matrix}(n-j+1)^{\beta}\left[\begin{matrix}2(n-j)^{2}+(3\beta+10)(n-j)\\+(2\beta^{2}+9\beta+12)\end{matrix}\right]\\-(n-j)^{\beta}\left[\begin{matrix}-2(n-j)^{2}+(5\beta+10)(n-j)\\+(6\beta^{2}+18\beta+12)\end{matrix}\right]\end{matrix}\right]
\end{aligned} \tag{320}$$

Let the functions be represented as

$$f\left(\alpha,r_i,t_{j-2},h\big(r_i,t_{j-2}\big)\right)=\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^{2}h\big(r_i,t_{j-2}\big)}{\partial r^{2}}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right)$$

$$+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{j-2})}{\lambda^{2}} \tag{321}$$

$$f\left(\alpha,r_i,t_{j-1},h\big(r_i,t_{j-1}\big)\right)=\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^{2}h\big(r_i,t_{j-1}\big)}{\partial r^{2}}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right)$$

$$+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-1})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h\big(r_i,t_{j-1}\big)}{\lambda^{2}} \tag{322}$$

$$
\begin{aligned}
f\left(\alpha, r_i, t_j, h\left(r_i, t_j\right)\right) = {} & \frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_j\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_j\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\
& + \frac{T}{S}\frac{1}{r}\frac{\partial h(r_i, t_j)}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r_i, t_j)}{\lambda^2} \qquad 323)
\end{aligned}
$$

Therefore, the final numerical solution can be given as,

$$
\begin{aligned}
h_i^{n+1} = {} & h_i^0 \\
& + \frac{\alpha(\Delta t)^\beta}{\Gamma(\beta+1)}\sum_{j=2}^{n} t_{j-2}^{\alpha-1}\left\{\begin{gathered}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_{j-2}\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_{j-2}\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\frac{1}{r}\frac{\partial h(r_i, t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r_i, t_{j-2})}{\lambda^2}\end{gathered}\right\} \\
& + \frac{\alpha(\Delta t)^\beta}{\Gamma(\beta+2)}\sum_{j=2}^{n}\left\{\begin{gathered} t_{j-1}^{\alpha-1}\left(\begin{gathered}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_{j-1}\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_{j-1}\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\frac{1}{r}\frac{\partial h\left(r_i, t_{j-1}\right)}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h\left(r_i, t_{j-1}\right)}{\lambda^2}\end{gathered}\right) \\ - t_{j-2}^{\alpha-1}\left(\begin{gathered}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_{j-2}\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_{j-2}\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\frac{1}{r}\frac{\partial h(r_i, t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r_i, t_{j-2})}{\lambda^2}\end{gathered}\right)\end{gathered}\right\} \\
& \times\left[\begin{gathered}(n-j+1)^\beta(n-j+3+2\beta) \\ -(n-j)^\beta n-j+3+3\beta)\end{gathered}\right] \\
& + \frac{\alpha(\Delta t)^\beta}{2\Gamma(\beta+3)}\sum_{j=2}^{n}\left(\begin{gathered} t_{j}^{\alpha-1}\left(\begin{gathered}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_j\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_j\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\frac{1}{r}\frac{\partial h\left(r_i, t_j\right)}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h\left(r_i, t_j\right)}{\lambda^2}\end{gathered}\right) \\ -2t_{j-1}^{\alpha-1}\left(\begin{gathered}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_{j-1}\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_{j-1}\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\frac{1}{r}\frac{\partial h\left(r_i, t_{j-1}\right)}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h\left(r_i, t_{j-1}\right)}{\lambda^2}\end{gathered}\right) \\ + t_{j-2}^{\alpha-1}\left(\begin{gathered}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h\left(r_i, t_{j-2}\right)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r_i^{-\alpha}\frac{\partial h\left(r_i, t_{j-2}\right)}{\partial r}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\frac{1}{r}\frac{\partial h(r_i, t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r_i, t_{j-2})}{\lambda^2}\end{gathered}\right)\end{gathered}\right) \\
& \times\left[\begin{gathered}(n-j+1)^\beta\left[2(n-j)^2 + (3\beta+10)(n-j) + \left(2\beta^2+9\beta+12\right)\right] \\ -(n-j)^\beta\left[-2(n-j)^2 + (5\beta+10)(n-j) + \left(6\beta^2+18\beta+12\right)\right]\end{gathered}\right] \qquad (324)
\end{aligned}
$$

7.1.2 IMPLEMENTATION OF CAPUTO FRACTAL-FRACTIONAL DERIVATIVE ON THE SELF-SIMILAR LEAKY AQUIFER EQUATION: SCENARIO 2

Another scenario was considered where the water is not flowing out of a small portion of a self-similar leaky aquifer but a normal leaky aquifer and the equation below was derived.

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\left(1+\frac{dr}{r}\right)+\frac{\partial h}{\partial r^{\alpha}}\left(\frac{1}{dr}-\frac{1}{r}\right)+\frac{h(r,t)}{\lambda^2} - \frac{1}{dr}\left[\frac{\partial h}{\partial r}\right] \tag{325}$$

Introducing the Caputo fractal-fractional derivative,

$$\frac{S}{T}\,{}_{0}^{C}D_t^{\beta,\alpha}h(r,t) = \frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\left(1+\frac{dr}{r}\right)+\frac{\partial h}{\partial r^{\alpha}}\left(\frac{1}{dr}-\frac{1}{r}\right)+\frac{h(r,t)}{\lambda^2} - \frac{1}{dr}\left[\frac{\partial h}{\partial r}\right] \tag{326}$$

Differentiating the single and double fractal derivative,

$$\frac{\partial h(r,t)}{\partial r^{\alpha}} = \frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha} \tag{327}$$

$$\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h(r,t)}{\partial r^{\alpha}}\right) = \frac{\partial}{\partial r}\left(\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right)\frac{r^{1-\alpha}}{\alpha} \tag{328}$$

$$\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h(r,t)}{\partial r^{\alpha}}\right) = \frac{\partial r^{2}h(r,t)}{\partial r^2}\left(\frac{r^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha} \tag{329}$$

The above equation can be simplified and given as,

$$f\big(\alpha,r,t,h(r,\tau)\big) = \frac{T}{S}\left[\frac{\partial r^{2}h(r,t)}{\partial r^2}\left(\frac{r^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha}\right]\left(1+\frac{dr}{r}\right)$$
$$+\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r}\right)-\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r,t)}{\lambda^2}\right] \tag{330}$$

Such that,

$${}_{0}^{C}D_t^{\beta,\alpha}h(r,t) \;=\; f\big(\alpha,r,t,h(r,\tau)\big) \tag{331}$$

$$
{}_{0}^{c}D_t^{\beta,\alpha}h(r,t) = \frac{T}{S}\left[\frac{\partial r^2 h(r,t)}{\partial r^2}\left(\frac{r^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha}\right]\left(1+\frac{dr}{r}\right)
$$

$$
+\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r}\right)-\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r,t)}{\lambda^2}\right] \tag{332}
$$

The functions can be represented as,

$$
f\left(\alpha, r_i, t_{j-2}, h\left(r_i, t_{j-2}\right)\right) = \frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_{j-2})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)
$$

$$
+\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-2}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-2}\right)}{\lambda^2}\right] \tag{333}
$$

$$
f\left(\alpha, r_i, t_{j-1}, h\left(r_i, t_{j-1}\right)\right) = \frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_{j-1})}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_{j-1})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)
$$

$$
+\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-1}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-1}\right)}{\lambda^2}\right] \tag{334}
$$

$$
f\left(\alpha, r_i, t_j, h\left(r_i, t_j\right)\right) = \frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_j)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_j)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)
$$

$$
+\frac{T}{S}\left[\frac{\partial h(r_i,t_j)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_j\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_j\right)}{\lambda^2}\right] \tag{335}
$$

Therefore the final numerical solution can be given as,

$$
\begin{aligned}
h_i^{n+1} &= h_i^0 \\
&+\frac{\alpha(\Delta t)^\beta}{\Gamma(\beta+1)}\sum_{j=2}^{n} t_{j-2}^{\alpha-1}\left\{\begin{matrix}\frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_{j-2})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-2}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-2}\right)}{\lambda^2}\right]\end{matrix}\right\} \\
&+\frac{\alpha(\Delta t)^\beta}{\Gamma(\beta+2)}\sum_{j=2}^{n}\left\{\begin{matrix} t_{j-1}^{\alpha-1}\begin{pmatrix}\frac{T}{S}\left[\frac{\partial r^2 h\left(r_i,t_{j-1}r,t\right)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\left(r_i,t_{j-1}r,t\right)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-1}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-1}\right)}{\lambda^2}\right]\end{pmatrix} \\ -t_{j-2}^{\alpha-1}\begin{pmatrix}\frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_{j-2})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-2}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-2}\right)}{\lambda^2}\right]\end{pmatrix}\end{matrix}\right\} \\
&\times\begin{bmatrix}(n-j+1)^\beta(n-j+3+2\beta) \\ -(n-j)^\beta n-j+3+3\beta)\end{bmatrix} \\
&+\frac{\alpha(\Delta t)^\beta}{2\Gamma(\beta+3)}\sum_{j=2}^{n}\begin{pmatrix} t_{j}^{\alpha-1}\begin{pmatrix}\frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_j)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_j)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_j)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_j\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_j\right)}{\lambda^2}\right]\end{pmatrix} \\ -2t_{j-1}^{\alpha-1}\begin{pmatrix}\frac{T}{S}\left[\frac{\partial r^2 h\left(r_i,t_{j-1}r,t\right)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\left(r_i,t_{j-1}r,t\right)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-1}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-1}\right)}{\lambda^2}\right]\end{pmatrix} \\ +t_{j-2}^{\alpha-1}\begin{pmatrix}\frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_{j-2})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\left(r_i,t_{j-2}\right)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\left(r_i,t_{j-2}\right)}{\lambda^2}\right]\end{pmatrix}\end{pmatrix} \\
&\quad\times\begin{bmatrix}(n-j+1)^\beta[2(n-j)^2+(3\beta+10)(n-j)+(2\beta^2+9\beta+12)] \\ -(n-j)^\beta[-2(n-j)^2+(5\beta+10)(n-j)+(6\beta^2+18\beta+12)]\end{bmatrix}
\end{aligned}
\tag{336}
$$

CHAPTER 8

8 DEFINITIONS OF FRACTAL-FRACTIONAL OPERATORS IN TERMS OF EXPONENTIAL DECAY LAW: CAPUTO FABRIZIO SENSE

Definition 1: Accede that $f(t)$is continuous on (a, b) and also fractal differentiable with order β then the fractal-fractional derivative of $f(t)$with order α in the Caputo-Fabrizio sense having exponential law type kernel has been written as,

$$ {}^{FFE}_{\ \ 0}D_t^{\alpha,\beta} f(t) = \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{dt^\beta} \exp\left[\frac{-\alpha}{1-\alpha}(t-\tau)\right] d\tau \qquad (337) $$

When,

$$ 0 < \alpha, \beta \leq n, M(0) = M(1) = 1 $$

$$ {}^{FFE}_{\ \ 0}D_t^{\alpha,\beta} f(t) = \frac{M(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{df(\tau)}{dt} \exp\left[\frac{-\alpha}{1-\alpha}(t-\tau)\right] d\tau \frac{t^{1-\alpha}}{\alpha} \qquad (338) $$

The more generalized version is given as,

$$ {}^{FFE}_{\ \ 0}D_t^{\alpha,\beta,\lambda} f(t) = \frac{M(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{d^\lambda f(\tau)}{dt^\beta} \exp\left[\frac{-\alpha}{1-\alpha}(t-\tau)\right] d\tau \quad (339) $$

When,

$0 < \alpha, \lambda, \beta \leq 1$

$$ {}^{FFE}_{\ \ 0}D_t^{\alpha,\beta,\lambda} f(t) = \frac{M(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{d^\lambda f(\tau)}{dt} \exp\left[\frac{-\alpha}{1-\alpha}(t-\tau)\right] d\tau \frac{t^{1-\alpha}}{\alpha} \quad (340) $$

Definition 2: Accede that $f(t)$ be continuous on (a, b) which are open intervals then the fractal-fractional integral of $f(t)$ with order α having exponential decay law type kernel is defined as follows,

$$ {}^{FFE}J_{0,t}^{\alpha,} f(t) = \frac{\alpha\beta}{\mathrm{M}(\alpha)} \int_0^t \tau^{\alpha-1} f(\tau) d\tau + \frac{\beta(1-\alpha)t^{\beta-1} f(t)}{M(\alpha)} \qquad (341) $$

8.1 NUMERICAL SCHEME: USING CAPUTO-FABRIZIO FRACTAL-FRACTIONAL DERIVATIVE

The method was developed using the Cauchy problem and implementing or introducing the Caputo-Fabrizio fractional derivative.

Let,

$$ {}_{0}^{FFE}D_{0,t}^{\alpha,\beta} h(t) = f\big(t, h(t)\big) \tag{342} $$

The integration will be given as,

$$ {}_{0}^{CF}D_{t}^{\alpha,} f(t) = \frac{1-\alpha}{\mathrm{M}(\alpha)} \beta t^{\beta-1} f\big(t, h(t)\big) + \frac{\alpha\beta}{\mathrm{M}(\alpha)} \int_0^t \tau^{\beta-1} h(\tau) d\tau \tag{343} $$

Integrating the equation above again, the following is obtained,

$$ h(t) - h(0) = \frac{1-\alpha}{\mathrm{M}(\alpha)} F\big(t, h(t)\big) + \frac{\alpha\beta}{\mathrm{M}(\alpha)} \int_0^t F\big(t, h(\tau)\big) d\tau \tag{344} $$

Where,

$$ F\big(t, h(t)\big) = \beta t^{\beta-1} f\big(t, h(t)\big) \tag{345} $$

At point,

$$ t_{n+1} = (n+1)\Delta t \tag{346} $$

and,

$$ h(t_{n+1}) - h(0) = \frac{1-\alpha}{\mathrm{M}(\alpha)} F\big(t_n, h(t_n)\big) + \frac{\alpha}{\mathrm{M}(\alpha)} \int_0^{t_{n+1}} F\big(t, h(\tau)\big) d\tau \tag{347} $$

At point,

$$ t_n = n\Delta t \tag{348} $$

and,

$$ h(t_n) - h(0) = \frac{1-\alpha}{\mathrm{M}(\alpha)} F\big(t_{n-1}, h(t_{n-1})\big) + \frac{\alpha}{\mathrm{M}(\alpha)} \int_0^{t_n} F\big(t, h(\tau)\big) d\tau \tag{349} $$

At $t_{n+1} - t_n$,

$$ h(t_{n+1}) - h(t_n) = \frac{1-\alpha}{\mathrm{M}(\alpha)} \left[F\big(t_n, h(t_n)\big) - F\big(t_{n-1}, h(t_{n-1})\big)\right] + \frac{\alpha}{\mathrm{M}(\alpha)} \int_{t_n}^{t_{n+1}} F\big(t, h(\tau)\big) d\tau \tag{350} $$

and,

$$h(t_{n+1})-h(t_n)=\frac{1-\alpha}{\mathrm{M}(\alpha)}\left[F\big(t_n,h(t_n)\big)-F\big(t_{n-1},h(t_{n-1})\big)\right]+\frac{\alpha}{\mathrm{M}(\alpha)}\int_{t_n}^{t_{n+1}}F\big(t,h(\tau)\big)d\tau \quad (351)$$

$F(t,h(t))$ was represented as a Newton Polynomial refer to equation (248). Replacing the Newton polynomial into the above equation,

$$\begin{aligned}h^{n+1}-h^n&=\frac{1-\alpha}{\mathrm{M}(\alpha)}\left[F\big(t_n,h(t_n)\big)-F\big(t_{n-1},h(t_{n-1})\big)\right]\\&+\frac{\alpha}{\mathrm{M}(\alpha)}\int_{t_n}^{t_{n+1}}\left\{\begin{matrix}F(t_{n-2},h^{n-2})\\+\dfrac{F\big(t_{n-1,}h^{n-1}\big)-F\big(t_{n-2,}h^{n-2}\big)}{\Delta t}(\tau-t_{n-2})\\+\dfrac{F\big(t_{n,}h^{n}\big)-2F\big(t_{n-1,}h^{n-1}\big)+F\big(t_{n-2,}h^{n-2}\big)}{2(\Delta t)^2}\\\times(\tau-t_{n-2})(\tau-t_{n-1})\end{matrix}\right\}d\tau\end{aligned} \quad (352)$$

The following is obtained,

$$\begin{aligned}h^{n+1}-h^n&=\frac{1-\alpha}{\mathrm{M}(\alpha)}\left[F\big(t_n,h(t_n)\big)-F\big(t_{n-1},h(t_{n-1})\big)\right]\\&+\frac{\alpha}{\mathrm{M}(\alpha)}\left\{\begin{matrix}F(t_{n-2},h^{n-2})\Delta t\\+\dfrac{F\big(t_{n-1,}h^{n-1}\big)-F\big(t_{n-2,}h^{n-2}\big)}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})d\tau\\+\dfrac{F\big(t_{n,}h^{n}\big)-2F\big(t_{n-1,}h^{n-1}\big)+F\big(t_{n-2,}h^{n-2}\big)}{2(\Delta t)^2}\\\times\displaystyle\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})(\tau-t_{n-1})d\tau\end{matrix}\right\}\end{aligned} \quad (353)$$

Therefore, the following equalities can be written as,

$$\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})d\tau=\frac{5}{2}(\Delta t)^2 \quad (354)$$

$$\int_{t_n}^{t_{n+1}}(\tau-t_{n-2})(\tau-t_{n-1})d\tau=\frac{23}{6}(\Delta t)^3 \quad (355)$$

The above scheme can be written as,

$$\begin{aligned}h^{n+1}-h^n&=\frac{1-\alpha}{\mathrm{M}(\alpha)}\left[F\big(t_n,h(t_n)\big)-F\big(t_{n-1},h(t_{n-1})\big)\right]\\&+\frac{\alpha}{\mathrm{M}(\alpha)}\sum_{j=2}^{n}\left\{\begin{matrix}F(t_{n-2},h^{n-2})\Delta t+\left[F\big(t_{n-1,}h^{n-1}\big)-F\big(t_{n-2,}h^{n-2}\big)\right]\dfrac{5}{2}\Delta t\\+\left[F\big(t_{n,}h^{n}\big)-2F\big(t_{n-1,}h^{n-1}\big)+F\big(t_{n-2,}h^{n-2}\big)\right]\dfrac{23\Delta t}{12}\end{matrix}\right\}\end{aligned} \quad (356)$$

$$h^{n+1} = h^n + \frac{1-\alpha}{\mathrm{M}(\alpha)}\left[F\left(t_n, h(t_n)\right) - F\left(t_{n-1}, h(t_{n-1})\right)\right] + \frac{\alpha}{\mathrm{M}(\alpha)}\left\{-\frac{4}{3}F\left(t_{n-1,}h^{n-1}\right)\Delta t + \frac{5}{12}F(t_{n-2}, h^{n-2})\Delta t + \frac{23}{12}F\left(t_{n,}h^n\right)\Delta t\right\} \tag{357}$$

Therefore the final numerical scheme is denoted as,

$$h^{n+1} = h^n + \frac{(1-\alpha)\beta}{\mathrm{M}(\alpha)}\left[t_n^{\beta-1}f\left(t_n, h(t_n)\right) - t_{n-1}^{\beta-1}f\left(t_{n-1}, h(t_{n-1})\right)\right]$$

$$+\frac{\alpha\beta}{\mathrm{M}(\alpha)}\left\{-\frac{4}{3}t_{n-1}^{\beta-1}f\left(t_{n-1,}h^{n-1}\right)\Delta t + \frac{5}{12}t_{n-2}^{\beta-1}f(t_{n-2}, h^{n-2})\Delta t + \frac{23}{12}t_n^{\beta-1}f\left(t_{n,}h^n\right)\Delta t\right\} \tag{358}$$

8.1.1 IMPLEMENTATION OF CAPUTO-FABRIZIO FRACTAL-FRACTIONAL DERIVATIVE ON THE SELF-SIMILAR LEAKY AQUIFER EQUATION SCENARIO 1

The two equations for self-similar leaky aquifer representing two different scenarios have been discretized in the previous chapter; however, instead of applying the Caputo Fractal-fractional derivative, the different derivative known as the Caputo-Fabrizio derivative will be applied. The discretized equation is given as,

$$ {}_{0}^{CF}D_t^{\alpha,\beta}h(r,t) = \frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r,t)}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right) + \frac{\partial h(r,t)}{\partial r}\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\right]\left(1+\frac{dr}{r}\right) + \frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r,t)}{\lambda^2} \tag{359}$$

The equation can be simplified as follows,

$$f\left(\alpha, r, t, h(r,\tau)\right) = \frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r,t)}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\frac{\partial h(r,t)}{\partial r}\right]\left(1+\frac{dr}{r}\right) + \frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r,t)}{\lambda^2} \tag{360}$$

Such that,

$$ {}_{0}^{CF}D_t^{\beta,\alpha}h(r,t) = f\left(\alpha, r, t, h(r,\tau)\right) \tag{361}$$

We recall that the Caputo-Fabrizio fractional derivative can be given as,

$$\frac{M(\alpha)}{1-\alpha}\frac{d}{dt^{\beta}}\int_{0}^{t}h(r,t)\,exp\left[\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau=F\big(\alpha,r,t,h(r,t)\big) \tag{362}$$

Differentiating in terms of β and multiplying by $\beta t^{\beta-1}$,

$$\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_{0}^{t}h(r,t)\,exp\left[\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau\frac{1}{\beta t^{\beta-1}}=F\big(\alpha,r,t,h(r,t)\big) \tag{363}$$

$$\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_{0}^{t}h(r,t)\,exp\left[\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau=F\big(\alpha,r,t,h(r,t)\big)\beta t^{\beta-1} \tag{364}$$

Integrating,

$$h(r,t)=\frac{1-\alpha}{M(\alpha)}\beta t^{\beta-1}F\big(\alpha,r,t,h(r,t)\big)+\frac{\beta\alpha}{M(\alpha)}\int_{0}^{t}\tau^{\beta-1}F\big(\alpha,r,t,h(r,t)\big)\,d\tau \tag{365}$$

At (r_i,t_{n+1}),

$$h_i^{n+1}=\frac{1-\alpha}{M(\alpha)}\beta t_{n+1}^{\beta-1}F\big(\alpha,r_i,t_{n+1},h(r_i,t_{n+1})\big)+\frac{\beta\alpha}{M(\alpha)}\int_{0}^{t_{n+1}}\tau^{\beta-1}F\big(\alpha,r_i,\tau,h(r_i,\tau)\big)\,d\tau \tag{366}$$

At (r_i,t_n),

$$h_i^{n}=\frac{1-\alpha}{M(\alpha)}\beta t_{n-1}^{\beta-1}F\big(\alpha,r_i,t_{n-1},h(r_i,t_{n-1})\big)+\frac{\beta\alpha}{M(\alpha)}\int_{0}^{t_{n}}\tau^{\beta-1}F\big(\alpha,r_i,\tau,h(r_i,\tau)\big)\,d\tau \tag{367}$$

At (r_i,t_{n+1})- (r_i,t_n),

$$h_i^{n+1}-h_i^{n}=\frac{1-\alpha}{M(\alpha)}\beta\left[t_{n+1}^{\beta-1}F(\alpha,r_i,t_n,h_i^n)-t_{n-1}^{\beta-1}F(\alpha,r_i,t_{n-1},h_i^{n-1})\right]$$

$$+\frac{\beta\alpha}{M(\alpha)}\int_{t_n}^{t_{n+1}}\tau^{\beta-1}F(\alpha,r_i,\tau,h(r,\tau))\,d\tau \tag{368}$$

Applying the numerical Caputo-Fabrizio fractal-fractional scheme, the resulting numerical solution is given as,

$$h_i^{n+1}-h_i^{n}=\frac{1-\alpha}{M(\alpha)}\beta\left[t_{n+1}^{\beta-1}F(\alpha,r_i,t_n,h_i^n)-t_{n-1}^{\beta-1}F(\alpha,r_i,t_{n-1},h_i^{n-1})\right]$$

$$+\frac{\alpha\beta}{\mathrm{M}(\alpha)}\left\{\begin{matrix} -\frac{4}{3}t_{n-1}^{\beta-1}F\big(\alpha,r_i,t_{n-1},h(r_i,t_{n-1})\big)\Delta t \\ +\frac{5}{12}t_{n-2}^{\beta-1}F\big(\alpha,r_i,t_{n-2},h(r_i,t_{n-2})\big)\Delta t \\ +\frac{23}{12}t_n^{\beta-1}F(\alpha,r_i,t_n,h(r_i,t_n))\Delta t \end{matrix}\right\} \tag{369}$$

$$h_i^{n+1}-h_i^n=\frac{1-\alpha}{M(\alpha)}\beta t_{n-1}^{\beta-1}\left\{\begin{matrix}\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\begin{matrix}\frac{\partial r^2 h(r_i,t_n)}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right)\\ +\frac{\partial h(r_i,t_n)}{\partial r}\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\end{matrix}\right]\left(1+\frac{dr}{r_i}\right)\\ +\frac{T}{S}\frac{1}{r_i}\frac{\partial h(r,t_n)}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_n)}{\lambda^2}\end{matrix}\right\}$$

$$-\frac{1-\alpha}{M(\alpha)}\beta t_{n-1}^{\beta-1}\left\{\begin{matrix}\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\begin{matrix}\frac{\partial r^2 h(r_i,t_{n-1})}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right)\\ +\frac{\partial h(r_i,t_{n-1})}{\partial r}\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\end{matrix}\right]\left(1+\frac{dr}{r_i}\right)\\ +\frac{T}{S}\frac{1}{r_i}\frac{\partial h(r,t_{n-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{n-1})}{\lambda^2}\end{matrix}\right\}$$

$$+\frac{\alpha\beta}{\mathrm{M}(\alpha)}\left\{\begin{matrix} -\frac{4\Delta t}{3}t_{n-1}^{\beta-1}\left[\begin{matrix}\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{n-1})}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\frac{\partial h(r_i,t_{n-1})}{\partial r}\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\ +\frac{T}{S}\frac{1}{r_i}\frac{\partial h(r,t_{n-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{n-1})}{\lambda^2}\end{matrix}\right]\\ +\frac{5\Delta t}{12}t_{n-2}^{\beta-1}\left[\begin{matrix}\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{n-2})}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\frac{\partial h(r_i,t_{n-2})}{\partial r}\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\ +\frac{T}{S}\frac{1}{r_i}\frac{\partial h(r,t_{n-2})}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{n-2})}{\lambda^2}\end{matrix}\right]\\ +\frac{23\Delta t}{12}t_n^{\beta-1}\left[\begin{matrix}\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_n)}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\frac{\partial h(r_i,t_n)}{\partial r}\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\ +\frac{T}{S}\frac{1}{r_i}\frac{\partial h(r,t_n)}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_n)}{\lambda^2}\end{matrix}\right]\end{matrix}\right\}$$

(370)

8.1.2 IMPLEMENTATION OF CAPUTO-FABRIZIO FRACTAL-FRACTIONAL DERIVATIVE ON THE SELF-SIMILAR LEAKY AQUIFER EQUATION SCENARIO 2

The second discretized equation in terms of Caputo-Fabrizio derivative is given as,

$$\begin{aligned} {}^{CF}_{0}D_t^{\alpha,\beta} h(r,t) \; = \; & \frac{T}{S}\left[\frac{\partial r^2 h(r,t)}{\partial r^2}\left(\frac{r^{2-2\alpha}}{\alpha^2}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r} r^{1-2\alpha}\right]\left(1+\frac{dr}{r}\right) \\ & + \frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r}\right) - \frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\right]\left(\frac{1}{dr}\right) + \frac{T}{S}\left[\frac{h(r,t)}{\lambda^2}\right] \end{aligned} \tag{371}$$

The right-hand side of the above equation can be simplified by replacing,

$$\begin{aligned} f\big(\alpha, r, t, h(r,\tau)\big) = & \frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r,t)}{\partial r^2}\left(\frac{r^{1-\alpha}}{\alpha}\right) + \left(\frac{1-\alpha}{\alpha}\right) r^{-\alpha}\frac{\partial h(r,t)}{\partial r}\right]\left(1+\frac{dr}{r}\right) \\ & + \frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha} + \frac{T}{S}\frac{h(r,t)}{\lambda^2} \end{aligned} \tag{372}$$

Such that,

$${}^{CF}_{0}D_t^{\beta,\alpha} h(r,t) \; = \; f\big(\alpha, r, t, h(r,\tau)\big) \tag{373}$$

Applying the numerical Caputo-Fabrizio fractal-fractional scheme, the resulting numerical solution is given as,

$$h_i^{n+1} - h_i^n = \frac{1-\alpha}{M(\alpha)}\beta t_{n-1}^{\beta-1} \left[\begin{array}{c} \frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_n)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_n)}{\partial r} r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\left[\frac{\partial h(r_i,t_n)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right) - \frac{T}{S}\left[\frac{\partial h(r_i,t_n)}{\partial r}\right]\left(\frac{1}{dr}\right) \\ + \frac{T}{S}\left[\frac{h(r_i,t_n)}{\lambda^2}\right] \end{array}\right]$$

$$-\frac{1-\alpha}{M(\alpha)}\beta t_{n-1}^{\beta-1} \left[\begin{array}{c} \frac{T}{S}\left[\frac{\partial r^2 h(r_i,t_{n-1})}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_{n-1})}{\partial r} r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ + \frac{T}{S}\left[\frac{\partial h(r_i,t_{n-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right) - \frac{T}{S}\left[\frac{\partial h(r_i,t_{n-1})}{\partial r}\right]\left(\frac{1}{dr}\right) + \frac{T}{S}\left[\frac{h(r_i,t_{n-1})}{\lambda^2}\right] \end{array}\right]$$

$$
\begin{aligned}
+\frac{\alpha\beta}{\mathrm{M}(\alpha)}\Bigg\{ & -\frac{4\Delta t}{3}t_{n-1}^{\beta-1}\left[\begin{array}{c} \frac{T}{S}\left[\frac{\partial r^{2}h(r_i,t_{n-1})}{\partial r^{2}}\left(\frac{r_i^{2-2\alpha}}{\alpha^{2}}\right)+\frac{(1-\alpha)}{\alpha^{2}}\frac{\partial h(r_i,t_{n-1})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{n-1})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h(r_i,t_{n-1})}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r_i,t_{n-1})}{\lambda^{2}}\right] \end{array}\right] \\
& +\frac{5\Delta t}{12}t_{n-2}^{\beta-1}\left[\begin{array}{c} \frac{T}{S}\left[\frac{\partial r^{2}h(r_i,t_{n-2})}{\partial r^{2}}\left(\frac{r_i^{2-2\alpha}}{\alpha^{2}}\right)+\frac{(1-\alpha)}{\alpha^{2}}\frac{\partial h(r_i,t_{n-2})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{n-2})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h(r_i,t_{n-2})}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r_i,t_{n-2})}{\lambda^{2}}\right] \end{array}\right] \\
& +\frac{23\Delta t}{12}t_{n}^{\beta-1}\left[\begin{array}{c} \frac{T}{S}\left[\frac{\partial r^{2}h(r_i,t_{n})}{\partial r^{2}}\left(\frac{r_i^{2-2\alpha}}{\alpha^{2}}\right)+\frac{(1-\alpha)}{\alpha^{2}}\frac{\partial h(r_i,t_{n})}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right) \\ +\frac{T}{S}\left[\frac{\partial h(r_i,t_{n})}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h(r_i,t_{n})}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r_i,t_{n})}{\lambda^{2}}\right] \end{array}\right] \Bigg\}
\end{aligned}
\tag{374}
$$

CHAPTER 9

9 FRACTAL-FRACTIONAL OPERATORS IN TERMS OF MITTAG-LEFFLER LAW: ATANGANA BALEANU SENSE

Definition 1: Accede that $f(t)$is continuous on (a, b) and also fractal differentiable with order β then the fractal-fractional derivative of $f(t)$ with order α e having Mittag-Leffler law type kernel has been defined as follows,

$$ {}^{FFM}_{\ \ 0}D_t^{\alpha,\beta} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt^{\beta}} \int_0^t f(\tau)\, E_\alpha \left[\frac{-\alpha}{1-\alpha} (t-\tau) \right] d\tau \qquad (375) $$

Where,

$$ AB(\alpha) = 1 - \alpha + \tfrac{\alpha}{\Gamma(\alpha)}, $$

$$ {}^{FFM}_{\ \ 0}D_t^{\alpha,\beta} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(\tau)\, E_\alpha \left[\frac{-\alpha}{1-\alpha} (t-\tau) \right] d\tau \frac{t^{1-\alpha}}{\alpha} \qquad (376) $$

The more generalized version is given as,

$$ {}^{FFM}_{\ \ 0}D_t^{\alpha,\beta,\lambda} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \frac{d^{\lambda}}{dt^{\beta}} \int_0^t f(\tau)\, E_\alpha \left[\frac{-\alpha}{1-\alpha} (t-\tau) \right] d\tau \qquad (377) $$

When,

$$ 0 < \alpha, \beta, \lambda \leq 1, $$

$$ {}^{FFM}_{\ \ 0}D_t^{\alpha,\beta} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \frac{d^{\lambda}}{dt} \int_0^t f(\tau)\, E_\alpha \left[\frac{-\alpha}{1-\alpha} (t-\tau) \right] d\tau \frac{t^{1-\alpha}}{\alpha} \qquad (378) $$

Definition 2: Accede that $f(t)$is continuous and fractal differentiable on (a, b) with order β then the fractal-fractional derivative of $f(t)$f(t) with order α in the Caputo sense having Mittag-Leffler law type kernel has been defined as follows,

$$ {}^{FFM}_{\ \ 0}D_t^{\alpha,\beta} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{dt^{\beta}} E_\alpha \left[\frac{-\alpha}{1-\alpha} (t-\tau) \right] d\tau \qquad (379) $$

Where,

$$ AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \qquad (380) $$

and,

$$ {}^{FFM}_{\ \ \ 0}D_t^{\alpha,\beta} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{dt} E_\alpha \left[\frac{-\alpha}{1-\alpha}(t-\tau) \right] d\tau \frac{t^{1-\alpha}}{\alpha} \quad (381) $$

The more generalized version is given as,

$$ {}^{FFM}_{\ \ \ 0}D_t^{\alpha,\beta,\lambda} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \int_0^t \frac{d^\lambda f(\tau)}{dt^\beta} E_\alpha \left[\frac{-\alpha}{1-\alpha}(t-\tau) \right] d\tau \quad (382) $$

When,

$0 < \alpha, \lambda, \beta \leq 1$,

Therefore,

$$ {}^{FFM}_{\ \ \ 0}D_t^{\alpha,\beta,\lambda} f(t) = \frac{AB(\alpha)}{\Gamma(1-\alpha)} \int_0^t \frac{d^\lambda f(\tau)}{dt} E_\alpha \left[\frac{-\alpha}{1-\alpha}(t-\tau) \right] d\tau \frac{t^{1-\alpha}}{\alpha} \quad (383) $$

Definition 3: Accede that $f(t)$ be continuous on (a, b) at an open interval then the fractal-fractional integral of $f(t)$ with order α having generalized Mittag-Leffler type is given as,

$$ {}^{FFM}J_{0,t}^{\alpha,\beta} f(t) = \frac{\alpha\beta}{\mathrm{AB}(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\alpha-1} f(\tau) d\tau + \frac{\beta(1-\alpha) t^{\beta-1} f(t)}{AB(\alpha)} \quad (384) $$

9.1 NUMERICAL SCHEME: USING ATANGANA-BALEANU FRACTAL-FRACTIONAL DERIVATIVE

The Cauchy problem was analyzed,

$$ {}^{FFM}_{\ \ \ 0}D_t^{\alpha,\beta} h(t) = f\left(t, \left(h(t)\right)\right) \quad (385) $$

Applying the Atangana-Baleanu fractal-fractional derivative on the equation above and integrating the equation,

$$ h(t) - h(0) = \frac{1-\alpha}{AB(\alpha)} \beta t^{\beta-1} f(t(h(t)) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \int_0^t \tau^{\beta-1} f(\tau, h(\tau))(t-\tau)^{\alpha-1} d\tau \quad (386) $$

Let,

$$ F\left(t, h(t)\right) = \beta t^{\beta-1} f(t(h(t)) \quad (387) $$

and,

$$ h(t) - h(0) = \frac{1-\alpha}{AB(\alpha)} F\left(t, h(t)\right) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \int_0^t F(\tau, h(\tau))(t-\tau)^{\alpha-1} d\tau \quad (388) $$

At the point $t_{n+1} = (n+1)\Delta t$,

$$h(t_{n+1}) - h(0) = \frac{1-\alpha}{AB(\alpha)} F\big(t_n, h(t_n)\big) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} F(\tau, h(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \quad (389)$$

This can also be given as,

$$\begin{aligned} h(t_{n+1}) &= h(0) + \tfrac{1-\alpha}{AB(\alpha)} F\big(t_n, h(t_n)\big) \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} F(\tau, h(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \end{aligned} \quad (390)$$

The Newton polynomial in equation (210) was substituted to approximate the following function $F\big(t, h(t)\big)$,

$$h^{n+1} = h^0 + \frac{1-\alpha}{\mathrm{AB}(\alpha)} F\big(t_n, h(t_n)\big) + \frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} \left\{ \begin{array}{c} F\big(t_{j-2}, h^{j-2}\big) \\ + \dfrac{F\big(t_{j-1,} h^{j-1}\big) - F\big(t_{j-2,} h^{j-2}\big)}{\Delta t} \big(\tau - t_{j-2}\big) \\ + \dfrac{F\big(t_{j,} h^{j}\big) - 2F\big(t_{j-1,} h^{j-1}\big) + F\big(t_{j-2,} h^{j-2}\big)}{2(\Delta t)^2} \\ \times \big(\tau - t_{j-2}\big)\big(\tau - t_{j-1}\big) \end{array} \right\} (t_{n+1}-\tau)^{\alpha-1} d\tau \quad (391)$$

This can be represented or simplified as follows,

$$h^{n+1} = h^0 + \frac{1-\alpha}{\mathrm{AB}(\alpha)} F\big(t_n, h(t_n)\big)$$

$$+ \frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} \left\{ \begin{array}{c} \displaystyle\int_{t_j}^{t_{j+1}} F\big(t_{j-2}, h^{j-2}\big)(t_{n+1}-\tau)^{\alpha-1} d\tau \\ + \displaystyle\int_{t_j}^{t_{j+1}} \frac{F\big(t_{j-1,} h^{j-1}\big) - F\big(t_{j-2,} h^{j-2}\big)}{\Delta t} \big(\tau - t_{j-2}\big)(t_{n+1}-\tau)^{\alpha-1} d\tau \\ + \displaystyle\int_{t_j}^{t_{j+1}} \frac{F\big(t_{j,} h^{j}\big) - 2F\big(t_{j-1,} h^{j-1}\big) + F\big(t_{j-2,} h^{j-2}\big)}{2(\Delta t)^2} \\ \times \big(\tau - t_{j-2}\big)\big(\tau - t_{j-1}\big)(t_{n+1}-\tau)^{\alpha-1} d\tau \end{array} \right\} \quad (392)$$

Therefore, the summations can be given as,

$$\begin{aligned} h^{n+1} = h^0 &+ \frac{1-\alpha}{\mathrm{AB}(\alpha)} F\big(t_n, h(t_n)\big) + \frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} F\big(t_{j-2}, h^{j-2}\big) \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} d\tau \\ &+ \frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)} \sum_{j=2}^{n} \frac{F\big(t_{j-1,} h^{j-1}\big) - F\big(t_{j-2,} h^{j-2}\big)}{\Delta t} \int_{t_j}^{t_{j+1}} \big(\tau - t_{j-2}\big)(t_{n+1}-\tau)^{\alpha-1} d\tau \end{aligned}$$

$$+\frac{\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha)}\sum_{j=2}^{n}\frac{F\left(t_{j,}h^{j}\right)-2F\left(t_{j-1,}h^{j-1}\right)+F\left(t_{j-2,}h^{j-2}\right)}{2(\Delta t)^{2}}$$

$$\times\int_{t_j}^{t_{j+1}}\times\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau \tag{393}$$

The integrals can be calculated and given as,

$$\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau=\frac{(\Delta t)^{\alpha}}{\alpha}\left[(n-j+1)^{\alpha}-(n-j)^{\alpha}\right] \tag{394}$$

$$\int_{t_j}^{t_{j+1}}\left(\tau-t_{j-2}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau=\frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)}\begin{bmatrix}(n-j+1)^{\alpha}(n-j+3+2\alpha)\\-(n-j+1)^{\alpha}(n-j+3+3\alpha)\end{bmatrix} \tag{395}$$

$$\int_{t_j}^{t_{j+1}}\left(\tau-t_{j-2}\right)\left(\tau-t_{j-1}\right)(t_{n+1}-\tau)^{\alpha-1}d\tau=\frac{(\Delta t)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}$$

$$\times\begin{bmatrix}(n-j+1)^{\alpha}\begin{bmatrix}2(n-j)^{2}+(3\alpha+10)(n-j)\\+(2\alpha^{2}+9\alpha+12)\end{bmatrix}\\-(n-j)^{\alpha}\begin{bmatrix}2(n-j)^{2}+(5\alpha+10)(n-j)\\+(6\alpha^{2}+18\alpha+12)\end{bmatrix}\end{bmatrix} \tag{396}$$

Replacing the calculated integrals in the equality above,

$$h^{n+1}=h^{0}+\frac{1-\alpha}{\mathrm{AB}(\alpha)}F\left(t_n,h(t_n)\right)$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+1)}\sum_{j=2}^{n}F\left(t_{j-2},h^{j-2}\right)\left[(n-j+1)^{\alpha}-(n-j)^{\alpha}\right]$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+2)}\sum_{j=2}^{n}\left[F\left(t_{j-1,}h^{j-1}\right)-F\left(t_{J-2,}h^{j-2}\right)\right]\times\begin{bmatrix}(n-j+1)^{\alpha}(n-j+3+2\alpha)\\-(n-j)^{\alpha}(n-j+3+3\alpha)\end{bmatrix}$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{2\mathrm{AB}(\alpha)\Gamma(\alpha+3)}\sum_{j=2}^{n}F\left(t_{j,}h^{j}\right)-2F\left(t_{j-1,}h^{j-1}\right)+F\left(t_{j-2,}h^{j-2}\right)$$

$$\times\begin{bmatrix}(n-j+1)^{\alpha}\left[2(n-j)^{2}+(3\alpha+10)(n-j)+(2\alpha^{2}+9\alpha+12)\right]\\-(n-j)^{\alpha}\left[2(n-j)^{2}+(5\alpha+10)(n-j)+(6\alpha^{2}+18\alpha+12)\right]\end{bmatrix} \tag{397}$$

Therefore, the final numerical scheme can be given as,

$$h^{n+1}=h^{0}+\frac{1-\alpha}{\mathrm{AB}(\alpha)}\beta t_{n}^{\beta-1}f\left(t_n,h(t_n)\right)$$

$$+\frac{\alpha\beta(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+1)}\sum_{j=2}^{n}t_{j-2}^{\beta-1}f\left(t_{j-2},h^{j-2}\right)\left[(n-j+1)^{\alpha}-(n-j)^{\alpha}\right]$$

$$
\begin{aligned}
&+\frac{\alpha\beta(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+2)}\sum_{j=2}^{n}\left[t_{j-1}^{\beta-1}f\left(t_{j-1,}h^{j-1}\right)-t_{j-2}^{\beta-1}f\left(t_{J-2,}h^{j-2}\right)\right] \\
&\times\begin{bmatrix}(n-j+1)^{\alpha}(n-j+3+2\alpha) \\ -(n-j)^{\alpha}(n-j+3+3\alpha)\end{bmatrix} \\
&+\frac{\alpha\beta(\Delta t)^{\alpha}}{2\mathrm{AB}(\alpha)\Gamma(\alpha+3)}\sum_{j=2}^{n}\left[t_{j}^{\beta-1}f\left(t_{j,}h^{j}\right)-2t_{j-1}^{\beta-1}f\left(t_{j-1,}h^{j-1}\right)+t_{j-2}^{\beta-1}f\left(t_{j-2,}h^{j-2}\right)\right] \\
&\times\begin{bmatrix}(n-j+1)^{\alpha}[2(n-j)^{2}+(3\alpha+10)(n-j)+(2\alpha^{2}+9\alpha+12)] \\ -(n-j)^{\alpha}[2(n-j)^{2}+(5\alpha+10)(n-j)+(6\alpha^{2}+18\alpha+12)]\end{bmatrix}
\end{aligned}
\tag{398}
$$

9.1.1 IMPLEMENTATION OF ATANGANA BALEANU FRACTAL-FRACTIONAL DERIVATIVE ON THE SELF-SIMILAR LEAKY AQUIFER EQUATION SCENARIO 1

The two equations for self-similar leaky aquifer representing two different scenarios have been discretized in the previous chapters, were the Caputo and the Caputo-Fabrizio fractal fractional derivative were introduced in the equations. In this chapter the Atangana Baleanu Fractional derivative will be introduced,

$$
\begin{aligned}
{}_{0}^{AB}D_{t}^{\beta,\alpha}h(r,t) \;=\; &\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^{2}h(r,t)}{\partial r^{2}}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\frac{\partial h(r,t)}{\partial r}\right]\left(1+\frac{dr}{r}\right) \\
&+\frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r,t)}{\lambda^{2}}
\end{aligned}
\tag{399}
$$

The equation can be simplified and given as,

$$
\begin{aligned}
f\big(\alpha,r,t,h(r,\tau)\big)=&\frac{T}{S}\frac{r^{1-\alpha}}{\alpha}\left[\frac{\partial r^{2}h(r,t)}{\partial r^{2}}\left(\frac{r^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r^{-\alpha}\frac{\partial h(r,t)}{\partial r}\right]\left(1+\frac{dr}{r}\right) \\
&+\frac{T}{S}\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r,t)}{\lambda^{2}}
\end{aligned}
\tag{400}
$$

Such that,

$$
{}_{0}^{AB}D_{t}^{\beta,\alpha}h(r,t) \;=\; f\big(\alpha,r,t,h(r,\tau)\big) \tag{401}
$$

As presented before, the above equation is converted into an integral equation by applying the Atangana-Baleanu fractional integral.

$$h(r,t)-h(r,0)=\left\{\begin{array}{c}\dfrac{1-\alpha}{AB(\alpha)}f\big(\alpha,r,t,h(r,t)\big)\\ +\dfrac{\alpha}{AB(\alpha)\Gamma(\alpha)}\displaystyle\int_{0}^{t}(t-l)^{\alpha-1}f\big(\alpha,r,l,h(r,l)\big)dl\end{array}\right\} \tag{402}$$

We evaluate the above equation at the point r_i, t_{n+1}, such that the function $h(r,t)$ will be represented as,

$$\begin{aligned} h(r_i,t_{n+1}) &= h(r_i,0)+\frac{1-\alpha}{AB(\alpha)}f\big(r_i,t_n,h(r_i,t_n)\big)\\ &\quad +\frac{\alpha}{AB\ (\alpha)\Gamma(\alpha)}\sum_{j=2}^{n}\int_{t_j}^{t_{j+1}}f(\alpha,r_i,\tau,h(r_i,\tau)(t_{n+1}-\tau)^{\alpha-1}\,d\tau \end{aligned} \tag{403}$$

Following the scheme presented earlier which was derived using the Newton polynomials and the Cauchy problem the numerical solution will be given as,

$$\begin{aligned} h^{n+1} &= h^0+\frac{1-\alpha}{\mathrm{AB}(\alpha)}\beta t_n^{\beta-1}f(\alpha,r_i,t_n\,,h_i^n)\\ &\quad +\frac{\alpha\beta(\Delta t)^\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha+1)}\sum_{j=2}^{n}t_{j-2}^{\beta-1}f\left(\alpha,r_i,t_{j-2},h\big(r,t_{j-2}\big)\right)\left[(n-j+1)^\alpha-(n-j)^\alpha\right]\\ &\quad +\frac{\alpha\beta(\Delta t)^\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha+2)}\sum_{j=2}^{n}\begin{bmatrix} t_{j-1}^{\beta-1}f\left(\alpha,r_i,t_{j-1},h\big(r,t_{j-1}\big)\right)\\ -t_{j-2}^{\beta-1}f\left(\alpha,r_i,t_{j-2},h\big(r,t_{j-2}\big)\right)\end{bmatrix}\begin{bmatrix}(n-j+1)^\alpha(n-j+3+2\alpha)\\ -(n-j)^\alpha(n-j+3+3\alpha)\end{bmatrix}\\ &\quad +\frac{\alpha\beta(\Delta t)^\alpha}{2\mathrm{AB}(\alpha)\Gamma(\alpha+3)}\sum_{j=2}^{n}\begin{bmatrix} t_j^{\beta-1}f\left(\alpha,r_i,t_j,h\big(r,t_j\big)\right)\\ -2t_{j-1}^{\beta-1}f\left(\alpha,r_i,t_{j-1},h\big(r,t_{j-1}\big)\right)\\ +t_{j-2}^{\beta-1}f\left(\alpha,r_i,t_{j-2},h\big(r,t_{j-2}\big)\right)\end{bmatrix}\\ &\quad \times\begin{bmatrix}(n-j+1)^\alpha\begin{bmatrix}2(n-j)^2+(3\alpha+10)(n-j)\\ +(2\alpha^2+9\alpha+12)\end{bmatrix}\\ -(n-j)^\alpha\begin{bmatrix}2(n-j)^2+(5\alpha+10)(n-j)\\ +(6\alpha^2+18\alpha+12)\end{bmatrix}\end{bmatrix} \end{aligned} \tag{404}$$

The functions $f\left(\alpha,r_i,t_j,h\big(r_i,t_j\big)\right)$, $f\left(\alpha,r_i,t_{j-1},h\big(r_i,t_{j-1}\big)\right)$ $and\ f\left(\alpha,r_i,t_{j-2},h\big(r_i,t_{j-2}\big)\right)$ were previously defined and represented in chapter six. Therefore, the final numerical solution can be given as,

$$h^{n+1}=h^{0}+\frac{1-\alpha}{\mathrm{AB}(\alpha)}\beta t_j^{\beta-1}\left[\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_j)}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_j)}{\partial r}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_j)}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_j)}{\lambda^2}\end{array}\right]$$

$$+\frac{\alpha\beta(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+1)}\sum_{j=2}^{n}t_{j-2}^{\beta-1}\left\{\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\begin{array}{c}\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)\\ +\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_{j-2})}{\partial r}\end{array}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}\\ +\frac{T}{S}\frac{h(r_i,t_{j-2})}{\lambda^2}\end{array}\right\}\left([(n-j+1)^{\alpha}-(n-j)^{\alpha}]\right)$$

$$+\frac{\alpha\beta(\Delta t)^{\alpha}}{\mathrm{AB}(\alpha)\Gamma(\alpha+2)}\sum_{j=2}^{n}\left[\begin{array}{c}t_{j-1}^{\beta-1}\left\{\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{j-1})}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_{j-1})}{\partial r}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-1})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{j-1})}{\lambda^2}\end{array}\right\}\\ -t_{j-2}^{\beta-1}\left\{\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_{j-2})}{\partial r}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{j-2})}{\lambda^2}\end{array}\right\}\end{array}\right]$$

$$\times\left[\begin{array}{c}(n-j+1)^{\alpha}(n-j+3+2\alpha)\\ -(n-j)^{\alpha}(n-j+3+3\alpha)\end{array}\right]$$

$$+\frac{\alpha\beta(\Delta t)^{\alpha}}{2\mathrm{AB}(\alpha)\Gamma(\alpha+3)}\sum_{j=2}^{n}\left[\begin{array}{c}t_{j}^{\beta-1}\left\{\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{j})}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_{j})}{\partial r}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{j})}{\lambda^2}\end{array}\right\}\\ -2t_{j-1}^{\beta-1}\left\{\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{j-1})}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_{j-1})}{\partial r}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-1})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{j-1})}{\lambda^2}\end{array}\right\}\\ +t_{j-2}^{\beta-1}\left\{\begin{array}{c}\frac{T}{S}\frac{r_i^{1-\alpha}}{\alpha}\left[\frac{\partial r^2 h(r_i,t_{j-2})}{\partial r^2}\left(\frac{r_i^{1-\alpha}}{\alpha}\right)+\left(\frac{1-\alpha}{\alpha}\right)r_i^{-\alpha}\frac{\partial h(r_i,t_{j-2})}{\partial r}\right]\\ \left(1+\frac{dr}{r_i}\right)+\frac{T}{S}\frac{1}{r}\frac{\partial h(r_i,t_{j-2})}{\partial r}\frac{r_i^{1-\alpha}}{\alpha}+\frac{T}{S}\frac{h(r_i,t_{j-2})}{\lambda^2}\end{array}\right\}\end{array}\right]$$

$$\times\left[\begin{array}{c}(n-j+1)^{\alpha}[2(n-j)^2+(3\alpha+10)(n-j)+(2\alpha^2+9\alpha+12)]\\ -(n-j)^{\alpha}[2(n-j)^2+(5\alpha+10)(n-j)+(6\alpha^2+18\alpha+12)]\end{array}\right]\tag{405}$$

9.1.2 IMPLEMENTATION OF ATANGANA BALEANU FRACTAL-FRACTIONAL DERIVATIVE ON THE SELF-SIMILAR LEAKY AQUIFER EQUATION SCENARIO 2

In chapter six the equation representing the water flowing in through a self-similar leaky aquifer however out of a normal leaky aquifer was presented and differentiated. For scenario 2 we will also use the concept of the fading memory applying the Atangana–Baleanu equation.

$$
{}_{0}^{c}D_{t}^{\beta,\alpha}h(r,t) = \frac{T}{S}\left[\frac{\partial r^{2}h(r,t)}{\partial r^{2}}\left(\frac{r^{2-2\alpha}}{\alpha^{2}}\right)+\frac{(1-\alpha)}{\alpha^{2}}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha}\right]\left(1+\frac{dr}{r}\right)
$$

$$
+\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r}\right)-\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r,t)}{\lambda^{2}}\right] \tag{406}
$$

The right-hand side of the above equation can be simplified as,

$$
f\big(\alpha,r,t,h(r,\tau)\big) = \frac{T}{S}\left[\frac{\partial r^{2}h(r,t)}{\partial r^{2}}\left(\frac{r^{2-2\alpha}}{\alpha^{2}}\right)+\frac{(1-\alpha)}{\alpha^{2}}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha}\right]\left(1+\frac{dr}{r}\right)
$$

$$
+\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r}\right)-\frac{T}{S}\left[\frac{\partial h(r,t)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h(r,t)}{\lambda^{2}}\right] \tag{407}
$$

Such that,

$$
{}_{0}^{AB}D_{t}^{\beta,\alpha}h(r,t) = f\big(\alpha,r,t,h(r,\tau)\big) \tag{408}
$$

As presented before, the above equation can be converted into an integral equation by applying the Atangana-Baleanu fractional integral.

$$
h(r,t)-h(r,0) = \left\{\begin{matrix} \dfrac{1-\alpha}{AB(\alpha)}f\big(\alpha,r,t,h(r,t)\big) \\ +\dfrac{\alpha}{AB(\alpha)\Gamma(\alpha)}\displaystyle\int_{0}^{t}(t-l)^{\alpha-1}f\big(\alpha,r,l,h(r,l)\big)dl \end{matrix}\right\} \tag{409}
$$

We evaluate the above equation at the point r_i, t_{n+1}, such that the function $h(r,t)$ will be represented as,

$$h(r_i,t_{n+1})=h(r_i,0)+\frac{1-\alpha}{AB(\alpha)}f\big(r_i,t_n,h(r_i,t_n)\big)$$

$$+\frac{\alpha}{AB\ (\alpha)\Gamma(\alpha)}\sum_{j=2}^{n}\int_{t_j}^{t_{j+1}}f(\alpha,r_i,\tau,h(r_i,\tau)(t_{n+1}-\tau)^{\alpha-1}\,d\tau \tag{410}$$

The three functions were previously defined in chapter six. Therefore,

$$h^{n+1}=h^0+\frac{1-\alpha}{\mathrm{AB}(\alpha)}\beta t_n^{\beta-1}\left[\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_j\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_j\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_j\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_j\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_j\big)}{\lambda^2}\right]\end{array}\right]$$

$$+\frac{\alpha\beta(\Delta t)^\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha+1)}\sum_{j=2}^{n}t_{j-2}^{\beta-1}\left\{\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_{j-2}\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_{j-2}\big)}{\lambda^2}\right]\end{array}\right\}$$

$$\times[(n-j+1)^\alpha-(n-j)^\alpha]$$

$$+\frac{\alpha\beta(\Delta t)^\alpha}{\mathrm{AB}(\alpha)\Gamma(\alpha+2)}\sum_{j=2}^{n}\left[\begin{array}{l}t_{j-1}^{\beta-1}\left\{\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_{j-1}\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_{j-1}\big)}{\lambda^2}\right]\end{array}\right\}\\-t_{j-2}^{\beta-1}\left\{\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_{j-2}\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_{j-2}\big)}{\lambda^2}\right]\end{array}\right\}\end{array}\right]$$

$$\times\left[\begin{array}{l}(n-j+1)^\alpha(n-j+3+2\alpha)\\-(n-j)^\alpha(n-j+3+3\alpha)\end{array}\right]$$

$$+\frac{\alpha\beta(\Delta t)^\alpha}{2\mathrm{AB}(\alpha)\Gamma(\alpha+3)}\sum_{j=2}^{n}\left[\begin{array}{l}t_{j}^{\beta-1}\left\{\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_{j}\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_{j}\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j}\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j}\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_{j}\big)}{\lambda^2}\right]\end{array}\right\}\\-2t_{j-1}^{\beta-1}\left\{\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_{j-1}\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-1}\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_{j-1}\big)}{\lambda^2}\right]\end{array}\right\}\\+t_{j-2}^{\beta-1}\left\{\begin{array}{l}\frac{T}{S}\left[\frac{\partial r^2h\big(r_i,t_{j-2}\big)}{\partial r^2}\left(\frac{r_i^{2-2\alpha}}{\alpha^2}\right)+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}r_i^{1-2\alpha}\right]\left(1+\frac{dr}{r_i}\right)\\+\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right]\left(\frac{1}{dr}-\frac{1}{r_i}\right)-\frac{T}{S}\left[\frac{\partial h\big(r_i,t_{j-2}\big)}{\partial r}\right]\left(\frac{1}{dr}\right)+\frac{T}{S}\left[\frac{h\big(r_i,t_{j-2}\big)}{\lambda^2}\right]\end{array}\right\}\end{array}\right]$$

$$\times\left[\begin{array}{l}(n-j+1)^\alpha[2(n-j)^2+(3\alpha+10)(n-j)+(2\alpha^2+9\alpha+12)]\\-(n-j)^\alpha[2(n-j)^2+(5\alpha+10)(n-j)+(6\alpha^2+18\alpha+12)]\end{array}\right] \tag{411}$$

CHAPTER 10

10 NUMERICAL SIMULATIONS

10.1 CAPUTO NUMERICAL FIGURES AND INTERPRETATION

The new generalized equation for leaky aquifer which was derived from local differential operators was analyzed and the stability was assessed. The equation was derived using the fractal operator; however, it did not take into account memory effect and long-term interaction. The numerical simulations of the new models based on power law are presented in this chapter. The figures were configured through software known as the MATLAB. Figure 4-8 represents cone of depression as function of space and as function of time in figure 9-13. The space and the time were kept constant in figure 14-18 and the time-space plots are given. In figure 19-23 cone of depression was expressed as a function of time space solution and the contour plots are given in figure 24-28.

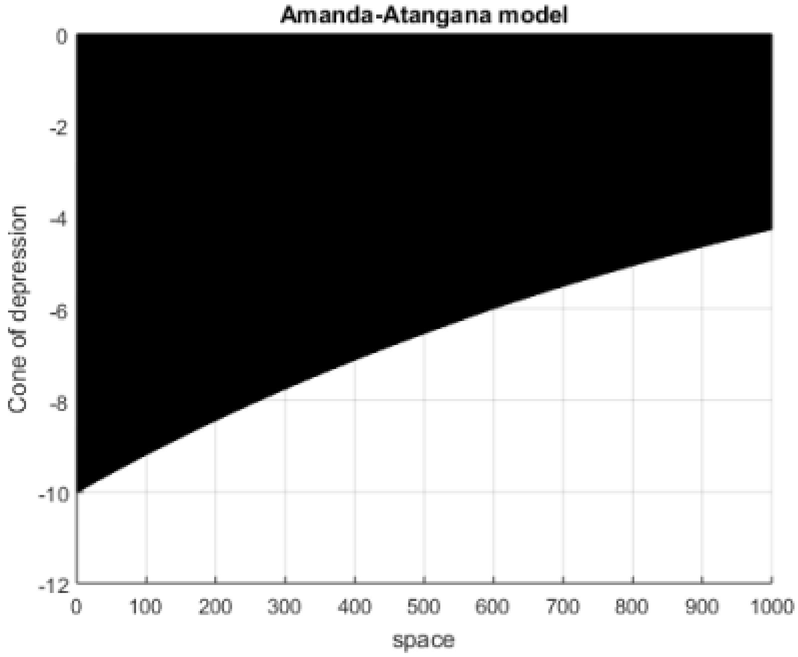


Figure 4: Cone of depression at order 0.04 in space

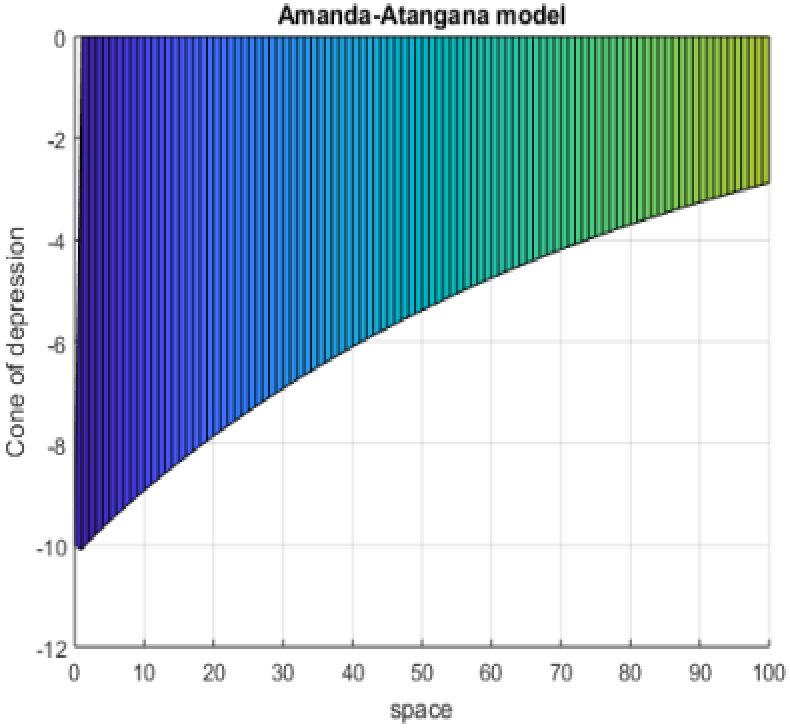


Figure 5: Cone of depression at order 0.4 in space

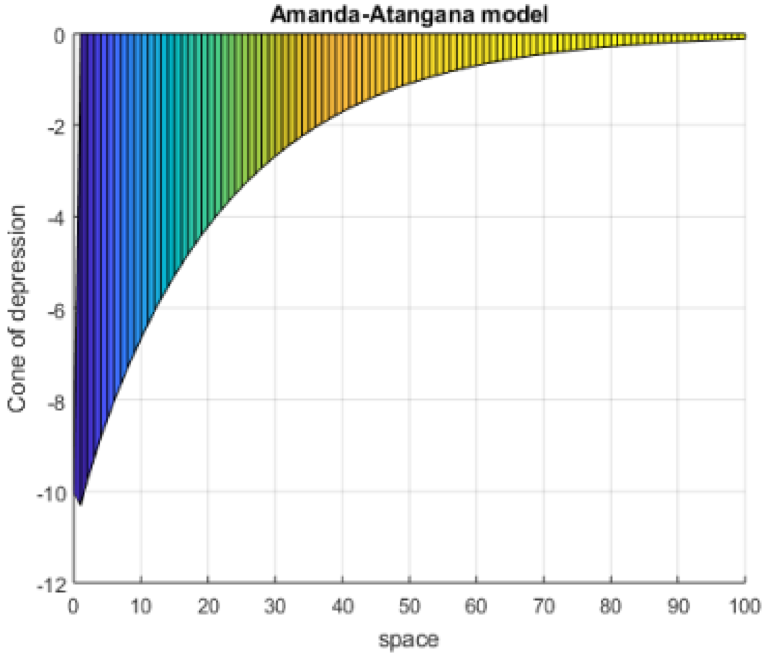


Figure 6: Cone of depression at order 0.7 in space

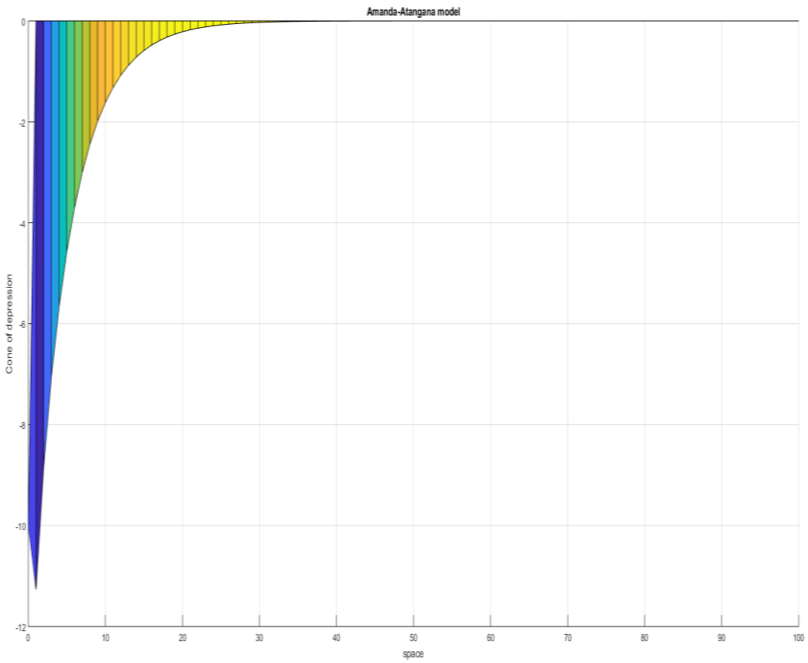


Figure 7: Cone of depression at order 0.9 in space

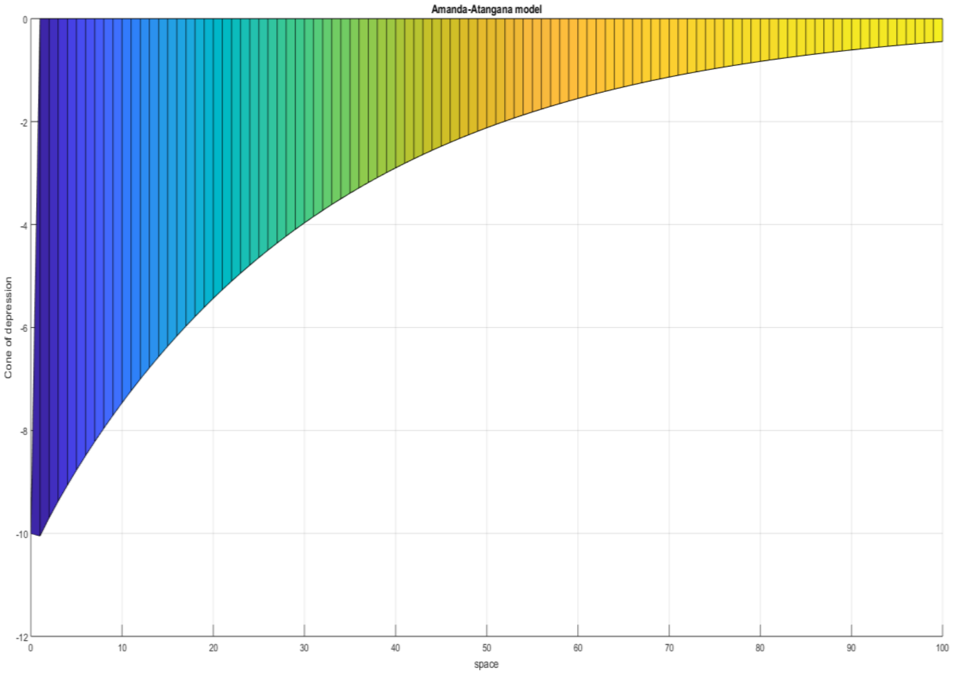
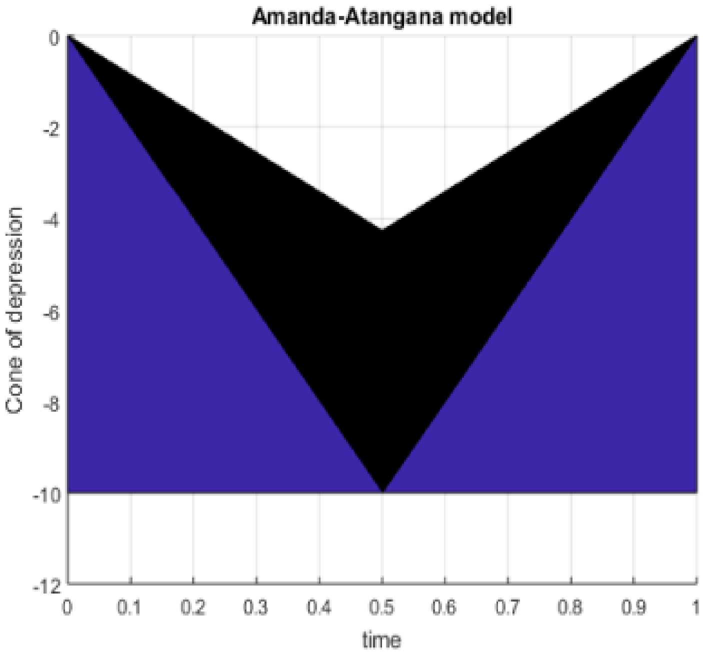


Figure 8: Cone of depression at order 1 in space



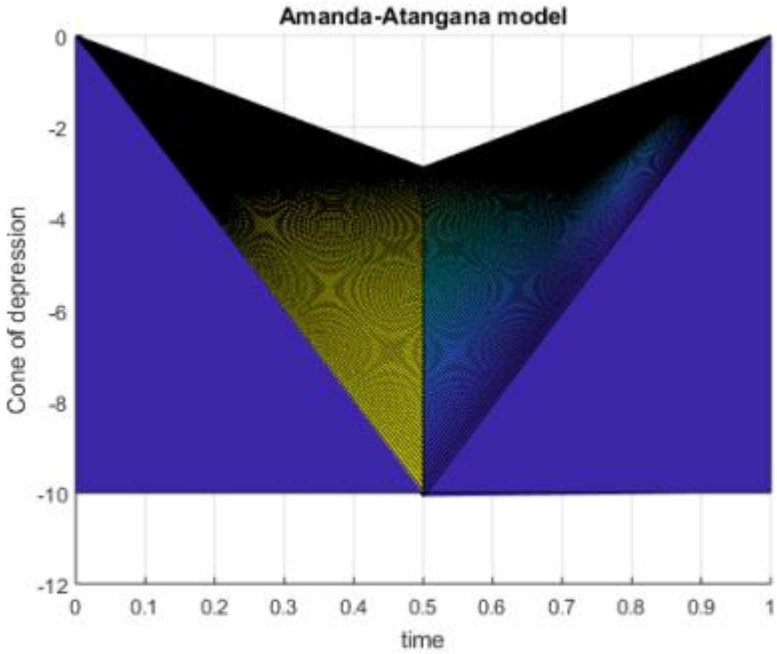
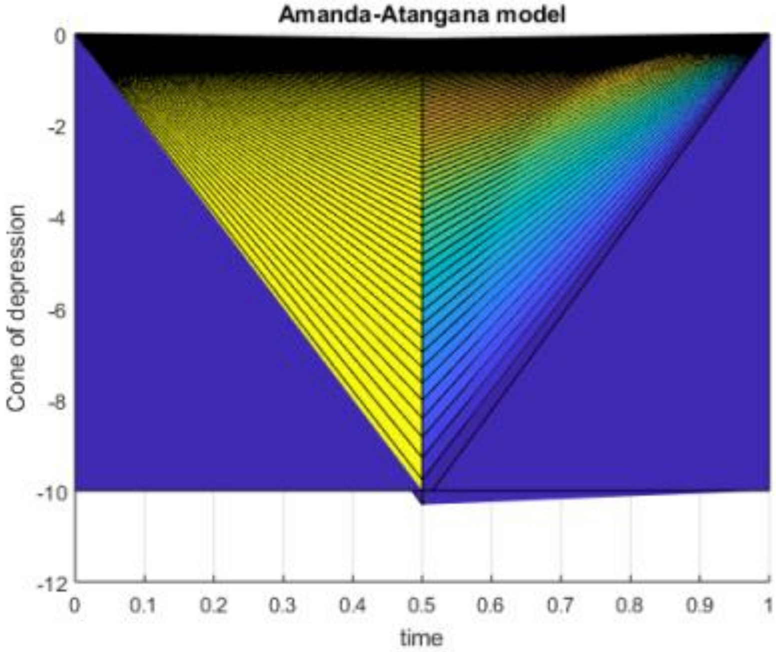


Figure 9: Cone of depression for order 0.04 in time	Figure 10: Cone of depression at order 0.4 in time



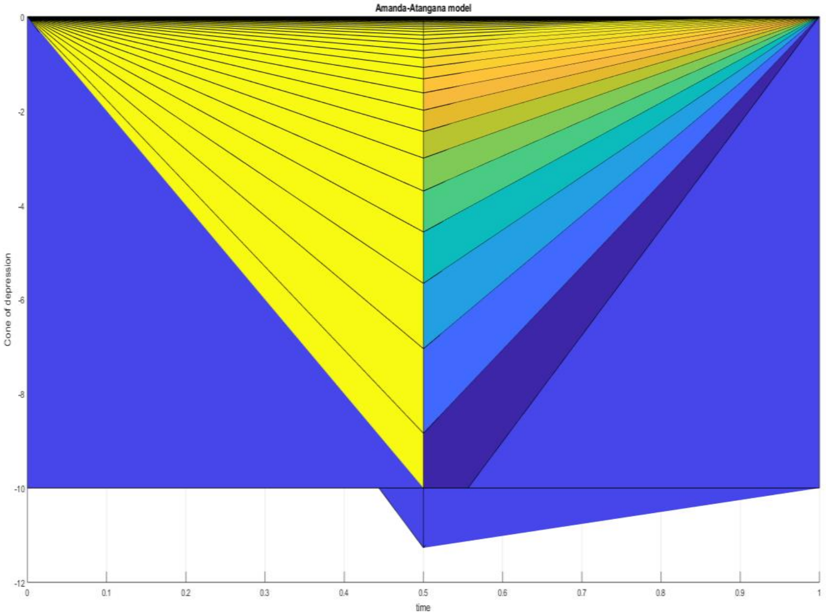


Figure 11: Cone of depression at order 0.7 in time	Figure 12: Cone of depression at order 0.9 time

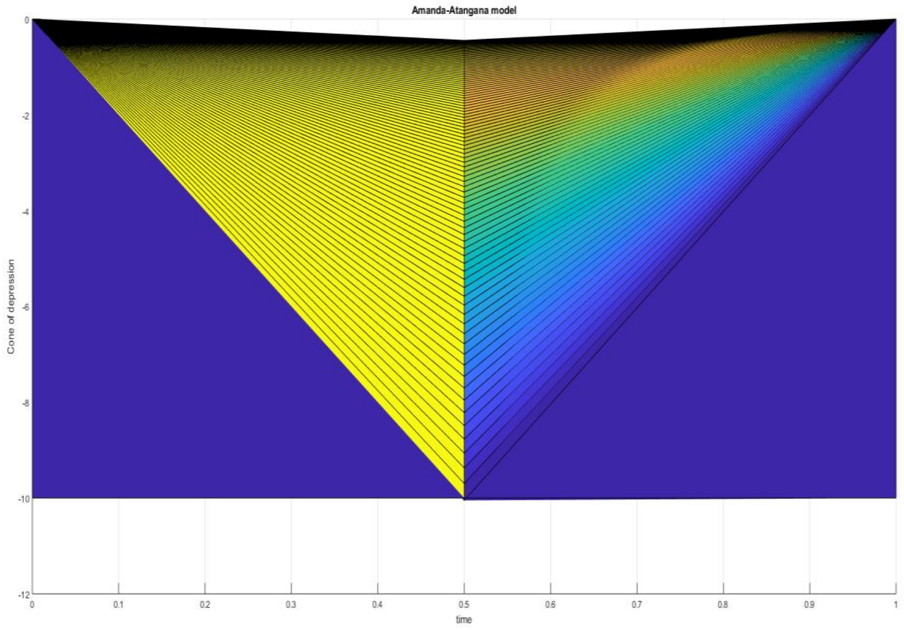
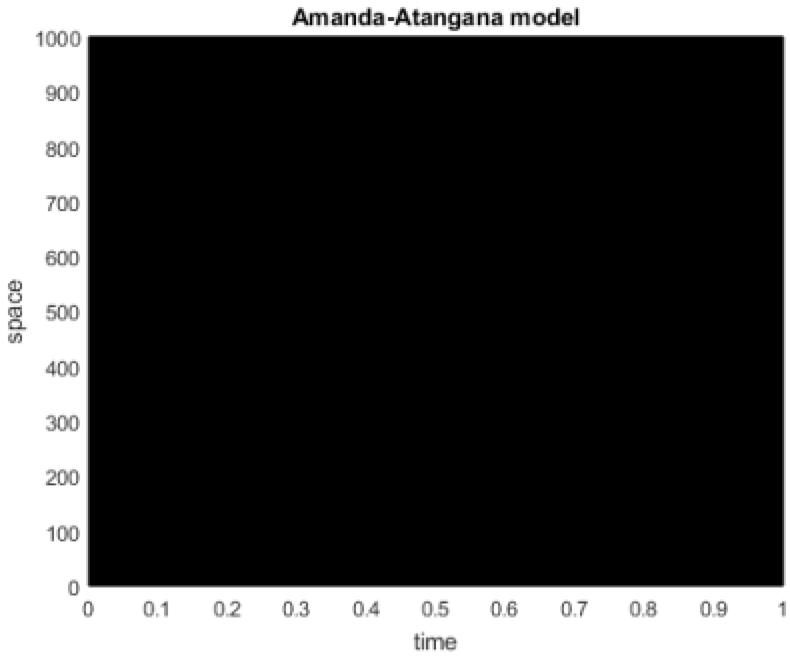


Figure 13: Cone of depression at order 1 in time



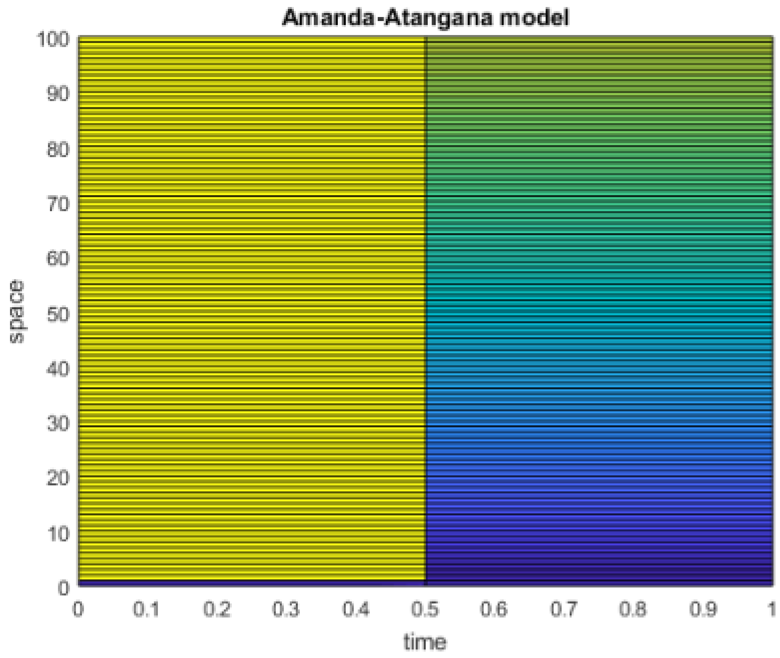


Figure 14: Amanda-Atangana model for order 0.04 time space solution	Figure 15: Amanda-Atangana model for order 0.4 time space solution

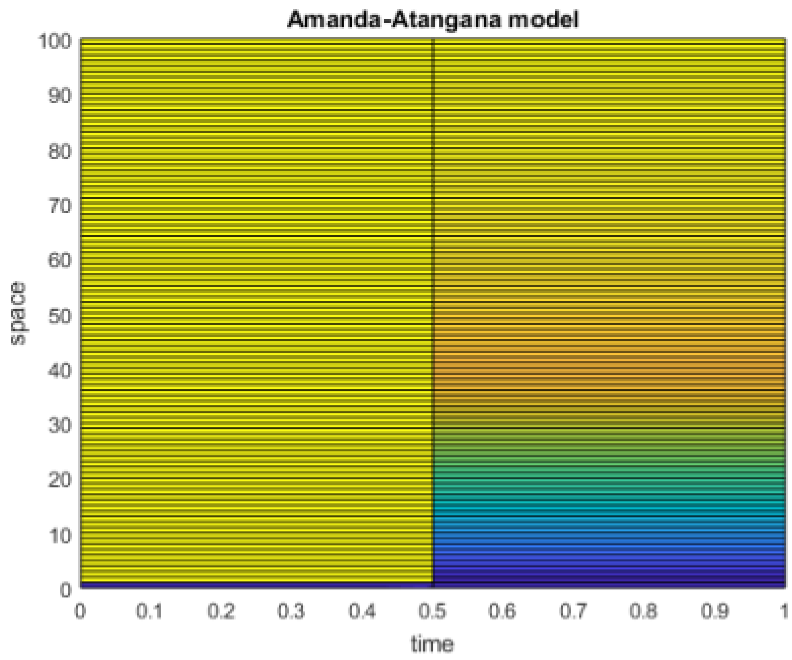


Figure 16: Amanda-Atangana model for order 0.7 time space solution

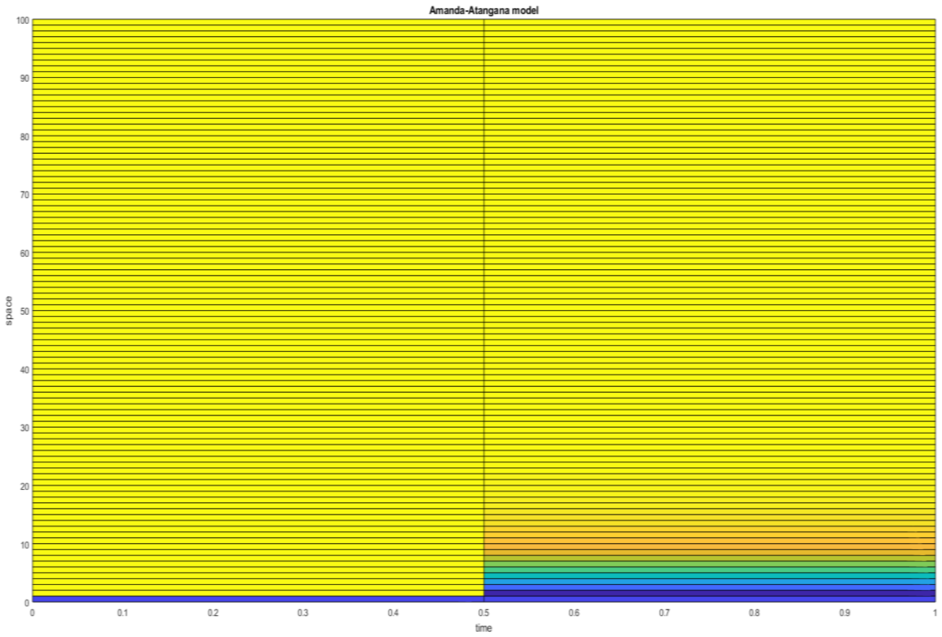


Figure 17: Amanda-Atangana model for order 0.9 time space solution

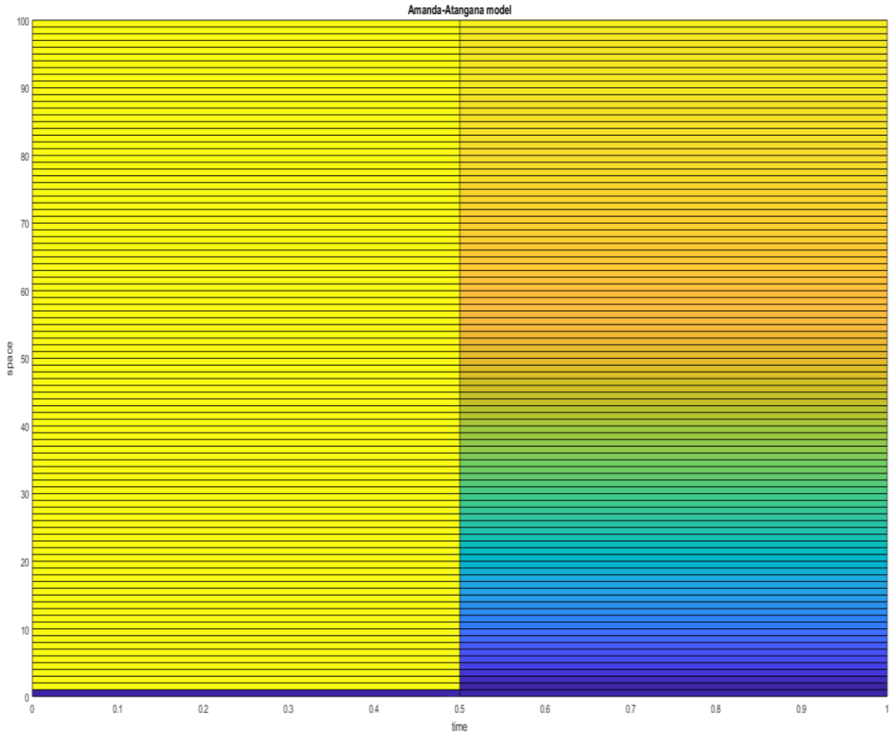


Figure 18: Amanda-Atangana model for order 1 time space solution

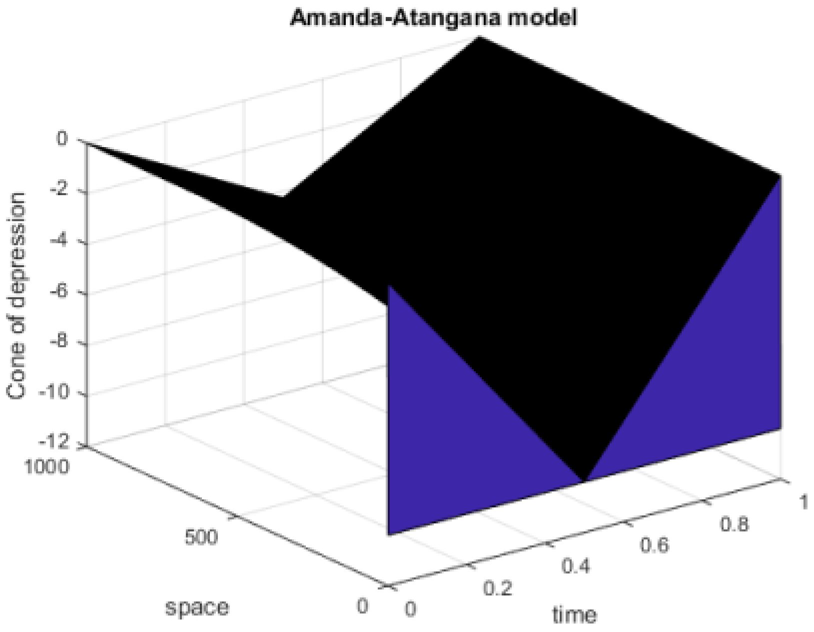


Figure 19: Cone of depression for order 0.04 time space solution

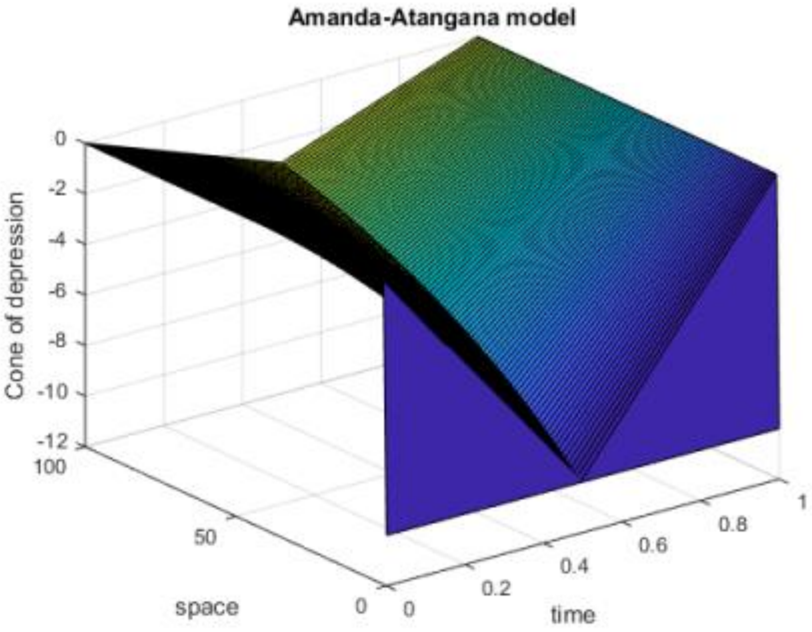
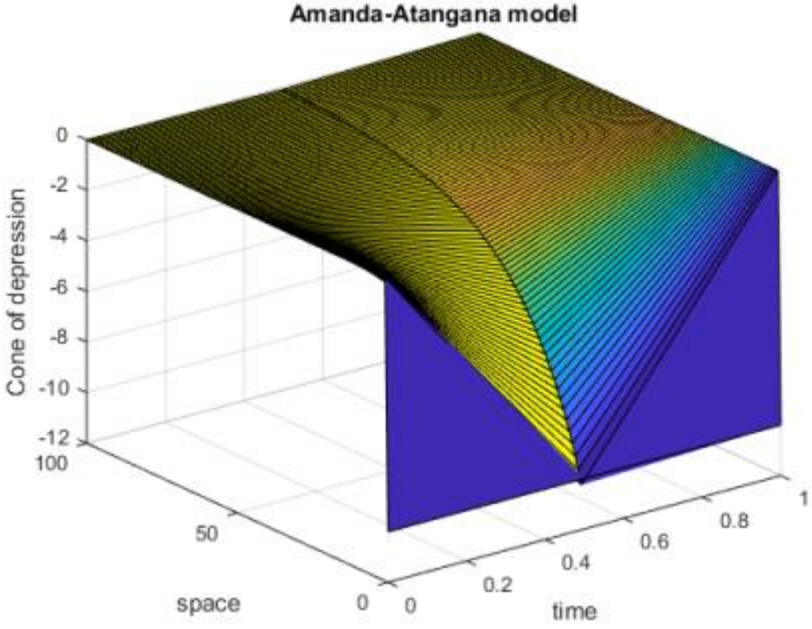


Figure 20: Cone of depression for order 0.4 time space solution



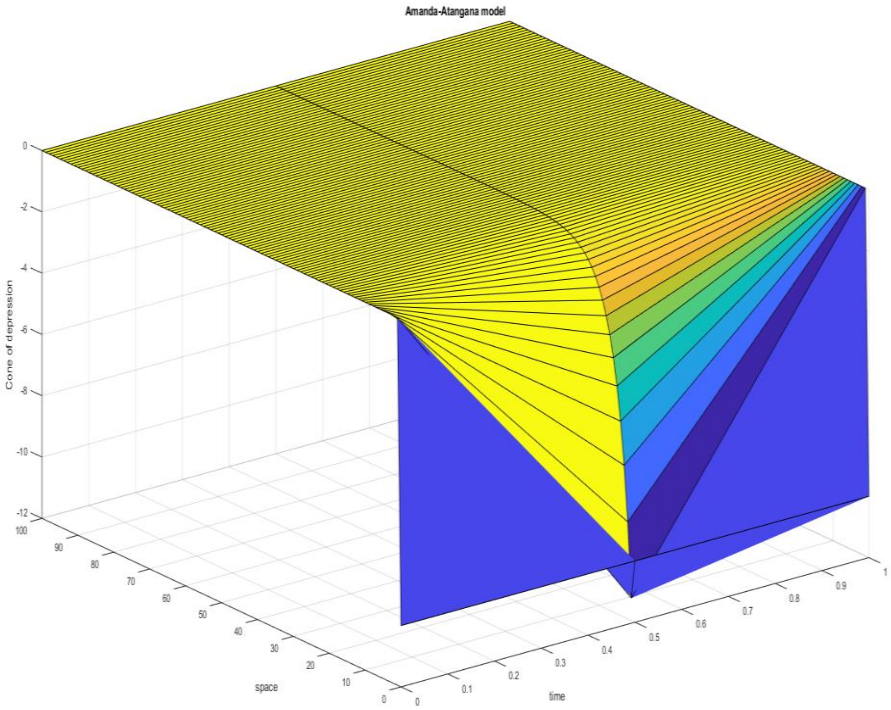


Figure 21: Cone of depression for order 0.7 time space solution	Figure 22: Cone of depression for order 0.9 time space solution

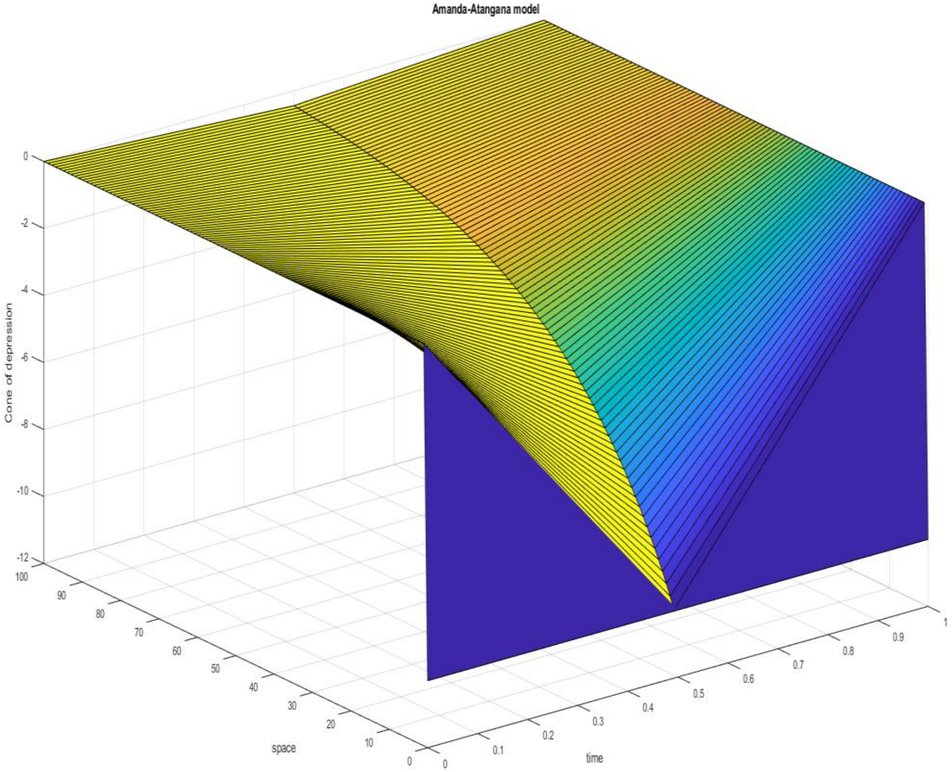


Figure 23: Cone of depression for order 1 time space solution

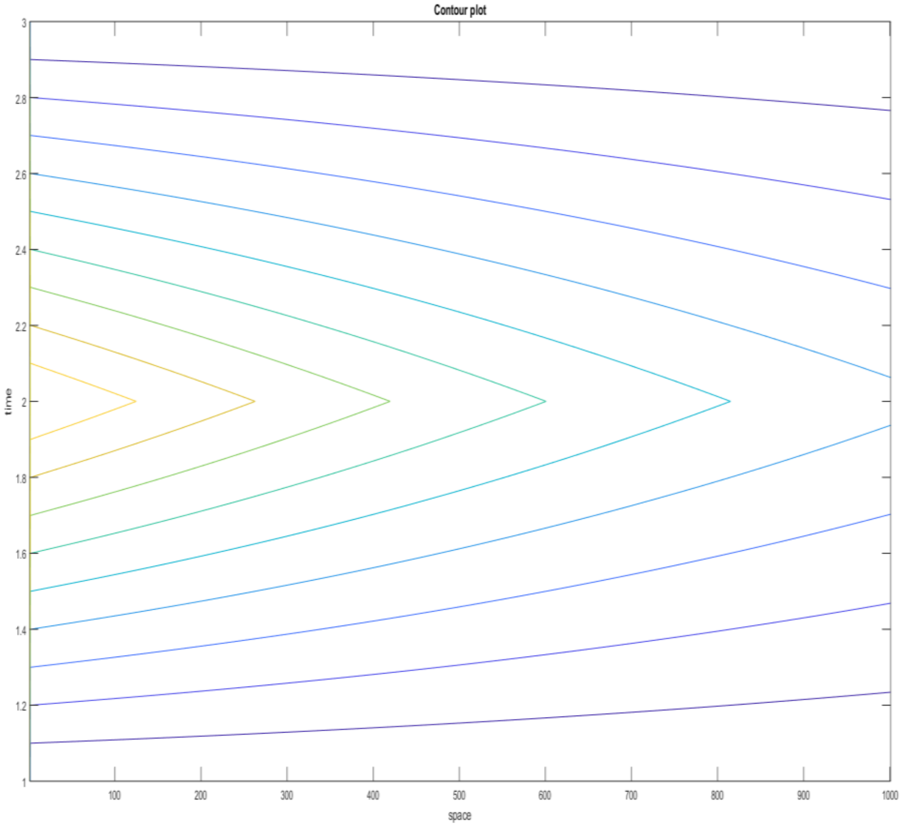


Figure 24: Contour plot at 0.04 time space solution

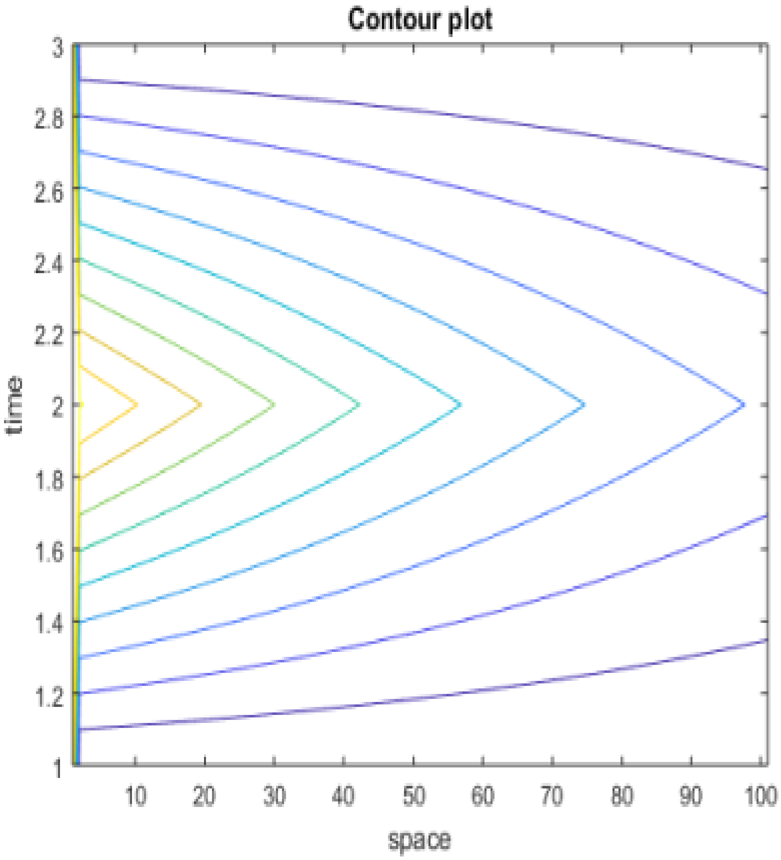


Figure 25: Contour plot at 0.4 time space solution

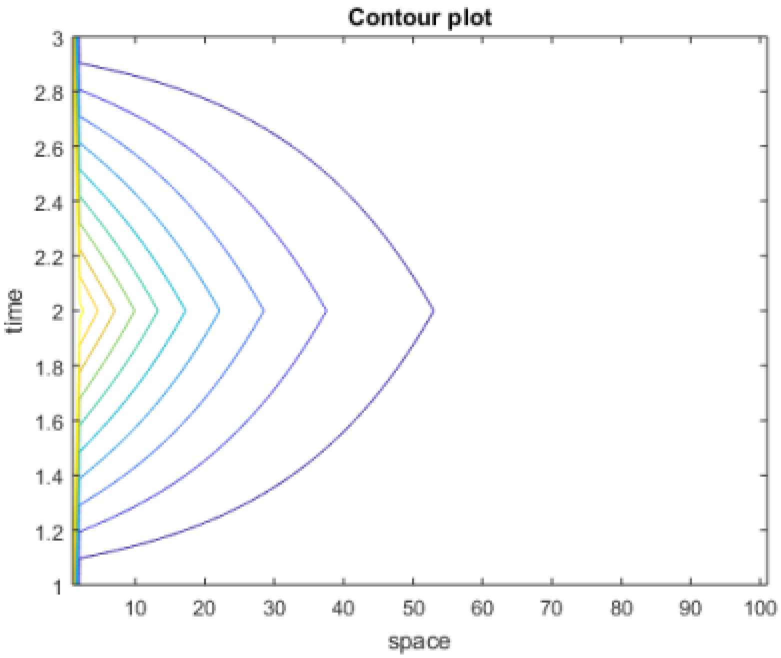


Figure 26: Contour plot at 0.7 time space solution

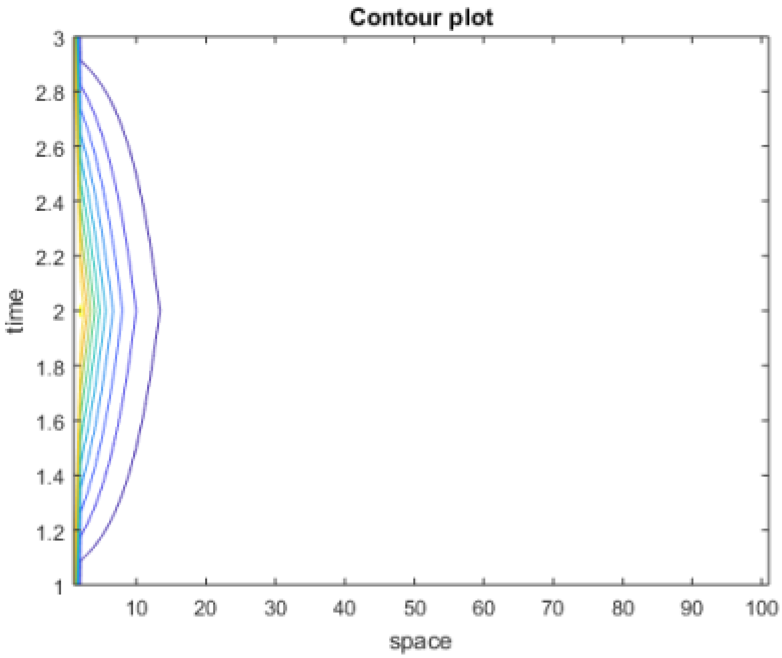


Figure 27: Contour plot at 0.9 time space solution

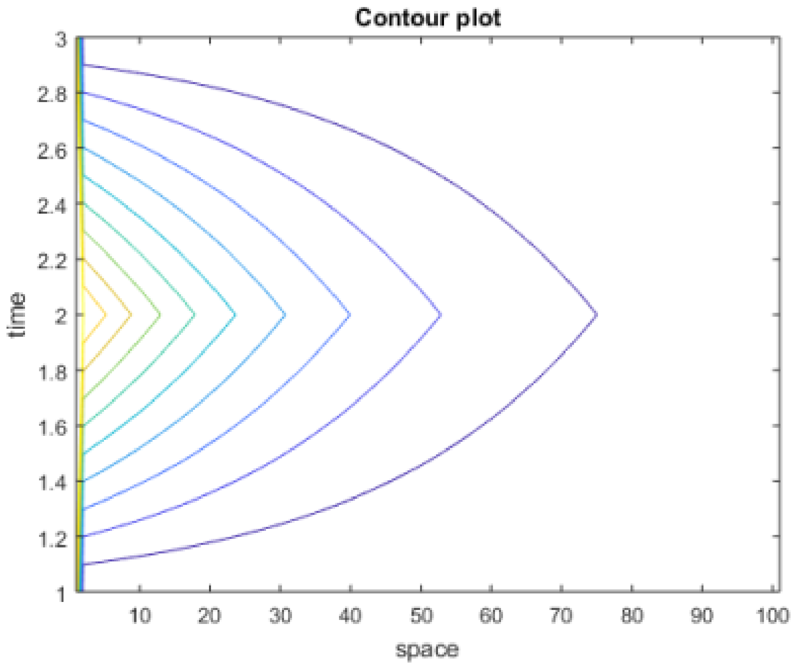


Figure 28: Contour plot at 1 time space solution

The cone of depression decreases as the time increases and decreases as the space decreases. From the simulations it is depicted that the lower the order the higher the cone of depression. When the order is 0.04 the cone of depression is high, as compared to when the order is 0.4. When the order is been increased to 0.7 and 0.9 the cone of depression is declining. At order one the cone of depression is normal. We can simply summaries that at order 0.04 and 0.4 the water is flowing through a sand or silt-sand type of soil. At order 0.7 the type of soil can be silt and at 0.9 the type of soil would be expected to be clay. At order 1 the water flow is normal as it is depicted in the figures. This indicates that the water flow is occurring through the fractures. The contour plots indicated that the memory is fully captured at order 0.7. This interpretation is presented in literature that a permeable material is capable of releasing more water and the cone of depression will be high. As groundwater flows within a geological formation, it encounters different formations with different porosity and permeability. We recall that the higher the porosity the large extent of water to spread out and the flow velocity or permeability will be very low. A clear example in nature is the experiment of the toilet paper and a towel. The water flow within a toilet paper will be high and within a towel will be low due to their permeability.

10.2 CAPUTO-FABRIZIO NUMERICAL FIGURES AND INTERPRETATION

The numerical simulations of the new models based on exponential law are presented. The numerical simulation for cone of depression as function of space are depicted in figure 29-33 and as function of time in figure 34-38. The space and the time were kept constant in figure 39-43 and the time-space plots are given. In figure 44-48 cone of depression was expressed as a function of time space solution and the contour plots are given in figure 49-53.

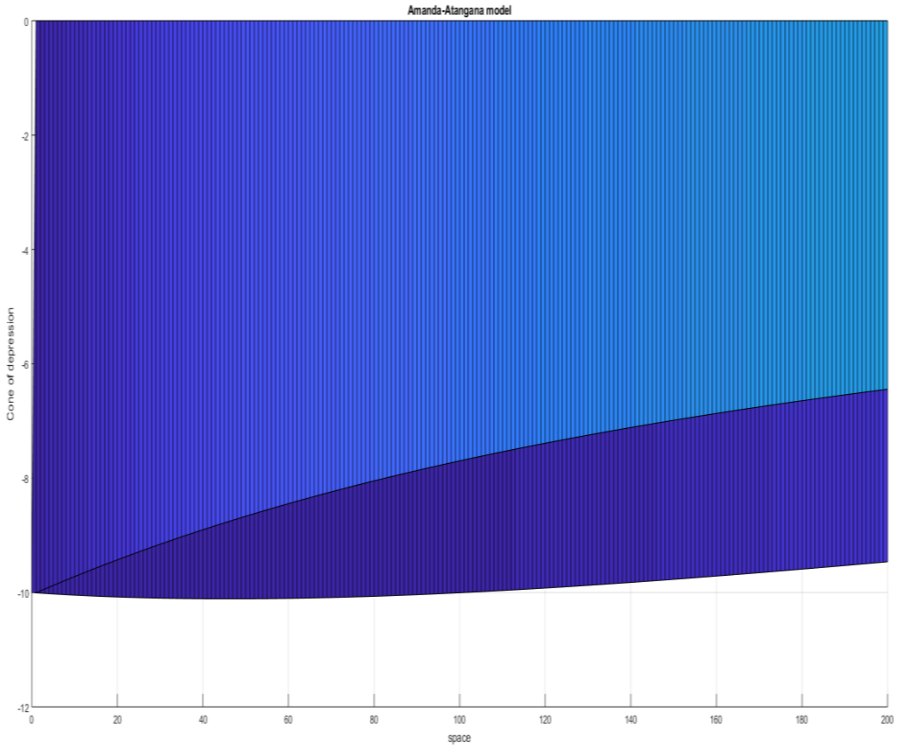


Figure 29: Cone of depression at order 0.03 in space

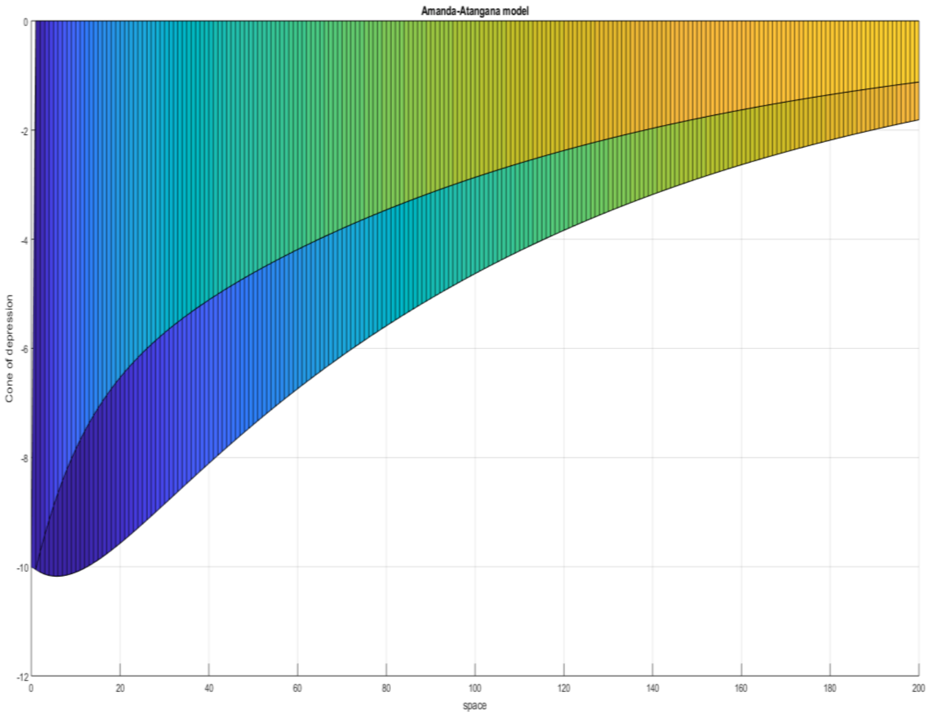


Figure 30: Cone of depression at order 0.3 in space

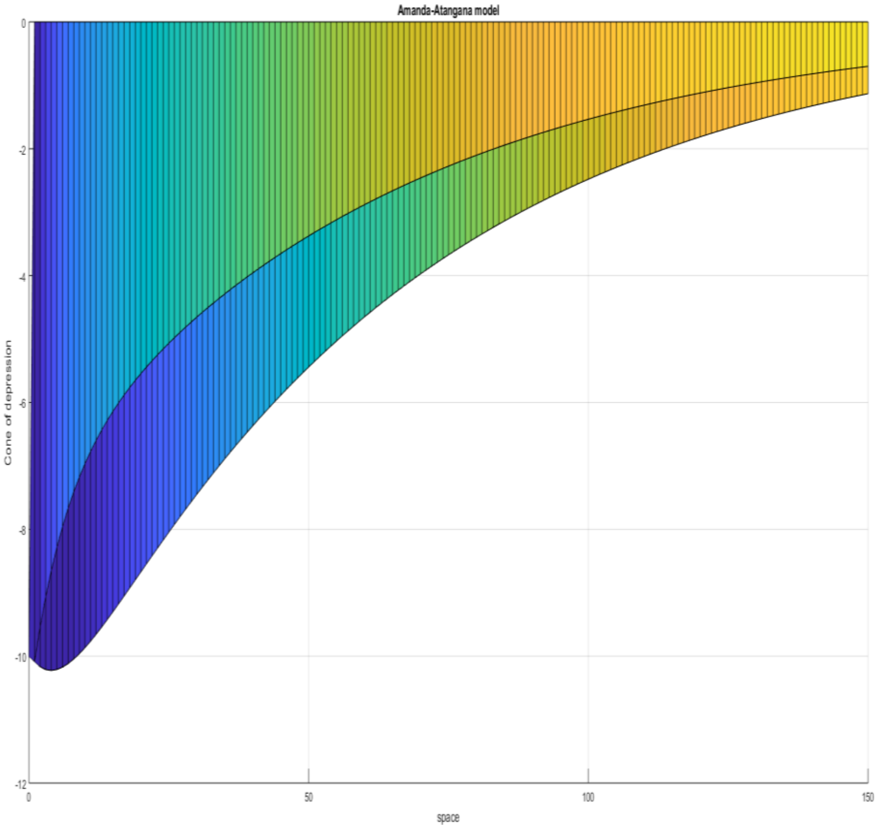


Figure 31: Cone of depression at order 0.5 in space

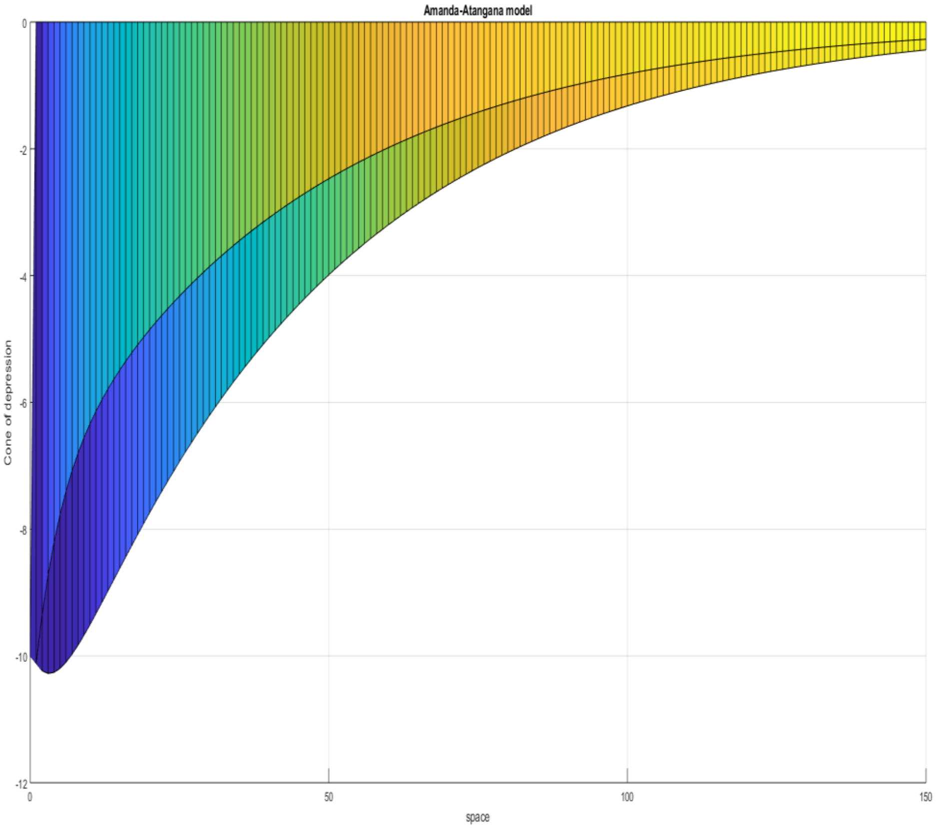


Figure 32: Cone of depression at order 0.7 in space

‘

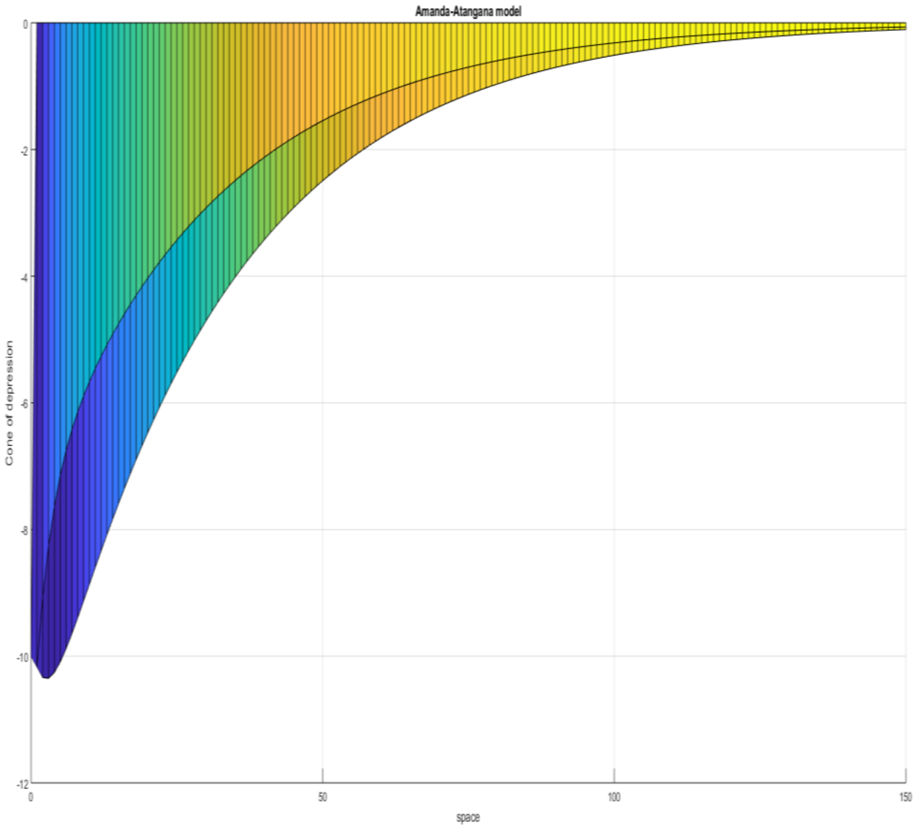
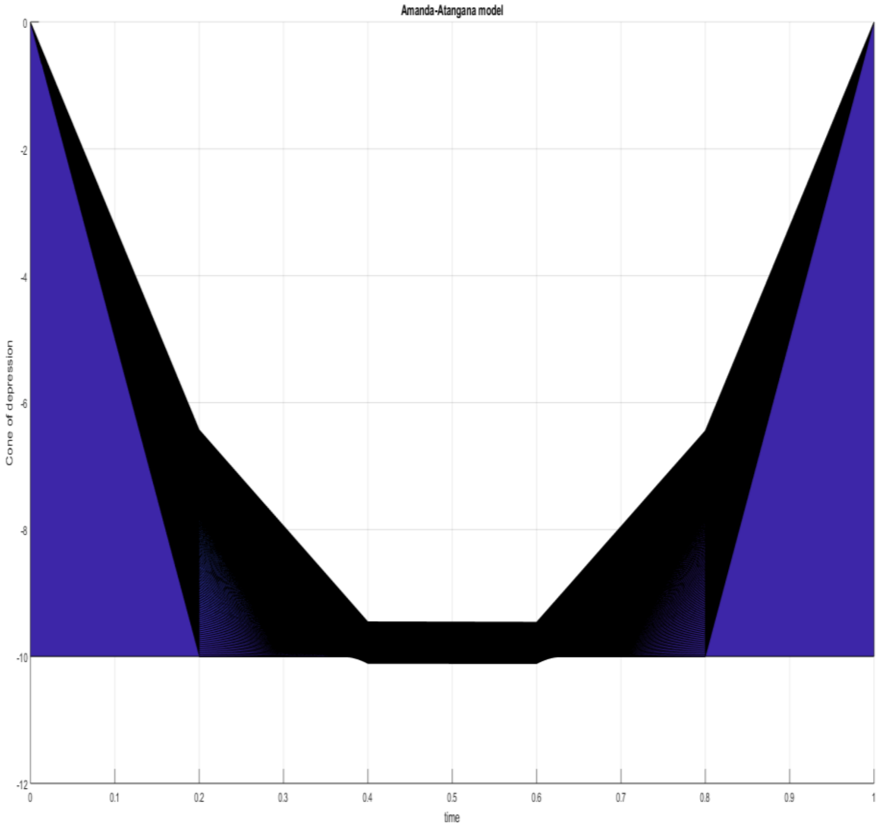


Figure 33: Cone of depression at order 1 in space



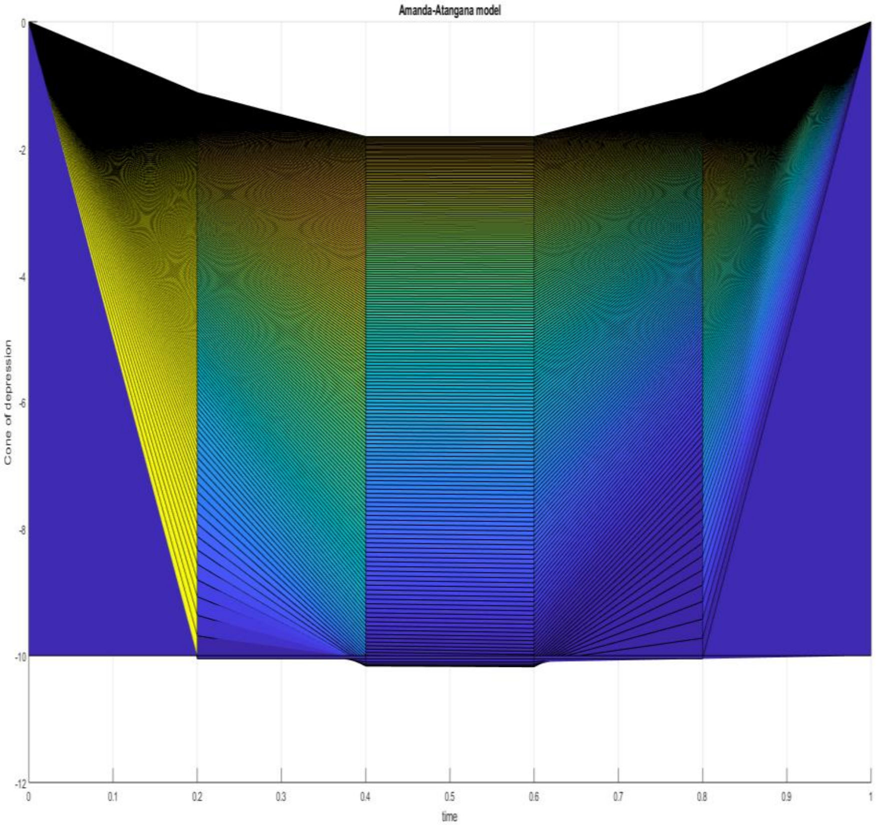
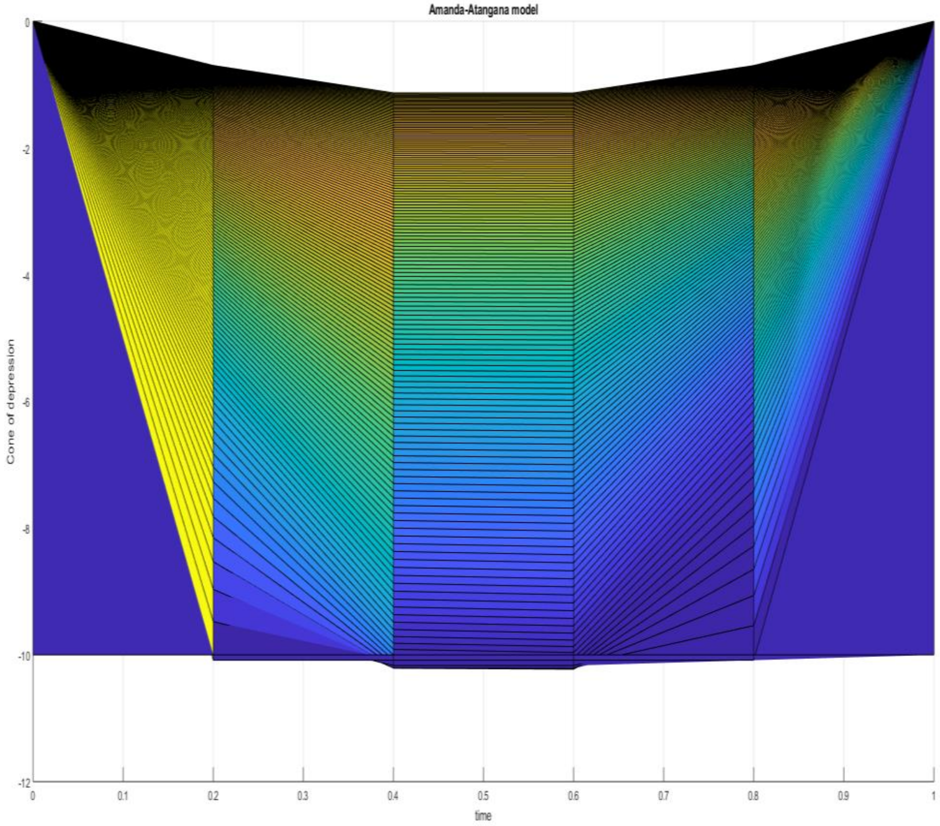


Figure 34: Cone of depression at order 0.03 in time	Figure 35: Cone of depression at order 0.3 in time



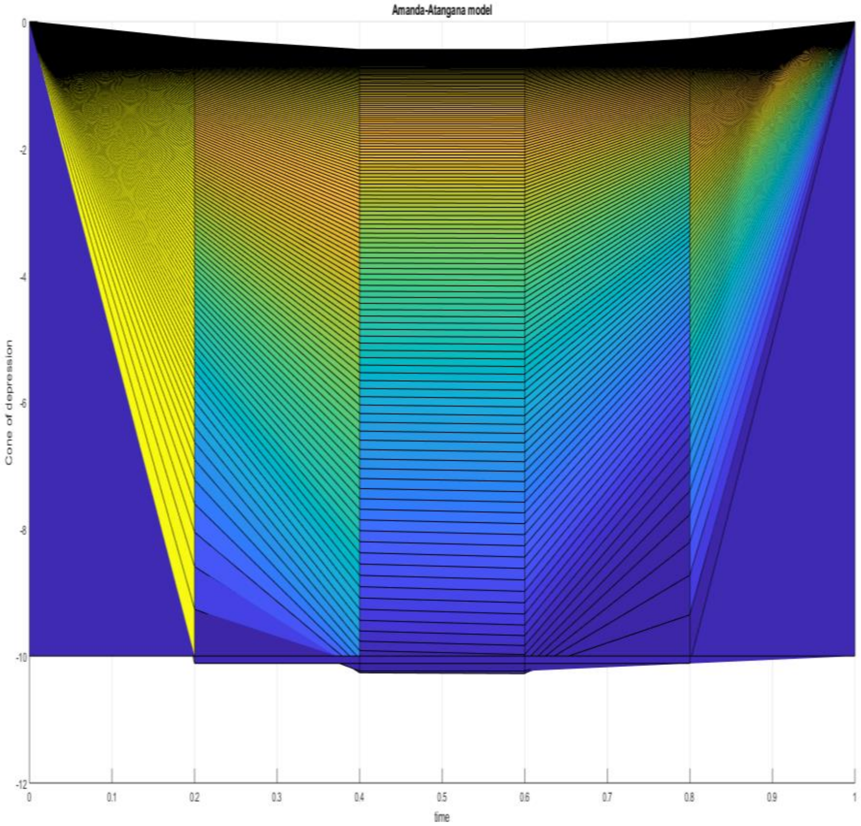


Figure 36: Cone of depression at order 0.5 in time	Figure 37: Cone of depression at order 0.7 in time

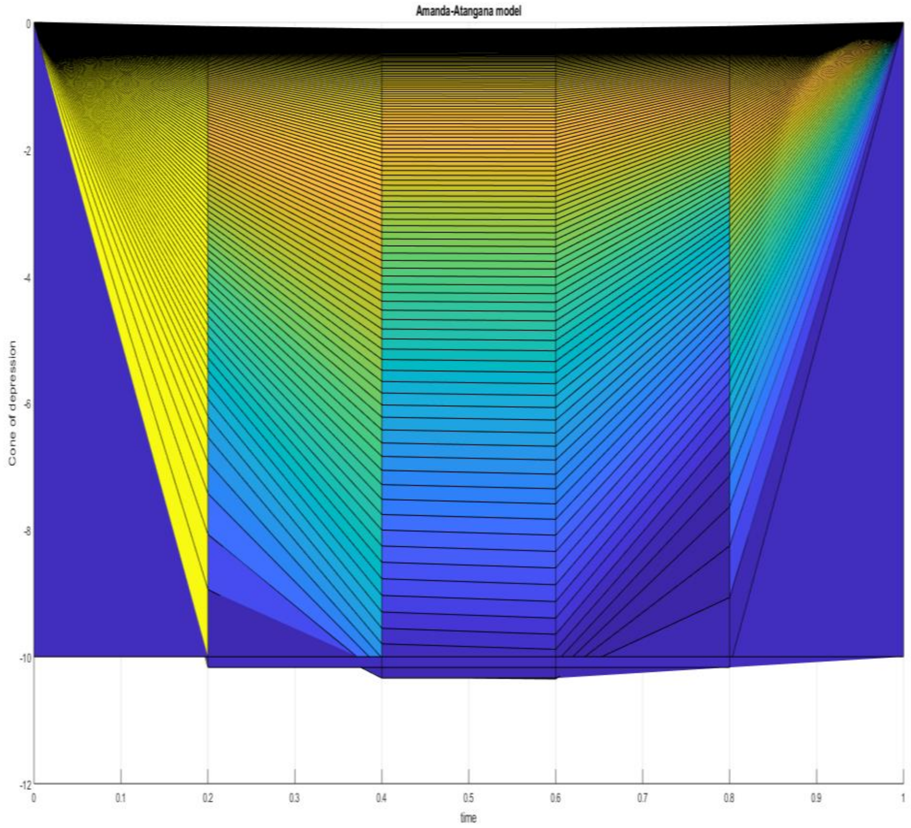
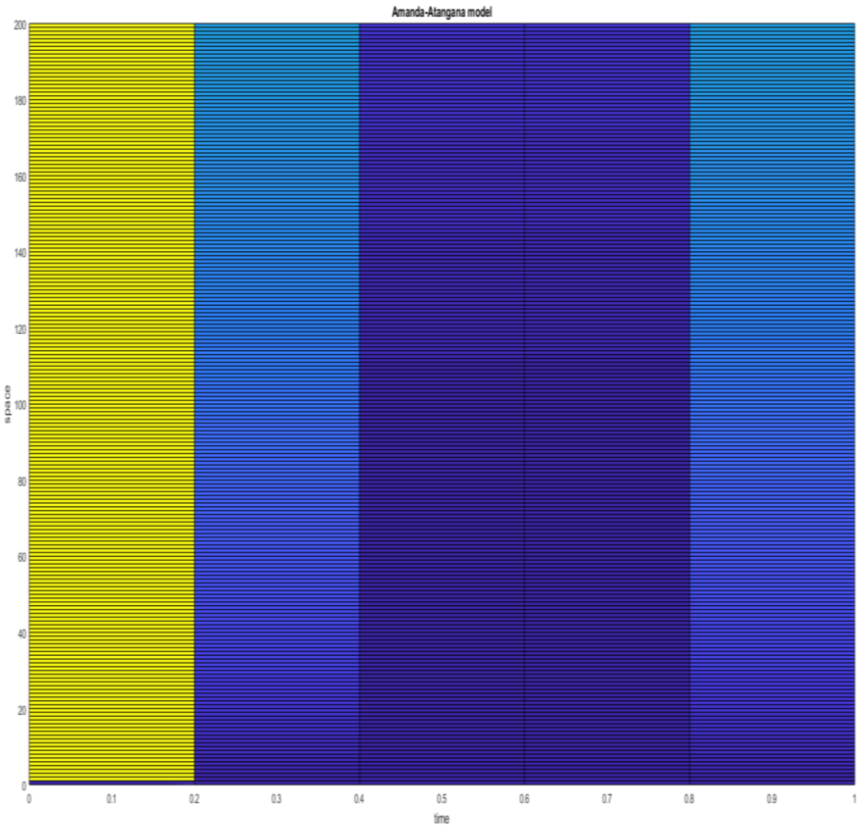


Figure 38: Cone of depression at order 1 in time



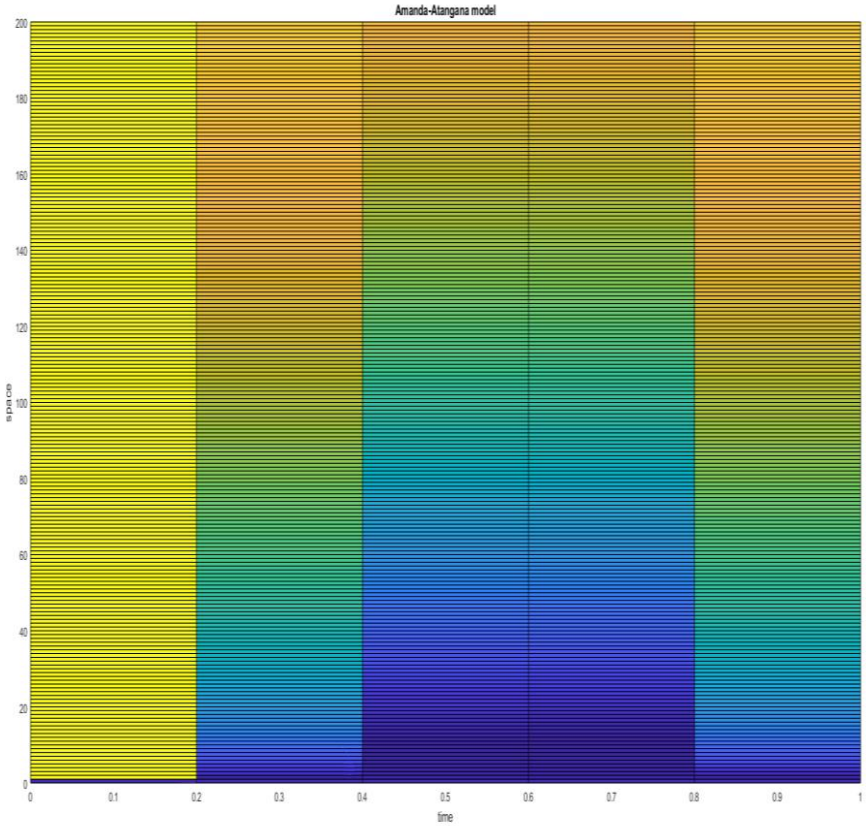
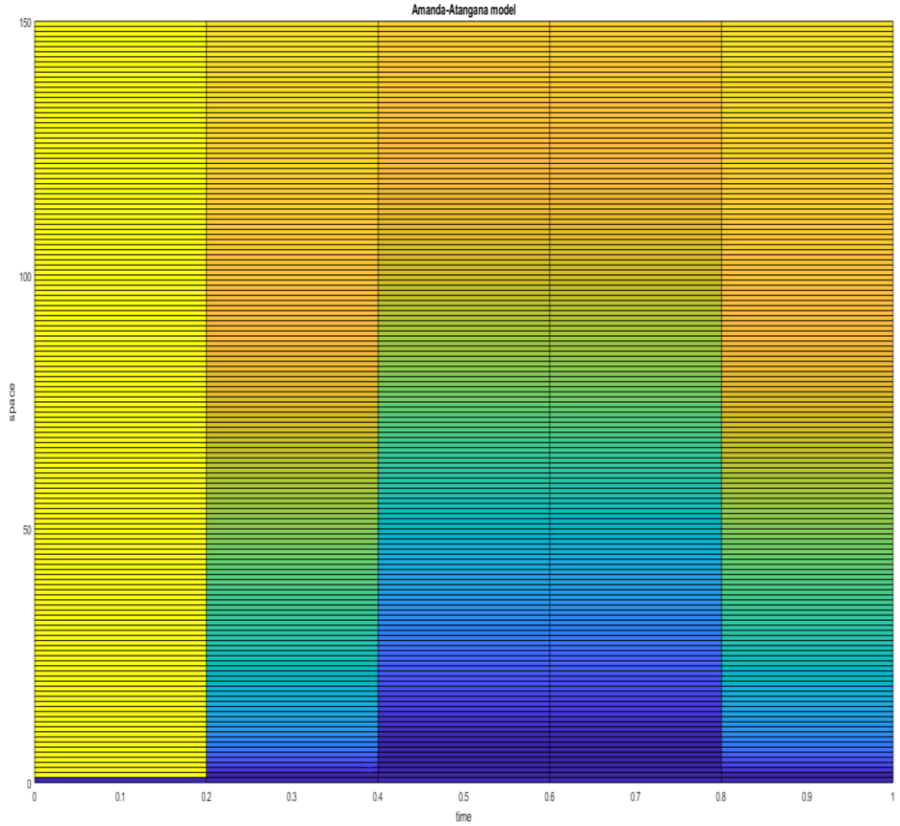


Figure 39: Amanda-Atangana model for order 0.03 time space solution	Figure 40: Amanda-Atangana model for order 0.3 time space solution



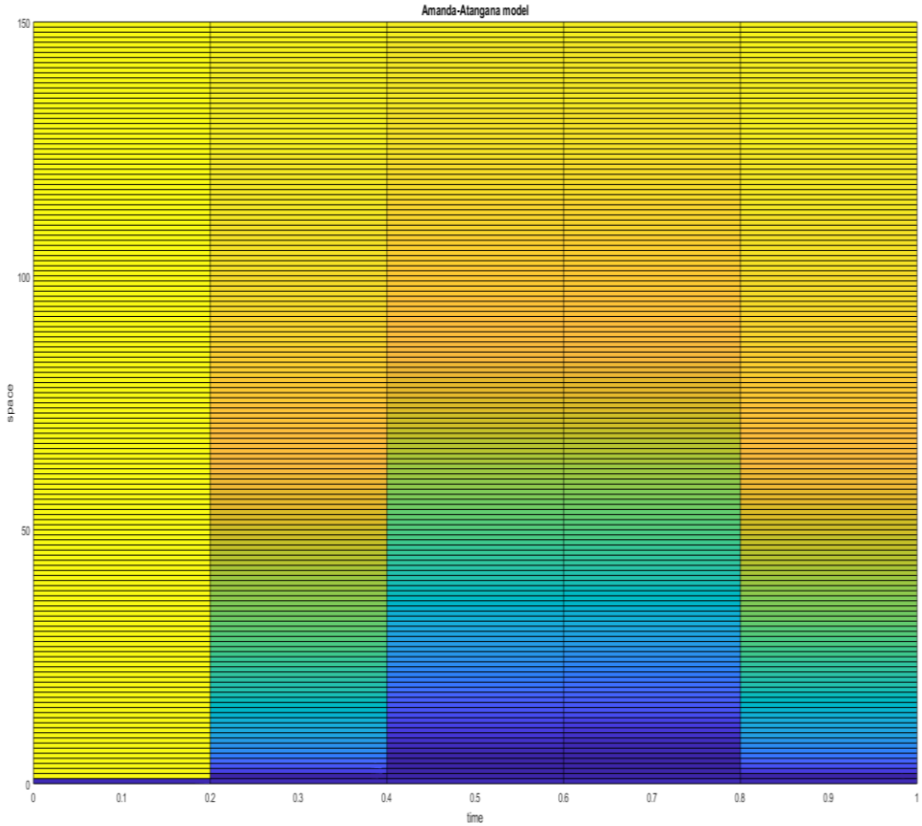


Figure 41: Amanda-Atangana model for order 0.5 time space solution

Figure 42: Amanda-Atangana model for order 0.7 time space solution

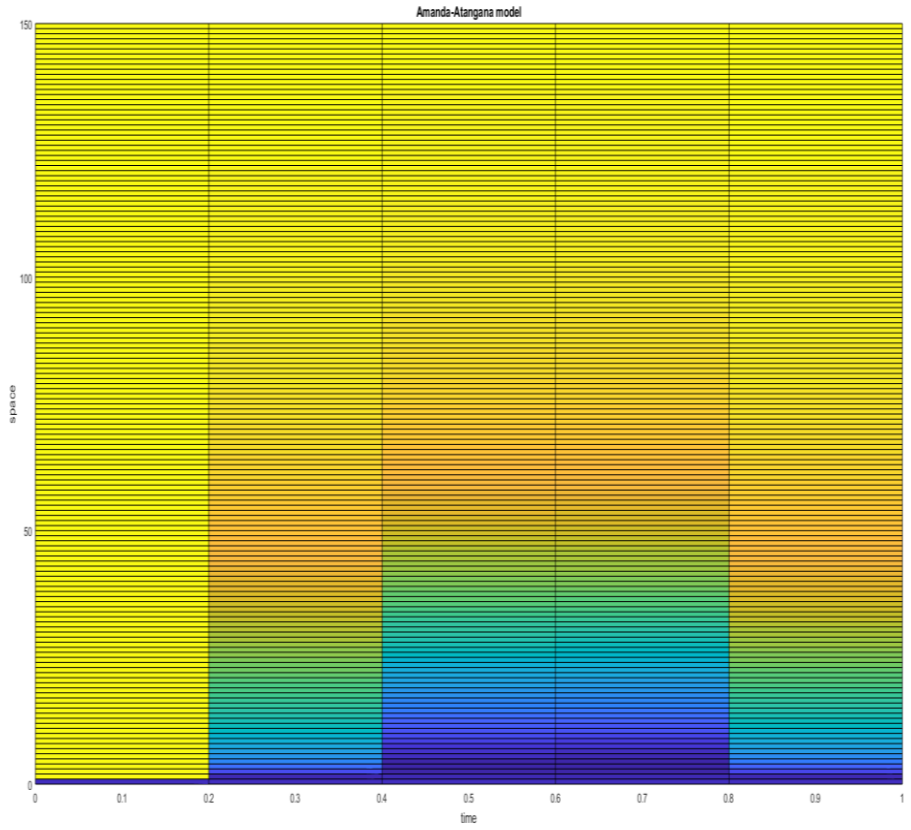


Figure 43: Amanda-Atangana model for order 1 time space solution

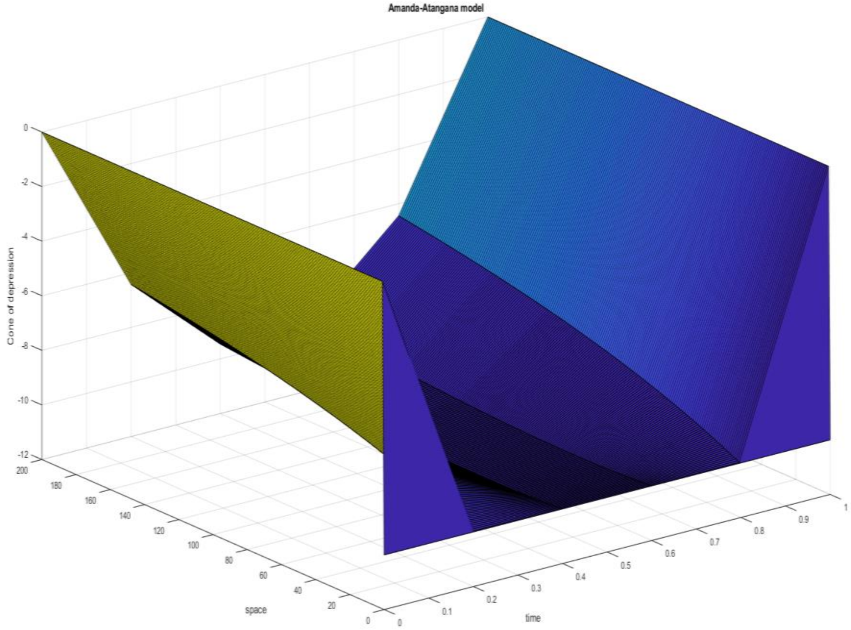


Figure 44: Cone of depression for order 0.03 time space solution

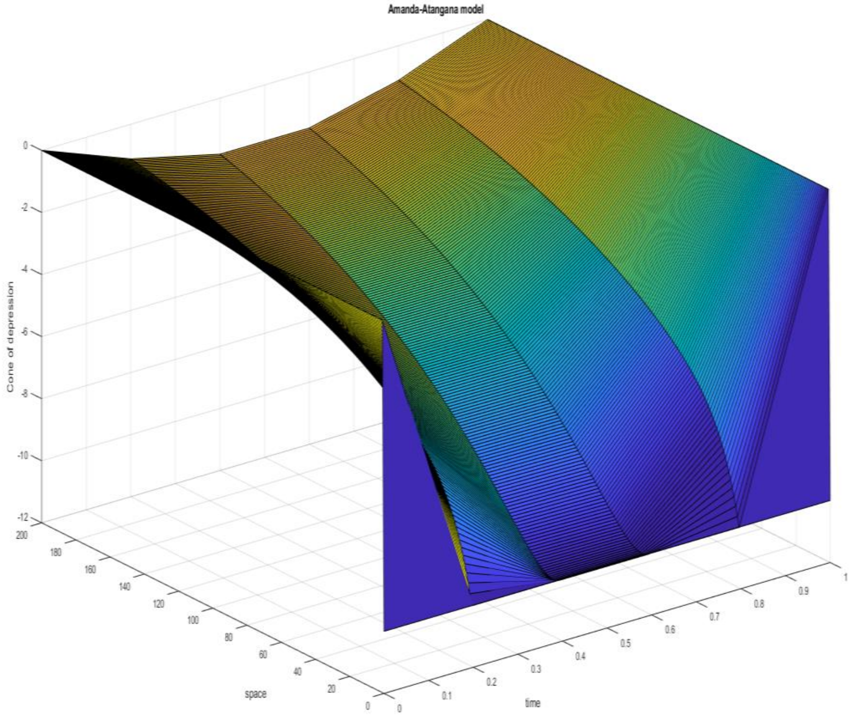
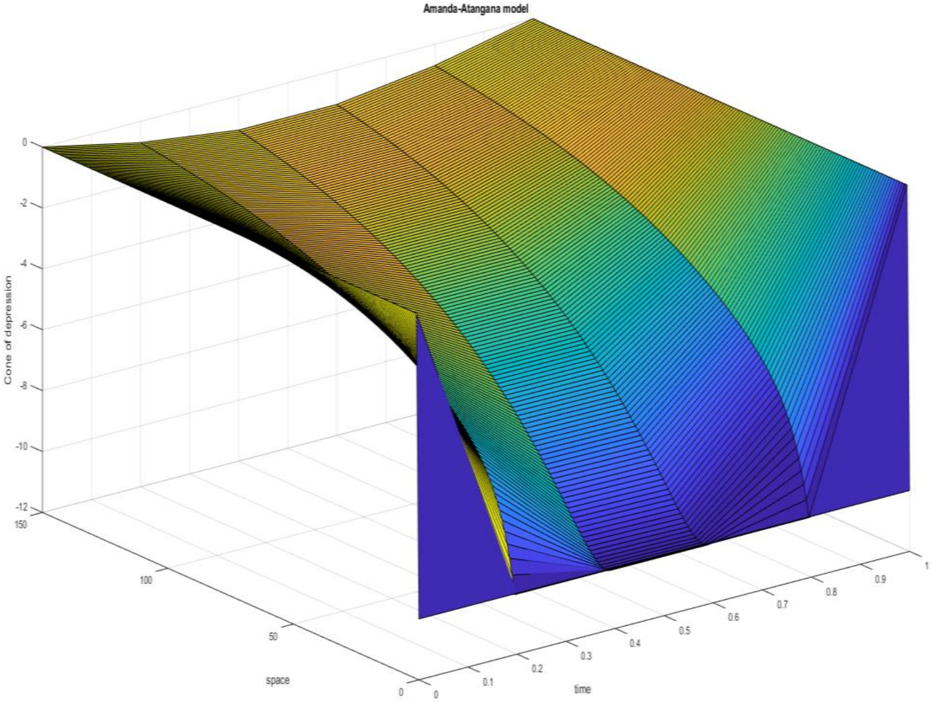


Figure 45: Cone of depression for order 0.3 time space solution



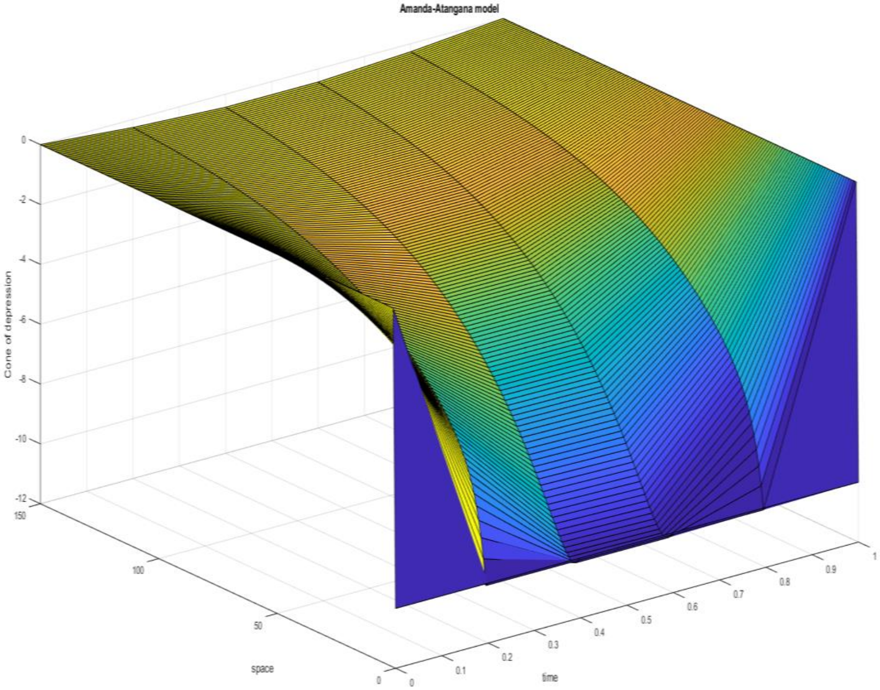


Figure 46: Cone of depression for order 0.5 time space solution	Figure 47: Cone of depression for order 0.7 time space solution

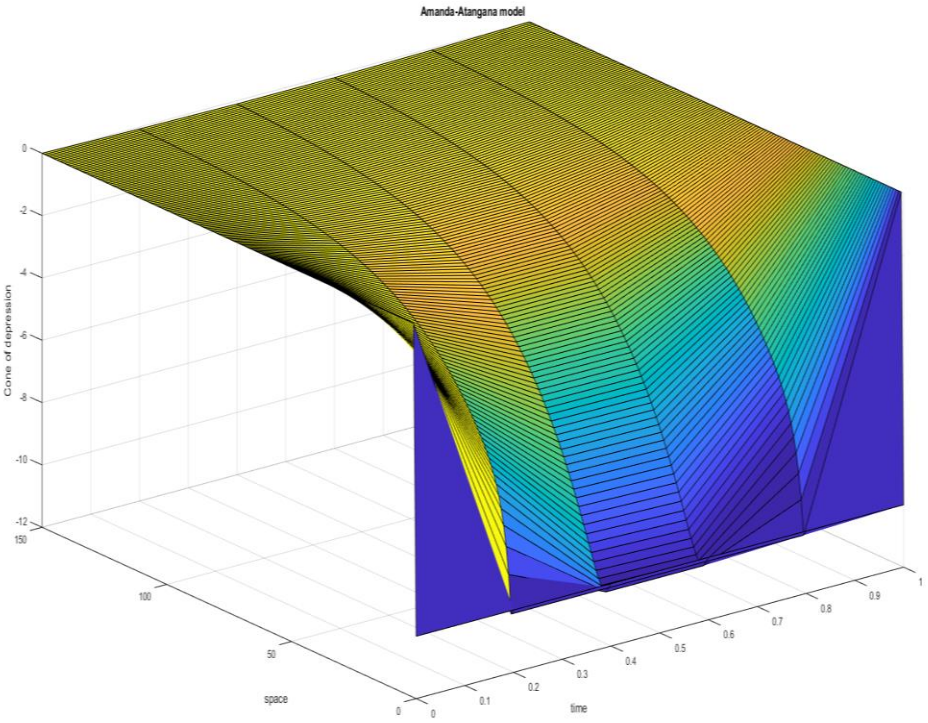


Figure 48: Cone of depression for order 1 time space solution

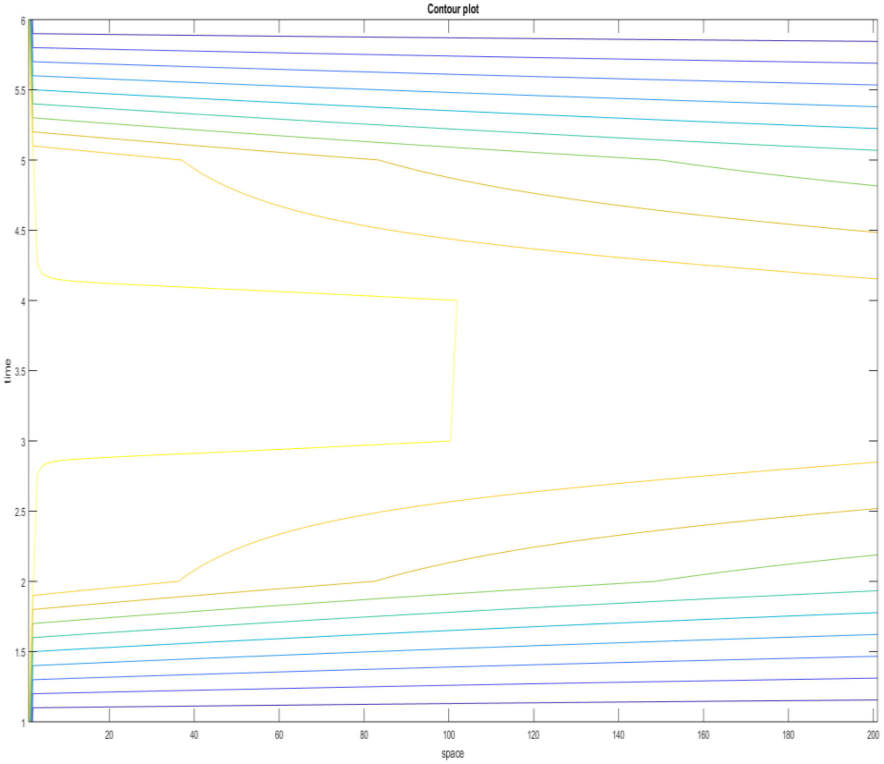


Figure 49: Contour plot at 0.03 time space solution

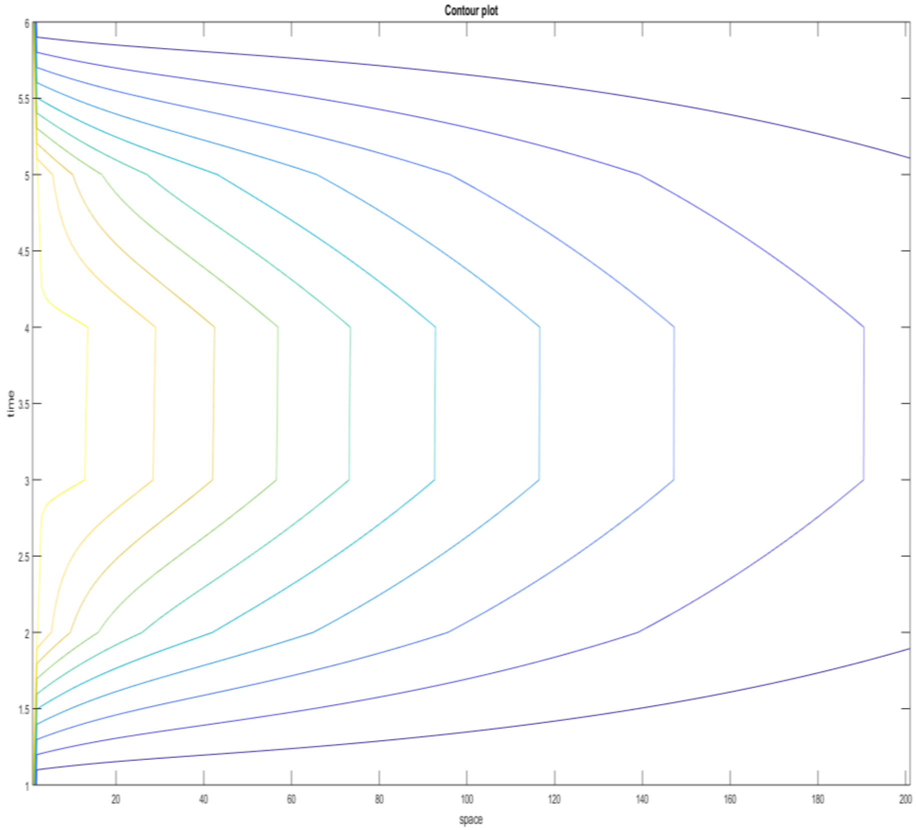
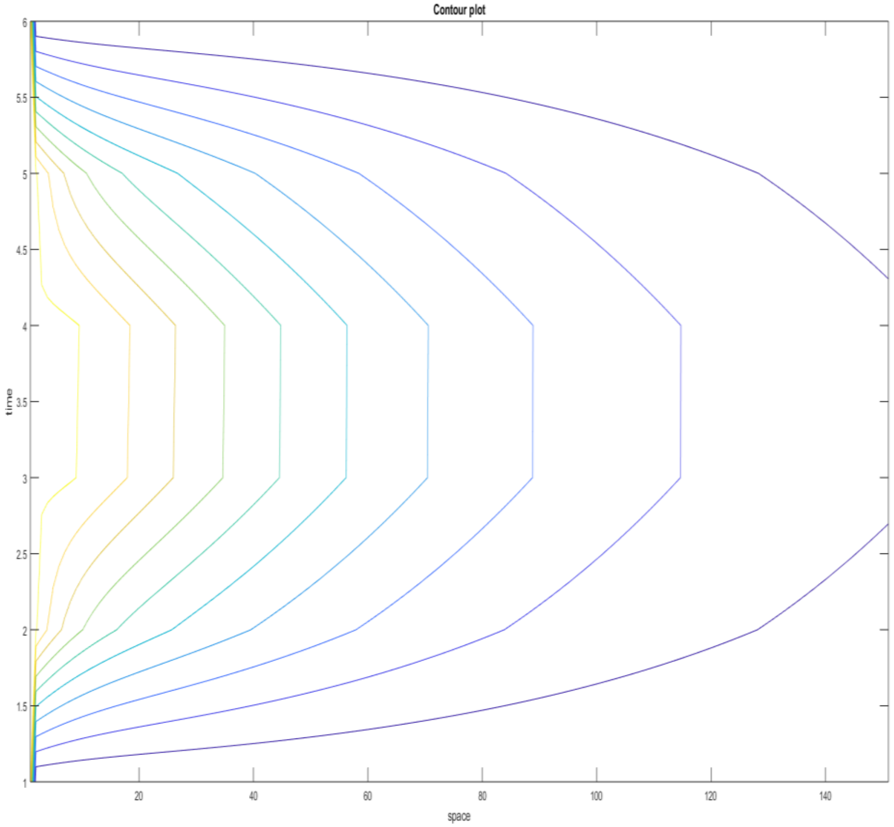


Figure 50: Contour plot at 0.3 time space solution



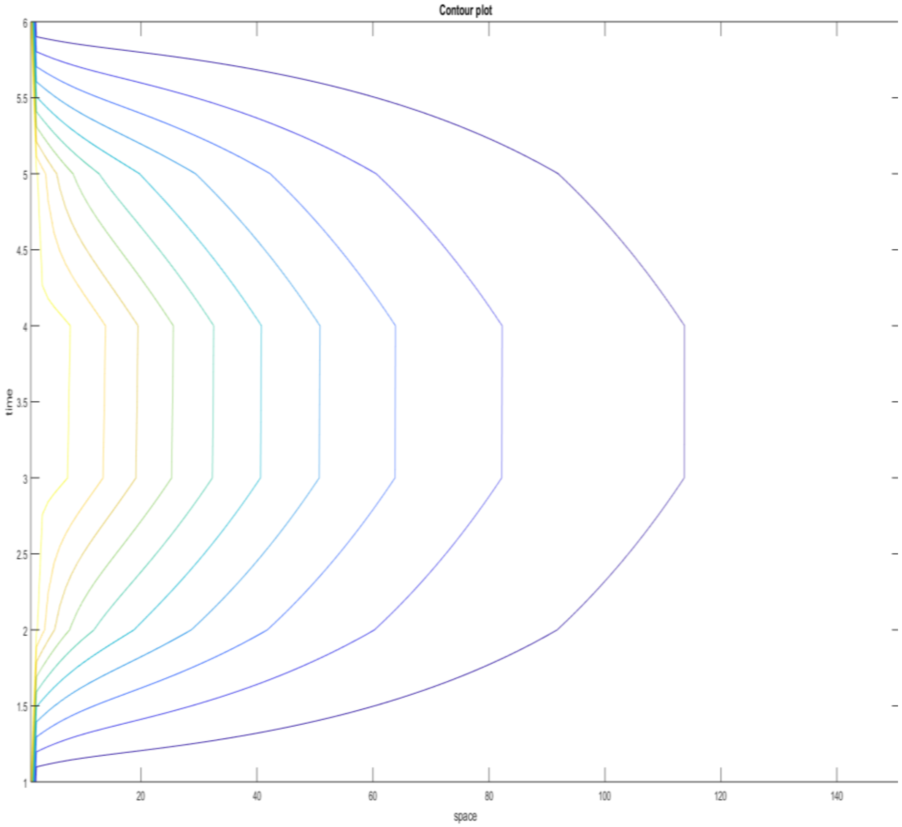


Figure 51: Contour plot at 0.5 time space solution	Figure 52: Contour plot at 0.7 time space solution

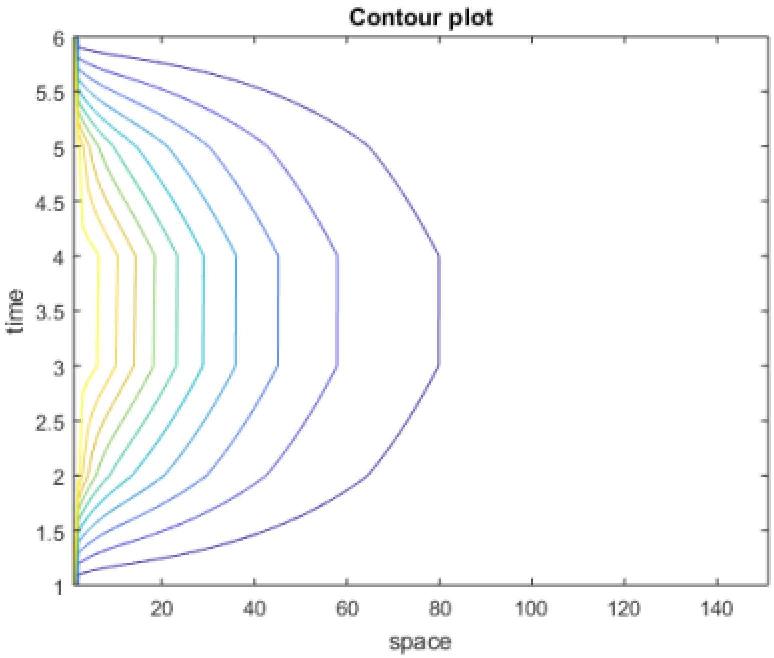


Figure 53: Contour plot at 1 time space solution

As opposed to Caputo the cone of depression is too high. The time against cone of depression figure at order 0.03 indicates a huge abstraction of water as the cone of depression is too high. The Caputo-Fabrizio figures indicate that the pores within the aquifer are larger and the friction is less so that is able to flow freely within the aquifer. The contour plot at order 1 represents a fractured geological formation and the normal flow within an aquifer. At order 0.7 the flow is too low; this indicates that the aquitard is clay as it does not allow a significant amount of water to be abstracted from the aquifer.

CHAPTER 11

11. CONCLUSION

One of the huge responsibilities of humankind is to protect nature and the environment. However, to reach the position for which humankind can claim to be in control of their environment, they must consider the three major steps. They must observe, predict and analyse. In the last decades, researchers have intensively used the concept of differentiation and integration based on the rate of change to model world problems. These operators have been used in many situations to model the flow of subsurface water within the geological formations called leaky aquifers. While these models have been intensively applied to determine the aquifer's parameters, it was clear that the used model was applicable only in the case of non-memory process where the geological formation is considered homogeneous. Nature on the other hand suggest that, the geological formation is not homogeneous, therefore modeling such flow using classical differential and integral operators lead to inadequate results. Therefore, either the aquifers parameters are highly estimated or underestimated. Additionally to this limitation, some simplifications were made while deriving the groundwater flow model within a leaky aquifer. Amanda and Atangana solved this problem; their mathematical equation was able to include the scale factor of the aquifer. However their equation was based on local differentiation operators, therefore a mathematical equation with fractal operators was proposed. To further capture more complexities of the geological formation, the concept of non-local operators is adopted with three different laws, the one based on power law, exponential decay law and the generalized Mittag-Leffler function. An additional extension was done as the classical nonlocal fractional differential and integral operators were replaced with fractal-fractional operators. The numerical simulations obtained from these models offer us with four different classes of flow that can be really observed in real world problems of which are:

(i) These equations can normally be used in multi-steps case where the aquifer is divided in multi-parts each part with same geological formation.

(ii) The class presents the flow with fading velocity, typical example of flow within a leaky elastic aquifer where the matrix soil changes from highly transmissivity to lower.

(iii) The flow changes from normal to fading

(iv) The flow with self-similar feature is represented

Beyond the classical model and the normal flow which can be observed in homogeneous media, a situation where the leaky aquifer matrix soil has the same property everywhere was presented.

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