MODELLING A CONVERSION OF A CONFINED TO AN UNCONFINED AQUIFER FLOW

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Submitted in fulfilment of the requirements for the degree

Magister Scientiae in Geohydrology

At the
Faculty of Natural and Agricultural Sciences
(Institute for Groundwater Studies)
At the
University of the Free State

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January 2019

DECLARATION

I Awodwa Magingi, declare that **MODELLING A CONVERSION OF A CONFINED TO AN UNCONFINED AQUIFER FLOW** is my independent work and that it has not been previously submitted for any qualification or examination in any other tertiary institution. I further declare that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Full name: Awodwa Magingi

Date: January 2019

ACKNOWLEDGEMENTS

Immense gratitude is given to God Almighty for surrounding and providing me with a platform to work with great, knowledgeable people who did not only provide academic coaching but also social and spiritual intelligence. Sincere appreciation is further conveyed to my supervisor, Professor Abdon Atangana for his generosity, patience, time and assistance with information of meaningful contribution towards the depth of understanding and advancement in academic proficiency regarding all the methods used for completion of this study.

Appreciation of assistance is also given to the Institute for Groundwater Studies for allowing me to do my MSc studies. Lastly, I would like to recognize auspicious moral support from my parents (Nomfusi Magingi and Roseman Magingi), my siblings, my friends, and my IGS and DWS colleagues.

ABSTRACT

As the human population increases, there is also an increase in water demand for water supply. Due to this increase in demand, groundwater is likely to be over-abstracted to meet the requirements that cannot be met by surface water resources alone. With minimal knowledge and understanding of the different aquifers, the over-abstraction can easily change the natural state of an aquifer, most likely from a confined aquifer conditions to an unconfined aquifer. It is because of this demand that many confined aquifers have been reported to be pumped intensively thus being converted to unconfined aquifers. While some researchers have devoted their attention to understanding this conversion, even also to predict it, we have to point out that several aspects have not been touched in the last decades. We shall point out that the understanding or the prediction of a given natural problem starts with good construction of a mathematical model, so far the studies done were based on local differential operator. It is important to recall that, classical differential operators have been recognized to only predict physical problems following processes with no memory. However, while dealing with the groundwater flow problem, it is important to include the effect of heterogeneity which of course cannot be captured with classical operators. Very recently, some new concepts of differentiation have been suggested, and are called non-local operators, they are able to capture the flow within heterogeneous media, and even the flow is able to follow the Brownian motion and even the random walk. The newly introduced mathematical operator has the ability to describe statistical setting like the Gaussian distribution. More precisely, the operator can capture normal and sub-diffusion, with the crossover in waiting time distribution that ranges from exponential decay law to power law. The aim of this thesis was to analyze the existing model especially the nonlinear one using some newly introduced numerical schemes that have been recognized to be very efficient and powerful mathematical tools. Secondly, the aim was to extend the existing model using the newly introduced differential operator known as the Atangana-Baleanu derivative to provide a numerical scheme that can be used to solve such a model, present the condition under which the scheme is stable and finally the presentation of numerical simulations for different values of alpha using a software package called MATLAB is highlighted and discussed.

LIST OF GREEK NOTATIONS

α	Alpha
β	Beta
∅	Phi
$\int$	Definite Integral
Δ	Delta
e	Exponential function
Γ	Gamma
∞	Infinity
λ	Lambda
$\mathcal{L}$	Laplace Transform operator
∂	Partial Derivative operator
τ	Tau
π	Pi
ρ	Rho
$\sum$	Summation

LIST OF ABBREVIATIONS

ABBREVIATION	**ABBREVIATION IN FULL**
AB	Adam Bashforth
BE	Backward Euler
IWRM	Integrated Water Resource Management
DWS	Department of Water and Sanitation
PDE	Partial Differential Equations
MATLAB	Matrix Laboratory
s	Drawdown
FE	Forward Euler
t^{α}	Fractal
h	Hydraulic head
ODE	Ordinary Differential Equation
Q	Pumping Rate/Discharge rate
r	Radial distance
K	Vertical Hydraulic Conductivity
S	Storativity
b	Thickness
∂_t	Time derivative
T	Transmissivity

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CHAPTER 1: INTRODUCTION

1.1 Background

Technically, water as an essential resource is considered as a renewable resource. However, this does not mean this resource cannot be depleted (Gray, 2012). Freshwater demand which includes groundwater demand is becoming an important issue in most parts of the world because of the increase in its usage. With the impacts of global warming on water resources and the drastic increase in population rate, it is hard to meet the freshwater demands with the existing surface water resources and therefore intervention measures such as the development of groundwater resources have to be conducted. Groundwater importance has been gradually increasing over the past decades, however, in certain cases the dynamics that govern groundwater are not fully understood and as a result of that, it leads to mismanagement of groundwater resources. The drought seasons experienced in some parts of Southern Africa including, Cape Town city of South Africa, due to global warming have resulted to treatment of groundwater as an 'emergency drought solution' and that includes drilling and abstraction of groundwater in both upper unconfined and commonly, lower confined aquifers. Wang and Zhan, (2009) as well as Xiao (2014) state that groundwater abstraction from both confined and unconfined zones has led to the mixing of the two aquifers around the world.

With the increase in water requirements for domestic use, industrial, agricultural use and even recreational and environmental activities, groundwater is likely to be over-abstracted to meet the demands that cannot be met by surface water resources alone. With minimal knowledge and understanding of the different aquifers, the over-abstraction can easily change the natural state of an aquifer, most likely from a confined aquifer to an unconfined. It is because of this demand that many confined aquifers have been reported to be pumped intensively thus being converted to unconfined aquifers (Wang and Zhan, 2009). It is therefore crucial to understand this conversion as it helps in the management of groundwater resources. A confined aquifer can be changed to an unconfined aquifer when the pumping rate is extremely high or the pumping period is long (Wang and Zhan, 2009).

A number of studies within hydrogeology have been conducted to determine and calculate groundwater flow properties such as hydraulic conductivity, storativity, and hydraulic head in a single confined or unconfined aquifer. Specific storage and hydraulic conductivity are the two hydraulic properties that are usually determined from groundwater pump testing field data. These parameters are based on the quantitative description of groundwater flow.

In the estimation of properties that govern groundwater flow, it becomes complex to determine the properties for an aquifer system that is both confined and unconfined; in such cases, the complexities lead to more improved accuracy in modelling groundwater flow that is useful in groundwater management (Wang *et al*, 2009). Similarly to any model, groundwater modelling can be loosely defined as a simplified representational version of a real-world, complex groundwater system (Bedient *et al*., 1997). Due to large amounts of data such as pump testing data, that need to be used to calculate and estimate hydraulic parameters in the real complex world, a number of analytical solutions were used to develop the type curves. The hydraulic parameters are extrapolated from the type curves by the graphical fitting of the observed data to the type curves (Hantush, 1964; Yeh & and Chang, 2013). Theis (1935) published a solution of transient flow and since then, researchers have developed a number of analytical solutions of which most of them have assumptions that are rare to find in the real world, such an aquifer with homogenous formation properties. In consideration of complexities and heterogeneity in geologic formation of most aquifers in the ideal world, numerical solutions are then derived for approximate solutions.

1.2 Problem Statement

Existing literature on conversion of confined aquifers to unconfined aquifers has increased which points to the increasing demand of groundwater and essentially the importance of understanding groundwater resources. The Department of Water and Sanitation (DWS) has raised conjunctive use of water resources and this conjunctive use involves amongst other sources, groundwater resources. A number of water sources can be used conjunctively for water supply to a community, for farming or any other water use; however, for an effective and sustainable scheme, the various water sources need to be managed differently. Management of water resources involves an understanding of their development, how they are operated and ultimately, these two factors guide how the resources should be maintained to ensure their sustainability.

Integrated Water Resource Management (IWRM) is a popular concept that emerged a few years ago in South Africa; it is described as management of water resources conjunctively. There are quite a number of factors that contribute towards a successful process of IWRM; one factor that is of paramount importance is to understand all the water resources involved. It is because of this importance that groundwater resources need to be understood thoroughly. Mathematical models have been used to understand this conversion however, the research conducted previously focused on the use of classical operators. Classical differential operators have been recognized to only predict physical problems following processes with no memory. This study therefore investigates the confined-unconfined aquifer flow through the use of the newly developed non-classical operators that capture heterogeneity as the effect of heterogeneity plays a role in groundwater flow problems. The main aims of groundwater modelling include amongst others is predicting or estimation the future of the system using its current and the history of a groundwater system, to generally understand the groundwater system and in most cases to predict aquifer parameters (Reilly and Harbough, 2004). It is for the former reason that in this study, the focus will be on the use of non-classical operators with memory to accommodate the past of a system.

1.3 Aims and objectives

As mentioned in the background, some studies have been conducted and solutions have been developed to understand the conversion of confined-unconfined aquifer conditions. However, the previous studies focused on classical operators, therefore the overarching aim of this MSc study is to analyse the existing model especially the nonlinear one using some newly introduced numerical schemes that have been recognized to be very efficient and powerful mathematical tools. Secondly, the aim is to extend the existing model using the newly introduced differential operator known as the Atangana-Baleanu derivative to provide a numerical scheme that can be used to solve such model, present the condition under which the scheme is stable. The following list contains the objectives of this MSc study:

1. To review existing literature on confined-unconfined flow
2. To review existing mathematical models for confined-unconfined aquifer conditions
3. To review certain mathematical methods for solving Partial Differential Equations
4. Find a suitable numerical solution for confined-unconfined conversion using various methods
5. Modify the classical model using the Atangana-Baleanu derivative
6. Do a numerical simulation and subsequently compare with field data

1.4 Study Framework

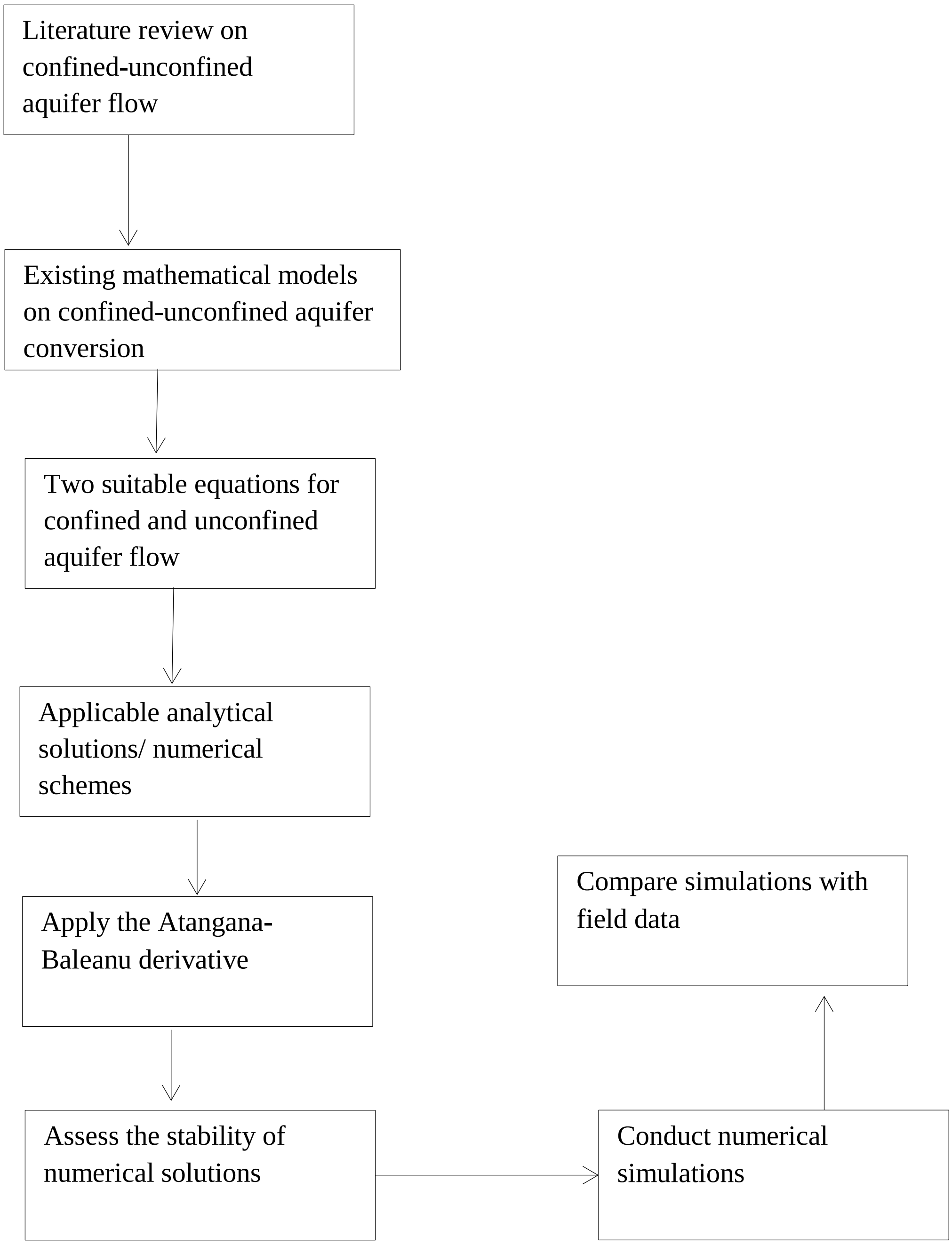

Figure 1: Structure of the study

1.5 Study outline

This MSc study is made up of five chapters. Chapter 1 is the study introductory chapter which provides a background of the study. Aims and objectives of the dissertation are also highlighted in this chapter and lastly, it provides an overview of the study structure, how the dissertation will unfold. Chapter 2 is a collection of literature review of existing studies conducted on conversion of confined-unconfined aquifer flow. This chapter addresses a general flow in a confined and in an unconfined aquifer; boundary of conversion between the confined and unconfined zone, existing mathematical models of the conversion boundary between the confined and unconfined zone and it also provides insight of existing numerical methods that can be used to solve partial differential equations. The models or groundwater flow equations to be used for this study are indicated in chapter 2.

Chapter 3 provides an application of few numerical schemes mentioned in chapter 2 such as the Crank-Nicolson method, to obtain numerical solutions. The stability of the solution is assessed using Von Neumann Stability analysis after modification through the application of the Atangana-Baleanu derivative. In chapter 4, the mathematical simulations are conducted. Chapter 5 as the last chapter of the dissertation provides a thorough discussion on achievements of study objectives and it ultimately provides the conclusion of the dissertation.

CHAPTER 2: LITERATURE REVIEW

2.1 Groundwater flow within a confined and unconfined aquifer

2.1.1 Confined and Unconfined aquifers

By definition, an aquifer refers to a saturated permeable geological layer or unit located in the subsurface; this geologic unit should be able to transmit significant quantities of water when subject to normal hydraulic gradients and this type of water is called groundwater (WRC, 2003). It is common and important to distinguish between an aquifer, aquitard and an aquiclude. The latter refers to a saturated geological layer that is incapable of transferring significant quantities of water due to its low transmissivity under normal hydraulic gradients. The term aquitard is commonly defined as a less permeable (compared to an aquifer) geological formation that consists of beds that are permeable however, their permeability maybe enough for groundwater flow but not permeable enough for development of production boreholes (Freeze and Cherry, 1979). Aquifers are categorised into porous and fractured aquifers and this classification is based merely on the lithological units of their geological formation and its arrangement.

Aquifers are further categorized into confined, semi-confined and unconfined aquifer based on the overlying and underlying geological units. A confined aquifer is defined as an aquifer that is usually located in deeper depths than an unconfined aquifer; this aquifer is always underlain and overlain by an impermeable layer shown in the schematic diagram (Fig. 2). The groundwater of this type of an aquifer is subjected under pressure that is greater than the atmospheric pressure and as a result of this high pressure, wells drilled into a confined aquifer have groundwater levels that exceed the ground surface and these wells are called artesian wells. A semi-confined aquifer refers to a geologic unit that has aquitards as overlying and underlying layers or when one of the layers is an aquitard and the other is an aquiclude. In contrast to a confined aquifer, an unconfined aquifer can be described as an aquifer that has a direct connection with the atmosphere and therefore has an atmospheric pressure. An unconfined aquifer does not have an impermeable rock bounding the top of the above the aquifer and due this absence of the impermeable layer, the water levels fluctuate.

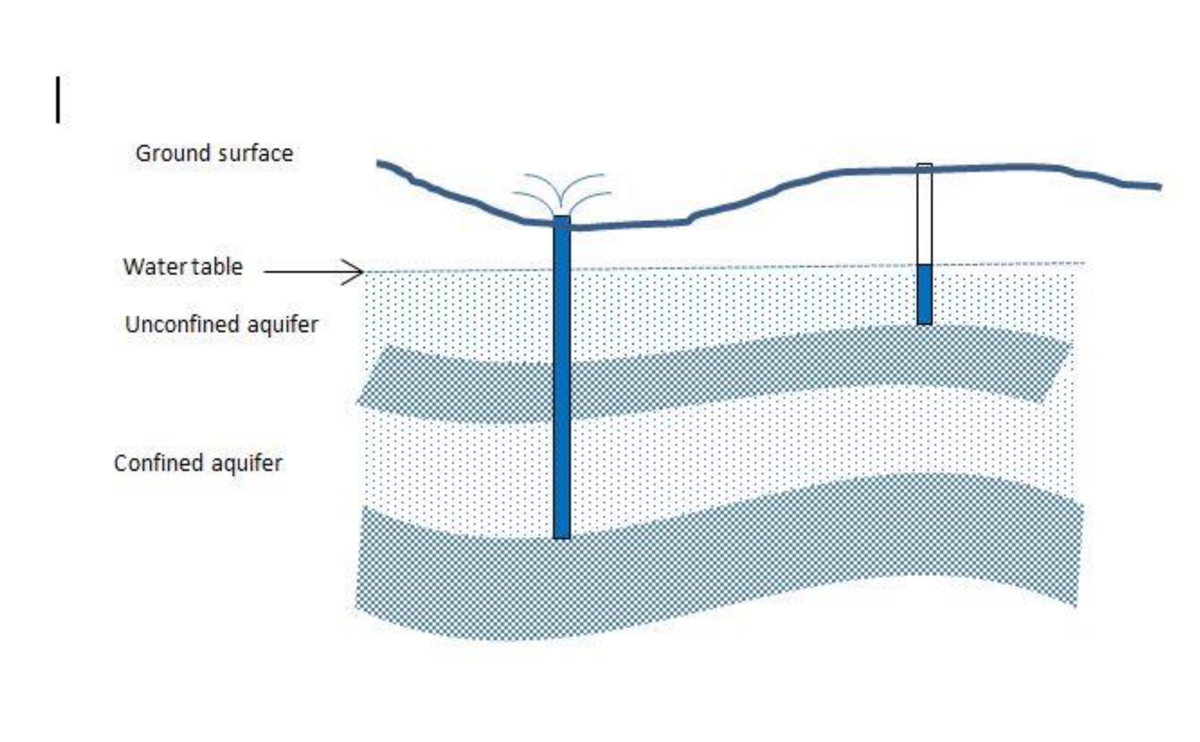

Figure 2: Schematic diagram showing the difference between a confined and an unconfined aquifer

2.1.2 Groundwater flow within a confined and unconfined aquifer

The behaviour of groundwater within an aquifer system during the process of groundwater movement is governed by characteristics of the water itself and the geologic formation through which it flows. Under natural, normal conditions, groundwater is expected to flow from areas of high hydraulic head to areas of low hydraulic head (Barackman and Brusseau, 2002). Due to the fact that for an aquifer to be classified as confined, it has to be bounded aquifer by impermeable formation(s), below and above the aquifer and that leads to a very limited groundwater inflow and discharge from the aquifer. This groundwater is therefore fairly protected from ground surface contamination and drought impacts than groundwater in an unconfined aquifer. Groundwater recharge into confined aquifers is through rain or stream water infiltrating the rock layer at a fairly significant distance away from the confined aquifer itself. Isotope dating reveals that groundwater age in these types of aquifer usually ranges to thousands of years

The flow of groundwater into an aquifer is commonly described using Darcy's law (1856) which serves as a tool for groundwater flow modelling. Most solutions use Darcy's law as a governing law of groundwater flow in developing more accurate analytical solutions. The law takes into account the fact that groundwater flow depends merely on hydraulic gradient.

$$Q = -KA\frac{dh}{dl} \qquad (1.1)$$

Q = Discharge rate

K – hydraulic conductivity

A – cross-sectional area of flow

dh/dl – hydraulic gradient (i)

dh = h1-h2

The well-known equation for Darcy's Law was developed based on the interpretation that the rate of flow through a porous medium such as an aquifer increases with an increase in cross-sectional area that is perpendicular to the flow itself and is also proportional to the head loss measured per unit length in the direction of flow.

The flow of groundwater into an unconfined aquifer can be categorized into continuity equation which defines that the volume of water recharged should be equal to the volume of water discharged, however in reality, most aquifers are heterogeneous and therefore some of the water that enters the system takes longer to be discharged.

The governing equations that describe groundwater flow are normally derived from a combination of Darcy's Law which is a transport law and continuity equation also known as the law of mass balance (Sato & Iwasa, 2000). The general governing equation for a confined aquifer can be written as:

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right) = S_c \frac{\partial h}{\partial t} \qquad (1.2)$$

Where T_x and T_y are the horizontal transmissivity elements, h is the hydraulic head, S_c represents storage coefficient and t symbolises time. The governing equation for unconfined aquifer is then equation (1.2) with replacement of the horizontal components T_x and T_y with K_x and K_y which are the horizontal components of hydraulic conductivity because it becomes complicated to determine transmissivity if the aquifer thickness continuously changes with continuous abstraction of groundwater. h is the saturated thickness in the unconfined aquifer and the storage coefficient is also replaced by specific yield represented by S_y; according to (Anderson and Woessner, 1992), the equation then becomes a nonlinear equation also known as the Boussinesq equation

$$\frac{\partial}{\partial x}\left(K_x h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial h}{\partial y}\right) = S_y \frac{\partial h}{\partial t} \qquad (1.3a)$$

Equation 1.3a can be re-arranged to equation 1.3b

$$\frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial h}{\partial y}\right) = \frac{S_y}{K} \frac{\partial h}{\partial t} \qquad (1.3b)$$

2.2 Interface between the confined and unconfined aquifer zone

The conversion of a confined aquifer to an unconfined aquifer is observed usually when there is groundwater over-abstraction in wells that penetrate the confined aquifer (Springer and Bair, 1992). This over-pumping occurs generally when the pumping rate is extremely high or the pumping period is extremely long and the conversion therefore occurs when groundwater abstraction results in piezometric surface to decrease until it is below the bottom of the top confining bed in the area of the pumping well (Hu and Chen, 2008; Wang and Zhan, 2009). For effective groundwater resource management, it is of paramount importance to understand the natural state of different aquifers as that guides the operating rules of a certain aquifer. A lot of studies have been conducted for this confined-unconfined conversion; these studies will be briefly explained in the section 1.6 below. Most studies have highlighted and explained the conversion of confined-unconfined aquifer conditions due to either a pumping borehole or a group of pumping boreholes. In addition to confined-unconfined conversion by pumping wells, Wang *et al*. (2009) state that another form of conversion that is less commonly investigated occurs when the hydraulic head changes at the surface-groundwater boundary, commonly of an aquifer with a stage where the river level rises mostly due to floods.

The confined-unconfined flow is commonly observed in surface water (usually from rivers) - groundwater interaction areas (Urbano *et al.,* 2006; Wang *et al.,* 2009). In a study conducted by Wang and Zhan (2009) about the conversion of confined to unconfined aquifer conditions, it is concluded that the critical conversion time, which is the time it takes for the conversion to occur; is directly affected by the initial hydraulic head and the time it takes for the pumping well to dry up is usually larger than this conversion time. As groundwater is out of sight, the existence of groundwater modelling using practical field data assists in making this precious resource understandable. Due to availability of groundwater models, the conversion interface between confined and unconfined aquifers can be better explained by mathematical groundwater models. Generally, the distance from the pumping well to the point of conversion depends on the rate of abstraction; the distance will be large for a slow pumping rate and shorter for a high pumping rate. Additionally, the saturated thickness of the confined aquifer in the already converted unconfined zone decreases with groundwater abstraction in the unconfined zones and it remains constant in the 'still' confined zones (Hu and Chen, 2008).

2.3 Existing mathematical model of the conversion interface between the confined and unconfined zone

A number of studies have been conducted and a number of solutions have been developed to understand the conversion of confined to unconfined aquifer. The confined-unconfined aquifer conversion investigations that have been carried out include amongst others, Moench and Prickett solutions and the Chen model.

Amongst other researchers, Rushton and Wedderburn (1971) developed an electrical analogue solution which is a use of resistance-capacitance electrical analogue to analyse the conversion of confined to unconfined aquifer flow behaviour. As mentioned above, each solution has its assumptions and for this solution, the storativity in the confined aquifer is replaced by the specific yield of the unconfined aquifer. Moench and Prickett (1972) came up with a mathematical solution which is called an MP model; the model was developed for the confined-unconfined conversion in a transient flow by assuming a constant transmissivity in the unconfined zone. The MP model was derived based on the analogous case of heat flow in cylindrical symmetry where there is occurrence of freezing or melting. A finite-element numerical solution was then developed by Elango and Swaminathan, 1980 for the transient confined-unconfined flow and it was restricted to analysing a conversion flow that is steady-state.

In 2006, Chen *et al* developed an analytical solution for the steady flow conversion of confined to unconfined aquifer induced by a group of abstraction wells. An approximate solution for the transient confined-unconfined flow is explained by the Chen model (Hu and Chen, 2008). The solution is described according to the Girinskii's potential function which is a potential of a steady-state groundwater flow in a horizontally layered porous medium. The Girinskii's function is therefore utilized to depict a varying transmissivity of the unconfined region in the Chen model. The development of both the Chen model and the MP has facilitated the understanding of the transient steady flow from confined to an unconfined aquifer and a number of investigations about confined-unconfined aquifer conversion are then compared to these two models.

For conversion of a confined to an unconfined aquifer flow in transient state, a semi-numerical solution flow was derived by Wang and Zhan (2009). The numerical solution considered the change of transmissivity as well as storativity that takes place during the conversion of confined-unconfined and solved the nonlinearity of unconfined flow by the Runge-Kutta method. This semi-numerical solution is explained in detail below.

2.3.1 Rushton and Wedderburn 1971

Rushton and Wedderburn the time variant behaviour of the change in a confined aquifer condition to unconfined aquifer conditions through a resistance-capacitance electrical analogue. In the numerical solution, the storativity of the confined zone is replaced by the specific yield of the unconfined aquifer zone. They further mentioned that the Theis equation should not be used to analyse pumping test results if the aquifer becomes unconfined during pumping.

2.3.2 Moench and Prickett, 1972- The MP model

This model is commonly known as the MP solution; it is an analytical solution derived for a confined aquifer converting to an unconfined aquifer at the constant rate of discharge**.** This model was developed to help understand aquifers in terms of determining their hydraulic properties such as transmissivity, specific yield and storativity of confined aquifers undergoing conversion to unconfined conditions. As one of the first methods to be developed, most solutions that were derived after this MP model are usually compared to it. In this model, the transmissivity in the unconfined zone is assumed to be constant which may have a huge error in cases where there is variation in transmissivity (Wang and Zhan, 2009). As indicated in figure 2 below, the flow is determined through transmissivity and storativity of the confined zone before conversion, prior to pumping; during the confined to unconfined conversion, the flow is then controlled by transmissivity and specific yield of the unconfined zone. R represents the radial distance between the confined zone and the unconfined zone, r is the radial distance from a pumping or abstraction well to an observation well, H represents the initial hydraulic head; b is the confined aquifer thickness.

In comparison to other models, the MP model also has assumptions to be considered, amongst the assumptions aquifer is assumed to be homogeneous, isotropic and of uniform thickness, pumping rate is constant and flow is unsteady.

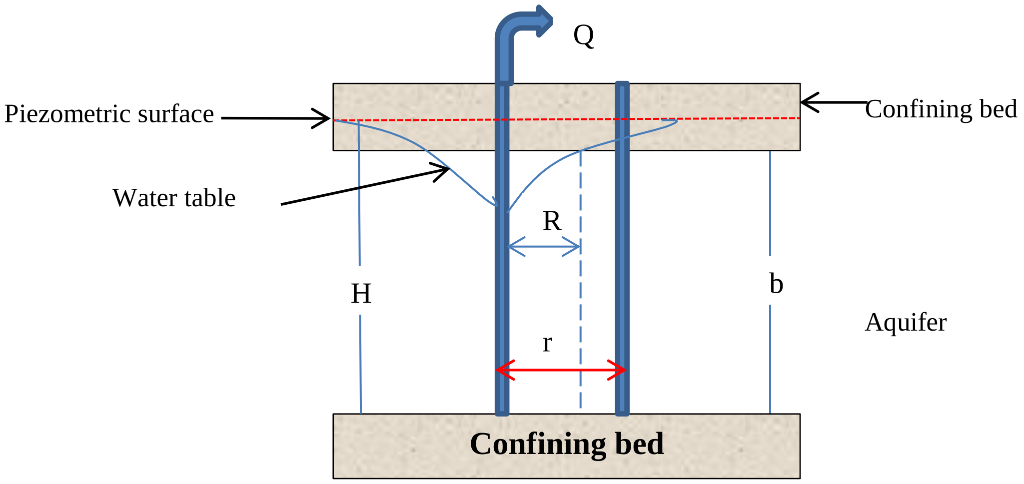

Figure 3: Schematic illustration of the MP model.

2.3.3 Bear 1972

Jacob Bear, in his book, states that Girinskii's Potential is one equation that has been used to describe confined-unconfined flow on a local scale as it was developed to locate an exact confined-unconfined boundary zone in a mixed aquifer. The assumptions for this Girinkii's Potential include amongst others the assuming that the aquifer is actually uniformly horizontal and flow conditions are steady-state.

2.3.4 Elango and Swaminathan, 1980

Elango and Swaminathan used a finite-element method with four-sided mixed-curved isoperimetric elements to develop a finite-element numerical solution for the conversion of confined-unconfined flow in a transient state. The model was based on the Dupuit's assumptions, and it was restricted to analysing a flow in a steady-state. Based on the results obtained to test the model, Elango and Swaminathan indicated that their solution could be useful in prediction of the occurrence of unconfined flow conditions in an initially confined aquifer that has over-abstracted wells and the model could assist in estimating the confined-unconfined zone interface.

2.3.5 Chen *et al.*, 2006

Chen and others derived an exact analytical solution for a steady-state confined–unconfined flow induced by a number of pumping wells. An application of an analytical approach based on the Girinskii's potential function was used for analysis of the steady state groundwater flow towards the group of pumping wells in a confined aquifer that is being converted to unconfined aquifer conditions.

2.3.6 Wang and Zhan, 2009

According to (Wang and Zhan, 2009), their model is a modification and acts as improvement of the MP model as it considers the unsteady flow of the unconfined aquifer. The model also considers the nonlinearity of the unconfined flow and solves it by the Runge-Kutta method. On their study, Wang and Zhan indicated that the conversion of a confined aquifer to an unconfined aquifer flow is solely dependent on 3 dimensionless parameters, namely the dimensionless pumping rate which they represented by q_D which is directly affected by the pumping rate and inversely proportional to the product of transmissivity of the aquifer and its thickness (1.4), the storativity ratio of confined-unconfined aquifer represented by a_D and lastly, the ratio of initial hydraulic head of the aquifer over its aquifer thickness f_i. Additionally, instead of using the unconfined transmissivity in the definition of the Girinskii's potential which is specially designed for steady-state confined aquifer to unconfined flow, this model used the Boltzmann transform with dimensionless parameters for the transient flow. The Runge-Kutta method was then used to solve the nonlinear equation of the unconfined zone. Wang and Zhan compared their solution to the MP model and the Chen model; the comparison showed that their model can be used to estimate the conversion time and critical drying time which helps in assessing possibilities of drying up of a pumping well.

The aim of this study is based on the two equations (1.5) and (1.6) as stipulated by Wang and Zhan, (2009). Consider a pumping well, fully penetrating a confined aquifer with thickness smaller than the initial hydraulic head and the abstraction of groundwater is at a constant rate. As the pumping period increases, the hydraulic head drops converting the vicinity around the pumping well from confined to unconfined conditions. With continuous pumping, the piezometric decreases to below the confining unit thus converting the flow conditions from confined to unconfined.

$$q_D = \frac{Q}{2\pi KB^2} \tag{1.4}$$

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right), \quad 0 < h \leq B \tag{1.5}$$

$$S_c \frac{\partial h}{\partial t} = \frac{KB}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right), \quad h \geq B \tag{1.6}$$

Theis equation will be used as an analytical solution for equation 1.5 and the solution is as follows: since transmissivity is a product of a confined aquifer thickness (B) and hydraulic conductivity (K) then equation 1.6 can be rewritten as 1.7. Theis (1935) derived this equation 1.9 based on unsteady flows towards a well penetrating a confined aquifer. *Sc* represents the storage coefficient in the confined aquifer, *r* is the radial distance with *t* being the time since the commencement of pumping and *h* is the head, accordingly.

$$S_c \frac{\partial h}{\partial t} = \frac{T}{r} \frac{\partial}{\partial r}\left(r \frac{\partial h}{\partial r}\right) \tag{1.7}$$

$$\frac{S_c}{T} \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} \tag{1.8}$$

This equation describes a groundwater flow within a confined aquifer taking time into consideration. It can determine storativity and transmissivity using pump testing data. The Theis analytical solution (1.9) is described as "a solution of a non-equilibrium flow equation in radial coordinates based on the analogy between groundwater flow and heat condition assuming that the well is replaced by a mathematical sink of constant strength" (McElwee, 2012). In the solution S is the dimensionless storage coefficient, Q is the discharge, T is the transmissivity, t is the time since the beginning of pumping, r is the radial observation distance from the pumped well and Theis called W(u) a well function.

$$S(r,t) = \frac{Q}{4\pi T} \int_u^{\infty e^{-x}} \frac{e^{-x}}{x} dx = \frac{Q}{4\pi T} W(u), \ldots u = \frac{r^2 S}{4Tt} \tag{1.9}$$

The type of data required to use the Theis solution include drawdown against time data at an observation well, distance measured from the pumping well to the observation well, as well as the pumping rate of the well. With the required data, certain assumptions have to be considered so that the Theis equation can be applicable and these assumptions include the following:

- It is assumed that the aquifer is a non-leaky confined
- The flow is unsteady/transient
- Before pumping, there is no slope, meaning the piezometric surface is horizontal
- It is assumed that the aquifer has an infinite area
- Another assumption is that the aquifer is homogeneous, isotropic and uniform in thickness over the area affected by abstraction
- There rate of pumping is constant

- The pumping well fully penetrates the confined aquifer,
- The water released from storage is discharged instantly with a decrease hydraulic head,
- The storage in the well is neglected because of the assumption that the well diameter is small

Due to complexities of unconfined groundwater unsteady flow, equation 1.5 expanded below does not have an existing analytical solution, therefore some of the numerical schemes for solving partial differential equations that will be explained in detail in section 2.4 below, will be applied in order to obtain an approximate numerical solution.

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right), \quad 0 < h \le B$$

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right) + rh\frac{\partial^2 h}{\partial r^2}$$

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r}\left[\left(h + \frac{\partial hr}{\partial r}\right)\frac{\partial h}{\partial r} + rh\frac{\partial^2 h}{\partial r^2}\right]$$

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r}\left[h\frac{\partial h}{\partial r} + r\left(\frac{\partial h}{\partial r}\right)^2 + rh\frac{\partial^2 h}{\partial r^2}\right]$$

$$S_y \frac{\partial h}{\partial t} = \frac{Kh}{r}\frac{\partial h}{\partial r} + K\left(\frac{\partial h}{\partial r}\right)^2 + Kh\frac{\partial^2 h}{\partial r^2} \tag{1.10}$$

2.4 Numerical method for solving partial differential equations

Differential equations are commonly known as equations can describe almost all systems that are evolving or under evolution. These equations are very common in the field of science and engineering. Many mathematicians have studied the nature of these equations for quite a number of years and developed solution methods (Soderlind and Arevalo, 2008). Most of the time, the natural systems are enormous that an application of an analytical solution is not manageable. Sometimes the systems described by differential equations become so complex to use an analytical solution. Numerical methods accompanied by computer simulations become important because of the size and the complexity of these systems. The differential equations can be ordinary (ODE), partial (PDE) and non-linear.

A partial differential equation (PDE) can be described as an equation that involves a dependent variable which is also known as the ‘unknown function’ and some of its partial derivatives in relation to two or more variables that are independent. The equations can be classified to elliptic ($if\ \delta > 0$), hyperbolic (δ< 0) and parabolic($if\ \delta = 0$). Numerical schemes for solving PDEs are often used when there is no exact analytical solution and they also provide a systematic approximation of exact solutions. The methods applied to solve PDEs are categorized into explicit and implicit and the main numerical methods include finite difference methods (FD), finite element methods (FE) and spectral methods.

Runge–Kutta methods are one of the old, explicit iterative methods, with the well-known method called the **Euler Method** as one of them. The Euler method is very useful when discretising temporarily for the approximate solutions of ordinary differential equations. The Euler method is also described as a first-order method, which points to the direct proportionality of the error per step also known as the local error, to the square size of the step, and the error at a given time also known as the global error is directly proportional to the size of the step. The explicit scheme is an unstable scheme, having an error that would grow exponentially. This means that after a few time steps the numerical solution will not have significance to the true solution, and so the time step becomes a restriction. Alternatively, the implicit scheme is unconditionally stable, and so any grid dimension can be selected for simulation. This means there is no restriction to the time steps. The implicit scheme however requires simultaneous solutions of a set of linear equations; and as a result, computational time increases

2.4.1 The Forward Euler and Backward Euler methods

2.4.1.1 Forward Euler/Explicit method

The Forward Euler method is part of the Runge-Kutta methods which can be defined as the first order numerical method that is used to solve Ordinary Differential Equations with a given or known initial value. This method is good to use to get long term behaviour solutions and for trends. In this method, generally you choose whether you use a uniform step or non-uniform step size.

Given as:

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{1.11}$$

The solution increases from t_n to t_{n+1}=t_{n+h}. The FE method is based on a truncated Taylor series expansion which is a standard method to perform a series expansion of a function about a point , meaning that if y is expanded in the $t = t_n$ then we get

$$y(t_n + h) = y_n + hf(t_n, y_n) + O(h^2) \quad (1.12)$$

The Forward Euler method is described as a simple method to implements however; an error is depicted at every time-step due to the truncation of the Taylor series which is a Local Truncation Error (LTE); this drawback is caused by the restrictions on the time step size to ensure numerical stability (Zeltkevic, 1998). The LTE for the FE method is $O(h^2)$ and LTE is different from global error g_n..The local error in FE method decreases as the square of the size of the step and the error at a given time decreases linearly with the size of the step (Amen and Bilokon, 2004). The local error is defined as the difference between the true solution $y(t_{n+1})$ and the approximation after one time step, assuming that (t_n, y_n) is a point on the graph of the true solution. The error at a given time is described as a difference between the true solution and a computed solution. It is said that in most cases, the true solution is not known and therefore the global error cannot be evaluated.

For the solution to be adhered the so-called Courant-Friedrichs-Lewy condition must be fulfilled and it states that when a function has a space discretization, a time step that is bigger than the modelling quantity should be omitted.

2.4.1.2 Backward Euler's Method (BE)/ implicit method

A backward Euler technique is an implicit analogue of the forward Euler method. In contrast to FE, the BE methods are expensive to implement to non-linear problems because y_{n+1} is given only in terms of an implicit equation. BE obtains a solution by approximating an equation involving both the current stage of time and the later stage of time. BE is widely used more than FE, despite its computational costs because it has more stability than FE although they both have local and global errors

Given as:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \quad (1.13)$$

2.4.2 Crank-Nicolson Method

Crank–Nicolson method is one of the finite difference methods; it can be defined as an implicit numerical method which basically finds an approximate solution for the current state and the later state. This method is based on the trapezoidal rule giving a second-order convergence in time. The Trapezoidal Rule can be defined as a simple average of the Forward and Backward Euler method. This rule estimates the region beneath the graph and calculates its area. In this Crank-

Nicolson method, the first and second partial derivatives are replaced with the backward and central differences, respectively.

Since the Crank-Nicolson solution is based on a scheme that equates to a central difference approximation, this means that the spatial derivatives are solved in the centre of two time periods. The Crank-Nicolson scheme is very much similar to the implicit scheme, and so it is associated with the same disadvantages. Moreover, the Crank-Nicolson scheme gives a better approximation to the exact solution for small change in time and it converges quicker than the explicit and implicit schemes (LeVeque, 2005). This method has been used where partial differential equations describing flow and transport of dissolved organic compounds in a 2D domain (Khebchareon, 2012). Additionally to some of Crank-Nicolson uses, in (Shahraiyni & Ataie-Ashtiani, 2012) it was used with the fully implicit and Runge-Kutta schemes for modelling pollutant migration associated with infiltration.

$$\frac{h_i^n - h_i^{n-1}}{\delta t} = \frac{1}{2}\frac{h_{i-1}^n + h_{i+1}^n - 2h_i^n}{(\Delta x)^2} + \frac{1}{2}\frac{h_{i-1}^{n+1} + h_{i+1}^{n+1} - 2h_i^{n+1}}{(\Delta)x^2} F_i^{n+\frac{1}{2}} \tag{1.14}$$

2.4.3 Adams methods

As opposed to one step methods, Adams methods are multistep methods. The well-known types of Adams methods include the explicit type which is called the Adams-Bashforth (AB) method and the implicit type which is known as the Adams-Moulton (AM) method. All the aforementioned Adams methods are based on approximation of the integral with a polynomial within the interval $(t_n, (t_{n+1})$. They use transient and steady state information to develop a numerical solution of a complex function in a form of (t_{n+1}). According to (Zeltkevic, 1998) there are first order AB and AM methods which are basically the forward and backward Euler methods respectively. He states that the second order AB method is given by (1.15)

$$y_{n+1} = y_{n+} + \frac{3h}{2}\ (f(\mathrm{t}_{n+}, \mathrm{y}_n) - f(\mathrm{t}_{n-1}, \mathrm{y}_{n-1})) \tag{1.15}$$

The AB method can then be rewritten as equation (1.16) as previously mentioned that it is basically a forward Euler method with the AM being the backward Euler method.

$$y(t_{n+2}) - y(t_{n+1}) = \frac{3h}{2}\ f\big(\mathrm{t}_{n+1}, \mathrm{y}(\mathrm{t}_{n+1})\big) - \frac{\mathrm{h}}{2} f(\mathrm{t}_n, \mathrm{y}(\mathrm{t}_n)) \tag{1.16}$$

The derivation of the AB numerical scheme is as follows:

Given the first order ODE function (1.17)

$$y'(t) = f\big(t, y(t)\big) \tag{1.17}$$

Integrated from t_{n+1} to t_{n+2} to obtain (1.18)

$$y(t_{n+2}) - y(t_{n+1}) = \int_{t_{n+1}}^{t_{n+2}} f\big(\tau, y(\tau)\big)d\tau \tag{1.18}$$

For a two-step f is represented by its linear interpolating polynomial at points $\tau = t_n$ and $\tau = t_{n+1}$.

Let h=$t_{n+1} - t_n$

$$\mathrm{P}(\tau) = \frac{\tau - t_{n+1}}{t_n - t_{n+1}} f(\mathrm{t_n}, \mathrm{y}(\mathrm{t_n})) + \frac{\tau - \mathrm{t_n}}{\mathrm{t_{n+1}} - \mathrm{t_n}} f\big(\mathrm{t_{n+1}}, \mathrm{y}(\mathrm{t_{n+1}})\big) \tag{1.19}$$

$$\mathrm{P}(\tau) = \frac{t_{n+1} - \tau}{h} f(\mathrm{t_n}, \mathrm{y}(\mathrm{t_n})) + \frac{\tau - \mathrm{t_n}}{h} f\big(\mathrm{t_{n+1}}, \mathrm{y}(\mathrm{t_{n+1}})\big) \tag{1.20}$$

An integral of equation (1.20) becomes (1.21)

$$\int_{t_{n+1}}^{t_{n+2}} f\big(\tau, y(\tau)\big)d\tau \approx \int_{t_{n+1}}^{t_{n+2}} p(\tau)d\tau \tag{1.21}$$

$$\int_{t_{n+1}}^{t_{n+2}} f\big(\tau, y(\tau)\big)d\tau = \int_{t_{n+1}}^{t_{n+2}} \left[\frac{t_{n+1} - \tau}{h} f(\mathrm{t_n}, \mathrm{y}(\mathrm{t_n})) + \frac{\tau - \mathrm{t_n}}{h} f\big(\mathrm{t_{n+1}}, \mathrm{y}(\mathrm{t_{n+1}})\big)\right] d\tau \tag{1.22}$$

$$\int_{t_{n+1}}^{t_{n+2}} f\big(\tau, y(\tau)\big)d\tau = \frac{f(\mathrm{t_n}, \mathrm{y}(\mathrm{t_n}))}{h} \int_{t_{n+1}}^{t_{n+2}} [t_{n+1} - \tau]\, d\tau + \frac{f\big(\mathrm{t_{n+1}}, \mathrm{y}(\mathrm{t_{n+1}})\big)}{h} \int_{t_{n+1}}^{t_{n+2}} [\tau - \mathrm{t_n}]\ d\tau \tag{1.23}$$

Integration assumptions for $\int_{t_{n+1}}^{t_{n+2}} [t_{n+1} - \tau]\, d\tau$: Let y= $t_{n+1} - \tau$ and $dy = -d\tau$

When

$$\tau = t_{n+1}\ and\ y = 0\ or\ \tau = t_{n+2}\ and\ y = t_{n+1} - t_{n+2}$$

Then

$$\int_{0}^{t_{n+1}-t_{n+2}} -ydy = \int_{-(t_{n+1}-t_{n+2})}^{0} ydy \tag{1.24}$$

Let h= $t_{n+1} - t_{n+2}$ and substitution into equation (1.24) to obtain:

$$\int_{-h}^{0} ydy = \left[\frac{y^2}{2}\right]_{-h}^{0} = -\frac{h^2}{2} \tag{1.25}$$

When the Integration assumptions for $\int_{t_{n+1}}^{t_{n+2}} [\tau - \mathrm{t_n}]\, d\tau$ changes to y=$\tau - \mathrm{t_n}$ and $dy = -d\tau$

If:

$$\tau = t_{n+1}\ and\ y = t_{n+1} - t_n /\ \tau = t_{n+2}\ and\ y = t_{n+2} - t_n$$

Then

$$\int_{t_{n+1}-t_n}^{t_{n+2}-t_n} -ydy = \int_{t_{n+2}-t_n}^{t_{n+1}-t_n} ydy \tag{1.26}$$

$$\int_{h}^{2h} ydy = \left[\frac{y^2}{2}\right]_{h}^{2h} = \frac{(2h)^2}{2} - \frac{h^2}{2} = \frac{3h^2}{2} \tag{1.27}$$

Substituting the integrated equations back, to get (1.28)

$$\int_{t_{n+1}}^{t_{n+2}} f(\tau, y(\tau))d\tau = -\frac{f(\mathrm{t_n}, \mathrm{y(t_n)})}{h}\frac{h^2}{2} + \frac{f(\mathrm{t_{n+1}}, \mathrm{y(t_{n+1})})}{h}\frac{3h^2}{2} \tag{1.28}$$

Then simplified by cancelling out *h* to get the AB equation (1.29)

$$\boldsymbol{y(t_{n+2}) - y(t_{n+1}) = \frac{3h}{2}\, f(\mathbf{t_{n+1}}, \mathbf{y(t_{n+1})}) - \frac{\mathbf{h}}{2} f(\mathbf{t_n}, \mathbf{y(t_n)})} \tag{1.29}$$

For the scope of this study, the focus will not be on this AB method but probably on the newly developed scheme by Atangana and Batogna which is a modification of the AB method to accommodate PDEs. The AB numerical method was initially developed to assess Ordinary Differential Equations (ODEs) and it cannot solve PDEs. (Atangana and Batogna, 2017) modified this efficiently common AB method and developed a numerical scheme that can solve PDEs with both local and non-local operators. The Atangana-Batogna is explained in the next section.

2.4.4 New Two-Step Laplace Adams-Bashforth Method (Atangana-Gnitchogna numerical scheme)

As mentioned in 2.4.3 above, Atangana and Gnitchogna analysed a gap in the powerful AB numerical scheme. The AB method has been proved to be very useful and useful for differential equations that contain classical derivatives and those with non-integer order derivatives (Atangana and Gnitchogna, 2017; Alkahtani, 2017). It was initially developed for only ODEs, therefore cannot be applied to PDEs that have both local and non-local operators; because of this gap, Atangana and Gnitchogna modified the AB method and combined it with the Laplace transform into a PDE to transform it to an ODE that can be analysed in Laplace space. The resulting equation was continued to be solved in a Laplace space through the use of the AB method and a is provided in inverse-form, taking back the space into the real space in order to be able to approximate differential equations that contain both local and non-local operators.

In their paper, Atangana and Gnitchonga used a general PDE with integer order to approximate the numerical solution that can take into account the non-classical equations. The method commences with a general partial differential equation (1.30) and in order to make it an ordinary differential equation, Laplace-transform (1.31) which is a system that depends on algebra to solve Linear Differential Equations is applied to get (1.32)

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t) + Nu(x,t) \tag{1.30}$$

Laplace Transform is given as

$$\mathcal{L}\{f(t)\} = F(s) \tag{1.31}$$

Where $F(s)$is the Laplace transform and $f(t)$ together with $F(s)$ are called a Laplace transform pair. Apply Laplace integral into equation (1.31):

$$\mathcal{L}\left(\frac{\partial u(x,t)}{\partial t}\right) = \mathcal{L}(Lu(x,t) + Nu(x,t)\ ) \tag{1.32}$$

The integral of Laplace is used in equation 1.32 so that there is only one variable remaining to be solved.

$$\frac{d}{dt}\big(u(p,t)\big) = \mathcal{L}(Lu(x,t) + Nu(x,t)\ \ ) \tag{1.33}$$

Equation (1.33) can be simplified to simple and similar terms

$$\frac{d}{dt}\big(u(t)\big) = F(u,t) \tag{1.34}$$

Where $\big(u(t)\big)$ is $\big(u(p,t)\big)$ and $F(u,t)$ is $\mathcal{L}(Lu(x,t) + Nu(x,t)\ )$

Now they introduced the theorem of calculus results in the following integrated equation (integrated in both sides), where integration boundaries are t and 0:

$$\int_0^t \frac{d}{dt}\big(u(t)\big) = \int_0^t F(u,\tau)\, d\tau \tag{1.35}$$

$$u(t) - u(0) = \int_0^t F(u,\tau)\, d\tau \tag{1.36}$$

$$u(t) = u(0) + \int_0^t F(u,\tau)\, d\tau \tag{1.37}$$

The boundary conditions can be given also as $t = (t_n)$ and $t = (t_{n+1})$, equation will be given with the following two equations:

When $t = (t_n)$

$$u(t_n) = u(0) + \int_0^{t_n} F(u,\tau) d\tau \tag{1.38}$$

When and $t = (t_{n+1})$

$$u(t_{n+1}) = u(0) + \int_0^{t_{n+1}} F(u,\tau) d\tau \tag{1.39}$$

$$u(t_{n+1}) - u(t_n) = \left[u(0) + \int_0^{t_{n+1}} F(u,\tau) d\tau\right] - \left[u(0) + \int_0^{t_n} F(u,\tau) d\tau\right] \tag{1.40}$$

Conventions: applying Reversing limit integral into Equation (1.19)

$$u(t_{n+1}) - u(t_n) = \left[u(0) + \int_0^{t_{n+1}} F(u,\tau) d\tau\right] + \left[u(0) + \int_{t_n}^{0} F(u,\tau) d\tau\right] \tag{1.41}$$

Addition of integration intervals and subtraction of like terms into Equation (1.35)

$$u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} F(u,\tau) d\tau \tag{1.42}$$

The AB method uses Lagrange Polynomial Method (LPM) to supplement the new proposal; LPM is used for polynomial interpolation, meaning that this method is used for verification in theoretical arguments to support what is studied. $F(u,t)$ is approximated using the LPD method.

Given as:

$$P(t) \approx F(u,t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F(u, t_n) + \frac{t - t_n}{t_{n-1} - t_n} F(u, t_{n-1}) \tag{1.43}$$

$$P(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F(u, t_{n-1}) \tag{1.44}$$

Equation (1.42) can be given as

$$u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} \left[\frac{t - t_{n-1}}{t_n - t_{n-1}} F_n + \frac{t - t_n}{t_{n-1} - t_n} F(u, t_{n-1}) \right] dt \tag{1.45}$$

$$u(t_{n+1}) - u(t_n) = \frac{F_n}{t_n - t_{n-1}} \int_{t_n}^{t_{n+1}} (t - t_{n-1}) dt + \frac{F_{n-1}}{t_{n-1} - t_n} \int_{t_n}^{t_{n+1}} (t - t_n) \tag{1.46}$$

Integrating Equation (1.21 in section 2.4.3 above) and Let h be $h = t_n - t_{n-1}$

$$u(t_{n+1}) - u(t_n) \tag{1.47}$$

$$= \frac{F_n}{t_n - t_{n-1}} \left[\frac{t^2}{2} - t t_{n-1} \right]_{t_n}^{t_{n+1}} + \frac{F_{n-1}}{t_{n-1} - t_n} \left[\frac{t^2}{2} - t t_n \right]_{t_n}^{t_{n+1}}$$

$$u(t_{n+1}) - u(t_n) = \frac{F_n}{h} \left[\frac{t^2}{2} - t t_{n-1} \right]_{t_n}^{t_{n+1}} + \frac{F_{n-1}}{(-h)} \left[\frac{t^2}{2} - t t_n \right]_{t_n}^{t_{n+1}} \tag{1.48}$$

Substituting the boundaries into Equation (1.25)

$$\begin{aligned} u(t_{n+1}) - u(t_n) & \\ &= \frac{F_n}{h} \left[\left(\frac{t_{n+1}^2}{2} - \frac{t_n^2}{2} \right) - (t_{n+1} t_{n-1} - t_n t_{n-1}) \right] \\ &- \frac{F_{n-1}}{h} \left[\left(\frac{t_{n+1}^2}{2} - \frac{t_n^2}{2} \right) - (t_{n+1} t_n - t_n t_n) \right] \end{aligned} \tag{1.49}$$

$$u(t_{n+1}) - u(t_n)$$
$$= \frac{F_n}{h}\left[\frac{1}{2}[(t_{n+1} - t_n)(t_{n+1} + t_n)] - t_{n-1}(t_{n+1} - t_n)\right] \qquad (1.50)$$
$$- \frac{F_{n-1}}{h}\left[\frac{1}{2}[(t_{n+1} - t_n)(t_{n+1} + t_n)] - t_n(t_{n+1} - t_n)\right]$$

It should be noted that h=$t_{n+1} - t_n$

$$u(t_{n+1}) - u(t_n) = \frac{F_n}{h}\left(\frac{h}{2}(t_{n+1} + t_n) - ht_{n-1}\right) - \frac{F_{n-1}}{h}\left(\frac{h}{2}(t_{n+1}t_n) - ht_n\right) \qquad (1.51)$$

Then h cancels out to obtain:

$$u(t_{n+1}) - u(t_n) = F_n - F_{n-1}\left(\frac{1}{2}(t_{n+1}t_n) - t_n\right) \qquad (1.52)$$

$$u(t_{n+1}) - u(t_n)$$
$$= F_n\left[\frac{1}{2}\big((n+1)h + nh\big) - (n-1)h\right]$$
$$- F_{n-1}\left[\frac{1}{2}\big((n+1)h + nh\big) - nh\right] \qquad (1.53)$$

$$u(t_{n+1}) - u(t_n) = F_n\left(\frac{nh}{2} + \frac{h}{2} + \frac{nh}{2} - nh + h\right) \quad - F_{n-1}\left(\frac{nh}{2} + \frac{h}{2} + \frac{nh}{2} - nh\right) \qquad (1.54)$$

$$u(t_{n+1}) - u(t_{n+1}) = F_n - F_{n-1}\left(nh - nh + \frac{h}{2}\right) \qquad (1.55)$$

$$u(t_{n+1}) - u(t_n) = F_n - F_{n-1}\left(\frac{h}{2}\right) \qquad (1.56)$$

2.4.5 Galerkin methods

Galerkin methods are a range of finite element methods used for converting differential equations with the time-dependent boundary to weak formulations. The methods are used for elliptical and parabolic PDEs as well as hyperbolic. The Galerkin type techniques can be useful in approximation of numerical solutions of both linear and non-linear problems (Douglas and Dupont, 2006). There is Ritz-Galerkin method amongst other Galerkin methods as well as the Sinc-Galerkin method which can be used to approximate solutions of nonlinear problems.

CHAPTER 3: DERIVATION OF NUMERICAL SOLUTIONS

A background and applicability of a number of numerical schemes has been stipulated in the previous chapter. Analytical methods and numerical methods are often used to understand systems however; analytical methods are mostly used in understanding the nature of the studied phenomena whenever such solutions are possible. The analytical methods reveal how characteristics of flow domains or conditions of a boundary exist in the background of a phenomenon (Sato, 2000). In conducting groundwater flow studies, the application of analytical methods is mostly impossible due to heterogeneity of the domain and complexities involved in the boundary. In addition to this, there are difficulties in analysis of groundwater flow to the nonlinearity in groundwater flow in phreatic zone, density flow and unsaturated flow. It is for these reasons that numerical methods become very useful in understanding complex differential equations and they are therefore solved using numerical solutions. Numerical models include a number of uncertainties which are defined as limited assurance, which means due to limited existing information, the future state of a system can be predicted but it cannot be accurately defined (JiChun and XianKui 2013). In this chapter, application of certain numerical schemes will be conducted to get numerical solutions for equation (1.5) and equation (1.6) which are the groundwater flow equation in an unconfined zone and zone, respectively.

3.1 New Two-Step Laplace Adams-Bashforth Method (Atangana-Gnitchogna numerical scheme)

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right), \quad 0 < h \leq B$$

$$S_y \frac{\partial h}{\partial t} = \frac{Kh}{r}\frac{\partial h}{\partial r} + K\left(\frac{\partial h}{\partial r}\right)^2 + Kh\frac{\partial^2 h}{\partial r^2}$$

We start by solving the second non-linear partial differential equation via the method suggested in section 2.4.4.

Following the steps involved the scheme; we rearrange equation 1.10 by dividing on both sides by S_y.

Given in time:

$$\frac{\partial h}{\partial t} = \frac{K}{S_y}\frac{h}{r}\frac{\partial h}{\partial r} + \frac{K}{S_y}\left(\frac{\partial h}{\partial r}\right)^2 + \frac{K}{S_y} h \frac{\partial^2 h}{\partial r^2} \tag{2.1}$$

Given in space:

$$h(x_i, t_{n+1}) - h(x_i, t_n) = \frac{3}{2}\Delta t F_i^n - \frac{\Delta t}{2} F_i^{n-1} \tag{2.2}$$

$$h_i^{n+1} - h_i^n = \frac{3}{2}\Delta t\, F_i(h_i, t_n) - \frac{\Delta t}{2} F_i(h_i, t_{n-1}) \tag{2.3}$$

Using equation (2.1); F (h, t) is defined as follows:

$$F(h,t) = \frac{K}{S_y}\frac{h}{r}\frac{\partial h}{\partial r} + \frac{K}{S_y}\left(\frac{\partial h}{\partial r}\right)^2 + \frac{K}{S_y} h \frac{\partial^2 h}{\partial r^2} \tag{2.4}$$

Now the definition of F_i^n is:

$$F_i^n = \frac{K}{S_y}\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)\frac{h_i^n}{r_i} + \frac{K}{S_y}\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)^2 + \frac{K}{S_y} h_i^n \left(\frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right) \tag{2.5}$$

The definition of F_i^{n-1} in terms of equation (2.1) is

$$F_i^{n-1} = \frac{K}{S_y}\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)\frac{h_i^{n-1}}{r_i} + \frac{K}{S_y}\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)^2 + \frac{K}{S_y} h_i^{n-1} \left(\frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2}\right) \tag{2.6}$$

Now we substitute equation (2.1) into equation (2.3) using the defined equations (2.5) and (2.6) to obtain

$$h_i^{n+1} - h_i^n = \frac{3}{2}\Delta t \left[\frac{K}{S_y}\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)\frac{h_i^n}{r_i} + \frac{K}{S_y}\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)^2 + \frac{K}{S_y} h_i^n \left(\frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right) \right]$$

$$-\frac{\Delta t}{2}\left[\frac{K}{S_y}\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)\frac{h_i^{n-1}}{r_i} + \frac{K}{S_y}\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)^2 + \frac{K}{S_y} h_i^{n-1} \left(\frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2}\right)\right] \quad (2.7)$$

One can notice that from equation (2.7), like terms cannot be grouped and therefore no stability analysis can be performed.

3.2 Crank-Nicolson numerical scheme

As mentioned in chapter 2 above, Crank-Nicolson numerical scheme is one of the highly appropriate methods for numerical simulations because of its stability.
Given equation (1.10), the definition of the Crank-Nicolson numerical scheme is as follows:

$$\frac{\partial h}{\partial t} = \frac{h_i^{n+1} - h_i^n}{\Delta t} \quad (3.1)$$

Definition of this numerical scheme at a particular time is given as follows:

$$h = \frac{h_i^{n+1} + h_i^n}{2} \quad (3.2)$$

Definition of this numerical scheme for 1st order derivative:

$$\frac{\partial h}{\partial r} = \frac{1}{2}\left[\frac{h_{i+1}^{n+1} - h_{i-1}^{n+1}}{2\Delta r} + \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta r}\right] \quad (3.3)$$

his numerical scheme for the 2nd order derivative:

$$\frac{\partial^2 h}{\partial r^2} = \frac{1}{2}\left[\frac{h_{i+1}^{n+1} - 2\,h_i^{n+1} + h_{i-1}^{n+1}}{(\Delta r)^2} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2}\right] \quad (3.4)$$

Now using the Crank-Nicolson method, substituting equation (1.10) into equation the definitions of Crank-Nicolson numerical method to get a solution of the unconfined flow

$$S_y \frac{\partial h}{\partial t} = \frac{Kh}{r}\frac{\partial h}{\partial r} + K\left(\frac{\partial h}{\partial r}\right)^2 + Kh\frac{\partial^2 h}{\partial r^2}$$

$$\frac{h(r_{i,}t_{n+1}) - h\,(r_{i,}t_n)}{\Delta t} = \frac{K}{r}\frac{h_i^{n+1} + h_i^n}{2}\frac{1}{2}\left[\frac{h_{i+1}^{n+1} - h_{i-1}^{n+1}}{\Delta r} + \frac{h_{i+1}^{n} + h_{i-1}^{n}}{\Delta r}\right] + K\left(\frac{1}{2}\left[\frac{h_{i+1}^{n+1} - h_{i-1}^{n+1}}{\Delta r} + \frac{h_{i+1}^{n} + h_{i-1}^{n}}{\Delta r}\right]\right)^2 + K\frac{h_i^{n+1} + h_i^n}{2}\frac{1}{2}\left[\frac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{(\Delta r)^2} + \frac{h_{i+1}^{n} - 2h_i^{n} + h_{i-1}^{n}}{(\Delta r)^2}\right] \quad (3.5)$$

Similarly to equation (2.7), stability analysis cannot be conducted as there is limited grouping of like-terms in equation (3.5)

3.3 Fractional differentiation and Application of the Atangana-Baleanu in Caputo sense derivative (ABC method)

In section 3.1 and 3.2 above, one will notice the application of the new two-step Adams-Bashforth method as modified by Atangana-Gnitchonga as well as the application of the well-known Crank-Nicolson method to obtain a numerical solution. However, it is noticeable that from the groundwater flow equation (1.10), the application of the methods mentioned above deviates from the normal expected numerical solution whose stability could be tested using the Von Neumann stability analysis. This deviation which shows complexities requires modelling of the system with new ways. Fractional calculus is known as a framework to deal with complex systems (Tateishi *et al.*, 2017) and literature shows that fractional differentiation has been and can be successfully used to model complex existing world problems in the field of science including Earth Sciences. With the theory gathered from literature, one can therefore state that fractional differentiation can be used to understand the conversion of a confined to unconfined flow. However, it is of utmost importance to understand which fractional order derivatives should be applied to model the complexity and the reality of the system such that it provides the memory effect which has a significant effect on field data used.

It is imperative for one to note that the permeable geological formations that can transmit large quantities of water, called aquifers are mostly nexus systems and they can be a compound of both heterogeneity and anisotropy. In fractional differentiation, it has been believed for fairly a long that the complex nature can be only represented mathematically using one function, which is known as the power law **x−α**. Power law owes its importance to the Riemann-Liouville power-law differential operator which has been the only commonly used operator to model physical problems. In cases like complex heterogeneous aquifers with transforming flow from confined to unconfined, the power law and the decay law cannot yield correct results when modelling such systems therefore an applicable operator that is suitable for differentiation must be used. Not so long ago, researchers have conducted studies and proposed new fractional-time operators, and these are the Caputo-Fabrizio and the Atangana-Baleanu which are then defined by non-singular memory kernel.

After the newly developed operators, one can confidently state that according to recent literature, the common fractional order derivatives of fluids are the singular Riemann-Liouville derivative, the two non-singular operators, namely Caputo-Fabrizio derivative and the Atangana-Baleanu derivative. The definition of these three derivatives will be given in detail below including their relation with index law which is defined as being useful for classical mechanic natural systems located in homogeneous medium at a given state (Tateishi *et al.*, 2017; Atangana and Gomez-Aguilar, 2018b). The literature and understanding the differences between these three fractional derivatives is obtained from the studies conducted to show their applicability in real-world problems.

3.3.1 Riemann-Liouville derivative

This is one of the old derivatives known in the industry which satisfies the index law. The commutativity of this derivative shows that this operator lacks the needed force to pass from one scale to another as it is scale invariant (Tateishi *et al.*, 2017; Atangana and Gomez-Aguilar, 2018a; Atangana and Gomez-Aguilar, 2018b;) . The probability distribution of this derivative is non-Gaussian only and randomness cannot be described as it lacks the power.

Definition:

$$ {}_{\alpha}D_t^{\alpha} f(\mathrm{t}) = \frac{d^n}{d^{tn}} D_t^{-(n-\alpha)} f(\mathrm{t}) \qquad (4.1)$$

3.3.2 Caputo-Fabrizio derivative

This derivative accommodates heterogeneity at variant scales and it improves the effect of memoy by avoiding singularity (non-singular operator). The exact solution satisfies very well all the principles of semi-group, however, the Caputo– Fabrizio fractional derivative does not meet the semi-group principle. The waiting time distribution of this fractional derivative is exponential with its probability distribution crossing over from non-Gaussian to Gaussian with steady state.

Definition:

$$ {}_{0}^{CF}D_t^{\alpha} f(\mathrm{t}) = \frac{M(\alpha)}{1-\alpha} \int_{\alpha}^{t} f'(x) exp\left[-\alpha \frac{t-x}{1-\alpha}\right] d(x) \qquad (4.2) $$

3.3.3 Atangana-Baleanu derivative

Similar to the Caputo-Fabrizio derivative, the Atangana-Baleanu derivative is a non-singular operator that addresses heterogeneity at different scales. It is scale variant and when compared to the aforementioned, the Atangana–Baleanu fractional differential operator satisfies the index law but only when the fractional order are the same. Furthermore, the Atangana-Baleanu fractional derivative which is based on the Mittag-Leffler function (4.8) is associated with a non-singular and non-local function, which allows addressing of the groundwater flow behaviour in geologic units that show effects of viscoelasticity. About the Atangana-Baleanu fractional derivative, one can stipulate some facts as well presented and explained in (Tateishi *et al.*, 2017; Atangana and Gomez-Aguilar, 2018a), some of the facts about this fractional derivative include:

- Atangana-Baleanu fractional derivative is deterministic and stochastic as it is comparable to the Riemann-Liouville one and Brownian motion, respectively.
- It has a Gaussian to non-Gaussian crossover with regards to a probability distribution
- The waiting time distribution for the Atangana-Baleanu fractional is a crossover from stretched exponential to the power law.
- The mean square displacement for the Atangana-Baleanu fractional derivative is a usual to sub-diffusion crossover.

Definitions: The definition of Atangana-Baleanu fractional derivative in Riemann-Liouville sense (ABR) is as follows: Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$

$$ {}_{0}^{ABR}D_t^{\alpha}\{f(\mathrm{t})\} = \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_{0}^{t} f(x) E_{\alpha}\left(-\alpha \frac{(t-x)^{\alpha}}{1-\alpha}\right) dx \qquad (4.3) $$

And the Laplace transform of this equation is represented as:

$$ \mathcal{L}\left\{D_t^{\alpha}\{f(\mathrm{t})\}\right\}(s) = \frac{AB(\alpha)}{1-\alpha} \mathcal{L}\left[\frac{d}{dt} \int_{0}^{t} f(x) E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right) dx\right](s) \qquad (4.4) $$

Then the Atangana-Baleanu fractional derivative in the Caputo sense (**ABC)** is as follows: Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$

$$ {}_{0}^{ABC}D_t^{\alpha}\{f(\mathrm{t})\} = \frac{AB\ (\alpha)}{1-\alpha}\int_0^t \frac{d}{dt}\ f(x)E_\alpha\ \left(-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right)dx \tag{4.5}$$

The Laplace transform of the Atangana-Baleanu fractional derivative in Caputo sense is:

$$ \mathcal{L}\left\{{}_{0}^{ABC}D_t^{\alpha}\{f(\mathrm{t})\}\right\}(s) = \frac{AB\ (\alpha)}{1-\alpha}\mathcal{L}\left[\int_0^t \tfrac{d}{dt}f(x)E_\alpha\left(-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right)dx\right](s) \tag{4.6}$$

The Atangana-Beleanu fractional integral of order α of a function (t) is given as:

$$ {}_{0}^{AB}\Gamma_t^{\alpha}\{f(\mathrm{t})\} = \frac{1-\alpha}{AB(\alpha)}f(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\ f(x)dx \tag{4.7}$$

3.3.4 Mittag-Leffler function

Mittag-Leffler function is a known as a good filter that supersedes the exponential and power law functions because it can capture non-locality and it avoids the singularity. This function is defined as a good, convenient function which is good for long time behavior. Thus any fractional derivative that is based on this function acts a very useful tool in assessing the complex problems in the real world, and that is the Atangana-Baleanu derivative.

Equation 4.8

$$ \mathrm{E}_\alpha(-\mathrm{t}^\alpha) = \sum_{n=0}^{\infty}\frac{(-t^\alpha)^n}{\Gamma(\alpha n+\beta)}\quad \alpha > 0 \text{ and } \beta > 0 \tag{4.8}$$

3.3.5 Application of the ABC

The Caputo-Fabrizio derivative is one of the good fractional derivatives as mentioned and proved in a number of papers; however, as good and appropriate as it is, it cannot be applied over the Atangana-Baleanu derivative because although they both have a probability distribution of Gaussian to non-Gaussian, the former has a steady state between the transitions. For the scope of this study “modeling conversion of a confined to an unconfined aquifer flow”, the type of flow under investigation is a transient state. As mentioned above, the Atangana-Baleanu derivative is based on the Mittag-Leffler function and because of Mittag-Leffler memory it is therefore able to differentiate between dynamical systems taking place at variant scales *without steady state*. It is for the aforementioned reasons that the ABC method will be applied to provide a better sense of understanding the conversion of a confined flow to an unconfined flow.

The initial equations for the confined and unconfined zone are as follows (5.1) for confined and (5.2) for the unconfined zone:

$$S_c \frac{\partial h}{\partial t} = \frac{KB}{r} \frac{\partial}{\partial r}\left(r \frac{\partial h}{\partial r}\right), \quad h \geq B \tag{5.1}$$

$$S_y \frac{\partial h}{\partial t} = \frac{K}{r} \frac{\partial}{\partial r}\left(rh \frac{\partial h}{\partial r}\right), \quad 0 < h \leq B \tag{5.2}$$

Then we substitute (5.1) with equation (4.5) which is the definition of the Atangan-Baleanu in Caputo sense (ABC) to obtain:

$$ {}^{ABC}_{0}D_t^{\alpha}\{f(\mathrm{t})\} = \frac{AB\ (\alpha)}{1-\alpha} \int_0^t \frac{d}{dt}\ f(x) E_{\alpha}\ \left[-\frac{\alpha}{1-\alpha}(t_n - x)^{\alpha}\right] dx \tag{5.3}$$

$$Sc\left[\frac{AB\ (\alpha)}{1-\alpha} \int_0^t \frac{\partial h}{\partial x}\ (r,x) E_{\alpha}\ \left[-\frac{\alpha}{1-\alpha}(t_n - x)^{\alpha}\right] dx\ \right] = \frac{KB}{r} \frac{\partial}{\partial r}\left(r \frac{\partial h}{\partial r}\right) \tag{5.4}$$

$$h \geq B$$

Then we substitute (5.2) with equation (4.5) as well to obtain

$$S_y\left[\frac{AB\ (\alpha)}{1-\alpha} \int_0^t \frac{\partial h}{\partial x}(r,x) E_{\alpha}\ \left(\frac{-\alpha}{1-\alpha}(t - x)^{\alpha}\right) dx\ \right] = \frac{K}{r} \frac{\partial}{\partial r}\left(rh \frac{\partial h}{\partial r}\right), \quad 0 < h \leq B \tag{5.5}$$

Now we discretize equation (5.4) using the Atangana-Baleanu in Caputo sense to obtain:

$$ {}^{ABC}_{0}D_t^{\alpha} h(r,t) = \frac{AB\ (\alpha)}{1-\alpha} \int_0^t \frac{d}{dt}\ f(x) E_{\alpha}\ \left[-\frac{\alpha}{1-\alpha}(t_n - x)^{\alpha}\right] dx \tag{5.6}$$

$$ {}^{ABC}_{0}D_t^{\alpha} h(r,t) = \frac{AB\ (\alpha)}{1-\alpha} \int_0^t \frac{\partial h}{\partial x}\ (r_i,t) E_{\alpha}\ \left[-\frac{\alpha}{1-\alpha}(t_n - x)^{\alpha}\right] dx \tag{5.7}$$

$$ {}^{ABC}_{0}D_t^{\alpha} h(r,t) = \frac{AB\ (\alpha)}{1-\alpha} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{h_i^{j+1} - h_i^j}{\Delta t}\ \ E_{\alpha}\ \left[-\frac{\alpha}{1-\alpha}(t_n - x)^{\alpha}\right] dx \tag{5.8}$$

$$ {}^{ABC}_{0}D_t^{\alpha} h(r,t) = \frac{AB\ (\alpha)}{1-\alpha} \sum_{j=0}^{n} \frac{h_i^{j+1} - h_i^j}{\Delta t}\ \ \int_{t_j}^{t_{j+1}} E_{\alpha}\ \left[-\frac{\alpha}{1-\alpha}(t_n - x)^{\alpha}\right] dx \tag{5.9}$$

$$= \frac{AB\ (\alpha)}{1-\alpha}\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^j}{\Delta t}\ \delta_{n,j}^{\alpha} \tag{5.10}$$

Where $\delta_{n,j}^{\alpha}$ is:

$$\delta_{n,j}^{\alpha} = \int_{t_j}^{t_{j+1}} E_\alpha\left(-\alpha\frac{(t_n-x)^\alpha}{1-\alpha}\right)dx \tag{5.11}$$

$$\delta_{n,j}^{\alpha} = (t_n-t_{j+1})E_\alpha\left[\frac{\alpha}{1-\alpha}(t_n-t_{j+1})\right]-(t_n-t_j)E_\alpha\left[\frac{\alpha}{1-\alpha}(t_n-t_j)\right] \tag{5.12}$$

Now substitution into the main equation in the confined zone

$$Sc\left[\frac{AB\ (\alpha)}{1-\alpha}\int_0^t\frac{\partial h}{\partial x}\ (r,x)E_\alpha\ \left[-\frac{\alpha}{1-\alpha}(t_n-x)^\alpha\right]dx\ \right]=\frac{KB}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) \tag{5.13}$$

$$\frac{S_c}{T}\left[\frac{AB\ (\alpha)}{1-\alpha}\int_0^t\frac{\partial h}{\partial x}\ (r,x)E_\alpha\ \left[-\frac{\alpha}{1-\alpha}(t_n-x)^\alpha\right]dx\right]\ =\frac{1}{r}\frac{\partial h}{\partial r}+\frac{\partial^2 h}{\partial r^2} \tag{5.14}$$

$$\begin{aligned}&\frac{S_c}{T}\left[\frac{AB\ (\alpha)}{1-\alpha}\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^j}{\Delta t}\ \ \int_{t_j}^{t_{j+1}}E_\alpha\ \left(-\frac{\alpha}{1-\alpha}(t_n-x)^\alpha\right)dx\right]\\&=\frac{1}{r_i}\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}+\frac{h_{i+1}^{n}-h_{i-1}^{n}}{\Delta r}\right]\\&+\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-2\ h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n}}{(\Delta r)^2}\right]\end{aligned} \tag{5.15}$$

Now we apply the Crank-Nicolson method into equation (5.13)

$$\frac{S_c}{T}\left[\frac{AB\ (\alpha)}{1-\alpha}\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^j}{\Delta t}\quad (t_n-t_{j+1})E_\alpha\left[\frac{\alpha}{1-\alpha}(t_n-t_{j+1})\right]\right.$$

$$\left.-(t_n-t_j)E_\alpha\left[\frac{\alpha}{1-\alpha}(t_n-t_j)\right]\right]$$

$$=\frac{1}{r_i}\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}+\frac{h_{i+1}^{n}-h_{i-1}^{n}}{\Delta r}\right]$$

$$+\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-2\,h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n}}{(\Delta r)^2}\right] \tag{5.16}$$

Now we apply the Crank-Nicolson method into equation (5.14) which is the ABC in the unconfined zone to obtain (5.17)

$$S_y\left[\frac{AB\ (\alpha)}{1-\alpha}\int_0^t\frac{\partial h}{\partial x}(r,x)E_\alpha\ \left(\frac{-\alpha}{1-\alpha}(t-x)^\alpha\right)dx\ \right]=\frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right)$$

$$S_y\left[\frac{AB\ (\alpha)}{1-\alpha}\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^j}{\Delta t}\ \delta_{n,j}^{\alpha}\ \right]$$

$$=\quad\frac{K}{r}\frac{h_i^{n+1}+h_i^n}{2}\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}\right.$$

$$\left.+\frac{h_{i+1}^{n}+h_{i-1}^{n}}{\Delta r}\right]K\left(\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}+\frac{h_{i+1}^{n}+h_{i-1}^{n}}{\Delta r}\right]\right)^2$$

$$+K\frac{h_i^{n+1}+h_i^n}{2}\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-\ 2h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n}}{(\Delta r)^2}\right] \tag{5.17}$$

3.1 Von Neumann Stability analysis

Von Neumann stability analysis which is also known as the Fourier stability analysis is a way of assessing the stability of a finite difference scheme as a PDE. It is therefore used to assess if the numerical solutions are stable. The analysis of stability includes Fourier's series and the finite difference scheme is then classified as stable if the errors sustained at a discrete time step are not spread throughout the simulation (Seta and Takahashi, 2001). In this section, a stability analysis will be conducted for the numerical solution of the confined zone and that is equation (5.15) in section 3.3.5 above.

$$\frac{S_c}{T}\left[\frac{AB\ (\alpha)}{1-\alpha}\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\ \delta_{n,j}^{\alpha}\right]$$

$$= \frac{1}{r_i}\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}+\frac{h_{i+1}^{n}-h_{i-1}^{n}}{\Delta r}\right]$$

$$+\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-2\,h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n}}{(\Delta r)^2}\right] \tag{5.18}$$

Fourier's series in space is given as:

$$\rho(x,t)=\sum_{n}\hat{\rho}(t)e^{ik_m x} \tag{5.19}$$

Now equation (5.18) can be written as follows:

$$\frac{S_c}{T}\left(\frac{h_i^{n+1}-h_i^{n}}{\Delta t}\ \delta_{n,n}^{\alpha}\right)+\frac{S_c}{T}\left[\sum_{j=0}^{n-1}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\ \delta_{n,j}^{\alpha}\right]$$

$$=\frac{1}{2r_i}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}+\frac{h_{i+1}^{n}-h_{i-1}^{n}}{\Delta r}\right] \tag{5.20}$$

$$+\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-2\,h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n}}{(\Delta r)^2}\right]$$

$$\frac{S_c h_i^{n+1}}{T\Delta t}\delta_{n,n}^{\alpha}=\frac{S_c h_i^{n}}{T\Delta t}\delta_{n,n}^{\alpha}-\frac{S_c}{T}\left[\sum_{j=0}^{n-1}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\ \delta_{n,j}^{\alpha}\right]+\frac{1}{2r_i}\left[\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}+\frac{h_{i+1}^{n}-h_{i-1}^{n}}{\Delta r}\right]$$

$$+\frac{1}{2}\left[\frac{h_{i+1}^{n+1}-2\,h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{h_{i+1}^{n}-2h_i^{n}+h_{i-1}^{n}}{(\Delta r)^2}\right] \tag{5.21}$$

$$+h_i^{n+1}\left\{\frac{S_c}{T\Delta t}\delta_{n,n}^{\alpha}+\frac{1}{(\Delta r)^2}\right\}$$

$$+h_i^{n}\left\{\frac{S_c}{T\Delta t}\delta_{n,n}^{\alpha}-\frac{1}{(\Delta r)^2}\right\}$$

$$+h_{i+1}^{n+1}\left\{\frac{1}{2r_i\Delta r}+\frac{1}{2(\Delta r)^2}\right\}$$

$$+h_{i-1}^{n+1}\left\{\frac{1}{2r_i\Delta r}+\frac{1}{2(\Delta r)^2}\right\}$$

$$+h_{i+1}^{n}\left\{\frac{1}{2r_i\Delta r}+\frac{1}{2(\Delta r)^2}\right\}$$

$$+h_{i-1}^{n}\left\{\frac{1}{2r_i\Delta r}+\frac{1}{2(\Delta r)^2}\right\}$$

$$-\frac{S_c}{T}\left[\sum_{j=0}^{n-1}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\ \delta_{n,j}^{\alpha}\right]$$

$$ah_i^{n+1}=bh_i^{n}+ch_{i+1}^{n+1}+dh_{i-1}^{n+1}+eh_{i+1}^{n}+fh_{i-1}^{n}-\frac{S_c}{T}\left[\sum_{j=0}^{n-1}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\ \delta_{n,j}^{\alpha}\right] \tag{5.22}$$

$$a\hat{\rho}_{n+1}e^{ik_m x}=b\hat{\rho}_n e^{ik_m x}+c\hat{\rho}_{n+1}e^{ik_m(x+\Delta x)}+d\hat{\rho}_{n+1}e^{ik_m(x-\Delta x)}$$
$$+e\hat{\rho}_n e^{ik_m(x+\Delta x)}+f\hat{\rho}_n e^{ik_m(x-\Delta x)}-\left[\sum_{j=0}^{n-1}\frac{\hat{\rho}_{j+1}e^{ik_m x}-\hat{\rho}_j e^{ik_m x}}{\Delta t}\delta_{n,j}^{\alpha}\right] \tag{5.23}$$

Then take multiple out to get:

$$a\hat{\rho}_{n+1}e^{ik_m x}=b\hat{\rho}_n e^{ik_m x}$$
$$+c\hat{\rho}_{n+1}e^{ik_m x}e^{ik_m\Delta x}+d\hat{\rho}_{n+1}e^{ik_m x}\ e^{ik_m-\Delta x}+e\hat{\rho}_n e^{ik_m x}\ e^{ik_m\Delta x}$$
$$+\ f\hat{\rho}_n e^{ik_m x}\ e^{ik_m\Delta x}-\left[\sum_{j=0}^{n-1}\frac{\hat{\rho}_{j+1}e^{ik_m x}-\hat{\rho}_j e^{ik_m x}}{\Delta t}\delta_{n,j}^{\alpha}\right] \tag{5.24}$$

Then divide both sides by e^{ixk_m} to obtain:

$$a\hat{\rho}_{n+1}=b\hat{\rho}_n+c\hat{\rho}_{n+1}e^{ik_m\Delta x}+d\hat{\rho}_{n+1}\ e^{-ik_m\Delta x}+e\hat{\rho}_n\ e^{ik_m\Delta x}+\ f\hat{\rho}_n\ e^{ik_m\Delta x}$$
$$-\sum_{j=0}^{n-1}\left[\frac{\hat{\rho}_{j+1}-\hat{\rho}_j}{\Delta t}\delta_{n,j}^{\alpha}\right] \tag{5.25}$$

$$\left(a-ce^{ik_m\Delta x}-de^{-ik_m\Delta x}\right)\hat{\rho}_{n+1}$$
$$=\left(b+ee^{ik_m\Delta x}+fe^{ik_m\Delta x}\right)\hat{\rho}_n\ -\sum_{j=0}^{n-1}\left[\frac{\hat{\rho}_{j+1}-\hat{\rho}_j}{\Delta t}\delta_{n,j}^{\alpha}\right] \tag{5.26}$$

If n=0 then:

(5.27)

$$\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right)\hat{\rho}_1 = \left(b + ee^{ik_m\Delta x} + fe^{ik_m\Delta x}\right)\hat{\rho}_0$$

And the condition required translates to:

$$\left|\frac{\hat{\rho}_1}{\hat{\rho}_0}\right| < 1$$

$$\frac{\hat{\rho}_1}{\hat{\rho}_0} = \frac{\left(b + ee^{-ik_m\Delta x} + fe^{ik_m\Delta x}\right)}{\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right)}$$

$$\left|\frac{\left(b + ee^{-ik_m\Delta x} + fe^{ik_m\Delta x}\right)}{\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right)}\right| < 1 \tag{5.28}$$

$e^{i\theta} = cos\theta + isin\theta$ and $e^{-i\theta} = cos\theta - isin\theta$

$$\left|\frac{b + 2c\ cos\theta}{a - 2c\ cos\theta}\right| < 1$$

If $a - 2c\ cos\theta > 0$

$$\left|\frac{b + 2c\ cos\theta}{a - 2c\ cos\theta}\right| < 1$$

$$b + 2c\ cos\theta < a - 2c\ cos\theta$$

$$b + 4\ cos\theta < a$$

Remembering that $a = \frac{S_c}{T\Delta t}\delta^{\alpha}_{n,n} + \frac{1}{(\Delta r)^2}$ and $b = \frac{S_c}{T\Delta t}\delta^{\alpha}_{n,n} - \frac{1}{(\Delta r)^2}$ then we get:

$$\frac{S_c}{T\Delta t}\delta^{\alpha}_{n,n} - \frac{1}{(\Delta r)^2} + 4cos\theta < \frac{S_c}{T\Delta t}\delta^{\alpha}_{n,n} + \frac{1}{(\Delta r)^2} \tag{5.29}$$

$$4cos\theta < \frac{2}{(\Delta r)^2}$$

$$cos\theta < \frac{1}{2(\Delta r)^2}$$

This therefore means that $\theta < cos^{-1}\left(\frac{1}{2(\Delta r)^2}\right)$

We assume that $\forall i > 0$ the formula is correct, then we want to verify at $n + 1$

$$\hat{\rho}_{n+1}\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right) = \hat{\rho}_n\left(b + ee^{ik_m\Delta x} + fe^{ik_m\Delta x}\right) - \sum_{j=0}^{n-1}\left(\frac{\hat{\rho}_{j+1} - \hat{\rho}_j}{\Delta t}\right)\delta_{n,j}^{\alpha}$$

$$\sum_{j=0}^{n-1}\left(\frac{\hat{\rho}_{j+1} - \hat{\rho}_j}{\Delta t}\right)\delta_{n,j}^{\alpha} = \frac{\hat{\rho}_{n-1} + \hat{\rho}_1}{\Delta t}\,\delta_{n,n-1}^{\alpha}$$

$$\hat{\rho}_{n+1}\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right) = \hat{\rho}_n\left(b + ee^{ik_m\Delta x} + fe^{ik_m\Delta x}\right) - \frac{\hat{\rho}_{n-1} + \hat{\rho}_1}{\Delta t}\delta_{n,n-1}^{\alpha} \quad (5.30)$$

$$|\hat{\rho}_{n+1}|\left|\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right)\right| = \left|\hat{\rho}_n\left(b + ee^{ik_m\Delta x} + fe^{ik_m\Delta x}\right) - \frac{\hat{\rho}_{n-1}}{\Delta t}\delta_{n,n-1}^{\alpha} + \frac{\hat{\rho}_1}{\Delta t}\delta_{n,1}^{\alpha}\right|$$

$$(a+b) \le |a| + |b|$$

$$|\hat{\rho}_{n+1}|\left|\left(a - ce^{ik_m\Delta x} - de^{-ik_m\Delta x}\right)\right| \le |\hat{\rho}_0|\left|\left(b + ee^{ik_m\Delta x} + fe^{ik_m\Delta x}\right)\right| + \left|\frac{\hat{\rho}_{n-1}}{\Delta t}\delta_{n,n-1}^{\alpha}\right| + \left|\frac{\hat{\rho}_1}{\Delta t}\delta_{n,1}^{\alpha}\right| \quad (5.31)$$

Then now using the recursive hypothesis, then we get:

$$|\hat{\rho}_{n+1}||a - 2c\,cos\theta| < |\hat{\rho}_0||b + 2c\cos\theta| + |\hat{\rho}_0|\frac{\delta_{n,n-1}^{\alpha}}{\Delta t} + |\hat{\rho}_0|\frac{\delta_{n,1}^{\alpha}}{\Delta t} \quad (5.32)$$

$$|\hat{\rho}_{n+1}||a - 2c\,cos\theta| < |\hat{\rho}_0|\left(b + 2c\cos\theta + \frac{\delta_{n,n-1}^{\alpha}}{\Delta t} + \frac{\delta_{n,1}^{\alpha}}{\Delta t}\right) \quad (5.33)$$

$$\frac{|\hat{\rho}_{n+1}|}{|\hat{\rho}_0|} < \frac{b + 2c\cos\theta + \frac{\delta_{n,n-1}^{\alpha}}{\Delta t} + \frac{\delta_{n,1}^{\alpha}}{\Delta t}}{a - 2c\,cos\theta} < 1 \quad (5.34)$$

$$\frac{S_c}{T\Delta t}\delta_{n,n}^{\alpha} - \frac{1}{(\Delta r)^2} + 4\,cos\theta < \frac{S_c}{T\Delta t}\delta_{n,n}^{\alpha} + \frac{1}{(\Delta r)^2} + \frac{\delta_{n,n-1}^{\alpha}}{\Delta t} + \frac{\delta_{n,1}^{\alpha}}{\Delta t} \quad (5.35)$$

$$cos\theta < \frac{1}{2(\Delta r)^2} + \frac{\delta_{n,n-1}^{\alpha}}{4\Delta t} + \frac{\delta_{n,1}^{\alpha}}{4\Delta t}$$

$$\theta < cos^{-1}\left(\frac{1}{2(\Delta r)^2} + \frac{\delta_{n,n-1}^{\alpha}}{4\Delta t} + \frac{\delta_{n,1}^{\alpha}}{4\Delta t}\right) \quad (5.36)$$

Therefore it can be conclusion can be taken that the numerical scheme is stable when

$$\boldsymbol{\theta < cos^{-1}\left(\frac{1}{2(\Delta r)^2} + \frac{\delta_{n,n-1}^{\alpha}}{4\Delta t} + \frac{\delta_{n,1}^{\alpha}}{4\Delta t}\right)}$$

3.2 Application of the Atangana-Batogna numerical scheme on confined zone

As opposed to the application of the classical AB numerical scheme explained and used in section 3.1 above, in this section we will apply the *non-classic* Atangana-Batogna numerical scheme which is given as follows:

$$u_{n+1} - u_n = \frac{h^\alpha}{\Gamma(\alpha)}\left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)f_n - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)f_{n-1}\right] \quad (6.1)$$

The derivation of this method is as follows: considering a general PDE

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = Lu(x,t) + Nu(x,t) \quad (6.2)$$

Considering that L symbolizes a linear operator and N being the nonlinear operator then applying the Laplace-transform on both sides of the equation to obtain:

$$\mathcal{L}\left(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right) = \mathcal{L}\big(Lu(x,t) + Nu(x,t)\big) \quad (6.3)$$

Taken into Caputo type fractional partial derivative, equation 6.3 will then be:

$${}_a^CD_t^\alpha u(p,t) = \mathcal{L}\big(Lu(x,t) + Nu(x,t)\big) \quad (6.4)$$

$${}_a^CD_t^\alpha u(t) = F\,(u,t) \quad (6.5)$$

Where $u(t) = u(p,t)$ and $F\,(u,t) = \mathcal{L}\big(Lu(x,t) + Nu(x,t)\big)$

With the application of the Caputo fractional Integral operator on the equation to obtain

$$u(t) - u(t_0) = \frac{1}{\Gamma\alpha}\int_0^t (t-\tau)^{\alpha-1}F(u,\tau)d\tau \quad (6.6)$$

Then when $t = t_{n+1}$

$$u(t_{n+1}) = u_0 + \frac{1}{\Gamma\alpha}\int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1}F(u,\tau)d\tau \quad (6.7)$$

When $t = t_n$

$$u(t_n) = u_0 + \frac{1}{\Gamma\alpha}\int_0^{t_n}(t_n - \tau)^{\alpha-1}F(u,\tau)d\tau \tag{6.8}$$

Then equation 6.7 and 6.8 result to:

$$u_{n+1} - u_n = \frac{1}{\Gamma\alpha}\left[\int_0^{t_{n+1}}(t_{n+1} - \tau)^{\alpha-1}F\,(u,\tau)d\tau - \int_0^{t_n}(t_n - \tau)^{\alpha-1}F\,(u,\tau)d\tau\right] \tag{6.9}$$

$$\int_0^{t_{n+1}}(t_{n+1} - \tau)^{\alpha-1}F(u,\tau)d\tau = \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1} - \tau)^{\alpha-1}F(u,\tau)d\tau \tag{6.10}$$

Then approximation of $F(u,t)$with a Lagrange polynomial below

$$P(t) \approx F(u,t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}F(u,t_n) + \frac{t - t_n}{t_{n-1} - t_n}F(u,t_{n-1}) \tag{6.11}$$

$$P(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}F_n + \frac{t - t_n}{t_{n-1} - t_n}F_{n-1} \tag{6.12}$$

The expression of the first fractional integral in equation (6.9) is given as follows:

$$\int_0^{t_{n+1}}(t_{n+1} - \tau)^{\alpha-1}F(u,\tau)d\tau =$$

$$\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1} - t)^{\alpha-1}\left(\frac{t - t_{n-1}}{t_n - t_{n-1}}F_n + \frac{t - t_n}{t_{n-1} - t_n}F_{n-1}\right)dt \tag{6.13}$$

$$\begin{aligned}\int_0^{t_{n+1}}(t_{n+1} - \tau)^{\alpha-1}F(u,\tau)d\tau \\ = \sum_{j=0}^{n}\left[\frac{F_n}{t_n - t_{n-1}}\int_{t_j}^{t_{j+1}}(t_{n+1} - t)^{\alpha-1}(t - t_{n-1})\,dt\right. \\ \left. + \frac{F_{n-1}}{t_{n-1} - t_n}\int_{t_j}^{t_{j+1}}(t_{n+1} - t)^{\alpha-1}\,(t - t_n)dt\right]\end{aligned} \tag{6.14}$$

$$= \sum_{j=0}^{n} \left[\frac{F_n}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1})\, dt - \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_n) dt \right] \quad (6.15)$$

And they implemented the change of variable in the following manner: let $y = t_{n+1} - t, dt = -dy,$ $and\ t = t_{n+1} - y$

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1})\, dt = \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} y^{\alpha-1} (t_{n+1} - y - t_{n-1})\, dy \quad (6.16)$$

$$= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y^{\alpha} - 2hy^{\alpha-1})\, dy \quad (6.17)$$

$$= \frac{1}{\alpha+1} [y^{\alpha+1}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} - \frac{2h}{\alpha} [y^{\alpha}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} \quad (6.18)$$

$$= \frac{1}{\alpha+1} \left((t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right) - \frac{2h}{\alpha} \left((t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right. \quad (6.19)$$

And on the other side

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (t - t_{n-1})\, dt = -\int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} y^{\alpha-1} (t_{n+1} - y - t_n)\, dy \quad (6.20)$$

$$= \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} (y^{\alpha} - hy^{\alpha-1})\, dy \quad (6.21)$$

$$= \frac{1}{\alpha+1} [y^{\alpha+1}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} - \frac{h}{\alpha} [y^{\alpha}]_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} \quad (6.22)$$

$$= \frac{1}{\alpha+1} \left((t_{n+1} - t_{j+1})^{\alpha+1} - (t_{n+1} - t_j)^{\alpha+1} \right) - \frac{h}{\alpha} \left((t_{n+1} - t_{j+1})^{\alpha} - (t_{n+1} - t_j)^{\alpha} \right. \quad (6.23)$$

Following this will be:

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} F(u,\tau) d\tau$$

$$= \frac{F_n}{h}\left\{\frac{1}{\alpha+1}\sum_{j=0}^{n}\left[\left(\left(t_{n+1}-t_{j+1}\right)^{\alpha+1}-\left(t_{n+1}-t_j\right)^{\alpha+1}\right)\right]\right.$$

$$\left.-\frac{2h}{\alpha}\sum_{j=0}^{n}[((t_{n+1}-t_{j+1})^{\alpha}-(t_{n+1}-t_j)^{\alpha})]\right\} \quad (6.24)$$

$$-\frac{F_{n-1}}{h}\left\{\frac{1}{\alpha+1}\sum_{j=0}^{n}\left[\left(\left(t_{n+1}-t_{j+1}\right)^{\alpha+1}-\left(t_{n+1}-t_j\right)^{\alpha+1}\right)\right]\right.$$

$$\left.-\frac{h}{\alpha}\sum_{j=0}^{n}[((t_{n+1}-t_{j+1})^{\alpha}-(t_{n+1}-t_j)^{\alpha})]\right\}$$

$$=\frac{F_n}{h}\left(\frac{1}{\alpha+1}(-(t_{n+1}-t_0)^{\alpha+1}) \;-\frac{2h}{\alpha}\left(-\left(t_{n+1}-t_{j+1}\right)^{\alpha}\right)\right)$$

$$-\frac{F_{n-1}}{h}\left(\frac{1}{\alpha+1}(-(t_{n+1}-t_0)^{\alpha+1}) \;-\frac{h}{\alpha}\left(-\left(t_{n+1}-t_j\right)^{\alpha}\right)\right). \quad (6.25$$

$$=\frac{F_n}{h}\left(\frac{-(n+1)^{\alpha+1}h^{\alpha+1}}{\alpha+1} + \frac{2h(n+1)^{\alpha}h^{\alpha}}{\alpha}\right)$$

$$-\frac{F_{n-1}}{h}\left(\frac{-(n+1)^{\alpha+1}h^{\alpha+1}}{\alpha+1} + \frac{h(n+1)^{\alpha}h^{\alpha}}{\alpha}\right). \quad (6.26)$$

$$= h^{\alpha}\left[\left(\frac{2(n+1)^{\alpha}}{\alpha}-\frac{(n+1)^{\alpha+1}}{\alpha+1}\right)F_n-\left(\frac{(n+1)^{\alpha}}{\alpha}-\frac{(n+1)^{\alpha+1}}{\alpha+1}\right)F_{n-1}\right]. \quad (6.27)$$

The evaluation of the second fractional integral in equation (6.9) is as follows:

$$\int_0^{t_n}(t_n-\tau)^{\alpha-1}F\,(u,\tau)d\tau = \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n-t)^{\alpha-1}\left(\frac{t-t_{n-1}}{t_n-t_{n-1}}F_n+\frac{t-t_n}{t_{n-1}-t_n}F_{n-1}\right)dt$$

$$\int_0^{t_n}(t_n-\tau)^{\alpha-1}F\,(u,\tau)d\tau \quad (6.28)$$

$$= \frac{F_n}{h}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n - t)^{\alpha-1}(t - t_{n-1})\,dt - \frac{F_{n-1}}{h}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n - t)^{\alpha-1}\,(t - t_n)dt$$

Similarly to equation (6.16), the change of variable is implemented as follows: let $y = t_n - t, dt = -dy,\ and\ t = t_n - y$

$$\int_0^{t_n}(t_n - \tau)^{\alpha-1}F\,(u,\tau)d\tau =$$

$$\frac{F_n}{h}\sum_{j=0}^{n-1}\int_{t_n-t_j}^{t_n-t_{j+1}} -(y)^{\alpha-1}(t - t_{n-1} - y)\,dy - \frac{F_{n-1}}{h}\sum_{j=0}^{n-1}\int_{t_n-t_j}^{t_n-t_{j+1}} y^{\alpha}\,dy$$

$$\int_0^{t_n}(t_n - \tau)^{\alpha-1}F\,(u,\tau)d\tau =$$

$$\frac{F_n}{h}\sum_{j=0}^{n-1}\int_{t_n-t_j}^{t_n-t_{j+1}}(y^{\alpha} - hy^{\alpha-1})\,dy - \frac{F_{n-1}}{h}\sum_{j=0}^{n-1}\int_{t_n-t_j}^{t_n-t_{j+1}} y^{\alpha}\,dy \tag{6.29}$$

$$\int_0^{t_n}(t_n - \tau)^{\alpha-1}F\,(u,\tau)d\tau = \frac{F_n}{h}\sum_{j=0}^{n-1}\left[\frac{y^{\alpha+1}}{\alpha+1} - \frac{h}{\alpha}y^{\alpha}\right]_{t_n-t_j}^{t_n-t_{j+1}} - \frac{F_{n-1}}{h}\sum_{j=0}^{n-1}\left[\frac{y^{\alpha+1}}{\alpha+1}\right]_{t_n-t_j}^{t_n-t_{j+1}} \tag{6.30}$$

$$\int_0^{t_n}(t_n - \tau)^{\alpha-1}F\,(u,\tau)d\tau$$
$$= \frac{F_n}{h}\sum_{j=0}^{n-1}\left(\frac{(t_n - t_{j+1})^{\alpha+1}}{\alpha+1} - \frac{h}{\alpha}(t_n - t_{j+1})^{\alpha} - \frac{(t_n - t_j)^{\alpha+1}}{\alpha+1} - \frac{h}{\alpha}(t_n - t_j)^{\alpha}\right)$$
$$- \frac{F_{n-1}}{h}\sum_{j=0}^{n-1}\left(\frac{(t_n - t_{j+1})^{\alpha+1}}{\alpha+1} - \frac{(t_n - t_j)^{\alpha+1}}{\alpha+1}\right) \tag{6.31}$$

$$= \frac{F_n}{h}\left\{\sum_{j=0}^{n-1}\left(\frac{\left(t_n - t_{j+1}\right)^{\alpha+1}}{\alpha+1} - \frac{\left(t_n - t_j\right)^{\alpha+1}}{\alpha+1}\right) - \frac{h}{\alpha}\sum_{j=0}^{n-1}\left(\left(t_n - t_{j+1}\right)^{\alpha} - \left(t_n - t_j\right)^{\alpha}\ \right)\right\}$$

$$- \frac{F_{n-1}}{h}\sum_{j=0}^{n-1}\left(\frac{\left(t_n - t_{j+1}\right)^{\alpha+1}}{\alpha+1} - \frac{\left(t_n - t_j\right)^{\alpha+1}}{\alpha+1}\right) \tag{6.32}$$

$$= \frac{F_n}{h}\left\{-\frac{(t_n - t_0)^{\alpha+1}}{\alpha+1} - \frac{h}{\alpha}\left(-(t_n - t_0)^{\alpha}\ \right)\right\} - \frac{F_{n-1}}{h(\alpha+1)}\left(-(t_n - t_0)^{\alpha+1}\right)$$

$$= \frac{F_n}{h}\left(\frac{-n^{\alpha+1}h^{\alpha+1}}{\alpha+1} + \frac{n^{\alpha}h^{\alpha+1}}{\alpha}\right) + \frac{n^{\alpha+1}h^{\alpha+1}}{h(\alpha+1)}F_{n-1} \tag{6.33}$$

This therefore means that:

$$\int_0^{t_n}(t_n - \tau)^{\alpha-1}F\ (u,\tau)d\tau = h^{\alpha}\left(\left(\frac{n^{\alpha}}{\alpha} - \frac{n^{\alpha+1}}{\alpha+1} + \right)F_n + \frac{n^{\alpha+1}}{(\alpha+1)}F_{n-1}\right) \tag{6.34}$$

Now equation (6.9) can be rewritten, substituting in the results to get:

$$u_{n+1} - u_n = \frac{h^{\alpha}}{\Gamma\alpha}\left[\left(\frac{2(n+1)^{\alpha}}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1}\right)F_n - \left(\frac{(n+1)^{\alpha}}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1}\right)F_{n-1}\right.$$

$$\left. - \left(\left(\frac{n^{\alpha}}{\alpha} - \frac{n^{\alpha+1}}{\alpha+1} + \right)F_n + \frac{n^{\alpha+1}}{(\alpha+1)}F_{n-1}\right)\right] \tag{6.35}$$

Equation (6.35) above can be simplified into equation (6.1) that will be used for both confined and unconfined zone. Atangana-Batogna, further elucidate in detail that if $\alpha = 1$, then we recover the CLASSIC Adam-Bashforth numerical scheme.

Equation (1.8) is substituted equation (6.1) with its derivation shown above.

$$\frac{S_c}{T}\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}$$

$$\frac{S_c}{T}(h_i^{n+1} - h_i^n)$$
$$= \frac{h^\alpha}{\Gamma(\alpha)}\left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right) f_i^n - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right) f_i^{n-1}\right]$$

Using equation (1.8) the definition of f_i^n and the definition of f_i^{n-1} are given as follows

$$F_i^n = \frac{T}{S_c}\left[\frac{1}{r_i}\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right] \quad (6.36)$$

$$F_i^{n-1} = \frac{T}{S_c}\left[\frac{1}{r_i}\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2}\right] \quad (6.37)$$

Now we substitute equation (1.8) into equation (6.1) using the defined equations (6.2) and (6.3) to get (6.4):

$$h_i^{n+1} - h_i^n = \frac{\Delta t^\alpha}{\Gamma(\alpha)}\left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)\frac{T}{S_c}\left[\frac{1}{r_i}\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right] - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)\frac{T}{S_c}\left(\frac{1}{r_i}\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2}\right)\right]$$

$$h_i^{n+1} = h_i^n\left\{1 - \frac{\Delta t^\alpha}{\Gamma(\alpha)}\left(\frac{4T(n+1)^\alpha - 2Tn^\alpha}{\alpha S_c(\Delta r)^2}\right) - \frac{\Delta t^\alpha}{\Gamma(\alpha)}\left(\frac{2Tn^{\alpha+1} - 2T(n+1)^{\alpha+1}}{(\alpha+1)S_c(\Delta r)^2}\right)\right\}$$

$$+h_{i+1}^{n}\left\{\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)\right.$$
$$\left.+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2Tn^{\alpha+1}-2T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_{i-1}^{n}\left\{\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{-2T(n+1)^{\alpha}+Tn^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{-Tn^{\alpha+1}+T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)\right.$$
$$\left.+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2Tn^{\alpha+1}-2T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_{i+1}^{n-1}\left\{\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)\right.$$
$$+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2Tn^{\alpha+1}-2T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)$$
$$-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)$$
$$\left.-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_{i-1}^{n-1}\left\{\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{2T(n+1)^{\alpha}-Tn^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}\right)\right.$$
$$+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)$$
$$\left.-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_{i}^{n-1}\left\{\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{-4T(n+1)^{\alpha}+2Tn^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}\right)+\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{-2Tn^{\alpha+1}+2T(n+1)^{\alpha+1}}{(\alpha+1)\,S_{c}(\Delta r)^{2}}\right)\right.$$
$$\left.-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{-2T(n+1)^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}-\frac{2Tn^{\alpha+1}+2T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_{i+1}^{n-2}\left\{-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)\right.$$
$$\left.-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_{i-1}^{n-2}\left\{-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{2\Delta rS_{c}r_{i}\alpha}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)2\Delta rS_{c}r_{i}}\right)\right.$$
$$\left.-\frac{\Delta t^{\alpha}}{\Gamma(\alpha)}\left(\frac{T(n+1)^{\alpha}}{\alpha S_{c}(\Delta r)^{2}}+\frac{Tn^{\alpha+1}-T(n+1)^{\alpha+1}}{(\alpha+1)S_{c}(\Delta r)^{2}}\right)\right\}$$

$$+h_i^{n-2}\left\{\frac{\Delta t^\alpha}{\Gamma(\alpha)}\left(\frac{2T(n+1)^\alpha}{\alpha S_c(\Delta r)^2}-\frac{2Tn^{\alpha+1}+2T(n+1)^{\alpha+1}}{(\alpha+1)S_c(\Delta r)^2}\right)\right\}$$

$$h_i^{n+1}=Ah_i^n+Bh_{i+1}^n+Ch_{i-1}^n+Dh_{i+1}^{n-1}+Eh_{i-1}^{n-1}+Fh_i^{n-1}+Gh_{i+1}^{n-2}+Hh_{i-1}^{n-2}+Ih_i^{n-2} \quad (6.38)$$

$$\hat{\rho}_{n+1}e^{ik_mx}=$$
$$A\hat{\rho}_ne^{ik_mx}+B\hat{\rho}_ne^{ik_m(x+\Delta x)}+C\hat{\rho}_ne^{ik_m(x-\Delta x)}+D\hat{\rho}_{n-1}e^{ik_m(x+\Delta x)}+E\hat{\rho}_{n-1}e^{ik_m(x-\Delta x)}+$$
$$F\hat{\rho}_{n-1}e^{ik_mx}+G\hat{\rho}_{n-2}e^{ik_m(x+\Delta x)}+H\hat{\rho}_{n-2}e^{ik_m(x-\Delta x)}+I\hat{\rho}_{n-2}e^{ik_mx} \quad (6.39)$$

Then we take the multiple out to obtain:

$$\begin{aligned}&\hat{\rho}_{n+1}e^{ik_mx}\\&=A\hat{\rho}_ne^{ik_mx}\\&+B\hat{\rho}_ne^{ik_mx}e^{ik_m\Delta x}+C\hat{\rho}_ne^{ik_mx}e^{-ik_m\Delta x}+D\hat{\rho}_{n-1}e^{ik_mx}e^{ik_m\Delta x}+E\hat{\rho}_{n-1}e^{ik_mx}e^{-ik_m\Delta x}\\&++F\hat{\rho}_{n-1}e^{ik_mx}+G\hat{\rho}_{n-2}e^{ik_mx}e^{ik_m\Delta x}+H\hat{\rho}_{n-2}e^{ik_mx}e^{-ik_m\Delta x}+I\hat{\rho}_{n-2}e^{ik_mx}\end{aligned} \quad (6.40)$$

Divide both sides by e^{ik_mx} to get:

$$\hat{\rho}_{n+1}=A\hat{\rho}_n+B\hat{\rho}_ne^{ik_m\Delta x}+C\hat{\rho}_ne^{-ik_m\Delta x}+D\hat{\rho}_{n-1}e^{ik_m\Delta x}+E\hat{\rho}_{n-1}e^{-ik_m\Delta x}$$
$$+F\hat{\rho}_{n-1}+G\hat{\rho}_{n-2}e^{ik_m\Delta x}+H\hat{\rho}_{n-2}e^{-ik_m\Delta x}+I\hat{\rho}_{n-2}$$

$$\begin{aligned}\hat{\rho}_{n+1}&=\big(A+Be^{ik_m\Delta x}\\&+Ce^{-ik_m\Delta x}\big)\hat{\rho}_n+\big(De^{ik_m\Delta x}+Ee^{-ik_m\Delta x}+F\big)\hat{\rho}_{n-1}+\big(Ge^{ik_m\Delta x}\\&+He^{-ik_m\Delta x}+I\big)\hat{\rho}_{n-2}\end{aligned} \quad (6.41)$$

Let $k_m\Delta x=\theta$

$e^{i\theta}=cos\theta+isin\theta$ and $e^{-i\theta}=cos\theta-isin\theta$

$e^{i\theta}+e^{-i\theta}=cos\theta+isin\theta+cos\theta-isin\theta=2cos\theta$

$$\hat{\rho}_{n+1}=[A+B(cos\theta+isin\theta)$$
$$+C(cos\theta-isin\theta)]\hat{\rho}_n+[D(cos\theta+isin\theta)+E(cos\theta-isin\theta)+F]\hat{\rho}_{n-1}$$
$$+[G(cos\theta+isin\theta)+H(cos\theta-isin\theta)+I]\hat{\rho}_{n-2}$$

$$\hat{\rho}_{n+1} = [A + (B + C)cos\theta + i(B - C)sin\theta]\hat{\rho}_n + [(D + E)cos\theta + i(D - E)sin\theta + F]\hat{\rho}_{n-1} + [(G + H)cos\theta + i(G - H)sin\theta + I]\hat{\rho}_{n-2} \tag{6.42}$$

Now we shall prove the stability using the inductive approach on the natural number. For $n = 0$, $\hat{\rho}_{-1}$and $\hat{\rho}_{-2}$ is not applicable and therefore we have:

$$\hat{\rho}_1 = [A + (B + C)cos\theta + (B - C)isin\theta]\hat{\rho}_0$$

$$\frac{\hat{\rho}_1}{\hat{\rho}_0} = [A + (B + C)cos\theta + i(B - C)sin\theta]$$

$$\frac{|\hat{\rho}_1|}{|\hat{\rho}_0|} < 1 \Rightarrow |[A + (B + C)cos\theta + i(B - C)sin\theta]| < 1 \tag{6.43}$$

$$(A + (B + C)cos\theta)^2 + i(B - C)^2 sin^2\theta \ < 1 \tag{6.44}$$

$$A^2 + 2A(B + C)cos\ \theta + (B + C)^2 cos\ ^2\theta + (B - C)^2 sin^2\theta < 1$$

$$A^2 + 2A(B + C)cos\ \theta + (B^2 + 2BC + C^2)\ cos\ ^2\theta + (B^2 - 2BC + C^2)\ sin^2\theta < 1$$

$$A^2 + 2A(B + C)cos\ \theta + B^2(cos\ ^2\theta + sin^2\theta) + 2BC\ (cos\ ^2\theta - sin^2\theta)\ + C^2(cos\ ^2\theta + sin^2\theta) < 1$$

$$A^2 + 2A(B + C)cos\ \theta + B^2 + C^2 + 2BC\ (1 - 2sin^2\theta)\ < 1 \tag{6.46}$$

Therefore the 1st condition is given in equation (6.47) below:

$$A^2 + B^2 + C^2\ + 2A(B + C)cos\ \theta < 1 + 2BC(2sin^2\theta - 1) \tag{6.47}$$

We assume that the formula is true for $\forall\ n > 0$, then we verify at $n + 1$

$$\hat{\rho}_{n+1} = \{[A + (B + C)cos\theta + (B - C)isin\theta]\hat{\rho}_n + [(D + E)cos\theta + (D - E)isin\theta + F]\hat{\rho}_{n-1} + [(G + H)cos\theta + (G - H)isin\theta + I]\hat{\rho}_{n-2}\} \tag{6.48}$$

Let $\mathrm{A}_1 = A + (B + C)cos\theta$; $\mathrm{B}_1 = (B - C)sin\theta$; $\mathrm{A}_2 = F + (D + E)cos\theta$; $\mathrm{B}_2 = (D - E)sin\theta$; $\mathrm{A}_3 = (G + H)cos\theta + I$; $\mathrm{B}_3 = (G - H)sin\theta$ then we will have:

$$|\hat{\rho}_{n+1}| < |A_1 + iB_1||\hat{\rho}_n| + |A_2 + iB_2||\,\hat{\rho}_{n-1}| + |A_3 + iB_3||\hat{\rho}_{n-2}| \tag{6.49}$$

Given the stability condition in equation (6.47) above,

$$|\hat{\rho}_{n+1}| < \; |A_1 + iB_1||\hat{\rho}_0| + |A_2 + iB_2||\,\hat{\rho}_0| + |A_3 + iB_3||\hat{\rho}_0| \tag{6.50}$$

$$\frac{|\hat{\rho}_{n+1}|}{|\hat{\rho}_0|} < |(A_1 + iB_1) + (A_2 + iB_2) + (A_3 + iB_3)| < 1 \tag{6.51}$$

$$\frac{|\hat{\rho}_{n+1}|}{|\hat{\rho}_0|} < |(A_1 + A_2 + A_3) + i(B_1 + B_2 + B_3)| < 1 \tag{6.52}$$

$$\sqrt{(A_1 + A_2 + A_3)^2 + (B_1 + B_2 + B_3)^2} < 1 \tag{6.53}$$

$$(A_1 + A_2 + A_3)^2 + (B_1 + B_2 + B_3)^2 < 1 \tag{6.54}$$

Now it can be concluded that this numerical scheme, the non-classical Atangana-Batogna is stable if and only if:

$$\boldsymbol{A^2 + B^2 + C^2 \; + 2A(B + C)\cos\theta < 1 + 2BC(2sin^2\theta - 1)} \textbf{ and}$$

$$\boldsymbol{(A_1 + A_2 + A_3)^2 + (B_1 + B_2 + B_3)^2 < 1}$$

Remembering that $\mathrm{A}_1 = A + (B + C)$; $\mathrm{A}_2 = F + (D + E)cos\theta$; $\mathrm{A}_3 =$

$$\mathrm{B}_1 = (B - C)sin\theta; \mathrm{B}_2 = (D - E)sin\theta; \mathrm{B}_3 = (G - H)sin\theta$$

Then equation (6.54) can be rewritten as:

$$\left[(A + (B + C)cos\theta) + (F + (D + E)cos\theta) + \left((G + H)cos\theta + I\,\right)\right]^2 + [(B - C)sin\theta + (D - E)sin\theta + (G - H)sin\theta]^2 < 1$$

3.3 APPLICATION OF THE ATANGANA-BATOGNA NUMERICAL SCHEME ON UNCONFINED ZONE

In this section, we apply the non-classical Atangana-Batogna numerical scheme in the unconfined zone as applied to the confined zone in section 3.5 above.

The definition of F_{n} and $F_{\mathrm{n-1}}$ is given in equation (2.5) and (2.6) now we substitute this equation of the unconfined zone $\frac{\partial h}{\partial t} = \frac{K}{S_y}\frac{h}{r}\frac{\partial h}{\partial r} + \frac{K}{S_y}\left(\frac{\partial h}{\partial r}\right)^2 + \frac{K}{S_y}h\frac{\partial^2 h}{\partial r^2}$ into equation (6.1) using the two defined equations.

$$u_{n+1} - u_n = \frac{h^\alpha}{\Gamma(\alpha)}\left[\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)f_n - \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)f_{n-1}\right]$$

$$\begin{aligned} h_i^{n+1} - h_i^n = \frac{t^\alpha}{\Gamma(\alpha)}\Bigg[\Bigg(&\frac{2(n+1)^\alpha - n^\alpha}{\alpha} \\ &+ \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\Bigg)\left(\frac{K}{S_y}\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)\frac{h_i^n}{r_i}\right. \\ &+ \frac{K}{S_y}\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)^2 \\ &\left.+ \frac{K}{S_y}h_i^n\left(\frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right)\right) \\ &- \Bigg(\frac{(n+1)^\alpha}{\alpha} \\ &+ \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\Bigg)\left(\frac{K}{S_y}\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)\frac{h_i^{n-1}}{r_i}\right. \\ &+ \frac{K}{S_y}\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)^2 \\ &\left.+ \frac{K}{S_y}h_i^{n-1}\left(\frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2}\right)\right)\Bigg] \end{aligned} \tag{7.1}$$

$$
\begin{aligned}
h_i^{n+1} - h_i^n = \frac{t^\alpha}{\Gamma(\alpha)} \Bigg[& \Bigg(\frac{2Kh_i^n (n+1)^\alpha - Kn^\alpha}{\alpha S_y r_i} \\
& + \frac{Kh_i^n n^{\alpha+1} - Kh_i^n (n+1)^{\alpha+1}}{(\alpha+1) K S_y r_i} \Bigg) \Bigg(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} \Bigg) \\
& + \Bigg(\frac{2K(n+1)^\alpha - Kn^\alpha}{\alpha S_y r_i} \\
& + \frac{Kn^{\alpha+1} - K(n+1)^{\alpha+1}}{(\alpha+1) K S_y r_i} \Bigg) \Bigg(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} \Bigg)^2 \\
& + \Bigg(\frac{Kh_i^n 2(n+1)^\alpha - Kh_i^n n^\alpha}{S_y \alpha} \\
& + \frac{Kh_i^n n^{\alpha+1} - Kh_i^n (n+1)^{\alpha+1}}{S_y (\alpha+1)} \Bigg) \Bigg(\frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} \\
& + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} \Bigg) \\
& - \Bigg(\frac{Kh_i^{n-1} (n+1)^\alpha}{\alpha S_y r_i} + \frac{Kh_i^{n-1} n^{\alpha+1} - Kh_i^{n-1} (n+1)^{\alpha+1}}{(\alpha+1) \alpha S_y r_i} \Bigg) \Bigg(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} \\
& + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r} \Bigg) \\
& - \Bigg(\frac{Kh_i^{n-1} (n+1)^\alpha}{\alpha S_y} \\
& + \frac{Kh_i^{n-1} n^{\alpha+1} - Kh_i^{n-1} (n+1)^{\alpha+1}}{(\alpha+1) \alpha S_y} \Bigg) \Bigg(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r} \Bigg)^2 \\
& - \Bigg(\frac{Kh_i^{n-1} (n+1)^\alpha}{\alpha S_y} \\
& + \frac{Kh_i^{n-1} n^{\alpha+1} - Kh_i^{n-1} (n+1)^{\alpha+1}}{(\alpha+1) S_y} \Bigg) \Bigg(\frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2} \\
& + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2} \Bigg) \Bigg]
\end{aligned}
\tag{7.2}
$$

$$h_i^{n+1} - h_i^n = \frac{t^\alpha}{\Gamma(\alpha)}\Bigg[\left(\frac{2Kh_i^n(n+1)^\alpha - Kn^\alpha}{\alpha S_y r_i} + \frac{Kh_i^n n^{\alpha+1} - Kh_i^n(n+1)^{\alpha+1}}{(\alpha+1)KS_y r_i}\right)\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r}\right.$$

$$\left. + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)$$

$$+ \left(\frac{2K(n+1)^\alpha - Kn^\alpha}{\alpha S_y r_i}\right.$$

$$\left. + \frac{Kn^{\alpha+1} - K(n+1)^{\alpha+1}}{(\alpha+1)KS_y r_i}\right)\left(\frac{h_{i+1}^n - h_{i-1}^n}{4\Delta r} + \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right)^2$$

$$+ \left(\frac{Kh_i^n 2(n+1)^\alpha - Kh_i^n n^\alpha}{S_y \alpha}\right.$$

$$\left. + \frac{Kh_i^n n^{\alpha+1} - Kh_i^n(n+1)^{\alpha+1}}{S_y(\alpha+1)}\right)\left(\frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{(\Delta r)^2} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right)$$

$$- \left(\frac{Kh_i^{n-1}(n+1)^\alpha}{\alpha S_y r_i} + \frac{Kh_i^{n-1} n^{\alpha+1} - Kh_i^{n-1}(n+1)^{\alpha+1}}{(\alpha+1)\alpha S_y r_i}\right)\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r}\right.$$

$$\left. + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)$$

$$- \left(\frac{Kh_i^{n-1}(n+1)^\alpha}{\alpha S_y}\right.$$

$$\left. + \frac{Kh_i^{n-1} n^{\alpha+1} - Kh_i^{n-1}(n+1)^{\alpha+1}}{(\alpha+1)\alpha S_y}\right)\left(\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{4\Delta r} + \frac{h_{i+1}^{n-2} - h_{i-1}^{n-2}}{4\Delta r}\right)^2$$

$$- \left(\frac{Kh_i^{n-1}(n+1)^\alpha}{\alpha S_y} + \frac{Kh_i^{n-1} n^{\alpha+1} - Kh_i^{n-1}(n+1)^{\alpha+1}}{(\alpha+1)S_y}\right)\left(\frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{(\Delta r)^2}\right.$$

$$\left. + \frac{h_{i+1}^{n-2} - 2h_i^{n-2} + h_{i-1}^{n-2}}{(\Delta r)^2}\right)\Bigg]$$

CHAPTER 4: NUMERICAL SIMULATIONS

This chapter entails presentations and discussions of the numerical simulations for the numerical solutions obtained in chapter 3. There are many numerical methods used for simulation of complex problems, among which the finite difference method (FD) and the finite element method (FE) are the most commonly used. There is quite a number of existing software packages that are used in numerical modelling, however in this study, we have used Matrix laboratory commonly known as MATLAB. MATLAB is a software package developed by MathWorks, it allows computation, data analysis, development of algorithms, simulation and modelling, and produces graphical displays and graphical user interfaces (Knight, 2000). In order to successfully present numerical simulations, we develop mathematical code using the numerical schemes presented in chapter 3 to simulate numerical solution for different values of alpha. Theoretical parameter values are used to obtain simulations and numerical coding is as follows:

Numerical coding

```
clc
clear
close all

Lmax = 100;
Tmax = 100;
maxt = 10;
h = Tmax/maxt;
n_spaces = 10;
dx = (Lmax/n_spaces);

%Parameter
alpha1 = 10;
Sc =0.0001;
Sy = 0.001;
B = 2;
T = 400;
K = 40;

Initial condition
for i = 1:(n_spaces+1)
    x(i) = (i-1)*dx;
    H(i,1) = 1;
end

%Vector
for n=1:maxt+1
    t(n) = (n-1)*h;
end

a1 = 1/(alpha1*(dx^2));
a2 = 1/(alpha1*dx);
```

```
a3 = Sy/(alpha1*T*dx);
a4 = T/(Sc*dx);
a5 = T/(Sc*(dx^2));

alpha = 0.53;
Ma = 1;

for n=0:maxt-1

for j=2:n_spaces

 if 0 < H(j+1,n+1) && H(j+1,n+1) <= B
           dh = a1*(H(j-1,n+1)-2*H(j,n+1)+H(j+1,n+1))+a2*(H(j,n+1)-
H(j+1,n+1))+a3*(H(j,n+1)-H(j+1,n+1))
  end

 if H(j+1,n+1) > B
           dh = a4*(H(j,n+1)-H(j+1,n+1))+a5*(H(j-1,n+1)-
2*H(j,n+1)+H(j+1,n+1)); end
        H(j,n+2) = H(j,n+1) + (((1-alpha)/(Ma)) + ((3*alpha*h)/(2*Ma)))*dh -
...
                              (((1-alpha)/(Ma)) + ((alpha*h)/(2*Ma)))*dh;
    end

end

figure
surf(x',t,H')
xlabel('space')
ylabel('time')
zlabel('h[r,t]')

figure
contour(H)
xlabel('space')
ylabel('time')
```

The hydraulic parameter values used to obtain the numerical simulations and contour plots of the numerical solutions in figure 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13 below include a Storage coefficient of 0.0001, a specific yield of 0.001, hydraulic conductivity value of 40 m/d with the transmissivity value equals to 400m^2/d and an aquifer thickness of 2m. Figure 4 and figure 6 shows numerical simulations for $\alpha = 1$ and 0.8, respectively; however, we notice how the hydraulic head (h) as a function of space and time, changes when the value of alpha decreases; this gradual change of h is depicted in figure 8, 10 and 12.

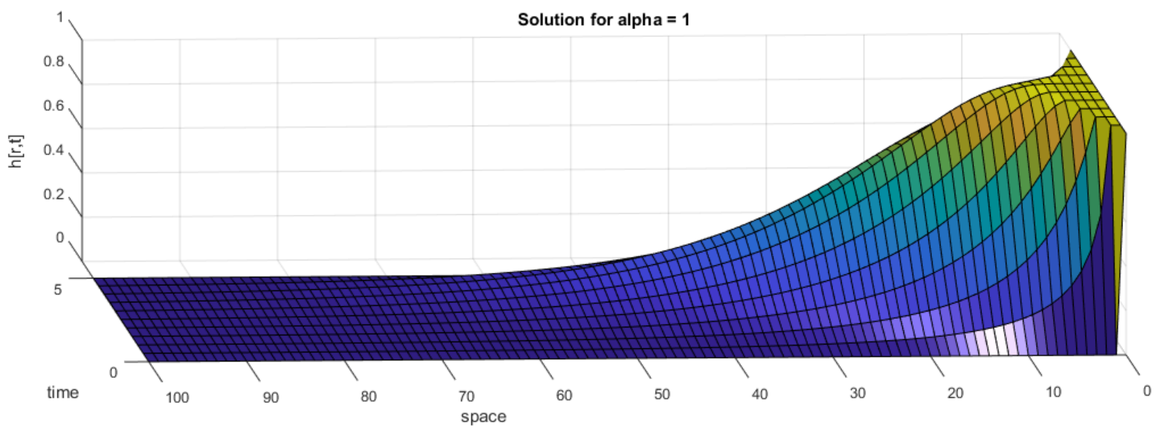

Figure 4: Numerical simulation for alpha equals 1 ($\alpha = 1$)

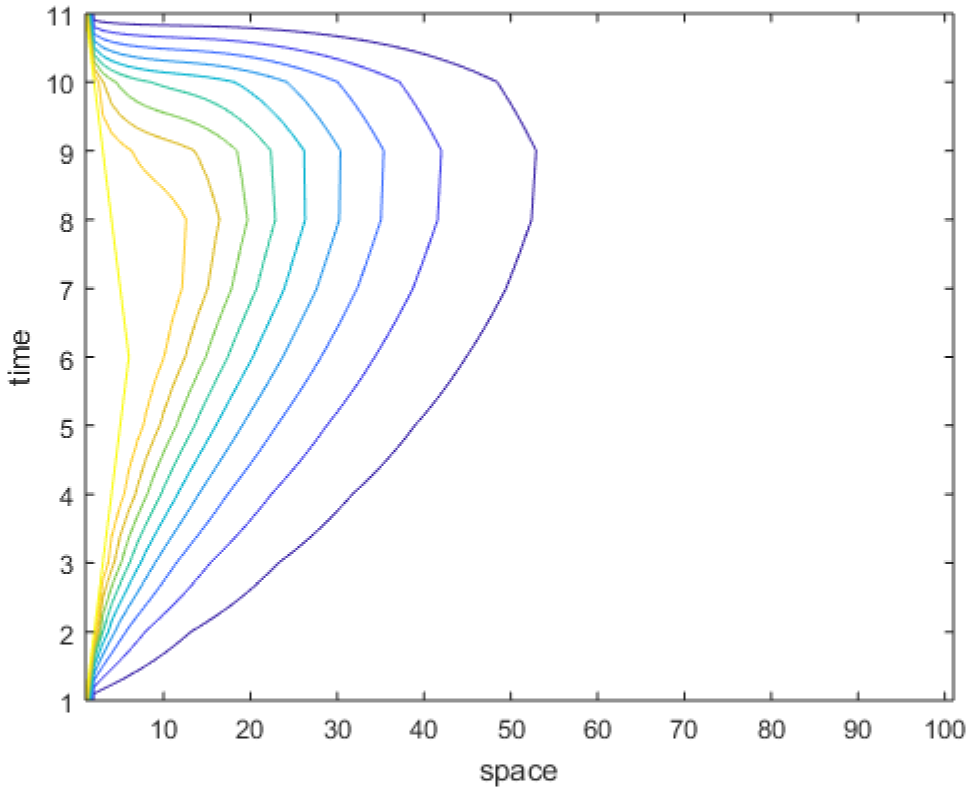

Figure 5: Contour plot of numerical solution for alpha = 1

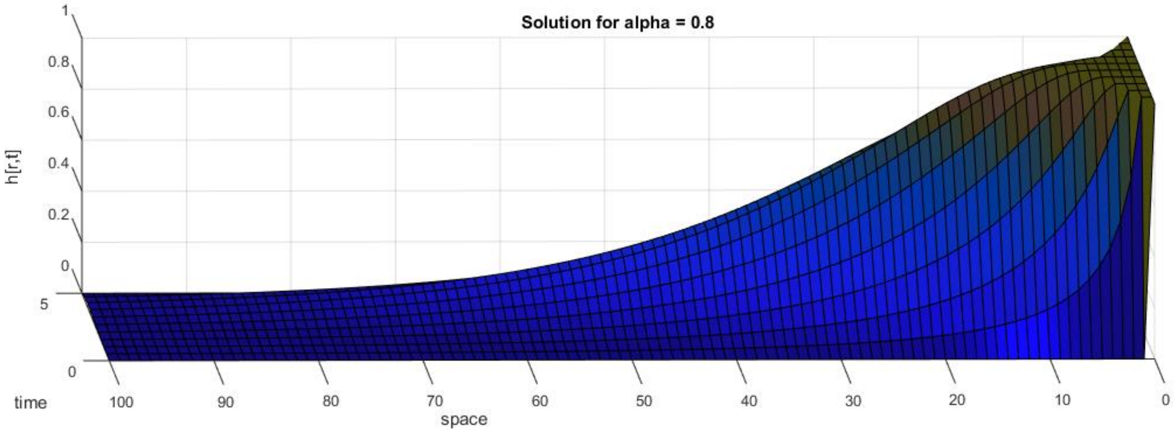

Figure 6: Numerical simulation for alpha equals 0.8 ($\alpha = 0.8$)

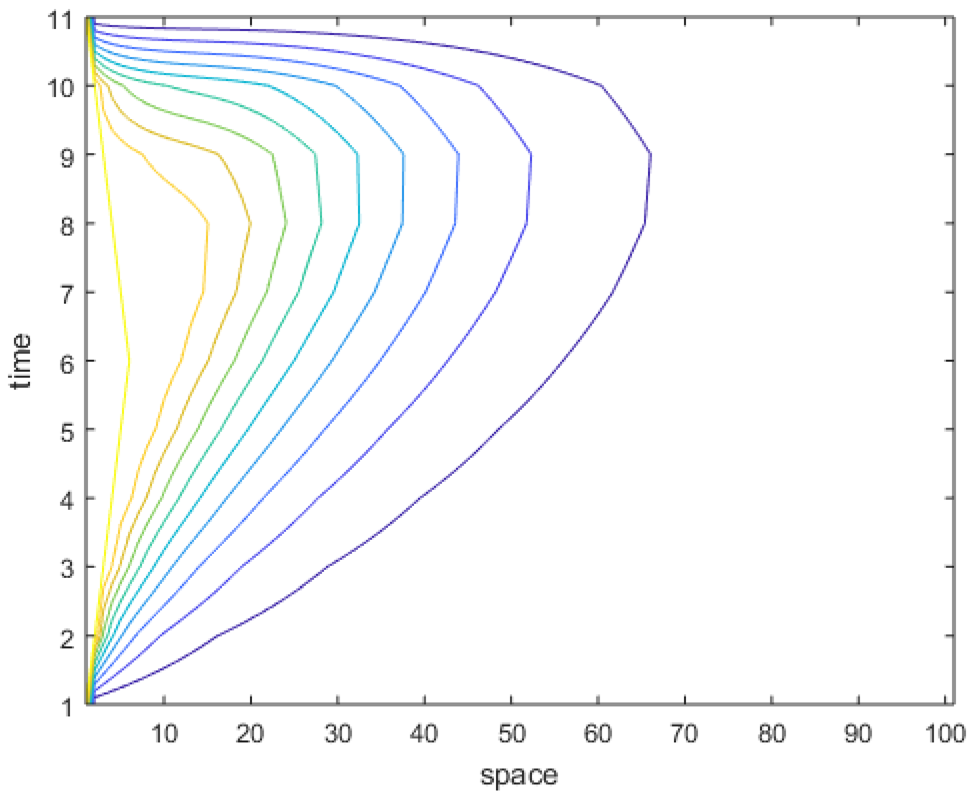

Figure 7: Contour plot of numerical solution for alpha = 0.8

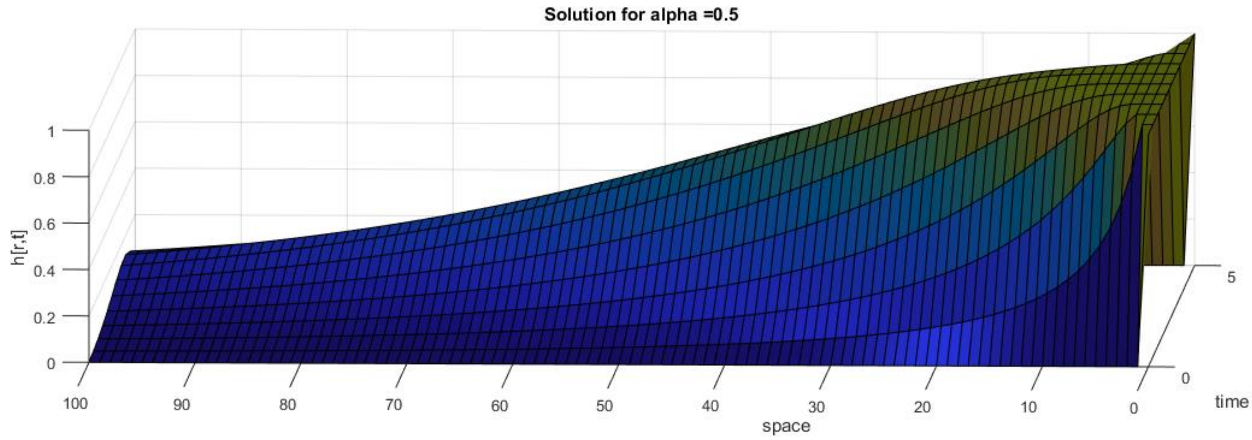

Figure 8: Numerical simulation for alpha equals 0.5

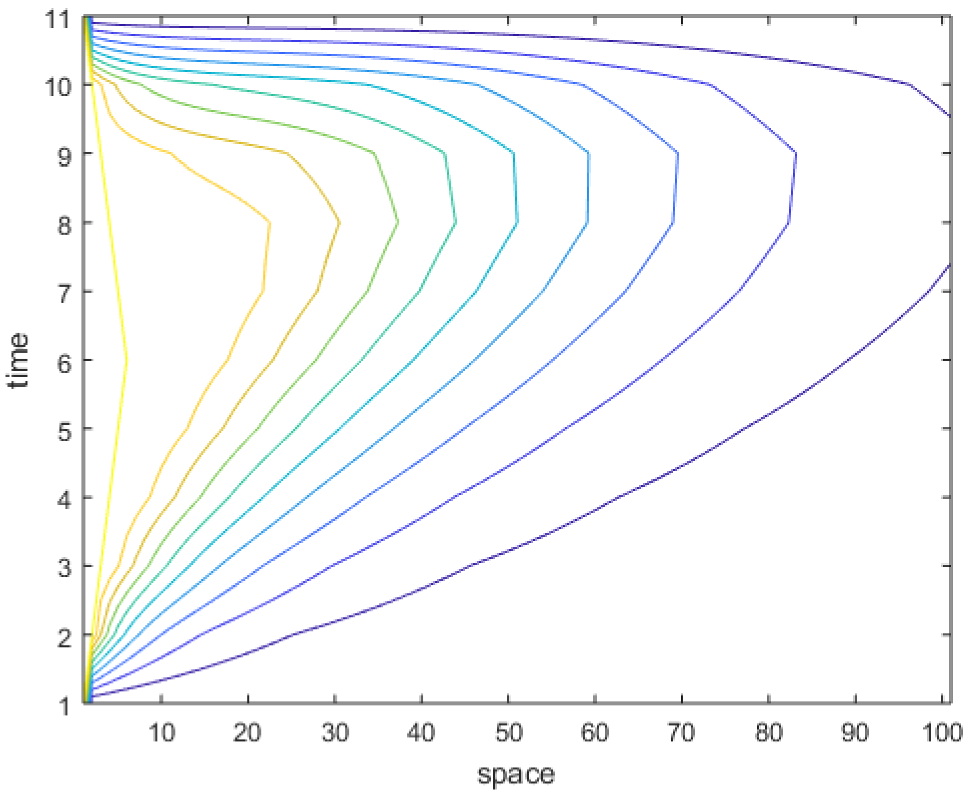

Figure 9: Contour plot of numerical solution for alpha = 0.5

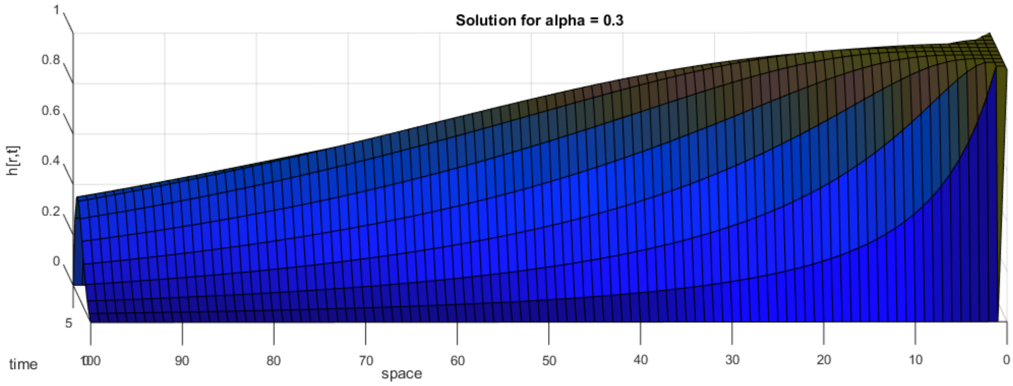

Figure 10: Numerical simulation for alpha equals 0.3

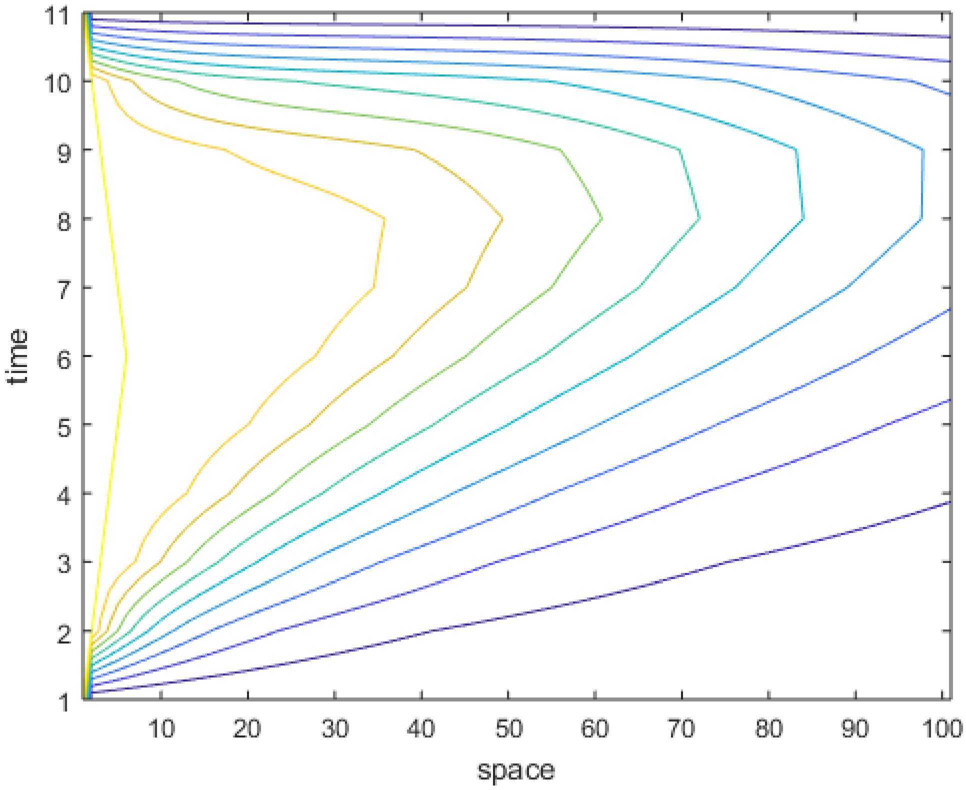

Figure 11 : Numerical simulation for alpha equals 0.3

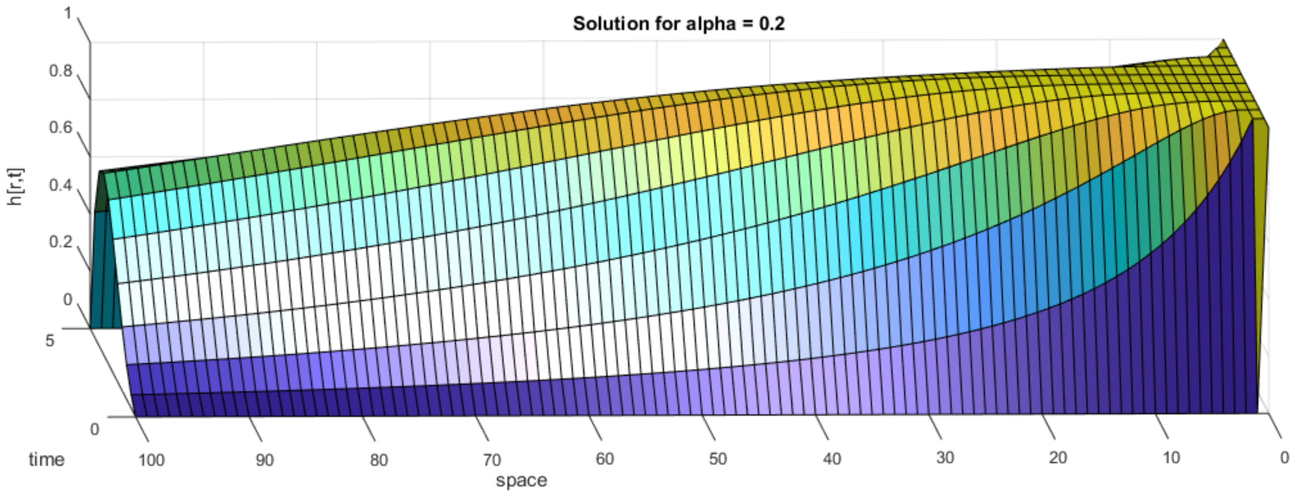

Figure 12: Numerical simulation for alpha equals 0.2

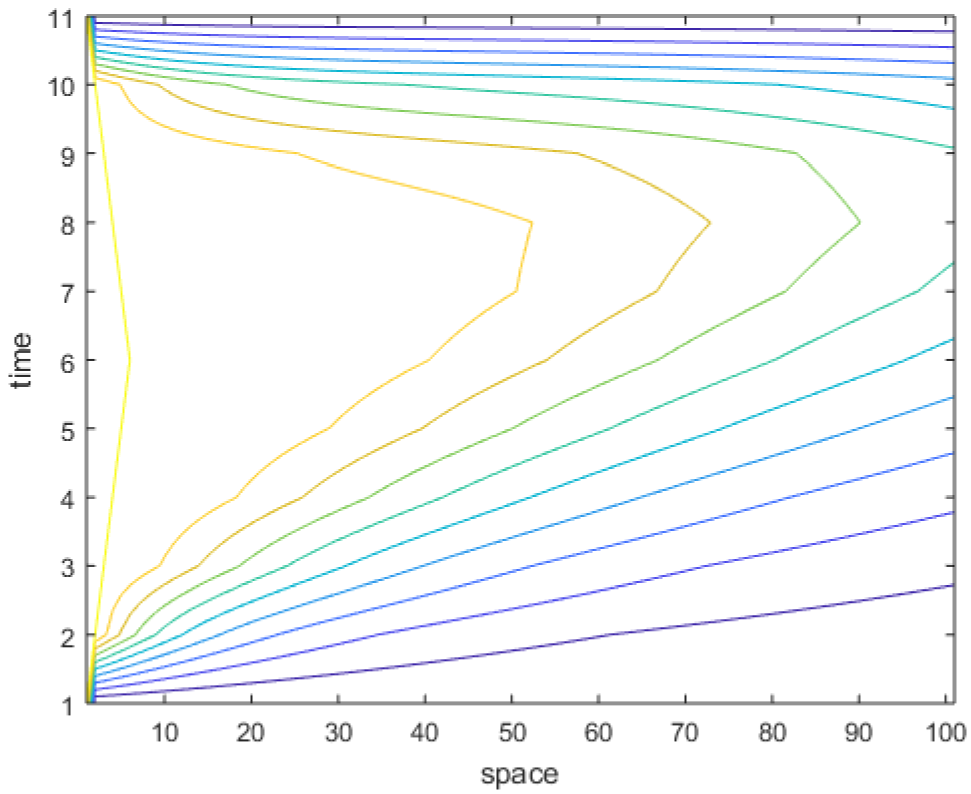

Figure 13: Numerical simulation for alpha equals 0.2

We take a closer look at the contour plots for different values of alpha; we notice the change in water within the aquifer as shown by the hydraulic head. The water level decreases as the value of alpha increases from 0.2 to 1, a decrease in water level means the water table becomes deeper and the distance between the bottom of the top confining layer and the piezometric surface is becoming smaller. These contour plots of numerical simulations serve as a 2D support of the aforementioned numerical simulations. Figure 8, 10 and 12 depict a gradual increase in hydraulic head as the value of alpha decreases from 0.5 to 0.2 and this is also shown by the increase in the water level. This points out to what is expected in the real world as captured by the Atangana-Baleanu non-local operator.

CHAPTER 5: CONCLUSION

To capture heterogeneities underlying the sub-surface formation within which the groundwater flows has been a worry among researchers working in this field in the last years. We shall recall that many countries nowadays rely on the fresh water found within the sub-surface, for industrial, domestics and agricultural purposes. Thus this source of fresh water needs to be monitored and protected, the monitoring is achieved via mathematical models, more importantly, their abstraction have to be fully monitored as this can cause damages within the aquifers. Mathematical models used differential operators and one can list four types in the literature. The first type based on the concept of rate of changed has been intensively used in the last decades. However, from well-established research, it was revealed that, this operator cannot handle or capture heterogeneities found in the geological formation called aquifer.

One of the big challenges is perhaps to model the conversion from confined to unconfined aquifer during intensive abstraction. Few research have been done in this direction, however, much work still needs to be done to fully understand this conversion.

We have reviewed the existing model, and pointed out their limitations. We presented a detailed study of numerical analysis of the nonlinear case using a novel numerical scheme that uses the fundamental theorem of classical calculus and the well-established Lagrange polynomial and the Laplace transform operator. We presented detailed stability analysis and also the convergence. Due to visco-elasticity of the geological formation and also other heterogeneities, that define the flow of water within the sub-surface, we argue that, the existing model cannot capture such a flow and thus we replaced the local differential operator by non-local with Mittag-Leffler function. The new model has the advantages that, it can capture flow following the power law and that following the decay law. The new model is also able to capture flow with Gaussian and non-Gaussian distribution and finally it can replicate very efficiently normal and sub-diffusion.

In addition, observations from the numerical simulations presented for different values of alpha show a true reflection of the actual physical groundwater system when using the Atangana-Baleanu operator in Caputo sense more than the system depicted when using the classical operators.

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