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Fiducial Inference based on Order Statistics in Location-Scale and Log-Location-Scale Families

by

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Declaration of Authorship

I, Petrus Tweuthigilwa Iiyambo, hereby declare that this work, submitted for the degree of Doctor of Philosophy in Statistics, at the University of the Free State, is my own original work and has not previously been submitted, for degree purposes or otherwise, to any other institution of higher learning. I further declare that all sources cited or quoted are indicated and acknowledged by means of a comprehensive list of references. Copyright hereby cedes to the University of the Free State.

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Abstract

Both exact and approximate statistical inference for the parameters and the quantiles of location-scale, log-location-scale and location-scale-shape families of distributions are usually derived from likelihood-based methods. However, parameter estimation using exact and approximate maximum likelihood-based methods can be difficult and may require extensive programming, especially when dealing with censored samples. In some cases, maximum likelihood estimation based on censored samples may encounter convergence problems. Alternative methods of parameter estimation in location-scale, log-location-scale and location-scale-shape distributions have been developed by many researchers.

In this thesis we develop exact rank-based conventional and fiducial generalized methods of inference for location and scale parameters of distributions belonging to the location-scale and log-location-scale families using the generalized least squares approach. Furthermore, we propose rank-based fiducial generalized methods of inference for location, scale and shape parameters of distributions belonging to location-scale-shape families of distributions using rank-based, iterative generalized least squares methods, and a Gibbs sampler. We compare through simulation and practical applications the results, inter alia the coverage probabilities and average lengths of conventional and fiducial generalized confidence intervals for the parameters and quantiles of location-scale, log-location-scale, and location-scale-shape distributions, obtained using our proposed methods of inference with alternative methods existing in literature, for example, exact and approximate maximum likelihood-based methods.

For the cases of location-scale and log-location-scale families of distributions involving one or two-sample situations, our simulation results show that the

proposed exact rank-based (conventional and fiducial generalized) methods are very competitive with exact and approximate maximum likelihood-based methods in terms of relative lengths of confidence intervals for the model parameters, parameter contrasts and quantiles of distribution. Moreover, rank-based methods produce confidence intervals for the model parameters, parameter contrasts and quantiles of distribution with good properties. In terms of practical application, our proposed rank-based methods produce confidence intervals for the model parameters, parameter contrasts and quantiles of distribution that are very close to the confidence intervals calculated using exact maximum likelihood-based methods.

When calculating rank-based fiducial generalized confidence intervals for the model parameters and quantiles of location-scale-shape family of distributions (when the shape parameter $\xi > 0$), using iterative generalized least squares approach based on Gibbs sampler algorithm, two different parametrizations, namely the θ and θ^* parametrizations are investigated. Our simulation results show that the θ^* parametrization produces, overall, better and more stable estimates compared to θ parametrization. Finally, results of illustrative examples of data modelled by the Generalized Extreme Value distribution for the case when $\xi > 0$, are comparable to the results based on the same data obtained using Bayesian methods.

Abbreviations and notations

Unless stated otherwise, the following abbreviations and notations were commonly used in this thesis:

AVL	average length
BLUEs	best linear unbiased estimators
CCPQ	conditional conventional pivotal quantity
cdf	cumulative distribution function
CFGPQ	conditional fiducial generalized pivotal quantity
CI	confidence interval
CP	coverage probability
CPQ	conventional pivotal quantity
e.g.	for example
FGCI	fiducial generalized confidence interval
FGCR	fiducial generalized confidence region
FGI	fiducial generalized inference
FGPQ	fiducial generalized pivotal quantity
GCI	generalized confidence interval
GEV	Generalized Extreme Value
GLM	general linear model
GLS	generalized least squares

GP	Generalized Pareto
GPQ	generalized pivotal quantity
GPV	generalized p-value
GTV	generalized test variable
i.i.d.	independent and identically distributed
LLS	log-location-scale
LM	likelihood moment
LS	location-scale
LSS	location-scale-shape
mgf	moment generating function
ML	maximum likelihood
MLEs	maximum likelihood estimates
MOM	methods of moments
NR	Newton-Raphson
pdf	probability density function
POME	principal of maximum entropy
POT	peaks-over-threshold
PQ	pivotal quantity
PWM	probability weighted moments
RHS	right hand side

SCB	simultaneous confidence band
SCR	simultaneous confidence region
$\log N$	Lognormal distribution
$B(\cdot)$	complete beta function
$Cov(\cdot)$	covariance matrix of a random vector
$E(\cdot)$	operator of mathematical expectation
Exp	Exponential distribution
Gum	Gumbel distribution
H	linear predictor matrix
$L(\cdot)$	likelihood function
$LLogist$	Log-Logistic distribution
$Logist$	Logistic distribution
N	Normal distribution
$\mathbb{R}$	a set of real numbers
U	standard Uniform variate
$Unif$	Uniform distribution
V	covariance matrix of Z or $\log(Z^{-1})$
$Var(\cdot)$	variance of a random variable
$Weib$	Weibull distribution
X	intercept matrix of a GLM

Y	a vector of the order statistics of the sample of size n
Z	a vector of the order statistics of the standardized random variate
d	difference of two p −quantiles of the distribution
e	scaled deviations of Z from $E(Z)$
$l(\cdot)$	log likelihood function
n	sample size
p	probability associated with the quantile of distribution
α	level of significance
δ	difference of two location parameters
η	quantile of the distribution
μ	location parameter
ξ	shape parameter
π	failure probability
ρ	ratio of two scale parameters
σ	scale parameter
$\mathcal{R}_{(\cdot)}(\cdot)$	FGPQ of a parameter or quantile of distribution
$Q_{(\cdot)}(\cdot)$	CPQ of a parameter or quantile of distribution
Y_*	a scalar continuous random variable
Z_*	a scalar standard random variate associated with Y_*

y_*	a scalar observation associated with Y_*
z_*	a scalar observation associated with Z_*
$\hat{\mu}(\cdot)_{ML}$	maximum likelihood based estimator for μ
$\hat{\mu}(\cdot)$	rank-based GLS estimator for μ
$\hat{\sigma}(\cdot)_{ML}$	maximum likelihood based estimator for σ
$\hat{\sigma}(\cdot)$	rank-based GLS estimator for σ
$\in$	an element of
$\backsim$	distributed as

Overview

Rank-based methods of estimation and inference for the parameters and quantiles in location-scale, log-location-scale and location-scale-shape families of distributions have many applications especially in statistical problems involving lifetime data analysis. Although various exact and approximate frequentist and Bayesian methods of inference exist in literature, they may not be applicable to some of the statistical problems. Furthermore, when the frequentist and Bayesian methods of inference are applicable, their applicability may involve, for example, computational complexity. In this thesis, an alternative method of inference to frequentist and Bayesian methods are explored. Specifically, the fiducial generalized methods of inference are based on the concept of fiducial inference introduced by R.A. Fisher (1920s and 1930s). Further development of fiducial inference by other researchers after the 1930s, for example, Weerahandi (1989, 1993 and 2004) and Iyer and Petterson (2002), have greatly broadened the applicability of fiducial inference. Initially, in this thesis we present methods of generalized fiducial inference for location-scale (LS) and log-location-scale (LLS) families of distribtutions, the distributions with two parameters, a location and a scale parameter. Inference is done using rank-based and maximum likelihood-based fiducial generalized pivotal quantities (FGPQs).

In an attempt to extend fiducial inference to locations-scale-shape (LSS) families of distributions (distributions with three parameters) we present a new approach to fiducial inference, in order to reduce the computational complexity when dealing with high-dimensional parameter distributions. Therefore, in this thesis, we introduce the concept of Conditional Fiducial Generalized Pivotal Quantities (CFGPQs) which allow one to reduce high-dimensional parameter problems to a lower-dimensional parameter problem.

The concept of CFGPQs allows us to, for example, to calculate FGCIs for the model parameters and quantiles of distributions in the LSS setting, inter alia, for the GEV, GP and three-parameter Weibull distributions. Through the concept of CFGPQs, we associate conditional fiducial distributions with the parameters of a statistical model. Using those conditional fiducial distributions, the marginal distribution of model parameters can be determined through a Gibbs sampler (that is, Monte Carlo simulations). This procedure is analogous to methods for the calculation of marginal posterior distributions in a Bayesian context.

A summary of the organization of chapters in this thesis is as follows:

Chapter 1 provides a general introduction to the research topic, specifically, the history of the concept of fiducial inference and subsequent related developments of the fiducial idea. Unlike in the context of Bayesian inference, fiducial inference does not require a prior distribution for the parameters in order to carry out inference on model parameters. Although the concept of fiducial inference was initially not generally accepted, further research has since broadened its applicability and acceptance in various challenging statistical problems. The definitions of conventional and fiducial generalized PQs, and of conditional fiducial generalized PQs are presented in Chapter 1. Lastly, Chapter 1 presents the objectives and research design of the thesis.

Chapter 2 presents the definitions of LS and LLS families of distributions. Among others, the Normal, Logistic, Uniform and Gumbel distributions belong to LS families, and the Log-Normal, Log-Logistic, Pareto and Weibull distributions belong to LLS families. Furthermore, for each LS and LLS distribution, we express the standard variate associated with a random variable in terms of standard uniform variate because we simulate from those standard distributions.

In Chapter 3, methods of parameter estimation in the LS and LLS families, namely rank-based GLS and ML-based methods, are discussed. A review of the literature on methods of estimation of the parameters in LS and LLS families is presented.

Chapter 4 presents, for the one sample problem, the rank-based conventional PQs and fiducial generalized PQs for model parameters and quantiles, as well as ML-based conventional PQs and fiducial generalized PQs in LS and LLS families. We can calculate confidence intervals for the parameters and quantiles through simulation of the corresponding conventional and fiducial generalized PQs. Furthermore, a simulation study is carried out to compare the average length of confidence intervals based respectively on rank-based and an ML-based PQs. Lastly, the proposed methods of inference are applied to the real data example.

In Chapter 5, for the two-sample problem, rank and ML-based conventional and fiducial generalized inference in LS and LLS families, are discussed. Similarly to Chapter 4, a simulation study is carried out to compare the coverage and average length of confidence intervals based respectively on rank-based and an ML-based FGPQs in the two sample problem. An illustrative example is also presented.

Chapter 6 presents methods for fiducial inference for the LSS family of distributions. Among others, distributions that belong in this class are the GEV, GP, and three-parameter Weibull distributions. Initially, rank-based, iterative generalized least squares estimation of the model parameters in the LSS family is discussed. Thereafter, CFGPQs for the model parameters are derived. Confidence intervals for the model parameters in LSS family can then be carried out through simulation of the marginal fiducial distribution of the CFGPQs by using Gibbs sampler. A simulation study of the coverage and average length of such confidence intervals is presented. Computational problems, and successes and failures of the proposed methodology are discussed. An illustrative example is presented.

Chapter 7 provides a summary of the overall results and conclusions, and sketches open problems and avenues for further research.

Chapter 1 - Introduction

1.1. Background of Order Statistics

Statistical methodologies involving order statistics and their functions have many practical applications. The random variables or observed random variables of interest ranked according to their relative magnitudes are called order statistics. For example, consider an experiment involving the athletic competition of ten athletes running a distance of 100 meters. Then, the observed times (in seconds) that each athlete takes to complete the 100 meters distance are order statistics. That is, the observed times are ranked naturally from the fastest athlete, namely the winner, followed by the runner up, to the slowest or last runner.

Historically, Fisher and Tippett (1928) first investigated the distribution of order statistics of a sample. In their pioneering paper, they derived the limiting distributions of the smallest or largest order statistics of an independent sample drawn from a general probability distribution. Since then, the results obtained by Fisher and Tippett (1928) did not see further development for about 30 years until Gumbel (1958) modified Fisher and Tippett (1928)'s results in his book. Gumbel's book has since become an important reference with regard to applications of extreme value theory, especially to engineering problems. Furthermore, it has given rise to the discovery of the Gumbel distribution, widely used in extreme value theory to model the smallest or largest order statistics of a sample. The text of Beirlant et al (2004) provides a comprehensive coverage of the theory and a wide range of applications of the extreme value theory. In addition, extensive discussion of various methods of estimation based on complete and censored samples of order statistics appear in the text of Balakrishnan and Cohen (1991).

We can denote the order statistics of the sample as follows:

If $Y_1, Y_2, \dots, Y_n$ denote an independent and identically distributed (i.i.d.) random sample from the distribution of a continuous random variable Y_*, then the random sample ordered in ascending order of magnitude can be written as

$$Y_{(1)}, Y_{(2)}, \dots, Y_{(n)} \tag{1.1}$$

where $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(n)}$. Similarly to equation (1.1) we write the realizations of the random sample $Y_1, Y_2, \dots, Y_n$, denoted by $y_1, y_2, \dots, y_n$ ordered in ascending order of magnitude as

$$y_{(1)}, y_{(2)}, \dots, y_{(n)} \tag{1.2}$$

Using the notation above, the minimum of Y_i, where $i = 1, 2, \dots, n$, is given by

$$Y_{(1)} = min(Y_1, Y_2, \dots, Y_n) \tag{1.3}$$

and the maximum of Y_i can be written as

$$Y_{(n)} = max(Y_1, Y_2, \dots, Y_n) \tag{1.4}$$

Well-known examples of the functions of random variables based on order statistics are $Y_{(1)}$, $Y_{(n)}$ and the range of the sample obtained as $Y_{(n)} - Y_{(1)}$ among others. The order statistics $Y_{(1)}$ and $Y_{(n)}$ are known as 'extremes' (Balakrishnan and Cohen, p. 2), and their distributional properties and a wide range of applications are investigated mainly in studies that deal with extreme value theory.

As will be seen in the remainder of this thesis, order statistics can be used to derive point and interval estimates of the location and scale parameters of location-scale families of distributions. Some of these methods can be extended to location-scale-shape distributions.

1.2. Background of Location-Scale, Log-Location-Scale and Location-Scale-Shape Familes of Distributions

In this section, we discuss the background of location-scale (LS), log location-scale (LLS) and location-scale-shape (LSS) families of distributions. Estimation and inference for the parameters and quantiles of these families of distributions are widely treated in the literature, for example, by Nkurunziza and Chen (2011), Balakrishnan and Kateri (2008), He and Lawless (2005), and Hong et al. (2009, 2010). The distributions in the LS and LLS family have two unknown parameters, namely a location and a scale parameter. Thus, estimation of both parameters is generally required. Similarly, in LSS families, three parameters are unknown, namely a shape parameter in addition to a location and scale parameter. However, in some cases one or more of the parameters of these distributions might be assumed known. For example, in a location problem (that is, the location parameter is unknown but the scale parameter is assumed known or fixed) or a scale problem (that is, the scale parameter is unknown but the location parameter is assumed known or fixed). Furthermore, if for example, the shape parameter in a LSS family is assumed known or fixed, then essentially we have a location-scale problem.

Examples of distributions that belong to the LS family are the Normal, Logistic, Uniform and Gumbel distributions; examples of distributions that belong to the LLS family are the Log-Normal, Log-Logistic, Weibull and Pareto distributions; the three-parameter Weibull, three-parameter Log-Normal, Generalized Extreme Value and Generalized Pareto distributions are examples of LSS families. The formal definitions of LS, LLS and LSS families of distributions are presented in Sections 2.2, 2.3 and 6.1 respectively.

Estimation and inference for parameters and quantiles of LS, LLS and LSS families of distributions can in principle be based on maximum likelihood (ML) method. However, parameter estimation using ML can be difficult, especially in the presence of censored data. Other challenges of using the ML method are that maximum likelihood estimators (MLEs) may not exist and that in the absence of closed form expressions for the estimators, it may require extensive programming (Balakrishnan and Kateri, 2008). Thus, it seems worthwhile to explore alternative methods of parameter estimation and inference.

In this study we will explore methods of estimation and inference based on order statistics. In the case of LS and LLS families of distributions, the location and scale parameters can be estimated using generalized least squares (GLS) estimation. This method of estimation, which we will refer to as the 'rank-based method', is based on a general linear model for the vector of order statistics of a sample, with, in general, non-diagonal variance-covariance matrix. The efficiency of the rank-based method is evaluated by comparing the estimates obtained using this method against ML-based estimates. It is noteworthy that the rank-based method of parameter estimation can very easily be applied to certain types of censored samples. ML-based inference for model parameters and quantiles in LS and LLS family using censored samples is, for example, discussed by Lawless (2003, pp. 218-235)).

In the case of three-parameter families of distributions, iterative generalized least squares estimation approach can be used. Through this approach, we estimate different combinations of two parameters as for LS families, while keeping the third parameter fixed at the iteration in question. A similar approach forms the basis of a new concept of fiducial inference, namely conditional fiducial generalized inference which is introduced in the next section.

1.3. Fiducial Inference

In this section, we discuss the concept of fiducial inference and related developments that gave rise to concepts of fiducial generalized pivotal quantities (FGPQs), conditional FGPQs (CFGPQs) and fiducial generalized confidence intervals (FGCIs).

1.3.1. History

Fiducial inference can be applied to statistical problems where frequentist and Bayesian inference may not be applicable or not be successful. Before the concept of fiducial inference, which is a subclass of generalized inference, was introduced, exact (non-Bayesian) methods of inference for statistical problems involving nuisance parameters were generally not available (Weerahandi, 1993). An example of such a situation is the Behrens-Fisher problem (Hannig et al., 2006) of comparing the means of two independent normal distributions with unequal variances. Another example is the comparison of means or quantiles of two independent Exponential distributions. In both cases, conventional pivotal quantities (CPQs) for the parameter contrast of interest, which might be used in constructing confidence intervals, are not available. In contrast, FGPQs can be used to construct FGCIs, for example for the difference in mean parameters (Behrens-Fisher problem) and difference in quantiles of the distributions (two independent Exponential distributions).

Historically, the concept of fiducial inference was first introduced by R.A. Fisher (1922, 1925, 1930, and 1935). Initially, Fisher introduced the concept of a fiducial distribution of a parameter, and proposed that the fiducial distribution replaces the Bayesian posterior distribution, when estimating a confidence interval of the parameter. Subsequently, the concept was developed through the work of other researchers, for example, D.A.S. Fraser in the 1960s, G.N. Wilkinson in the 1970s, David and Stone in the early 1980s. While Fraser, Wilkinson, David and Stone made great advances to fiducial inference, the concept was not generally accepted by other researchers during that time because some of “Fisher’s bold claims” about the properties of fiducial distributions were thought not to apply to statistical problems involving nuisance parameters. In other words, for one-parameter distributions, Fisher’s fiducial confidence intervals for the parameter were efficient relative to conventional confidence intervals. However, while for the multi-parameter case, Fisher's approach produced confidence intervals whose coverage probabilities were relatively close to nominal values, they did not conform to frequentist property of exactness in the case of repeated sampling. For further references on the history of fiducial inference, the reader is referred to the papers of Hannig (2009) and Hannig et al. (2016).

The work on fiducial inference and related ideas was developed further in the late 1980s and early 1990s. Tsui and Weerahandi (1989) introduced the new concepts of generalized P-values and generalized test variables (GTVs). His work led to new concepts of generalized pivotal quantities (GPQs) and generalized confidence intervals (GCIs) published in his subsequent papers, namely Weerahandi (1993, 2004). Following the work of Weerahandi (1993), methods for construction of GCIs were developed for a vast array of applications. For the various applications, the reader is referred to numerous references cited by Hannig, Iyer and Patterson (2006, Section 1).

Further developments in the area of fiducial inference were made through papers of Iyer and Patterson (2002), Hannig, Iyer and Patterson (2006) and Hannig (2009). Iyer and Patterson (2002), and Hannig, Iyer and Patterson (2006) demonstrated that the concepts of fiducial inference and generalized inference are closely related, and thus those authors unified the two concepts; they called the unified concept *generalized fiducial inference* (GFI). By unifying the two concepts, they transferred randomness from the data to the parameter space, using an inverse of a data generating equation without the use of Bayes Theorem (Hannig et al., 2016).

A major advance was made by Hannig, Iyer and Patterson (2006) by proposing a special class of GPQs, namely FGPQs. Hannig, Iyer and Patterson (2006) described general methods for constructing FGPQs, and, by extension, for construction of FGCIs. Subsequently, Hannig (2009) presented a very general method for construction of FGPQs that further broadens the applicability of fiducial inference. For examples, the general methods of Hannig, Iyer and Patterson (2006) and Hannig (2009) have been used successfully in the derivation of methods for fiducial inference in complex and challenging applications such as linear mixed models (Hannig and Iyer, 2008; Cisewski and Hannig, 2012), the largest mean of a multivariate distribution (Wandler and Hannig, 2011), and inference for the generalized Pareto distribution (Wandler and Hannig, 2012).

However, specifically in the last chapter of the present thesis, we pursue a different route in an attempt to widen the applicability and reduce the computational complexity of fiducial inference. In order to achieve these goals, we introduce the concept of conditional FGPQs (CFGPQs). Through the concept of CFGPQs, we associate conditional fiducial distributions with the parameters of a statistical model. By analogy with methods for the calculation of marginal posterior distributions in a Bayesian context (Gelfand and Smith, 1990), the marginal fiducial distribution of the model parameters can be determined by Monte Carlo simulation, using the Gibbs sampler, from the relevant conditional fiducial distributions associated with CFGPQs. Essentially, the concept of CFGPQ allows one to reduce a high-dimensional parameter problem to a lower dimensional or even single-parameter problem.

In this thesis, the concept of CFGPQs allows one to tackle fairly challenging problems in statistical inference. For example, using the concept of CFGPQs, one can calculate GCIs for the parameters and quantiles of distributions in the LSS setting, inter alia, three-parameter Weibull, Generalized Extreme Value, and Generalized Pareto distributions. However, let it be noted at this stage already: inference for LSS families based on CFGPQs has its own computational problems, and the approach is not in all cases successful.

1.3.2. Definition of Fiducial Generalized Pivotal Quantity

Generally, we define a pivotal quantity (PQ) as a statistic whose distribution does not depend on any unknown parameters. In this thesis, the acronym CPQ stands for conventional pivotal quantity, as opposed to a fiducial generalized pivotal quantity (FGPQ). Formally, we define a (conventional) pivotal quantity as follows:

> Definition 1.1: Conventional Pivotal Quantity (Weerahandi, 1995, p. 19):
>
> A random variable $Q = \varrho(Y, \theta)$, a function of the data Y and the parameter θ of the underlying distribution, is a pivotal quantity if the distribution of Q is independent of the parameter.

Furthermore, in the manner of Weerahandi (2004, pp. 18-20) (and similar to the definition of Hannig, Iyer and Patterson, 2006) we define a fiducial generalized pivotal quantity (FGPQ) as follows:

> Definition 1.2: Fiducial Generalized Pivotal Quantity (Hannig, Iyer and Patterson, 2006):
>
> Let S be a k-dimensional random vector whose distribution is indexed by a parameter or a vector parameter $\xi = (\theta, \eta, \zeta)'$ of dimension p. Here we are interested in making inferences about the q-dimensional sub-vector θ of ξ, $1 \leq q \leq p$. Furthermore, let s be an observed value of S. Then, a FGPQ for θ, denoted by $R_\theta(s, S, \xi)$, is a function of (s, S, ξ) with the following properties:
>
> > FGPQ1: The distribution of $R_\theta(s, S, \xi)$ is free of ξ.
> >
> > FGPQ2: For every allowable s, $R_\theta(s, s, \xi) = \theta$.

1.3.3. **Definition of Conditional Fiducial Generalized Pivotal Quantity**

Although many FGPQs exist where no CPQs are available, there remain situations where, a FGPQ for θ does not exist, or if a FGPQ for θ exists, it may not be obvious how to construct it. However, in many situations it may be possible to construct a FGPQ for a sub-vector θ, conditional on the $r-$dimensional sub-vector ζ of ξ, where $1 \leq r \leq p-q$. Thus, we define a CFGPQ as follows:

Definition 1.3: Conditional Fiducial Generalized Pivotal Quantity:

Let S be a k-dimensional random vector whose distribution is indexed by a parameter or a vector parameter $\xi = (\theta, \eta, \zeta)'$ of dimension p. Here we are interested in making inferences about the q-dimenstional sub-vector θ of ξ, $1 \leq q \leq p$. Furthermore, let s be an observed value of S. Then, a CFGPQ for θ, conditional on ξ, denoted by $R_{\theta|\xi}(s, S, \xi)$ is a function of (s, S, ξ) with the following properties:

CFGPQ1: The distribution of $R_{\theta|\xi}(s, S, \xi)$ is free of (θ, η).

CFGPQ2: For every allowable s, $R_{\theta|\xi}(s, s, \xi) = \theta$.

Specific examples of rank-based conventional and fiducial generalized pivotal quantities for testing hypotheses and construction of confidence intervals for parameters, quantiles and tail probabilities in the LS and LLS families, are discussed in detail in Section 4.2. Similarly, specific examples of maximum likelihood-based conventional and fiducial generalized pivotal quantities are discussed in detail in Section 4.3.

1.3.4. **Fiducial Generalized Confidence Intervals**

In Section 1.2.2, we presented a general definition of the FGPQ. In Section 4.2, we will derive rank-based FGPQs for the parameters, quantiles and tail probabilities of the distributions in the LS and LLS families, for the one sample problem. Similarly, ML-based FGPQs for the parameters and quantiles of those distributions are presented in Section 4.3. In Section 4.4, we use the distributions of rank-based FGPQs derived in Section 4.2 to construct the corresponding FGCIs. The FGCIs are obtained by taking $\alpha/2$ quantiles of the distribution of FGPQs as lower limits and $(1-\alpha/2)$ quantiles of the distribution of FGPQs as upper limits. ML-based FGCIs can be obtained in a similar manner; for more details, see Section 5.4. Similarly, confidence intervals based on CPQs (when such CPQs are available) can be obtained through inversion of CPQs, as is well known.

More generally, we might be required to make inferences about the difference in location parameters from two independent samples, or the difference of two quantiles. For example, we might consider the problem of obtaining a confidence interval for the difference of location parameters from two independent normal distributions with unequal variances (that is, the Behrens-Fisher problem). We note that a conventional pivotal quantity for this problem is not available (Hannig, Iyer and Patterson, 2006). The solution to this problem can be derived using the fiducial argument discovered by Behrens in 1929 and later developed further by Fisher in 1935. In this thesis, we construct rank and ML-based FGCIs for the ratio of scale parameters, difference of location parameters, and difference of two quantiles, for the distributions in LS and LLS families. Thus, we discuss fiducial generalized inference for LS and LLS distributions involving two independent samples.

Regarding the two-sample problem for LS and LLS distributions, rank-based FGPQs for the ratio of scale parameters, difference of location parameters, difference of two quantiles and log-odds ratio of tail probabilities, are presented in Sections 5.2.2, 5.2.3, 5.2.4, and 5.2.5 respectively; whereas ML-based FGPQs for the same contrasts are presented in Sections 5.3.1, 5.3.2, 5.3.3, and 5.3.4

respectively. As is the case with one sample problem, when FGPQs derived from two independent samples are available, FGCIs can be obtained directly through simulation of the distribution of respective FGPQs. The general algorithm for calculating the FGCIs for the difference of two location parameters and difference of two quantiles of the distribution is presented in Section 5.4.

1.4. Objectives

Primary objectives:

The primary objectives of this thesis are as follows:

- To develop exact rank-based conventional and fiducial generalized methods of estimation and inference for location-scale, log-location-scale and location-scale-shape families of distributions.
- To compare by simulation and practical application the performance of the proposed rank-based conventional and fiducial generalized methods of estimation and inference for location-scale, log-location-scale and location-scale-shape families of distributions with that based on likelihood methods.

Secondary objectives:

The secondary objectives of this thesis are as follows:

For the case of location-scale and log-location-scale families of distributions: one-sample problem

- To compare by simulation the average lengths of confidence intervals for σ, μ and quantiles (η) for the Normal, Logistic, Uniform, Pareto and Weibull distributions obtained using rank-based CPQs and FGPQs with those obtained using ML-based CPQs and FGPQs.

- To identify through simulation the most efficient rank-based CPQ or FGPQ for the scale parameter σ among the three rank-based CPQs or FGPQs for σ proposed.
- To illustrate the use of exact rank-based conventional and fiducial generalized methods of estimation and inference with real life data examples.

For the case of location-scale and log-location-scale families of distributions: two-sample problem

- To compare through simulation the coverage and average lengths of rank-based FGCIs for the difference of location parameters and difference of quantiles of two independent samples of Normal, Logistic, Uniform, Pareto and Weibull distributions, with FGPQs for the difference of location parameters and difference of quantiles obtained using ML-based method: for the cases when $\sigma_1 = \sigma_2$; and when $\sigma_1 \neq \sigma_2$.
- To illustrate the use of rank-based fiducial generalized methods of estimation and inference with a real life data example.

For the case of location-scale-shape family of distributions

- To develop fiducial generalized rank-based methods of estimation and inference for the model parameters and quantiles of Generalized Extreme Value, Generalized Pareto and three-parameter Weibull distributions, using iterative generalized least squares (and the Gibbs sampler) based respectively on the
    - Theta and
    - Theta star parametrizations
- To evaluate through simulation the coverage and average length of FGCIs for the model parameters and quantiles of the Generalized Extreme Value, Generalized Pareto and three-parameter Weibull distributions.

    - To determine through simulation the most efficient parametrization in terms of empirical coverage and average length of FGCIs for the model parameters and quantiles of the Generalized Extreme Value, Generalized Pareto and three-parameter Weibull distributions.
- To illustrate, with real life data examples, the use of rank-based Gibbs sampler algorithm.

1.5. Research Design

This is an empirical study. Thus, simulation studies are used to assess coverage probabilities and average lengths of conventional and fiducial generalized confidence intervals for the parameters, parameter contrasts and quantiles of the distribution in LS, LLS and LSS families of distributions. The proposed methods of inference for model parameters and quantile of distributions are applied to examples of real data.

Chapter 2 - Location-Scale and Log-Location-Scale Families of Distributions

2.1. Introduction

This chapter provides background on LS and LLS families of distributions. As was mentioned in Section 1.1, for LS and LLS families of distributions we are concerned with estimation and inference problems about the location and scale parameters of the distribution, and about other quantities which are functions of those parameters, such as quantiles and tail probabilities. In general, statistical methods based on order statistics have been investigated extensively in literature, using different approaches; see, for example, Ene and Karahasan (2016) and the references cited therein. For this thesis, the pioneering paper of Lloyd (1952) is of interest in the context of applying the method of least squares estimation to the order statistics of samples drawn from LS and LLS families.

The general forms of LS and LLS families can be defined in terms of the cumulative distribution functions (cdfs) as is shown below in equations (2.2.1) and (2.3.1) respectively. Specifically, in Sections 2.2 and 2.3 we discuss some examples of distributions that belong to the LS and LLS families. For Sections 2.2 and 2.3 below, the main references are Escobar et al. (2009), and Hong et al. (2010).

2.2. Location-Scale Family of Distributions

The distributions that belong to the LS family have two parameters, namely the location and scale parameters. The general form of location-scale family of distributions is presented as follows:

A scalar continuous random variable Y_* belongs to the LS family of distributions if its cumulative distribution function can be written in the form

$$F_{Y_*}(y_*;\theta) = \Phi\left(\frac{y_* - \mu}{\sigma}\right) \tag{2.2.1}$$

where the support of the distribution may be on the whole real line. Moreover, the vector of parameters θ is given by

$$\theta = (\mu, \sigma)'$$

where μ and σ are, respectively, the location and scale parameters. Furthermore, the function $\Phi(\cdot)$ is continuous for all values of y_* in the support of the distribution, and $\Phi(\cdot)$ is a monotone non-decreasing function of its argument. Lastly, the values of $\Phi(\cdot)$ lie in the interval $[0,1]$.

We define the standard random variate Z_* associated with Y_* as

$$Z_* = \frac{Y_* - \mu}{\sigma} \tag{2.2.2}$$

The cdf of Z_* is given by

$$F_{Z_*}(z_*;\theta) = \Phi(z_*) \tag{2.2.3}$$

where

$$\theta = (0, 1)'$$

Thus, the distribution of Z_* does not depend on any unknown parameters.

In this thesis, unless indicated otherwise, a scalar random variable is denoted by an upper case letter with subscript $*$, as in Y_* and Z_*; a scalar observation associated with a scalar random variable, say Y_*, is denoted by a lower case letter with subscript $*$, as in y_*.

Balakrishnan et al. (2008, Proof of Theorem 2.3, Section 2.3) showed that the generalized least squares estimators based on order statistics, discussed in this thesis, are equivariant (see Definition 1.4). That is, the estimators are invariant under any scalar linear transformation, as is defined below:

> Definition 1.4: Equivariant estimators (Balakrishnan et al., 2008, Section 1):
>
> Let $1_n = (1, \dots, 1)'$ be the $n-$dimensional vector of ones and $Y = (Y_1, \dots, Y_n)'$ be a random sample of observations of size n. Then, for the location-scale model (2.2.1), an estimator of the location parameter $\hat{\mu}(Y)$ is referred to as an equivariant estimator of μ if $\hat{\mu}(d \cdot Y + c \cdot 1) = d \cdot \hat{\mu}(Y) + c$ for $d > 0, c \in \mathbb{R}$.
>
> Similarly, for the location-scale model (2.2.1), an estimator of the scale parameter $\hat{\sigma}(Y)$ is referred to as an equivariant estimator of σ if $\hat{\sigma}(d \cdot Y + c \cdot 1_n) = d \cdot \hat{\sigma}(Y)$ for $d > 0, c \in \mathbb{R}$.

As is shown in Sections 2.2.1 through 2.2.4, the Normal, Logistic, Uniform and Gumbel distributions belong to the location-scale family.

2.2.1. Normal Distribution

As a result of the central limit theorem, the Normal distribution is the most important and useful distribution in the fields of probability theory and statistics. In this thesis, we denote the Normal distribution as follows:

Let Y_* follow a Normal distribution with parameters μ and σ^2, denoted by $Y_* \sim N(\mu, \sigma^2)$. The parameters μ and σ^2 are, respectively, the mean and variance of the distribution. Then the cdf of Y_* is given by

$$F_{Y_*}(y_*; \theta) = \Phi\left(\frac{y_* - \mu}{\sigma}\right) \tag{2.2.4}$$

where

$$\Phi(\mathrm{z}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} exp\left[-\frac{t^2}{2}\right] dt$$

and its pdf is given by

$$f_{Y_*}(y_*; \theta) = \frac{1}{\sqrt{2\pi\sigma^2}}\ exp\left[-\frac{1}{2}\left(\frac{y_* - \mu}{\sigma}\right)^2\right]$$

where μ and the support of the distribution are on the whole real line, whereas σ is always positive. Thus μ and σ are respectively the location and scale parameters of the distribution.

The standard variate Z_*, given by

$$Z_* = \frac{Y_* - \mu}{\sigma}$$

follows the standard Normal $N(0, 1)$ distribution, which does not depend on any unknown parameters. The standard random variate can also be written as

$$Z_* = \Phi^{-1}\left\{\Phi\left(\frac{Y_* - \mu}{\sigma}\right)\right\} = F^{-1}(U) = \Phi^{-1}(U) \tag{2.2.5}$$

where

$$F^{-1}(U) = \Phi^{-1}(U)$$

is the inverse of the cdf $\Phi(Y_*)$ of the standard Normal $N(0, 1)$ distribution. Furthermore, U is the standard Uniform variate.

2.2.2. **Logistic Distribution**

The shape of the Logistic distribution is similar to that of the Normal distribution. However, the Logistic distribution has relatively heavier tails (Lawless, 2003, p. 24). Let Y_* follow a Logistic distribution. Then its cdf and pdf are respectively given by

$$F_{Y_*}(y_*;\theta) = \frac{1}{1 + exp\left\{-\left(\frac{y_* - \mu}{\sigma}\right)\right\}} \qquad (2.2.6)$$

and

$$f_{Y_*}(y_*;\theta) = \frac{\sigma^{-1} exp\left(-\frac{y_* - \mu}{\sigma}\right)}{\left\{1 + exp\left(-\frac{y_* - \mu}{\sigma}\right)\right\}^2}$$

where the support of the distribution is on the whole real line. Thus μ and σ are respectively the location and scale parameters of the distribution.

The standard random variate is given by

$$Z_* = \frac{Y_* - \mu}{\sigma}$$

which can also be expressed as

$$Z_* = -\log\left\{\left(F_{Y_*}(Y_*;\theta)\right)^{-1} - 1\right\} = F^{-1}(U) = -\log(U^{-1} - 1) \quad (2.2.7)$$

where

$$F^{-1}(U) = -\log(U^{-1} - 1) = \log\left(\frac{U}{1 - U}\right)$$

is the inverse of the cdf $F_{Z_*}(z_*;0,1)$ of the standard Logistic $Logist(0,1)$ distribution.

2.2.3. **Uniform Distribution**

The standard Uniform distribution can be used in simulations of data to generate pseudo random values of other standard distributions where the inverse of the cdf of such standard distributions can be expressed in closed form. See for example, equations (2.2.5) and (2.2.7) above. Let a scalar continuous random variable Y_* follow a Uniform distribution with parameters a and b, denoted by $Y_* \sim Unif(a, b)$. Then the cdf and pdf of Y_* are given by

$$F_{Y_*}(y_*; \theta) = \frac{y_* - a}{b - a} \tag{2.2.8}$$

and

$$f_{Y_*}(y_*; \theta) = \frac{1}{b - a}$$

respectively, where the support of the distribution lies in the interval $[a, b]$.

If $Y_* \sim Unif(a, b)$ then

$$(Y_* - a) \sim Unif(0, b - a)$$

and

$$\frac{Y_* - a}{b - a} \sim Unif(0, 1)$$

Thus the standard random variate is given by

$$Z_* = \frac{Y_* - a}{b - a} = \frac{Y_* - \mu}{\sigma} \tag{2.2.9}$$

where $\mu = a$ and $\sigma = b - a$ are the location and scale parameters of the distribution, and the distribution of Z_* does not depend on any unknown parameters. Hence $Y_* \sim Unif(a, b)$ is a member of location-scale family. Equation (2.2.9) can formally be expressed as

$$Z_* = F^{-1}(U) = U \tag{2.2.10}$$

where $F^{-1}(U) = U$ is the inverse of the cdf $F_{Z_*}(z_*; 0, 1)$ of the standard Uniform $Unif(0, 1)$ distribution.

The standard Uniform distribution has the property which states that if U has a Uniform distribution in the interval $[0, 1]$, then $1 - U$ also has a Uniform distribution in the same interval.

2.2.4. Gumbel Distribution

The practical application of Gumbel distribution arises in extreme value theory where it is used to model extremes. For specific examples of applications, refer to the text of Rinne (2009, p. 109). In extreme value theory, the Gumbel distribution is known as the Extreme Value 'Type-I-minimum' distribution or the Extreme Value 'Type-I-maximum' distribution, referring to modelling the smallest or the largest values respectively. The Extreme Value 'Type-I-maximum' distribution can be obtained from Extreme Value 'Type-I-minimum' distribution by taking the negative values of Extreme Value 'Type-I-minimum' distribution. The Gumbel distribution is closely related to the Weibull distribution in the sense that in literature, it is known as the Log-Weibull distribution (Lawless, 2003, p. 20). Extreme Value 'Type-I-minimum' and Extreme Value 'Type-I-maximum' distributions are denoted as follow:

Let a scalar continuous random variable Y_* follow a Gumbel (minimum) distribution with parameters μ and σ, denoted by $Y_* \sim Gum(\mu, \sigma)$ with cdf

$$F_{Y_*}(y_*; \theta) = exp\left[-exp\left\{\left(\frac{y_* - \mu}{\sigma}\right)\right\}\right] \qquad (2.2.11)$$

and pdf

$$f_{Y_*}(y_*; \theta) = \frac{1}{\sigma} exp\left[\left(\frac{y_* - \mu}{\sigma}\right) - exp\left\{\left(\frac{y_* - \mu}{\sigma}\right)\right\}\right]$$

where μ and the support of the distribution are on whole real line, and σ is always positive. Thus μ and σ are respectively the location and scale parameters of the distribution.

Then the standard random variate

$$Z_* = \frac{Y_* - \mu}{\sigma}$$

has a standard $Gum(0, 1)$ distribution, which is parameter free.

Alternatively, the standard random variate Z_* can be written as

$$Z_* = F^{-1}\left[F_{Y_*}(Y_*; \theta)\right] = \log\left[-\log\left\{F_{Y_*}(Y_*; \theta)\right\}\right] = \log[-\log(U)] \quad (2.2.12)$$

where $F^{-1}(U) = \log[-\log(U)]$ is the inverse of the cdf $F_{Z_*}(z_*; 0, 1)$ of the standard Gumbel (minimum), namely $Gum(0, 1)$ distribution.

Since the random variable $-\log(U)$ has a standard $Exp(1)$ Exponential distribution, the standard Gumbel random variate $Z_* = \log[-\log(U)]$ follows the distribution of the logarithm of a standard Exponential variate.

Similarly to the cdf (2.2.11) and pdf of the Gumbel (minimum) distribution, the cdf and pdf of the Gumbel (maximum) distribution are, respectively given by

$$F_{Y_*}(y_*; \theta) = exp\left[-exp\left\{-\left(\frac{y_* - \mu}{\sigma}\right)\right\}\right] \quad (2.2.13)$$

and

$$f_{Y_*}(y_*; \theta) = \frac{1}{\sigma} exp\left[-\left(\frac{y_* - \mu}{\sigma}\right) - exp\left\{-\left(\frac{y_* - \mu}{\sigma}\right)\right\}\right]$$

In this case, the standard random variate Z_* follows the distribution of the negative of the logarithm of a standard Exponential variate, namely the negative of the logarithm of an $Exp(1)$ variate. Hence, Z_* can be expressed in terms of a standard Uniform random variate U as

$$Z_* = F^{-1}(U) = -\log[-\log(U)] \qquad (2.2.14)$$

2.3. Log-Location-Scale Family of Distributions

In contrast to a location–scale family, a positive scalar random variable W_* belongs to the log-location-scale family of distributions if $Y_* = \log(W_*)$ belongs to the location-scale family of distributions (Hong, Escobar and Meeker, 2010). Thus, if W_* belongs to the log-location-scale family of distributions, the cdf of $W_* = exp(Y_*)$ can be written as

$$F_{W_*}(w_*;\theta^*) = \Phi\left[\frac{\log(w_*) - \mu^*}{\sigma^*}\right] \qquad (2.3.1)$$

where $w_* > 0$.

It can be noted here that equation (2.3.1) derives from a Y_* distributed as in equation (2.2.4). Examples of distributions that belong to the log-location-scale family are the Log-Normal, Log-Logistic, two-parameter Weibull (or simply Weibull) and Pareto distributions.

2.3.1. Log-Normal Distribution

The practical application of Log-Normal distribution arises in diverse fields, for example in engineering and medical sciences (Lawless, 2003, p. 21).

The Log-Normal distribution can be defined as follows:

The positive scalar continuous random variable W_* follows a Log-Normal distribution with parameters μ and σ^2, denoted by $W_* \sim \log N(\mu, \sigma^2)$, if $Y_* = \log(W_*)$ follows a Normal distribution with mean μ and variance σ^2. The cdf of Y_* is given in equation (2.2.4) above. Therefore, similarly to (2.2.4), the cdf and pdf of W_* are, respectively, given by

$$F_{W_*}(w_*;\theta) = \Phi\left[\frac{\log(w_*) - \mu}{\sigma}\right] \tag{2.3.2}$$

and

$$f_{w_*}(w_*;\theta) = \frac{1}{\sqrt{2\pi\sigma^2 w_*^2}} exp\left\{-\frac{1}{2}\left[\frac{\log(\, w_*) - \mu}{\sigma}\right]^2\right\}$$

where w_* and σ are always positive and μ is on the whole real line. Thus μ and σ are respectively the location and scale parameters of the distribution.

The standard random variate Z_*, whose distribution does not depend on μ and σ^2, is given by

$$Z_* = \frac{\log(W_*) - \mu}{\sigma}$$

The standard variate can also be written as

$$Z_* = \Phi^{-1}\left\{\Phi\left[\frac{\log(W_*) - \mu}{\sigma}\right]\right\} = F^{-1}(U) = \Phi^{-1}(U) \tag{2.3.3}$$

where $F^{-1}(U) = \Phi^{-1}(U)$ is the inverse of the cdf of the standard Normal $N(0,1)$ distribution. Thus Z_* follows a standard Normal $N(0,1)$ distribution.

2.3.2. **Log-Logistic Distribution**

Examples of practical applications of Log-Logistic distribution are the modelling of lifetime data and logistic regression modelling. The Log-Logistic distribution can be defined as follows: A positive scalar continuous random variate W_* follows a Log-Logistic distribution with parameters μ^* and σ^*, denoted by $W_* \sim LLogist(\mu^*, \sigma^*)$, if $Y_* = \log(W_*)$ follows a Logistic distribution. Thus the cdf and pdf of W_* are, respectively, given by

$$F_{W_*}(w_*;\theta^*) = 1 - \left[1 + \left(\frac{w_*}{\mu^*}\right)^{\sigma^*}\right]^{-1} = \frac{(w_*/\mu^*)^{\sigma^*}}{1 + [(w_*/\mu^*)^{\sigma^*}]} \tag{2.3.4}$$

and

$$f_{W_*}(w_*;\theta^*) = \frac{(\sigma^*/\mu^*)(w_*/\mu^*)^{\sigma^*-1}}{[1+(w_*/\mu^*)^{\sigma^*}]^2}$$

where the support of the distribution, and the parameters μ^*, and σ^* are all positive. Then the standard random variate is given by

$$Z_* = \log\left[\left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right] = \frac{\log(W_*) - \log(\mu^*)}{1/\sigma^*} = \frac{\log(W_*) - \mu}{\sigma}$$

where $\mu = \log(\mu^*)$ and $\sigma = 1/\sigma^*$ which are respectively the location and scale parameters of the distribution. The standard random variate Z_* follows a standard $Logist(0,1)$ distribution, or equivalently, the distribution of the logarithm of a standard $LLogist(1,1)$ variate.

Alternatively, Z_* can be expressed in terms of a standard Uniform random variate U as follows:

Since $F_{W_*}(W_*;\theta^*) = U$ then

$$1 - \left[1 + \left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right]^{-1} = U$$

$$\left[1 + \left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right]^{-1} = 1 - U$$

$$1 + \left(\frac{W_*}{\mu^*}\right)^{\sigma^*} = \frac{1}{1-U}$$

$$\left(\frac{W_*}{\mu^*}\right)^{\sigma^*} = \frac{U}{1-U}$$

Thus

$$Z_* = \log\left[\left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right] = \log\left(\frac{U}{1-U}\right) = -\log\left(\frac{1-U}{U}\right) \qquad (2.3.5)$$

where $F^{-1}(U) = U/(1-U)$ is the inverse of the cdf of the standard $Logist(0,1)$ distribution.

2.3.3. Weibull Distribution

The Weibull distribution has been applied in diverse fields of application, for example demography, survival analysis, reliability theory, extreme value theory, and industrial engineering. For references detailing discussions of various applications, see for example, the texts of Lawless (2003, p. 18) and Rinne (2009, p. 275). In extreme value theory, the Weibull distribution is known as 'Extreme Value Type III' distribution.

Different types of Weibull distribution exist in literature. However, the traditional types are the three-parameter, two-parameter and one-parameter Weibull distributions. Thus, the Weibull distribution can be related to other distributions as special cases, for example, the Exponential, Gamma and Generalized Extreme Value distributions.

For the detailed discussion about the four-parameter and five-parameter Weibull distributions, see for example, Rinne (2009, pp. 158-166).

In this thesis, unless specified otherwise, the Weibull distribution will refer to the scale-shape version of the two-parameter Weibull distribution. It is by far the most often used in practice (Rinne, 2009, p. 35). We define the Weibull distribution as follows (Lawless, 2003, p. 18):

Let a continuous and positive scalar random variable W_* follow a Weibull distribution with parameters μ^* and σ^*, denoted by $W_* \sim Weib(\mu^*, \sigma^*)$. Then, the cdf and pdf of W_* are given by

$$F_{W_*}(w_*; \theta^*) = 1 - exp\left[-\left(\frac{w_*}{\mu^*}\right)^{\sigma^*}\right] \qquad (2.3.6)$$

and

$$f_{W_*}(w_*;\theta^*) = \frac{\sigma^*}{\mu^*}\left(\frac{w_*}{\mu^*}\right)^{\sigma^*-1} exp\left[-\left(\frac{w_*}{\mu^*}\right)^{\sigma^*}\right]$$

$$= \frac{\sigma^*}{(\mu^*)^{\sigma^*}} {w_*}^{\sigma^*-1} exp\left[-\left(\frac{w_*}{\mu^*}\right)^{\sigma^*}\right]$$

where the support of the distribution, the parameters μ^*, and σ^* are all always positive.

The standard random variate is given by

$$Z_* = \log\left[\left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right] = \frac{\log(W_*) - \log(\mu^*)}{1/\sigma^*} = \frac{\log(W_*) - \mu}{\sigma}$$

where $\mu = \log(\mu^*)$ and $\sigma = 1/\sigma^*$ are respectively the location and scale parameters of the distribution.

Alternatively, Z_* can be expressed in terms of a standard Uniform random variate U as follows:

Since $F_{W_*}(W_*;\theta^*) = U$ then

$$1 - exp\left[-\left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right] = U$$

$$exp\left[-\left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right] = 1 - U$$

$$\left(\frac{W_*}{\mu^*}\right)^{\sigma^*} = -\log(1-U)$$

Thus

$$Z_* = \log\left(\frac{W_*}{\mu^*}\right)^{\sigma^*} = \log[-\log(1-U)] \qquad (2.3.7)$$

where $F^{-1}(U) = -\log(1-U)$ is the inverse of the cdf of the standard $Weib(1,1)$ distribution. Thus, the standard random variate Z_* follows the distribution of the logarithm of a standard $Weib(1,1)$ variate (which is the distribution of the logarithm of a standard Exponential, namely $Exp(1)$ variate since the standard $Weib(1,1)$ variate has a standard $Exp(1)$ distribution).

2.3.4. Pareto Distribution

The Pareto distribution is widely used in various fields of application to model phenomena with skewed distributions. In Economics and Finance, for example, it is used to model the distributions of income and wealth (Reiss and Thomas, 2007). The Pareto distribution is related to other distributions, such as the Generalized Pareto, Exponential and Log-Normal distributions. For example, the Pareto distribution is a special case of the Generalized Pareto distribution where the location parameter is equal to zero. We define the Pareto distribution as follows (Villaseňor-Alva and González-Estrada, 2009):

Let a continuous and positive scalar random variable W_* follow a Pareto distribution with parameters μ^* and σ^*, denoted by $W_* \sim Pareto(\mu^*, \sigma^*)$. Then the cdf and pdf of W_* are, respectively, given by

$$F_{W_*}(w_*;\theta^*) = 1 - \left(\frac{\mu^*}{w_*}\right)^{\sigma^*} \tag{2.3.8}$$

and

$$f_{W_*}(w;\theta^*) = \frac{\sigma^*(\mu^*)^{\sigma^*}}{(w_*)^{\sigma^*+1}} = \sigma^*(\mu^*)^{\sigma^*}(w_*)^{-\sigma^*-1}$$

where the support of the distribution lies in the interval $[\mu^*, \infty)$ and the parameters μ^* and σ^* are always positive.

The standard random variate is given by

$$Z_* = \log\left[\left(\frac{W_*}{\mu^*}\right)^{\sigma^*}\right] = \frac{\log(W_*) - \log(\mu^*)}{1/\sigma^*} = \frac{\log(W_*) - \mu}{\sigma}$$

where $\mu = \log(\mu^*)$ and $\sigma = 1/\sigma^*$ are respectively the location and scale parameters of the distribution.

Alternatively, Z_* can be expressed in terms of a standard Uniform random variate U as follows:

Since $F_{W_*}(W_*; \theta^*) = U$ then

$$1 - \left(\frac{\mu^*}{W_*}\right)^{\sigma^*} = U$$

$$\left(\frac{\mu^*}{W_*}\right)^{\sigma^*} = 1 - U$$

$$\log\left(\frac{\mu^*}{W_*}\right)^{\sigma^*} = \log(1 - U)$$

Thus

$$Z_* = \log\left(\frac{W^*}{\mu_*}\right)^{\sigma^*} = -\log(1 - U) \tag{2.3.9}$$

where $F^{-1}(U) = (1 - U)^{-1}$ is the inverse of the cdf of the standard $Pareto(1, 1)$ distribution. Thus, the standard random variate Z_* follows the distribution of the logarithm of a standard $Pareto(1, 1)$ variate (which is the standard Exponential distribution, namely $Exp(1)$).

Chapter 3 - Estimation for Location-Scale and Log-Location-Scale Distributions

In this chapter, Section 3.1 presents a literature review on the existing methods of parameter estimation for LS and LLS families of distributions. The derivation of the GLS estimators for the location and scale parameters based on order statistics in LS and LLS families of distributions is presented in Section 3.2. Exact and approximate expressions for the mean and covariance matrix of the order statistics of standardized random variates are presented in Sections 3.3 and 3.5 respectively. Lastly, the ML estimators for the location and scale parameters in the LS and LLS families of distributions are presented in Section 3.6.

3.1. Literature Review

A great deal of research has been done on the various methods of estimation of the location and scale parameters in LS and LLS families of distributions. A number of methods for estimation of the location and scale parameters in the LS and LLS families of distributions and applicable references are listed below:

General least squares: Lloyd (1952), Gupta (1952), Gupta et al. (1967), Blom (1958, 1962), Downton (1953), Winer (1963), Hall (1975), Chan et al. (1971), Gupta and Gnanadesikan (1966), Chan and Cheng (1988), Balakrishnan and Cohen (1991, Sections 4.4 through 4.9), Sajeevkumar and Thomas (2005), Balakrishnan and Papadatos (2002), and Sajeevkumar and Thomas (2010).

Asymptotically best linear unbiased: Blom (1958), D'Agostino and Lee (1976).

Best linear equivariant: Balakrishnan et al. (2008).

Best linear invariant: Mann and Fertig (1973) and Mann (1971, 1967).

Probability weights: Ene and Karahasan (2016).

Method of moments: Schafer and Sheffield (1973).

Method of estimation based on the linear and polynomial coefficients: Downton (1966).

Simple method: Bain and Antle (1967), Gumbel (1958), and Menon (1963).

Optimal asymptotic methods: Balakrishnan and Cohen (1991, Sections 7.2 through 7.4), Bennett (1952), Jung (1955, 1962), Ogawa (1951, 1962), Dixon (1957, 1960), and Raghunandanan and Srinivasan (1970, 1971).

Exact Maximum likelihood: Balakrishnan and Cohen (1991, Sections 5.2 through 5.8), Malik (1970), Harter and Moore (1965, 1967), Gajjar and Khatri (1969), Murthy and Swartz (1975), Billmann et al. (1972), Wilson and Worcester (1943), Berkson (1957), Plackett (1958), and Harter and Moore (1967).

Approximate maximum likelihood: The text of Balakrishnan and Cohen (1991, Sections 6.1 through 6.7) and all the references cited in Sections 6.1 through 6.7 in the text.

The general least squares method for estimating the location and scale parameters in LS families of distributions was first introduced by Llyod (1952). Llyod's (1952) method of estimation is based on the least squares theorem of

Gauss and Markoff that is discussed in the paper of Aitken (1935). Llyod (1952) used the general least squares theory to derive the best (minimum variance) linear unbiased estimators for the location and scale parameters based on a complete sample of order statistics of a sample from a LS distribution. In addition, Llyod (1952) also derived closed form expressions for the variances and covariance of the least squares estimators for the location and scale parameters. Llyod (1952, Section 7) showed that the variance of least squares estimator for the location parameter based on order statistics is always less than or equal to the variance of the sample mean; and he investigated and provided the required conditions under which such variance is strictly less than the variance of the sample mean in the case of symmetric LS families of distributions. Similarly, Downton (1953) obtained the required conditions under which the variance of the estimator for the location parameter for non-symmetrical LS distributions is strictly less than the variance of the sample mean.

In the practical sense, Llyod's (1952) method requires the evaluation of the expected values, variances, and inverse of covariance matrix of order statistics of a complete sample from the standard LS distributions. The expected values and the inverse of the covariance matrix of the order statistics of standard LS variates, essentially, provide the weights for the order statistics of a complete sample of observations from the LS families of distributions, and estimates of the location and scale parameters are then obtained as linear functions of the order statistics.

Even though Llyod's (1952) method of least squares estimation introduced more than six decades ago could be applied generally to LS families of distributions, the difficulty of its general applicability was argued by other researchers. For example, Gupta (1952) argued that the covariance matrix of order statistics of standard LS variates may be difficult to evaluate (Balakrishnan and Cohen, 1991,

p. 94). Furthermore, even if the covariance matrix of standardized LS variates were available, its inverse may be difficult to obtain especially when dealing with large sample sizes. To overcome this difficulty, Gupta (1952) suggested that the covariance matrix of the standardized LS variates be replaced by an identity matrix of the same dimension such that the inverse of such an identity matrix becomes also an identity matrix. Gupta (1952) referred to the process of replacing the covariance matrix by an identity matrix as the simplified linear unbiased estimation of the location and scale parameters in LS families of distributions. Balakrishnan and Cohen (1991, pp. 98-99) demonstrated that Gupta's (1952) method of simplified linear unbiased estimation is relatively competitive with the method of the best linear unbiased estimation, specifically in the case of Type-II censored samples drawn from the Normal distribution.

In contrast to Gupta's (1952) method of the simplified linear unbiased estimation of the location and scale parameters, Blom (1958, 1962) proposed a different simplification approach to Llyod's (1952) method of least squares estimation of the location and scale parameters. Blom's (1958, 1962) method of simplified linear estimation, which can be applied generally to the LS families of distributions, is based on an asymptotic approximation of the covariance matrix of order statistics of the standardized variates. Similarly to Gupta's (1952) method of simplified linear estimation, Blom's (1958, 1962) method also uses the exact expected values of order statistics of the standardized variates. However, if the exact expected values of order statistics of the standardized variates are not available, an asymptotic approximation for the expected values may be used (Balakrishnan and Cohen, 1991, p. 100). The estimators for the location and scale parameters in LS families of distributions obtained using Gupta's (1952) and Blom's (1958, 1962) simplified linear methods are, respectively, referred to as, "unbiased nearly best linear estimators" and "nearly unbiased, nearly linear estimators" in the literature (Balakrishnan and Cohen, 1991, p. 100). Estimation of the location and scale parameters using the method of asymptotic approximation has an advantage over the method of best linear unbiased estimation. Asymptotic approximation-based estimation of the parameters does not require, as shown by Balakrishnan and Cohen (1991, p. 104), the inverse of the covariance matrix of order statistics of the standardized variates. However, the asymptotic estimators of location and scale parameters may not be good, in particular in the case of dealing with small and censored sample sizes (Balakrishnan and Cohen, 1991, p. 104).

Other related methods of estimation for the location and scale parameters in LS families of distributions are the various approaches applied to optimal linear estimation based on selected order statistics described by Balakrishnan and Cohen (1991, pp. 215-255). These approaches include the optimal asymptotic estimators of the location and scale parameters of LS distributions based on the weighted sums of probability functions and a sample of order statistics (Balakrishnan and Cohen, 1991, pp. 226-227) and derived by Bennett (1952). Jung (1955, 1962) derived the optimal asymptotic estimators for the location and scale parameters using a linear estimator that is expressed in terms of a continuous differentiable probability function bounded in the interval (0, 1); refer to Balakrishnan and Cohen (1991, pp. 229-233) for more details about Jung's (1955, 1962) approach. Ogawa (1951, 1962) extended the work based on the Gauss-Markov Theorem by deriving the optimally and asymptotically best linear estimators for the location and scale parameters of LS distributions based on selected order statistics that minimise the variances of these estimators (Balakrishnan and Cohen, 1991, p. 235). Dixon (1957, 1960) derived simplified linear estimators for the location (one estimator) and scale (two estimators) parameters of the Normal distribution. Firstly, the simplified linear estimator for the location parameter is based on quasi-midrange of order statistics of a sample of the Normal distribution (Balakrishnan and Cohen, 1991, p. 249). Secondly, the first simplified linear estimator of the scale parameter is based on the quasi-range of order statistics of a sample of the Normal distribution

(Balakrishnan and Cohen, 1991, p. 251). Thirdly, the second simplified linear estimator (with smallest variance) of the scale parameter is based on the sample range of the Normal distribution (Balakrishnan and Cohen, 1991, p. 252). Similarly to Dixon's (1957, 1960) simplified linear estimators for the location and scale parameters of the Normal distributions, Raghunandanan and Srinivasan (1970, 1971) derived simplified linear estimators of the location and scale parameters for the Logistic and two-parameter Exponential distributions based on symmetrically Type-II censored samples (Balakrishnan and Cohen, 1991, p. 255).

Following the advancement of the general theory of least squares method of estimation for the location and scale parameters in LS families of distributions during 1950's and 1960's, various studies have since focused on its practical applications. For example, Gupta et al. (1967, Section 5) obtained the best linear unbiased estimates for the location and scale parameters of the Logistic distribution based on real data of a double censored sample in the tails. The best linear unbiased estimates obtained by Gupta et al. (1967) are efficient and competitive with the results based on the Normal distribution obtained by Sarhan and Greenberg (1962). Other researchers who also made a significant contribution to the application of least squares method of estimation are, for example, Gupta (1952), Winer (1963), Hall (1975), Balakrishnan and Cohen (1991, pp. 83-118), Chan et al. (1971), Gupta and Gnanadesikan (1966), and Chan and Cheng (1988). For example, Gupta (1952, pp. 270-271) obtained best linear estimates for the location and scale parameters of the Normal distribution based a Type-II censored sample (three censored observations) of size 10; whereas Winer (1963, p. 464) obtained estimates for the location and scale parameters of two-parameter Exponential distribution based on a singly (one censored observation) Type-II censored sample of size 5.

A further development of best linear unbiased estimation of the location and scale parameters based on order statistics in LS families of distributions was formulated in 2000's. Balakrishnan and Papadatos (2002) and Sajeevkumar and Thomas (2005, 2010) derived the best linear unbiased estimators of common location and common scale parameters of different LS distributions based on order statistics. Estimation of the common location and common scale parameters in their papers are based on spacings of the pooled sample obtained by combining all observations of individual samples from several LS distributions into a single sample.

The general theory of asymptotic estimation of the location and scale parameters of LS families of distributions was discussed by Blom (1958). Other researchers have since applied the concept of asymptotic estimation to specific LS distributions. For example, Gupta and Gnanadesikan (1966), Chan et al. (1971), Jung (1955), and D'Agostino and Lee (1976) have asymptotically approximated the best linear unbiased estimates for the location and scale parameters of the Logistic distribution, either based on the complete or otherwise censored samples.

Other methods of estimation of the location and scale parameters in LS families of distribution, related to Llyod's (1952) method, are best linear equivariant, best linear invariant, and method of estimation based on linear and polynomial coefficients. Balakrishnan et al. (2008) derived the best linear equivariant estimators (estimators that minimise the standardized mean squared error) of the location and scale parameters using two different approaches. Firstly, the best linear equivariant estimators are based on an original sample of order statistics from a LS distribution, and secondly on the joint sample of original values and predicted future observations. The highlight of Balakrishnan et al.'s (2008) results is, perhaps, that the best linear equivariant estimators based on the joint sample were the same as the estimators obtained using the original sample; and this finding turned out to be the same as the finding based on the best linear unbiased estimators established by Doganaksoy and Balakrishnan (1997) (Balakrishnan et al., 2008, p. 231). The best linear invariant estimators for the location and scale parameters are estimators that minimise the mean squared error invariant under linear transformations of the location and scale parameters (Mann and Fertig, 1973, p. 88). These estimators are linear functions of best linear unbiased estimators and based on sums of the weights of order statistics. Mann (1967) tabulated the weights for obtaining best linear invariant estimates for the location and scale parameters of Gumbel or Extreme Value distribution based on Type-II censored samples of sizes 2 through 15. Mann and Fertig (1973) used the weights obtained by Mann (1967) to calculate the confidence bounds and tolerance bounds of the Weibull distribution. Similarly to Mann’s (1967) tables of weights, Mann (1971) tabulated the weights for obtaining best linear invariant estimates for the location and scale parameters of the Gumbel distribution based on progressive censored samples of sizes 2 through 6. However, the principal disadvantage of using best linear equivariant and best linear invariant methods of estimation is that estimators obtained based on these methods have larger variances relative to the variances of best linear unbiased estimators obtained by Llyod (1952).

The methods of estimation of the location and scale parameters discussed so far in this section are based on linear functions of the parameters and order statistics of the sample. However, nonlinear methods of estimation of the location and scale parameters in the LS families of distributions are also available in literature. For example, Ene and Karahasan (2014) proposed the unbiased and nonlinear estimators of the location (one estimator) and scale (two estimators) parameters for symmetric LS distributions based on complete and symmetric double Type-II censored samples. An estimator for the location parameter and the first estimator for the scale parameter are expressed in terms of the ratios of the weighted sums of the probability weights and order statistics of the sample (Ene and Karahasan, 2014, p. 8); whereas the second estimator of the scale parameter is based on ratios of the weighted sums of the probability weights and order statistics sample quasi ranges (Ene and Karahasan, 2014, p. 9). Ene and Karahasan (2014) evaluated the performance of the three proposed estimators against that of the best linear unbiased estimators for the Normal, Logistic and Laplace distributions through simulations and real data examples. Overall, the results obtained through simulation and real data example show that the proposed estimators are competitive with those obtained using the best linear unbiased methods of estimation.

Furthermore, other choices of methods of estimation for the location and scale parameters in LS families of distribution that have also been investigated are methods of moments, simple method, linear and polynomial coefficients-based method and ML-based methods. A good reference for the estimation of the location and scale parameters using the method of moments is, for example, Schafer and Sheffield (1973). Bain and Antle (1967) derived the simple estimators for the location and scale parameters of the Gumbel distribution. Downton (1966) obtained the estimators for the location and scale parameters of the Normal and Gumbel distributions based on linear and polynomial weights of order statistics of the sample. Although the estimators by Downton (1966) are efficient for the Normal and Gumbel distributions, they cannot be applied to other LS distributions. Another difficulty with Downton's (1966) method of estimation is that the omega matrix is ill-conditioned such that it may be difficult to achieve sufficient accuracy in the final results (Balakrishnan and Cohen, 1991, p. 119).

Various studies have investigated the methods of ML estimation for the location and scale parameters in LS and LLS families of distributions, despite the fact that in some situations, for example censored samples, such estimators may not be optimal due to convergence problems resulting from computational difficulties.

However, in general, the ML estimators can be modified to either minimise considerably or completely eliminate the convergence problems, in order to make them at least nearly optimal (Balakrishnan and Cohen, 1991, p. 121). The general definition of the ML estimation of the location and scale parameters in LS and LLS families of distributions, based on a complete sample, is presented later in Section 3.6.1. Furthermore, the ML estimation for the location and scale parameters of the examples of distributions belonging to the LS and LLS families based on complete samples are discussed, in detail, in Sections 3.6.2 through 3.6.7. However, we give a brief summary of some of the references in which the methods of estimation using ML for the location and scale parameters are discussed. The exact and approximate methods of ML estimation of the location and scale parameters based on complete and censored samples are well documented in the text of Balakrishnan and Cohen (1991, Chapters 5 and 6). For example, the explicit ML estimators for the location and scale parameters of the two-parameter Exponential and Rayleigh distributions based on complete samples were derived by Balakrishnan and Cohen (1991, p. 141). In addition, Balakrishnan and Cohen (1991, p. 148) derived the explicit ML estimators for the location and scale parameters of the left truncated Normal distribution. Due to symmetry property of the Normal distribution about its population mean, the ML estimators for the location and scale parameters of the left truncated Normal distribution can also be applied to the case of right truncated Normal distribution. For the explicit ML estimators for the location and scale parameters of the Normal distribution based on a complete sample, see for example, Balakrishnan and Cohen (1991, p. 147). However, when the ML estimation is based on Type-I and Type-II, left and right, censored samples, the explicit estimators for the location and squared scale parameters were derived by Balakrishnan and Cohen (1991, p. 153). Malik (1970) obtained the explicit ML estimators for the location and scale parameters of the Pareto distribution based on order statistics of the sample. In addition, various researches have investigated the methods of obtaining the ML estimators for the location and scale parameters of various LS and LLS families of distributions using progressive censored samples. For example, Cohen (1963) obtained the ML estimators for the location and scale parameters of the Normal and two-parameter Exponential distributions, under the assumption that these parameters remain unchanged at each stage of censoring. Furthermore, Gajjar and Khatri (1969) obtained the explicit estimators for the Log-Normal and Logistic distributions.

For some LS and LLS distributions, explicit ML estimators for the location and scale parameters are not available because the likelihood functions or log likelihood functions associated with such distributions cannot be solved explicitly for the location and scale parameters. Examples of such distributions are Logistic, Log-Logistic, Gumbel and Weibull distributions. Approximate ML estimators for location and scale parameters of these distributions can be derived such that the resulting estimators are explicit functions of order statistics of the sample and that they have optimal properties (Balakrishnan and Cohen, 1991, p. 161). For more details about the approximate ML estimators for the location and scale parameters of the one-parameter Rayleigh, Normal, Logistic, and Gumbel distributions among others, based on a general type-II censored sample, see for example, the text of Balakrishnan and Cohen (1991, pp. 161-213) and all the references cited in the pages 161 through 213 of the text.

Alternatively, when ML estimation of the location and scale parameters of LS and LLS distributions cannot provide the explicit estimators, iterative procedures can be used to obtain computationally the estimates in the cases when complete and censored samples are considered. However, in some situations iterative procedures can be computationally difficult when convergence problems are encountered. The choice of suitable starting or initial values might be a problem. Various iterative procedures have been developed in the literature. For example, the Newton-Raphson (NR) method, named after the English mathematicians, Isaac Newton (1643-1727) and Joseph Raphson (~1648-~1715), is commonly used in the optimization of statistical problems. For related work on iterative procedures associated with the Logistic distribution, Wilson and Worcester (1943) discussed an iterative procedure for obtaining the ML estimators for the location and scale parameters of the Logistic based on complete samples. In addition, Plackett (1958) discussed an iterative procedure for obtaining the ML estimators for the location and scale parameters based on singly right censored samples. In the same manner, Harter and Moore (1967) discussed an iterative procedure for the ML estimators based on doubly censored samples.

3.2. Generalized Least Squares Estimation

The GLS method of estimation has been widely applied, specifically in estimation and inference problems involving linear models. The location and scale parameters of LS and LLS families of distributions can be estimated using GLS based on a general linear model (with non-diagonal variance-covariance matrix) for the order statistics from an i.i.d. sample. Specifically, the joint estimator for the location and scale parameters of LS and LLS families of distributions can be derived as described in the following (see, for example, Mann, Schafer and Singpurwalla, 1974; and Llyod, 1952).

We refer back to equation (2.2.2), which defines the standard random variate Z_* from a location-scale family as

$$Z_* = \frac{Y_* - \mu}{\sigma}$$

Solving the above equation for Y_* we can write

$$Y_* = \mu + \sigma \cdot Z_* \tag{3.2.1}$$

Now let $Y_1, Y_2, \dots, Y_n$ be an i.i.d. random sample from the distribution of Y_*, and let

$$Z_i = \frac{Y_i - \mu}{\sigma}; \quad i = 1, 2, \dots, n$$

be the corresponding standardized random variates as shown in equation (2.2.2). Furthermore, let $Y = \left[Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}\right]'$ be the vector of the order statistics of the sample $Y_1, Y_2, \dots, Y_n$, and similarly let $Z = \left[Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}\right]'$ be the vector of the order statistics of the corresponding standardized random variates $Z_1, Z_2, \dots, Z_n$. Then, using equation (3.2.1), the vector Y can be written as

$$Y = \mu \cdot \mathbf{1}_n + \sigma \cdot Z \tag{3.2.2}$$

where $\mathbf{1}_n = (1, 1, \dots, 1)'$.

The expectation and covariance matrix of Y are given by

$$E(Y) = \mu \cdot \mathbf{1}_n + \sigma \cdot E(Z) \tag{3.2.3}$$

and

$$Cov(Y) = \sigma^2 \cdot Cov(Z) = \sigma^2 \cdot V \tag{3.2.4}$$

respectively. Here $E(Z)$ and $V = Cov(Z)$ are the expected value and covariance matrix of the order statistics of an i.i.d. sample of size n from the distribution of Z_*, respectively. We note that the quantities $E(Z)$ and $V = Cov(Z)$ are known, or at any rate can be calculated, because the distribution of Z is parameter free.

Writing the scaled deviations of the standardized observations from their mean as $e = \sigma \cdot [Z - E(Z)]$, and using equations (3.2.2), (3.2.3) and (3.2.4) leads to the following general linear model (GLM) for Y:

$$Y = \mu \cdot \mathbf{1}_n + \sigma \cdot E(Z) + e; \quad Cov(e) = \sigma^2 \cdot V \tag{3.2.5}$$

In matrix notation, model (3.2.5) can be written as

$$Y = X\theta + e; \qquad Cov(e) = \sigma^2 \cdot V \tag{3.2.6}$$

where X is an $n \times 2$ matrix, and is given by

$$X = [\mathbf{1}_n : E(Z)] = \begin{bmatrix} 1 & E\{Z_{(1)}\} \\ \vdots & \vdots \\ 1 & E\{Z_{(n)}\} \end{bmatrix} \tag{3.2.7}$$

Under model (3.2.6) the generalized least squares (GLS) estimator $\hat{\theta}(Y)$ for $\theta = (\mu, \sigma)'$ is given by

$$\hat{\theta}(Y) = (X'V^{-1}X)^{-1}X'V^{-1}Y = HY \tag{3.2.8}$$

where

$$H = (X'V^{-1}X)^{-1}X'V^{-1} \tag{3.2.9}$$

Clearly, $\hat{\theta}(Y) = [\hat{\mu}(Y), \hat{\sigma}(Y)]'$ in equation (3.2.8) is the best (minimum variance) linear unbiased estimator for $\theta = (\mu, \sigma)'$.

3.3. Exact Expressions for the Mean and Covariance Matrix of the Order Statistics of Standardized Random Variates

For some LS and LLS distributions investigated in this thesis, closed form expressions for the mean and covariance matrix of order statistics of the standardized random variates are available. Thus, exact expected values and covariance matrix of such random variates can be calculated. However, if the closed form expressions of the mean and covariance matrix of order statistics of standardized random variates are unknown, then the values of such quantities can be calculated through simulation to any desired precision.

For example, let $Z = [Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}]'$ be a vector of order statistics of standardized random variates $Z_1, Z_2, \dots, Z_n$ from a Normal distribution with cdf and pdf, respectively, given by

$$F_{Z_*}(z_*; \theta) = \Phi(z_*) \tag{3.3.1}$$

and

$$f_{Z_*}(z_*; \theta) = \frac{1}{\sqrt{2\pi}} exp(-z_*^2/2)$$

where $\theta = (0, 1)'$. The support of the distribution of Z_* is on the whole real line. The exact expressions for $E[Z_{(i)}]$, $Var[Z_{(i)}]$ and $Cov[Z_{(i)}, Z_{(j)}]$ for the standard Normal distribution have been derived by various researchers, and the values of $E[Z_{(i)}]$, $Var[Z_{(i)}]$ and $Cov[Z_{(i)}, Z_{(j)}]$ have been tabulated for selected sample sizes; see for example, Ruben (1954) and Harter (1961).

In Sections 3.3.1 through 3.3.3, we present closed form expressions for the mean and covariance matrix of order statistics of the standardized random variates for the Uniform, Logistic, and Weibull distributions.

3.3.1. Closed Form Expressions for Uniform Distribution

As shown in (2.2.10), the standard random variate $Z_* = U$ follows a standard Uniform distribution in the interval $[0, 1]$. Thus, the order statistics of an i.i.d. random sample of size n from the standard Uniform distribution $Unif(0, 1)$, are denoted by the vector

$$Z = [Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}]'$$

which can be written as

$$\mathcal{U} = [U_{(1)}, U_{(2)}, \dots, U_{(n)}]'$$

The expressions of expected values (or the first moments) and variances of order statistics of standardized random variates from the Uniform distribution are, respectively, derived by Balakrishnan and Cohen (1991, pp. 30-31) as

$$E(\mathcal{U}) = \{E[U_{(1)}], E[U_{(2)}], \dots, E[U_{(n)}]\}'$$

$$= \left[\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}\right]'$$

$$= [\beta_{(1)}, \beta_{(2)}, \cdots, \beta_{(n)}]' \tag{3.3.2}$$

where $\beta_{(i)} = i/(n+1)$ for $i = 1, 2, \dots, n$ is the expected value of the ith order statistic $U_{(i)} = Z_{(i)}$ from the Uniform distribution, and

$$Var(\mathcal{U}) = Var[U_{(i)}]$$

$$= E[U_{(i)}{}^2] - \beta_{(i)}{}^2$$

$$= \frac{i(n+1-i)}{(n+2)\,(n+1)^2}$$

$$= \frac{\beta_{(i)}[1-\beta_{(i)}]}{n+2} \tag{3.3.3}$$

where $E[U_{(i)}{}^2]$ is the second moment of $U_{(i)}$.

For $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$, the covariance of $U_{(i)}$ and $U_{(j)}$ where $1 \leq i < j \leq n$, is given by

$$V_{ij} = Cov[U_{(i)}, U_{(j)}]$$

$$= E[U_{(i)}U_{(j)}]$$

$$= \frac{i(j+1)}{(n+1)\,(n+2)} \tag{3.3.4}$$

In terms of $\beta_{(i)}$, the covariance of $U_{(i)}$ and $U_{(j)}$ (3.3.4) can be written as

$$V_{ij} = \frac{\beta_{(i)}\left[1 - \beta_{(i)}\right]}{n + 2} \tag{3.3.5}$$

where $\beta_{(i)}$ is defined in (3.3.2).

Furthermore, the covariance matrix of order statistics of standardized random variates based on a sample of size n can be written as

$$Cov(U) = V_U$$

3.3.2. **Closed Form Expressions for Logistic Distribution**

The cdf and pdf of the Logistic distribution are given, respectively, in and below equation (2.2.6). In the manner of Balakrishnan and Cohen (1991, p. 12), if $Y = \left[Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}\right]'$ is a vector of order statistics of an i.i.d. continuous sample $Y_1, Y_2, \dots, Y_n$ with common cdf $F_{Y_*}[y_*; \theta]$ and pdf $f_{Y_*}[y_*; \theta]$, then the marginal pdf of $Y_{(i)}$, where $i = 1, 2, \dots, n$, is given by

$$f_{Y_{(i)}}\left[y_{(i)}; \theta\right] = \frac{n!}{(i-1)!\,(n-i)!}\left\{F_{Y_*}\left[y_{(i)}; \theta\right]\right\}^{i-1} \times$$

$$\left\{1 - F_{Y_*}\left[y_{(i)}; \theta\right]\right\}^{n-i} f_{Y_*}\left[y_{(i)}; \theta\right] \tag{3.3.6}$$

where $F_{Y_*}\left[y_{(i)}; \theta\right]$ and $f_{Y_*}\left[y_{(i)}; \theta\right]$ are, respectively, the cdf and pdf of Y_* in and below equation (2.2.6) at point $y_{(i)}$. The support of the distribution of $Y_{(i)}$ is on the whole real line. Based on the cdf and pdf in and below equation (2.2.6), let $Z = \left[Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}\right]'$ be a vector of order statistics of standardized variates $Z_1, Z_2, \dots, Z_n$ from a Logistic distribution with cdf and pdf, respectively, given by

$$F_{Z_*}(z_*; \theta) = \frac{1}{1 + exp(-z_*)} \tag{3.3.7}$$

and

$$f_{Z_*}(z_*; \theta) = \frac{exp(-z_*)}{[1 + exp(-z_*)]^2}$$

where $\theta = (0, 1)'$. Using the marginal pdf of $Y_{(i)}$ in (3.3.6), the moments of $Z_{(i)}$ for a Logistic distribution can be obtained from the moment generation function (mgf) of $Z_{(i)}$ as follows (see, for example, Balakrishnan and Cohen (1991, pp. 38-39))

$$M_{Z_{(i)}}(t) = E\{exp[tZ_{(i)}]\}$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \int_{-\infty}^{\infty} exp[tz_*] \left\{\frac{1}{1 + exp[-z_*]}\right\}^{i-1} \times$$

$$\left\{\frac{exp[-z_*]}{1 + exp[-z_*]}\right\}^{n-i} \frac{exp[-z_*]}{\{1 + exp[-z_*]\}^2} dz_* \tag{3.3.8}$$

The right hand side (RHS) of integral (3.3.8) can be simplified by transforming z_* into a uniform u, namely by letting $u = 1/\{1 + exp[-z_*]\}$ such that

$$exp[z_*] = \frac{u}{1-u}$$

and that

$$dz_* = \frac{\{1 + exp[-z_*]\}^2}{exp[-z_*]}$$

where

$$exp[-z_*] = \frac{1-u}{u}$$

Thus, after the transformation and making the suitable substitution, the mgf of $Z_{(i)}$ in (3.3.8) simplifies to

$$M_{Z_{(i)}}(t) = \frac{\Gamma(i+t)\Gamma(n-i+1-t)}{\Gamma(i)\Gamma(n-i+1)} \tag{3.3.9}$$

The mgf of $Z_{(i)}$ in (3.3.9) is, then, used to generate the first moment of $Z_{(i)}$ about the origin, namely the expected value of $Z_{(i)}$, and other higher moments of $Z_{(i)}$, as

$$E\left[Z_{(i)}\right] = \frac{d}{dt} M_{Z_{(i)}}(t)\bigg|_{t=0}$$

$$= \frac{\Gamma'(i)}{\Gamma(i)} - \frac{\Gamma'(n-i+1)}{\Gamma(n-i+1)}$$

$$= \psi(i) - \psi(n-i+1) \tag{3.3.10}$$

where the function

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

$$= \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the 'psi' or 'digamma' function.

Similarly to (3.3.10), the second moment of $Z_{(i)}$ can be derived as

$$E\left[Z_{(i)}^2\right] = \frac{d^2}{dt^2} M_{Z_{(i)}}(t)\bigg|_{t=0}$$

$$= \frac{\Gamma''(i)}{\Gamma(i)} - 2\frac{\Gamma'(i)}{\Gamma(i)}\frac{\Gamma'(n-i+1)}{\Gamma(n-i+1)} + \frac{\Gamma''(n-i+1)}{\Gamma(n-i+1)}$$

$$= \{\psi'(i) + [\psi(i)]^2\} - 2\psi(i)\,\psi(n-i+1) +$$

$$\{\psi'(n-i+1) + [\psi(n-i+1)]^2\}$$

$$= \psi'(i) + \psi'(n-i+1) + [\psi(i) - \psi(n-i+1)]^2$$

$$(3.3.11)$$

where, similarly to (3.3.10), the function

$$\psi'(x) = \frac{d}{dx}\psi(x)$$

$$= \frac{d^2}{dx^2}\ln\Gamma(x)$$

is the derivative of the 'psi or digamma' function and known as the 'trigamma' function. The variance of $Z_{(i)}$ is, then derived using the second moment of $Z_{(i)}$ in (3.3.11) and expected value of $Z_{(i)}$ in (3.3.10) as

$$Var\left[Z_{(i)}\right] = E\left[Z_{(i)}^2\right] - \left\{E\left[Z_{(i)}\right]\right\}^2$$

$$= \psi'(i) + \psi'(n-i+1) + [\psi(i) - \psi(n-i+1)]^2$$

$$-[\psi(i)\,\psi(n-i+1)]^2$$

$$= \psi'(i) + \psi'(n-i+1) \qquad (3.3.12)$$

Furthermore and similarly to (3.3.10), (3.3.11) and (3.3.12), the covariance of $Z_{(i)}$ and $Z_{(j)}$ where $1 \le i < j \le n$ can be derived and its expression is given by

$$Cov\left[Z_{(i)}, Z_{(j)}\right] = E\left[Z_{(j)}^2\right]$$

$$+\sum_{r=i}^{j-1}\sum_{s=1}^{r-1}\left[\begin{matrix}(-1)^{r+i}\binom{r-1}{i-1}\binom{n}{r}\binom{j-i-r+s}{s}\mathrm{B}(s, n-r+1)\\ \times E\left\{\check{Z}_{(j+s-r)}\right\}\end{matrix}\right] +$$

$$\binom{n}{r}\sum_{r=0}^{j-i+1}\left\{(-1)^r\binom{n-i}{r}\frac{1}{i+r}\left[\begin{matrix}-\psi'(n-j+1) + \\ \begin{Bmatrix}\psi(n-j+1) - \\ \psi(n-i+1)\end{Bmatrix}\begin{Bmatrix}\psi(j-i+r) - \\ \psi(n-j+1)\end{Bmatrix}\end{matrix}\right]\right\}$$

$$(3.3.13)$$

where $\check{Z}_{(j+s-r)}$ is the $(j+s-r)$ order statistic of standardized variates sampled of sample size $(n+s-r)$ and $\mathrm{B}(s, n-r+1)$ is a complete beta function given by

$$\mathrm{B}(s, n-r+1) = \frac{\Gamma(s)\Gamma(n-r+1)}{\Gamma(s+n-r+1)}$$

Clearly, the digamma and trigamma functions depend on the position of order statistics of the standardized variates and the sample size. Thus, the exact values of $E[Z_{(i)}]$, $E[Z_{(i)}^2]$, $Var[Z_{(i)}]$, and $Cov[Z_{(i)}, Z_{(j)}]$ can be determined for any sample of a given size drawn from a Logistic distribution. The values of the bigamma and trigamma functions of order statistics of the standardized variates from a Logistic distribution have been computed for different sample sizes. For example, Balakrishnan and Manlik (1990) tabulated the values of the means, variances and covariances of order statistics of the standardized variates from a Logistic distribution for sample sizes up to $n = 50$ (Balakrishnan and Cohen, 1991, p. 40).

3.3.3. **Closed Form Expressions for Weibull Distribution**

The cdf and pdf of the Weibull distribution are given, respectively, in and below equation (2.3.6) of Section 2.3.3. Let $W = [W_{(1)}, W_{(2)}, \dots, W_{(n)}]'$ be the vector of order statistics of a random sample of size n from the Weibull distribution. Furthermore, let $W_* = W_1, W_2, \dots, W_n$ be a random sample of size n from the Weibull distribution with cdf and pdf, respectively, given by

$$F_{W_*}(w_*; \theta^*) = 1 - exp(-w_*) \qquad (3.3.14)$$

and

$$f_{W_*}(w_*; \theta^*) = exp(-w_*)$$

where $\theta^* = (1, 1)'$. The support of the distribution of W_* is always positive. The k ($k \geq 1$) moment of $W_{(i)}$, $i = 1, 2, \dots, n$ about the origin can be derived as follows (Balakrinshnan and Cohen, 1991, p. 48):

$$E[W_{(i)}^k] = \int_{-\infty}^{\infty} w_*^k \, f_{w_{(i)}}[w_{(i)}; \theta^*] dw_*$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \int_{-\infty}^{\infty} w_*^k \left[F_{W_*}\left(w_{(i)};\theta^*\right)\right]^{i-1} \left[1 - F_{W_*}\left(w_{(i)};\theta^*\right)\right]^{n-i} \times$$

$$f_{W_*}[w_*;\theta^*]dw_*$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \int_{0}^{\infty} w_*^k \left[1 - exp(-w_*)\right]^{i-1} \times$$

$$[exp(-w_*)]^{n-i} exp(-w_*) dw_*$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \sum_{r=0}^{i-1} (-1)^r \binom{i-1}{r} \times$$

$$\int_{0}^{\infty} exp[-(n-i+r+1)w_*]\, w_*^k dw_*$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \Gamma(1+k) \times$$

$$\sum_{r=0}^{i-1} (-1)^r \binom{i-1}{r} \Big/ (n-i+r+1)^{1+k} \qquad (3.3.15)$$

Thus, using the moment of $W_{(i)}$ (3.3.15), the variance of $W_{(i)}$ for $i = 1, 2, \dots, n$ can be obtained as

$$Var\left[W_{(i)}\right] = E\left[W_{(i)}^2\right] - \left\{E\left[W_{(i)}\right]\right\}^2 \qquad (3.3.16)$$

Similarly to moment of $W_{(i)}$ (3.3.15), the covariance of $W_{(i)}$ and $W_{(j)}$ can be derived for $1 \le i < j \le n$. Initially, it is convenient to determine the joint pdf of $W_{(i)}$ and $W_{(j)}$ given by

$$f_{W_{(i)}W_{(j)}}\left[w_{(i)}, w_{(j)};\theta^*\right] = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \times$$

$$\left\{F_{W_{(i)}}\left[w_{(i)};\theta^*\right]\right\}^{i-1} \left\{F_{W_{(j)}}\left[w_{(j)};\theta^*\right] - \left[w_{(i)};\theta^*\right]\right\}^{j-i-1} \times$$

$$\left\{1 - F_{W_{(j)}}\left[w_{(j)};\theta^*\right]\right\}^{n-j} f_{W_{(i)}}\left[w_{(i)};\theta^*\right] f_{W_{(j)}}\left[w_{(j)};\theta^*\right]$$

(3.3.17)

where $-\infty < w_{(i)} < w_{(j)} < \infty$. Using (3.3.17), the expression for the cross product moment of $W_{(i)}$ and $W_{(j)}$ is given by

$$E\left[W_{(i)}W_{(j)}\right] = \int_0^\infty \int_0^\infty w_{(i)}w_{(j)}\, f_{W_{(i)}W_{(j)}}\left[w_{(i)}, w_{(j)}; \theta^*\right] dw_{(i)}dw_{(j)}$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \times$$

$$\sum\nolimits_{r=0}^{i-1}\sum\nolimits_{s=0}^{j-i-1}(-1)^{j-i-1-s+r}\binom{i-1}{r}\binom{j-i-1}{s} \times$$

$$\phi(r+s+\ 1, n-i-s) \tag{3.3.18}$$

where for $a \geq b$ such that $a/(a+b)$ lies in the uniform interval $[0,1]$, the function $\phi(a,b)$ is known as Lieblein's ϕ-function given by

$$\phi(a,b) = \frac{1}{(ab)^2}\mathrm{IB}_{a/(a+b)}[2,2] \tag{3.3.19}$$

where $\mathrm{IB}_p(b_1, b_2)$ is known as Karl Pearson's (1935) incomplete beta function. Finally, using (3.3.18) the covariance between $W_{(i)}$ and $W_{(j)}$ where $1 \leq i < j \leq n$ can be derived as

$$Cov\left[W_{(i)}, W_{(j)}\right] = E\left[W_{(i)}W_{(j)}\right] - E\left[W_{(i)}\right]E\left[W_{(j)}\right] \tag{3.3.20}$$

3.4. Calculation of the Expected Value and Covariance Matrix of Z by Simulation

Evaluating the GLS estimator $\hat{\theta}(Y)$ for θ using the vector of order statistics of the random sample of observations $Y_1, Y_2, \dots, Y_n$ (refer to equation (3.2.8)) in Section 3.2 requires the calculation of the values of $E(Z)$ and covariance matrix $V = Cov(Z)$. When the closed form expressions of these quantities are not available, the values of $E(Z)$ and V can be simulated as described by the following algorithm.

Algorithm 1: Calculation of $E(Z)$ and $Cov(Z)$ by simulation:

1. Simulate N independent standardized random samples $Z_1, Z_2, \ldots, Z_N$ of size n from the distribution of Z_*.
2. Sort each of the N simulated standardized random samples in ascending order, thereby obtaining the order statistics $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ of the simulated values within each random sample. We denote the sorted samples by $\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_N$. That is, the sample $\mathcal{Z}_i$, where $i = 1, 2, \ldots, N$ contains the order statistics $Z_{(1)i}, Z_{(2)i}, \ldots, Z_{(n)i}$ for the ith sorted sample.
3. Calculate the expected value of order statistics of standardized random variates as the average
$$E(Z) = \frac{1}{N}\sum_{i=1}^{N} \mathcal{Z}_i \tag{3.4.1}$$
4. Calculate the covariance matrix of order statistics of the standardized random variates as
$$\begin{aligned} V &= Cov(Z) \\ &= \frac{1}{N}\sum_{i=1}^{N} [\mathcal{Z}_i - E(Z)]\,[\mathcal{Z}_i - E(Z)]' \end{aligned} \tag{3.4.2}$$

The simulated values of $E(Z)$ and inverse of the covariance matrix V, namely V^{-1} can then be used to estimate the location and scale parameters of various LS and LLS distributions. In addition, the simulated values of $E(Z)$ and V^{-1} can be used to simulate the values of rank-based conventional pivotal quantities and fiducial generalized pivotal quantities as presented in Chapters 4 and 5.

3.5. Approximations of Mean and Covariance Matrix of the Order Statistics of Standardized Random Variates

The approximations of expressions for the mean and covariance matrix of order statistics of the standardized variates drawn from an arbitrary continuous distribution can be obtained by using the method of David and Johnson (1954) (Balakrishnan and Cohen, 1991, p. 68). We note that approximations of the expected value and covariance matrix of order statistics of the standardized variates from an arbitrary continuous distribution, using David and Johnson's (1954) method, require the transformation of order statistics of the standardized variates into the standardized Uniform order statistics. David and Johnson (1954) showed that in general, it is easier to obtain the moments of order statistics variates for continuous distributions, because the pdf of the ith order statistic is equivalent to the pdf of the two-parameter Beta distribution (Balakrishnan and Cohen, 1991, p. 12). As a result of this property, it is well known that a standard two-parameter Beta distribution is the same as a standard Uniform distribution.

David and Johnson's (1954) method of approximation of the expected value, variance and covariance matrix of the standardized order statistics from an arbitrary continuous distribution, can be summarized as follows (Balakrishnan and Cohen, 1991, p. 69):

Initially, it is convenient to obtain the first, second, and other higher order derivatives of the transformation function $G[F^{-1}(U)]$ with respect to U, where U is a standard Uniform variate. These derivatives are then evaluated at the expected values of order statistics of the Uniform distribution, namely at $E\left[U_{(i)}\right] = \beta_{(i)}$. In this thesis, however, only the first and second order derivatives of the function $G[F^{-1}(U)]$ are required to obtain the approximations of $E(Z)$ and $Cov(Z)$ up to the third term of Taylor series expansion about the point $E\left[U_{(i)}\right] = \beta_{(i)}$, as shown below (Balakrishnan and Cohen, 1991, p. 69):

The first order derivative of the function $G[F^{-1}(U)]$ is given by

$$\frac{d}{dU} G[F^{-1}(U)] = G'[F^{-1}(U)] \cdot \frac{d}{dU}[F^{-1}(U)]$$

$$= G'[F^{-1}(U)] \cdot \frac{1}{f[F^{-1}(U)]} \qquad (3.5.1)$$

where $G'(\cdot)$ denotes the first order derivative of the function $G(\cdot)$ and $d/dU\,[F^{-1}(U)]$ can be obtained as

$$\frac{d}{dU}[F^{-1}(U)] = \frac{1}{F'[F^{-1}(U)]}$$

$$= \frac{1}{f'[F^{-1}(U)]}$$

where $F'(\cdot) = f(\cdot)$.

Similarly to (3.5.1), the second order derivative of $G[F^{-1}(U)]$ can be obtained by determining the second order derivative of the function $G[F^{-1}(U)]$ as

$$\frac{d^2}{dU^2} G[F^{-1}(U)] = \frac{d}{dU}\left\{G'[F^{-1}(U)] \cdot \frac{1}{f[F^{-1}(U)]}\right\}$$

$$= G''[F^{-1}(U)] \cdot \frac{1}{f[F^{-1}(U)]} \cdot \frac{1}{f[F^{-1}(U)]} - G'[F^{-1}(U)] \times$$

$$\cdot \qquad \frac{1}{\{f[F^{-1}(U)]\}^2} \cdot f'[F^{-1}(U)] \cdot \frac{1}{f[F^{-1}(U)]}$$

$$= \frac{G''[F^{-1}(U)] \cdot f[F^{-1}(U)] - G'[F^{-1}(U)] \cdot f'[F^{-1}(U)]}{\{f[F^{-1}(U)]\}^3}$$

$$(3.5.2)$$

3.5.1. **Approximation of Expected Value of Z**

In Sections 3.5.1 and 3.5.2, the general principle of David and Johnson's (1954) method described in Section 3.5 can be used to derive the approximate expressions of $E(Z)$ and $Cov(Z)$ for an arbitrary continuous distribution. Based on the probability integral transformation, the David and Johnson's (1954) method explicitly uses the cdf $U = F_{Z_*}(Z_*;\theta)$ to transform the order statistics $Z_{(i)}, i = 1, 2, \dots, n$ from a LS distribution with a cdf and pdf, $F_{Z_*}(z_*;\theta)$ and $f_{Z_*}(z_*;\theta)$ respectively, into the Uniform order statistics $U_{(i)}$ whose LS distribution has a cdf $F_{U_*}(u_*;\theta)$ and a pdf $f_{U_*}(u_*;\theta)$ where $\theta = (0, 1)'$. Here, the values 0 and 1, denote respectively the lower and upper boundaries of the standard Uniform distribution as presented in Section 2.2.3. Thus, by finding the inverse of the transformation, namely finding the inverse of the cdf $F_{U_*}(u_*;\theta)$ leads to

$$\begin{aligned} Z_{(i)} &= F^{-1}\left[U_{(i)}\right] \\ &= G\left[U_{(i)}\right] \end{aligned} \tag{3.5.3}$$

Then, when a Taylor series expansion is applied to the function (3.5.3) at the point $E\left[U_{(i)}\right] = \beta_{(i)}$ up to the fourth term of the series leads to (refer to Balakrishnan and Cohen (1991, p. 69))

$$\begin{aligned} Z_{(i)} &= \sum_{n=0}^{\infty} \frac{G_i^{(n)}\left[U_{(i)} - \beta_{(i)}\right]^n}{n!} \\ &= G_i + \frac{G_i'\left[U_{(i)} - \beta_{(i)}\right]}{1!} + \frac{G_i''\left[U_{(i)} - \beta_{(i)}\right]^2}{2!} + \frac{G_i'''\left[U_{(i)} - \beta_{(i)}\right]^3}{3!} + \cdots \\ &= G_i + G_i'\left[U_{(i)} - \beta_{(i)}\right] + \frac{G_i''\left[U_{(i)} - \beta_{(i)}\right]^2}{2} + \frac{G_i'''\left[U_{(i)} - \beta_{(i)}\right]^3}{6} + \cdots \end{aligned} \tag{3.5.4}$$

where $G_i = G[\beta_{(i)}]$, $G'_i = d/dU\, G(U)|_{U=\beta_{(i)}}$, and similarly to the first derivative, G''_i, G'''_i, ..., are equivalent to successive (i.e. second, third, … ,) derivatives of the function $G(U)$ evaluated at point $U = \beta_{(i)}$.

Following the Taylor series expansion of the inverse transformation function (3.5.3), the expression for the approximate $E[Z_{(i)}]$ can now be obtained by taking expectation on both sides of equation (3.5.4) as (Balakrishnan and Cohen, 1991, p. 69)

$$E[Z_{(i)}] = E\{G(F^{-1}[U_{(i)}])\}$$

$$\approx G\{F^{-1}(E[U_{(i)}])\} + \frac{Var[U_{(i)}]}{2} \cdot \frac{d^2}{dU_{(i)}^2} G(F^{-1}[U_{(i)}]) \bigg|_{U_{(i)}=\beta_{(i)}}$$

$$= G\{F^{-1}[\beta_{(i)}]\} + \frac{\beta_{(i)}[1-\beta_{(i)}]}{2(n+2)} \times \cdot$$

$$\frac{G''[F^{-1}(\beta_{(i)})] \cdot f[F^{-1}(\beta_{(i)})] - G'[F^{-1}(\beta_{(i)})] \cdot f'[F^{-1}(\beta_{(i)})]}{\{f[F^{-1}(\beta_{(i)})]\}^3}$$

(3.5.5)

where the expressions for $\beta_{(i)}$'s are given in (3.3.2).

3.5.2. Approximation of Covariance Matrix of Z

Similarly to obtaining (3.5.5), the expression for approximate $Var[Z_{(i)}]$ can be obtained using the following steps (for a detailed discussion, refer to Balakrishnan and Cohen (1991, p. 69)):

1. Use equation (3.5.4) to apply a Taylor series expansion to $Z_{(i)}^2$.

2. Then, take the expectation of a Taylor series expansion of $Z_{(i)}^2$ obtained in the first step.

3. Lastly, the approximate $Var[Z_{(i)}]$ is obtained by subtracting the expression of $\{E[Z_{(i)}]\}^2$, namely equation (3.5.5) raised to the power of two, from the expectation of a Taylor series expansion of $Z_{(i)}^2$.

Thus, the expression for approximate variance of $Z_{(i)}$ is given by

$$Var[Z_{(i)}] \approx Var[U_{(i)}] \cdot d_{(i)}^2 \tag{3.5.6}$$

where the expression for $d_{(i)}, i = 1, 2, \dots, n$ is given by

$$d_{(i)} = \frac{d}{dU_{(i)}} G\{F^{-1}[U_{(i)}]\}\bigg|_{U_{(i)}=\beta_{(i)}}$$

$$= \frac{G'\{F^{-1}[\beta_{(i)}]\}}{f\{F^{-1}[U_{(i)}]\}} \tag{3.5.7}$$

and the expression for $Var[U_{(i)}]$ is given in (3.3.3). In terms of equations (3.3.3) and (3.5.7), the approximation of $Var[Z_{(i)}]$ in (3.5.6) is the same as

$$Var[Z_{(i)}] = \frac{\beta_{(i)}[1-\beta_{(i)}]}{n+2} \cdot \left(\frac{G'\{F^{-1}[\beta_{(i)}]\}}{f\{F^{-1}[U_{(i)}]\}}\right)^2 \tag{3.5.8}$$

Furthermore, for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$, where $1 \leq i < j \leq n$ and similarly to obtaining (3.5.8), the expression for approximate $Cov[Z_{(i)}, Z_{(j)}]$ can be obtained using the following steps (for a detailed discussion, refer to Balakrishnan and Cohen (1991, p. 70)):

1. Use equation (3.5.4) to expand a Taylor series expansion to $Z_{(i)}Z_{(j)}$.

2. Then, take the expectation of a Taylor series expansion of $Z_{(i)}Z_{(j)}$ obtained in the first step.
3. Lastly, the approximate $Cov[Z_{(i)}, Z_{(j)}]$ is obtained by subtracting the expression of the product $E[Z_{(i)}] \cdot E[Z_{(j)}]$ from the expectation of a Taylor series of $Z_{(i)}Z_{(j)}$.

Thus, the expression for approximate covariance matrix for Z is given by

$$Cov(Z) = Cov\{G[F^{-1}(U)]\}$$
$$\approx DCov(U)D$$
$$= DV_U D \qquad (3.5.9)$$

where $D = diag\left[d_{(1)}, d_{(2)}, \dots, d_{(n)}\ \right]$ is a diagonal matrix of dimension n by n, of which the diagonal entries are given in (3.5.7), for $i = 1, 2, \dots, n$; and V_U is a covariance matrix of order statistics of standardized variates from a Uniform distribution of sample size n, of which the entries, namely the variances of $U_{(i)}$'s and covariance between $U_{(i)}$'s and $U_{(j)}$'s for $1 \leq i < j \leq n$, are defined in (3.3.3) and (3.3.4) respectively.

3.6. Maximum Likelihood Estimation

In Section 3.61 below we present a general definition of method of the ML estimation for the vector parameter of arbitrary distribution. We then present in Sections 3.6.2 through 3.6.7 the ML estimation of location and scale parameters of examples of distributions that belong to LS and LLS families.

3.6.1. Definition for Maximum Likelihood Estimation

In some situations, the method of maximum likelihood is used for parameter estimation problems where explicit expressions for the maximum likelihood estimators as functions of the observations exist. However, when such explicit expressions do not exist, it may be possible to obtain numerically the maximum likelihood estimators for the parameters by using various methods of optimization. In most cases, methods of optimization can handle the maximization of complex log-likelihood functions, which the classical ML estimation procedures cannot. For a detailed discussion of optimization methods for maximum likelihood, refer to, for example, Lawless (2003, p. 555).

MLE for the location and scale parameters of LS and LLS families of distributions can be defined as follows:

Let $x = (x_1, x_2, \dots, x_n)'$ be a vector of n independent observations (not necessarily arranged in ascending order of magnitude) sampled randomly from a distribution with cdf $F_X(x;\theta)$ and joint pdf $f_X(x;\theta)$. Here, θ is a vector of unknown parameters of the distribution $F_X(\cdot)$. The maximum likelihood estimate for θ based on x is denoted by $\hat{\theta}(x)_{ML} \in \Theta$, where Θ is the parameter space of θ, containing all possible combinations of the values of $\hat{\theta}(x)_{ML}$. In this thesis, unless stated otherwise, we use the subscript $(\cdot)_{ML}$ on the estimators $\hat{\theta}(\cdot)$ to distinguish them from the estimators based on rank-based GLS method. We assume that the analytic form of the cdf $F_X(\cdot)$ is known.

The likelihood function of θ based on observations x, namely $L(\theta;x)$, is the product of the marginal probability density functions (pdf's) and is written as

$$L(\theta;x) = \prod_{i=1}^{n} f_X(x_i;\theta) \tag{3.6.1}$$

In most cases, it is convenient to take the logarithm of the likelihood function, denoted by $l(\theta;x)$, and which is given by

$$l(\theta;x) = \log[L(\theta;x)] = \sum_{i=1}^{n} \log[f_X(x_i;\theta)] \tag{3.6.2}$$

The maximum likelihood estimate $\hat{\theta}(x)_{ML}$ is obtained by maximizing the likelihood function $L(\theta;x)$ or the logarithm of the likelihood function $l(\theta;x)$. To this end, one can determine the partial derivatives of $L(\theta;y)$ or $l(\theta;x)$ with respect to θ individually or simultaneously, and then equating the partial derivatives to zero. For any observed values $x = (x_1, x_2, \dots, x_n)'$, the maximum likelihood estimates (MLEs) of θ are denoted by $\hat{\theta}(x)_{ML}$.

When a closed form expression for $\hat{\theta}(x)_{ML}$ which maximizes the likelihood function (3.6.1) or log-likelihood (3.6.2) is not available, the maximum likelihood estimate $\hat{\theta}(x)_{ML}$ of θ can often be obtained through numerical optimization methods. For example, the iterative procedures can be used in such cases. Depending mainly on the shape of the likelihood functions or log-likelihood functions, various types of iterative methods have been developed in the literature. For example, NR, the system of non-linear equations, the direct maximization of the log-likelihood function, and a non-linear optimization are some of the examples of iterative methods. Thus, the shape of the likelihood functions or log-likelihood functions can be used as a criterion for selecting a suitable iterative method. As a result of the central limit theorem, the shape of many likelihood functions or log-likelihood functions tend to approximate quadratic functions. Thus, in this thesis, the NR iterative method, which is widely applied to many statistical maximization problems, is used.

3.6.2. MLE of θ for the Normal and Log-Normal Distributions

The explicit ML estimates for the location and scale parameters of the Normal and Log-Normal distributions based on complete samples are well known. For the Normal distribution, the explicit ML estimates for μ and σ are, respectively, given by (see, for example, Balakrishnan and Cohen, 1991, p. 147)

$$\hat{\mu}(y)_{ML} = \bar{y} = \sum_{i=1}^{n} y_i/n \tag{3.6.3}$$

and

$$\hat{\sigma}(y)_{ML} = [(n-1)s^2/n]^{1/2} \tag{3.6.4}$$

where $\bar{y}$ is the sample mean and s^2 is the sample variance, which is

$$s^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2/(n-1) \tag{3.6.5}$$

Similarly to (3.6.3) and (3.6.4), the explicit ML estimates for the location and scale parameters of the Log-Normal distribution based on a complete sample can be obtained as follows:

Based on the general definition of the LLS family of distributions in Section 2.3, let $W_1, W_2, \dots, W_n$ be an i.i.d. random sample of size n from the distribution of W_*.

Furthermore, let $w = (w_1, w_2, \dots, w_n)'$ be a vector of observations associated with a random sample $W_1, W_2, \dots, W_n$ of size n from any LLS distribution investigated in this thesis. Then, the explicit ML estimates for μ and σ of the Log-Normal distribution are, respectively, given by

$$\widehat{\mu^*}(w)_{ML} = \sum_{i=1}^{n} \log(w_i)/n \tag{3.6.6}$$

and

$$\widehat{\sigma^*}(w)_{ML} = \left\{\sum_{i=1}^{n} [\log(w_i) - \widehat{\mu^*}(w)_{ML}]^2/n\right\}^{1/2} \tag{3.6.7}$$

The ML estimates for the location and scale parameters of the Normal and Log-Normal distributions base on censored samples can be obtained using the iterative procedures. For the log-likelihood function for the location and scale parameters of these LS distributions and a detailed discussion, refer to Lawless (2003, p. 230).

3.6.3. MLE of θ^* for the Weibull Distribution

The MLE for the parameters of the Weibull distribution has been investigated by many researchers; see, for example, Balakrishnan and Kateri (2008), and the text of Rinne (2009, Chapter 11). It has been shown (see, for example, Rinne (2009, pp. 412-413)) that ML estimates for the parameters of the Weibull distribution based on a complete sample cannot be obtained explicitly from the equations of partial derivatives. To overcome this difficulty, an alternative approach to this method of estimation was proposed and is presented as follows.

Let w be a vector of the i.i.d. sample of observations drawn from the Weibull population with parameters μ^* and σ^* as defined earlier in Sections 2.3.3. and

3.6.2. Then, the log-likelihood function of the parameter vector $\theta^* = (\mu^*, \sigma^*)'$, given the data $w = (w_1, w_2, \dots, w_n)'$, can be given by

$$l(\theta^*; w) = n[\log(\sigma^*) - \sigma^* \log(\mu^*)] +$$

$$(\sigma^* - 1)\sum_{i=1}^{n} \log(w_i) - \sum_{i=1}^{n} (w_i/\mu^*)^{\sigma^*}$$

Thus, taking the partial derivatives of the log-likelihood function with respect to the parameters μ^* and σ^*, and equating the partial derivatives to zero lead to the estimating equations (3.6.8) and (3.6.9)

$$\frac{\partial l(\theta^*; w)}{\partial \mu^*} = -\frac{n\sigma^*}{\mu^*} + \frac{\mu^*}{\sigma^*}\sum_{i=1}^{n} (w_i/\mu^*)^{\sigma^*} = 0 \tag{3.6.8}$$

$$\frac{\partial l(\theta^*; w)}{\partial \sigma^*} = \frac{n}{\sigma^*} - n\log(\mu^*) + \sum_{i=1}^{n} \log(w_i) -$$

$$\sum_{i=1}^{n} (w_i/\mu^*)^{\sigma^*} \log(w_i/\mu^*) = 0 \tag{3.6.9}$$

Clearly, the partial derivative (estimating) equations (3.6.8) and (3.6.9) are non-linear in μ^* and σ^* respectively. Thus, they may be solved numerically using iterative methods. Alternatively, the approximate ML estimates for the parameters of the Weibull distribution, when available, may be used. However, using estimating equation for μ^* (3.6.8), it may be convenient to express μ^* in terms of the parameter σ^*, explicitly, so that only estimating equation for σ^* can be solved iteratively. That is,

$$\mu^* = \left[\sum_{i=1}^{n} (w_i)^{\sigma^*}/n\right]^{1/\sigma^*} \tag{3.6.10}$$

It follows from the explicit equation for μ^* (3.6.10) that

$$(\mu^*)^{\sigma^*} = \sum_{i=1}^{n} (w_i)^{\sigma^*}/n \tag{3.6.11}$$

Substituting the RHS of equation (3.6.10) for μ^*, and RHS of equation (3.6.11) for $(\mu^*)^{\sigma^*}$ in estimating equation (3.6.9) lead to the simplified estimating equation

$$\frac{1}{\sigma^*}+\frac{1}{n}\sum_{i=1}^{n}\log(w_i)-\frac{\sum_{i=1}^{n}\log(w_i)}{\sum_{i=1}^{n}(w_i)^{\sigma^*}}=0 \qquad (3.6.12)$$

The ML estimate for the parameter σ^*, namely $\widehat{\sigma^*}(w)_{ML}$ can then be obtained by using a NR iterative procedure. After $\widehat{\sigma^*}(w)_{ML}$ has been obtained iteratively, it is substituted for σ^* in explicit equation for μ^* (3.6.10) to obtain $\widehat{\mu^*}(w)_{ML}$.

Thus, the ML estimate for μ^*, which depends on $\widehat{\sigma^*}(w)_{ML}$, of the Weibull distribution is given by

$$\widehat{\mu^*}(w)_{ML}=\left[\sum_{i=1}^{n}(w_i)^{\widehat{\sigma^*}(w)_{ML}}/n\right]^{1/\widehat{\sigma^*}(w)_{ML}} \qquad (3.6.13)$$

Furthermore, in the manner of Rinne (2009, p. 416) and using the ML estimate for μ^* (3.6.13), let

$$g[\widehat{\sigma^*}(w)]=\frac{1}{\widehat{\sigma^*}(w)}+\frac{1}{n}\sum_{i=1}^{n}\log(w_i)-\frac{\sum_{i=1}^{n}\log(w_i)}{\sum_{i=1}^{n}(w_i)^{\widehat{\sigma^*}(w)}}$$

Then

$$g'[\widehat{\sigma^*}(w)]=\frac{\partial}{\partial\widehat{\sigma^*}(w)}g[\widehat{\sigma^*}(w)]$$

$$=\frac{\sum_{i=1}^{n}\log(w_i)\sum_{i=1}^{n}\log(w_i)\,(w_i)^{\widehat{\sigma^*}(w)}}{\left\{\sum_{i=1}^{n}(w_i)^{\widehat{\sigma^*}(w)}\right\}^2}-\frac{1}{[\widehat{\sigma^*}(w)]^2}$$

Thus, an NR iterative procedure for obtaining $\widehat{\sigma^*}(w)_{ML}$ can be summarized as follows

$$\widehat{\sigma^*}(w)_{c+1}=\widehat{\sigma^*}(w)_c-\frac{g[\widehat{\sigma^*}(w)_c]}{g'[\widehat{\sigma^*}(w)_c]} \qquad (3.6.14)$$

where $c=0,1,2,\ldots$ denotes the number of iterations. We present the NR algorithm for obtaining numerically the ML estimates for the scale and shape parameters of the Weibull distribution as follows:

Algorithm 2: Calculating $\widehat{\mu^*}(w)_{ML}$ and $\widehat{\sigma^*}(w)_{ML}$ for Weibull distribution

1. Let $w_1, w_2, \ldots, w_n$ be an i.i.d. random sample of observations of size n from a Weibull distribution, namely $Weib(\mu^*, \sigma^*)$.
2. Start with a suitable initial estimate $\widehat{\sigma^*}(w)_0$.
3. For $c = 0, 1, 2, \ldots$ substitute the value $\widehat{\sigma^*}(w)_c$ into equation (3.6.14) and then calculate $\widehat{\sigma^*}(w)_{c+1}$.
4. Repeat step 3 iteratively until the values of $\widehat{\sigma^*}(w)_c$ converge to $\widehat{\sigma^*}(w)_{ML}$.
5. After the convergence of $\widehat{\sigma^*}(w)_{ML}$ is achieved, substitute the ML estimate $\widehat{\sigma^*}(w)_{ML}$ for σ^* into the ML estimate for μ^* (3.6.13), thereby obtaining the ML estimate of μ^*.

Alternatively, the approximate ML estimates for the scale and shape parameters of the Weibull distribution based on a complete sample, when available, may be used. The approximate ML estimates for the location and scale parameters of the Gumbel distribution, which is the anti-log distribution of the Weibull, based on Type-II or doubly censored sample were derived by Balakrishnan and Cohen (1991, p. 189). Similarly, the log-likelihood function of the location and scale parameters of the Gumbel distribution, and the associated first and second order partial derivative equations of the likelihood function, based on a censored sample; standard survivor and pdf functions, are discussed by Lawless (2003, pp. 218-219).

3.6.4. MLE of θ for the Logistic Distribution and θ^* for the Log-Logistic Distribution

Let $y_1, y_2, \ldots, y_n$ be an i.i.d. random sample of observations drawn from the Logistic distribution with location and scale parameters μ and σ, respectively, as defined in Section 2.2.2. Then, the ML estimates for μ and σ, namely $\hat{\mu}(y)_{ML}$

and $\hat{\sigma}(y)_{ML}$, may be obtained by solving estimating equations (3.6.15) and (3.6.16) below (see, for example, Mahdi and Cenac (2006)):

$$\frac{\partial l(\theta; y)}{\partial \mu} = \frac{n}{\sigma} - \frac{2}{\sigma} \sum_{i=1}^{n} \left\{ \frac{exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]}{1 + exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]} \right\} = 0 \qquad (3.6.15)$$

and

$$\frac{\partial l(\theta; y)}{\partial \sigma} = \sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma^2}\right) - \frac{n}{\sigma} -$$

$$\frac{2}{\sigma^2} \sum_{i=1}^{n} \left\{ \frac{(y_i - \mu) exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]}{1 + exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]} \right\} = 0 \qquad (3.6.16)$$

Furthermore, estimating equations (3.6.15) and (3.6.16) can be simplified to estimating equations

$$\frac{n}{2} = \sum_{i=1}^{n} \left[\frac{1}{1 + exp\left(\frac{y_i - \mu}{\sigma}\right)} \right] \qquad (3.6.17)$$

and

$$\sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma}\right) \left\{ \frac{1 - exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]}{1 + exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]} \right\} = n \qquad (3.6.18)$$

respectively. Then, the ML estimates for μ and σ, namely $\hat{\mu}(y)_{ML}$ and $\hat{\sigma}(y)_{ML}$ based on a complete sample y can be obtained numerically from estimating equations (3.6.17) and (3.6.18) by using the NR iterative procedure.

Furthermore, the ML estimates for μ^* and σ^* of the Log-Logistic distribution based on a complete sample w, namely $\widehat{\mu^*}(w)_{ML}$ and $\widehat{\sigma^*}(w)_{ML}$, can be obtained by solving the following estimating equations for μ^* and σ^*, respectively (see, for example, Singh and Guo (1995)):

$$2\sum_{i=1}^{n}\left[\frac{\left(\frac{w_i}{\mu^*}\right)^{\sigma^*}}{1+\left(\frac{w_i}{\mu^*}\right)^{\sigma^*}}\right]=n \tag{3.6.19}$$

$$2\sigma^*\sum_{i=1}^{n}\left[\frac{\log\left(\frac{w_i}{\mu^*}\right)\left(\frac{w_i}{\mu^*}\right)^{\sigma^*}}{1+\left(\frac{w_i}{\mu^*}\right)^{\sigma^*}}\right]-\sigma^*\sum_{i=1}^{n}\log\left(\frac{w_i}{\mu^*}\right)-n=0 \tag{3.6.20}$$

As is the case with the Logistic distribution, the ML estimates $\widehat{\mu^*}(w)_{ML}$ and $\widehat{\sigma^*}(w)_{ML}$ for the Log-Logistic distribution based on the complete sample w can be obtained numerically by using the NR iterative procedure. Alternatively, the approximate ML estimates for the parameters of the Logistic and Log-Logistic distributions based on complete samples, when available, may be used. Balakrishnan and Cohen (1991, p. 179) derived the approximate ML estimates for the location and scale parameters of the Logistic distribution based on a doubly censored sample. For further related work, refer to Lawless (2003, p. 232), who derived the log-likelihood function for the location and scale parameters of the Logistic distribution, and the first and second partial derivatives of the log-likelihood function, based on the standard survivor and pdf functions and a censored sample.

3.6.5. MLE of θ for the Uniform Distribution

Let $y_1, y_2, \dots, y_n$ be an i.i.d. random sample of observations drawn from the Uniform distribution with the location and scale parameters $a=\mu$ and $b-a=\sigma$, respectively, as defined earlier in Section 2.2.3. Then, explicit ML estimates for a and $b-a$, namely $\hat{a}(y)_{ML}$ and $\widehat{b-a}(y)_{ML}$, can be obtained as follows:

The likelihood function of $\theta=(a,b)'$, given the data y can be given by

$$L(\theta;y)=\prod_{i=1}^{n}\frac{1}{b-a}=(b-a)^{-n}$$

and the log-likelihood function associated with the likelihood function can be given by

$$l(\theta; y) = \log[(b-a)^{-n}] = -n\log(b-a) \qquad (3.6.21)$$

The log-likelihood function (3.6.21) can be maximized by minimizing the value of $(b-a)$ under the constraint that $a \leq y_{(1)}$ and $b \geq y_{(n)}$, where $y_{(1)} = \min_{1\leq i\leq n}(y_i)$ and $y_{(n)} = \max_{1\leq i\leq n}(y_i)$. Thus, explicit ML estimates for a and b are, respectively, $\hat{a}(y)_{ML} = y_{(1)}$ and $\hat{b}(y)_{ML} = y_{(n)}$. However, explicit unbiased ML estimates for a and b are given by

$$\hat{a}(y)_{ML} = \frac{n-1}{n} y_{(1)} \qquad (3.6.22)$$

and

$$\hat{b}(y)_{ML} = \frac{n+1}{n} y_{(n)}$$

respectively. In terms of LS parametrization, the unbiased ML estimate for the scale parameter $\sigma = b - a$ is given by

$$\hat{\sigma}(y)_{ML} = \hat{b}(y)_{ML} - \hat{a}(y)_{ML} \qquad (3.6.23)$$

3.6.6. MLE of θ for the Gumbel Distribution

Let $y_1, y_2, \dots, y_n$ be an i.i.d. random sample of observations drawn from the Gumbel distribution with parameters μ and σ as defined in Section 2.2.4. Then, the respective ML estimates for the location and scale parameters μ and σ, namely $\hat{\mu}(y)_{ML}$ and $\hat{\sigma}(y)_{ML}$, can be obtained as follows (Mahdi and Cenac (2005)):

The likelihood function of $\theta = (\mu, \sigma)'$, given the data y is given by

$$L(\theta; y) = \sigma^{-n} exp\left[-\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)\right] exp\left\{-\sum_{i=1}^{n} exp\left[-\left(\frac{y_i-\mu}{\sigma}\right)\right]\right\}$$

and the log-likelihood function associated with the likelihood function is given by

$$l(\theta; y) = -n\log(\sigma) - \sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma}\right) -$$

$$\sum_{i=1}^{n} exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right] \quad (3.6.24)$$

The log-likelihood function (3.6.24) can be maximized by finding its partial derivatives with respect to μ and σ, and equating to zero, which lead, respectively, to estimating equations

$$\frac{\partial l(\theta; y)}{\partial \mu} = \frac{1}{\sigma}\left\{n - \sum_{i=1}^{n} exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right]\right\} = 0 \quad (3.6.25)$$

and

$$\frac{\partial l(\theta; y)}{\partial \sigma} = \sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma}\right) - \frac{n}{\sigma} -$$

$$\sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma^2}\right) exp\left[-\left(\frac{y_i - \mu}{\sigma}\right)\right] = 0 \quad (3.6.26)$$

where $\sigma \neq 0.$ Furthermore, estimating equations (3.6.25) and (3.6.26) can be simplified, respectively, to

$$\mu = \sigma\left\{\log(n) - \log\left[\sum_{i=1}^{n} exp\left\{-\left(\frac{y_i}{\sigma}\right)\right\}\right]\right\} \quad (3.6.27)$$

and

$$\bar{y} = \sigma + \frac{\sum_{i=1}^{n} y_i\, exp\left[-\left(\frac{y_i}{\sigma}\right)\right]}{\sum_{i=1}^{n} exp\left[-\left(\frac{y_i}{\sigma}\right)\right]} \quad (3.6.28)$$

where $\bar{y} = \sum_{i=1}^{n} y_i/n$ is a sample mean. Thus, the ML estimates for σ, namely $\hat{\sigma}(y)_{ML}$ for a Gumbel distribution can be obtained numerically by applying the NR iterative procedure to estimating equation (3.6.28). Then, the ML estimate for μ, namely $\hat{\mu}(y)_{ML}$ can be obtained implicitly by substituting $\hat{\sigma}(y)_{ML}$ for σ in equation (3.6.27). Thus,

$$\hat{\mu}(y)_{ML} = \hat{\sigma}(y)_{ML}\left\{\log(n) - \log\left[\sum_{i=1}^{n} exp\left\{-\left(\frac{y_i}{\hat{\sigma}(y)_{ML}}\right)\right\}\right]\right\} \qquad (3.6.29)$$

In the manner of Rinne (2009, p. 416) and using (3.6.28), let

$$g[\hat{\sigma}(y)] = \hat{\sigma}(y) + \frac{\sum_{i=1}^{n} y_i\, exp\left[-\left(\frac{y_i}{\hat{\sigma}(y)}\right)\right]}{\sum_{i=1}^{n} exp\left[-\left(\frac{y_i}{\hat{\sigma}(y)}\right)\right]} - \bar{y}$$

Then,

$$g'[\hat{\sigma}(y)] = \frac{\partial}{\partial\hat{\sigma}(y)} g[\hat{\sigma}(y)]$$

$$= 1 + \frac{0}{\left\{\sum_{i=1}^{n} exp\left[-\left(\frac{y_i}{\hat{\sigma}(y)}\right)\right]\right\}^2} = 1$$

Thus, an NR iterative procedure for obtaining $\hat{\sigma}(y)_{ML}$ can be summarized as follows

$$\hat{\sigma}(y)_{c+1} = \hat{\sigma}(y)_c - \frac{g[\hat{\sigma}(y)_c]}{g'[\hat{\sigma}(y)]}$$

$$= \hat{\sigma}(Y)_c - \frac{\sum_{i=1}^{n} y_i\, exp\left[-\left(\frac{y_i}{\hat{\sigma}(y)_c}\right)\right]}{\sum_{i=1}^{n} exp\left[-\left(\frac{y_i}{\hat{\sigma}(y)_c}\right)\right]} - \bar{y} \qquad (3.6.30)$$

where $c = 0, 1, 2, \ldots$ denotes the number of iterations. We present NR algorithm for obtaining numerically the ML estimates for the parameters of the Gumbel distribution as follows:

Algorithm 3: Calculating $\hat{\mu}(y)_{ML}$ and $\hat{\sigma}(y)_{ML}$ for a Gumbel distribution

1. Let $y_1, y_2, \ldots, y_n$ be an i.i.d. random sample of observations of size n from the Gumbel distribution, namely $Gumbel\ (\mu, \sigma)$.
2. Calculate the sample mean $\bar{y}$.
3. Start with a suitable initial estimate $\hat{\sigma}(y)_0$.

4. For $c = 0, 1, 2, \ldots$ substitute the value $\hat{\sigma}(y)_c$ into equation (3.6.30) and then calculate $\hat{\sigma}(y)_{c+1}$.
5. Repeat step 4 iteratively until the values of $\hat{\sigma}(y)_c$ converge to $\hat{\sigma}(y)_{ML}$.
6. After the convergence of $\hat{\sigma}(y)_{ML}$ is achieved, substitute the ML estimate $\hat{\sigma}(y)_{ML}$ for σ into the explicit equation for μ (3.6.27), thereby obtaining the ML estimate of μ.

Alternatively, the approximate ML estimates for the location and scale parameters of the Gumbel distribution based on a complete sample, when available, may be used. Balakrishnan and Cohen (1991, p. 189) derived the approximate ML estimates for the location and scale parameters of the Gumbel distribution based on a doubly censored sample. Similarly, Lawless (2003, pp. 218-219) discussed the log-likelihood function of the location and scale parameters of the Gumbel distribution, and the associated first and second order partial derivative equations of the likelihood function, based on a censored sample; standard survivor and pdf functions.

3.6.7. MLE of θ^* for the Pareto Distribution

Let w be an i.i.d. random sample of observations drawn from the Pareto distribution with the scale and shape parameters μ^* and σ^*, respectively, as defined in Sections 2.3.4. and 3.6.2. Then, the explicit ML estimates for μ^* and σ^*, namely $\widehat{\mu^*}(w)_{ML}$ and $\widehat{\sigma^*}(w)_{ML}$, can be obtained as follow.

The log-likelihood function for the Pareto distribution can be given by

$$l(\theta^*; w) = n\log(\sigma^*) + n\sigma^*\log(\mu^*) - (\sigma^* + 1)\sum_{i=1}^{n}\log(w_i) \qquad (3.6.31)$$

Clearly, the log-likelihood function (3.6.31) is a monotone increasing function of the scale parameter μ^* and, because the support of the Pareto distribution lies in the interval $[\mu^*, \infty)$, then the ML estimate for μ^* based on a complete sample w is given by

$$\widehat{\mu^*}(w)_{ML} = \min_{1 \le i \le n}(w_i) = w_{(1)} \tag{3.6.32}$$

The ML estimate for the shape parameter σ^* based on a complete sample w, namely $\widehat{\sigma^*}(w)_{ML}$, can be obtained by first determining the partial derivative of the log-likelihood function (3.6.31) with respect to σ^*, equating the resulting estimating equation to zero and then finally solve explicitly for σ^*.

Thus, the partial derivative of (3.6.31) with respect to σ^* leads to the estimating equation

$$\frac{\partial l(\theta^*; w)}{\partial \sigma^*} = n/\sigma^* + n\log(\mu^*) - \sum_{i=1}^{n} \log(w_i) = 0 \tag{3.6.33}$$

Solving the partial derivative (3.6.33) explicitly for σ^*, leads the ML estimate for σ

$$\widehat{\sigma^*}(w)_{ML} = n/\left[\sum_{i=1}^{n} \log(w_i) - n\log(\mu^*)\right] \tag{3.6.34}$$

The ML estimate for σ (3.1.34) can be simplified further to

$$\widehat{\sigma^*}(w)_{ML} = 1/\left[\overline{\log(w)} - \log(\mu^*)\right] \tag{3.6.35}$$

where

$$\overline{\log(w)} = \sum_{i=1}^{n} \log(w_i)/n \tag{3.6.36}$$

Thus, $\widehat{\sigma^*}(w)_{ML}$ can be obtained by substituting the ML estimate for μ^* (3.6.32), for μ^* in (3.6.35), namely

$$\widehat{\sigma^*}(w)_{ML} = 1/\left[\overline{\log(w)} - \log(\widehat{\mu^*}(w)_{ML})\right] \tag{3.6.37}$$

Chapter 4 - Inference for Location-Scale and Log-Location-Scale Distributions: One-Sample Problem

In this chapter we present firstly a literature review (Section 4.1) on inference for the location and scale parameters, quantiles of the distributions, parameter vector of the location and scale parameters, and tail probabilities based on order statistics of a single sample (complete or censored) and rank and/or ML-based methods. Secondly, we discuss the rank (Section 4.2) and ML-based (Section 4.3) CPQs and FGPQs for inference based on a single sample from LS and LLS distributions. Statistical inference using rank and ML-based CPQs and FGPQs are presented in Sections 4.4 and 4.5 respectively. Lastly, a simulation study involving one-sample problem and illustrative examples are presented in Sections 4.6 and 4.7 respectively.

4.1. Literature Review

A considerable range of literature is available on inference for the location and scale parameters and quantiles of the distribution from a single sample (one-sample problem). Bain (1972) derived a chi-square test for testing a hypothesis about the scale parameter of a Gumbel distribution using a simple and unbiased estimator from a censored sample. A similar test based on the ML estimator for the scale parameter of a Weibull distribution using a censored sample, when the shape parameter is assumed to be known, was derived by Harter and Moore (1965). Related work was done, for example, by Mann (1967, 1968, and 1970). Nkurunziza and Chen (2011) applied the method of generalized inference based on Pitman estimators to the location and scale parameters of the Normal, Cauchy and Logistic distributions; and special cases (see Nkurunziza and Chen (2011, Example 4.4, p. 226)) of the LS families of distributions, for which the ML estimators do not exist, were investigated by Gupta and Székey (1994). Through simulation, Nkurunziza and Chen (2011) obtained the coverage probabilities of a 95% confidence interval for the location and scale parameters of the Normal, Cauchy and Logistic distributions for sample sizes $n = 2, 5, 10$ and 100 and compared their results based on Pitman estimators with those based on ML methods. Furthermore, Nkurunziza and Chen (2011) demonstrated the performance of their method using real data examples.

Another study on one-sample based inference was carried out by Withers and Nadarajah (2014), who obtained exact and approximate coverage probabilities and average lengths of a 95% confidence interval for the scale parameter of the Exponential and Gamma distributions, for the location parameter of the Gumbel distribution, and the location and scale parameters of the Normal distribution. The results of Withers and Nadarajah (2014) are based on simulated samples of sizes $n = 2$ through 40 using the ML-based methods. Cheng and Iles (1983) derived the simultaneous confidence region (SCR) and subsequently the simultaneous confidence band (SCB) for the vector parameter $\theta = (\mu, \sigma)'$ in order to estimate the cdf of a general continuous distribution based on ML methods. Cheng and Iles (1983) applied this general method by considering the special cases of the LS distributions, namely the Normal, Log Normal, Gumbel and Weibull distributions. Through simulation, Cheng and Iles (1983) evaluated, using real data examples, their method based on a complete sample by comparing the 90% SCB for θ of the Normal and Weibull distributions with the SCB obtained by Kanofsky and Srinivasan (1972). Simulation results showed that a 90% SCB for θ obtained by Cheng and Iles (1983) is competitive with that of Kanofsky and Srinivasan's (1972).

4.2. Rank-Based Conventional and Fiducial Generalized Pivotal Quantities

In Sections 4.2.1 through 4.2.5 we discuss specific examples of rank-based conventional and fiducial generalized pivotal quantities for testing hypotheses and constructing confidence intervals for the parameters, quantiles of the distributions, and tail probabilities in the LS and LLS families of distributions.

4.2.1. CPQs and FGPQs for $\boldsymbol{\sigma}$

In Sections 4.2.1.1 through 4.2.1.3, we present CPQs and FGPQs for the parameters σ and σ^2.

4.2.1.1. CPQ and FGPQ for $\boldsymbol{\sigma}$ based on GLS estimator for $\boldsymbol{\sigma}$

As was shown in Section 3.2, equation (3.2.8), the GLS estimator for $\theta = (\mu, \sigma)'$ is given by

$$\hat{\theta}(Y) = (X'V^{-1}X)^{-1}X'V^{-1}Y = HY$$

where

$$H = (X'V^{-1}X)^{-1}X'V^{-1}$$

Thus the GLS estimator for σ is $L_2'\hat{\theta}(Y)$, where $L_2 = (0,1)'$, namely

$$\begin{aligned}\hat{\sigma}(Y) &= L_2'(X'V^{-1}X)^{-1}X'V^{-1}Y \\ &= L_2'HY \\ &= L_2'H(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\ &= \sigma \cdot L_2'HZ \qquad (4.2.1)\end{aligned}$$

since $H\mathbf{1}_n = (1,0)'$ and therefore $L_2'H\mathbf{1}_n = 0$. Based on (4.2.1), a CPQ for σ can be defined as follows:

$$Q_\sigma(Y,\theta) = \hat{\sigma}(Y)/\sigma$$

$$= L_2' HZ \tag{4.2.2}$$

The second equality in (4.2.2) shows that the distribution of $Q_\sigma(Y,\theta)$ is free of θ. Thus $Q_\sigma(Y,\theta)$ is a CPQ for σ.

Let y be an observation of Y. Based on (4.2.1), a FGPQ for σ can be defined as follows:

$$\mathcal{R}_\sigma(y,Y,\theta) = \frac{\hat{\sigma}(y)}{\hat{\sigma}(Y)/\sigma}$$

$$= \frac{L_2' Hy}{L_2' HZ} \tag{4.2.3}$$

The first equality in (4.2.3) shows that $\mathcal{R}_\sigma(y,Y,\theta)$ is a function of y, Y and σ, and that $\mathcal{R}_\sigma(y,y,\theta) = \sigma$ for all possible observations y of Y; thus property FGPQ2 is fulfilled. Furthermore, the second equality in (4.2.3) shows that the distribution of $\mathcal{R}_\sigma(y,Y,\theta)$, conditional on an observation y of Y, is free of θ since the quantity (4.2.3) is solely a function of the vector of ordered standard variates Z. This completes the proof that $\mathcal{R}_\sigma(y,Y,\theta)$ is a FGPQ for σ.

Comparing the CFQ (4.2.2) and the FGPQ (4.2.3) we note that, given an observation y, the two quantities are monotonic functions of each other. In that sense, the CFQ (and its distribution) is equivalent to the FGPQ.

4.2.1.2. CPQ and FGPQ for $\boldsymbol{\sigma^2}$ based on residual sum of squares

The generalized least squares residuals under model (3.2.5)/(3.2.6) are defined as

$$\hat{e}(Y) = [I - X(X'V^{-1}X)^{-1}X'V^{-1}]Y = NY \tag{4.2.4}$$

where

$$N = I - XH = I - X(X'V^{-1}X)^{-1}X'V^{-1} \tag{4.2.5}$$

Using equation (3.2.2) in (4.2.4) the least squares residuals can be written as

$$\begin{aligned} \hat{e}(Y) &= NY \\ &= N(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\ &= \sigma \cdot NZ \end{aligned} \tag{4.2.6}$$

since $N\mathbf{1}_n = 0$. Now the following residual sum of squares is considered

$$\begin{aligned} S(Y) &= \hat{e}(Y)'V^{-1}\hat{e}(Y) \\ &= Y'N'V^{-1}NY \\ &= \sigma^2 \cdot Z'N'V^{-1}NZ \\ &= \sigma^2 \cdot Z'MZ \end{aligned} \tag{4.2.7}$$

where

$$M = N'V^{-1}N = V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1} \tag{4.2.8}$$

Based on (4.2.7) a CPQ for σ^2 can be defined as follows:

$$\begin{aligned} \mathcal{Q}_{\sigma^2}(Y,\theta) &= S(Y)/\sigma^2 \\ &= Y'MY/\sigma^2 \\ &= Z'MZ \end{aligned} \tag{4.2.9}$$

As above, the third equality in (4.2.9) shows that the distribution of $\mathcal{Q}_{\sigma^2}(Y,\theta)$ does not depend on any unknown parameters, since the quadratic form is solely a function of the vector of ordered standard variates Z. Thus, $\mathcal{Q}_{\sigma^2}(Y,\theta)$ is a CPQ. The expected value of $\mathcal{Q}_{\sigma^2}(Y,\theta)$ is $n-2$ in the case of a Normal distribution.

Similarly, based in (4.2.7), a FGPQ for σ^2 can be defined as

$$\mathcal{R}_{\sigma^2}(y,Y,\theta) = \frac{S(y)}{S(Y)/\sigma^2}$$

$$= \frac{y'My}{Z'MZ} \qquad (4.2.10)$$

The first equality in (4.2.10) shows that $\mathcal{R}_{\sigma^2}(y, y, \theta) = \sigma^2$ for all possible observations y of Y; the second equality in (4.2.10) shows that the distribution of $\mathcal{R}_{\sigma^2}(y, Y, \theta)$, conditional on an observation y of Y, is free of θ. Thus, $\mathcal{R}_{\sigma^2}(y, Y, \theta)$ is a FGPQ for σ^2.

Similarly, based on (4.2.7), a CPQ for σ and a FGPQ for σ are given, respectively by

$$\mathcal{Q}_\sigma(Y, \theta) = \sqrt{Z'MZ} \qquad (4.2.11)$$

and

$$\mathcal{R}_\sigma(y, Y, \theta) = \sqrt{\frac{y'My}{Z'MZ}} \qquad (4.2.12)$$

4.2.1.3. Combined CPQ and FGPQ for $\boldsymbol{\sigma^2}$

The fact that there are two CPQs for σ (or σ^2), in (4.2.2) and (4.2.9), may suggest that information from these two quantities should be combined. The CPQ for σ^2 in (4.2.9) is based on the residual sum of squares from the GLM (3.2.5)/(3.2.6), while the CPQ for σ in (4.2.2) is based on the GLS estimator for σ. This suggests a combined CPQ based on the sum of the regression sum of squares due to the variable $E(Z)$ in model (3.2.5) and residual sum of squares $S(Y)$ in (4.2.7).

Let

$$\tilde{X}_2 = [I - 1_n(1_n'V^{-1}\mathbf{1}_n)^{-1}1_n'V^{-1}]\, E(Z) \qquad (4.2.13)$$

be the projection of $X_2 = E(Z)$ onto the space orthogonal to $\mathbf{1}_n$. Then the GLS estimator for σ can be written as

$$\hat{\sigma}(Y) = \left(\tilde{X}_2'V^{-1}\tilde{X}_2\right)^{-1}\tilde{X}_2'V^{-1}Y$$

$$= \sigma \cdot \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right)^{-1} \tilde{X}_2' V^{-1} Z \tag{4.2.14}$$

since $\tilde{X}_2' V^{-1} \mathbf{1}_n = 0$. Furthermore, in terms of $\tilde{X}_2$, the regression sum of squares due to the variable $E(Z)$ in model (3.2.5) can be written as

$$\tilde{S}(Y) = \hat{\sigma}^2(Y) \cdot \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right) = \hat{\sigma}^2(Y) / [(X'V^{-1}X)^{-1}]_{22}$$

where $[(X'V^{-1}X)^{-1}]_{22}$ is the second diagonal entry of the matrix $Var\left(\hat{\theta}\right) = (X'V^{-1}X)^{-1}$. Thus, using (4.2.14)

$$\begin{aligned}\tilde{S}(Y) &= \hat{\sigma}^2(Y) \cdot \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right) \\ &= Y' V^{-1} \tilde{X}_2 \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right)^{-1} \tilde{X}_2' V^{-1} Y \\ &= \sigma^2 \cdot Z' V^{-1} \tilde{X}_2 \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right)^{-1} \tilde{X}_2' V^{-1} Z \end{aligned} \tag{4.2.15}$$

The sum of $S(Y)$ in (4.2.7) and $\tilde{S}(Y)$ in (4.2.15) is then

$$\begin{aligned} S(Y) + \tilde{S}(Y) &= \sigma^2 \cdot \left\{ Z'MZ + \frac{(L_2' HZ)^2}{[(X'V^{-1}X)^{-1}]_{22}} \right\} \\ &= \sigma^2 \cdot Z' \left[M + V^{-1} \tilde{X}_2 \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right)^{-1} \tilde{X}_2' V^{-1} \right] Z \end{aligned} \tag{4.2.16}$$

Based on $S(Y) + \tilde{S}(Y)$ in (4.2.16), a CPQ for σ^2 can be defined as follows:

$$\begin{aligned} Q_{\sigma^2}(Y, \theta) &= \left[S(Y) + \tilde{S}(Y)\right] / \sigma^2 \\ &= Z' \left[M + V^{-1} \tilde{X}_2 \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right)^{-1} \tilde{X}_2' V^{-1} \right] Z \\ &= Z' \widetilde{M}\, Z \end{aligned} \tag{4.2.17}$$

where

$$\widetilde{M} = M + V^{-1} \tilde{X}_2 \left(\tilde{X}_2' V^{-1} \tilde{X}_2\right)^{-1} \tilde{X}_2' V^{-1} \tag{4.2.18}$$

The second equality in (4.2.17) shows that the distribution of $Q_{\sigma^2}(Y, \theta)$ is free of θ. Thus $Q_{\sigma^2}(Y, \theta)$ is a CPQ for σ^2.

Similarly, based on $S(Y) + \tilde{S}(Y)$ in (4.2.16), a combined FGPQ for σ^2 can be defined as

$$\mathcal{R}_{\sigma^2}(y, Y, \theta) = \frac{S(y) + \tilde{S}(y)}{\left[S(Y) + \tilde{S}(Y)\right]/\sigma^2}$$

$$= \frac{y'\left[M + V^{-1}\tilde{X}_2\left(\tilde{X}_2'V^{-1}\tilde{X}_2\right)^{-1}\tilde{X}_2'V^{-1}\right]y}{Z'\left[M + V^{-1}\tilde{X}_2\left(\tilde{X}_2'V^{-1}\tilde{X}_2\right)^{-1}\tilde{X}_2'V^{-1}\right]Z}$$

$$= \frac{y'\widetilde{M}y}{Z'\widetilde{M}Z} \tag{4.2.19}$$

where, as defined in (4.2.18)

$$\widetilde{M} = M + V^{-1}\tilde{X}_2\left(\tilde{X}_2'V^{-1}\tilde{X}_2\right)^{-1}\tilde{X}_2'V^{-1}$$

The first equality in (4.2.19) shows that $\mathcal{R}_{\sigma^2}(y, y, \theta) = \sigma^2$ for all possible observations y of Y; the second equality in (4.2.19) shows that the distribution of $\mathcal{R}_{\sigma^2}(y, Y, \theta)$, conditional on an observation y of Y, is free of θ. Thus $\mathcal{R}_{\sigma^2}(y, Y, \theta)$ is a FGPQ for σ^2.

Similarly to (4.2.11) and (4.2.12), based on $S(Y) + \tilde{S}(Y)$ in (4.2.16), a combined CPQ for σ and a combined FGPQ for σ are given, respectively by

$$\mathcal{Q}_\sigma(Y, \theta) = \sqrt{Z'\widetilde{M}\,Z} \tag{4.2.20}$$

and

$$\mathcal{R}_\sigma(y, Y, \theta) = \sqrt{\frac{y'\widetilde{M}y}{Z'\widetilde{M}Z}} \tag{4.2.21}$$

Again we can note that the pairs of CPQs and FGPQs (4.2.9) / (4.2.10) and (4.2.20) / (4.2.21) are equivalent, since they are monotonic functions of each other, given an observation y.

In Section 4.6.3.4, the efficiency of the ranked-based CPQs for σ (4.2.2), (4.2.11) and (4.2.20) and that of the rank-based FGPQs for σ (4.2.3), (4.2.12) and (4.2.21) is evaluated through a simulation study, to determine the most efficient rank-based pivotal quantity for σ among the three rank-based CPQs and FGPQs. In other words, the average length of confidence intervals for σ based on three rank-based CPQs for σ and/or three rank-based FGPQs for σ are calculated and the average lengths compared. Based on the results of the simulation study, a ranked-based pivotal quantity for σ associated with a confidence interval for σ that has the lowest average length among the three, is the most efficient. After the best and most efficient ranked-based pivotal quantity for σ has been identified, only this pivotal quantity will be used further for the remainder of this thesis, for example, for inference for the two-sample problem discussed below in Chapter 5. Specifically, the best and most efficient rank-based CPQ and/or FGPQ for σ will be used to calculate the confidence intervals for the ratio of scale parameters, difference of location parameters and difference of quantiles of the distributions in the case of a two-sample problem presented below in Sections 5.4 and 5.5.

4.2.1.4. Investigation of most efficient CPQ and FGPQ for σ

Above in Sections 4.2.1.1 through 4.2.1.3 we have presented three different CPQs and FGPQs for σ. In the simulation study described below in Section 4.6.3, we will investigate which of the three CPQs and FGPQs is the best and most efficient, that is, which leads to the shortest lengths of confidence intervals for σ and ratio of σ_1 and σ_2 (in the case of two-sample problem presented below in Chapter 5).

It will turn out that the CPQ and FGPQ for σ based on GLS estimator for σ (4.2.2 and 4.2.3, respectively) are the best and most efficient. Below, when deriving the FGPQs for μ in Section 4.2.2, we will continue to derive three versions based on three versions for sigma. From Section 4.2.3 onwards, we will only use the CPQ (4.2.2) and FGPQ (4.2.3) for sigma and derive only one version of CPQs and FGPQs for quantiles of the distribution and a linear combination of theta in order to avoid duplication and unnecessary detail.

4.2.1.5. Comparison of the distribution of CPQs and FGPQs for μ, σ and quantiles

Above in Section 4.2.1 and below in Sections 4.2.2 through 4.2.5, we have presented the CPQs for μ, σ, quantiles of the distribution and failure probability and their respective FGPQs. As noted above, the CPQs and their counterpart FGPQs are equivalent. Therefore, in the following chapters the CPQs, when they exist, will be used in simulation studies because they are slightly faster to simulate.

4.2.2. **CPQs and FGPQs for μ**

In Sections 4.2.2.1, 4.2.2.2 and 4.2.2.3 we present CPQs and FGPQs for μ based on GLS estimator for μ, residual sum of squares and combined CPQs and FGPQs for σ^2 in LS and LLS families of distributions.

4.2.2.1. CPQ and FGPQ for μ based on GLS estimator for σ

Similarly to the development in Section 4.2.1., it is noted that the GLS estimator for μ is $L_1'\hat{\theta}(Y)$, where $L_1 = (1,0)'$, namely

$$\begin{aligned}\hat{\mu}(Y) &= L_1'(X'V^{-1}X)^{-1}X'V^{-1}Y \\ &= L_1'HY \\ &= L_1'H(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\ &= \mu + \sigma \cdot L_1'HZ \qquad (4.2.22)\end{aligned}$$

since $H\mathbf{1}_n = (1,0)'$ and therefore $L_1' H \mathbf{1}_n = 1$. Based on (4.2.22), a conditional conventional pivotal quantity (CCPQ) for μ is defined as follows:

$$\mathcal{Q}_{\mu|\sigma}(Y,\theta) = \hat{\mu}(Y) - \mu$$

$$= \sigma \cdot L_1' HZ \tag{4.2.23}$$

The second equality in (4.2.23) shows that the distribution of $\mathcal{Q}_{\mu|\sigma}(Y,\theta)$, conditional on σ, is free of θ (that is, free of μ). Thus $\mathcal{Q}_{\mu|\sigma}(Y,\theta)$ is a CCPQ for μ.

Based on the conditional conventional pivotal quantity $\mathcal{Q}_{\mu|\sigma}(Y,\theta)$, a CPQ for μ is obtained by dividing (4.2.23) by $\sigma \cdot L_2' HZ$ (see (4.2.2)), thus eliminating the parameter σ. The resulting CPQ for μ is given by

$$\mathcal{Q}_{\mu}(Y,\theta) = \frac{\mathcal{Q}_{\mu|\sigma}(Y,\theta)}{\sigma \ \cdot \mathcal{Q}_{\sigma}(Y,\theta)}$$

$$= \frac{\hat{\mu}(Y) - \mu}{L_2' HY}$$

$$= \frac{L_1' HZ}{L_2' HZ} \tag{4.2.24}$$

The FGPQ for μ is derived as follows: Similarly to the definition of CPQ for μ (see (4.2.24)), based on (4.2.22) the conditional fiducial pivotal quantity (CFGPQ) for μ given σ is defined as follows:

$$\mathcal{R}_{\mu|\sigma}(y,Y,\theta) = \hat{\mu}(y) - [\hat{\mu}(Y) - \mu]$$

$$= \hat{\mu}(y) - \sigma \cdot L_1' HZ$$

$$= L_1' Hy - \sigma \cdot L_1' HZ$$

$$= L_1' H(y - \sigma \cdot Z) \tag{4.2.25}$$

The first equality in (4.2.25) shows that $\mathcal{R}_{\mu|\sigma}(y,y,\theta) = \mu$ for all possible observations y of Y (and for all σ); the second equality in (4.2.25) shows that the distribution of $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$, conditional on an observation y of Y and on σ, is free of θ (that is, free of μ). Thus, $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$ is a CFGPQ for μ.

Furthermore, based on the CFGPQ $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$ in (4.2.25) a FGPQ for μ is obtained by replacing the parameter σ in (4.2.25) by a FGPQ for σ, namely FGPQ (4.2.3). The resulting FGPQ for μ is given by

$$\begin{aligned} \mathcal{R}_{\mu}(y, Y, \theta) &= L_1' H \left[y - \mathcal{R}_{\sigma}(y, Y, \theta) \cdot Z \right] \\ &= L_1' H \left(y - \frac{L_2' H y}{L_2' H Z} \cdot Z \right) \end{aligned} \tag{4.2.26}$$

4.2.2.2. CPQ and FGPQ for μ based on residual sum of squares

Similarly to the presentation of CPQ and FGPQ for μ based on GLS estimator for σ (see (4.2.24) and (4.2.26)), the CPQ and FGPQ for μ based on the residual sum of squares are, respectively given by

$$\mathcal{Q}_{\mu}(Y, \theta) = \frac{L_1' H Z}{\sqrt{Z' H Z}} \tag{4.2.27}$$

and

$$\mathcal{R}_{\mu}(y, Y, \theta) = L_1' H \left(y - \frac{L_2' H y}{\sqrt{Z' H Z}} \cdot Z \right) \tag{4.2.28}$$

4.2.2.3. CPQ and FGPQ for μ based on combined CPQ and FGPQ for σ^2

Again, similarly to as presented in Sections 4.2.2.1 and 4.2.2.2, and based on the conditional conventional pivotal quantity $\mathcal{Q}_{\mu|\sigma}(Y, \theta)$, a CPQ for μ is obtained by dividing (4.2.23) by the square root of $\sigma^2 \cdot \mathcal{Q}_{\sigma^2}(Y, \theta) = \sigma^2 \cdot Z' \widetilde{M}\, Z$ (see (4.2.17)), thus eliminating the parameter σ. The resulting CPQ for μ is given by

$$\mathcal{Q}_{\mu}(Y, \theta) = \frac{\mathcal{Q}_{\mu|\sigma}(Y, \theta)}{\sigma \cdot \sqrt{\mathcal{Q}_{\sigma^2}(Y, \theta)}}$$

$$= \frac{\hat{\mu}(Y) - \mu}{\sqrt{S(Y) + \tilde{S}(Y)}}$$

$$= \frac{L_1' HZ}{\sqrt{Z^{*\prime} \widetilde{M} Z^*}} \tag{4.2.29}$$

where $\widetilde{M}$ has been defined in (4.2.18). In (4.2.29), Z^* denotes an independent copy of the standard variate Z.

The FGPQ for μ is derived as follows: Similarly to the definition of CPQ for μ (see (4.2.29)), based on (4.2.22) the conditional fiducial pivotal quantity (CFGPQ) for μ given σ is defined as follows:

$$\mathcal{R}_{\mu|\sigma}(y, Y, \theta) = \hat{\mu}(y) - [\hat{\mu}(Y) - \mu]$$

$$= \hat{\mu}(y) - \sigma \cdot L_1' HZ$$

$$= L_1' Hy - \sigma \cdot L_1' HZ$$

$$= L_1' H(y - \sigma \cdot Z) \tag{4.2.30}$$

The first equality in (4.2.30) shows that $\mathcal{R}_{\mu|\sigma}(y, y, \theta) = \mu$ for all possible observations y of Y (and for all σ); the second equality in (4.2.30) shows that the distribution of $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$, conditional on an observation y of Y and on σ, is free of θ (that is, free of μ). Thus, $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$ is a CFGPQ for μ.

Furthermore, based on the CFGPQ $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$ in (4.2.30) a FGPQ for μ is obtained by replacing the parameter σ in (4.2.30) by a FGPQ for σ. The resulting FGPQ for μ is given by

$$\mathcal{R}_{\mu}(y, Y, \theta) = L_1' H \left[y - \sqrt{\mathcal{R}_{\sigma^2}(y, Y, \theta)} \cdot Z \right]$$

$$= L_1' H \left(y - \sqrt{\frac{y' \widetilde{M} y}{Z^{*\prime} \widetilde{M} Z^*}} \cdot Z \right) \tag{4.2.31}$$

where $\widetilde{M}$ has been defined in (4.2.18). In (4.2.31), Z^* denotes an independent copy of the standard variate Z.

4.2.3. CPQs and FGPQs for a Linear Combination $\mathbf{L'\theta}$ of $\boldsymbol{\theta}$

In this section, we derive the CPQ and FGPQ for a linear combination $L'\theta$ of θ, and for a quantile of the distribution of Y_*.

4.2.3.1. CPQ for a linear combination $\boldsymbol{L'\theta}$ of $\boldsymbol{\theta}$

Let $L = (l_1, l_2)'$. Using equation (3.2.8) a CPQ for a linear combination $L'\theta$ of θ can be derived from the GLS estimator $L'\hat{\theta}(Y)$ for $L'\theta$ as follows:

$$\begin{aligned}\hat{\theta}(Y) &= L'HY \\ &= L'H(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\ &= L'H\{\mu \cdot \mathbf{1}_n + \sigma \cdot E(Z) + \sigma \cdot [Z - E(Z)]\} \\ &= L'H\{X\theta + \sigma \cdot [Z - E(Z)]\} \\ &= L'\theta + \sigma \cdot L'H[Z - E(Z)] \ (4.2.32) \qquad (4.2.32)\end{aligned}$$

since $HX\theta = \theta$. Then a CCPQ for $L'\theta$ given σ can be defined as follows:

$$\begin{aligned}Q_{L'\theta|\sigma}(Y,\theta) &= L'\left[\hat{\theta}(Y) - \theta\right] \\ &= \sigma \cdot L'H[Z - E(Z)] \qquad (4.2.33)\end{aligned}$$

Based on the CCPQ $Q_{L'\theta|\sigma}(Y,\theta)$, a CPQ for $L'\theta$ can be obtained by dividing (4.2.33) by $Q_\sigma(Y,\theta) = L_2'HZ$ (see 4.2.2), thus eliminating the parameter σ. The resulting CPQ for $L'\theta$ is given by

$$Q_{L'\theta}(Y,\theta) = \frac{Q_{L'\theta|\sigma}(Y,\theta)}{\sigma \cdot Q_\sigma(Y,\theta)}$$

$$= \frac{L'\left[\hat{\theta}(Y) - \theta\right]}{L_2' HZ}$$

$$= \frac{L'H[Z - E(Z)]}{L_2' HZ} \tag{4.2.34}$$

It is noted that in the derivation of $Q_{L'\theta}(Y, \theta)$, it can be assumed that $l_1 \neq 0$. If $l_1 = 0$, then $Q_{L'\theta}(Y, \theta)$ in (4.2.34) is a another CPQ for σ.

4.2.3.2. CPQ for a $\boldsymbol{p}$ *quantile of the distribution*

For $0 \leq p \leq 1$, the p quantile η_p of the distribution of Y_* is given by $\eta_p = \mu + \sigma \cdot z_p$, where z_p is the p quantile of the distribution of the standard variate Z_*. Thus, the p quantile is a linear combination of the parameters μ and σ of the distribution of Y_*, namely

$$\eta_p = \mu + \sigma \cdot z_p = L_p' \theta$$

with $L_p = \left(1, z_p\right)'$. A CPQ for the linear combination $L_p'\theta$ of θ is given by (4.2.34), when L in (4.2.34) is of the form $L = L_p = \left(1, z_p\right)'$.

Equivalently, a CPQ for the p quantile η_p of the distribution of Y_* can be derived as follows:

The generalized least squares estimator $\hat{\eta}_p(Y)$ of η_p is given by

$$\begin{aligned}
\hat{\eta}_p(Y) &= L_p' HY \\
&= L_p' H(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\
&= L_p' H\left[\left(\mu + \sigma \cdot z_p\right) \cdot \mathbf{1}_n + \sigma \cdot \left(Z - z_p \cdot \mathbf{1}_n\right)\right] \\
&= L_p' H\left[\eta_p \cdot \mathbf{1}_n + \sigma \cdot \left(Z - z_p \cdot \mathbf{1}_n\right)\right] \\
&= \eta_p + \sigma \cdot L_p' H\left(Z - z_p \cdot \mathbf{1}_n\right)
\end{aligned} \tag{4.2.35}$$

since $H\mathbf{1}_n = (1,0)'$ and therefore $L_p' H\mathbf{1}_n = 1$. A CCPQ is defined as

$$\mathcal{Q}_{\eta_p|\sigma}(Y,\theta) = \hat{\eta}_p(Y) - \eta_p$$

$$= \sigma \cdot L_p' H\big(Z - z_p \cdot \mathbf{1}_n\big) \tag{4.2.36}$$

Based on the CCPQ $\mathcal{Q}_{\eta_p|\sigma}(Y,\theta)$, a CPQ for η_p can be obtained by dividing (4.2.36) by $\mathcal{Q}_\sigma(Y,\theta) = L_2'HZ$ (4.2.2), thus eliminating the parameter σ. The resulting CPQ for η_p is given by

$$\mathcal{Q}_{\eta_p}(Y,\theta) = \frac{\mathcal{Q}_{\eta_p|\sigma}(Y,\theta)}{\sigma \,\cdot\, \mathcal{Q}_\sigma(Y,\theta)}$$

$$= \frac{\hat{\eta}_p(Y) - \eta_p}{L_2'HZ}$$

$$= \frac{L_p' H\big(Z - z_p \cdot \mathbf{1}_n\big)}{L_2'HZ} \tag{4.2.37}$$

It is noted that CPQs (4.2.34) and (4.2.37) are identical, when $L = L_p$ in (4.2.34).

As is noted earlier in this Section 4.2.3.2, for $0 \le p \le 1$, the p quantile η_p of the distribution of Y_* is given by $\eta_p = \mu + \sigma \cdot z_p$, where z_p is the p quantile of the distribution of the standard variate Z_*. Thus, the p quantile is a linear combination of the parameters μ and σ of the distribution of Y_*, so that it is of particular interest to derive FGPQs for linear combinations $L_p'\theta$ of θ. In the next section, we use two different approaches to derive FGPQs for linear combinations $L_p'\theta$ of θ as follows:

4.2.3.3. FGPQ for a linear combination $\boldsymbol{L}'\boldsymbol{\theta}$ of $\boldsymbol{\theta}$ derived by "plug-in" principle

We wish to derive a FGPQ for a linear combination $L'\theta$ of θ, namely for

$$L'\theta = l_1 \cdot \mu + l_2 \cdot \sigma \tag{4.2.38}$$

where $L = (l_1, l_2)'$. In general, a FGPQ for $L'\theta$ in (4.2.38) can be obtained by replacing the unknown parameters μ and σ with suitable FGPQs for those parameters. We refer to this method of deriving FGPQs as the "plug-in" principle; Hannig et al. (2006b, Theorem 3) give a formal proof of principle, but call it the "two-stage approach". Thus, a FGPQ for $L'\theta$ in (4.2.38) can be written as

$$\mathcal{R}_{L'\theta}(y, Y, \theta) = l_1 \cdot \mathcal{R}_\mu(y, Y, \theta) + l_2 \cdot \mathcal{R}_\sigma(y, Y, \theta) \tag{4.2.39}$$

where $\mathcal{R}_\mu(y, Y, \theta)$ and $\mathcal{R}_\sigma(y, Y, \theta)$ have been defined in (4.2.26) and (4.2.3), respectively.

However, it is noted that the FGPQ $\mathcal{R}_\sigma$ appears twice in (4.2.39), namely in the first term of the sum, since the FGPQ $\mathcal{R}_\mu$ for μ implicitly involves $\mathcal{R}_\sigma$ (see the first equality of (4.2.26)), and in the second term of the sum, where $\mathcal{R}_\sigma$ appears explicitly. Now $\mathcal{R}_\sigma$ is a function of the standard variate Z (see (4.2.3)), and it may not be immediately clear whether the two appearances of $\mathcal{R}_\sigma$ in (4.2.39) should be associated with identical or independent copies of Z.

In order for this question to be answered, the FGPQ $\mathcal{R}_{L'\theta}(y, Y, \theta)$ for $L'\theta$ is derived via the CFGPQ $\mathcal{R}_{L'\theta|\sigma}(y, Y, \theta)$ for $L'\theta$ conditional on σ. $\mathcal{R}_{L'\theta|\sigma}(y, Y, \theta)$ is obtained by replacing μ in (4.2.38) by the CFGPQ $\mathcal{R}_{\mu|\sigma}$ in (4.2.30):

$$\begin{aligned} \mathcal{R}_{L'\theta|\sigma}(y, Y, \theta) &= l_1 \cdot \mathcal{R}_{\mu|\sigma} + l_2 \cdot \sigma \\ &= l_1 \cdot L_1' H(y - \sigma \cdot Z) + l_2 \cdot \sigma \\ &= l_1 \cdot L_1' H y - \sigma \cdot (l_1 \cdot L_1' H Z - l_2) \end{aligned} \tag{4.2.40}$$

Now a FGPQ for $L'\theta$ is obtained by replacing the parameter σ in (4.2.40) by the FGPQ for σ in (4.2.3):

$$\begin{aligned} \mathcal{R}_{L'\theta}(y, Y, \theta) &= l_1 \cdot L_1' H y - \mathcal{R}_\sigma(y, Y, \theta) \cdot (l_1 \cdot L_1' H Z - l_2) \\ &= l_1 \cdot \mathcal{R}_\mu + l_2 \cdot \mathcal{R}_\sigma(y, Y, \theta) \\ &= l_1 \cdot L_1' H y - \frac{L_2' H y}{L_2' H Z} \cdot (l_1 \cdot L_1' H Z - l_2) \end{aligned} \tag{4.2.41}$$

Therefore, the FGPQ $\mathcal{R}_{L'\theta}(y, Y, \theta)$ for $L'\theta$ in (4.2.41) is obtained by replacing both instances of σ in (4.2.40) by $\mathcal{R}_\sigma$.

It is noted that if $l_1 = 0$, equation (4.2.39) yields a FGPQ for $l_2 \cdot \sigma$ based solely on the FGPQ (4.2.3) for σ.

4.2.3.4. FGPQ for a linear combination $\boldsymbol{L'\theta}$ of $\boldsymbol{\theta}$ derived from GLS estimator for $\boldsymbol{L'\theta}$

Alternatively to (4.2.39)/(4.2.41), a FGPQ for $L'\theta$ can be derived from the GLS estimator $L'\hat{\theta}(Y)$ for $L'\theta$ as follows:

$$\begin{aligned} L'\hat{\theta}(Y) &= L'HY \\ &= L'H(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\ &= L'H\{\mu \cdot \mathbf{1}_n + \sigma \cdot E(Z) + \sigma \cdot [Z - E(Z)]\} \\ &= L'H\{X\theta + \sigma \cdot [Z - E(Z)]\} \\ &= L'\theta + \sigma \cdot L'H[Z - E(Z)] \end{aligned} \tag{4.2.42}$$

since $HX\theta = \theta$. Then a CFGPQ $\mathcal{R}_{L'\theta|\sigma}(y, Y, \theta)$ for $L'\theta$ conditional on σ is

$$\begin{aligned} \mathcal{R}_{L'\theta|\sigma}(y, Y, \theta) &= L'\hat{\theta}(y) - \left[L'\hat{\theta}(Y) - L'\theta\right] \\ &= L'\hat{\theta}(y) - \sigma \cdot L'H[Z - E(Z)] \\ &= L'H\{y - \sigma \cdot [Z - E(Z)]\} \end{aligned} \tag{4.2.43}$$

Again, the parameter σ in (4.2.43) is replaced by the FGPQ for σ in (4.2.3):

$$\begin{aligned} \mathcal{R}_{L'\theta}(y, Y, \theta) &= L'H\{y - \mathcal{R}_\sigma(y, Y, \theta) \cdot [Z - E(Z)]\} \\ &= L'H\left\{y - \frac{L_2'Hy}{L_2'HZ} \cdot [Z - E(Z)]\right\} \end{aligned} \tag{4.2.44}$$

4.2.3.5. FGPQs for a $\boldsymbol{p}$ *quantile of the distribution*

As noted earlier in Section 4.2.3.2, the p quantile η_p of the distribution of Y_* is given by $\eta_p = \mu + \sigma \cdot z_p$, where z_p is the p quantile of the distribution of the standard variate Z_*. Thus, η_p can be written as the linear combination $\eta_p = L_p'\theta$ of θ, where $L_p = \left(1, z_p\right)'$. Therefore, two FGPQs for η_p are obtained when L in (4.2.41) and (4.2.44), respectively, is replaced by L_p; namely

$$\mathcal{R}_{\eta_p}(y, Y, \theta) = L_1'Hy - \frac{L_2'Hy}{L_2'HZ} \cdot \left(L_1'HZ - z_p\right) \tag{4.2.45}$$

and

$$\mathcal{R}_{\eta_p}(y, Y, \theta) = L_p'H\left\{y - \frac{L_2'Hy}{L_2'HZ} \cdot [Z - E(Z)]\right\} \tag{4.2.46}$$

4.2.4. **CPQs and FGPQs for θ**

In Sections 4.2.4.1 through 4.2.4.4 below, we derive the CPQs and FGPQs for the two-dimensional parameter vector θ by using different approaches.

4.2.4.1. CPQ for $\boldsymbol{\theta}$ *derived from GLS estimator for* $\boldsymbol{\theta}$

Using equation (3.2.8), the generalized least squares estimator $\hat{\theta}(Y)$ of θ as

$$\begin{aligned}
\hat{\theta}(Y) &= HY \\
&= H(\mu \cdot \mathbf{1}_n + \sigma \cdot Z) \\
&= H\{\mu \cdot \mathbf{1}_n + \sigma \cdot E(Z) + \sigma \cdot [Z - E(Z)]\} \\
&= H\{X\theta + \sigma \cdot [Z - E(Z)]\} \\
&= \theta + \sigma \cdot [Z - E(Z)] \qquad (4.2.47)
\end{aligned}$$

The following CPQ for θ can be defined:

$$Q_\theta(Y,\theta) = \left\{\left[\hat{\theta}(Y) - \theta\right]' X'V^{-1}X\left[\hat{\theta}(Y) - \theta\right]\right\}/\sigma^2$$

$$= [(HY - \theta)'X'V^{-1}X(HY - \theta)]/\sigma^2$$

$$= [Z - E(Z)]'V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}[Z - E(Z)] \qquad (4.2.48)$$

Clearly, the distribution of the quadratic form $Q_\theta(Y,\theta)$ does not depend on any unknown parameters, since the quadratic form is solely a function of the ordered standard variates Z. Thus, $Q_\theta(Y,\theta)$ is a CPQ.

4.2.4.2. CPQ for $\boldsymbol{\theta}$ based on a ratio of CPQs for $\boldsymbol{\theta}$ and $\boldsymbol{\sigma^2}$

Alternatively to CPQ (4.2.48), a CPQ for θ can be defined as the ratio of the CPQ (4.2.48) for θ and of the square of CPQ (4.2.2) for σ:

$$\tilde{Q}_\theta(Y,\theta) = \frac{Q_\theta(Y,\theta)}{Q_{\sigma^2}(Y,\sigma)}$$

$$= \frac{\left[\hat{\theta}(Y) - \theta\right]' X'V^{-1}X\left[\hat{\theta}(Y) - \theta\right]/\sigma^2}{[\hat{\sigma}(Y)]^2/\sigma^2}$$

$$= \frac{(HY - \theta)'X'V^{-1}X(HY - \theta)}{(L_2'HZ)^2}$$

$$= \frac{[Z - E(Z)]'V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}[Z - E(Z)]}{(L_2'HZ)^2} \qquad (4.2.49)$$

Clearly, the distribution of $\tilde{Q}_\theta(Y,\theta)$ does not depend on any unknown parameters, since it is solely a function of the ordered standard variates Z. Thus, $\tilde{Q}_\theta(Y,\theta)$ is a CPQ for θ.

4.2.4.3. FGPQ for $\boldsymbol{\theta}$ derived by "plug-in" principle

We derive a FGPQ for θ. In general, a FGPQ for $\theta = (\mu,\sigma)'$ can be obtained by replacing the unknown parameters μ and σ with suitable FGPQs for those parameters ("plug-in" principle). Thus, a FGPQ for θ can be written as

$$\mathcal{R}_\theta(y,Y,\theta) = \begin{pmatrix} \mathcal{R}_\mu(y,Y,\theta) \\ \mathcal{R}_\sigma(y,Y,\theta) \end{pmatrix}$$

$$= \begin{pmatrix} L_1'Hy - \mathcal{R}_\sigma(y,Y,\theta)\cdot L_1'HZ \\ \mathcal{R}_\sigma(y,Y,\theta) \end{pmatrix}$$

$$= \begin{pmatrix} L_1'Hy - \dfrac{L_2'Hy}{L_2'HZ}\cdot L_1'HZ \\ \dfrac{L_2'Hy}{L_2'HZ} \end{pmatrix} \qquad (4.2.50)$$

where $\mathcal{R}_\mu(y,Y,\theta)$ and $\mathcal{R}_\sigma(y,Y,\theta)$ have been defined in (4.2.26) and (4.2.3), respectively.

4.2.4.4. FGPQ for $\boldsymbol{\theta}$ *based on GLS estimator for* $\boldsymbol{\theta}$

Alternatively to (4.2.50), a GFPQ for θ can be derived from a GLS estimator $\hat{\theta}(Y)$ for θ:

$$\hat{\theta}(Y) = HY$$

$$= H(\mu\cdot\mathbf{1}_n + \sigma\cdot Z)$$

$$= H\{\mu\cdot\mathbf{1}_n + \sigma\cdot E(Z) + \sigma\cdot[Z - E(Z)]\}$$

$$= H\{X\theta + \sigma\cdot[Z - E(Z)]\}$$

$$= \theta + \sigma\cdot H[Z - E(Z)] \qquad (4.2.51)$$

since $HX\theta = \theta$. Then

$$\mathcal{R}_{\theta|\sigma}(y,Y,\theta) = \hat{\theta}(y) - \left[\hat{\theta}(Y) - \theta\right]$$

$$= \hat{\theta}(y) - \sigma\cdot H[Z - E(Z)]$$

$$= H\{y - \sigma\cdot H[Z - E(Z)]\} \qquad (4.2.52)$$

Again, the parameter σ in (4.2.52) is replaced by the FGPQ for σ in (4.2.3):

$$\mathcal{R}_\theta(y,Y,\theta) = H\{y - \mathcal{R}_\sigma(y,Y,\theta)\cdot H[Z - E(Z)]\}$$

$$= H\left\{y - \frac{L_2'Hy}{L_2'HZ} \cdot H[Z - E(Z)]\right\} \tag{4.2.53}$$

4.2.5. **FGPQs for a Failure Probability**

Below in Sections 4.2.5.1 and 4.2.5.2, we derive two FGPQs for a failure probability by using two different approaches.

4.2.5.1. FGPQ for a failure probability derived by "plug-in" principle

The failure probability π at "time" y_e is given by

$$\pi = F_Y(y_e;\theta) = \Phi\left(\frac{y_e - \mu}{\sigma}\right) \tag{4.2.54}$$

Let

$$\varsigma = \Phi^{-1}(\pi) = \frac{y_e - \mu}{\sigma} \tag{4.2.55}$$

A FGPQ for ς can be obtained by replacing the unknown parameters μ and σ in (4.2.55) with suitable FGPQs for those parameters ("plug-in" principle). Thus, a FGPQ for ς can be written as

$$\mathcal{R}_\varsigma(y, Y, \theta) = \frac{y_e - \mathcal{R}_\mu}{\mathcal{R}_\sigma}$$

$$= \frac{y_e - L_1'H[y - \mathcal{R}_\sigma(y, Y, \theta) \cdot Z]}{\mathcal{R}_\sigma(y, Y, \theta)}$$

$$= \frac{y_e - L_1'H\left(y - \frac{L_2'Hy}{L_2'HZ} \cdot Z\right)}{\frac{L_2'Hy}{L_2'HZ}} \tag{4.2.56}$$

A FGPQ for π is then given by

$$\mathcal{R}_\pi(y, Y, \theta) = \Phi\left[\mathcal{R}_\varsigma(y, Y, \theta)\right] \tag{4.2.57}$$

4.2.5.2. *FGPQ for a failure probability based on CPQ for a linear combination* $\boldsymbol{L'\theta}$ *of* $\boldsymbol{\theta}$

Alternatively to (4.2.56) and (4.2.57), a FGPQ for the failure probability can be based on CPQ $\mathcal{Q}_{L'\theta}(Y,\theta)$ for a linear combination $L'\theta$ of θ, namely (see equation (4.2.34))

$$\mathcal{Q}_{L'\theta}(Y,\theta) = \frac{L'\hat{\theta}(Y) - L'\theta}{L_2'HZ}$$

$$= \frac{L'H[Z - E(Z)]}{L_2'HZ} \tag{4.2.58}$$

Let $L_\varsigma = (1,\varsigma)'$. From (4.2.55), it follows that $L_\varsigma'\theta = \mu + \varsigma \cdot \sigma = y_e$. L and $L'\theta$ in the numerator of (4.2.58) are now replaced with L_ς and y_e, respectively, to obtain

$$\mathcal{Q}_\varsigma(Y,\theta) = \mathcal{Q}_{L_\varsigma'\theta}(Y,\theta) = \frac{L_\varsigma'\hat{\theta}(Y) - y_e}{L_2'HZ}$$

$$= \frac{L_\varsigma'H[Z - E(Z)]}{L_2'HZ} \tag{4.2.59}$$

As the third equality of (4.2.59) shows, the distribution of $\mathcal{Q}_{L_\varsigma'\theta}(Y,\theta)$ depends only on the parameter ς. In the manner of Hannig, Iyer and Patterson (2006, Example 5), let $F(x;\varsigma)$ be the cdf of $\mathcal{Q}_\varsigma(Y,\theta)$, evaluated at x. Then $F(x;\varsigma) = t\epsilon(0,1)$, and $Q(t;x) = \varsigma$ is defined as the inverse of F when F is viewed as a function of ς. Therefore, a FGPQ for ς is given by

$$\mathcal{R}_\varsigma = Q\{F[\mathcal{Q}_\varsigma(Y,\theta);\varsigma];\mathcal{Q}_\varsigma(y,\theta)\} = Q[U;\mathcal{Q}_\varsigma(y,\theta)] \tag{4.2.60}$$

where $\mathcal{Q}_\varsigma(y,\theta)$ is the observed value of $\mathcal{Q}_\varsigma(Y,\theta)$ and $U\sim$Uniform$(0,1)$.

4.3. Maximum Likelihood-Based Conventional and Fiducial Generalized Pivotal Quantities

Rank-based CPQs and FGPQs were presented in Sections 4.2.1 through 4.2.5. In Sections 4.3.1 and 4.3.2 we present the CPQs and FGPQs for the parameters μ and σ, and quantiles of the distribution based on ML method.

Approximate ML-based inference for the parameters μ, σ and for quantiles of the distribution based on Wald-Type and likelihood ratio methods are not investigated in this thesis. However, for the detailed discussion about the approximate inference for μ, σ and quantiles of the distribution, see for example, Lawless (2003, pp. 213-216).

ML-based CPQs and FGPQs for σ, μ and a p $(0 \leq p \leq 1)$ quantile of the distribution can be derived in the manner of Lawless (2003, p. 217) as presented in Sections 4.3.1 and 4.3.2 below.

4.3.1. CPQs and FGPQs for σ and μ Based on ML Estimators for σ and μ

CPQs and FGPQs for σ and μ based on ML estimators for σ and μ can be defined as follows:

Similar to the GLS estimator for σ in (4.2.1) and GLS estimator for μ in (4.2.22), it can be shown that for any LS parameterization (2.2.1) or (2.3.1), the ML estimators for σ and μ are, respectively, given by

$$\hat{\sigma}(Y)_{ML} = \sigma \cdot \hat{\sigma}(Z)_{ML} \tag{4.3.1}$$

and

$$\hat{\mu}(Y)_{ML} = \mu + \sigma \cdot \hat{\mu}(Z)_{ML} \tag{4.3.2}$$

Proof: (Lawless, 2003, p. 562).

Let $Y = (Y_1, Y_2, \dots, Y_n)'$ be a vector of an independent random sample $Y_1, Y_2, \dots, Y_n$ of size n drawn from any LS distribution with a cdf of the form (2.2.1). Furthermore, let $Z = (Z_1, Z_2, \dots, Z_n)'$ be the vector of the standardized variates $Z_1, Z_2, \dots, Z_n$. As shown in (3.2.2), Y can be expressed in terms of μ and σ as

$$Y = \mu \cdot \mathbf{1}_n + \sigma \cdot Z$$

where $\mathbf{1}_n = (1, 1, \dots, 1)'$. Then, the ML functions for μ and σ based on Z and Y are, respectively, given by

$$L(\mu, \sigma; z) = \left[\prod_{i=1}^{n} g(z_i)\right] \tag{4.3.3}$$

and

$$L(\mu, \sigma; y) = \sigma^{-n}\left[\prod_{i=1}^{n} g\left(\frac{y_i - \mu}{\sigma}\right)\right] \tag{4.3.4}$$

It is clear from equations (4.3.3) and (4.3.4) that $\hat{\mu}(Y)_{ML}$ and $\hat{\sigma}(Y)_{ML}$ are invariant estimators because $\hat{\mu}(Z)_{ML}$ and $\hat{\sigma}(Z)_{ML}$ maximize (4.3.3) only if $\hat{\mu}(Y)_{ML} = \mu + \sigma \cdot \hat{\mu}(Z)_{ML}$ and $\hat{\sigma}(Y)_{ML} = \sigma \cdot \hat{\sigma}(Z)_{ML}$.

Based on (4.3.1), a CPQ for σ based on ML estimator for σ can be defined as

$$\begin{aligned} Q_\sigma(Y, \theta)_{ML} &= \frac{\hat{\sigma}(Y)_{ML}}{\sigma} \\ &= \frac{\sigma \cdot \hat{\sigma}(Z)_{ML}}{\sigma} \\ &= \hat{\sigma}(Z)_{ML} \end{aligned} \tag{4.3.5}$$

The third equality in (4.3.5) shows that the distribution of $Q_\sigma(Y, \theta)_{ML}$ does not depend on any unknown parameters since (4.3.5) is equivalent to the ML estimator for σ based on the vector of standard variates Z. Thus $Q_\sigma(Y, \theta)_{ML}$ is a pivotal quantity and its exact distribution can be determined easily through simulation.

Similarly to (4.2.3), let y be an observation of Y. Based on (4.2.3), a FGPQ for σ based on ML method can be defined as follows:

$$\mathcal{R}_{\sigma}(y,Y,\theta) = \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Y)_{ML}/\sigma}$$

$$= \frac{\hat{\sigma}(y)_{ML}}{\sigma \cdot \hat{\sigma}(Z)_{ML}/\sigma}$$

$$= \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Z)_{ML}} \tag{4.3.6}$$

Similarly to (4.3.5), based on (4.3.2), a CPQ for μ based on ML estimator for μ can be defined as

$$\mathcal{Q}_{\mu}(Y,\theta)_{ML} = \frac{\hat{\mu}(Y)_{ML} - \mu}{\hat{\sigma}(Y)_{ML}}$$

$$= \frac{\mu + \sigma \cdot \hat{\mu}(Z)_{ML} - \mu}{\sigma \cdot \hat{\sigma}(Z)_{ML}}$$

$$= \frac{\hat{\mu}(Z)_{ML}}{\hat{\sigma}(Z)_{ML}} \tag{4.3.7}$$

Similarly to (4.3.5), the third equality in (4.3.7) shows that the distribution of $\mathcal{Q}_{\mu}(Y,\theta)_{ML}$ does not depend on any unknown parameters since (4.3.7) is solely a function of ML estimators for μ and σ based on the standard variate Z_*, where $Z_* \sim LS(0,1)$. Thus, $\mathcal{Q}_{\mu}(Y,\theta)_{ML}$ is a pivotal quantity and its exact distribution can be determined easily through simulation.

Similarly to (4.2.25) and (4.2.26), a FGPQ for μ based on ML method can be defined as follows:

$$\mathcal{R}_{\mu|\sigma}(y,Y,\theta) = \hat{\mu}(y)_{ML} - [\hat{\mu}(Y)_{ML} - \mu]$$

$$= \hat{\mu}(y)_{ML} - [\mu + \sigma \cdot \hat{\mu}(Z)_{ML} - \mu]$$

$$= \hat{\mu}(y)_{ML} - \sigma \cdot \hat{\mu}(Z)_{ML} \tag{4.3.8}$$

Based on CFGPQ $\mathcal{R}_{\mu|\sigma}(y, Y, \theta)$ in (4.3.8), a FGPQ for μ is obtained by replacing the parameter σ in (4.3.8) by a FGPQ for σ in (4.3.6). Thus,

$$\mathcal{R}_{\mu}(y, Y, \theta) = \hat{\mu}(y)_{ML} - \mathcal{R}_{\sigma}(y, Y, \theta) \cdot \hat{\mu}(Z)_{ML}$$

$$= \hat{\mu}(y)_{ML} - \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Z)_{ML}} \cdot \hat{\mu}(Z)_{ML} \tag{4.3.9}$$

4.3.2. CPQ and FGPQ for p Quantile of the Distribution

As noted earlier, for $0 \leq p \leq 1$, the p quantile of the distribution of Y_* is given by $\omega_p = \mu + \sigma \cdot z_p$, where z_p is the p quantile of the distribution of the standard variate Z_*.

A CPQ for ω_p can be defined as follows:

$$\mathcal{Q}_{\omega_p}\left(Y, \omega_p\right)_{ML} = \frac{\hat{\mu}(Y)_{ML} - \omega_p}{\hat{\sigma}(Y)_{ML}}$$

$$= \frac{\mu + \sigma \cdot \hat{\mu}(Z)_{ML} - \mu - \sigma \cdot z_p}{\sigma \cdot \hat{\sigma}(Z)_{ML}}$$

$$= \frac{\hat{\mu}(Z)_{ML} - z_p}{\hat{\sigma}(Z)_{ML}} \tag{4.3.10}$$

The third equality in (4.3.10) shows that the distribution of $\mathcal{Q}_{\omega_p}\left(Y, \omega_p\right)_{ML}$ does not depend on any unknown parameters since (4.3.10) is solely a function of ML based estimators for μ and σ, and of z_p.

Similarly to (4.2.45), a FGPQ for ω_p is given by

$$\mathcal{R}_{\omega_p}(y, Y, \theta) = \hat{\mu}(y)_{ML} - \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Z)_{ML}} \cdot \left[\hat{\mu}(Z)_{ML} - z_p\right] \tag{4.3.11}$$

4.4. Statistical Inference Using Rank-Based Conventional and Fiducial Generalized Pivotal Quantities

In this section we present hypothesis tests and confidence intervals (CIs) using rank-based conventional and fiducial generalized pivotal quantities for the parameters μ and σ, linear combination $L'\theta$ of θ, p quantile of the distribution, failure probability, and for the vector parameter θ.

4.4.1. Hypothesis Tests for $\boldsymbol{\sigma}$

In this section we construct a hypothesis test for σ based on the most efficient CPQ for σ (4.2.2), that is, a CPQ based on GLS estimators as follows:

Let y be an observation of the random variable Y. The null hypothesis to be tested is given by

$$H_0: \sigma = \sigma_0 \tag{4.4.1}$$

against the alternative

$$H_A: \sigma \neq \sigma_0$$

A test statistic for H_0 in (4.4.1), based on the CPQ (4.2.2) for σ, is given by

$$\begin{aligned} Q_\sigma(y, \sigma_0) &= \hat{\sigma}(y)/\sigma_0 \\ &= L_2' H y/\sigma_0 \end{aligned} \tag{4.4.2}$$

As shown in Section 4.2.1.1., the statistic $Q_\sigma(Y, \sigma_0)$ in (4.4.2), under H_0, has the same distribution as the CPQ

$$Q_\sigma(Y, \theta) = L_2' H Z$$

Exact quantiles of the distribution of $Q_\sigma(Y, \theta)$ can be obtained through simulation. Let $Q_\sigma(Y, \theta)_{\alpha/2}$ and $Q_\sigma(Y, \theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1-\alpha/2)$

quantiles of the distribution of $\mathcal{Q}_\sigma(Y,\theta)$. Then, H_0 in (4.4.1) is rejected if $\mathcal{Q}_\sigma(y,\sigma_0) < \mathcal{Q}_\sigma(Y,\theta)_{\alpha/2}$ or if $\mathcal{Q}_\sigma(y,\sigma_0) > \mathcal{Q}_\sigma(Y,\theta)_{1-\alpha/2}$.

4.4.2. Confidence Intervals for σ

In this section we present confidence intervals for σ based on CPQ and FGPQ for σ derived from GLS estimators.

A confidence interval for σ can be obtained by inverting the CPQ (4.2.2), namely $\mathcal{Q}_\sigma(Y,\theta)$. Let $\mathcal{Q}_\sigma(Y,\theta)_{\alpha/2}$ and $\mathcal{Q}_\sigma(Y,\theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\mathcal{Q}_\sigma(Y,\theta)$. Then

$$\begin{aligned} 1-\alpha &= P\left[\mathcal{Q}_\sigma(Y,\theta)_{\alpha/2} \le \mathcal{Q}_\sigma(Y,\theta) \le \mathcal{Q}_\sigma(Y,\theta)_{1-\alpha/2}\right] \\ &= P\left[\mathcal{Q}_\sigma(Y,\theta)_{\alpha/2} \le \frac{\hat{\sigma}(Y)}{\sigma} \le \mathcal{Q}_\sigma(Y,\theta)_{1-\alpha/2}\right] \\ &= P\left[\frac{\hat{\sigma}(Y)}{\mathcal{Q}_\sigma(Y,\theta)_{1-\alpha/2}} \le \sigma \le \frac{\hat{\sigma}(Y)}{\mathcal{Q}_\sigma(Y,\theta)_{\alpha/2}}\right] \end{aligned} \tag{4.4.3}$$

Thus a $100\ (1-\alpha)$ CI for σ based on GLS CPQ (4.2.2) is given by

$$\left[\frac{\hat{\sigma}(Y)}{\mathcal{Q}_\sigma(Y,\theta)_{1-\alpha/2}};\ \frac{\hat{\sigma}(Y)}{\mathcal{Q}_\sigma(Y,\theta)_{\alpha/2}}\right] \tag{4.4.4}$$

Alternatively to (4.4.4), a CI for σ can be obtained from the distribution of FGPQ (4.2.3), namely from the distribution of $\mathcal{R}_\sigma(y,Y,\theta)$. Let $\mathcal{R}_\sigma(y,Y,\theta)_{\alpha/2}$ and $\mathcal{R}_\sigma(y,Y,\theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\mathcal{R}_\sigma(y,Y,\theta)$. Then a $100(1-\alpha)$ CI for σ based on $\mathcal{R}_\sigma(y,Y,\theta)$ is given by

$$\left[\mathcal{R}_\sigma(y,Y,\theta)_{\alpha/2};\mathcal{R}_\sigma(y,Y,\theta)_{1-\alpha/2}\ \right] \tag{4.4.5}$$

We note that CIs for σ (4.4.4) and (4.4.5) are identical.

4.4.3. Hypothesis Test for μ

The null hypothesis to be tested is given by

$$H_0: \mu = \mu_0 \tag{4.4.6}$$

against the alternative

$$H_A: \mu \neq \mu_0$$

A test statistic for H_0 in (4.4.6), based on the CPQ (4.2.24) for μ, is given by

$$\mathcal{Q}_\mu(y, \mu_0) = \frac{\hat{\mu}(y) - \mu_0}{\hat{\sigma}(y)}$$

$$= \frac{L_1' H y - \mu_0}{L_2' H y} \tag{4.4.7}$$

As shown in Section 4.2.2, the statistic $\mathcal{Q}_\mu(Y, \mu_0)$, under H_0, has the same distribution as the CPQ

$$\mathcal{Q}_\mu(Y, \theta) = \frac{L_1' H Z}{L_2' H Z}$$

Exact quantiles of the distribution of $\mathcal{Q}_\mu(Y, \theta)$ can be obtained through simulation. Let $\mathcal{Q}_\mu(Y, \theta)_{\alpha/2}$ and $\mathcal{Q}_\mu(Y, \theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the distribution of $\mathcal{Q}_\mu(Y, \theta)$. Then H_0 in (4.4.6) is rejected if $\mathcal{Q}_\mu(y, \mu_0) < \mathcal{Q}_\mu(Y, \theta)_{\alpha/2}$ or if $\mathcal{Q}_\mu(y, \mu_0) > \mathcal{Q}_\mu(Y, \theta)_{1-\alpha/2}$.

4.4.4. Confidence Interval for μ

In this section we present the confidence intervals for μ based on CPQ and FGPQ for σ derived from GLS estimators.

A CI for μ can be obtained by inverting the CPQ (4.2.24) for μ, namely $\mathcal{Q}_\mu(Y, \theta)$. Let $\mathcal{Q}_\mu(Y, \theta)_{\alpha/2}$ and $\mathcal{Q}_\mu(Y, \theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the distribution of $\mathcal{Q}_\mu(Y, \theta)$. Then

$$1 - \alpha = P\left[\mathcal{Q}_\mu(Y, \theta)_{\alpha/2} \leq \mathcal{Q}_\mu(Y, \theta) \leq \mathcal{Q}_\mu(Y, \theta)_{1-\alpha/2}\right]$$

$$= P\left[\mathcal{Q}_{\mu}(Y,\theta)_{\alpha/2} \leq \frac{\hat{\mu}(Y)-\mu}{\hat{\sigma}(Y)} \leq \mathcal{Q}_{\mu}(Y,\theta)_{1-\alpha/2}\right]$$

$$= P\begin{bmatrix}\hat{\mu}(Y) - \mathcal{Q}_{\mu}(Y,\theta)_{1-\alpha/2}\cdot\hat{\sigma}(Y) \leq \mu \\ \\ \leq \hat{\mu}(Y) - \mathcal{Q}_{\mu}(Y,\theta)_{\alpha/2}\cdot\hat{\sigma}(Y)\end{bmatrix} \tag{4.4.8}$$

Thus a $100(1-\alpha)$ CI for μ, based on GLS estimator CPQ for σ, is given by

$$\left[\hat{\mu}(y) - \mathcal{Q}_{\mu}(Y,\theta)_{1-\alpha/2}\cdot\hat{\sigma}(Y)\ ;\ \hat{\mu}(y) - \mathcal{Q}_{\mu}(Y,\theta)_{\alpha/2}\cdot\hat{\sigma}(Y)\right] \tag{4.4.9}$$

Alternatively to (4.4.9), a CI for μ can be obtained from the distribution of FGPQ (4.2.26), namely $\mathcal{R}_{\mu}(y,Y,\theta)$. Let $\mathcal{R}_{\mu}(y,Y,\theta)_{\alpha/2}$ and $\mathcal{R}_{\mu}(y,Y,\theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\mathcal{R}_{\mu}(y,Y,\theta)$. Then, a $100(1-\alpha)$ CI for μ based on $\mathcal{R}_{\mu}(y,Y,\theta)$ is given by

$$\left[\mathcal{R}_{\mu}(y,Y,\theta)_{\alpha/2};\ \mathcal{R}_{\mu}(y,Y,\theta)_{1-\alpha/2}\right] \tag{4.4.10}$$

We note that CIs (4.4.9) and (4.4.10) are identical.

4.4.5. Hypothesis Test for a Linear Combination $\boldsymbol{L'\theta}$ of $\boldsymbol{\theta}$

The null hypothesis to be tested is given by

$$H_0: L'\theta = \lambda_0 \tag{4.4.11}$$

against the alternative

$$H_A: L'\theta \neq \lambda_0$$

A test statistic for H_0 in (4.4.11), based on the CPQ (4.2.34) for $L'\theta$, is given by

$$\mathcal{Q}_{L'\theta}(y,\lambda_0) = \frac{L'\hat{\theta}(y)-\lambda_0}{\hat{\sigma}(y)}$$

$$= \frac{L'Hy-\lambda_0}{L_2'Hy} \tag{4.4.12}$$

As shown in Section 4.2.3.1, the statistic $\mathcal{Q}_{L'\theta}(Y, \lambda_0)$, under H_0, has the same distribution as the CPQ

$$\mathcal{Q}_{L'\theta}(Y, \theta) = \frac{L'H[Z - E(Z)]}{L_2'HZ}$$

Exact quantiles of the distribution of $\mathcal{Q}_{L'\theta}(Y, \theta)$ can be obtained through simulation. Let $\mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2}$ and $\mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the distribution of $\mathcal{Q}_{L'\theta}(Y, \theta)$. Then H_0 in (4.4.11) is rejected if $\mathcal{Q}_{L'\theta}(y, \lambda_0) < \mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2}$ or if $\mathcal{Q}_{L'\theta}(y, \lambda_0) > \mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2}$.

4.4.6. Confidence Interval for a Linear Combination $\boldsymbol{L'\theta}$ of $\boldsymbol{\theta}$

A CI for $L'\theta$ can be obtained by inverting the CPQ (4.2.34) for $L'\theta$, namely $\mathcal{Q}_{L'\theta}(Y, \theta)$. Let $\mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2}$ and $\mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the distribution of $\mathcal{Q}_{L'\theta}(Y, \theta)$. Then

$$\begin{aligned}
1 - \alpha &= P\left[\mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2} \leq \mathcal{Q}_{L'\theta}(Y, \theta) \leq \mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2}\right] \\
&= P\left[\mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2} \leq \frac{L'\hat{\theta}(Y) - L'\theta}{\hat{\sigma}(Y)} \leq \mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2}\right] \\
&= P\left[\begin{array}{c} L'\hat{\theta}(Y) - \mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2} \cdot \hat{\sigma}(Y) \\ \leq L'\theta \leq L'\hat{\theta}(Y) - \mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2} \cdot \hat{\sigma}(Y) \end{array}\right] \qquad (4.4.13)
\end{aligned}$$

Thus a $100(1 - \alpha)$ CI for $L'\theta$, based on CPQ for σ, is given by

$$\left[L'\hat{\theta}(y) - \mathcal{Q}_{L'\theta}(Y, \theta)_{1-\alpha/2} \cdot L_2'Hy;\ L'\hat{\theta}(y) - \mathcal{Q}_{L'\theta}(Y, \theta)_{\alpha/2} \cdot L_2'Hy\right] \qquad (4.4.14)$$

Alternatively to (4.4.14), a CI for $L'\theta$ can be obtained from the distribution of FGPQ (4.2.41)/(4.2.44), namely $\mathcal{R}_{L'\theta}(y, Y, \theta)$. Let $\mathcal{R}_{L'\theta}(y, Y, \theta)_{\alpha/2}$ and $\mathcal{R}_{L'\theta}(y, Y, \theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the distribution of $\mathcal{R}_{L'\theta}(y, Y, \theta)$. Then a $100(1 - \alpha)$ CI for $L'\theta$, based on $\mathcal{R}_{L'\theta}(y, Y, \theta)$, is given by

$$\left[\mathcal{R}_{L'\theta}(y, Y, \theta)_{\alpha/2};\ \mathcal{R}_{L'\theta}(y, Y, \theta)_{1-\alpha/2}\right] \tag{4.4.15}$$

We note that CIs (4.4.14) and (4.4.15) are identical.

4.4.7. Hypothesis Test for a p Quantile of the Distribution

For $0 \le p \le 1$, let η_p be the p quantile of the distribution of Y_*. The null hypothesis that is tested is given by

$$H_0: \eta_p = \eta_{p0} \tag{4.4.16}$$

against the alternative

$$H_A: \eta_p \neq \eta_{p0}$$

As noted in Section 4.2.3.2, the p quantile η_p is a linear combination of the parameters μ and σ of the distribution of Y_*, namely

$$\begin{aligned} \eta_p &= \mu + \sigma \cdot z_p \\ &= L_p'\theta \end{aligned} \tag{4.4.17}$$

where $L_p = (1, z_p)'$, and z_p is the p quantile of the distribution of Z_* (which, of course, is known for all p). Thus hypothesis (4.4.16) is equivalent to

$$H_0: L_p'\theta = \eta_{p0} \tag{4.4.18}$$

But the null hypothesis (4.4.18) is of the form (4.4.11), so that H_0 in (4.4.16) and (4.4.18) can be tested using the test statistic (4.4.12), where the terms L and λ_0 in (4.4.12) are replaced by $L_p = (1, z_p)'$ and η_{p0}, respectively. Explicitly, the test statistic for testing (4.4.16) and (4.4.18) is given by

$$\begin{aligned} \mathcal{Q}_{\eta_p}(y, \eta_{p0}) &= \frac{L_p'\hat{\theta}(y) - \eta_{p0}}{L_2'Hy} \\ &= \frac{L_p'Hy - \eta_{p0}}{L_2'Hy} \end{aligned} \tag{4.4.19}$$

4.4.8. **Confidence Interval for a p Quantile of the Distribution**

As shown in Section 4.4.7 above, we note that when L in (4.4.14) is replaced by L_p, then $L'\theta$ is specifically of the form $L_p = (1, z_p)'$. Therefore (4.4.14) is a $100(1-\alpha)$ CI for the p quantile $\eta_p = \mu + \sigma \cdot z_p = L'_p\theta$ of the distribution of Y_*. Explicitly, a $100(1-\alpha)$ CI for the p quantile η_p based on CPQ for σ derived from GLS estimators (4.2.37) is given by

$$\left[L'_p\hat{\theta}(y) - \mathcal{Q}_{L'_p\theta}(Y,\theta)_{1-\alpha/2} \cdot L'_2Hy;\ L'_p\hat{\theta}(y) - \mathcal{Q}_{L'_p\theta}(Y,\theta)_{\alpha/2} \cdot L'_2Hy\right] \tag{4.4.20}$$

Alternatively to (4.4.20), a $100(1-\alpha)$ CI for the p quantile η_p based on FGPQs (4.2.45) and (4.2.46) is given by

$$\left[\mathcal{R}_{\eta_p}(y,Y,\theta)_{\alpha/2};\ \mathcal{R}_{\eta_p}(y,Y,\theta)_{1-\alpha/2}\right] \tag{4.4.21}$$

4.4.9. **Hypothesis Test for a Failure Probability**

The failure probability π_e at some y_e is given by

$$\begin{aligned}\pi_e &= F_Y(\,y_e;\theta) \\ &= \Phi\left(\frac{y_e-\mu}{\sigma}\right)\end{aligned} \tag{4.4.22}$$

The null hypothesis to be tested is

$$H_0\colon \pi_e = \pi_{e0} \tag{4.4.23}$$

against the alternative

$$H_A\colon \pi_e \neq \pi_{e0}$$

Let

$$\begin{aligned}\zeta_e &= \Phi^{-1}(\pi_e) \\ &= \Phi^{-1}[F_Y(\,y_e;\theta)]\end{aligned}$$

$$= \frac{y_e - \mu}{\sigma} \tag{4.4.24}$$

and similarly,

$$\zeta_{e0} = \Phi^{-1}(\pi_{e0}) \tag{4.4.25}$$

Then the null hypothesis in (4.4.23) is equivalent to

$$H_0: \zeta_e = \frac{y_e - \mu}{\sigma} = \zeta_{e0} \tag{4.4.26}$$

which in turn is equivalent to

$$H_0: \mu + \sigma \cdot \zeta_{e0} = y_e \tag{4.4.27}$$

against

$$H_0: \mu + \sigma \cdot \zeta_{e0} \neq y_e$$

But the null hypothesis (4.4.27) is of the form (4.4.11), so that H_0 in (4.4.23)/(4.4.26)/(4.4.27) can be tested using the test statistic (4.4.12), where the terms L and λ_0 in (4.4.12) are replaced by $L_e = (1, \lambda_{e0})'$ and y_e, respectively. Explicitly, the test statistic for testing (4.4.23)/(4.4.26)/(4.4.27) is given by

$$\mathcal{Q}_{L'_e\theta}(y, y_e) = \frac{L'_e\hat{\theta}(y) - y_e}{L'_2Hy}$$

$$= \frac{L'_eHy - y_e}{L'_2Hy} \tag{4.4.28}$$

4.4.10. Confidence Interval for a Failure Probability

An exact CI for the failure probability π_e is obtained by inversion of the exact test described in Section 4.4.9. Thus a two-sided $100(1-\alpha)$ CI for π_e is given by all values π_{e0} for which the null hypothesis (4.4.23)/(4.4.26)/(4.4.27) is not rejected at two-sided significance level α.

Explicitly, let π_{e1} and π_{e2} respectively be the lower and upper limits of the $100(1-\alpha)$ CI for π_e. Furthermore, let $\zeta_{e1} = \Phi^{-1}(\pi_{e1})$ and $\zeta_{e2} = \Phi^{-1}(\pi_{e2})$, and $L_{e1} = (1, \lambda_{e1})'$ and $L_{e2} = (1, \lambda_{e2})'$. Then ζ_{e1} and ζ_{e2} satisfy the following equalities

$$\frac{L'_{e1}\hat{\theta}(y) - y_e}{L'_2 Hy} = Q_{L'_{e1}\theta}(Y, \theta)_{\alpha/2} \tag{4.4.29}$$

and

$$\frac{L'_{e2}\hat{\theta}(y) - y_e}{L'_2 Hy} = Q_{L'_{e2}\theta}(Y, \theta)_{1-\alpha/2} \tag{4.4.30}$$

where $Q_{L'_{e1}\theta}(Y, \theta)_{\alpha/2}$ and $Q_{L'_{e2}\theta}(Y, \theta)_{1-\alpha/2}$ are the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $Q_{L'_{e1}\theta}(Y, \theta)$ and $Q_{L'_{e2}\theta}(Y, \theta)$, respectively. Once ζ_{e1} and ζ_{e2} are determined from (4.4.29) and (4.4.30), then the lower and upper limits of the $100(1-\alpha)$ CI for π_e are given by $\pi_{e1} = \Phi(\zeta_{e1})$ and $\pi_{e2} = \Phi(\zeta_{e2})$.

Since the distributions of $Q_{L'_{e1}\theta}(Y, y_e)$ relevant in (4.4.23) and of $Q_{L'_{e2}\theta}(Y, y_e)$ relevant in (4.4.30), and thus the quantiles $Q_{L'_{e1}\theta}(Y, \theta)_{\alpha/2}$ and $Q_{L'_{e2}\theta}(Y, \theta)_{1-\alpha/2}$ depend on the null values π_{e1} and π_{e2} (through $\zeta_{e1} = \Phi^{-1}(\pi_{e1})$ and $\zeta_{e2} = \Phi^{-1}(\pi_{e2})$), a closed-form expression for the exact CI for π_e is not available. However, the upper and lower confidence limits can be calculated fast and in a straightforward manner, for example through bisection.

4.4.11. Hypothesis Tests for $\boldsymbol{\theta}$

The null hypothesis to be tested is

$$H_0: \theta = \theta_0 \tag{4.4.31}$$

against the alternative

$$H_A: \theta \neq \theta_0$$

Two options for testing H_0 in (4.4.31) are available and thus, are presented in the following two sections.

4.4.11.1. Hypothesis test for $\boldsymbol{\theta}$ based on GLS estimator CPQ for $\boldsymbol{\theta}$

A test statistic for H_0 in (4.4.31), based on the CPQ (4.2.48) for θ, is given by

$$Q_\theta(y,\theta_0) = \left\{\left[\hat{\theta}(y) - \theta_0\right]' X'V^{-1}X\left[\hat{\theta}(y) - \theta_0\right]\right\}/\sigma_0^2$$

$$= [(Hy - \theta_0)'X'V^{-1}X(Hy - \theta_0)]/\sigma_0^2 \qquad (4.4.32)$$

As shown in Section 4.2.4.1, under H_0 the statistic $Q_\theta(Y,\theta_0)$ has the same distribution as the statistic

$$Q_\theta(y,\theta) = [Z - E(Z)]'V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}[Z - E(Z)]$$

Exact quantiles of the distribution of $Q_\theta(Y,\theta_0)$ can therefore be obtained by simulation from the distribution of Z. Let $Q_\theta(Y,\theta)_{\alpha/2}$ and $Q_\theta(Y,\theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $Q_\theta(Y,\theta)$. Then H_0 in (4.4.31) is rejected if $Q_\theta(y,\theta_0) < Q_\theta(Y,\theta)_{\alpha/2}$ or if $Q_\theta(y,\theta_0) > Q_\theta(Y,\theta)_{1-\alpha/2}$.

4.4.11.2. Hypothesis test for $\boldsymbol{\theta}$ based on the ratio of CPQ for $\boldsymbol{\theta}$ and of CPQ for $\boldsymbol{\sigma}^2$

Alternatively to (4.4.32), H_0 can be tested using a test statistic based on the CPQ (4.2.49) for θ:

$$\tilde{Q}_\theta(y,\theta_0) = \frac{Q_\theta(y,\ \theta_0)}{Q_{\sigma^2}(y,\ \sigma_0)}$$

$$= \frac{\left[\hat{\theta}(y) - \theta_0\right]' X'V^{-1}X\left[\hat{\theta}(y) - \theta_0\right]/\sigma_0^2}{(L_2'HZ)^2/\sigma_0^2}$$

$$= \frac{(Hy - \theta_0)'X'V^{-1}X(Hy - \theta_0)}{L_2'HZ} \qquad (4.4.33)$$

As shown in Section 4.2.4.2, under H_0 the statistic $\tilde{Q}_\theta(Y,\theta_0)$ has the same distribution as the statistic

$$\tilde{Q}_\theta(Y,\theta) = \frac{[Z-E(Z)]'V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}[Z-E(Z)]}{L_2'HZ}$$

Exact quantiles of the distribution of $\tilde{Q}_\theta(Y,\theta_0)$ can therefore be obtained by simulation from the distribution of Z. Let $\tilde{Q}_\theta(Y,\theta)_{\alpha/2}$ and $\tilde{Q}_\theta(Y,\theta)_{1-\alpha/2}$ be the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\tilde{Q}_\theta(Y,\theta)$. Then H_0 in (4.4.31) is rejected if $\tilde{Q}_\theta(y,\theta_0) < \tilde{Q}_\theta(Y,\theta)_{\alpha/2}$ or if $\tilde{Q}_\theta(y,\theta_0) > \tilde{Q}_\theta(Y,\theta)_{1-\alpha/2}$.

4.4.12. Confidence Ellipsoids for $\boldsymbol{\theta}$

In this section we derive explicitly using different approaches, two versions of confidence ellipsoids for θ. Exact simultaneous confidence regions (SCRs), in other words, confidence ellipsoids, for θ can be obtained by inversion of the CPQs described in Sections 4.2.4.1 and 4.2.4.2.

4.4.12.1. Confidence ellipsoid for $\boldsymbol{\theta}$ based on GLS estimator CPQ for $\boldsymbol{\theta}$

An SCR for θ is obtained by inverting the CPQ (4.2.48) for θ. Let $Q_\theta(Y,\theta)_{1-\alpha}$ be the $(1-\alpha)$ quantile of the distribution of $Q_\theta(Y,\theta)$. Then

$$\begin{aligned} 1-\alpha &= P\{Q_\theta(Y,\theta) \le Q_\theta(Y,\theta)_{1-\alpha}\} \\ &= P\left\{\left[\hat{\theta}(Y)-\theta\right]' X'V^{-1}X\left[\hat{\theta}(Y)-\theta\right]/\sigma^2 \le Q_\theta(Y,\theta)_{1-\alpha}\right\} \\ &= P\left\{\left[\hat{\theta}(Y)-\theta\right]' X'V^{-1}X\left[\hat{\theta}(Y)-\theta\right] \le \sigma^2 \cdot Q_\theta(Y,\theta)_{1-\alpha}\right\} \qquad (4.4.34) \end{aligned}$$

Thus a $100(1-\alpha)$ SCR for θ is given by

$$SCR(\theta) = \left\{\theta \middle| \left[\hat{\theta}(y)-\theta\right]' X'V^{-1}X\left[\hat{\theta}(y)-\theta\right] \le \sigma^2 \cdot Q_\theta(Y,\theta)_{1-\alpha}\right\} \qquad (4.4.35)$$

As is shown in Appendix G (R. Schall, working paper, 2012), the SCR for θ in (4.4.35) can be calculated as follows:

Let $DD' = X'V^{-1}X$ be the Cholesky decomposition of $X'V^{-1}X$, such that

$$D = \begin{pmatrix} d_1 & 0 \\ d_2 & d_3 \end{pmatrix} \tag{4.4.36}$$

Then an SCR for θ is given by all $\theta = (\mu, \sigma)'$ which satisfy the following two conditions:

$$\hat{\sigma}(y) \cdot \frac{d_3}{d_3 + \sqrt{Q_\theta(Y,\theta)_{1-\alpha}}} \leq \sigma \leq \hat{\sigma}(y) \cdot \frac{d_3}{d_3 - \sqrt{Q_\theta(Y,\theta)_{1-\alpha}}} \tag{4.4.37}$$

and

$$\hat{\mu}(y) - \left\{\sqrt{\sigma^2 \cdot Q_\theta(Y,\theta)_{1-\alpha} - d_3^2[\hat{\sigma}(y) - \sigma]^2} - d_2[\hat{\sigma}(y) - \sigma]\right\}/d_1$$
$$\leq \mu \leq \hat{\mu}(y) + \left\{\sqrt{\sigma^2 \cdot Q_\theta(Y,\theta)_{1-\alpha} - d_3^2[\hat{\sigma}(y) - \sigma]^2} + d_2[\hat{\sigma}(y) - \sigma]\right\}/d_1 \tag{4.4.38}$$

4.4.12.2. Confidence ellipsoid for $\boldsymbol{\theta}$ based on the ratio of CPQ for $\boldsymbol{\theta}$ and of CPQ for $\boldsymbol{\sigma^2}$

Alternatively to the SCR (4.4.35), an SCR for θ is obtained by inverting the CPQ (4.2.49) for θ. Let $\tilde{Q}_\theta(Y,\theta)_{1-\alpha}$ be the $(1-\alpha)$ quantile of the distribution of $\tilde{Q}_\theta(Y,\theta)$. Then

$$\begin{aligned} 1-\alpha &= P\{\tilde{Q}_\theta(Y,\theta) \leq \tilde{Q}_\theta(Y,\theta)_{1-\alpha}\} \\ &= P\left\{\left[\hat{\theta}(Y) - \theta\right]' X'V^{-1}X\left[\hat{\theta}(Y) - \theta\right]/L_2'HZ \leq \tilde{Q}_\theta(Y,\theta)_{1-\alpha}\right\} \\ &= P\left\{\left[\hat{\theta}(Y) - \theta\right]' X'V^{-1}X\left[\hat{\theta}(Y) - \theta\right] \leq L_2'HZ \cdot \tilde{Q}_\theta(Y,\theta)_{1-\alpha}\right\} \end{aligned} \tag{4.4.39}$$

Thus a $100(1-\alpha)$ SCR for θ is given by

$$SCR(\theta) = \left\{\theta \middle| \left[\hat{\theta}(y) - \theta\right]' X'V^{-1}X\left[\hat{\theta}(y) - \theta\right] \leq L_2'HZ \cdot \tilde{Q}_\theta(Y,\theta)_{1-\alpha}\right\} \tag{4.4.40}$$

In the same manner as for the SCR (4.4.35), it can be shown that an SCR for θ is given by all $\theta = (\mu, \sigma)'$ which satisfy the following two conditions:

$$\hat{\sigma}(y) - \sqrt{L_2' HZ \cdot \tilde{Q}_\theta(Y, \theta)_{1-\alpha}}/d_3 \le \sigma \le \hat{\sigma}(y) + \sqrt{L_2' HZ \cdot \tilde{Q}_\theta(Y, \theta)_{1-\alpha}}/d_3 \tag{4.4.41}$$

and

$$\hat{\mu}(y) - \left\{\sqrt{L_2' HZ \cdot \tilde{Q}_\theta(Y, \theta)_{1-\alpha} - d_3^2[\hat{\sigma}(y) - \sigma]^2} - d_2[\hat{\sigma}(y) - \sigma]\right\}/d_1$$
$$\le \mu \le \hat{\mu}(y) + \left\{\sqrt{L_2' HZ \cdot \tilde{Q}_\theta(Y, \theta)_{1-\alpha} - d_3^2[\hat{\sigma}(y) - \sigma]^2} + d_2[\hat{\sigma}(y) - \sigma]\right\}/d_1 \tag{4.4.42}$$

4.4.13. Simultaneous Confidence Bands for the CDF of the Distribution

Based on the SCRs for θ described in Sections 4.4.12.1 and 4.4.12.2, simultaneous confidence bands (SCBs) for the cdf of Y_* can be constructed as described by Cheng and Iles (1983); see also Hong, Escobar and Meeker (2010, Section 1.2).

4.5. Statistical Inference using ML-Based Conventional Pivotal Quantities

In Sections 4.5.1 through 4.5.6 we present the hypothesis tests and confidence intervals for the parameters μ and σ, and for the p quantile of the distribution using ML-based CPQs.

4.5.1. **Hypothesis Test for σ**

Let y be an observation of the random variable Y. Equivalently to (4.4.1), the null hypothesis that is tested is given by

$$H_0: \sigma = \sigma_0 \tag{4.5.1}$$

against the alternative

$$H_A: \sigma \neq \sigma_0$$

A test statistic for H_0 in (4.5.1) based on the CPQ (4.3.5) for σ is given by

$$Q_\sigma(y, \sigma_0)_{ML} = \frac{\hat{\sigma}(y)_{ML}}{\sigma_0} \tag{4.5.2}$$

As shown in Section 4.3.1, the test statistic $Q_\sigma(Y, \sigma_0)_{ML}$, under H_0, has the same distribution as the CPQ

$$Q_\sigma(Y, \theta)_{ML} = \hat{\sigma}(Z)_{ML}$$

Exact quantiles of the distribution of $Q_\sigma(Y, \theta)_{ML}$ can be obtained through simulation from the distribution of Z_*. Let $Q_\sigma(Y, \theta)_{ML\,(\alpha/2)}$ and $Q_\sigma(Y, \theta)_{ML\,(1-\alpha/2)}$ be $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $Q_\sigma(Y, \theta)_{ML}$. Then H_0 in (4.5.1) is rejected if $Q_\sigma(y, \sigma_0)_{ML} < Q_\sigma(Y, \theta)_{ML\,(\alpha/2)}$ or if $Q_\sigma(y, \sigma_0)_{ML} > Q_\sigma(Y, \theta)_{ML\,(1-\alpha/2)}$.

4.5.2. **Confidence Interval for σ**

In the manner of Lawless (2003, p. 564), exact CI for σ can be obtained by inverting the CPQ (4.3.5) for σ, namely $Q_\sigma(Y, \theta)_{ML}$. Let $Q_\sigma(Y, \theta)_{ML\,(\alpha/2)}$ and $Q_\sigma(Y, \theta)_{ML\,(1-\alpha/2)}$ be $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $Q_\sigma(Y, \theta)_{ML}$.

Then

$$1 - \alpha = P\left[Q_\sigma(Y, \theta)_{ML\,(\alpha/2)} \leq Q_\sigma(Y, \theta)_{ML} \leq Q_\sigma(Y, \theta)_{ML\,(1-\alpha/2)}\right]$$

$$= P\left[Q_\sigma(Y,\theta)_{ML\,(\alpha/2)} \leq \frac{\hat{\sigma}(Y)_{ML}}{\sigma} \leq Q_\sigma(Y,\theta)_{ML\,(1-\alpha/2)}\right]$$

$$= P\left[\frac{\hat{\sigma}(Y)_{ML}}{Q_\sigma(Y,\theta)_{ML\,(1-\alpha/2)}} \leq \sigma \leq \frac{\hat{\sigma}(Y)_{ML}}{Q_\sigma(Y,\theta)_{ML\,(\alpha/2)}}\right] \tag{4.5.3}$$

Thus a $100(1-\alpha)$ exact CI for σ is given by

$$\left[\frac{\hat{\sigma}(y)_{ML}}{Q_\sigma(Y,\theta)_{ML\,(1-\alpha/2)}};\ \frac{\hat{\sigma}(y)_{ML}}{Q_\sigma(Y,\theta)_{ML\,(\alpha/2)}}\right] \tag{4.5.4}$$

4.5.3. Hypothesis Test for $\boldsymbol{\mu}$

Equivalently to (4.4.6), the null hypothesis that is tested is given by

$$H_0: \mu = \mu_0 \tag{4.5.5}$$

against the alternative

$$H_A: \mu \neq \mu_0$$

A test statistic for H_0 in (4.5.5), based on the CPQ (4.3.7) for μ, is given by

$$Q_\mu(y,\mu_0)_{ML} = \frac{\hat{\mu}(y)_{ML} - \mu_0}{\hat{\sigma}(y)_{ML}} \tag{4.5.6}$$

As shown in Section 4.3.1, the test statistic $Q_\mu(Y,\mu_0)_{ML}$, under H_0, has the same distribution as the CPQ

$$Q_\mu(Y,\theta)_{ML} = \frac{\hat{\mu}(Z)_{ML}}{\hat{\sigma}(Z)_{ML}}$$

Exact quantiles of the distribution of $Q_\mu(Y,\theta)_{ML}$ can be obtained through simulation from the distribution of Z_*. Let $Q_\mu(Y,\theta)_{ML\,(\alpha/2)}$ and $Q_\mu(Y,\theta)_{ML\,(1-\alpha/2)}$ be $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $Q_\mu(Y,\theta)_{ML}$. Then H_0 in (4.5.5) is rejected if $Q_\mu(y,\mu_0)_{ML} < Q_\mu(Y,\theta)_{ML\,(\alpha/2)}$ or if $Q_\mu(y,\mu_0)_{ML} > Q_\mu(Y,\theta)_{ML\,(1-\alpha/2)}$.

4.5.4. **Confidence Interval for μ**

Similarly to obtaining the CI for σ in (4.5.4), exact confidence interval for μ can be obtained by inverting the CPQ (4.3.7) for μ, namely $\mathcal{Q}_\mu(Y,\theta)_{ML}$. Let $\mathcal{Q}_\mu(Y,\theta)_{ML\,(\alpha/2)}$ and $\mathcal{Q}_\mu(Y,\theta)_{ML\,(1-\alpha/2)}$ be $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\mathcal{Q}_\mu(Y,\theta)_{ML}$.

Then

$$1-\alpha = P\left[\mathcal{Q}_\mu(Y,\theta)_{ML\,(\alpha/2)} \le \mathcal{Q}_\mu(Y,\theta)_{ML} \le \mathcal{Q}_\mu(Y,\theta)_{ML\,(1-\alpha/2)}\right]$$

$$= P\left[\mathcal{Q}_\mu(Y,\theta)_{ML\,(\alpha/2)} \le \frac{\hat{\mu}(Y)_{ML}-\mu}{\hat{\sigma}(Y)_{ML}} \le \mathcal{Q}_\mu(Y,\theta)_{ML\,(1-\alpha/2)}\right]$$

$$= P\begin{bmatrix}\hat{\mu}(Y)_{ML} - \mathcal{Q}_\mu(Y,\theta)_{ML\,(1-\alpha/2)}\cdot\hat{\sigma}(Y)_{ML} \le \mu \\ \le \hat{\mu}(Y)_{ML} - \mathcal{Q}_\mu(Y,\theta)_{ML\,(\alpha/2)}\cdot\hat{\sigma}(Y)_{ML}\end{bmatrix} \tag{4.5.7}$$

Thus a $100(1-\alpha)$ exact CI for μ is given by

$$\begin{bmatrix}\hat{\mu}(y)_{ML} - \mathcal{Q}_\mu(Y,\theta)_{ML\,(1-\alpha/2)}\cdot\hat{\sigma}(y)_{ML}; \\ \hat{\mu}(y)_{ML} - \mathcal{Q}_\mu(Y,\theta)_{ML\,(\alpha/2)}\cdot\hat{\sigma}(y)_{ML}\end{bmatrix} \tag{4.5.8}$$

4.5.5. **Hypothesis Tests for p Quantile of the Distribution**

For $0 \le p \le 1$, let ω_p be the p quantile of the distribution of Y_*. The null hypothesis that is tested is given by

$$H_0\colon \omega_p = \omega_{p0} \tag{4.5.9}$$

against the alternative

$$H_A\colon \omega_p \neq \omega_{p0}$$

A test statistic for H_0 in (4.5.9) based on the CPQ (4.3.10) for ω_p is

$$\mathcal{Q}_{\omega_p}(y,\omega_p)_{ML} = \frac{\hat{\mu}(y)_{ML}-\omega_{p0}}{\hat{\sigma}(y)_{ML}} \tag{4.5.10}$$

As shown in Section 4.3.2, the test statistic $\mathcal{Q}_{\omega_p}(Y, \omega_{p0})_{ML}$, under H_0, has the same distribution as the CPQ

$$\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML} = \frac{\hat{\mu}(Z)_{ML} - z_p}{\hat{\sigma}(Z)_{ML}}$$

Exact quantiles of the distribution of $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML}$ can be obtained through simulation from the distribution of Z_*. Let $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(\alpha/2)}$ and $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(1-\alpha/2)}$ be $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML}$. Then H_0 in (4.5.9) is rejected if $\mathcal{Q}_{\omega_p}(y, \omega_{p0})_{ML} < \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(\alpha/2)}$ or if $\mathcal{Q}_{\omega_p}(y, \omega_{p0})_{ML} > \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(1-\alpha/2)}$.

4.5.6. Confidence Intervals for p Quantile of the Distribution

Exact confidence interval for the p quantile of the distribution, namely ω_p can be constructed as follows.

The CI for $\omega_p = \mu + \sigma \cdot z_p$ can be obtained by inverting the CPQ (4.3.10) for ω_p, namely $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML}$. Let $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(\alpha/2)}$ and $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(1-\alpha/2)}$ be $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML}$.

Then

$$\begin{aligned} 1-\alpha &= P\left[\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(\alpha/2)} \le \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML} \le \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(1-\alpha/2)}\right] \\ &= P\left[\mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(\alpha/2)} \le \frac{\hat{\mu}(Y)_{ML} - \omega_p}{\hat{\sigma}(Y)_{ML}} \le \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(1-\alpha/2)}\right] \\ &= P\begin{bmatrix} \hat{\mu}(Y)_{ML} - \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(1-\alpha/2)} \cdot \hat{\sigma}(Y)_{ML} \le \omega_p \\ \le \hat{\mu}(Y)_{ML} - \mathcal{Q}_{\omega_p}(Y, \omega_p)_{ML\,(\alpha/2)} \cdot \hat{\sigma}(Y)_{ML} \end{bmatrix} \end{aligned} \tag{4.5.11}$$

Thus a $100(1-\alpha)$ exact CI for ω_p is given by

$$\begin{bmatrix} \hat{\mu}(y)_{ML} - \mathcal{Q}_{\omega_p}\left(Y, \omega_p\right)_{ML\,(1-\alpha/2)} \cdot \hat{\sigma}(y)_{ML}; \\ \hat{\mu}(y)_{ML} - \mathcal{Q}_{\omega_p}\left(Y, \omega_p\right)_{ML\,(\alpha/2)} \cdot \hat{\sigma}(y)_{ML} \end{bmatrix} \tag{4.5.12}$$

4.6. Simulation Study: One-Sample Problem

In Section 4.6.1 we describe the calculation, by simulation, of the expected value and inverse of the covariance matrix of order statistics of an i.i.d. sample from the Normal, Logistic, Uniform, Pareto and Weibull distributions. Similarly, the calculation, by simulation, of the lower and upper quantiles of the distribution of CPQs for the location and scale parameters and quantiles of the distribution for these LS and LLS families of distributions is described in Section 4.6.2. In Section 4.6.3, we describe a simulation study for determining the coverage probabilities and average lengths of confidence intervals for the location and scale parameters and quantiles of the distribution involving a one-sample. We discuss the summary of the results of a simulation study in Section 4.6.3.4 below.

All program code for the simulation studies, and example program code of each method included with this thesis, unless stated otherwise, was written using MATLAB R2013a software. Only selected program code is presented in this thesis at Appendix H. However, all other program code is available electronically on request.

4.6.1. Calculation of the Expected Value and Inverse of Covariance Matrix of Z in LS and LLS Families

The program code of the calculation of the expectation $E(Z)$ and the inverse of the covariance matrix $Cov(Z)$ is presented at Appendix H1 (Code H1.1 through

H1.5) below. The rationale, design and results of this simulation study are presented as follows:

4.6.1.1. Rationale of simulation

This simulation was carried out to calculate the $E(Z)$ and the inverse of $Cov(Z)$ of order statistics of an i.i.d. sample of size n from standard LS and LLS distributions. The values of these two quantities, namely $E(Z)$ and the inverse of $Cov(Z)$, are required for the calculation of the lower and upper quantiles of the distribution of rank-based CPQs and FGPQs for the parameters μ and σ, and for quantiles of the distribution for the Normal, Logistic, Uniform, Pareto and Weibull distributions. Furthermore, the values of $E(Z)$ and inverse of $Cov(Z)$ are required for hypotheses testing to calculate the value of the test statistics, and for the calculation of confidence intervals for the parameters μ and σ, and for quantiles of the distribution using rank-based CPQs and FGPQs.

4.6.1.2. Design of simulation

For this simulation, five programs were written to calculate the values of $E(Z)$ and inverse of $Cov(Z)$ for the Normal, Logistic, Uniform, Weibull and Pareto distributions. The values of the $E(Z)$ and inverse of $Cov(Z)$ for these distributions were calculated using $S = 100000$ simulated independent samples of selected sizes $n = 10$ and $n = 25$ from the standard Normal, Logistic, Uniform, Weibull and Pareto distributions based on the LS parametrization. Furthermore, the values of the $E(Z)$ and inverse of $Cov(Z)$ for each distribution were calculated using the Algorithm 1 presented earlier in Section 3.4.

4.6.1.3. Results of simulation

For sample sizes $n = 10$ and $n = 25$, the values of five vectors of the $E(Z)$ of dimensions 1 by 10 and 1 by 25, and inverses of five $Cov(Z)$ of dimensions 10 by 10, and 25 by 25, respectively, were calculated for the standard Normal, Logistic, Uniform, Weibull, and Pareto distributions through simulation and have been stored away.

4.6.2. Calculation of Lower and Upper Quantiles of the Distribution of CPQs for μ, σ and η

Below we present the rationale, design, and results of this simulation as follows:

4.6.2.1. Rationale of simulation

We calculate through simulation the lower and upper quantiles of the distribution of rank and ML-based CPQs for μ, σ and quantile of the distribution η in LS and LLS families of distributions. We recall from Sections 4.4 through 4.5 that the values of the lower and upper quantiles of the distribution of rank and ML-based CPQs, which we refer to in this thesis, respectively, as $\alpha/2$ and $1-\alpha/2$ quantiles, are required to calculate the confidence intervals for μ, σ and η. In addition, the values of the $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of rank and ML-based CPQs are required for hypotheses testing to decide whether or not to reject a specified null hypothesis.

4.6.2.2. Design of simulation

For sample sizes $n = 10$ and $n = 25$, five programs were written to calculate the values of $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of rank and ML-based CPQs for the Normal, Logistic, Uniform, Weibull, and Pareto distributions. The values of $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of these rank and ML-based CPQs for the Normal, Logistic, Uniform, Weibull and Pareto distributions were calculated using $S = 1000000$ simulated independent samples of sizes $n = 10$ and $n = 25$ from the standard LS distributions and for the levels of significance α of $0.1, 0.05$ and 0.01.

4.6.2.3. Results of simulation

Rank-based $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of μ, σ and η for the Normal, Logistic, Uniform, Weibull and Pareto distributions are similar to their ML-based counterparts. The results of $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of rank and ML-based CPQs for μ, σ and η of the Normal, Logistic, Uniform, Weibull and Pareto distributions are presented in Appendix A1 (Tables A1.1 through A1.15 for $n=10$) and in Appendix A2 (Tables A2.1 through A2.15 for $n=25$).

4.6.3. **Simulation Study: Coverage Probabilities and Average Lengths of Confidence Intervals for $\boldsymbol{\mu}, \boldsymbol{\sigma}$ and $\boldsymbol{\eta}$**

In this simulation study we evaluate the performance of the proposed rank-based methods of inference against the ML-based methods. Specifically, we calculate the coverage probabilities and average lengths of confidence intervals for μ, σ and η in the LS and LLS families of distributions using rank and ML-based CPQs. Below we present the objectives, design and results of this simulation study as follows:

4.6.3.1. Objectives of simulation study

- Firstly, we calculate the average length of confidence intervals for μ, σ and η using rank and ML-based CPQs:
    - To determine the “best” (most efficient) CPQ and/or FGPQ for σ in terms of average length of confidence intervals for μ, σ and η based on the respective CPQ and/or FGPQ for σ.
    - Compare the efficiency of rank and ML-based methods in terms of average length of confidence intervals for μ, σ and η and the coverage probabilities.

    - To determine whether confidence intervals for μ and η calculated using two independent copies of the standard variate Z have shorter average length than confidence intervals based on a single copy of Z.
- Secondly, since the rank and ML-based methods for the one-sample problem are exact, the coverage probability of confidence intervals must be exact if the program code is correct. Hence, we determine if the simulated coverage probabilities are equal to the specified nominal probabilities, which serves as a validation check for the programming.

4.6.3.2. Design of simulation study

For each distribution investigated in this thesis (Normal, Logistic, Uniform, Weibull and Pareto distributions), a program code was written to calculate the coverage probabilities and average lengths of confidence intervals for μ, σ and η based on rank and ML-CPQs, at confidence levels $(1-\alpha/2)$ of 0.90, 0.95, and 0.99 and $S = 1000000$ simulated samples of sizes of $n = 10$ and $n = 25$.

4.6.3.3. Results of simulation study

The results of the coverage probabilities and average lengths of confidence intervals for μ, σ and η of the Weibull distribution ($n = 25$) are presented below (Tables 4.6.3.1 through 4.6.3.3). However, the results of a simulation study ($n = 10$ and $n = 25$) for the Normal, Logistic, Uniform and Pareto distributions are presented in Appendix B1 (Tables B1.1 through B1.15 for $n = 10$) and Appendix B2 (Tables B2.1 through B2.12 for $n = 25$) in order to avoid too many tables from appearing in the text.

Table 4.6.3.1: Weibull distribution: Coverage probability and average length of confidence intervals for σ based on four CPQs at specified nominal confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1[1]	σ_2[2]	σ_3[3]	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.9007	0.9003	0.9003	0.9012	0.9503	0.9503	0.9508	0.9504	0.9901	0.9901	0.9904	0.9900
Length	0.5565	0.8116	0.5900	0.5551	0.6729	0.9841	0.7149	0.6708	0.9184	1.3409	0.9762	0.9155

[1] Subscript 1used in Tables (4.6.3.1 through 4.6.3.), A1, A2, B1 and B2 refers to the calculations based on the GLS estimator for σ.

[2] Subscript 2 used un Tables (4.6.3.1 through 4.6.3.), A1, A2, B1 and B2 refers to the calculations based on the residual sum of squares (RSS) for σ.

[3] Subscript 3 used un Tables (4.6.3.1 through 4.6.3.), A1, A2, B1 and B2 refers to the calculations based on the combined information of the GLS estimator for σ and RSS for σ.

Table 4.6.3.2: Weibull distribution: Coverage probability and average length of confidence intervals for μ based on eight CPQs at specified nominal confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9000	0.9002	0.9006	0.9006	0.9509	0.9506	0.9505	0.9507	0.9902	0.9895	0.9900	0.9902
Length	0.7165	0.7446	0.7203	0.7161	0.8660	0.9117	0.8709	0.8654	1.1777	1.2742	1.1892	1.1761
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9003	0.8999	0.9002	0.9007	0.9492	0.9491	0.9496	0.9505	0.9889	0.9886	0.9886	0.9901
Length	0.7192	0.7465	0.7227	0.7165	0.8699	0.9154	0.8756	0.8646	1.1922	1.2914	1.2039	1.1754

Table 4.6.3.3: Weibull distribution: Coverage probability and average length of confidence intervals for η based on eight CPQs at specified nominal confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.9007	0.9008	0.9009	0.9003	0.9508	0.9506	0.9508	0.9507	0.9907	0.9903	0.9905	0.9909
Length	0.8655	0.8982	0.8694	0.8630	1.0479	1.1034	1.0549	1.0446	1.4390	1.5539	1.4500	1.4342
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.9051	0.9035	0.9050	0.9002	0.9515	0.9502	0.9443	0.9509	0.9859	0.9873	0.9862	0.9909
Length	0.8583	0.8917	0.8624	0.8626	1.0386	1.0932	1.0184	1.0461	1.4197	1.5395	1.4337	1.4382

4.6.3.4. Discussion of results of simulation study

- Comparison of three proposed rank-based CPQs for σ with a ML-based CPQ for σ:
    - Earlier in this chapter (see Section 4.2), three rank-based CPQs for σ were proposed, namely the CPQ for σ based on GLS estimator for σ (4.2.2) referred to, for example, in Table 4.6.3.1 as $Q_{\sigma_1}(Y,\theta)$, CPQ for σ based on residual sum of squares (4.2.11) referred to, for example, in Table 4.6.3.1 as $Q_{\sigma_2}(Y,\theta)$, and CPQ for σ based on combined CPQs for σ (4.2.20) referred to, for example, in Table 4.6.3.1 as $Q_{\sigma_3}(Y,\theta)$.
    - The simulation results of the average lengths of confidence intervals for the scale parameter σ of the Normal, Logistic, Uniform, Weibull and Pareto distributions show that, when using rank-based CPQs or FGPQs for σ, the most efficient rank-based CPQ or FGPQ for the scale parameter σ is a CPQ or FGPQ based on GLS method, namely CPQ for σ (4.2.2) or FGPQ for σ (4.2.3). This finding applies to both sample sizes $n = 10$ and $n = 25$.
    - As is shown in Table 4.6.3.1 (presented above in the text) and in Appendix B2 (Tables B2.1, B2.4, B2.7 and B2.10), the average lengths of confidence intervals for the scale parameter σ calculated using rank-based CPQ for σ_1 are shorter than those calculated using the rank-based CPQs for σ_2 and σ_3 at all confidence levels investigated in this thesis, namely 0.90, 0.95, and 0.99. This finding is confirmed when we consider the average length of confidence intervals for μ and η. Thus, the rank-based CPQ for σ derived from GLS estimators will be used for further inferences involving a two-sample problem and for a three-

parameter problem discussed later in Chapters 5 and 6, respectively. This finding also applies to the case when $n = 10$.

- Comparison of relative performance of the rank and ML-based methods:
    - The simulation results suggest that rank-based methods are very competitive with ML-based methods in terms of relative length of confidence intervals for the model parameters μ and σ and 0.975 quantiles of distributions for the cases when $n = 10$ and $n = 25$.
    - The average lengths of confidence intervals calculated using the most efficient rank-based method and those calculated using the ML-based methods, are practically equal, with a slight advantage for ML-based methods.
- Choice of using two independent copies of standard variate Z when calculating the average length of confidence intervals for μ and η:
    - We recall that in the rank and ML-based CPQs for μ (rank (4.2.24), (4.2.27), and (4.2.29); and ML (4.3.7)), the standard variate Z appears in the numerator and denominator. This leads to the question of whether we should use the standard variate Z in both the numerator and denominator (one copy of standard variate Z) when calculating the rank and ML-based CPQs for μ and η of the Normal, Logistic, Uniform, Pareto and Weibull distributions, or alternatively we should use the standard variate Z in the numerator and an independent copy of Z, namely Z^* in the denominator (two independent copies of standard variate Z)? In other words, does using two independent copies of standard variate Z lead to $\alpha/2$ and $1 - \alpha/2$ quantiles of the distribution of CPQs (4.2.24), (4.2.27), (4.2.29) and (4.3.7) that are significantly different from those calculated when one copy of standard variate Z is used?

- The simulation results show that, in general, the $\alpha/2$ and $1 - \alpha/2$ quantiles of the distribution of four rank-based CPQs for the location parameter μ ((4.2.24), (4.2.27), (4.2.29) and (4.3.7)) obtained using one copy of standard variate Z are practically the same as those obtained using two independent copies of standard variate Z. This finding is true for the cases of the Normal, Logistic, Uniform, Pareto and Weibull distributions (see Appendix A1 (Tables A1.2, A1.5, A1.8, A1.11 and A1.14, for $n = 10$); and Appendix A2 (Tables A2.2, A2.5, A2.8, A2.11 and A2.14, for $n = 25$)).
- In contrast, the simulation results suggest that, in general, the distance between the $\alpha/2$ and $1 - \alpha/2$ quantiles of the distribution of three ranked-based CPQs for η calculated using one copy of standard variate Z is shorter than the distance between the quantiles when the rank-based CPQ for η is based on two independent copies of standard variate Z (see Appendix A1 (Tables A1.3, A1.6, A1.9, A1.12 and A1.15, for $n = 10$); and Appendix A2 (A2.3, A2.6, A2.9, A2.12 and A2.15, for $n = 25$)). However, the same finding does not apply to ML-based CPQ for η because the $\alpha/2$ and $1 - \alpha/2$ quantiles of the distribution of ML-based CPQs for η remain generally the same.
- <u>Conclusion:</u>
  - In view of the discussion above, the simulation results show that confidence intervals calculated from rank and ML-based CPQs or FGPQs for μ and η using a single copy of the standard variate Z have either approximately the same average length as, or are shorter than confidence intervals calculated from CPQs or FGPQs for μ and η using two independent copies of the standard variate Z. Refer,

for example, to Tables 4.6.3.2 and 4.6.3.3 (in the text), and Appendix B2 (Tables B2.2, B2.3, B2.5, B2.6, B2.8, B2.9, B2.11 and B2.12). This finding is also consistent with a sample size $n = 10$. Thus it is recommended to use only a single copy of the standard variate Z for calculation of these confidence intervals.

- Checking for the exactness of the coverage:
  - Both the rank and ML-based methods investigated for the one-sample problem in this chapter are exact. As can be expected, the simulation results show that the simulated coverage probabilities of confidence intervals for the parameters μ and σ and for 0.975 quantiles of distributions are very close to the specified nominal probabilities (0.90, 0.95, and 0.99) for both cases when one copy of the standard variate Z and two independent copies of standard variate Z are used. This finding validates our programming.

4.7. Illustrative Examples: One-Sample Problem: Weibull Distributions

In this section, we evaluate the performance of the proposed rank-based methods of inference against ML-based methods using two real data examples provided in the text of Lawless (2003, p. 240). This data represent the levels of voltage at which failures occurred in two independent types of electrical cable (Type I and Type II) insulation after specimens were subjected to an increasing voltage stress in a laboratory experiment. For each type of electrical cable insulation, twenty (20) specimens were considered and the failure voltages (in kilovolts per millimeter) were recorded. The ranked failure voltages (in increasing order of magnitude) of the two types are given in Table 4.7.1 below.

Table 4.7.1: Levels of voltage (in kilovolts per millimeter) for Type I and Type II cable insulations

Type I	32.0	35.4	36.2	39.8	41.2	43.3	45.5	46.0	46.2	46.4
Insulation	46.5	46.8	47.3	47.3	47.6	49.2	50.4	50.9	52.4	56.3
Type II	39.4	45.3	49.2	49.4	51.3	52.0	53.2	53.2	54.9	55.5
Insulation	57.1	57.2	57.5	59.2	61.0	62.4	63.8	64.3	67.3	67.7

Stone and Lawless (1979) showed that the levels of voltage at which failures occur in each type of cable insulation follow a Weibull distribution with a common shape parameter $\sigma^* = \sigma_1^*$[4] $= \sigma_2^*$[5]. For the one-sample problem described in this chapter, we have analyzed the data for Type I and Type II insulations separately. However, for the two-sample problem presented in Chapter 5, we have analyzed the data set under the assumption that the failure voltages for Type I and Type II insulations cables follow the Weibull distributions with possibly unequal shape parameters σ_1^* and σ_2^* respectively.

The results of the analysis of the data for the two samples are presented in Table 4.7.2 below. The analysis of the two samples is based on the fact that the Weibull distribution belongs to the log-location-scale family of distributions as was presented in Section 2.3.3 above.

[4] Subscript 1 used in Sections 4.7 and 5.6 refers to Type I.
[5] Subscript 2 used in Sections 4.7 and 5.6 refers to Type II.

Table 4.7.2: Log-Weibull distributions real data examples: Fiducial generalized confidence intervals (FGCIs) for the parameters μ_1, μ_2, σ_1 and σ_2 based on the proposed rank and ML-based FGPQs at specified nominal confidence levels ($S = 10000$ simulated standard samples).

$n_1 = 20$; $\hat{\mu}_{1R} = 3.8688$; $\hat{\mu}_{1ML} = 3.8666$; $\hat{\sigma}_{1R} = 0.1116$; $\hat{\sigma}_{1ML} = 0.1066$						
	Nominal confidence level					
FGCI	0.90		0.95		0.99	
	R	ML	R	ML	R	ML
FGCI for μ_1	[3.8223, 3.9124]	[3.8224, 3.9121]	[3.8112, 3.9208]	[3.8114, 3.9203]	[3.7902, 3.9412]	[3.7903, 3.9404]
FGCI for σ_1	[0.0855, 0.1555]	[0.0849, 0.1543]	[0.0813, 0.1662]	[0.0810, 0.1648]	[0.0733, 0.1916]	[0.0730, 0.1903]
$n_2 = 20$; $\hat{\mu}_{2R} = 4.0824$; $\hat{\mu}_{2ML} = 4.0796$; $\hat{\sigma}_{2R} = 0.1144$; $\hat{\sigma}_{2ML} = 0.1094$						
FGCI for μ_2	[4.0347, 4.1270]	[4.0343, 4.1269]	[4.0233, 4.1357]	[4.0230, 4.1347]	[4.0018, 4.1566]	[4.0013, 4.1554]
FGCI for σ_2	[0.0876, 0.1594]	[0.0871, 0.1584]	[0.0834, 0.1704]	[0.0831, 0.1691]	[0.0752, 0.1965]	[0.0749, 0.1954]

- <u>Rank and ML-based point estimates of the model parameters for the two Log Weibull samples</u>
  - Sample 1 rank and ML-based point estimates of the parameters μ_1 and σ_1 are $\hat{\mu}_{1R} = 3.8688$ and $\hat{\mu}_{1ML} = 3.8666$; and $\hat{\sigma}_{1R} = 0.1116$ and $\hat{\sigma}_{1ML} = 0.1066$ respectively. The ML-based point estimates obtained by Lawless (2003, p. 241) are $\hat{\mu}_{1ML} = 3.867$ and $\hat{\sigma}_{1ML} = 0.107$.
  - Similarly, sample 2 results based on the proposed methods are $\hat{\mu}_{2R} = 4.0824$ and $\hat{\mu}_{2ML} = 4.0796$; and $\hat{\sigma}_{2R} = 0.1144$ and $\hat{\sigma}_{2ML} = 0.1094$. Sample 2 ML-based point estimates obtained by Lawless (2003, p. 241) are $\hat{\mu}_{2ML} = 4.080$ and $\hat{\sigma}_{2ML} = 0.109$. Thus, these results suggest that the rank-based estimates of the parameters are very close to the ML estimates.
- <u>Rank and ML-based FGCIs of the parameters for the two Log Weibull samples</u>
  - In terms of relative lengths of FGCIs of the parameters μ_1, μ_2, σ_1 and σ_2, the results show that rank-based methods produced FGCIs of μ_1, μ_2, σ_1 and σ_2 that are very close to the ML-based FGCIs at 0.90, 0.95 and 0.99 confidence levels. This conclusion validates the general conclusion from the results of simulation study concerning the relative performance of the rank and ML-based methods; refer to the third bullet of Section 4.6.3.4 above.

Chapter 5 - Conventional and Fiducial Generalized Inference for Location-Scale and Log-Location-Scale Distributions: Two-Sample Problem

This chapter presents a literature review, conventional pivotal quantities, where they exist, and fiducial generalized pivotal quantities for the two-sample problem in LS and LLS distributions, using both rank and ML methods. Furthermore, in this chapter we present methods for statistical inference for comparing two independent samples from LS and LLS distributions using fiducial generalized inference. Simulation studies and their results are presented for obtaining the lower and upper quantiles of CPQs for the ratio of scale parameters, and for determining the coverage probabilities and average lengths of fiducial generalized confidence intervals for the ratio of scale parameters, difference of location parameters and difference of two quantiles are also presented. We note that in this and subsequent chapters only CPQs and FGPQs based on the GLS estimator for σ (the "most efficient" CPQ or FGPQ, as was discussed in Section 4.6.3.4) are presented.

5.1. Literature Review

The well-known Behrens-Fisher problem (Behrens, 1929 and Fisher, 1935) has motivated consideration of statistical inference for comparing two independent samples in LS and LLS families of distributions. This problem in its original form involves the construction of exact confidence intervals for the difference in location parameters from two independent Normal samples, without making the assumption that the scale or variance parameters are equal for the two samples. Possible solutions to such statistical problems were introduced on the basis of the concepts of generalized p-values (GPVs), see, for example, Weerahandi (1987) and Tsui and Weerahandi (1989), and, subsequently, generalized confidence intervals (GCIs) (Weerahandi, 1993). Since then, a considerable range of research work on statistical inference involving two-sample problems in LS and LLS families of distributions using various approaches has appeared in the literature.

For example, Wu et al. (2002) used ML based methods, namely signed log-likelihood ratio and modified signed log-likelihood ratio statistics to compare by hypothesis testing the population means (location and scale parameters contrast) of two independent Lognormal distributions. In addition, Wu at al. (2002) derived a method for calculating confidence intervals for the ratio of population means (locations parameters contrast) of two independent Lognormal distributions. Wu et al. (2002) evaluated the performance of their methods through simulation studies and real-life examples by obtaining the coverage probabilities and average lengths of nominal 90% confidence intervals for the population means of two independent Lognormal distributions based on a range of smaller sample size, and compared their results with the results based on a Z-score method obtained by Zhou et al. (1997).

Hannig et al. (2006) used the method of fiducial generalized inference to derive simultaneous confidence intervals for all possible combinations of ratios of the population means of at least three independent Lognormal distributions (i.e. pairwise multiple comparisons). The performance of the FGI method was evaluated through a simulation study based on small, medium and large sample sizes by obtaining the exact and asymptotic coverage probabilities of 95% FGCIs for the pairs of ratios of the population means of independent Lognormal distributions. Similarly, Kharrati-Kapaei et al. (2013) applied the method of fiducial generalized inference to obtain coverage probabilities and average lengths of simultaneous fiducial generalized confidence intervals for the successive differences of location parameters of several independent two-parameter Exponential distributions when the scale parameters are not equal.

Nkurunziza and Chen (2011) developed a method constructing the generalized confidence interval and hypothesis test for the difference of location parameters of LS families of distributions using minimum risk equivariant estimation of the location and scale parameters of the distributions. Nkurunziza and Chen's (2011) method of inference is based on a generalization of the ML-based methods of inference for the difference in location parameters of LS families of distributions. Recently, Zakerzadeh and Jafari (2015) developed exact methods of carrying out conventional and generalized inference for various parameter contrasts of two independent Weibull distributions based on record values, which essentially may be considered as order statistics, and ML methods. Specifically, Zakerzadeh and Jafari (2015) derived a method for hypothesis testing and conventional constructing confidence interval for the ratio of shape parameters of two Weibull distributions. Furthermore, they used the generalized methods to develop a generalized pivotal quantity for testing hypotheses and constructing generalized confidence intervals for the ratio of scale parameters of two Weibull distributions when the shape parameters are equal and when they are not equal. Similarly, Zakerzadeh and Jafari (2015) applied their method of generalized inference to stress-strength reliability, which they expressed as a function of scale and shape parameters of two independent Weibull distributions. Finally, the efficiency of the developed methods for the natural logarithm of the ratio of scale parameters and stress-strength reliability was evaluated through the simulation studies and real life example in terms of obtaining coverage probabilities and average lengths of 95% generalized confidence intervals.

5.2. Rank-Based CPQ and FGPQs

As is well-known, with regard to the ratio of the scale parameters of two independent samples from LS and LSS distributions, namely $\rho = \sigma_1/\sigma_2$, there exists a CPQ (see Section 5.2.1). Furthermore, a FGPQ for ρ is presented in Section 5.2.2. Finally, FGPQ for the difference of location parameters of two independent samples from LS and LSS distributions, namely a FGPQ for $\delta = \mu_1 - \mu_2$ and for the difference of two quantiles, namely $d = \eta_{1p} - \eta_{2p}$ are presented in Sections 5.2.3 and 5.2.4 respectively.

In order to derive CPQs and FGPQs, respectively based on ranks and ML methods, for inference on two independent samples from LS and LSS distributions, we use the following notation:

Let $Y_{11}, Y_{12}, \dots, Y_{1n_1}$ be an i.i.d. random sample from the distribution of Y_{1*}, (where that distribution is from an LS or LLS family), and let

$$Z_{1i} = \frac{Y_{1i} - \mu_1}{\sigma_1}; \qquad i = 1, 2, \dots, n_1$$

be the corresponding standardized random variates associated with Y_{1*}.

Furthermore, let $Y_1 = \left[Y_{(11)}, Y_{(12)}, \dots, Y_{(1n_1)}\right]'$ be the vector of the order statistics of the sample $Y_{11}, Y_{12}, \dots, Y_{1n_1}$, and similarly let $Z_1 = \left[Z_{(11)}, Z_{(12)}, \dots, Z_{(1n_1)}\right]'$ be the vector of the order statistics of the corresponding standardized variates $Z_{11}, Z_{12}, \dots, Z_{1n_1}$.

Similarly, let $Y_{21}, Y_{22}, \dots, Y_{2n_1}$ be an i.i.d. random sample from the distribution of Y_{2*}, (where that distribution is from the same LS or LLS family as the distribution of Y_{1*}), and let

$$Z_{2i} = \frac{Y_{2i} - \mu_2}{\sigma_2}; \qquad i = 1, 2, \dots, n_2$$

be the corresponding standardized random variates associated with Y_{2*}.

Furthermore, let $Y_2 = \left[Y_{(21)}, Y_{(22)}, \dots, Y_{(2n_2)}\right]'$ be the vector of the order statistics of the sample $Y_{21}, Y_{22}, \dots, Y_{2n_2}$, and similarly let $Z_2 = \left[Z_{(21)}, Z_{(22)}, \dots, Z_{(2n_2)}\right]'$ be the vector of the order statistics of the corresponding standardized variates $Z_{21}, Z_{22}, \dots, Z_{2n_2}$.

Now we let $y_1$be an observation of Y_1 and y_2 be an observation of Y_2.

5.2.1. CPQ for the Ratio of Scale Parameters

A CPQ for σ based on the GLS estimator for σ from a single sample is given in (4.2.2), namely

$$\begin{aligned} Q_\sigma(Y, \theta) &= \hat{\sigma}(Y)/\sigma \\ &= L_2' H Z \end{aligned}$$

Using (4.2.2), let $Q_{\sigma_1}(Y_1, \theta_1)$ and $Q_{\sigma_2}(Y_2, \theta_2)$ denote the CPQs for σ_1 and σ_2, respectively. Then a CPQ for ρ is given by

$$\begin{aligned} Q_\rho &= Q_{\sigma_1}(Y_1, \theta_1)/Q_{\sigma_2}(Y_2, \theta_2) \\ &= \frac{\hat{\sigma}(Y_1)/\sigma_1}{\hat{\sigma}(Y_2)/\sigma_2} \\ &= \frac{L_2' H Z_1}{L_2' H Z_2} \end{aligned} \tag{5.2.1}$$

5.2.2. **FGPQ for the Ratio of Scale Parameters**

We refer back to (4.2.3), which is a FGPQ for σ based on GLS estimator for σ from a single sample as

$$\mathcal{R}_\sigma(y, Y, \theta) = \frac{L_2' H y}{L_2' H Z}$$

Similarly, let $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ be FGPQs for σ_1 and σ_2, respectively, from two independent samples. Then a FGPQ for ρ is given by

$$\begin{aligned}\mathcal{R}_\rho &= \mathcal{R}_{\sigma_1}(y_1, Y_1, \theta_1)/\mathcal{R}_{\sigma_2}(y_2, Y_2, \theta_2) \\ &= \frac{L_2' H y_1 / L_2' H Z_1}{L_2' H y_2 / L_2' H Z_2} \\ &= \frac{L_2' H y_1}{L_2' H Z_1} \cdot \frac{L_2' H Z_2}{L_2' H y_2} \qquad (5.2.2)\end{aligned}$$

5.2.3. **FGPQ for the Difference of Location Parameters**

We note in this section that a CPQ for $\delta = \mu_1 - \mu_2$ does not exist in general, in particular when $\sigma_1 \neq \sigma_2$ with the scale parameters unknown. Therefore, only a FGPQ for δ is presented. We recall that when only one sample is available, a FGPQ for μ is given in (4.2.26) by

$$\begin{aligned}\mathcal{R}_\mu(y, Y, \theta) &= L_1' H\,[y - \mathcal{R}_\sigma(y, Y, \theta) \cdot Z] \\ &= L_1' H \left(y - \frac{L_2' H y}{L_2' H Z} \cdot Z\right)\end{aligned}$$

Similarly to (4.2.26), let $\mathcal{R}_{\mu_1}$ and $\mathcal{R}_{\mu_2}$ be FGPQs for μ_1 and μ_2, respectively, from two independent samples. Then the FGPQ for δ is given by

$$\mathcal{R}_\delta = \mathcal{R}_{\mu_1} - \mathcal{R}_{\mu_2}$$

$$= L'_1 H\left(y_1 - \frac{L'_2 H y_1}{L'_2 H Z_1}\cdot Z_1\right) - L'_1 H\left(y_2 - \frac{L'_2 H y_2}{L'_2 H Z_2}\cdot Z_2\right) \qquad (5.2.3)$$

5.2.4. FGPQ for the Difference of Two p-Quantiles

As stated in Section 5.2.3, a CPQ for d does not exist either. We refer back to (4.2.45) and (4.2.46), which give the FGPQ for the p-quantile of the distribution when only one sample is available as

$$\mathcal{R}_{\eta_p}(y, Y, \theta) = L'_1 H y - \frac{L'_2 H y}{L'_2 H Z}\cdot\left(L'_1 H Z - z_p\right)$$

and

$$\mathcal{R}_{\eta_p}(y, Y, \theta) = L'_p H\left\{y - \frac{L'_2 H y}{L'_2 H Z}\cdot[Z - E(Z)]\right\}$$

Let $\mathcal{R}_{\eta_{1p}}$ and $\mathcal{R}_{\eta_{2p}}$ be FGPQs for η_{1p} and η_{2p}, respectively, from two independent samples. Then, using (4.2.45) and (4.2.46), the FGPQ for the difference of two p-quantiles of the distribution is given by

$$\mathcal{R}_d = \mathcal{R}_{\eta_{1p}}(y_1, Y_1, \theta_1) - \mathcal{R}_{\eta_{2p}}(y_2, Y_2, \theta_2)$$

$$= \left\{L'_1 H y_1 - \tfrac{L'_2 H y_1}{L'_2 H Z_1}\cdot\left(L'_1 H Z_1 - z_p\right)\right\} - \left\{L'_1 H y_2 - \tfrac{L'_2 H y_2}{L'_2 H Z_2}\cdot\left(L'_1 H Z_2 - z_p\right)\right\}$$

(5.2.4)

and

$$\mathcal{R}_d = L'_p H\left\{y_1 - \frac{L'_2 H y_1}{L'_2 H Z_1}\cdot[Z_1 - E(Z)]\right\} - L'_p H\left\{y_2 - \frac{L'_2 H y_2}{L'_2 H Z_2}\cdot[Z_2 - E(Z)]\right\}$$

(5.2.5)

5.2.5. FGPQ for the Log-Odds Ratio of Tail Probabilities

In this section we present a FGPQ for the log-odds ratio of tail probabilities based on the FGPQ for σ (4.2.3). We recall from (4.2.57) that the FGPQ for failure probability π at "time" y_e is given by

$$\mathcal{R}_{\pi}(y, Y, \theta) = \Phi\left[\mathcal{R}_{\varsigma}(y, Y, \theta)\right]$$

where

$$\mathcal{R}_{\varsigma}(y, Y, \theta) = \frac{y_e - L_1' H\left(y - \frac{L_2' H y}{L_2' H Z} \cdot Z\right)}{\frac{L_2' H y}{L_2' H Z}}$$

Let $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$ be FGPQs for failure probabilities π_1 and π_2 at "time" y_e, respectively. Then, using (4.2.56) and (4.2.57), it is clear that $\mathcal{R}_{\varsigma_1}$ and $\mathcal{R}_{\varsigma_2}$ are FGPQs for ς_1 and ς_2 at "time" y_e, respectively. Furthermore, let $q_1 = 1 - \mathcal{R}_{\pi_1}$ and $q_2 = 1 - \mathcal{R}_{\pi_2}$. Using (4.2.57), a FGPQ for the log-odds ratio of tail probabilities based on the FGPQ for σ (4.2.3) is given by

$$\begin{aligned}\mathcal{R}_{\tau} &= \log\left\{\frac{\mathcal{R}_{\pi_1}(y_1, Y_1, \theta_1)/q_1}{\mathcal{R}_{\pi_2}(y_2, Y_2, \theta_2)/q_2}\right\} \\ &= \log\left[\mathcal{R}_{\pi_1}(y_1, Y_1, \theta_1)\right] + \log(q_1) - \log\left[\mathcal{R}_{\pi_2}(y_2, Y_2, \theta_2)\right] - \log(q_2)\end{aligned} \tag{5.2.6}$$

where in terms of (4.2.57)

$$\mathcal{R}_{\pi_1}(y_1, Y_1, \theta_1) = \Phi\left[\mathcal{R}_{\varsigma_1}(y_1, Y_1, \theta_1)\right] \tag{5.2.7}$$

and

$$\mathcal{R}_{\pi_2}(y_2, Y_2, \theta_2) = \Phi\left[\mathcal{R}_{\varsigma_2}(y_2, Y_2, \theta_2)\right] \tag{5.2.8}$$

whereas in terms of (4.2.57)

$$\mathcal{R}_{\varsigma_1}(y_1, Y_1, \theta_1) = \frac{y_e - L_1' H \left(y_1 - \frac{L_2' H y_1}{L_2' H Z_1} \cdot Z_1 \right)}{\frac{L_2' H y_1}{L_2' H Z_1}} \tag{5.2.9}$$

and

$$\mathcal{R}_{\varsigma_2}(y_2, Y_2, \theta_2) = \frac{y_e - L_1' H \left(y_2 - \frac{L_2' H y_2}{L_2' H Z_2} \cdot Z_2 \right)}{\frac{L_2' H y_2}{L_2' H Z_2}} \tag{5.2.10}$$

5.3. Maximum Likelihood-Based CPQ and FGPQs

In this section, we present the CPQ and FGPQ for ρ and FGPQs for δ and d based on maximum likelihood method.

5.3.1. CPQ for the Ratio of Scale Parameters

A CPQ for σ based on ML estimator for σ using a single sample is given in (4.3.5) as

$$\begin{aligned} \mathcal{Q}_\sigma(Y, \theta)_{ML} &= \frac{\hat{\sigma}(Y)_{ML}}{\sigma} \\ &= \frac{\sigma \cdot \hat{\sigma}(Z)_{ML}}{\sigma} \\ &= \hat{\sigma}(Z)_{ML} \end{aligned}$$

Similarly to (4.3.5), let $\mathcal{Q}_{\sigma_1}(Y_1, \theta_1)_{ML}$ and $\mathcal{Q}_{\sigma_2}(Y_2, \theta_2)_{ML}$ denote CPQs for σ_1 and σ_2, respectively. Then a CPQ for ρ is given by

$$\begin{aligned} \mathcal{Q}_\rho &= \mathcal{Q}_{\sigma_1}(Y_1, \theta_1)_{ML} / \mathcal{Q}_{\sigma_2}(Y_2, \theta_2)_{ML} \\ &= \frac{\hat{\sigma}(Y_1)_{ML} / \sigma_1}{\hat{\sigma}(Y_2)_{ML} / \sigma_2} \end{aligned}$$

$$= \frac{\hat{\sigma}(Z_1)_{ML}}{\hat{\sigma}(Z_2)_{ML}} \tag{5.3.1}$$

5.3.2. FGPQ for the Ratio of Scale Parameters

A FGPQ for σ based on ML estimator for σ using a single sample is given in (4.3.6) as

$$\mathcal{R}_\sigma(y, Y, \theta) = \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Y)_{ML}/\sigma}$$

$$= \frac{\hat{\sigma}(y)_{ML}}{\sigma \cdot \hat{\sigma}(Z)_{ML}/\sigma}$$

$$= \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Z)_{ML}}$$

Similarly to (4.3.6), let $\mathcal{R}_{\sigma_1}(y_1, Y_1, \theta_1)$ and $\mathcal{R}_{\sigma_2}(y_2, Y_2, \theta_2)$ denote the FGPQs for σ_1 and σ_2, respectively. Then, a FGPQ for ρ based on ML is given by

$$\mathcal{R}_\rho = \mathcal{R}_{\sigma_1}(y_1, Y_1, \theta_1)/\mathcal{R}_{\sigma_2}(y_2, Y_2, \theta_2)$$

$$= \frac{\hat{\sigma}(y_1)_{ML}}{\hat{\sigma}(Y_1)_{ML}/\sigma_1} \Big/ \frac{\hat{\sigma}(y_2)_{ML}}{\hat{\sigma}(Y_2)_{ML}/\sigma_2}$$

$$= \frac{\hat{\sigma}(y_1)_{ML}}{\hat{\sigma}(Z_1)_{ML}} \cdot \frac{\hat{\sigma}(Z_2)_{ML}}{\hat{\sigma}(y_2)_{ML}} \tag{5.3.2}$$

5.3.3. FGPQ for the Difference of Location Parameters

A FGPQ for μ based on ML estimator for μ using a single sample is given in (4.3.9) as

$$\mathcal{R}_\mu(y, Y, \theta) = \hat{\mu}(y)_{ML} - \mathcal{R}_\sigma(y, Y, \theta) \cdot \hat{\mu}(Z)_{ML}$$

$$= \hat{\mu}(y)_{ML} - \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Z)_{ML}} \cdot \hat{\mu}(Z)_{ML}$$

Similarly to (4.3.9), let $\mathcal{R}_{\mu_1}(y_1, Y_1, \theta_1)$ and $\mathcal{R}_{\mu_2}(y_2, Y_2, \theta_2)$ denote the FGPQs for μ_1 and μ_2, respectively. Then, a FGPQ for δ based on ML is given by

$$\mathcal{R}_\delta = \mathcal{R}_{\mu_1}(y_1, Y_1, \theta_1) - \mathcal{R}_{\mu_2}(y_2, Y_2, \theta_2)$$

$$= \left[\hat{\mu}(y_1)_{ML} - \frac{\hat{\sigma}(y_1)_{ML}}{\hat{\sigma}(Z_1)_{ML}} \cdot \hat{\mu}(Z_1)_{ML}\right] - \left[\hat{\mu}(y_2)_{ML} - \frac{\hat{\sigma}(y_2)_{ML}}{\hat{\sigma}(Z_2)_{ML}} \cdot \hat{\mu}(Z_2)_{ML}\right]$$

(5.3.3)

5.3.4. FGPQ for Difference of Two p-Quantiles

A FGPQ for ω_p based on ML estimator for ω_p using a single sample is given in (4.3.11) as

$$\mathcal{R}_{\omega_p}(y, Y, \theta) = \hat{\mu}(y)_{ML} - \frac{\hat{\sigma}(y)_{ML}}{\hat{\sigma}(Z)_{ML}} \cdot \left[\hat{\mu}(Z)_{ML} - z_p\right]$$

Based on (4.3.11), let $\mathcal{R}_{\omega 1_p}(y_1, Y_1, \theta_1)$ and $\mathcal{R}_{\omega 2_p}(y_2, Y_2, \theta_2)$ denote FGPQs for $\omega 1_p$ and $\omega 2_p$ respectively. Then, a FGPQ for d based on maximum likelihood method is given by

$$\mathcal{R}_d = \mathcal{R}_{\omega 1_p}(y_1, Y_1, \theta_1) - \mathcal{R}_{\omega 2_p}(y_2, Y_2, \theta_2)$$

$$= \left\{\hat{\mu}(y_1)_{ML} - \frac{\hat{\sigma}(y_1)_{ML}}{\hat{\sigma}(Z_1)_{ML}} \cdot \left[\hat{\mu}(Z_1)_{ML} - z_p\right]\right\} -$$

$$\left\{\hat{\mu}(y_2)_{ML} - \frac{\hat{\sigma}(y_2)_{ML}}{\hat{\sigma}(Z_2)_{ML}} \cdot \left[\hat{\mu}(Z_2)_{ML} - z_p\right]\right\}$$

(5.3.4)

5.4. Conventional and Fiducial Generalized Confidence Intervals for Differences, Ratios and Log-Odds Ratios

In this section we describe the general principle of how to obtain conventional and fiducial generalized confidence intervals for differences, ratios and log-odds ratios presented earlier in Sections 5.2 and 5.3.

When CPQs for the ratio of scale parameters, namely $\mathcal{R}_\rho$ exist as is the case in (5.2.1) and (5.3.1), the conventional confidence intervals can be obtained by inverting the respective CPQs.

However, when only FGPQs are available, as is the case in (5.2.3), (5.2.4), (5.2.5), (5.2.6), (5.3.3) and (5.3.4), fiducial generalized confidence intervals can be obtained by calculating the confidence intervals through the simulation of FGPQs $\mathcal{R}_\rho$, $\mathcal{R}_\tau$, and $\mathcal{R}_\delta$ and $\mathcal{R}_d$.

Following is a sketch of the algorithm for calculating fiducial generalized confidence intervals for δ and d based on rank- and ML-based FGPQs:

Algorithm 4: Calculation of fiducial generalized confidence intervals for the difference of location parameters and difference of two quantiles of the distribution

1. Draw two independent samples of observations of sizes n_1 and n_2 from a specified standard LS or LLS distribution.
2. Sort the values of the samples drawn in Step 1 in ascending order of magnitude, thus obtaining the order statistics of the samples.
3. For each ordered sample, calculate the GLS estimates of the location and scale parameters.

4. For each unordered sample, calculate the unbiased ML estimates of the location and scale parameters.
5. Using the GLS and unbiased ML estimates of the location and scale parameters calculated in Steps 3 and 4, respectively, calculate the values of rank and ML-based FGPQs for $\delta = \mu_1 - \mu_2$ and $d = \eta_1 - \eta_2$.
6. Save the values of FGPQs calculated in Step 5.
7. Repeat Steps 1 through 6 J number of times. The $(1 - \alpha)$ FGCIs for $\delta = \mu_1 - \mu_2$ and $d = \eta_1 - \eta_2$ are given as the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the respective distributions of simulated FGPQs.

5.5. Simulation Studies: Two-Sample Problem

In Sections 5.5.1 and 5.5.2 two simulation studies for the two-sample problem are described. Firstly, the simulations for calculating the lower and upper quantiles of CPQs for the ratio of two scale parameters are presented in Section 5.5.1. Secondly, a simulation study for obtaining the coverage probabilities and average lengths of fiducial generalized confidence intervals for the ratio of scale parameters, difference of location parameters and difference of two quantiles of the distribution is presented in Section 5.5.2.

5.5.1. Calculation of Lower and Upper Quantiles of CPQs for ρ through Simulation

Since the CPQs for the ratio of the scale parameters based on rank and ML methods are available, we present in this section the simulations for calculating the lower and upper quantiles of such CPQs for the Normal, Logistic, Uniform, Pareto and Weibull distributions. The lower and upper quantiles of CPQs for the ρ (ratio of scale parameters) when $n_1 = n_2 = 10$ are presented in Appendix C1 (Tables C1.1 through C1.5). Similarly, the lower and upper quantiles of ρ when

$n_1 = n_2 = 25$ and are presented in Appendix C2 (Tables C2.1 through C2.5). These quantiles are used to calculate the coverage probabilities and average lengths of conventional confidence intervals for the ratio of scale parameters discussed in Section 5.5.2.3 and presented in Appendix D1 (Tables D1.1 through D1.5 (for $n_1 = n_2 = 10$)) and Appendix D2 (Tables D2.1 through D2.5 (for $n_1 = n_2 = 25$)).

5.5.1.1. Objectives of simulation study

The objective of these simulations is to calculate the lower and upper quantiles of rank and ML-based CPQs for the ratio of scale parameters for the Normal, Logistic, Uniform, Pareto and Weibull distributions.

5.5.1.2. Design of simulation study

Five programs were written to calculate the lower and upper quantiles of a CPQ for the ratio of scale parameters based on order statistics of the samples and a CPQ for the ratio of scale parameters based on ML methods, at the level of significance $\alpha = 0.1, 0.05$ and 0.01. The lower and upper quantiles of these two CPQs for $\rho = \sigma_1/\sigma_2$ were calculated using $S = 1000000$ simulated samples of sizes $n_1 = n_2 = 25$ drawn from the Normal, Logistic, Uniform, Pareto and Weibull distributions. The calculation of lower and upper quantiles of CPQs for ρ was repeated for sample sizes $n_1 = n_2 = 10$.

5.5.1.3. Results of simulation study

The results of the lower and upper quantiles of two CPQs for ρ at levels of significance $\alpha = 0.1, 0.05$ and 0.01 for the Normal, Logistic, Uniform, Pareto and Weibull distributions are presented in Appendix C1 (Tables C1.1 through C1.5 for $n_1 = n_2 = 10$) and Appendix C2 (Tables C2.1 through C2.5 for $n_1 = n_2 = 25$).

5.5.2. Simulation Studies: Coverage Probabilities and Average Lengths of Conventional Confidence Intervals for ρ and Fiducial Generalized Confidence Intervals for δ and d

We present in this section two simulations. Firstly, a simulation for calculating the coverage probabilities and average lengths of conventional confidence intervals for the ratio of scale parameters. Secondly, a simulation for calculating the fiducial generalized confidence intervals for the difference of location parameters, and difference of two quantiles of the distribution for the Normal, Logistic, Uniform, Pareto and Weibull distributions.

5.5.2.1. Objectives of simulation studies

The objectives of these simulations, respectively, are:

- To calculate the coverage probabilities and average lengths of conventional confidence intervals for the ratio of scale parameters based on rank and ML CPQs.
- To calculate the coverage probabilities and average lengths of fiducial generalized confidence intervals for the difference of location parameters and difference of two quantiles of the distribution based on rank and ML FGPQs for the Normal, Logistic, Uniform, Pareto and Weibull distributions.

5.5.2.2. Design of simulation studies

For the first simulation, five programs were written to calculate the coverage probabilities and average lengths of conventional confidence intervals for the ratio of scale parameters of the Normal, Logistic, Uniform, Pareto and Weibull distributions based on the rank and ML CPQs. Similarly, for the second simulation, five programs were written to calculate the coverage probabilities and average lengths of fiducial generalized confidence intervals for the difference of location parameters and difference of two quantiles of the distribution for the Normal, Logistic, Uniform, Pareto and Weibull distributions based on rank and ML FGPQs. For both cases, the coverage probabilities and average lengths of confidence intervals were calculated using $S = 100000$ simulated samples of observations of sample sizes $n_1 = n_2 = 10$, and $n_1 = n_2 = 25$, nominal confidence levels $1 - \alpha = 0.90, 0.95$ and 0.99 and for the cases of equal and unequal scale parameters of two independent samples from LS and LLS distributions.

Following is a sketch of the algorithm for calculating the coverage probabilities and average lengths of fiducial generalized confidence intervals for δ and d based on rank- and ML-based FGPQs:

Algorithm 5: Calculation of coverage probabilities and average lengths of confidence intervals for the ratio of scale parameters, difference of location parameter, and difference of quantiles:

1. Draw two independent samples of observations of sizes n_1 and n_2 from a specified LS or LLS distribution.
2. Sort the values of the samples drawn in Step 1 in ascending order of magnitude, thus obtaining the order statistics of the samples.
3. For each ordered sample, calculate the GLS estimates of the location and scale parameters.
4. For each (unordered) sample, calculate the unbiased ML estimates of the location and scale parameters.
5. Using the samples in Steps 1 and 2, and GLS and ML estimates calculated in Steps 3 and 4, calculate the confidence intervals for $\delta = \mu_1 - \mu_2$ and $d = \eta_1 - \eta_2$ as was described above in Algorithm 4.
6. Repeat Steps 1 through 5 J number of times.
7. Calculate the average length and observed coverage of the J confidence intervals calculated in Steps 1 through 6.

5.5.2.3. Results of simulation studies

The results of the coverage probabilities and average lengths of conventional confidence intervals for $\rho = \sigma_1/\sigma_2$ of the Normal, Logistic, Uniform, Pareto and Weibull distributions based on two CPQs (rank and ML-based) are presented in Appendix D1 (Tables D1.1 through D1.5 for $n_1 = n_2 = 10$) and Appendix D2 (Tables D2.1 through D2.5 for $n_1 = n_2 = 25$). Furthermore, the results of the coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ and $d = \eta_1 - \eta_2$ for the Normal, Logistic, Uniform, Pareto and Weibull distributions based on two FGPQs (rank and ML-based) are presented in Appendix E1 (Tables E1.1 through E1.10 for $n_1 = n_2 = 10$) and Appendix E2 (Tables E2.1 through E2.10 for $n_1 = n_2 = 25$).

5.5.2.4. Discussion of result of simulation studies:

- The results of both simulations show that the average lengths of confidence intervals for the ratio of scale parameters, difference of location parameters, and difference of two 0.975 quantiles of the distribution calculated using rank-based methods are practically the same as those calculated using ML-based methods for the case of equal scale parameters of two independent samples from LS and LLS distributions.
- This finding is confirmed when the case of unequal scale parameters of two independent samples is considered: in this case, too, the average lengths of confidence intervals for the ratio of scale parameters, difference of location parameters, and difference of two 0.975 quantiles of the distribution calculated using rank-based methods are practically the same as those calculated using ML-based methods.
- As was case with simulations of one sample problem, the observed coverage at all three confidence levels is practically equal to the

respective nominal coverage. This finding can only serve as a validation of the simulation program for the ratio of scale parameters, since this confidence interval method is of course known to be exact. However, the statement that the confidence interval methods for the difference of location parameters, and for the difference of two 0.975 quantiles are exact (or at least nearly exact), is not trivial.

5.6. Illustrative Example: Two-Sample Problem: Weibull Distributions

In this section, we apply the fiducial rank and ML-based methods of inference using the data set of Type I and Type II cable-insulation failures (in kilovolts per millimeter) of two Weibull distributions presented in Lawless (2003, p. 240). For the data set, see Table 4.7.1 and the background to the data set is described in Section 4.7 above.

Under the assumption that the two samples were drawn from Weibull distributions with possibly unequal shape parameters (or equivalently the two samples were drawn from Extreme Value distributions with unequal scale parameters), we estimate the ratio of two scale parameters ($\rho = \sigma_1/\sigma_2$), difference of two location parameters ($\delta = \mu_1 - \mu_2$) and difference of two 0.975 quantiles of the distribution ($d = \eta_1 - \eta_2$).

Furthermore, we calculated, using the two sample data, the 90%, 95% and 99% FGCIs for ρ, δ and d based on rank and ML-based FGPQs. The results of the analysis for the two-sample are presented in Table 5.6.1 below:

Table 5.6.1: Log-Weibull distribution real data – failure voltages (in kilovolts per millimetre) example: Fiducial generalized confidence intervals (FGCIs) for the parameters $\rho = \sigma_1/\sigma_2$, $\delta = \mu_1 - \mu_2$ and $d = \eta_1 - \eta_2$ based on rank and ML-based FGPQs at specified nominal confidence levels ($S = 10000$ pairs of simulated standard samples)

$n_1 = n_2 = 20$; $\hat{\rho}_R = 0.9755$; $\hat{\rho}_{ML} = 0.9744$; $\hat{\delta}_R = -0.2136$; $\hat{\delta}_{ML} = -0.2130$; $\hat{\eta}_1 = 4.0145$; $\hat{\eta}_2 = 4.2318$; $\hat{d} = -0.2173$						
	Nominal confidence level					
FGCI	0.90		0.95		0.99	
	R	ML	R	ML	R	ML
FGCI for ρ	[0.6358, 1.4837]	[0.6384, 1.4814]	[0.5862, 1.6209]	[0.5842, 1.6183]	[0.4940, 1.9074]	[0.4945, 1.8923]
FGCI for δ	[-.2765, -.1484]	[-.2758, -.1485]	[-.2907, -.1349]	[-.2901, -.1353]	[-.3164, -.1082]	[-.3155, -.1087]
FGCI for d	[-.2964, -.1379]	[-.2948, -.1388]	[-.3138, -.1222]	[-.3126, -.1221]	[-.3539, -.0809]	[-.3520, -.0804]

- <u>Rank and ML-based point estimates of ρ, δ and d for two Log Weibull samples</u>
  - The rank and ML-based estimates of $\mu_1, \mu_2, \sigma_1,$ and σ_2 were calculated and presented in Section 4.7. See Table 4.7.2. For the two-sample problem the rank-based point estimates of the parameters ρ, and δ were calculated, respectively, as $\hat{\rho}_R = 0.1116/0.1144 = 0.9755$ and $\hat{\delta}_R = 3.8688 - 4.0824 = -0.213$. Similarly, the ML-based point estimates of the parameters ρ and δ were be calculated as $\hat{\rho}_{ML} = 0.1066/0.1094 = 0.9744$ and $\hat{\delta}_{ML} = 3.8666 - 4.0796 = -0.2130$ respectively. Lastly, the point estimate of the difference of two 0.975 quantiles of the distribution was calculated as $\hat{d} = 4.0145 - 4.2318 = -0.2173$.
  - ML-based point estimates for ρ and δ presented in Lawless (2003, p. 241) are $\hat{\rho}_{ML} = 0.107/0.109 = 0.982$ and $\hat{\delta}_{ML} = 3.867 - 4.080 = -0.213$ respectively. Clearly, the results in Lawless (2003, p. 241) are very similar to those based on the proposed rank and ML-based methods.
- <u>Rank and ML-based FGCIs estimates of ρ, δ and d for two Log Weibull samples</u>
  - In terms of relative lengths of FGCIs of the parameters ρ, δ and d, the results show that rank-based methods produced FGCIs of ρ, δ and d that are practically the same as those based on ML-based methods at 0.90, 0.95 and 0.99 confidence levels, with a slight advantage for ML-based methods. This conclusion agrees with the general conclusion of the results of simulation studies discussed in Section 5.5.2.4, the first and second bullets above.
  - The 90% CIs for δ based on approximate likelihood ratio procedure, and approximate standard normal pivotal quantity for

δ presented in Lawless (2003, p. 241) are (-0.272, -0.157) and (-0.272, -0.154) respectively. Clearly, these 90% CIs for δ are similar to the 90% CIs for δ based on the proposed rank and ML methods presented in Table 5.6.1 above.

Chapter 6 - Location-Scale-Shape Family of Distributions

6.1. Definition of Location-Scale-Shape Family

This chapter provides the background on Location-scale-shape (LSS) family of distributions. It is concerned with estimation and inference problems involving the three unknown parameters of the LSS distributions, namely the location, scale and shape parameters; the chapter is also concerned with the estimation of and inference for quantiles of LSS distributions. An LSS family of distributions is defined as follows:

A scalar continuous random variable Y_* belongs to an LLS family of distributions if its cumulative distribution function (cdf) can be written in the form

$$F_{Y_*}(y_*;\theta^*) = \Phi\left\{\left[1+\xi\left(\frac{y_*-\mu_*}{\sigma_*}\right)\right]^{-1/\xi}\right\} \qquad (6.1.1)$$

where the support of the distribution may be on the whole real line. The vector parameter of (6.1.1) $\theta^*$is given by

$$\theta^* = (\mu_*, \sigma_*, \xi)'$$

where μ_*, σ_* and ξ are, respectively, the location, scale and shape parameters of the distribution. As is the case for LS families of distributions presented in Chapter 2 above, the function $\Phi(\cdot)$ in equation (6.1.1) is continuous for all real values of y_* and it is a monotone non-decreasing function of y_*, given θ. Furthermore, the values of the function $\Phi(\cdot)$ given in (6.1.1) lie in the interval $[0,1]$.

As is shown below, when $\xi \neq 0$ it may be convenient to write

$$1 + \xi\left(\frac{y_* - \mu_*}{\sigma_*}\right) = \frac{y_* - (\mu_* - \sigma_*/\xi)}{\sigma_*/\xi}$$

which suggests the re-parametrization of the distribution as

$$\mu = \mu_* - \sigma_*/\xi; \qquad \sigma = \sigma_*/\xi \tag{6.1.2}$$

Under re-parametrization (6.1.2), Y_* belongs to an LSS family of distributions if its cdf has the form

$$F_{Y_*}(y_*;\theta) = \Phi\left[\left(\frac{y_* - \mu}{\sigma}\right)^{-1/\xi}\right] \tag{6.1.3}$$

In this case, the vector parameter is given by

$$\theta = (\mu, \sigma, \xi)'$$

where μ, σ and ξ are, respectively, the location, scale and shape parameters of the distribution. Thus, the scalar continuous random variable Y_* is said to have a LSS(θ) distribution.

Using (6.1.3) we define the standard random variate Z_* associated with Y_* as

$$Z_* = \left(\frac{Y_* - \mu}{\sigma}\right)^{-1/\xi} \tag{6.1.4}$$

so that the cdf of Z_* is given by

$$F_{Z_*}(z_*;\theta) = \Phi(\, z_*)$$

where

$$\theta = (0, 1, 1)'$$

such that $F_{Z_*}(z_*;\theta) = \Phi(z_*)$ does not depend on any unknown parameters.

<u>Limiting case $\xi = 0$:</u>

Raising both sides of equation (6.1.4) to the power $-\xi$, and solving for Y_*, we obtain an LS model

$$Y_* = \mu + \sigma \cdot {Z_*}^{-\xi} \tag{6.1.5}$$

We note that the distribution of $Z_*^{-\xi}$ is degenerate as $\xi \to 0$. Thus, using re-parametrization (6.1.2), we write equation (6.1.5) in terms of the original (μ_*, σ_*) parametrization as follows

$$\begin{aligned} Y_* &= \mu_* - \frac{\sigma_*}{\xi} + \frac{\sigma_*}{\xi} \cdot Z_*^{-\xi} \\ &= \mu_* + \sigma_* \cdot \left(\frac{Z_*^{\xi} - 1}{\xi} \right) \\ &= \mu_* + \sigma_* \cdot \left(\frac{(Z_*^{-1})^{\xi} - 1}{\xi} \right) \end{aligned} \tag{6.1.6}$$

Using the last inequality in equation (6.1.6) we can note that term $\left[\left(Z_*^{\xi} - 1\right)/\xi\right]$ is the Box-Cox power transformation of the variate Z_*^{ξ}. Thus $\lim_{\xi \to 0} \left[\left(Z_*^{-\xi} - 1\right)/\xi\right] = -\log(Z_*) = \log(Z_*^{-1})$.

<u>Theorem 1</u>

If the distribution of Y_* is an LSS family with parameter vector $\theta^* = (\mu_*, \sigma_*, \xi\ )'$, the limiting distribution of Y_*, as $\xi \to 0$, exists and has the following properties:

1. The limiting distribution of Y_* is an LS family with location parameter μ_* and scale parameter σ_*.
2. The standard variate of the limiting LS family is given by $W_* = \log(Z_*^{-1}) = -\log(Z_*)$, where Z_* is the standard variate of the LSS family.

As is shown in Sections 6.1.1 through 6.1.3, examples of distributions that belong to the LSS family include the Generalized Extreme Value (GEV), Generalized Pareto (GP) and three-parameter Weibull distributions.

Parametrizations: In the following we will repeatedly refer to the two parametrizations of LSS families outlined above. The parametrization $\theta = (\mu, \sigma, \xi)'$, see equations (6.1.2)/(6.1.5) we call the θ parametrization, while the parametrization $\theta^* = (\mu_*, \sigma_*, \xi)'$, see equation (6.1.1)/(6.1.6) we call the θ^* parametrization.

6.1.1 Generalized Extreme Value Distribution

The Generalized Extreme Value distribution has been used widely in extreme value theory as the limiting distribution of block maxima. Of special interest is usually inference about the shape parameter ξ, which is known as a tail index in the literature, and inference about quantiles of the distribution. We define the GEV distribution as follows:

Let a scalar continuous random variable Y_* follow a GEV distribution parameterised by θ^*, denoted by $Y_* \sim GEV(\mu_*, \sigma_*, \xi)$. Then, the cdf and pdf of $GEV(\theta^*)$ are, respectively, given by

$$F_{Y_*}(y_*; \theta^*) = exp\left\{-\left[1 + \xi\left(\frac{y_* - \mu_*}{\sigma_*}\right)\right]^{-1/\xi}\right\} \tag{6.1.7}$$

and

$$f_{Y_*}(y_*; \theta^*) = \frac{1}{\sigma_*}\left[1 + \xi\left(\frac{y_* - \mu_*}{\sigma_*}\right)\right]^{(-1/\xi)-1} exp\left\{-\left[1 + \xi\left(\frac{y_* - \mu_*}{\sigma_*}\right)\right]^{-1/\xi}\right\}$$

The location parameter μ_* and shape parameter ξ of the GEV distribution lie in the interval $(-\infty, \infty)$. Furthermore, the scale parameter σ_* is positive.

When $\xi \neq 0$, under re-parametrization (6.1.2) the random variable Y_* follows a Generalized Extreme distribution $GEV(\theta)$ with the cdf and pdf given by

$$F_{Y_*}(y_*; \theta) = exp\left[-\left(\frac{y_* - \mu}{\sigma}\right)^{-1/\xi}\right] \tag{6.1.8}$$

and

$$f_{Y_*}(y_*;\theta) = \frac{1}{\sigma \cdot \xi}\left[\frac{y_* - \mu}{\sigma}\right]^{(-1/\xi)-1} exp\left\{-\left[\left(\frac{y_* - \mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

respectively. Furthermore, for the $GEV(\theta)$ distribution, the cdf of the standard variate Z_* in (6.1.4), and the form of $\Phi(\cdot)$ in (6.1.8), are given by

$$F_{Z_*}(z_*;\theta) = \Phi(z_*) = exp(-z_*)$$

Since the random variate $F_{Z_*}(Z_*)$ has a Uniform distribution, we can write

$$exp(-Z_*) = U$$

where U is a random variable with a Uniform distribution in the interval [0,1]. Alternatively solving for Z_* we obtain

$$Z_* = -\log(U)$$

Thus the standard variate Z_* has a standard Exponential distribution.

Using equation (6.1.8) we can write the general random variate Y_* in terms of the standard variate Z_* as

$$Y_* = \mu + \sigma \cdot Z_*^{-\xi}$$

where $Z_* = -\log(U)$.

Case 1: $\xi > 0$

Initially we assume $\xi > 0$, in which case the support of the distribution is the interval $[\mu_* - \sigma_*/\xi, \infty)$. Thus, the support of the distribution is bounded below when $\xi > 0$. Similarly, the support of the re-parametrized $GEV(\theta)$ distribution, in terms of μ, is the interval $[\mu, \infty]$, when $\xi > 0$. Thus, any estimate of μ should be less than the smallest order statistic $y_{(1)}$ when $\xi > 0$.

Case 2: $\xi = 0$

For $\xi = 0$, it follows from Theorem 1 that the distribution of Y_* is an LS family with standard variate

$$W_* = \frac{Y_* - \mu_*}{\sigma_*} = -\log(Z_*) = -\log[-\log(U)] = \log\left(\frac{1}{-\log(U)}\right) \tag{6.1.9}$$

Thus the distribution of W_* follows the distribution of the logarithm of the inverse of standards Exponential variate with cdf

$$F_{W_*}(w_*) = exp[-exp(-w_*)]$$

where the vector parameter $\theta^* = (\mu_*, \sigma_*)'$.

Furthermore, the cdf of Y_*, for $\xi = 0$, is given by

$$F_{Y_*}(y_*;\ \theta^*) = exp\left[-exp\left(-\frac{y_* - \mu_*}{\sigma_*}\right)\right] \tag{6.1.10}$$

which is the cdf of the Gumbel distribution.

Case 3: $\xi < 0$

When $\xi < 0$, the support of the $GEV(\theta^*)$ distribution is bounded above, and is in the interval $(-\infty,\ \mu_* - \sigma_*/\xi]$. Furthermore, in terms of re-parametrization (6.1.2), the support of the GEV distribution is in the interval $(-\infty,\ \mu]$, when $\xi < 0$. Thus, any estimate of μ should be greater than the largest order statistic $y_{(n)}$ when $\xi < 0$.

6.1.2. Generalized Pareto Distribution

The Generalized Pareto (GP) distribution is a family of various distributions such as the Pareto, Exponential, Uniform, and sub-models of Beta distributions (Villaseňor-Alva and González-Estrada, 2009). This important family of distributions is widely used in many applications of extreme value analysis. See, for example, the text of Reiss and Thomas (2007, Chapter 5). For example, the GP distribution can be used to model excesses over a threshold. The GP distribution was used to select a suitable threshold and treatment of time-series dependence (Smith, 2001). The GP distribution can be defined as follows:

Let a scalar continuous random variable Y_* follow a GP distribution parameterised by θ^*, and denoted by $Y_* \sim GP(\theta^*)$ distribution. Here the vector parameter $\theta^* = (\mu_*, \sigma_*, \xi)'$. Then, the cdf and pdf of the $GP(\theta^*)$ distribution are, respectively, given by

$$F_{Y_*}(y_*; \theta^*) = 1 - \left[1 + \frac{\xi(y_* - \mu_*)}{\sigma_*}\right]^{-1/\xi} \tag{6.1.11}$$

and

$$f_{Y_*}(y_*; \theta^*) = \frac{1}{\sigma_*}\left[1 + \frac{\xi(y_* - \mu_*)}{\sigma_*}\right]^{-(1/\xi+1)}$$

The location parameter μ_* and shape parameter ξ of the $GP(\theta^*)$ distribution lie in the interval $(-\infty, \infty)$. Furthermore, the scale parameter σ_* is always positive.

As we have done in Section 6.1.1 above, when $\xi \neq 0$ we can employ re-parametrization (6.1.2) so that the random variable Y_* follows a Generalized Pareto $GP(\theta)$ distribution with the cdf and pdf, respectively, given by

$$F_{Y_*}(y_*; \theta) = 1 - \left(\frac{y_* - \mu}{\sigma}\right)^{-1/\xi} \tag{6.1.12}$$

and

$$f_{Y_*}(y_*; \theta) = \frac{1}{\sigma \cdot \xi}\left(\frac{y_* - \mu}{\sigma}\right)^{-(1/\xi+1)}$$

For the $GP(\theta)$ distribution, the cdf of the standard variate Z_* in (6.1.4), and the form of $\Phi(\cdot)$ in (6.1.8), are given by

$$F_{Z_*}(z_*; \theta) = \Phi(z_*) = 1 - z_* \tag{6.1.13}$$

Since the standard random variable $F_{Z_*}(Z_*)$ has a Uniform distribution, we can write

$$1 - Z_* = U$$

where U is a random variable with a Uniform distribution on the interval [0,1]. Alternatively, the standard random variate Z_* can be expressed in terms of a standard Uniform variate U as

$$Z_* = \Phi^{-1}(z_*) = F^{-1}(U) = 1 - U$$

where $F^{-1}(U) = 1 - U$ is the cdf of the standard Generalized Pareto $GP(0,1,1)$ distribution. Thus, Z_* follows the distribution of the standard Uniform variate.

Using equation (6.1.12) we can write the general random variate Y_* in terms of the standard variate Z_* as

$$Y_* = \mu + \sigma \cdot Z_*^{-\xi}$$

where $Z_* = (1 - U)$.

Case 1: $\xi > 0$

When $\xi > 0$ the support of the distribution is the interval $[\mu_*, \infty)$. Thus, any estimate of μ_* should be less than or equal to the smallest order statistic $y_{(1)}$ when $\xi > 0$. In terms of re-parametrization (6.1.2), the support of the distribution is the interval $[\mu + \sigma, \infty)$, when $\xi > 0$.

Case 2: $\xi = 0$

For $\xi = 0$, it follows from Theorem 1 that the distribution of Y_* is an LS family with standard variate

$$W_* = \frac{Y_* - \mu_*}{\sigma_*} = -\log(Z_*) = -\log(U) \qquad (6.1.14)$$

Thus, the distribution of W_* is the standard Exponential distribution, with cdf

$$F_{W_*}(w_*) = 1 - exp(-w_*)$$

Furthermore, the cdf of Y_*, for $\xi = 0$, is given by

$$F_{Y_*}(y_*; \theta^*) = 1 - exp\left(-\frac{y_* - \mu_*}{\sigma_*}\right) \tag{6.1.15}$$

where $y_* \geq \mu_*$. Alternatively, W_* follows the distribution of the logarithm of the inverse of a standard Uniform variate.

Case 3: $\xi < 0$

When $\xi < 0$, the support of the $GP(\theta^*)$ distribution is bounded both above and below, namely in the interval $[\mu_*, \mu_* - \sigma_*/\xi]$ when $\xi < 0$. In terms of the re-parametrization (6.1.2), the support of the $GP(\theta)$ distribution is the interval $[\mu - \sigma, \mu]$. In other words, any estimate of $\mu - \sigma$ should be less than or equal to the smallest order statistic $y_{(1)}$, and any estimate of μ should be greater than or equal to the largest order statistic $y_{(n)}$, when $\xi < 0$.

6.1.3. Three-Parameter Weibull Distribution

The three-parameter Weibull distribution is a generalization of the two-parameter Weibull distribution presented in Section 2.3.3 above. It has an additional parameter, namely the location parameter, compared to a two-parameter Weibull distribution. The three-parameter Weibull distribution is defined as follows (Nagatsuka et al., 2013, Section 1):

Let a scalar continuous random variable Y_* follow a three-parameter Weibull distribution parameterized by θ, denoted by $Y_* \sim Weib(\mu, \sigma, \xi)$ with the cdf and pdf of the distribution, respectively, given by

$$F_{Y_*}(y_*; \theta) = 1 - exp\left[-\left(\frac{y_* - \mu}{\sigma}\right)^{1/\xi}\right] \tag{6.1.16}$$

and

$$f_{Y_*}(y_*;\theta) = \frac{1}{\sigma\xi}\cdot\left(\frac{y_*-\mu}{\sigma}\right)^{1/\xi-1} exp\left[-\left(\frac{y_*-\mu}{\sigma}\right)^{1/\xi}\right]$$

where μ is the location parameter, $\sigma > 0$ the scale parameter and $\xi > 0$ the shape parameter. We note that Nagatsuka et al. (2013, Section 1) parametrize the distribution with $\beta = 1/\xi$. The support of the distribution is the interval (μ, ∞) when $\xi > 0$. In other words, any estimate of μ should be strictly smaller than the smallest order statistic $y_{(1)}$. We note that when $\mu = 0$, then the cdf (6.1.16) becomes a cdf of a two-parameter Weibull distribution.

Using re-parametrization (6.1.2) we can write the cdf (6.1.16) in terms of θ^* parametrization as

$$\begin{aligned} F_{Y_*}(y_*;\theta^*) &= 1 - exp\left[-\left(\frac{y_*-\mu}{\sigma}\right)^{1/\xi}\right] \\ &= 1 - exp\left\{-\left[1+\xi\left(\frac{y_*-\mu_*}{\sigma_*}\right)\right]^{1/\xi}\right\} \\ &= 1 - exp\left\{-\left\{\left[1+\xi\left(\frac{y_*-\mu_*}{\sigma_*}\right)\right]^{-1/\xi}\right\}^{-1}\right\} \\ &= 1 - exp\{-Z_*^{-1}\} \qquad (6.1.17) \end{aligned}$$

where the standard variate of Y_*, that is, Z_*, as in equation (6.1.4), is given by

$$\begin{aligned} Z_* &= \left(\frac{Y_*-\mu}{\sigma}\right)^{-1/\xi} \\ &= \left[1+\xi\left(\frac{y_*-\mu_*}{\sigma_*}\right)\right]^{-1/\xi} \qquad (6.1.18) \end{aligned}$$

Thus, for the three-parameter Weibull (θ^*) distribution, cdf of the standard variate Z_* in (6.1.18), and the function $\Phi(\cdot)$ in (6.1.1), have the form

$$F_{Z_*}(z_*;\theta^*) = \Phi(z_*) = 1 - exp(-z_*{}^{-1}) \qquad (6.1.19)$$

Thus, $F_{Z_*}(z_*;\theta^*)$ does not depend on any unknown parameters.

Since the random variable $F_{Z_*}(Z_*)$ has a Uniform distribution, we can write

$$1 - exp\left(-Z_*{}^{-1}\right) = U$$

where U is a random variable with a Uniform distribution in the interval [0, 1]. Solving for Z_* we obtain

$$Z_* = [-\log(1-U)]^{-1}$$

Thus, for the three-parameter Weibull (θ^*) distribution, the standard variate Z_* follows the distribution of the inverse of a standard Exponential variate.

Using equation (6.1.18) we can write the general random variate Y_* in terms of the standard variate Z_* as

$$Y_* = \mu + \sigma \cdot Z_*{}^{-\xi} \qquad (6.1.20)$$

where $Z_* = [-\log(1-U)]^{-1}$.

<u>Case: $\boldsymbol{\xi = 0}$</u>

For $\xi = 0$, it follows from Theorem 1 that the distribution of Y_* is an LS family with standard variate

$$W_* = \frac{Y_* - \mu_*}{\sigma_*} = -\log(Z_*) = -\log([-\log(U)]^{-1}) = \log[-\log(U)]$$

$$(6.1.21)$$

Thus the distribution of W_*follows the distribution of the logarithm of a standard Exponential variate, with cdf

$$F_{W_*}(w_*) = 1 - exp[-exp(w_*)]$$

Furthermore, the cdf of Y_*, for $\xi = 0$, is given by

$$F_{Y_*}(y_*) = 1 - exp\left[-exp\left(\frac{y_* - \mu_*}{\sigma_*}\right)\right] \tag{6.1.22}$$

We summarise in the following table the LSS distributions investigated in this thesis, the distribution of their respective standard variate and their distributions when $\xi = 0$.

Table 6.1: Comparison of the GEV, GP and three-parameter Weibull distribution in terms of the distribution of their standard variates.

Distribution	Distribution of a standard variate	Distribution when $\xi = 0$
Generalized Extreme Value	$Z_* = -\log(U) = \varkappa$ where $\varkappa \sim exp(1)$	$\Upsilon_* = -\log(Z_*)$ $= -\log(\varkappa)$ where $\varkappa \sim exp(1)$
Generalized Pareto	$Z_* = U$	$\Upsilon_* = -\log(U) = \varkappa$ where $\varkappa \sim exp(1)$
Three-parameter Weibull	$Z_* = [-\log(U)]^{-1}$ $= \varkappa^{-1}$ where $\varkappa \sim exp(1)$	$\Upsilon_* = -\log(\varkappa^{-1})$ $= \log(\varkappa)$ where $\varkappa \sim exp(1)$

6.2. Conditional Location-Scale Families of Distributions

In this section we present the concept of conditional LS families. Specifically, assuming that the random variable Y_*comes from a three-parameter LSS family, we write the random variable Y_*, or a suitable function of the random variable Y_*, as a linear function of two parameters, conditional on the third parameter. Subsequently, we use methods developed for (two-parameter) LS families to develop methods of generalized inference for three-parameter LSS families, using the concept of conditional generalized pivotal quantities (CFGPQs) defined in Sections 6.4.1 through 6.4.4 below. The concept of a conditional LS family can be presented as follows.

<u>Case 1: $\xi > 0$</u>

As was the case in Section 6.1 above, initially we assume $\xi > 0$. Taking the logarithm on both sides of equation (6.1.4), and solving for $\log(Y_* - \mu)$, we obtain

$$\log(Y_* - \mu) = \log(\sigma) - \xi \cdot \log(Z_*) \ = \ \log(\sigma) + \xi \cdot \log(Z_*^{-1}) \tag{6.2.1}$$

Since the distribution of $-\log(Z_*) = \log(Z_*^{-1})$ is free of unknown parameters, equation (6.2.1) shows that, conditional on μ, the random variable $(Y_* - \mu)$ belongs to an LLS family of distributions with location parameter $\log(\sigma)$ and scale parameter ξ.

Similarly, raising both sides of equation (6.1.4) to the power $-\xi$, and solving for Y_*, we obtain

$$Y_* = \mu + \sigma \cdot Z_*^{-\xi} \tag{6.2.2}$$

Since, conditional on ξ, the distribution of $Z_*^{-\xi}$ is free of unknown parameters, equation (6.2.2) shows that the random variable Y_*, conditional on ξ, belongs to an LS family of distributions with location parameter μ and scale parameter σ.

We note that the distribution of $Z_*^{-\xi}$ is degenerate as $\xi \to 0$. Thus, using re-parametrization (6.1.2), we write (6.2.2) in terms of the original (μ_*, σ_*) parametrization to obtain

$$Y_* = \mu_* - \frac{\sigma_*}{\xi} + \frac{\sigma_*}{\xi} \cdot Z_*^{-\xi}$$

$$= \mu_* + \sigma_* \cdot \left(\frac{Z_*^{-\xi} - 1}{\xi} \right) \qquad (6.2.3)$$

Again, conditional on ξ the distribution of $\left(Z_*^{-\xi} - 1\right)/\xi$ is free of unknown parameters, so that, conditional on ξ, the random variable Y_* belongs to an LS family of distributions with location parameter μ_* and scale parameter σ_*.

In summary:

1. Conditional on μ, the random variable $(Y_* - \mu)$ belongs to an LLS family of distributions with location parameter $\log(\sigma)$ and scale parameter ξ.
2. Conditional on ξ, the random variable Y_*, belongs to an LS family of distributions with location parameter μ and scale parameter σ. In the alternative parametrization, conditional on ξ, the random variable Y_* belongs to an LS family of distributions with location parameter μ_* and scale parameter σ_*.

Case 2: $\xi = 0$

We note that $\lim_{\xi \to 0} \left[\left(Z_*^{-\xi} - 1\right)/\xi\right] = -\log(Z_*) = \log(Z_*^{-1})$. Thus, for $\xi = 0$, we have

$$Y_* = \mu_* + \sigma_* \cdot \log(Z_*^{-1})$$

which is the an LS family, for example, the Gumbel distribution.

Case 3: $\xi < 0$

Similarly to obtaining model (6.2.1) above, we take the logarithm on both sides of equation (6.1.4), and with $\xi < 0$ we obtain

$$\log(Z_*) = -1/\xi \cdot [\log(\mu - Y_*) - \log(-\sigma)]$$

Solving for $\log(\mu - Y_*)$, we obtain

$$\log(\mu - Y_*) = \log(-\sigma) - \xi \cdot \log(Z_*) \qquad (6.2.4)$$

(Note that, with $\xi < 0$, $Y_* < \mu$ and $\sigma < 0$). Again, since the distribution of $\log(Z_*)$ does not depend on any unknown parameters, equation (6.2.4) shows that, conditional on μ, the random variable $(\mu - Y_*)$ belongs to an LS family of distributions with location parameter $\log(-\sigma)$ and scale parameter $-\xi$.

Alternatively, equation (6.2.4) can be written as

$$\log(\mu - Y_*) = \log(-\sigma) + \xi \cdot \log(Z_*^{-1}) \qquad (6.2.5)$$

Finally, we can note that equations (6.2.2) and (6.2.3) hold unchanged for the case $\xi < 0$.

6.3. Estimation for Location-Scale-Shape Family of Distributions

We start by presenting a literature review on the estimation of parameters and vector parameters of LSS distributions using rank and maximum likelihood-based methods in Section 6.3.1. In Section 6.3.2 we present a method of iterative generalized least squares estimation of the parameters of LSS family of distributions.

6.3.1. Literature Review

Generalized Extreme Value distribution

Various methods of parameter estimation have mostly been used in the literature to estimate the parameters of GEV distribution. For example, the probability-weighted moments (Hosking et al. (1985) and Dupuis (1999)); maximum likelihood (Hosking (1985), Prescott and Walden (1980, 1983), Martins and Stedinger (2000), Feng et al. (2007), Ahsanullah and Holland (1994), Ahsanullah (1996)); likelihood moments (Hosking (1985) and Martins and Stedinger (2000); and method of moments (Martins and Stedinger (2000)).

Generalized Pareto distribution:

The GP distribution has diverse areas of applications in literature, for example, engineering, insurance (e.g. extreme claims), economics and finance (e.g. extreme market movements), hydrology (e.g. extreme river flows) and climatology (e.g. extreme rain fall or temperatures) among others (de Zea Bermudez and Kotz (2010). The range of a shape parameter (or tail index) of the GP distribution plays an important role in determining the area of application. For example, hydrological phenomena tend to exhibit and be modelled by heavier-tailed version of a GP distribution. In extreme value theory, a GP distribution has been used to fit the distribution of peaks-over-threshold (POT); see, for example, Mackay et al. (2011) and references cited in Section 1 of their paper.

A great deal of research has been done on the various methods of estimation of the parameters of a GP distribution. Examples of methods of estimation include the maximum likelihood (ML) (Brabson and Palutikof, 2000; Mackay et al. (2011); Peng and Welsh (2001)), methods of moments (MOM) (Hoskins and Wallis (1987), probability weighted moments (PWM) (Hosking and Wallis (1987)), least squares (LS) (Kulldorff and Vannman (1973); Vannman (1976)) and likelihood moment (LM) (Zhang, 2007). Most methods of parameter estimation for the GP distribution in literature, according to our knowledge, appear to focus on the estimation of the scale and shape parameter of the distribution, conditional on threshold (location) parameter. However, in such cases predetermining the sufficiently high threshold before the actual parameter estimation appears to be subjective. Although overall, different methods of parameter estimation are ideal for different versions of GP distribution in terms of the range of shape parameter space. For example, MOM estimators only exist when the shape parameter is greater than -0.5, that is, the population mean and variance of GP distribution are not defined for the values of the shape parameter outside that range. On the contrary, least squares estimation is ideal for position shape parameter. For more detail on comparison of various methods of estimation, see de Zea Bermudez and Kotz (2010, Section 7).

Historically, the work on parameter estimation of GP distribution started in the early 1970s. Kulldorff and Vannman (1973) derived, using Gauss-Markoff theorem, the best linear unbiased estimators (BLUEs) of the location and scale parameters, conditional on the shape parameter based on order statistics of a sample. We note here that although the approach used by Kulldorff and Vannman (1973) is different, the general principle is similar to our method of using iterative generalized least squares estimation of the location parameter μ_* and scale parameter σ_* conditional on the shape parameter ξ as presented in Section 6.3.2.2 below. In addition, Kulldorff and Vannman (1973) also derived conditional BLUEs for the location and scale parameters of GP distribution. Furthermore, they derived the conditional asymptotic BLUEs for the location, scale, and location and scale parameters based on a censored sample. Similarly to Kulldorff and Vannman (1973), Vannman (1976) derived, using Gauss-Markoff theorem, the conditional BLUEs for location, scale, and location and scale parameters of GP distribution based on a censored sample. Since then, many researchers have applied various methods of estimation to obtain estimates of the parameters. Moreover, new methods of estimation were developed and their performance compared through Monte Carlo simulations with that of other methods. For example, Singh and Guo (1995) derived a method of estimation for the parameters of a GP distribution based on the principle of maximum entropy (POME) of the distribution. They then compared POME-based estimators with estimators based on ML, PWM and MOM. Zhang (2007) proposed a method of estimation of the parameters of a GP distribution based on likelihood moment (LM) and found that, under a specified condition of the shape parameter, estimators derived using LM have asymptotic properties equivalent to those obtained using the maximum likelihood-based method. Furthermore, Peng and Welsh (2001) derived the parameter estimators of GP distribution based on sample and population medians of the distribution.

<u>Three-parameter Weibull distribution</u>

Many authors have studied various methods of estimation of the parameters of three-parameter Weibull distribution. Examples of methods of estimation include the maximum likelihood, maximum product spacings, method of moments, least squares, and data transformation-based method. Nagatsuka at al. (2013, section 1) provide comprehensive overview of estimation methods for the parameters of three-parameter Weibull distribution.

6.3.2. Iterative Generalized Least Squares Estimation in LSS Family

In Sections 6.3.2.1 through 6.3.2.3 we present three generalized least squares methods of estimation in LSS family of distributions using rank-based methods.

6.3.2.1. Generalized least squares estimation of ξ, given μ

1. **<u>Case $\xi > 0$: Estimation of ξ and $\log(\sigma)$, given μ</u>**

For $\xi > 0$, to estimate the parameters $\log(\sigma)$ and ξ, conditional on μ, we consider equation (6.2.1): Let $Y_1, Y_2, \dots, Y_n$ be an i.i.d. random sample from the distribution of Y_*, and let

$$Z_i = \left(\frac{Y_i - \mu}{\sigma}\right)^{-1/\xi}, \qquad i = 1, 2, \dots, n \tag{6.3.1}$$

be the standardized variates in the manner of equation (6.1.4). Furthermore, let $Y = \left[Y_{(1),}Y_{(2)}, \dots, Y_{(n)}\right]'$ be the vector of the order statistics of the sample $Y_1, Y_2, \dots, Y_n$, and similarly let $Z^{-1} = \left[Z_{(1)}^{-1}, Z_{(2)}^{-1}, \dots, Z_{(n)}^{-1}\right]'$ be the vector of the order statistics of the inverse standard variates $Z_1^{-1}, Z_2^{-1}, \ldots, Z_n^{-1}$. Then we can write

$$\log(Y-\mu) = \log(\sigma) + \xi \cdot \log(Z^{-1}) \qquad (6.3.2)$$

> **Note:** Here (equation (6.3.2)) and in the following, the sum, difference or product of a vector and a scalar, such as $Y-\mu$, denotes the sum, difference or product of each component of the vector and the scalar, in the usual way. Similarly, all the vector-valued functions, such as $\log(Y-\mu)$, are evaluated component-wise.

The expectation and covariance matrix of $\log(Y-\mu)$ are respectively given by

$$E[\log(Y-\mu)] = \log(\sigma) + \xi \cdot E[\log(Z^{-1})] \qquad (6.3.3)$$

and

$$Cov[\log(Y-\mu)] = \xi^2 \cdot Cov[\log(Z^{-1})] = \xi^2 \cdot V \qquad (6.3.4)$$

Thus, $E[\log(Z^{-1})]$ and $V = Cov[\log(Z^{-1})]$ are respectively the expected value and covariance matrix of the order statistics of a sample of size n from the distribution of $\log(Z_*^{-1})$, namely the distribution of the logarithm of the inverse standard variate Z_*^{-1}.

Writing $\log(Y-\mu) = E[\log(Y-\mu)] + e$, and using (6.3.3) and (6.3.4), we have the following general linear model for $\log(Y-\mu)$:

$$\log(Y-\mu) = \log(\sigma) + \xi \cdot E[\log(Z^{-1})] + e; \qquad Cov(e) = \xi^2 \cdot V$$

$$(6.3.5)$$

In matrix notation model (6.3.5) can be written as

$$\log(Y-\mu) = X\binom{\log(\sigma)}{\xi} + e; \qquad Cov(e) = \xi^2 \cdot V \qquad (6.3.6)$$

where the $n \times 2$ matrix X is given by

$$X = \left[1_n : E[\log(Z^{-1})]\right] = \begin{pmatrix} 1 & E\left[\log\left(Z_{(1)}^{-1}\right)\right] \\ \vdots & \vdots \\ 1 & E\left[\log\left(Z_{(n)}^{-1}\right)\right] \end{pmatrix} \qquad (6.3.7)$$

Under model (6.3.6) the generalized least squares (GLS) estimator $\hat{\theta}(Y)$ for $\theta = (\log(\sigma), \xi)'$ when $\xi > 0$, is given by

$$\hat{\theta}(Y) = \begin{pmatrix} \widehat{\log(\sigma)}\,(Y) \\ \hat{\xi}(Y) \end{pmatrix}$$

$$= (X'V^{-1}X)^{-1}X'V^{-1}\log(Y - \mu) \qquad (6.3.8)$$

$$= H\log(Y - \mu)$$

where $H = (X'V^{-1}X)^{-1}X'V^{-1}$. Clearly, for given μ, $\widehat{\log(\sigma)}\,(Y)$ and $\hat{\xi}(Y)$ in equation (6.3.8) are the best (minimum variance) linear unbiased estimators for the parameters $\log(\sigma)$ and ξ.

2. **<u>Case $\xi < 0$: Estimation of ξ and $\log(-\sigma)$, given μ</u>**

Similarly to obtaining the estimates $\widehat{\log(\sigma)}\,(Y)$ and $\hat{\xi}(Y)$ conditional on μ in equation (6.3.8), we now consider the estimation of $\log(-\sigma)$ and ξ, conditional on μ, when $\xi < 0$, by using equation (6.2.5).

Again, we let $Y_1, Y_2, \dots, Y_n$ be an i.i.d. random sample from the distribution of Y_*, and let

$$Z_i = \left(\frac{Y_i - \mu}{\sigma}\right)^{-1/\xi}, \qquad i = 1, 2, \dots, n \qquad (6.3.9)$$

be the standardized variates in the manner of equation (6.1.4). Furthermore, let $Y = \left[Y_{(1),}Y_{(2)}, \dots, Y_{(n)}\right]'$ be the vector of the order statistics of the sample $Y_1, Y_2, \dots, Y_n$, and similarly, let $Z^{-1} = \left[Z_{(1)}^{-1}, Z_{(2)}^{-1}, \dots, Z_{(n)}^{-1}\right]'$ be the vector of the order statistics of the inverse standard variates $Z_1^{-1}, Z_2^{-1}, \dots, Z_n^{-1}$. Then using equation (6.2.5) we can write

$$\log(\mu - Y) = \log(-\sigma) + \xi \cdot \log(Z^{-1}) \qquad (6.3.10)$$

The identity (6.3.10) for order statistics can be motivated as follows: since the vector Y of order statistics is ordered from small to large, the vector $\log(\mu - Y)$ is ordered from large to small; similarly, since the vectors Z^{-1}, and thus $\log(Z^{-1})$ of order statistics are ordered from small to large, the vector $\xi \cdot \log(Z^{-1})$ is ordered from large to small, when $\xi < 0$. Thus the vectors on both sides of equation (6.3.10) are ordered from large to small.

The expectation and covariance matrix of $\log(\mu - Y)$ in equation (6.3.10) are, respectively, given by

$$E[\log(\mu - Y)] = \log(-\sigma) + \xi \cdot E[\log(Z^{-1})] \qquad (6.3.11)$$

and

$$Cov[\log(\mu - Y)] = \xi^2 \cdot Cov[\log(Z^{-1})] = \xi^2 \cdot V \qquad (6.3.12)$$

Thus $E[\log(Z^{-1})]$ and $V = Cov[\log(Z^{-1})]$ are respectively the expected value and covariance matrix of the order statistics of a sample of size n from the distribution of $\log(Z_*^{-1})$, namely the distribution of the logarithm of the inverse standard variate Z_*^{-1}.

Writing $\log(\mu - Y) = E[\log(\mu - Y)] + e$, and using equations (6.3.11) and (6.3.12), we have the following general linear model for $\log(\mu - Y)$:

$$\log(\mu - Y) = \log(-\sigma) + \xi \cdot E[\log(Z^{-1})] + e; \qquad Cov(e) = \xi^2 \cdot V$$

$$(6.3.13)$$

In matrix notation model (6.3.13) can be written as

$$\log(\mu - Y) = X \begin{pmatrix} \log(-\sigma) \\ \xi \end{pmatrix} + e; \qquad Cov(e) = \xi^2 \cdot V \qquad (6.3.14)$$

where the $n \times 2$ matrix X is given by

$$X = \left[1_n : E[\log(Z^{-1})]\right] = \begin{pmatrix} 1 & E\left[\log\left(Z_{(1)}^{-1}\right)\right] \\ \vdots & \vdots \\ 1 & E\left[\log\left(Z_{(n)}^{-1}\right)\right] \end{pmatrix} \quad (6.3.15)$$

Similarly, under model (6.3.14) the generalized least squares (GLS) estimator $\hat{\theta}(Y)$ for $\theta = (\log(-\sigma), \xi)'$ is given by

$$\hat{\theta}(Y) = \begin{pmatrix} \widehat{\log(-\sigma)}\,(Y) \\ \hat{\xi}(Y) \end{pmatrix}$$

$$= (X'V^{-1}X)^{-1}X'V^{-1}\log(\mu - Y) \quad (6.3.16)$$

$$= H\log(\mu - Y)$$

where $H = (X'V^{-1}X)^{-1}X'V^{-1}$. Clearly, for given μ, $\widehat{\log(-\sigma)}\,(Y)$ and $\hat{\xi}(Y)$ in equation (6.3.16) are the best (minimum variance) linear unbiased estimators for the parameters $\log(-\sigma)$ and ξ.

We note that the matrix H in (6.3.16) is the same as the matrix H in (6.3.8); estimators (6.3.8) and (6.3.16) differ only in the dependent variables, namely $\log(Y - \mu)$ in (6.3.8) but $\log(\mu - Y)$ in (6.3.16).

6.3.2.2. Generalized least squares estimation of μ_* and σ_*, given ξ

Here we consider equation (6.2.3) when $\xi \neq 0$. As in Section 6.3.2.1 above, let Y be the vector of order statistics from an i.i.d. sample of size n from the distribution of Y_*; furthermore let $\left(Z^{-\xi} - 1\right)/\xi = \left[\left(Z_{(1)}^{-\xi} - 1\right)/\xi, \ldots, \left(Z_{(n)}^{-\xi} - 1\right)/\xi\right]$ be the vector of the order statistics of the variates $\left(Z_1^{-\xi} - 1\right)/\xi, \ldots, \left(Z_n^{-\xi} - 1\right)/\xi$ from the distribution of $\left(Z_*^{-\xi} - 1\right)/\xi$.Then, similarly to equation (6.3.5), we obtain a general linear model for the vector of order statistics Y as

$$Y = \mu_* + \sigma_* \cdot E\left(\frac{Z^{-\xi} - 1}{\xi}\right) + e; \qquad Cov(e) = \sigma^2 \cdot V_\xi \qquad (6.3.17)$$

where $E\left[\left(Z^{-\xi} - 1\right)/\xi\right]$ and $V_\xi = Cov\left[\left(Z^{-\xi} - 1\right)/\xi\right]$ are respectively the expected value and covariance matrix of the vector of order statistics $\left(Z^{-\xi} - 1\right)/\xi$. In matrix notation model (6.3.17) can be written as

$$Y = X_\xi \begin{pmatrix} \mu_* \\ \sigma_* \end{pmatrix} + e; \qquad Cov(e) = \sigma^2 \cdot V_\xi \qquad (6.3.18)$$

where the $n \times 2$ matrix X_ξ is given by

$$X_\xi = \left[1_n : E\left[\left(Z^{-\xi} - 1\right)/\xi\right]\right] = \begin{pmatrix} 1 & E\left[\left(Z_{(1)}^{-\xi} - 1\right)/\xi\right] \\ \vdots & \vdots \\ 1 & E\left[\left(Z_{(n)}^{-\xi} - 1\right)/\xi\right] \end{pmatrix} \qquad (6.3.19)$$

Under model (6.3.18) the GLS estimator $\hat{\theta}(Y)$ for $\theta = (\mu_*, \sigma_*)'$ is given by

$$\begin{aligned} \hat{\theta}(Y) &= \begin{pmatrix} \widehat{\mu_*}(Y) \\ \widehat{\sigma_*}(Y) \end{pmatrix} \\ &= \left(X_\xi' V_\xi^{-1} X_\xi\right)^{-1} X_\xi' V_\xi^{-1} Y \qquad (6.3.20) \\ &= H_\xi Y \end{aligned}$$

where $H_\xi = \left(X_\xi' V_\xi^{-1} X_\xi\right)^{-1} X_\xi' V_\xi^{-1}$. For given ξ, $\widehat{\mu_*}(Y)$ and $\widehat{\sigma_*}(Y)$ in equation (6.3.20) are the best (minimum variance) linear unbiased estimates for the parameters μ_* and σ_*.

We note that a similar general principle for estimation was used by Kulldorff and Vannman (1973, Section 4).

6.3.2.3. Iterative generalized least squares estimation of μ, σ and ξ

The estimators presented by equations (6.3.8), (6.3.16) and (6.3.20) suggest that the parameters μ_*, σ_* and ξ of a $LSS(\theta^*)$ distribution can be estimated using the following iterative generalized least squares algorithms:

Algorithm 6a: Case $\xi > 0$: Iterative generalized least squares estimation of μ_*, σ_* and ξ

Let $y_1, y_2, \dots, y_n$ be a sample of observations from a $LSS(\theta^*)$ distribution, and let $y = [y_{(1)}, y_{(2)}, \dots, y_{(n)}]'$ be the vector of associated order statistics.

1. Choose an initial estimate $\xi^{(0)}$ for ξ.
2. For $i = 1, 2, \dots, M$ do Steps 2(a) and 2(b) until convergence:
   (a) Given the current estimate $\xi^{(i-1)}$, and using estimator (6.3.20), obtain GLS estimates $\mu_*^{(i)}$ and $\sigma_*^{(i)}$ for μ_* and σ_* as
   $$\begin{pmatrix} \mu_* \\ \sigma_* \end{pmatrix}^i = \left(X'_{\xi^{(i-1)}} V^{-1}_{\xi^{(i-1)}} X_{\xi^{(i-1)}}\right)^{-1} X'_{\xi^{(i-1)}} V^{-1}_{\xi^{(i-1)}} y$$
   and obtain the estimate $\mu^{(i)}$ for μ as
   $$\mu^{(i)} = \mu_*^{(i)} - \sigma_*^{(i)} / \xi^{(i-1)}$$
   (b) Given the current estimate $\mu^{(i)}$, and using estimator (6.3.8), obtain GLS estimates $\log(\sigma)^i$ and $\xi^{(i)}$ for $\log(\sigma)$ and ξ as
   $$\begin{pmatrix} \log(\sigma) \\ \xi \end{pmatrix}^{(i)} = (X'V^{-1}X)^{-1} X'V^{-1} \log\left(y - \mu^{(i)}\right)$$
   Note: We assume in this algorithm that all positive GLS estimates of ξ are greater than or equal to 0.1.

Algorithm 6a: Case $\xi < 0$: Iterative generalized least squares estimation of μ_*, σ_* and ξ

Again, let $y_1, y_2, \dots, y_n$ be a sample of observations from a $LSS(\theta^*)$ distribution, and let $y = [y_{(1)}, y_{(2)}, \dots, y_{(n)}]'$ be the vector of associated order statistics.

1. Choose an initial estimate $\xi^{(0)}$ for ξ.

2. For $i = 1, 2, \dots, M$ do Steps 2(a) and 2(b) until convergence:

   (a) Given the current estimate $\xi^{(i-1)}$, and using estimator (6.3.20), obtain GLS estimates $\mu_*^{(i)}$ and $\sigma_*^{(i)}$ for μ_* and σ_* as

$$\begin{pmatrix}\mu_* \\ \sigma_*\end{pmatrix}^{i} = \left(X'_{\xi^{(i-1)}} V^{-1}_{\xi^{(i-1)}} X_{\xi^{(i-1)}}\right)^{-1} X'_{\xi^{(i-1)}} V^{-1}_{\xi^{(i-1)}} y$$

   and obtain the estimate $\mu^{(i)}$ for μ as

$$\mu^{(i)} = \mu_*^{(i)} - \sigma_*^{(i)} / \xi^{(i-1)}$$

   (b) Given the current estimate $\mu^{(i)}$, and using estimator (6.3.16), obtain GLS estimates $\log(-\sigma)^{i}$ and $\xi^{(i)}$ for $\log(-\sigma)$ and ξ as

$$\begin{pmatrix}\log(-\sigma) \\ \xi\end{pmatrix}^{(i)} = (X'V^{-1}X)^{-1} X'V^{-1} \log\left(\mu^{(i)} - y\right)$$

Note: Here we do not pursue further the GLS estimation of the parameters of LSS families of distribution. Instead, in the following section we present methods for fiducial inference for LSS families of distributions that are inspired by the idea of iterative GLS estimation presented above. One output of the proposed fiducial inference method is the joint fiducial distribution of model parameters, from which not only GCIs for parameters can be obtained, but also parameter estimates (by taking, for example, the mean or the median of the fiducial distribution of the parameter of interest).

6.4. Conditional Fiducial Generalized Pivotal Quantities for LSS Family of Distributions Based on Ranks

We present below in Sections 6.4.1 through 6.4.5 rank-based conditional fiducial generalized pivotal quantities (CFGPQs) for the parameters and parameter vectors in LSS family of distributions. Computational problems in respect of the simulation of the linear predictors of the parameters μ_* and σ_* are presented in Section 6.4.6.

6.4.1. Conditional Fiducial Generalized Pivotal Quantities for ξ Based on Ranks

Below in Sections 6.4.1.1 and 6.4.1.2 we present two rank-based CFGPs for ξ, namely a CFGPQ for ξ, given μ; and a CFGPQ for ξ, given μ and σ respectively.

6.4.1.1. Conditional fiducial generalized pivotal quantity for ξ, given μ, based on ranks

1. **<u>Case $\xi > 0$</u>**

Using equation (6.3.8), with $L_2 = (0, 1)'$, the GLS estimator for ξ is given by

$$\hat{\xi}(Y) = L_2' H \log(Y - \mu)$$

$$= L_2' H[\log(\sigma) \cdot 1_n + \xi \cdot \log(Z^{-1})]$$

$$= \xi \cdot L_2' H \log(Z^{-1})$$

since $L_2' H 1_n = 0$. We define the following CFGPQ for ξ

$$\mathcal{R}_{\xi|\mu}(y, Y) = \frac{\hat{\xi}(y)}{\hat{\xi}(Y)/\xi}$$

$$= \frac{L_2' H \log(y - \mu)}{L_2' H \log(Y - \mu)/\xi}$$

$$= \frac{L_2' H \log(y - \mu)}{L_2' H \log(Z^{-1})} \tag{6.4.1}$$

Firstly, conditional on μ, the distribution of $\mathcal{R}_{\xi|\mu}$ in equation (6.4.1) does not depend on any unknown parameters (as can be seen from the last term in

(6.4.1)); secondly, the observed value of $\mathcal{R}_{\xi|\mu}$ (for any value μ) is equal to ξ (as can be seen from the last term in (6.4.1)). Thus, conditional on μ, $\mathcal{R}_{\xi|\mu}$ is a CFGPQ for ξ.

2. **Case $\xi < 0$**

Similarly, using equation (6.3.16), with $L_2 = (0, 1)'$, the GLS estimator for ξ is given by

$$\begin{aligned}\hat{\xi}(Y) &= L_2' H \log(\mu - Y) \\ &= L_2' H[\log(-\sigma) \cdot 1_n + \xi \cdot \log(Z^{-1})] \\ &= \xi \cdot L_2' H \log(Z^{-1})\end{aligned}$$

since $L_2' H 1_n = 0$. We define the following CFGPQ for ξ conditional on μ, as

$$\begin{aligned}\mathcal{R}_{\xi|\mu}(y, Y) &= \frac{\hat{\xi}(y)}{\hat{\xi}(Y)/\xi} \\ &= \frac{L_2' H \log(\mu - y)}{L_2' H \log(\mu - Y)/\xi} \\ &= \frac{L_2' H \log(\mu - y)}{L_2' H \log(Z^{-1})} \qquad (6.4.2)\end{aligned}$$

Again, conditional on μ the distribution of $\mathcal{R}_{\xi|\mu}$ in equation (6.4.2) does not depend on any unknown parameters, and secondly the observed value of $\mathcal{R}_{\xi|\mu}$ (for any value μ) is equal to ξ. Thus, conditional on μ, $\mathcal{R}_{\xi|\mu}$ is a CFGPQ for $\xi < 0$.

6.4.2. Conditional Fiducial Generalized Pivotal Quantity for $\log(\sigma)$, Given μ, Based on Ranks

1. **Case $\xi > 0$**

Using equation (6.3.8), with $L_1 = (1, 0)'$, the GLS estimator for $\log(\sigma)$ when $\xi > 0$, is given by

$$\widehat{\log(\sigma)}\,(Y) = L_1'(X'V^{-1}X)^{-1}X'V^{-1}\log(Y-\mu)$$

$$= L_1'H[\log(\sigma)\cdot 1_n + \xi\cdot\log(Z^{-1})]$$

$$= \log(\sigma) + \xi\cdot L_1'H\log(Z^{-1}) \qquad (6.4.3)$$

Since $H1_n = (1, 0)'$ and therefore $L_1'H1_n = 1$. Based on equation (6.4.3) we define the following CFGPQ for $\log(\sigma)$ given μ and ξ:

$$\mathcal{R}_{\log(\sigma)|\mu,\xi}(y, Y) = \widehat{\log(\sigma)}\,(y) - \left[\widehat{\log(\sigma)}\,(Y) - \log(\sigma)\right]$$

$$= \widehat{\log(\sigma)}\,(y) - \xi\cdot L_1'H\log(Z^{-1})$$

$$= L_1'H\log(y-\mu) - \xi\cdot L_1'H\log(Z^{-1})$$

$$= L_1'H[\log(y-\mu) - \xi\cdot\log(Z^{-1})] \qquad (6.4.4)$$

The first equality in equation (6.4.4) shows that $\mathcal{R}_{\log(\sigma)|\mu,\xi}(y, y) = \log(\sigma)$ for all possible observations y of Y (for all the values of μ and ξ); the second equality in equation (6.4.4) show that the distribution of $\mathcal{R}_{\log(\sigma)|\mu,\xi}$, conditional on an observation y of Y and on μ and ξ, is free of μ and ξ. Thus $\mathcal{R}_{\log(\sigma)|\mu,\xi}$ is a CFGPQ for $\log(\sigma)$.

Furthermore, we can obtain the CFGPQs for $\log(\sigma)$ given μ by replacing the parameter ξ in equation (6.4.4) by the CFGPQ $\mathcal{R}_{\xi|\mu}$ in equation (6.4.1) as

$$\mathcal{R}_{\log(\sigma)|\mu}(y, Y) = L_1'H\left[\log(y-\mu) - \mathcal{R}_{\xi|\mu}\cdot\log(Z^{-1})\right]$$

$$= L_1'H\left[\log(y-\mu) - \frac{L_2'H\log(y-\mu)}{L_2'H\log(Z^{-1})}\cdot\log(Z^{-1})\right] \qquad (6.4.5)$$

We note that the anti-log of equation (6.4.5) is a CFGPQ for σ, given μ, namely $\mathcal{R}_{\sigma|\mu} = exp\left(\mathcal{R}_{\log(\sigma)|\mu}\right)$.

2. Case $\xi < 0$

Using equation (6.3.16), with $L_1 = (1,0)'$, the GLS estimator for $\log(-\sigma)$ when $\xi < 0$, is given by

$$\widehat{\log(-\sigma)}\,(Y) = L_1'(X'V^{-1}X)^{-1}X'V^{-1}\log(\mu - Y)$$

$$= L_1'H[\log(-\sigma)\cdot 1_n + \xi\cdot\log(Z^{-1})]$$

$$= \log(-\sigma) + \xi\cdot L_1'H\log(Z^{-1}) \tag{6.4.6}$$

Since $H1_n = (1,0)'$ and therefore $L_1'H1_n = 1$. Based on equation (6.4.6) we define the following CFGPQ for $\log(-\sigma)$ given μ and ξ:

$$\mathcal{R}_{\log(-\sigma)|\mu,\xi}(y,Y) = \widehat{\log(-\sigma)}\,(y) - \left[\widehat{\log(-\sigma)}\,(Y) - \log(-\sigma)\right]$$

$$= \widehat{\log(-\sigma)}\,(y) - \xi\cdot L_1'H\log(Z^{-1})$$

$$= L_1'H\log(\mu - y) - \xi\cdot L_1'H\log(Z^{-1})$$

$$= L_1'H[\log(\mu - y) - \xi\cdot\log(Z^{-1})] \tag{6.4.7}$$

The first equality in equation (6.4.7) shows that $\mathcal{R}_{\log(-\sigma)|\mu,\xi}(y,y) = \log(-\sigma)$ for all possible observations y of Y (for all the values of μ and ξ); the second equality in equation (6.4.7) shows that the distribution of $\mathcal{R}_{\log(-\sigma)|\mu,\xi}$, conditional on an observation y of Y and on μ and ξ, is free of μ and ξ. Thus $\mathcal{R}_{\log(-\sigma)|\mu,\xi}$ is a CFGPQ for $\log(-\sigma)$.

Furthermore, we can obtain the CFGPQ for $\log(-\sigma)$ given μ by replacing the parameter ξ in equation (6.4.7) by the CFGPQs $\mathcal{R}_{\xi|\mu}$ in equation (6.4.2) as

$$\mathcal{R}_{\log(-\sigma)|\mu}(y,Y) = L_1'H\left[\log(\mu - y) - \mathcal{R}_{\xi|\mu}\cdot\log(Z^{-1})\right]$$

$$= L_1'H\left[\log(\mu - y) - \frac{L_2'H\log(\mu - y)}{L_2'H\log(Z^{-1})}\cdot\log(Z^{-1})\right] \tag{6.4.8}$$

We note that the anti-log of equation (6.4.8) is a CFGPQ for $-\sigma$, given μ, namely $\mathcal{R}_{-\sigma|\mu} = exp(\mathcal{R}_{\log(-\sigma)|\mu})$.

6.4.3. Conditional Fiducial Generalized Pivotal Quantities for σ_* and σ, Given ξ, Based on Ranks

In this section we present the rank-based CFGPQs for σ_*, given ξ. In addition, we derive the rank-based CFGPQ for σ, given $\xi > 0$ and CFGPQ for $-\sigma$, given $\xi < 0$.

Using equation (6.3.20), with $L_2 = (0, 1)'$, the GLS estimator for σ_* is given by

$$\begin{aligned}\widehat{\sigma_*}(Y) &= L_2' H_\xi Y \\ &= L_2' H_\xi \left(\mu_* \cdot 1_n + \sigma_* \cdot \left[\left(Z^{-\xi} - 1\right)/\xi\right]\right) \\ &= \sigma_* \cdot L_2' H_\xi \left[\left(Z^{-\xi} - 1\right)/\xi\right]\end{aligned}$$

since $L_2' H_\xi 1_n = 0$. We define the following CFGPQ for σ_*, given $\xi > 0$:

$$\begin{aligned}\mathcal{R}_{\sigma_*|\xi}(y, Y) &= \frac{\widehat{\sigma_*}(y)}{\widehat{\sigma_*}(Y)/\sigma_*} \\ &= \frac{L_2' H_\xi y}{L_2' H_\xi Y/\sigma_*} \\ &= \frac{L_2' H_\xi y}{L_2' H_\xi [(Z^{-\xi} - 1)/\xi]} \qquad (6.4.9)\end{aligned}$$

Firstly, conditional on ξ the distribution of $\mathcal{R}_{\sigma_*|\xi}$ does not depend on any unknown parameters, and secondly the observed value of $\mathcal{R}_{\sigma_*|\xi}$ (for any ξ) is equal to σ_*. Thus, conditional on ξ, $\mathcal{R}_{\sigma_*|\xi}$ is a CFGPQ for σ_*.

Furthermore, we obtain the CFGPQ for $\sigma = \sigma_*/\xi$, given ξ, as

$$\mathcal{R}_{\sigma|\xi}(y, Y) = \mathcal{R}_{\sigma_*|\xi}(y, Y)/\xi \qquad (6.4.10)$$

where $\mathcal{R}_{\sigma_*|\xi}(y, Y)$ is given in equation (6.4.9) above.

6.4.4. Conditional Fiducial Generalized Pivotal Quantities for $\boldsymbol{\mu_*}$ and $\boldsymbol{\mu}$, Given $\boldsymbol{\xi}$, Based on Ranks

We present in this section rank-based CFGPQs for μ_* given ξ; and rank-based CFGPQs for μ, given ξ.

Using equation (6.3.20), with $L_1 = (1, 0)'$, the GLS estimator for μ_* is given by

$$\begin{aligned}\widehat{\mu_*}(Y) &= L_1' H_\xi Y \\ &= L_1' H_\xi\big(\mu_* \cdot 1_n + \sigma_* \cdot \big[\big(Z^{-\xi} - 1\big)/\xi\big]\big) \\ &= \mu_* + \sigma_* \cdot L_1' H_\xi\big[\big(Z^{-\xi} - 1\big)/\xi\big]\end{aligned}$$

since $H_\xi 1_n = (1, 0)'$ and therefore $L_1' H_\xi 1_n = 1$. We define the following CFGPQ for μ_*, given ξ and σ_*:

$$\begin{aligned}\mathcal{R}_{\mu_*|\xi,\sigma_*}(y, Y) &= \widehat{\mu_*}(y) - [\widehat{\mu_*}(Y) - \mu_*] \\ &= \widehat{\mu_*}(y) - \sigma_* \cdot L_1' H_\xi\big[\big(Z^{-\xi} - 1\big)/\xi\big] \\ &= L_1' H_\xi y - \sigma_* \cdot L_1' H_\xi\big[\big(Z^{-\xi} - 1\big)/\xi\big] \\ &= L_1' H_\xi\big(y - \sigma_* \cdot \big[\big(Z^{-\xi} - 1\big)/\xi\big]\big) \qquad (6.4.11)\end{aligned}$$

The first equality in equation (6.4.11) shows that $\mathcal{R}_{\mu_*|\xi,\sigma_*}(y, y) = \mu_*$ for all possible observations y of Y (and for all values of ξ and σ_*); the second equality in equation (6.4.11) shows that the distribution of $\mathcal{R}_{\mu_*|\xi,\sigma_*}(y, Y)$, conditional on an observation y of Y, and conditional on ξ and σ_*, is free of any unknown parameters. Thus, $\mathcal{R}_{\mu_*|\xi,\sigma_*}(y, Y)$ is a CFGPQ for μ_*.

Furthermore, we obtain the CFGPQ for μ_* given ξ by replacing the parameter σ_* in equation (6.4.11) by the CFGPQ $\mathcal{R}_{\sigma_*|\xi}$ in equations (6.4.9) ("plug-in-principle"):

$$\mathcal{R}_{\mu_*|\xi}(y,Y) = L_1' H_\xi\left(y - \mathcal{R}_{\sigma_*|\xi} \cdot \left[\left(Z^{-\xi} - 1\right)/\xi\right]\right)$$

$$= L_1' H_\xi \left\{ y - \frac{L_2' H_\xi y}{L_2' H_\xi [(Z^{-\xi} - 1)/\xi]} \cdot \left[\left(Z^{-\xi} - 1\right)/\xi\right] \right\} \qquad (6.4.12)$$

Finally, we obtain the CFGPQ for $\mu = \mu_* - \sigma_*/\xi$, given ξ as

$$\mathcal{R}_{\mu|\xi}(y,Y) = \mathcal{R}_{\mu_*|\xi}(y,Y) - \mathcal{R}_{\sigma_*|\xi}(y,Y)/\xi \qquad (6.4.13)$$

where $\mathcal{R}_{\mu_*|\xi}(y,Y)$ is taken from equation (6.4.12), and $\mathcal{R}_{\sigma_*|\xi}(y,Y)$ from equation (6.4.9) above.

6.4.5. Gibbs Sampler for the Joint Fiducial Distribution of the Model Parameters

In Section 6.4.1 above we have derived the CFGPQs $\mathcal{R}_{\xi|\mu}$ for both cases, $\xi > 0$ and $\xi < 0$, and in Sections 6.4.12, 6.4.9, and 6.4.13 we have, respectively, derived CFGPQs $\mathcal{R}_{\mu_*|\xi}$ for μ_*, $\mathcal{R}_{\sigma_*|\xi}$ for σ_*, and $\mathcal{R}_{\mu|\xi}$ for μ. Thus we can determine the conditional fiducial distribution of ξ given μ, and the conditional fiducial distributions of μ_* given ξ, of σ_* given ξ, and of μ given ξ respectively. Based on these conditional distributions, and assuming that either $\xi > 0$ or $\xi < 0$, the joint distribution of ξ, μ_* and σ_*, and thus the marginal fiducial distributions of these parameters, can be determined using the Gibbs sampler algorithm as follows:

Algorithm 7: Gibbs sampler for the joint fiducial distribution of μ_*, σ_* and ξ

1. If $\xi > 0$ is assumed, choose an initial copy $\tilde{\xi}^{(0)} > 0$. If $\xi < 0$ is assumed, choose an initial copy $\tilde{\xi}^{(0)} < 0$.
2. For $i = 1, \dots, M$ do Steps 2(a) through 2(c) below:
   (a) Given the current copy $\tilde{\xi}^{(i-1)}$ of $\xi > 0$ or $\xi < 0$, draw the following random copies:
      i. a copy $\tilde{\sigma}_*^{(i)}$ of σ_* from the distribution of $\mathcal{R}_{\sigma_*|\xi=\tilde{\xi}^{(i-1)}}$, see equation (6.4.9) for both cases, $\xi > 0$, and $\xi < 0$.
      ii. a copy $\tilde{\mu}_*^{(i)}$ of μ_* from the distribution of $\mathcal{R}_{\mu_*|\xi=\tilde{\xi}^{(i-1)}}$, see equation (6.4.12) for both cases, $\xi > 0$, and $\xi < 0$.
   (b) Calculate $\tilde{\mu}^{(i)}$ of μ as $\tilde{\mu}^{(i)} = \tilde{\mu}^{*(i)} - \tilde{\sigma}^{*(i)} / \tilde{\xi}^{(i-1)}$, see equation (6.4.13) for both cases $\xi > 0$, and $\xi < 0$.
   (c) Given the current random copy $\tilde{\mu}^{(i)}$ of μ, draw a random copy $\tilde{\xi}^{(i)}$ of ξ from the distribution of $\mathcal{R}_{\xi|\mu=\tilde{\mu}^{(i)}}$, see equation (6.4.1) for $\xi > 0$, and equation (6.4.2) for $\xi < 0$.

The Monte-Carlo Markov chain of M draws $\left(\tilde{\mu}_*^{(i)}, \tilde{\sigma}_*^{(i)}, \tilde{\xi}^{(1)}\right), \dots, \left(\tilde{\mu}_*^{(M)}, \tilde{\sigma}_*^{(M)}, \tilde{\xi}^{(M)}\right)$ provides an estimate of the joint fiducial distribution of the parameters in the usual way.

Note: In Algorithm 7 either $\xi > 0$ or $\xi < 0$ is assumed, that is, in this form the methodology does not determine whether $\xi > 0$ or $\xi < 0$. The latter problem is not handled here and is the subject of future research.

6.4.6. Computational Problems

<u>**Simulation of the linear predictors μ_* and σ_* in the LSS family of distributions**</u>:

We note that FGCIs for $\mu_*, \mu, \sigma_*, \sigma, \xi$ and η of GEV, GP and three-parameter Weibull distributions are calculated using a Gibbs sampler through the simulation of CFGPQs for these parameters (and for functions of these parameters such as quantiles) as are defined in Sections 6.4.1 through 6.4.4 above. Furthermore, we note that the calculation of CFGPQs requires the calculation of the H_ξ matrix for the eventual calculation of the linear predictors for μ_* and σ_*. To this end, for GEV, GP and three-parameter Weibull distributions we simulate the H_ξ matrix for a grid of ξ values, assuming $\xi > 0$. For $\xi > 0$, the lower and upper limits were as chose as presented in the table below.

Table 6.4.1: Grid of ξ ($\xi > 0$) values for simulating H_ξ matrix for GEV, GP and three-parameter Weibull distributions

Distribution	Lower limit	Increment	Upper limit
Generalized Extreme Value	0.1	0.01	3.5
Generalized Pareto	0.01	0.01	2.5
Three-parameter Weibull	0.1	0.01	3.5

We note that in our experience using a lower or higher limit of the grid for ξ makes, in general, no practical difference with respect to the calculated FGCIs for the model parameters (as long as the true value of ξ can be assumed to lie well within the interval in question).

6.5. Fiducial Generalized Inference for Location-Scale-Shape Family of Distributions

We present in Section 6.5.1 below a brief literature review on fiducial generalized inference for the location, scale and shape parameters or a function of a parameter of the LSS distributions. Furthermore, the general principle of how to obtain fiducial generalized confidence intervals for the location, scale and shape parameters or a function of a parameter of the LSS distributions is described in Section 6.5.2.

6.5.1. Literature Review

We have noted above that considerable work on estimation of parameters in the LSS families of distributions using various methods such as maximum likelihood, method of moments, probability-weighted-moments, least squares, and likelihood moment, among others, has been done in the literature. However, thus far there appears to be limited literature on hypotheses tests and generalized confidence intervals for the location, scale and shape parameters of LSS families of distributions, especially using fiducial methods. Among the authors, Challenor and Carter (1983) derived a two-sided hypothesis test for testing whether the shape parameter of GEV distribution is equal to zero based on likelihood ratio methods. Alternatively, Challenor and Carter (1983) fitted a Gumbel distribution to the data. Similarly, Hosking (1984) proposed a modification to the likelihood test of Challenor and Carter (1983) and applied it to one-sided and two-sided hypothesis testing of the shape parameter of GEV distribution. Huang et al. (2013) derived a generalized likelihood test for the shape parameter of GEV distribution. Wandler and Hannig (2012) proposed fiducial generalized confidence intervals, using fiducial probability density functions of the parameters of high quantiles of the GP distribution, of the parameter and quantiles in the case when the threshold parameter is either known, or unknown and needs to be chosen. The performance of proposed methods was evaluated using Monte Carlo simulations and real life data example and compared with results based on Bayesian methods.

6.5.2. Fiducial Generalized Confidence Intervals for the Parameters of Location-Scale-Shape Distributions

Fiducial generalized confidence intervals (FGCIs) for the location, scale and shape parameters or a function of a parameter in LSS family of distributions can be obtained by calculating the FGCIs using a Gibbs sampler through the simulation of CFGPQs defined in Sections 6.4.1 through 6.4.4 above.

Summary of the simulation of FGCIs of model parameters and quantiles LSS distribution:

We present below a summary of the simulation of fiducial generalized confidence intervals for $\sigma^*, \sigma, \mu^*, \mu, \xi$ and η of the Generalized Extreme Value, Generalized Pareto and three-parameter Weibull distributions. As presented above, we note again here that the methods for calculating the FGCIs for the parameters and quantiles of the distribution can in principle be generally applied to any LSS family of distribution. When applying the general method to various distributions (LSS families), in principle, the only adaptation necessary is to specify the relevant standard distribution of the standard variate Z_* (see equation 6.1.4), from which we simulate the estimates of the H and H_ξ matrices needed in the calculation of the various CFGPQs.

- For the GEV distribution, the estimates of the H and H_ξ matrices are simulated using the standard variate Z_* from the standard Exponential distribution.
- For GP distribution, the estimates of H and H_ξ matrices are simulated using the standard variate Z_* from the standard Uniform distribution.
- For the three-parameter Weibull distribution, the estimates of the H and H_ξ matrices are simulated using the standard variate Z_* which is the inverse of a variate from the standard Exponential distribution.

Since the matrix H_ξ depends on ξ, it is simulated using a grid of ξ values, and we chose a lower limit of 0.1 (GEV and three-parameter Weibull distributions) and 0.01 (GP distribution) for the grid in the simulation studies described in Section 6.6 below. When the lower limit for the grid is chosen too low, the simulation trace for ξ can collapse and converge to the minimum value.

6.6. Simulation Studies: Location-Scale-Shape Family of Distributions

In Sections 6.6.1, 6.6.2 and 6.6.3 we present the simulation studies for calculating empirical coverages and average lengths of fiducial generalized confidence intervals for model parameters and quantiles of Generalized Extreme Value, Generalized Pareto and three-parameter Weibull distributions, respectively. The simulation studies are based on CFGPQs for $\sigma, \sigma^*, \mu, \mu^*, \xi$ and η using θ (theta) (see equations (6.1.2)/(6.1.5)) and θ^* (theta star) (see equations (6.1.1)/(6.1.6)) parametrizations and selected values of ξ.

6.6.1. Coverage Probabilities and Average Lengths of FGCIs for Model Parameters and Quantiles of Generalized Extreme Value Distribution

We present in this section the simulation study of calculating the coverage probabilities and average lengths of FGCIs for model parameters and quantiles of distribution for the Generalized Extreme Value distribution.

6.6.1.1. Objectives of simulation study

The objectives of this simulation study are:

- To evaluate, using rank-based fiducial generalized methods applied to the GEV distribution, the performance of Gibbs sampler.

- To evaluate and compare the empirical coverages and average lengths of FGCIs for model parameters and quantiles of the distribution obtained using θ parametrization with those based on θ^* parametrization.

6.6.1.2. Design of simulation study

We have written a number of programs to calculate the empirical coverages and average lengths of FGCIs for model parameters and 0.975 quantile of GEV distribution. For both parametrizations, the empirical coverages and average lengths of FGCIs were calculated for sample size $n = 50$ using the input parameters $\sigma = 1$ and $\mu = 0$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution, and specified values of positive shape parameter $\xi = 0.2, 0.25, 0.33, 0.5$, and 1. Furthermore, the simulation study was repeated for sample size $n = 25$. Schall and Ring (2011, Section 3.2) have shown that the simulation study of empirical coverage of 90% confidence interval for a model parameters or contrasts based on 4000 simulations produces a simulation standard error of the coverage probability of 0.005. Asimulation standard error of the coverage probability of 0.5% is very small, and thus it serves as the justification for the choice of using 4000 simulated samples.

6.6.1.3. Results of simulation study

We compare in this section the simulation results of 95% FGCIs for model parameters and 0.975 quantile of GEV distribution obtained using θ parametrization with those based on θ^* parametrization for $n = 50$ and $\xi = 0.2, 0.25, 0.33, 0.5,$ and 1 (see Tables 6.6.1.1 through 6.6.1.5 below). Full simulation results of 90%, 95% and 99% FGCIs for model parameters and 0.975 quantile of GEV distribution based on θ^* parametrization for sample sizes $n = 25$ and $n = 50$ are presented in Appendix F1 (Tables F1.1 through F1.5) below.

Table 6.6.1.1: Generalized Extreme Value distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.2$; $\eta_{0.975} = 2.0860$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0	0.9822
	AVL	0.1099	1.6588
σ^*	CP	0.0140	0.9593
	AVL	1.6541	0.1105
μ	CP	0	0.9832
	AVL	0.1253	1.6196
μ^*	CP	0	0.9543
	AVL	1.6149	0.1259
ξ	CP	0.9838	0.9818
	AVL	0.3547	0.3553
$\eta_{0.975}$	CP	0.9587	0.9410
	AVL	1.3851	1.2938

Table 6.6.1.2: Generalized Extreme Value distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.25$; $\eta_{0.975} = 2.5069$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0	0.9735
	AVL	0.1428	1.9958
σ^*	CP	0.0235	0.9540
	AVL	1.9868	0.1418
μ	CP	0	0.9762
	AVL	0.1597	1.9458
μ^*	CP	0	0.9453
	AVL	1.9358	0.1592
ξ	CP	0.9732	0.9795
	AVL	0.3974	0.3947
$\eta_{0.975}$	CP	0.9530	0.9430
	AVL	2.1465	1.9017

Table 6.6.1.3: Generalized Extreme Value distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.33$; $\eta_{0.975} = 3.3641$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

		Parametrization	
Parameter	Quantity	Theta	Theta star
σ	CP	0	0.9490
	AVL	0.1977	2.3477
σ^*	CP	0.0558	0.9490
	AVL	2.3462	0.1982
μ	CP	0	0.9483
	AVL	0.2135	2.2739
μ^*	CP	0	0.9493
	AVL	2.2729	0.2138
ξ	CP	0.9570	0.9503
	AVL	0.4621	0.4608
$\eta_{0.975}$	CP	0.9497	0.9263
	AVL	3.8334	3.4631

Table 6.6.1.4: Generalized Extreme Value distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.5$; $\eta_{0.975} = 6.2847$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0.0185	0.9343
	AVL	0.3392	2.3289
σ^*	CP	0.2488	0.9537
	AVL	2.3264	0.3402
μ	CP	0	0.9357
	AVL	0.3303	2.1883
μ^*	CP	0	0.9440
	AVL	2.1881	0.3318
ξ	CP	0.9275	0.9380
	AVL	0.5655	0.5673
$\eta_{0.975}$	CP	0.9240	0.9077
	AVL	11.7493	10.3809

Table 6.6.1.5: Generalized Extreme Value distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 1$; $\eta_{0.975} = 39.4979$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0.9575	0.9420
	AVL	0.9997	1.8769
σ^*	CP	0.9483	0.9530
	AVL	1.3253	1.0453
μ	CP	0	0.9343
	AVL	0.6900	1.4070
μ^*	CP	0	0.9503
	AVL	0.8710	0.7161
ξ	CP	0.9430	0.9393
	AVL	0.7684	0.7681
$\eta_{0.975}$	CP	0.9345	0.9050
	AVL	207.7658	180.5037

6.6.1.4. Discussion of results of simulation study for GEV distribution

Based on the results of simulation study presented in Tables 6.6.1.1 through 6.6.1.5 above, we conclude the following:

- Overall, the results of simulation study suggest that the Gibbs sampler algorithm using rank-based CFGPQs produces FGCIs with good coverage when the θ^* parametrization is applied compared to θ parametrization. Therefore, the θ^* parametrization is preferred.
- The coverage probabilities for the parameters μ^* and σ^* are close to the nominal 95%, but for the quantiles η the observed coverage is generally smaller than 95% and tends to be about 91% when ξ is close to 1. However, for smaller values of ξ, for example $\xi < 0.33$, the coverage of the guantiles η is satisfactory.
- However, the Gibbs sampler algorithm using rank-based CFGPQs does not work well in the case of GEV distribution for $\xi > 1$ because the draws from the conditional fiducial distributions of the various parameters become very unstable. To illustrate this instability, as an example, we have produced a plot of the trace for the FGPQ for the quantile $\eta_{0.975}$ using θ^* parametrization for the parameter combination $\sigma = 1,\ \mu = 0,\ \xi = 2$ and sample size $n = 50$. See Figure 6.6.1 below. For $\xi = 2$, the value of $\eta_{0.975}$ is 1560.1 which is very far away from the median estimate of 0.975 quantile of 17784 (median of the simulated fiducial distribution).
- In contrast, for $\xi < 1$, for example $\xi = 0.25$, the draws from the conditional fiducial distributions of the various parameters become generally stable and the trace for the quantile η looks better (see Figure 6.6.2 below).

Figure 6.6.1: Generalized Extreme Value distribution: Distribution of draws of $\eta_{0.975}$ quantile ($n = 50$; $S = 1000$ draws from the fiducial distribution; true $\eta_{0.975} = 1560.1$; $\sigma = 1$; $\mu = 0$; $\xi = 2$; median = 17784)

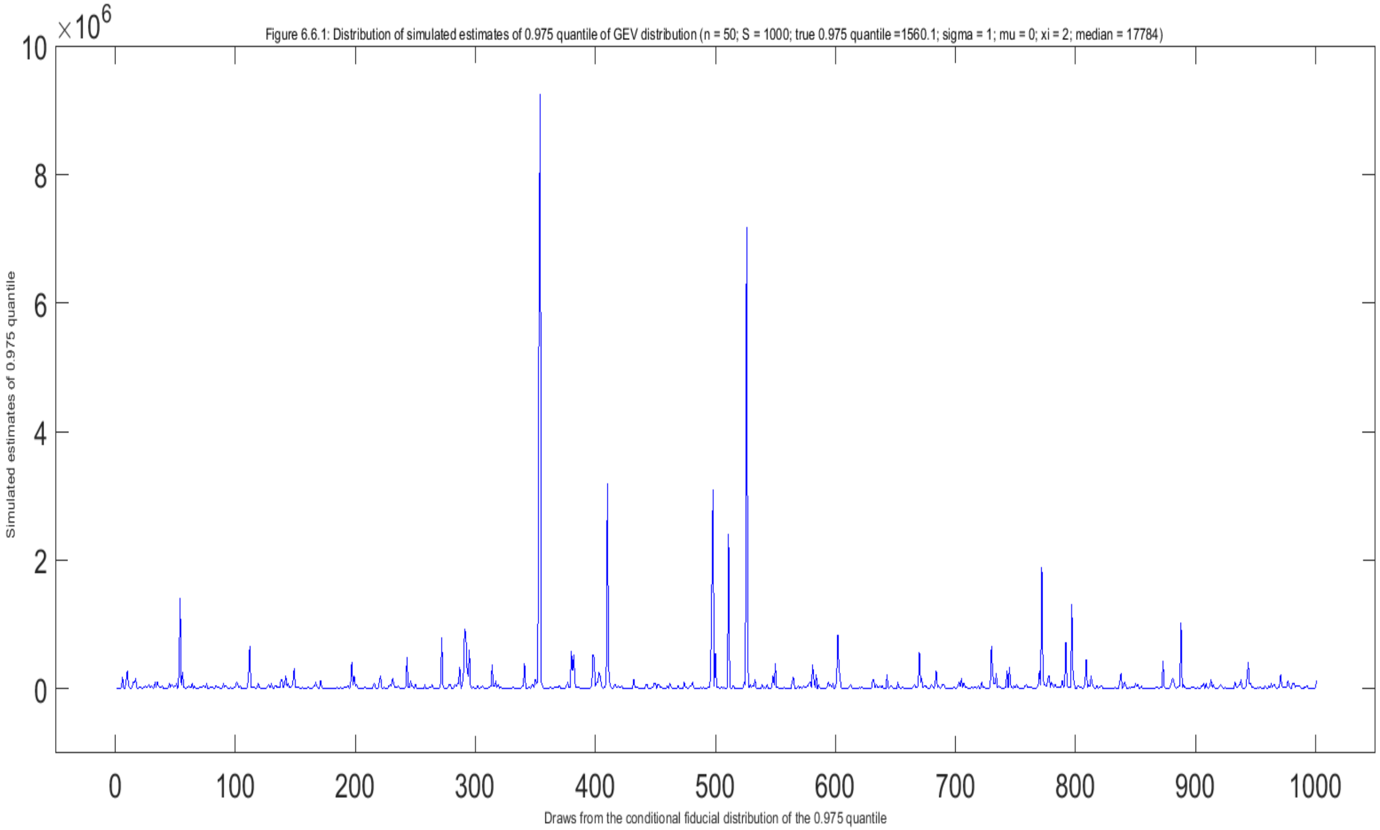
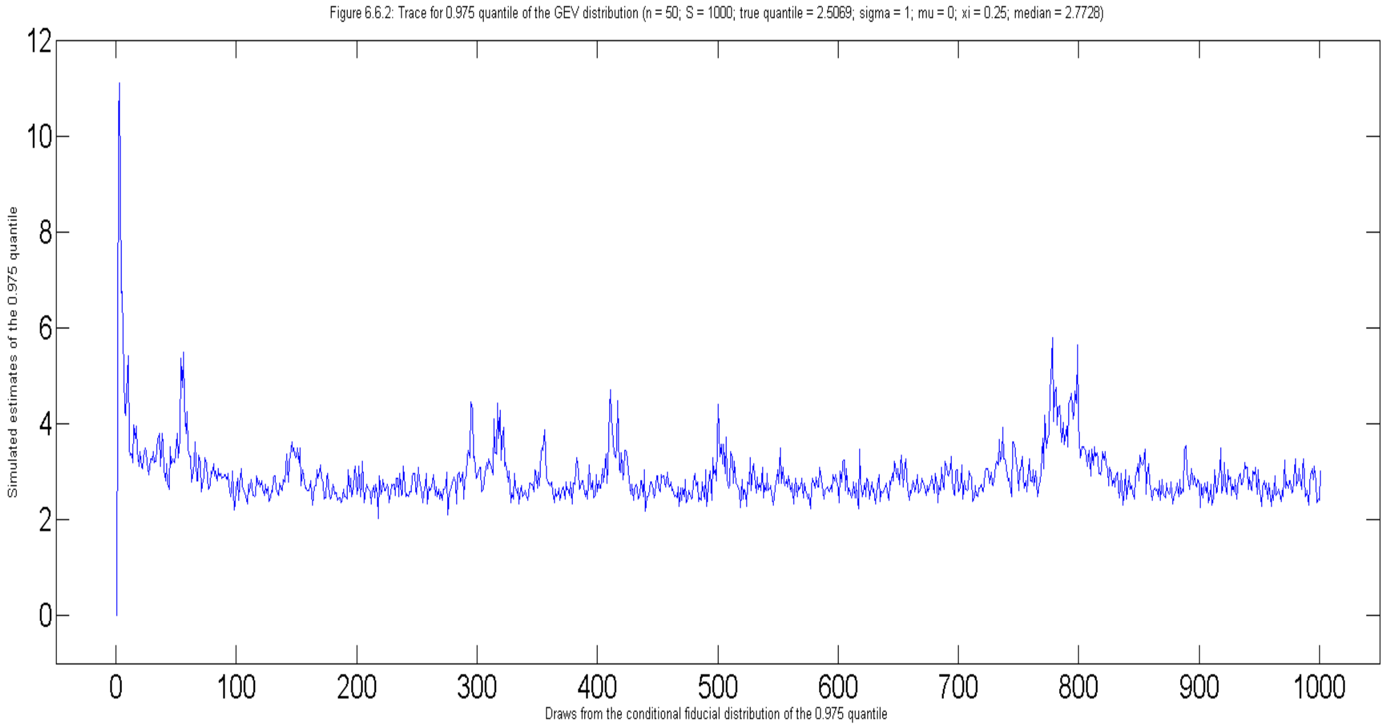


Figure 6.6.2: Generalized Extreme Value distribution: Distribution of draws of $\eta_{0.975}$ quantile ($n = 50$; $S = 1000$ draws from the fiducial distribution; $\sigma = 1$; $\mu = 0$; $\xi = 0.25$; median = 2.7728)



6.6.2. Coverage Probabilities and Average Lengths of FGCIs for Model Parameters and Quantiles of Generalized Pareto Distribution

We present in this section the simulation study of calculating the coverage probabilities and average lengths of FGCIs for model parameters and quantiles of distribution for the Generalized Pareto distribution.

6.6.2.1. Objectives of simulation study

The objectives of this simulation study are:

- To evaluate, using rank-based fiducial generalized methods applied to the GP distribution, the performance of Gibbs sampler.
- To compare the empirical coverages and average lengths of FGCIs for model parameters and quantiles of GP distribution obtained using θ parametrization with those based on θ^* parametrization.

6.6.2.2. Design of simulation study

We have written a number of programs to calculate the empirical coverages and average lengths of FGCIs for model parameters and 0.975 quantile of GP distribution. For both parametrizations, the empirical coverages and average lengths of FGCIs were calculated for sample size $n = 50$ using the input parameters $\sigma = 1$ and $\mu = 0$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution, and specified values of positive shape parameter $\xi = 0.05, 0.1, 0.2, 0.25, 0.33, 0.5$ and 1. Furthermore, the simulation study was repeated for sample size $n = 25$.

6.6.2.3. Results of simulation study

We compare in this section the simulation results of 95% FGCIs for model parameters and 0.975 quantile of GP distribution obtained using θ

parametrization with those based on θ^* parametrization for $n = 50$ and $\xi = 0.05$ and 0.1 (see Tables 6.6.2.1 and 6.6.2.2 below). Full simulation results of 90%, 95% and 99% FGCIs for model parameters and 0.975 quantile of GP distribution based on θ^* parametrization for sample sizes $n = 25$ and $n = 50$ are presented in Appendix F2 (see Tables F2.1 through F2.3 below).

Table 6.6.2.1: Generalized Pareto distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.05$; $\eta_{0.975} = 1.2025$; $S = 4000$ simulated samples of data; and $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0	0.9785
	AVL	0.0338	4.8069
σ^*	CP	0.1187	0.9537
	AVL	4.7778	0.0337
μ	CP	0	0.9788
	AVL	0.0037	4.8347
μ^*	CP	0	0.9465
	AVL	4.8068	0.0037
ξ	CP	0.9728	0.9788
	AVL	0.3823	0.3756
$\eta_{0.975}$	CP	0.9623	0.9360
	AVL	0.2297	0.1999

Table 6.6.2.2: Generalized Pareto distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.1$; $\eta_{0.975} = 1.4461$; $S = 4000$ simulated samples of data; and $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0	0.9802
	AVL	0.0719	9.4858
σ^*	CP	0.1543	0.9605
	AVL	9.5148	0.0721
μ	CP	0	0.9800
	AVL	0.0076	9.5492
μ^*	CP	0	0.9477
	AVL	9.5773	0.0076
ξ	CP	0.9830	0.9798
	AVL	0.4200	0.4274
$\eta_{0.975}$	CP	0.9633	0.9413
	AVL	0.5421	0.4829

6.6.2.4. Discussion of results of simulation study for GP distribution

Based on the results of simulation study, we conclude the following:

- The results of simulation study (see Tables 6.6.2.1 and 6.6.2.2 above) show that the Gibbs sampler algorithm using rank-based CFGPQs does not work in the case of GP distribution when the θ parametrization is applied. The draws from the conditional fiducial distributions of the various model parameters are very unstable and the coverage probabilities of GCIs are generally close to zero.
    - For illustration, as an example, we have produced a plot of the trace for the draws of the FGPQ for the parameter σ using θ parametrization for the parameter combination $\sigma = 1$, $\mu = 0$, $\xi = 0.15$ and sample size $n = 50$. See Figure 6.6.3 below.
- Furthermore, the results of simulation study suggest that the Gibbs sampler algorithm using rank-based CFGPQs produces FGCIs with, overall, good coverage when θ^* parametrization is applied in the case of smaller values of ξ. That is, for ξ values smaller than or equal to 0.2. Similarly to the case of GEV distribution, the coverage probabilities for the parameters μ^* and σ^* are close to the nominal 95%, but for the quantiles η the observed coverage is generally smaller than 95% but tends to be satisfactory, say about 94%, when ξ is close to 0.1.
- The full results of empirical coverages and average lengths of FGCIs for model parameters and quantiles of GP distribution in the case of θ^* parametrization for $\xi = 0.05, 0.1$ and 0.2 are presented in Appendix F2 (see Tables F2.1 through F2.3).
    - Figure 6.6.4 below illustrates that for $\xi > 0.2$, the draws from the conditional fiducial distributions of various parameters of GP distribution become very unstable for θ^* parametrization. This result is perhaps not surprising given the fact that the GP

distribution is a very heavy tailed distribution when the extreme value index is above 0.2.

Figure 6.6.3: Generalized Pareto distribution: Distribution of the draws of σ ($n = 50$; $S = 1000$ draws from the fiducial distribution; $\sigma = 1$; $\mu = 0$; $\xi = 0.15$; median = 0.3755)

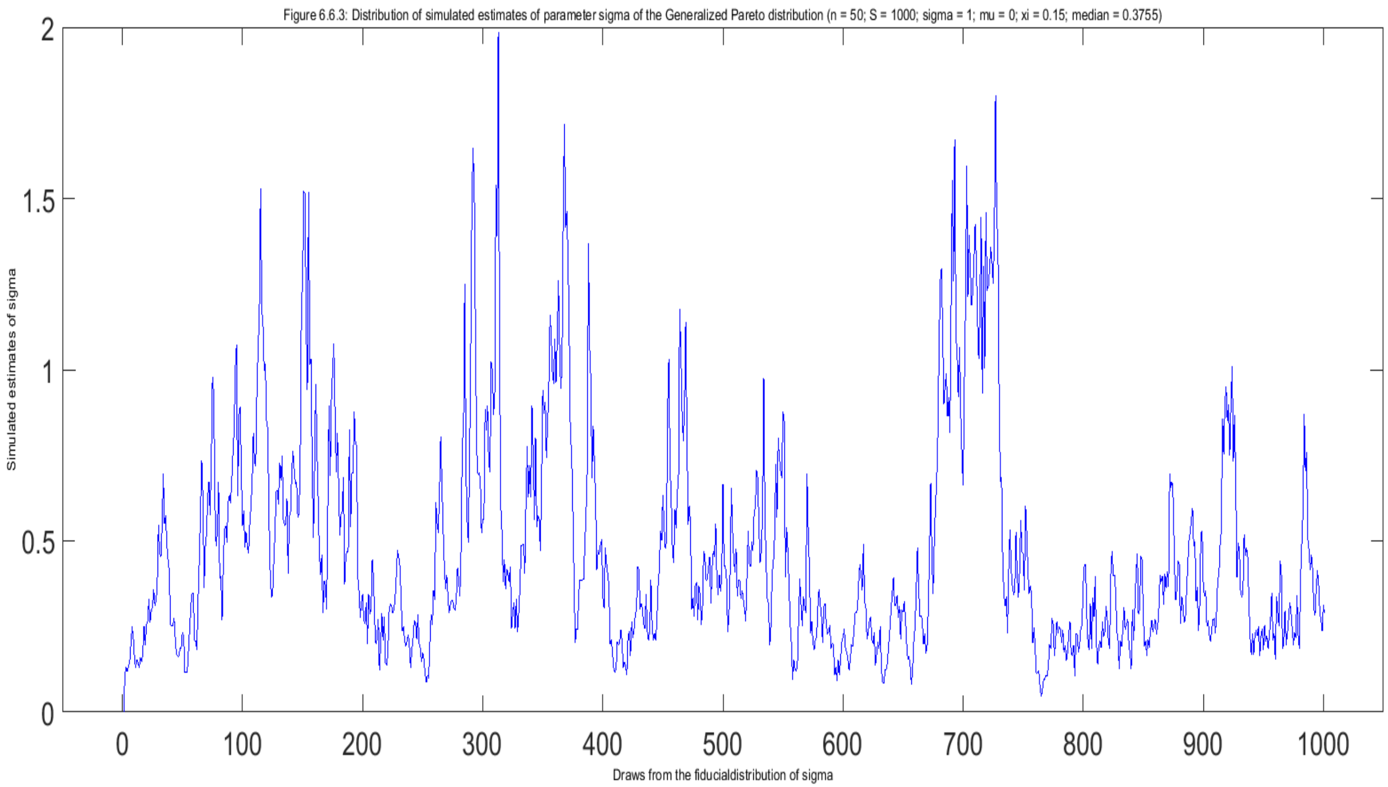
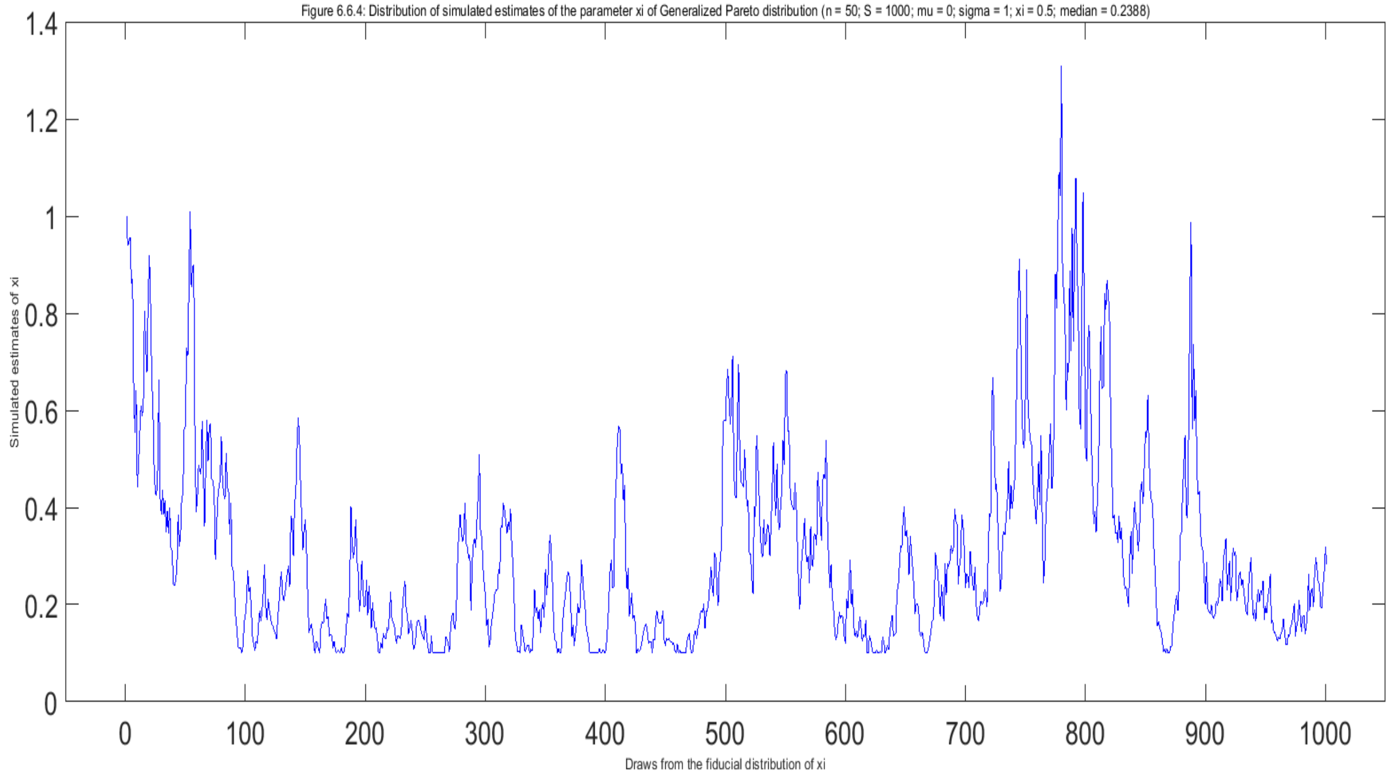


Figure 6.6.4: Generalized Pareto distribution: Distribution of the draws of ξ ($n = 50$; $S = 1000$ draws from the fiducial distribution; $\sigma = 1$; $\mu = 0$; $\xi = 0.5$; median = 0.2388)



6.6.3. Coverage Probabilities and Average Lengths of FGCIs for Model Parameters and Quantiles of Three-Parameter Weibull Distribution

We present in this section the simulation study of calculating the coverage probabilities and average lengths of FGCIs for model parameters and quantiles of distribution for the three-parameter Weibull distribution.

6.6.3.1. Objectives of simulation study

The objectives of this simulation study are:

- To evaluate, using rank-based fiducial generalized methods applied to three-parameter Weibull distribution, the performance of Gibbs sampler.
- To compare the empirical coverages and average lengths of FGCIs for model parameters and quantiles of three-parameter Weibull distribution obtained using θ parametrization with those based on θ^* parametrization.

6.6.3.2. Design of simulation study

We have written a number of programs to calculate the empirical coverages and average lengths of FGCIs for model parameters and 0.975 quantile of three-parameter Weibull distribution. For both parametrizations, the empirical coverages and average lengths of FGCIs were calculated for sample size $n = 50$ using the input parameters $\sigma = 1$ and $\mu = 0$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution, and specified values of positive shape parameter $\xi = 0.2, 0.25, 0.33, 0.5, 1$ and 2. Furthermore, the simulation study was repeated for sample size $n = 25$.

6.6.3.3. Results of simulation study

We compare in this section the simulation results of 95% FGCIs for model parameters and 0.975 quantile of three-parameter Weibull distribution obtained using θ parametrization with those based on θ^* parametrization for $n = 50$ and $\xi = 0.2, 0.25, 0.33, 0.5, 1$ and 2 (see Tables 6.6.3.1 through 6.6.3.6 below). Full simulation results of 90%, 95% and 99% FGCIs for model parameters and 0.975 quantile of three-parameter Weibull distribution, based on θ^* parametrization for sample sizes $n = 25$ and $n = 50$ are presented in Appendix F3 below (see Tables F3.1 through F3.6).

Table 6.6.3.1: Three-parameter Weibull distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.2$; $\eta_{0.975} = 1.2983$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

		Parametrization	
Parameter	Quantity	Theta	Theta star
σ	CP	0.9690	0.9720
	AVL	1.3294	1.3298
σ^*	CP	0.9643	0.9660
	AVL	0.0953	0.0953
μ	CP	0.9695	0.9710
	AVL	1.2973	1.2978
μ^*	CP	0.9595	0.9597
	AVL	0.1284	0.1252
ξ	CP	0.9705	0.9712
	AVL	0.2857	0.2881
$\eta_{0.975}$	CP	0.9860	0.9523
	AVL	0.2451	0.2029

Table 6.6.3.2: Three-parameter Weibull distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.25$; $\eta_{0.975} = 1.3859$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

		Parametrization	
Parameter	Quantity	Theta	Theta star
σ	CP	0.9587	0.9627
	AVL	1.5188	1.5198
σ^*	CP	0.9640	0.9650
	AVL	0.1183	0.1180
μ	CP	0.9577	0.9633
	AVL	1.4746	1.4754
μ^*	CP	0.9563	0.9540
	AVL	0.1581	0.1556
ξ	CP	0.9610	0.9648
	AVL	0.3189	0.3203
$\eta_{0.975}$	CP	0.9805	0.9497
	AVL	0.3221	0.2625

Table 6.6.3.3: Three-parameter Weibull distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.33$; $\eta_{0.975} = 1.5384$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

		Parametrization	
Parameter	Quantity	Theta	Theta star
σ	CP	0.9427	0.9433
	AVL	1.5688	1.5709
σ^*	CP	0.9575	0.9595
	AVL	0.1570	0.1572
μ	CP	0.9387	0.9445
	AVL	1.4975	1.5002
μ^*	CP	0.9530	0.9533
	AVL	0.2069	0.2051
ξ	CP	0.9370	0.9397
	AVL	0.3662	0.3686
$\eta_{0.975}$	CP	0.9760	0.9385
	AVL	0.4695	0.3775

Table 6.6.3.4: Three-parameter Weibull distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 0.5$; $\eta_{0.975} = 1.9206$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

		Parametrization	
Parameter	Quantity	Theta	Theta star
σ	CP	0.9410	0.9470
	AVL	0.9859	0.9940
σ^*	CP	0.9543	0.9533
	AVL	0.2481	0.2491
μ	CP	0.9445	0.9485
	AVL	0.8304	0.8392
μ^*	CP	0.9515	0.9517
	AVL	0.3094	0.3098
ξ	CP	0.9423	0.9475
	AVL	0.4114	0.4115
$\eta_{0.975}$	CP	0.9760	0.9367
	AVL	0.8877	0.6972

Table 6.6.3.5: Three-parameter Weibull distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 1$; $\eta_{0.975} = 3.6889$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

		Parametrization	
Parameter	Quantity	Theta	Theta star
σ	CP	0.9510	0.9493
	AVL	0.6449	0.6470
σ^*	CP	0.9455	0.9515
	AVL	0.6678	0.6709
μ	CP	0.9457	0.9497
	AVL	0.1161	0.1158
μ^*	CP	0.9495	0.9475
	AVL	0.6099	0.6124
ξ	CP	0.9587	0.9483
	AVL	0.5161	0.5165
$\eta_{0.975}$	CP	0.9780	0.9367
	AVL	3.4513	2.6667

Table 6.6.3.6: Three-parameter Weibull distribution: 95% CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs using theta and theta star parametrizations ($n = 50$; $\sigma = 1$; $\mu = 0$; $\xi = 2$; $\eta_{0.975} = 13.6078$; $S = 4000$ simulated samples of data; $J = 1000$ draws from the fiducial distribution)

Parameter	Quantity	Parametrization	
		Theta	Theta star
σ	CP	0.9527	0.9487
	AVL	1.3029	1.2991
σ^*	CP	0.9465	0.9483
	AVL	2.5703	2.5632
μ	CP	0.8472	0.8610
	AVL	0.0087	0.0087
μ^*	CP	0.9525	0.9485
	AVL	1.3016	1.2979
ξ	CP	0.9593	0.9543
	AVL	0.9268	0.9273
$\eta_{0.975}$	CP	0.9802	0.9545
	AVL	29.8563	23.5094

6.6.3.4. Discussion of simulation results of three-parameter Weibull distribution

Based on the results of simulation study presented in Tables 6.6.3.1 through 6.6.3.6 above, we conclude the following:

- Overall, the results of simulation study suggest that the Gibbs sampler algorithm using rank-based CFGPQs produces FGCIs with good coverage when θ^* parametrization is applied compared to θ parametrization. Therefore, as is the case with GEV and GP distributions, the θ^* parametrization is preferred.
- The coverage probability of both model parameters and quantiles is satisfactory, namely at least 94% (when rounded to full percent) in all cases.

6.7. Illustrative Example: GEV Distribution

In this section, we evaluate the performance of the proposed fiducial rank-based methods of inference using the real life data example. We apply our proposed rank-based methods of inference to an example of data set of extreme values observed from an environmental context. The data set represent wind-speed measurements recorded at different locations in Cape Town, South Africa, namely the harbour, airport and Robben Island. The monthly maximal wind gust measurement (in miles per hour) was recorded for 70 months at each location. This data set was also analysed by Beirlant et al. (2004, pp. 452-459) to illustrate the application of Bayesian methods in extreme value theory. In the context of our parametrization, the monthly maximal wind speed measurements from each location follow a Generalized Extreme Value distribution with the location parameter μ_*, scale parameter σ_* and shape parameter ξ.

For each location, we used the Gibbs sampler to calculate point estimates and 95% fiducial generalized confidence intervals for the model parameters and quantiles, based on the distribution of the simulated FGPQs for μ_*, σ_*, ξ and η (see Tables 6.7.1 through 6.7.3 below). The point estimates of the model parameters and quantiles were calculated as the averages of the simulated draws ($J = 10000$) from the fiducial distribution of FGPQs for model parameters and quantiles. We note that the calculations of the FGPQ for σ_*(see equation 6.4.9 above) and FGPQ for μ_* (see equation 6.4.12 above) depend on ξ, namely the linear predictors of μ_* and σ_* (that is, matrix H_ξ). Therefore, for this specific illustrative example, the H_ξ matrix based on sample size $n = 70$ and $I = 100000$ simulated samples from the standard Exponential distribution was simulated for the GEV distribution using the grid for ξ values with the lower limit of 0.01, with an increment of 0.01, and upper limit of 1.5. In our simulations from the fiducial distributions, the initial value chosen for the shape parameter ξ is 0.135≈0.14 which is the same as the one used by Beirlant at el. (2004, p. 454). Similarly to the results of illustrative example by Beirlant at el. (2004, Figures 11.12 through 11.14), we have also shown the plots of traces of 10000 simulated draws from the distribution of FGPQs for μ_*, σ_*, ξ and $\eta_{0.975}$ for the three locations (see Figures 6.7.1 through 6.7.12 below). Furthermore, for the three locations we presented the plots of the median quantile and 95% fiducial generalized confidence region (FGCR) for η at selected values of failure probability p, as a function of $1 - p$ (see Figures 6.7.13 through 6.7.15 below).

Table 6.7.1: Generalized Extreme Value distribution: Point estimates and 95% fiducial generalized confidence intervals for model parameters and quantile of the distribution based on four FGPQs using the Cape Town Harbour data ($n = 70$; $J = 10000$ draws from the fiducial distribution; lower limit of the grid for $\xi = 0.01$)

Parameter/Quantile	Point estimate	95% FGCI
σ_*	8.03 ($\sim$8)[6]	[6.39, 10.02]
μ_*	47.64 ($\sim$47.5)	[45.58, 49.90]
ξ	0.144 ($\sim$0.135)	[0.011, 0.354]
$\eta_{0.975}$	87.28	[76.26, 107.41]

[6] For Tables 6.7.1 through 6.7.3, the approximate point estimates for the parameters σ_*, μ_* and ξ obtained by Beirlant et al. (2004, pp. 452 - 457) are presented in round brackets.

Table 6.7.2: Generalized Extreme Value distribution: Point estimates and 95% fiducial generalized confidence intervals for model parameters and quantile of the distribution based on four FGPQs using the Cape Town Airport data ($n = 70$; $J = 10000$ draws from the fiducial distribution; lower limit of the grid for $\xi = 0.01$)

Parameter/Quantile	Point estimate	95% FGCI
σ_*	4.56 (~ 4.5)	[3.75, 5.55]
μ_*	45.23 (~ 45)	[44.10, 46.40]
ξ	0.051 (~ 0.135)	[0.010, 0.167]
$\eta_{0.975}$	63.73	[59.67, 69.58]

Table 6.7.3: Generalized Extreme Value distribution: Point estimates and 95% fiducial generalized confidence intervals for model parameters and quantile of the distribution based on four FGPQs using the Cape Town Robben Island data ($n = 70$; $J = 10000$ draws from the fiducial distribution; lower limit of the grid for $\xi = 0.01$)

Parameter/Quantile	Point estimate	95% FGCI
σ_*	4.47 (~ 4.5)	[3.69, 5.42]
μ_*	35.23 (~ 35)	[34.15, 36.37]
ξ	0.029 (~ 0.135)	[0.010, 0.108]
$\eta_{0.975}$	52.58	[48.95, 57.27]

Figure 6.7.1: Generalized Extreme Value distribution: Trace of σ_* ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town harbour data)

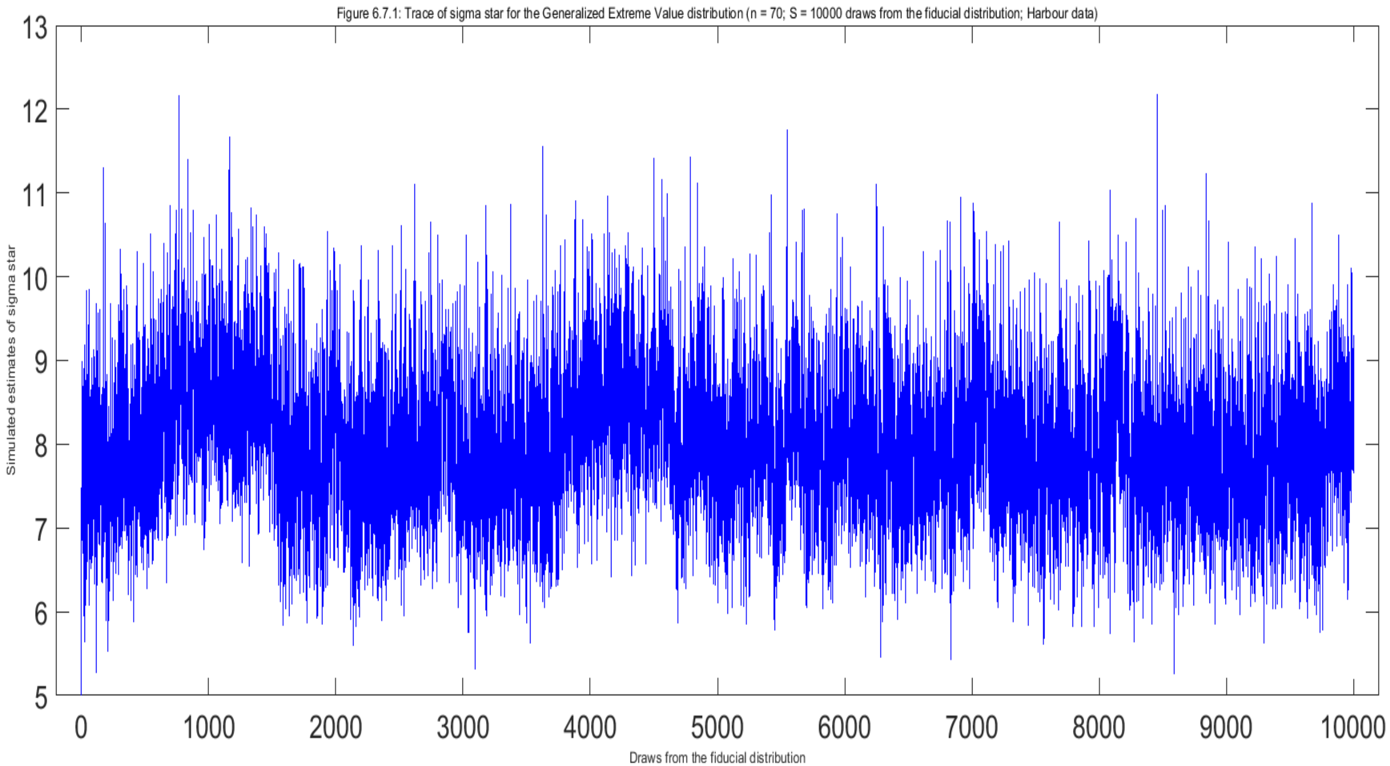


Figure 6.7.2: Generalized Extreme Value distribution: Trace of σ_* ***(***$n = 70; S = 10000$ ***draws from the fiducial distribution; Cape Town airport data)***

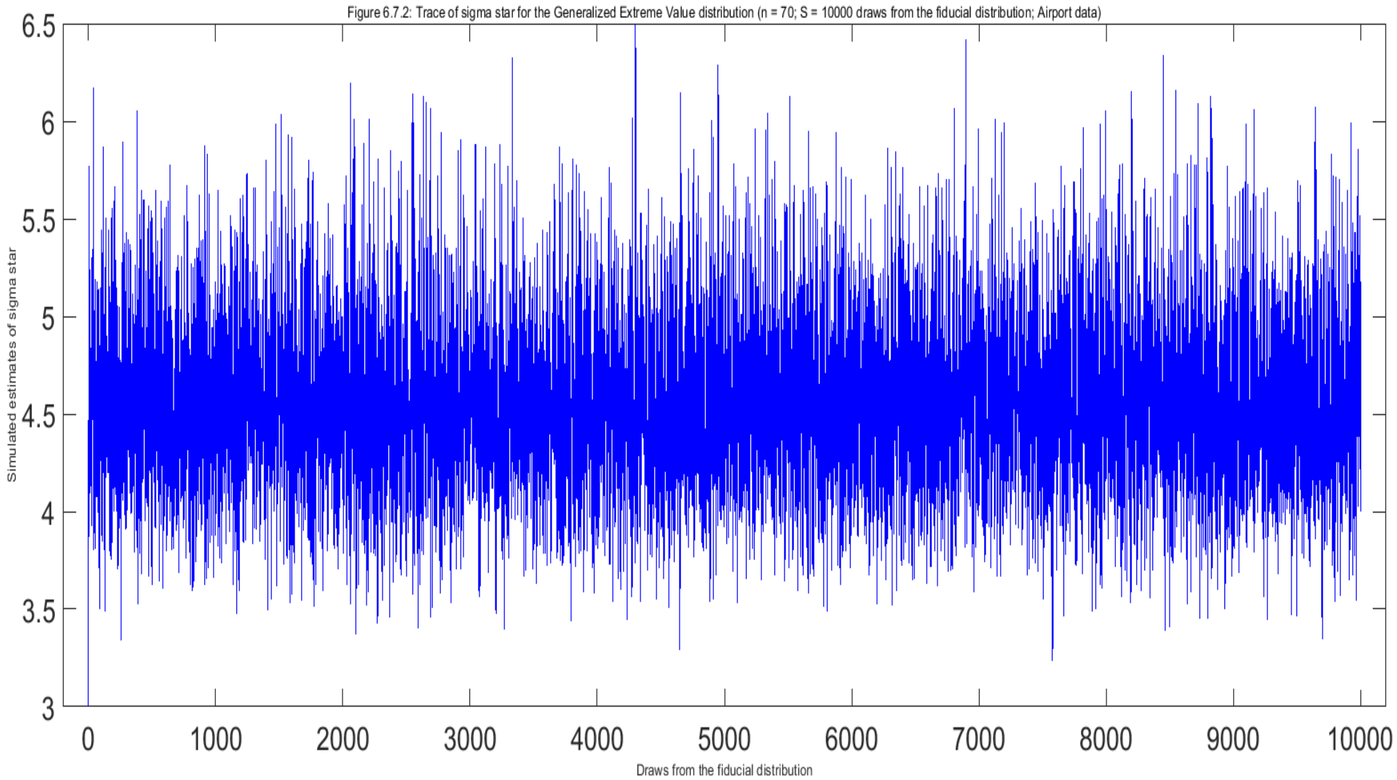


Figure 6.7.3: Generalized Extreme Value distribution: Trace of σ_* ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town Robben Island data)

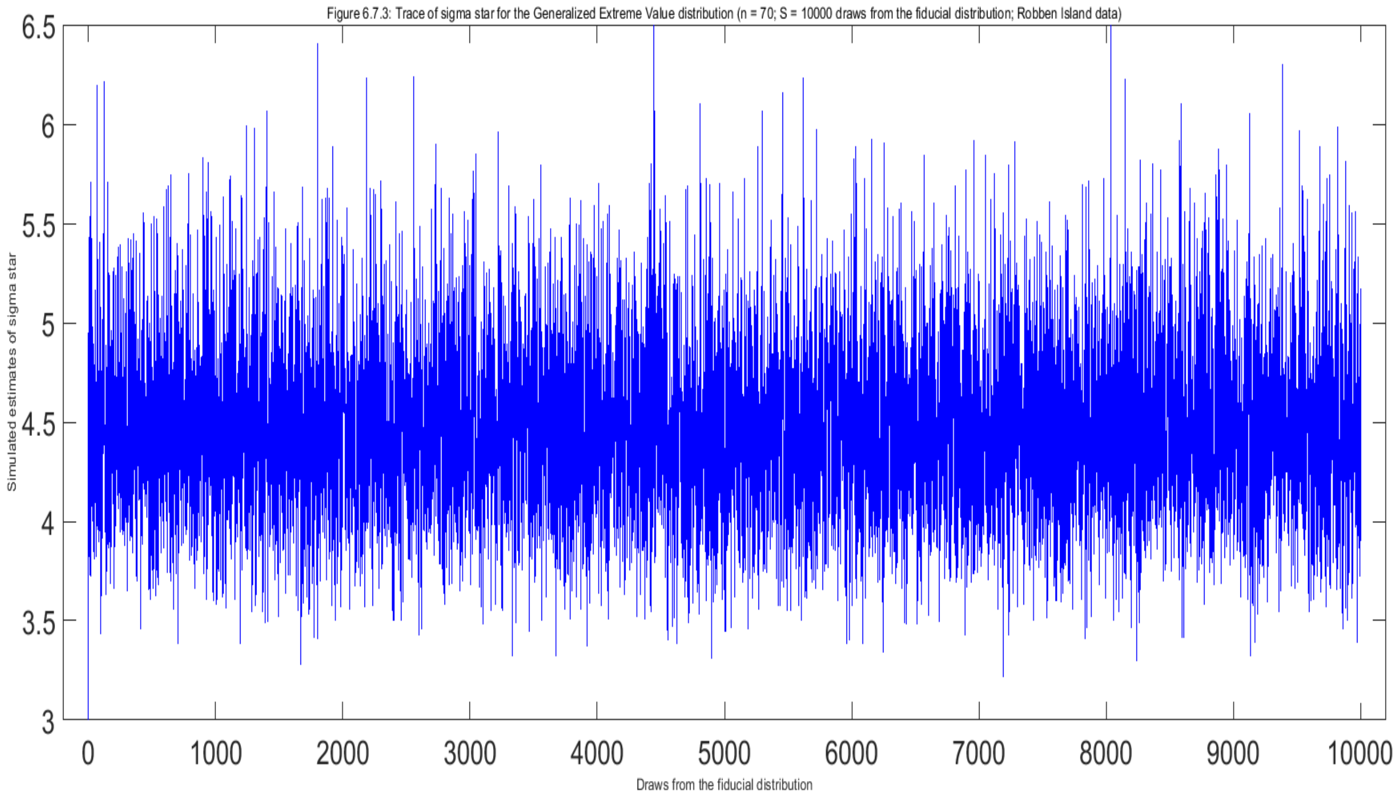


Figure 6.7.4: Generalized Extreme Value distribution: Trace of μ_* ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town harbour data)

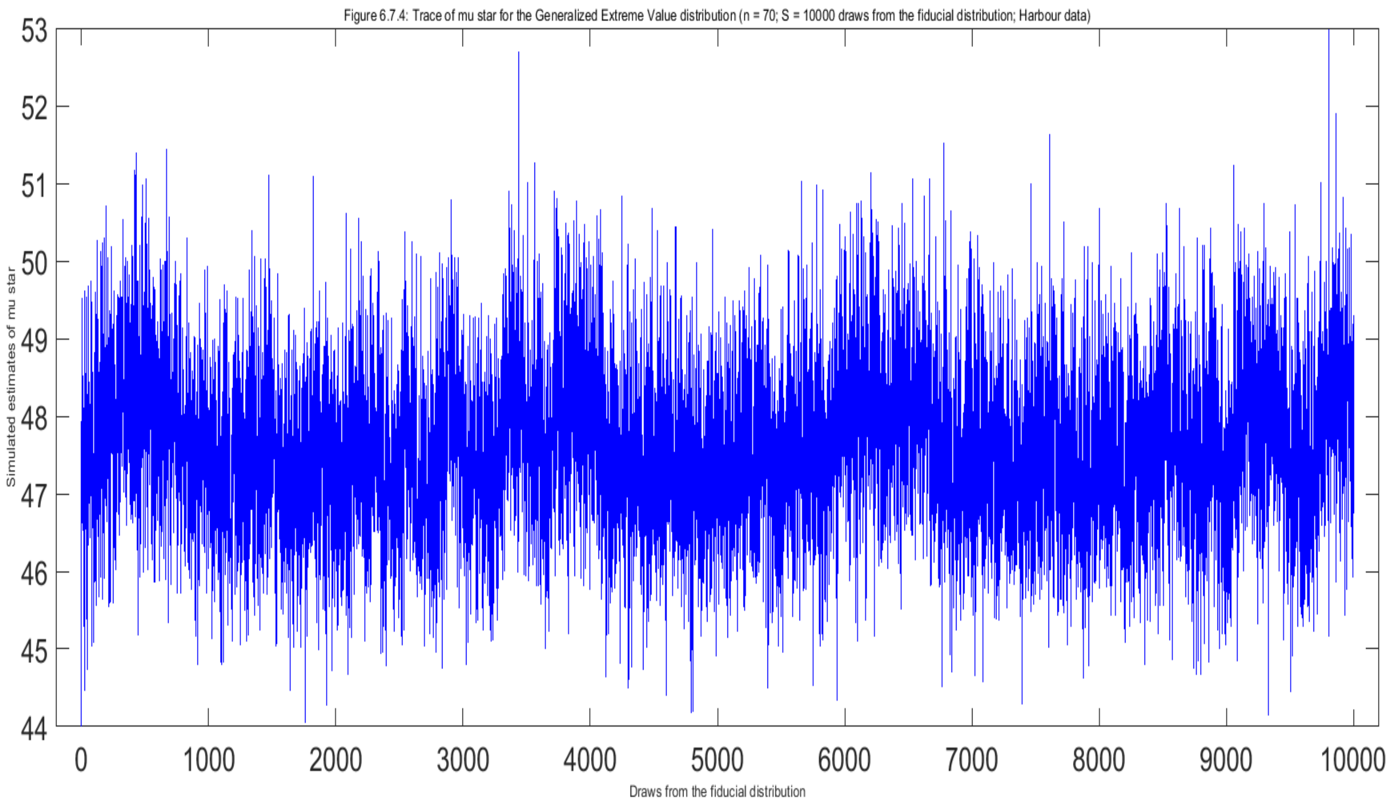


Figure 6.7.5: Generalized Extreme Value distribution: Trace of μ_* ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town airport data)

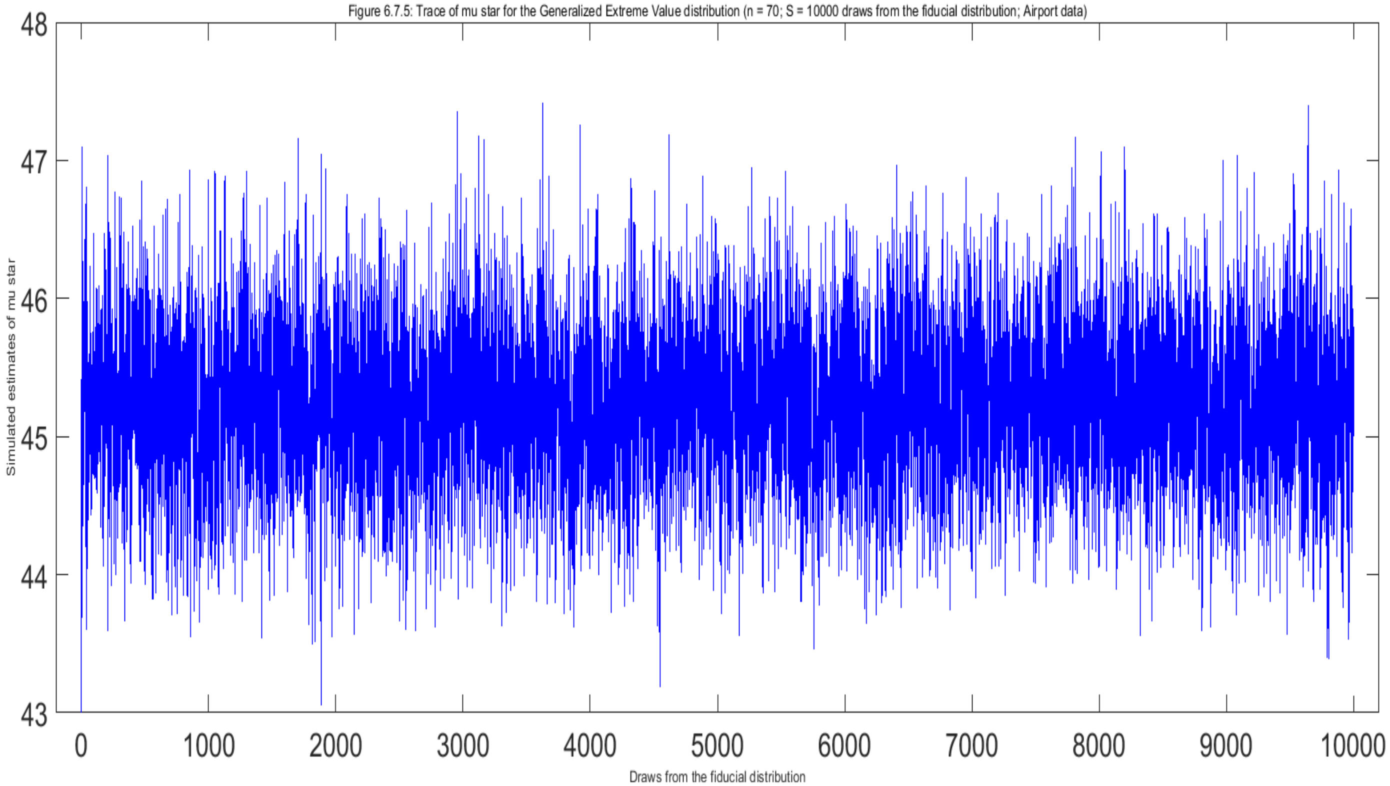


Figure 6.7.6: Generalized Extreme Value distribution: Trace of μ_* ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town Robben Island data)

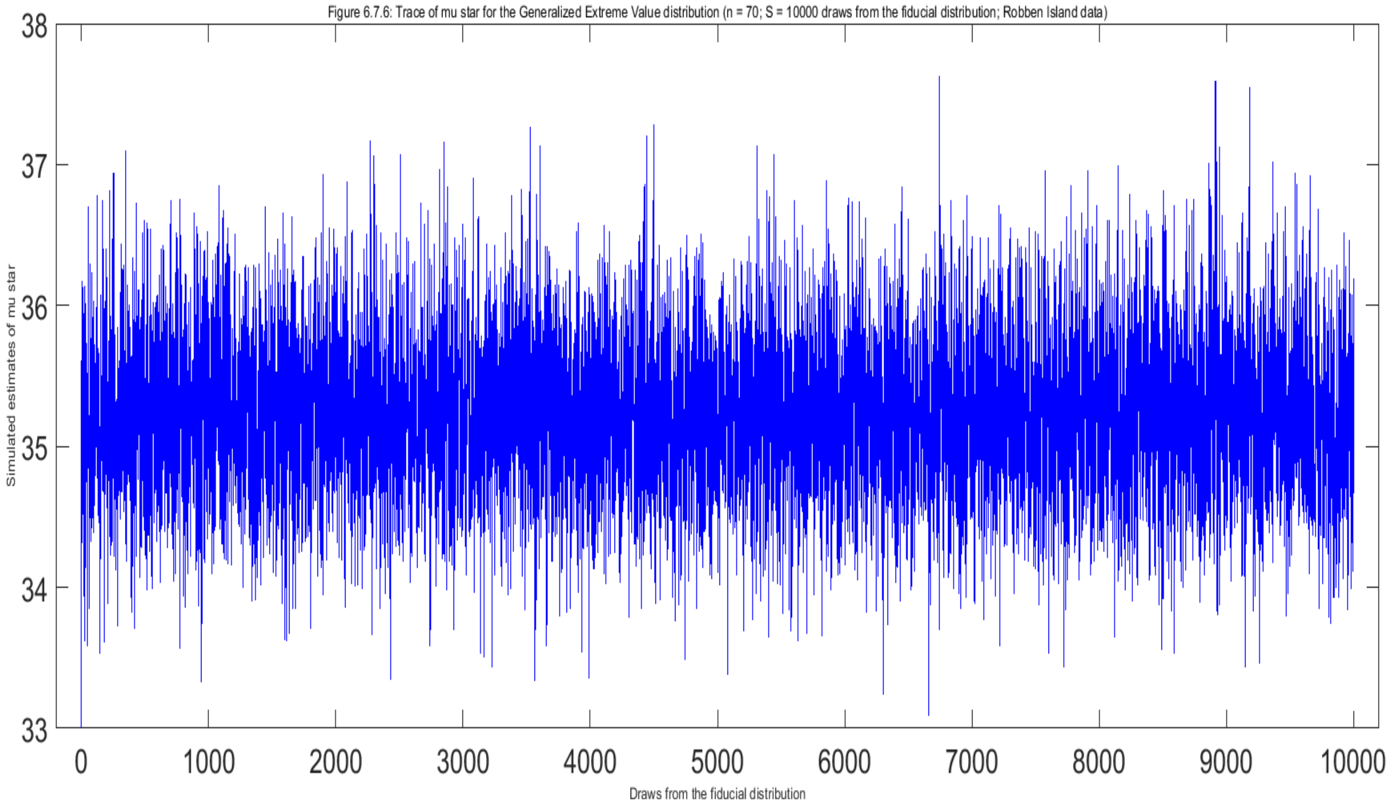


Figure 6.7.7: Generalized Extreme Value distribution: Trace of ξ ($n = 70$; $S = 10000$ draws from the fiducial distribution; Cape Town harbour data)

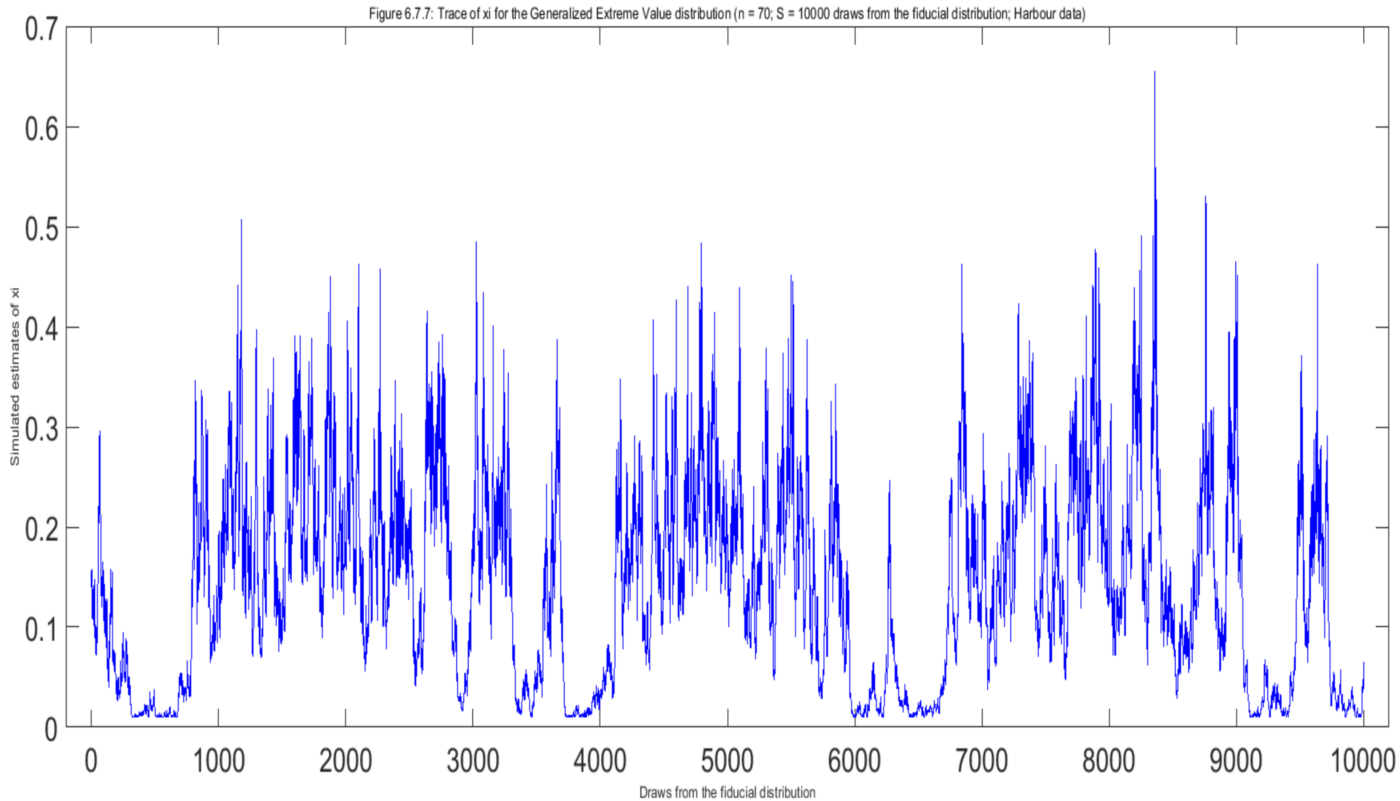


Figure 6.7.8: Generalized Extreme Value distribution: Trace of ξ ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town airport data)

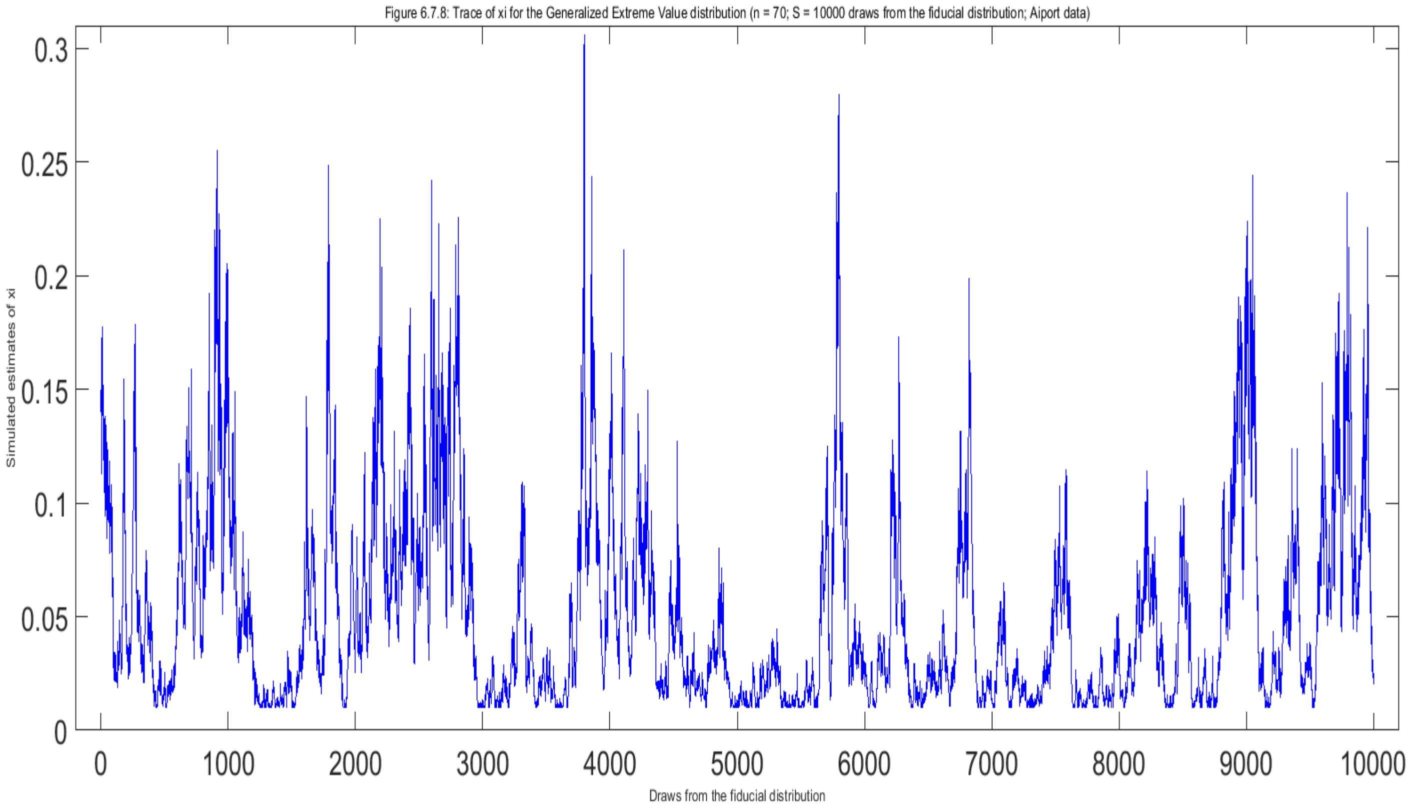


Figure 6.7.9: Generalized Extreme Value distribution: Trace of ξ ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town Robben Island data)

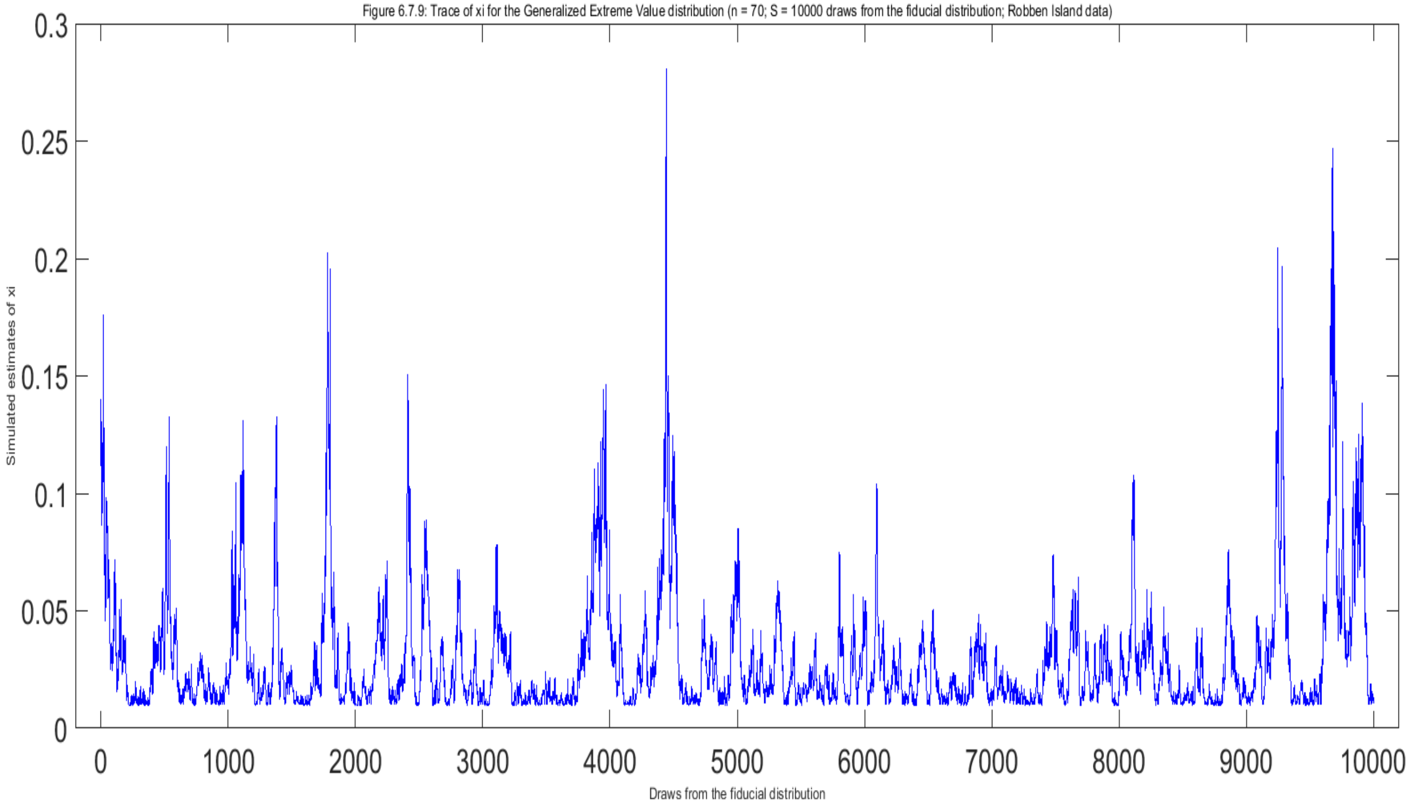


Figure 6.7.10: Generalized Extreme Value distribution: Trace of $\boldsymbol{\eta_{0.975}}$ ***(***$\boldsymbol{n = 70; S = 10000}$ ***draws from the fiducial distribution; Cape Town harbour data)***

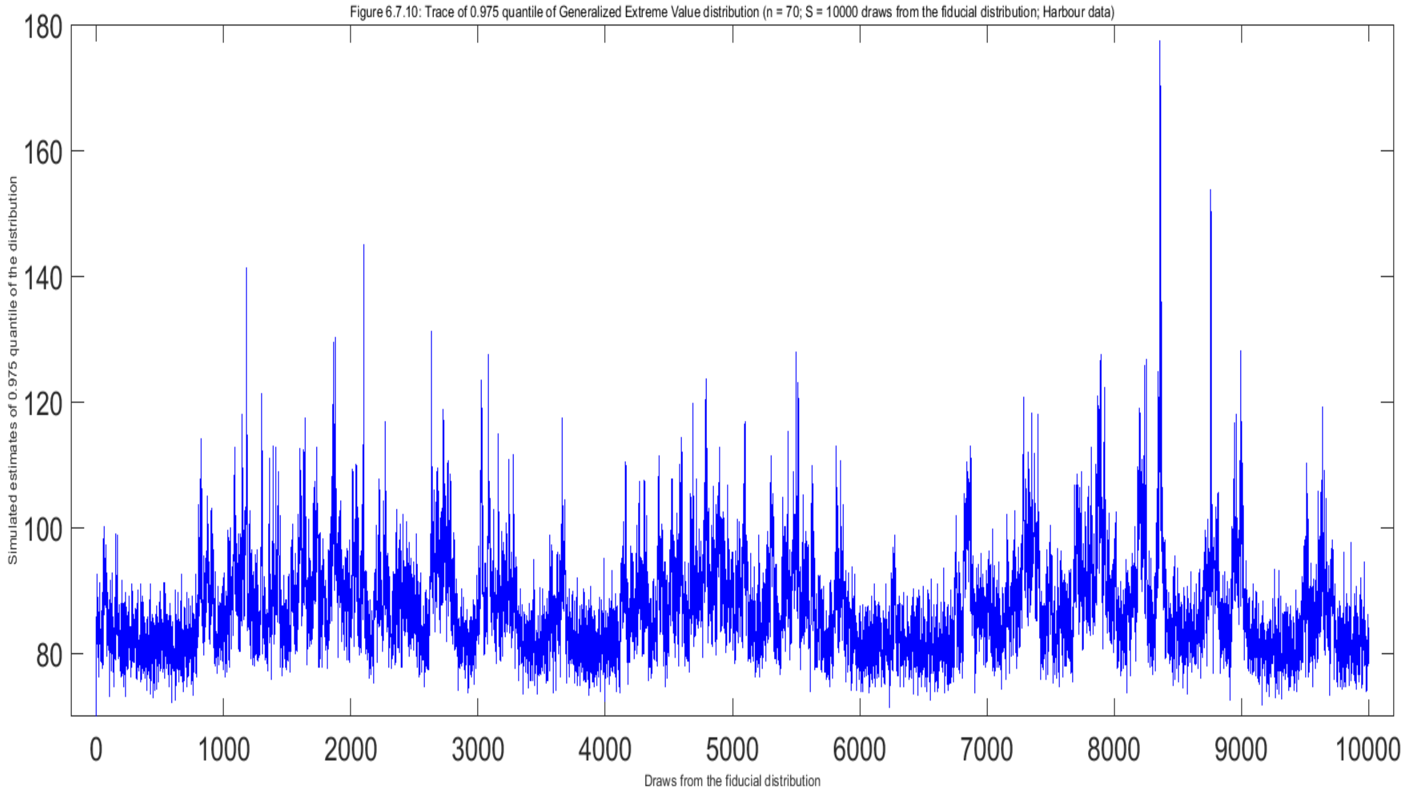


Figure 6.7.11: Generalized Extreme Value distribution: Trace of $\boldsymbol{\eta_{0.975}}$ ***($\boldsymbol{n = 70; S = 10000}$ draws from the fiducial distribution; Cape Town airport data)***

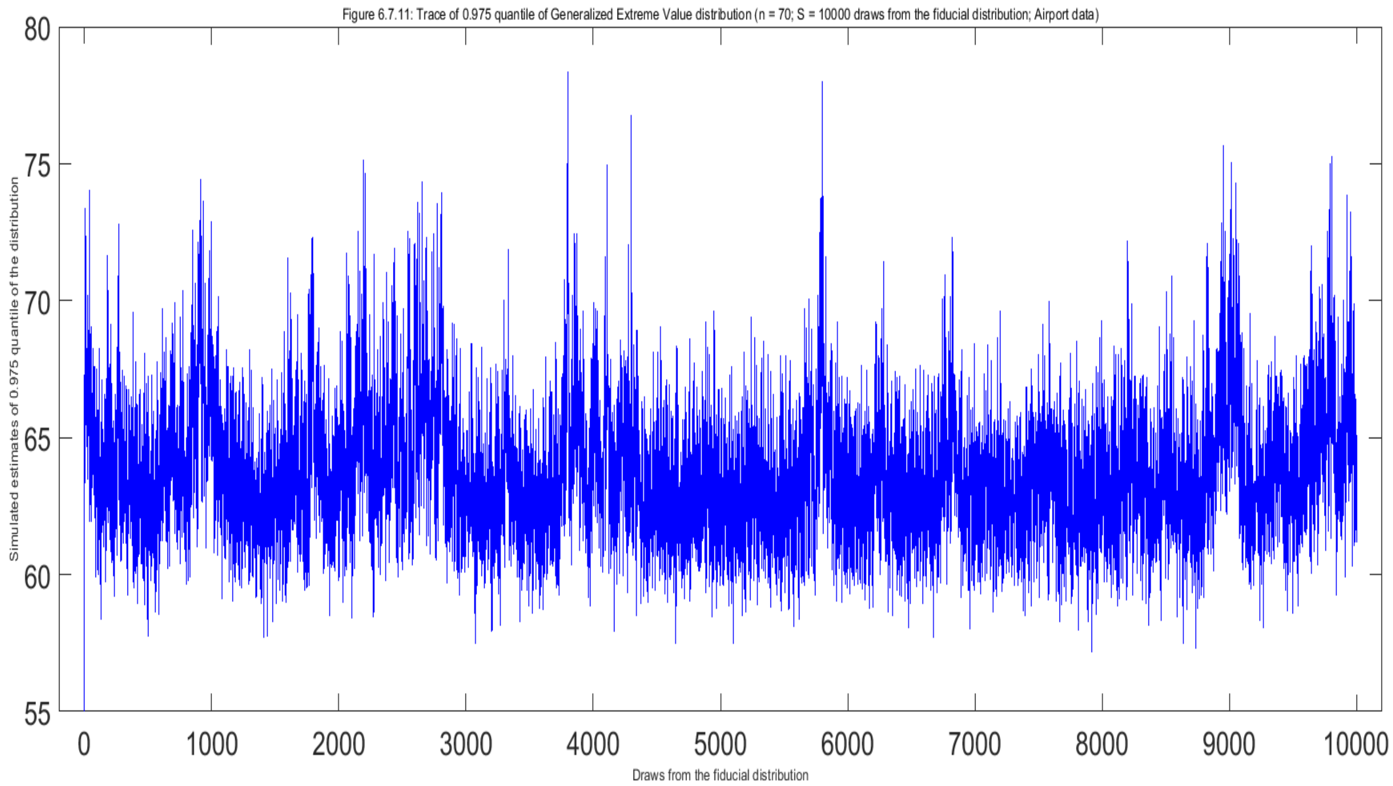


Figure 6.7.12: Generalized Extreme Value distribution: Trace of $\eta_{0.975}$ ($n = 70; S = 10000$ draws from the fiducial distribution; Cape Town Robben Island data)

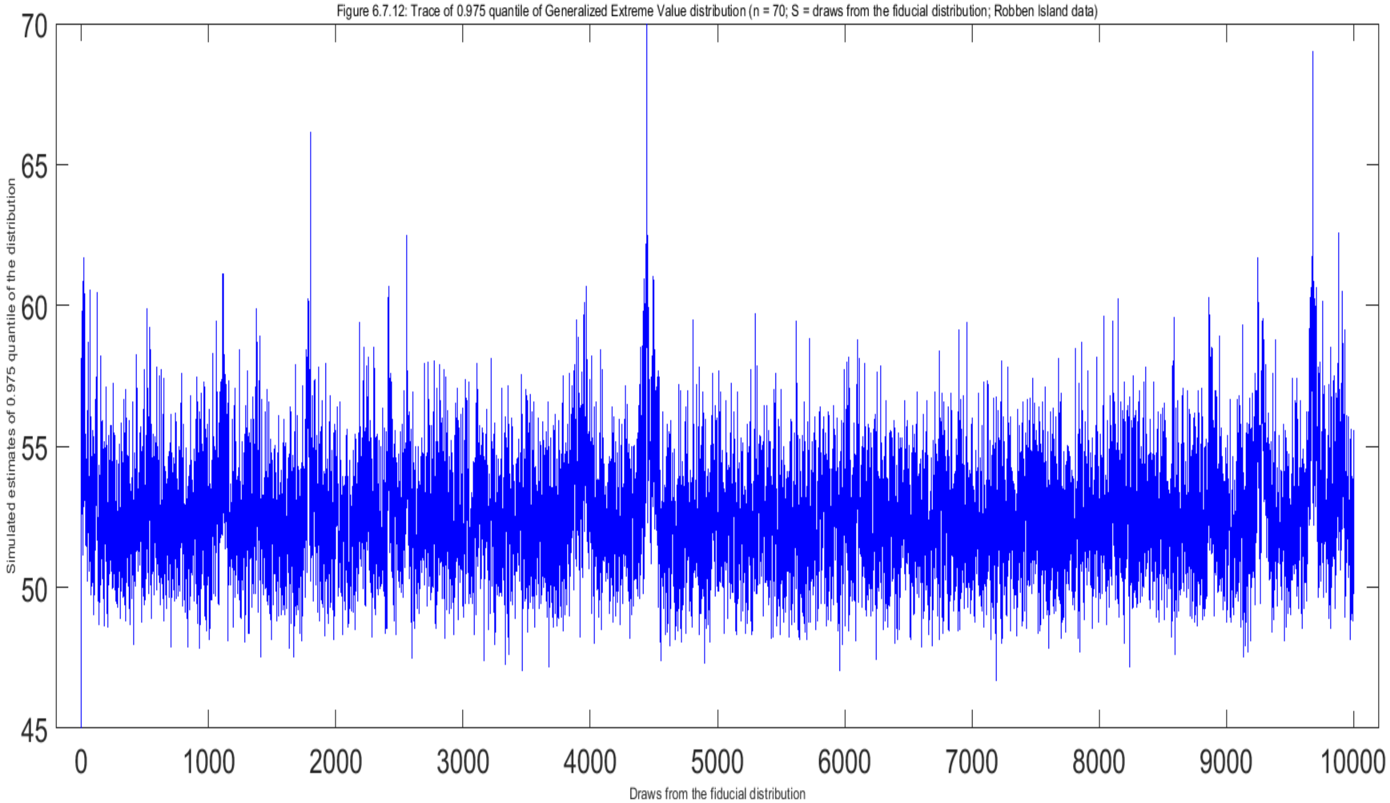


Figure 6.7.13: Generalized Extreme Value distribution: Median quantile (in miles per hour) and 95% fiducial generalized confidence region of quantiles of the distribution at selected values of failure probabilities p, as a function of $1 - p$ ($n = 70$; $S = 10000$ draws from the fiducial distribution; Cape Town harbour data)

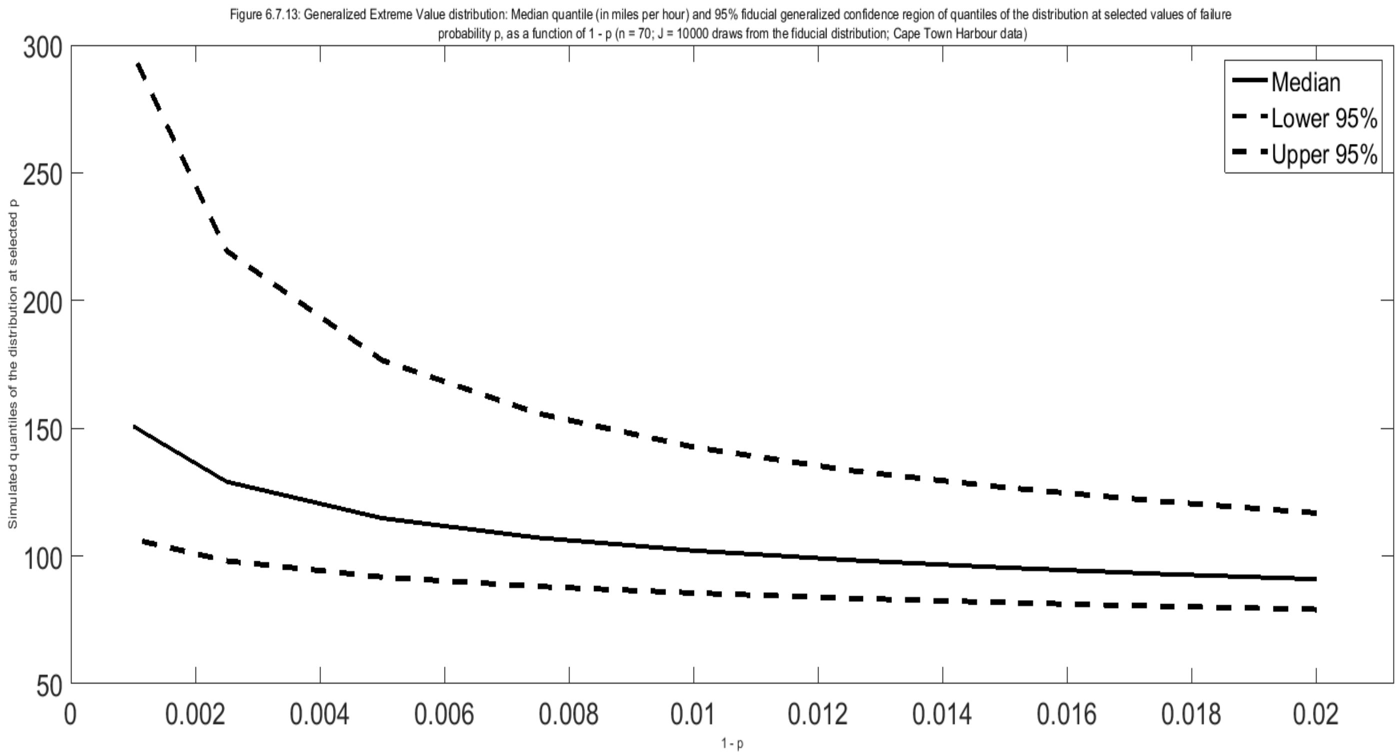


Figure 6.7.14: Generalized Extreme Value distribution: Median quantile (in miles per hour) and 95% fiducial generalized confidence region of quantiles of the distribution at selected values of failure probabilities p, as a function of $1 - p$ ($n = 70$; $S = 10000$ draws from the fiducial distribution; Cape Town airport data)

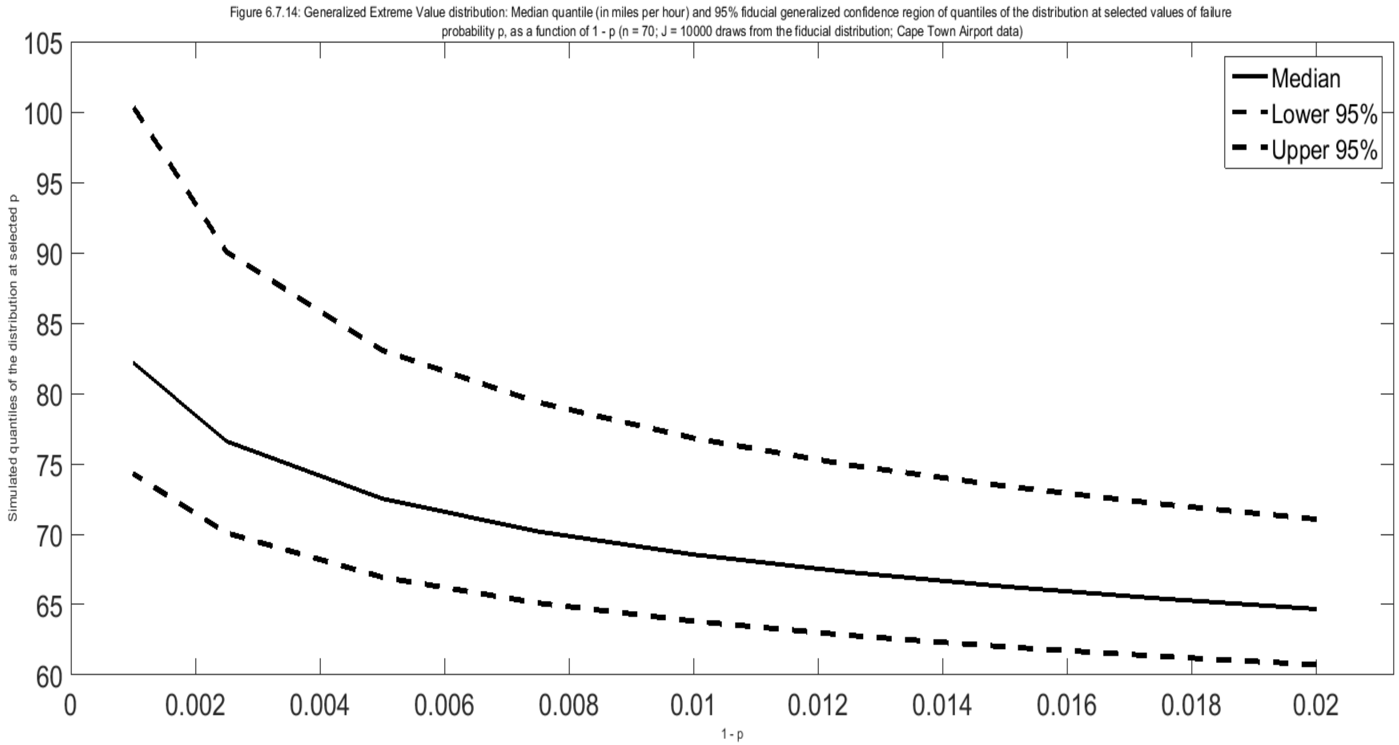
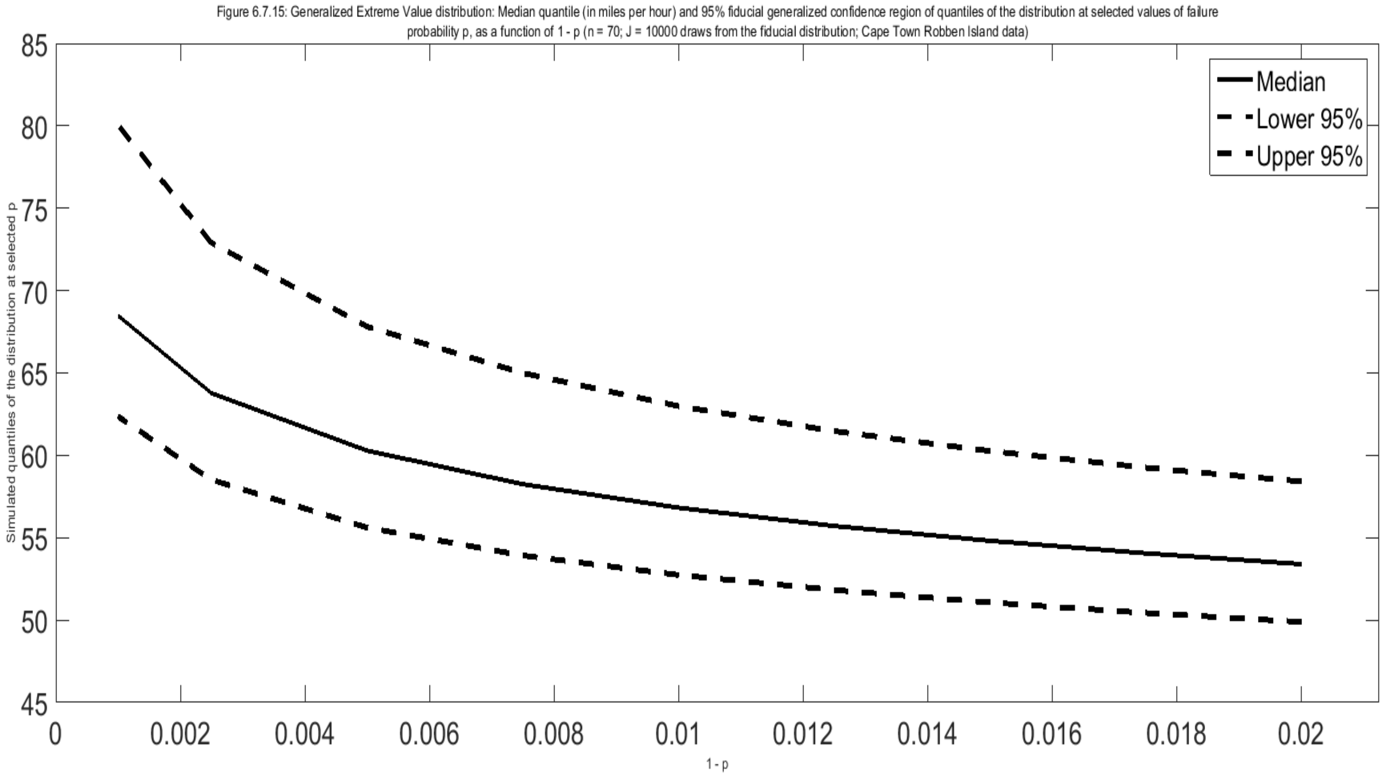


Figure 6.7.15: Generalized Extreme Value distribution: Median quantile (in miles per hour) and 95% fiducial generalized confidence region of quantiles of the distribution at selected values of failure probabilities p, as a function of $1 - p$ ($n = 70$; $S = 10000$ draws from the fiducial distribution; Cape Town Robben Island data)



The discussion of results of illustrative example

- The choice of using a lower or higher limit of the grid for ξ when simulating the H_ξ matrix
    - We stated in Section 6.5.2 above that a value of 0.01 was used as a lower limit of the grid for ξ when simulating the linear predictors for μ_* and σ_*, namely the H_ξ matrix.
    - It is important to state here that higher values of lower limit of the grid for ξ were also used to simulate H_ξ, however, they produced essentially similar results for the theta star parameters and quantiles of the distribution. That is, higher values of lower limit of the grid for ξ gave practically the same FGCIs and traces of μ_*, σ_* and η for the GEV distribution.
- The comparison of the distribution of FGPQs for the model parameters and quantiles of the three locations
    - Overall, the results of our proposed rank-based fiducial generalized inference methods agree well with the results from the Bayesian analysis of Beirlant et al. (2004, pp. 453-457). See Tables 6.7.1 through 6.7.3 above.
    - Similarly to the results from a Bayesian analysis of Beirlant et al. (2004, Figures 11.15 (a), (b) and (c)), our methods show that the distribution of maximal wind gust at Cape Town harbour has the highest extreme value index (see Figures 6.7.13 through 6.7.15 above).
    - Even though our FGCRs of the quantiles appear to be somewhat wider than the high probability density regions obtained by Beirlant at el. (2004), overall, results

produced by the proposed rank-based fiducial methods seem satisfactory.

Chapter 7 - Summary and Conclusion

We present, in this chapter, a summary and discussion of summary of the main results of this thesis (Section 7.1), and identify open problems and possible avenues for further research (Section 7.2).

7.1. Discussion

Location-scale and log-location-scale families of distributions: one-sample problem

- Our simulation results and results based on illustrative examples suggest that the proposed rank-based methods are competitive with ML-based methods in terms of relative length of confidence intervals for the model parameters μ and σ, and for quantiles of the distribution η. Refer to Tables 4.6.3.1 through 4.6.3.3 (for the simulation study) and 4.7.2 (for the illustrative example) in the text. Alternatively, refer to Appendix B1 (Tables B1.1 through B1.15), and Appendix B2 (Tables B2.1 through B2.12) for the simulation studies.
- Furthermore, our investigation through simulation shows that when using the proposed rank-based CPQs or FGPQs for σ, the most efficient rank-based CPQ or FGPQ for σ is a CPQ or FGPQ based on GLS method, namely the CPQ or FGPQ for σ_1. Refer to Tables 4.6.3.1 (in the text), Appendix B1 (Tables B1.1, B1.4, B1.7, B1.10 and B1.13) and Appendix B2 (Tables B2.1, B2.4, B2.7 and B2.10).
- Lastly, when calculating the confidence intervals for μ and η in LS and LLS families of distributions using rank and ML-based CPQs or FGPQs for μ and η, using one copy of the standard variate Z produces GCIs as good

(or slightly better) than using two independent copies, in terms of the average lengths of such confidence intervals.

Location-scale and log-location-scale families of distributions: two-sample problem

- For both cases of equal and unequal scale parameters of the two independent distributions, our simulation results and results based on illustrative example suggest that the proposed rank-based methods are competitive with ML-based methods in terms of relative length of confidence intervals for the ratio of scale parameters, difference of location parameters, and difference of quantiles of the distribution. Refer to Tables 5.6.1 for the illustrative example (in the text), Appendix D1 (Tables D1.1 through D1.5), Appendix D2 (Tables D2.1 through D2.5), Appendix E1 (Tables E1.1 through E1.10), and Appendix E2 (Tables E2.1 through E2.10) for the simulation study.
- Simulation results show that rank-based FGCIs for ρ, δ and d have very good coverage properties, with the observed coverage in simulation studies being very close to the nominal coverage in all cases. Refer to Appendix D1 (Tables D1.1 through D1.5), Appendix D2 (Tables D2.1 through D2.5), Appendix E1 (Tables E1.1 through E1.10), and Appendix E2 (Tables E2.1 through E2.10) for the simulation study. These results suggest that the FGCIs for ρ, δ and d in the two-sample problem are either exact or near exact.

Location-scale-shape family of distributions

Generalized Extreme Value distribution:

- Overall, simulation results show that the Gibbs sampler using rank-based CFGPQs produces rank-based FGCIs with generally good properties when the θ^* parametrization is used and when $\xi \leq 1$. Refer to Tables 6.6.1.1

through 6.6.1.5 in the text, and Appendix F1 (Tables F1.1 through F1.5). In addition, refer to Figure 6.6.2 in the text.

- Therefore, when calculating the rank-based FGCIs for the model parameters and quantiles of the distribution for the Generalized Extreme Value distribution using Gibbs sampler algorithm, the preferred parameterization is the θ^* parametrization.
    - This finding is demonstrated by the comparison of θ and θ^* parametrizations in terms of the empirical coverages and average lengths of rank-based FGCIs for the model parameters and quantiles of distribution in Tables 6.6.1.1 through 6.6.1.5 above.
- However, the Gibbs sampler using rank-based CFGPQs does not work well when $\xi > 1$. For $\xi > 1$, the draws from the conditional fiducial distributions of the parameters μ_*, σ_* and ξ for the Generalized Extreme Value distribution become very unstable, as is illustrated by Figure 6.6.1 above. It can be noted here that $\xi > 1$ for the Generalized Extreme Value distribution is quite rare in practice.

Generalized Pareto distribution:

- For the case of Generalized Pareto distribution, simulation results show that the Gibbs sampler using rank-based CFGPQs fails when θ parametrization is used to calculate FGCIs for the model parameters and quantile of the distribution.
- However, the θ^* parametrization produces FGCIs of model parameters and quantiles of the distribution with generally good properties for smaller values of ξ, that is, for ξ values smaller than or equal 0.2. Refer to Figures 6.6.3 and 6.6.4 above.

Three-parameter Weibull distribution:

- The θ^* parametrization produces FGCIs of model parameters and quantiles of three-parameter Weibull distribution with good properties for a wide range of values of ξ.

Overall conclusion for the case of three-parameter distributions:

- Overall, for the LSS distributions investigated here, when calculating FGCIs for the model parameters and quantiles of distribution based on the Gibbs sampler using rank-based CFGPQs, the θ^* parametrization is recommended.
- The proposed fiducial methodology works quite well overall for the three-parameter Weibull distribution, well for the Generalized Extreme Value distribution when ξ is not larger than 1, but well for the Generalized Pareto distribution only when ξ is not larger than 0.2.

7.2. Open Problems and Further Research

The concept of conditional fiducial generalized pivotal quantities, and their application for fiducial inference based on a Gibbs sampler, presented in this thesis is new. Our investigation through simulation and practical application of rank-based methods of fiducial generalized inference for the model parameters and quantiles of distribution of LSS families can handle cases when the shape parameter ξ is assumed to be positive. However, cases of rank-based fiducial inferences when ξ is negative, or when potentially both positive and negative values of ξ are allowed, have not been investigated in this thesis, and are the subject of further research. Thus, in this thesis, when determining the joint fiducial distribution of θ^* parameters and ξ of LSS distributions using the Gibbs sampler algorithm, we assume that ξ is positive. Methodology for determining whether ξ is positive or ξ is negative is not handled here, and it is an open problem for further research.

Furthermore, instead of using rank-based methods for fiducial inference in LSS families, the use of the corresponding ML-based methods can be investigated (although, based on our results for LS and LLS families, one could expect that the rank-based methods are competitive with the ML-based methods).

A major open research question is a comparison of the performance of the fiducial inference methods proposed here for LSS families, with Bayesian methods.

Lastly, exact rank-based iterative generalized least squares estimation of model parameters of LSS distributions for the cases when ξ is positive and ξ is negative, involving censored samples, is an open problem for further research.

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Appendices

Appendix A: LS and LLS families of distributions: Lower and upper quantiles of the distribution of CPQs for μ, σ and η

A1: LS and LLS families of distributions: Lower and upper quantiles of the distribution of CPQs for μ, σ and η for sample size $n = 10$

Table A1.1: Normal distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.6241	1.3931	3.0850	0.5467	1.4134	4.3413	7.1225	1.3016
0.05	0.5621	1.2233	2.7788	0.5200	1.4998	4.7729	7.5893	1.3802
0.01	0.4509	0.9284	2.2242	0.4174	1.6722	5.7146	8.5437	1.5378

Table A1.2*:* Normal distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.5644	-0.2272	-0.1137	-0.6106	0.5650	0.2272	0.1136	0.6129
0.05	-0.6967	-0.2880	-0.1405	-0.7533	0.6967	0.2876	0.1404	0.7553
0.01	-1.0022	-0.4352	-0.2019	-1.0840	1.0000	0.4361	0.2019	1.0826
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.5643	-0.2273	-0.1136	-0.6107	0.5639	0.2269	0.1135	0.6092
0.05	-0.6980	-0.2881	-0.1407	-0.7540	0.6956	0.2880	0.1403	0.7515
0.01	-1.0016	-0.4370	-0.2021	-1.0847	1.0013	0.4339	0.2017	1.0836

CPQs with * are based on both the original sample of the standard variate Z and an independent copy of the standard variate Z, namely Z^*.

Table A1.3: Normal distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-1.3552	-0.5344	-0.2713	-3.5858	0.7235	0.3023	0.1472	-1.3409
0.05	-1.7436	-0.6996	-0.3493	-4.0041	0.8298	0.3630	0.1695	-1.2262
0.01	-2.7065	-1.1372	-0.5441	-5.0478	1.0182	0.5002	0.2103	-1.0214
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-0.9859	-0.3984	-0.1988	-3.5843	1.0345	0.4159	0.2084	-1.3427
0.05	-1.2059	-0.5000	-0.2429	-4.0023	1.2872	0.5301	0.2593	-1.2278
0.01	-1.7104	-0.7495	-0.3452	-5.0588	1.8716	0.8146	0.3777	-1.0241

CPQs with * are based on both the original sample of the standard variate Z and an independent copy of the standard variate Z, namely Z^*.

Table A1.4: Logistic distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.5874	1.2973	2.5983	0.5526	1.4888	4.5149	6.8365	1.4006
0.05	0.5243	1.1333	2.3167	0.4938	1.6016	5.0352	7.4165	1.5053
0.01	0.4171	0.8521	1.8392	0.3921	1.8474	6.2233	8.7103	1.7333

Table A1.5: Logistic distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 10$; S; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.9797	-0.4005	-0.2189	-1.0399	0.9733	0.4011	0.2183	1.0348
0.05	-1.1968	-0.5043	-0.2685	-1.2718	1.1906	0.5044	0.2682	1.2661
0.01	-1.6937	-0.7523	-0.3810	-1.8055	1.7169	0.7659	0.3863	1.8187
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-1.0114	-0.4148	-0.2262	-1.0420	1.0126	0.4174	0.2272	1.0496
0.05	-1.2612	-0.5284	-0.2835	-1.2782	1.2628	0.5317	0.2834	1.2879
0.01	-1.8565	-0.8070	-0.4144	-1.8227	1.8671	0.8259	0.4231	1.8292

Table A1.6: Logistic distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-2.8303	-1.1345	-0.6301	-6.8971	1.4337	0.6010	0.3238	-2.3710
0.05	-3.6104	-1.4843	-0.8114	-7.7440	1.6610	0.7251	0.3768	-2.1336
0.01	-5.5285	-2.3987	-1.2500	-9.7690	2.0547	0.9997	0.4715	-1.7117
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-1.9505	-0.8042	-0.4380	-6.8992	2.2302	0.9141	0.5020	-2.3687
0.05	-2.3815	-1.0039	-0.5344	-7.7575	2.8619	1.1901	0.6416	-2.1245
0.01	-3.4015	-1.5286	-0.7696	-9.8872	4.3698	1.9383	0.9802	-1.7155

Table A1.7: Uniform distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.7389	1.5479	5.8104	0.7033	1.1766	4.1709	9.3786	1.0637
0.05	0.6761	1.3740	5.3143	0.6505	1.1903	4.5586	9.5793	1.0748
0.01	0.5565	1.0745	4.3492	0.5473	1.2081	5.4357	10.0516	1.0893

Table A1.8: Uniform distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$\mathcal{Q}_{\mu_1}(Y,\theta)$	$\mathcal{Q}_{\mu_2}(Y,\theta)$	$\mathcal{Q}_{\mu_3}(Y,\theta)$	$\mathcal{Q}_{\mu_{ML}}(Y,\theta)$	$\mathcal{Q}_{\mu_1}(Y,\theta)$	$\mathcal{Q}_{\mu_2}(Y,\theta)$	$\mathcal{Q}_{\mu_3}(Y,\theta)$	$\mathcal{Q}_{\mu_{ML}}(Y,\theta)$
0.10	-0.0860	-0.0379	-0.0109	0.0046	0.2346	0.0931	0.0297	0.3033
0.05	-0.0883	-0.0428	-0.0113	0.0023	0.3252	0.1310	0.0415	0.3807
0.01	-0.0902	-0.0548	-0.0117	0.00048529	0.5712	0.2379	0.0722	0.5762
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.1009	-0.0439	-0.0128	0.0046	0.1911	0.0759	0.0242	0.3036
0.05	-0.1117	-0.0511	-0.0142	0.0023	0.2488	0.1016	0.0316	0.3818
0.01	-0.1381	-0.0678	-0.0175	0.00046491	0.3750	0.1633	0.0473	0.5678

Table A1.9: Uniform distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-0.2287	-0.0902	-0.0290	-1.2167	0.0836	0.0369	0.0106	-0.8798
0.05	-0.3169	-0.1277	-0.0402	-1.3119	0.0859	0.0418	0.0110	-0.8721
0.01	-0.5589	-0.2336	-0.0708	-1.5654	0.0900	0.0529	0.0116	-0.8534
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-0.1855	-0.0732	-0.0235	-1.2142	0.0971	0.0423	0.0123	-0.8799
0.05	-0.2416	-0.0989	-0.0306	-1.3066	0.1071	0.0489	0.0136	-0.8723
0.01	-0.3680	-0.1582	-0.0467	-1.5628	0.1322	0.0657	0.0168	-0.8536

Table A1.10: Weibull distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$\mathcal{Q}_{\sigma_1}(Y,\theta)$	$\mathcal{Q}_{\sigma_2}(Y,\theta)$	$\mathcal{Q}_{\sigma_3}(Y,\theta)$	$\mathcal{Q}_{\sigma_{ML}}(Y,\theta)$	$\mathcal{Q}_{\sigma_1}(Y,\theta)$	$\mathcal{Q}_{\sigma_2}(Y,\theta)$	$\mathcal{Q}_{\sigma_3}(Y,\theta)$	$\mathcal{Q}_{\sigma_{ML}}(Y,\theta)$
0.10	0.5952	1.3196	2.7143	0.5500	1.4682	4.4703	6.9060	1.3529
0.05	0.5328	1.1531	2.4295	0.4925	1.5740	4.9863	7.4705	1.4498
0.01	0.4224	0.8698	1.9238	0.3906	1.7967	6.1636	8.6903	1.6564

Table A1.11: Weibull distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$\mathcal{Q}_{\mu_1}(Y,\theta)$	$\mathcal{Q}_{\mu_2}(Y,\theta)$	$\mathcal{Q}_{\mu_3}(Y,\theta)$	$\mathcal{Q}_{\mu_{ML}}(Y,\theta)$	$\mathcal{Q}_{\mu_1}(Y,\theta)$	$\mathcal{Q}_{\mu_2}(Y,\theta)$	$\mathcal{Q}_{\mu_3}(Y,\theta)$	$\mathcal{Q}_{\mu_{ML}}(Y,\theta)$
0.10	-0.5726	-0.2312	-0.1242	-0.6628	0.6346	0.2619	0.1386	0.6438
0.05	-0.7180	-0.2972	-0.1559	-0.8195	0.7802	0.3308	0.1708	0.8018
0.01	-1.0707	-0.4643	-0.2332	-1.2005	1.1186	0.5003	0.2456	1.1681
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.6348	-0.2579	-0.1381	-0.6645	0.5798	0.2378	0.1262	0.6417
0.05	-0.8055	-0.3348	-0.1754	-0.8204	0.7087	0.2982	0.1544	0.7999
0.01	-1.2053	-0.5281	-0.2632	-1.2003	1.0049	0.4465	0.2199	1.1577

Table A1.12: Weibull distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-1.0137	-0.4027	-0.2188	-2.5516	0.5442	0.2318	0.1200	-0.8699
0.05	-1.3238	-0.5358	-0.2862	-2.8862	0.6227	0.2779	0.1380	-0.7847
0.01	-2.1040	-0.8847	-0.4563	-3.7296	0.7670	0.3826	0.1719	-0.6286
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-0.7778	-0.3164	-0.1693	-2.5514	0.7499	0.3063	0.1634	-0.8699
0.05	-0.9753	-0.4071	-0.2127	-2.8831	0.9280	0.3885	0.2023	-0.7850
0.01	-1.4481	-0.6328	-0.3160	-3.7242	1.3359	0.5925	0.2934	-0.6295

Table A1.13: Pareto distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.5224	1.1392	2.0349	0.4704	1.6024	4.7167	6.6003	1.4427
0.05	0.4585	0.9766	1.7776	0.4129	1.7520	5.3364	7.2789	1.5775
0.01	0.3490	0.7081	1.3440	0.3150	2.0683	6.7456	8.8209	1.8639

Table A1.14: Pareto distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.0947	-0.0437	-0.0246	0.0057	0.2551	0.1052	0.0647	0.3945
0.05	-0.0973	-0.0496	-0.0259	0.0029	0.3552	0.1501	0.0902	0.5059
0.01	-0.0996	-0.0623	-0.0281	0.00055920	0.6171	0.2718	0.1578	0.7961
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.1472	-0.0635	-0.0376	0.0057	0.2304	0.0957	0.0584	0.3923
0.05	-0.1804	-0.0795	-0.0463	0.0028	0.3188	0.1354	0.0812	0.5056
0.01	-0.2620	-0.1219	-0.0677	0.00059433	0.5631	0.2498	0.1453	0.8011

Table A1.15: Pareto distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 10$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-3.2830	-1.3713	-0.8300	-7.6309	1.3572	0.5732	0.3452	-2.4788
0.05	-4.2458	-1.8120	-1.0838	-8.7009	1.5440	0.6892	0.3970	-2.2711
0.01	-6.6988	-3.0175	-1.7262	-11.4241	1.8592	0.9344	0.4863	-1.9232
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-2.0410	-0.8738	-0.5215	-7.6390	2.4768	1.0320	0.6280	-2.4732
0.05	-2.5077	-1.1008	-0.6413	-8.7085	3.2679	1.3910	0.8286	-2.2588
0.01	-3.6052	-1.6669	-0.9329	-11.4446	5.2720	2.3669	1.3546	-1.9187

A2: LS and LLS families of distributions: Lower and upper quantiles of the distribution of CPQs for μ, σ and η for sample size $n = 25$.

Table A2.1: Normal distribution: Lower and upper quantiles of four pivotal CPQs for σ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.7662	3.1717	6.3342	0.7439	1.2450	6.5397	10.5519	1.2069
0.05	0.7250	2.9461	5.9866	0.7039	1.2943	7.0040	11.0221	1.2547
0.01	0.6469	2.5405	5.3323	0.6281	1.3934	8.0298	11.9872	1.3494

Table A2.2: Normal distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.3393	-0.0751	-0.0407	-0.3492	0.3382	0.0749	0.0406	0.3490
0.05	-0.4092	-0.0920	-0.0492	-0.4212	0.4081	0.0916	0.0491	0.4211
0.01	-0.5541	-0.1279	-0.0668	-0.5703	0.5540	0.1278	0.0668	0.5714
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.3397	-0.0751	-0.0407	-0.3493	0.3381	0.0748	0.0406	0.3487
0.05	-0.4100	-0.0918	-0.0492	-0.4212	0.4082	0.0914	0.0490	0.4211
0.01	-0.5541	-0.1280	-0.0668	-0.5697	0.5543	0.1279	0.0667	0.5715

CPQs with * are based on both the original sample of the standard variate Z and an independent copy of the standard variate Z, namely Z^*.

Table A2.3: Normal distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-0.7080	-0.1554	-0.0848	-2.7481	0.4859	0.1090	0.0586	-1.5197
0.05	-0.8785	-0.1944	-0.1052	-2.9234	0.5649	0.1291	0.0682	-1.4386
0.01	-1.2496	-0.2834	-0.1500	-3.3045	0.7053	0.1687	0.0856	-1.2943
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-0.5837	-0.1294	-0.1294	-2.7461	0.5967	0.1319	0.0716	-1.5207
0.05	-0.7000	-0.1574	-0.1574	-2.9189	0.7225	0.1620	0.0868	-1.4396
0.01	-0.9360	-0.2168	-0.2168	-3.3052	0.9865	0.2272	0.1188	-1.2922

Table A2.4: Logistic distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.7356	3.0327	5.4216	0.7188	1.2935	6.7123	9.9142	1.2634
0.05	0.6919	2.7994	5.0901	0.6762	1.3578	7.2446	10.4751	1.3260
0.01	0.6107	2.3995	4.4806	0.5968	1.4895	8.4392	11.6772	1.4545

Table A2.5: Logistic distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.5922	-0.1324	-0.0794	-0.6032	0.5868	0.1310	0.0786	0.6035
0.05	-0.7114	-0.1611	-0.0955	-0.7258	0.7070	0.1601	0.0948	0.7262
0.01	-0.9571	-0.2238	-0.1290	-0.9770	0.9557	0.2232	0.1289	0.9810
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.5995	-0.1338	-0.0803	-0.6024	0.5930	0.1323	0.0794	0.6024
0.05	-0.7268	-0.1647	-0.0976	-0.7262	0.7205	0.1630	0.0967	0.7257
0.01	-0.9972	-0.2312	-0.1342	-0.9789	0.9905	0.2307	0.1335	0.9815

Table A2.6: Logistic distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-1.4733	-0.3273	-0.1970	-5.2552	0.9779	0.2195	0.1311	-2.7466
0.05	-1.8191	-0.4090	-0.2435	-5.6083	1.1394	0.2607	0.1535	-2.5814
0.01	-2.5879	-0.5969	-0.3478	-6.3962	1.4337	0.3428	0.1942	-2.2817
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-1.1724	-0.2620	-0.1570	-5.2525	1.2661	0.2825	0.1695	-2.7467
0.05	-1.3970	-0.3174	-0.1876	-5.6045	1.5621	0.3530	0.2097	-2.5821
0.01	-1.8586	-0.4356	-0.2505	-6.3935	2.2080	0.5112	0.2973	-2.2795

Table A2.7: Uniform distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.8927	3.4329	16.7069	0.9223	1.0677	6.3106	20.0767	1.0379
0.05	0.8627	3.2390	16.1461	0.8971	1.0726	6.7332	20.2329	1.0390
0.01	0.7996	2.8832	14.9675	0.8408	1.0788	7.6868	20.5763	1.0398

Table A2.8: Uniform distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.0365	-0.0085	-0.0019	0.0020	0.0842	0.0184	0.0045	0.1082
0.05	-0.0375	-0.0092	-0.0020	0.00097038	0.1149	0.0254	0.0061	0.1314
0.01	-0.0384	-0.0106	-0.0021	0.00019273	0.1907	0.0428	0.0102	0.1836
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.0382	-0.0089	-0.0020	0.0020	0.0776	0.0170	0.0041	0.1084
0.05	-0.0399	-0.0097	-0.0021	0.00097154	0.1031	0.0228	0.0055	0.1315
0.01	-0.0435	-0.0114	-0.0023	0.00019701	0.1606	0.0364	0.0086	0.1831

Table A2.9: Uniform distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-0.0823	-0.0180	-0.0044	-1.0305	0.0355	0.0082	0.0019	-0.8551
0.05	-0.1121	-0.0248	-0.0060	-1.0601	0.0366	0.0090	0.0020	-0.8314
0.01	-0.1849	-0.0416	-0.0099	-1.1318	0.0382	0.0104	0.0020	-0.7797
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-0.0760	-0.0166	-0.0041	-1.0301	0.0370	0.0086	0.0020	-0.8548
0.05	-0.1007	-0.0223	-0.0054	-1.0595	0.0386	0.0094	0.0021	-0.8315
0.01	-0.1562	-0.0356	-0.0083	-1.1321	0.0420	0.0110	0.0022	-0.7796

Table A2.10: Weibull distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.7460	3.0800	5.7154	0.7235	1.2768	6.6539	10.1059	1.2366
0.05	0.7028	2.8510	5.3743	0.6818	1.3350	7.1793	10.6423	1.2926
0.01	0.6222	2.4526	4.7492	0.6038	1.4543	8.3616	11.7919	1.4078

Table A2.11: Weibull distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.3443	-0.0765	-0.0445	-0.3702	0.3731	0.0835	0.0483	0.3696
0.05	-0.4168	-0.0937	-0.0538	-0.4447	0.4503	0.1022	0.0584	0.4494
0.01	-0.5680	-0.1314	-0.0737	-0.6001	0.6112	0.1424	0.0795	0.6150
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.3722	-0.0827	-0.0481	-0.3707	0.3479	0.0777	0.0450	0.3695
0.05	-0.4553	-0.1025	-0.0589	-0.4444	0.4157	0.0942	0.0539	0.4489
0.01	-0.6361	-0.1472	-0.0826	-0.5999	0.5577	0.1303	0.0725	0.6145

Table A2.12: Weibull distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-0.5111	-0.1125	-0.0658	-1.8875	0.3555	0.0805	0.0462	-0.9959
0.05	-0.6373	-0.1420	-0.0822	-2.0173	0.4120	0.0951	0.0537	-0.9381
0.01	-0.9259	-0.2098	-0.1193	-2.3135	0.5150	0.1241	0.0675	-0.8318
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-0.4379	-0.0976	-0.0566	-1.8875	0.4215	0.0940	0.0545	-0.9963
0.05	-0.5333	-0.1203	-0.0656	-2.0198	0.5067	0.1146	0.0656	-0.9390
0.01	-0.7345	-0.1710	-0.0955	-2.3186	0.6871	0.1598	0.0892	-0.8327

Table A2.13: Pareto distribution: Lower and upper quantiles of four CPQs for σ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$	$Q_{\sigma_1}(Y,\theta)$	$Q_{\sigma_2}(Y,\theta)$	$Q_{\sigma_3}(Y,\theta)$	$Q_{\sigma_{ML}}(Y,\theta)$
0.10	0.6891	2.8416	4.5332	0.6615	1.3580	6.9331	9.4285	1.3036
0.05	0.6407	2.5997	4.2022	0.6151	1.4391	7.5426	10.0762	1.3813
0.01	0.5529	2.1780	3.6056	0.5308	1.6050	8.9345	11.4739	1.5407

Table A2.14: Pareto distribution: Lower and upper quantiles of eight CPQs for μ at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$	$Q_{\mu_1}(Y,\theta)$	$Q_{\mu_2}(Y,\theta)$	$Q_{\mu_3}(Y,\theta)$	$Q_{\mu_{ML}}(Y,\theta)$
0.10	-0.0381	-0.0090	-0.0058	0.0021	0.0873	0.0195	0.0130	0.1327
0.05	-0.0391	-0.0098	-0.0060	0.0010	0.1191	0.0268	0.0178	0.1658
0.01	-0.0402	-0.0113	-0.0065	0.0002066	0.1977	0.0454	0.0296	0.2477
	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$	$Q^*_{\mu_1}(Y,\theta)$	$Q^*_{\mu_2}(Y,\theta)$	$Q^*_{\mu_3}(Y,\theta)$	$Q^*_{\mu_{ML}}(Y,\theta)$
0.10	-0.0468	-0.0108	-0.0070	0.0021	0.0839	0.0187	0.0125	0.1331
0.05	-0.0532	-0.0125	-0.0080	0.0011	0.1143	0.0258	0.0171	0.1660
0.01	-0.0673	-0.0162	-0.0102	0.00021019	0.1883	0.0434	0.0283	0.2472

Table A2.15: Pareto distribution: Lower and upper quantiles of eight CPQs for η at specified values of alpha (sample size $n = 25$; $J = 1000000$ draws from the distribution of CPQs)

	Quantile							
	$\alpha/2$				$1-\alpha/2$			
α	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$	$Q_{\eta_1}(Y,\theta)$	$Q_{\eta_2}(Y,\theta)$	$Q_{\eta_3}(Y,\theta)$	$Q_{\eta_{ML}}(Y,\theta)$
0.10	-1.6480	-0.3712	-0.2463	-5.5177	0.9636	0.2179	0.1441	-2.7975
0.05	-2.0483	-0.4674	-0.3070	-5.9340	1.1143	0.2571	0.1674	-2.6403
0.01	-2.9572	-0.6910	-0.4454	-6.8822	1.3771	0.3334	0.2093	-2.3662
	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$	$Q^*_{\eta_1}(Y,\theta)$	$Q^*_{\eta_2}(Y,\theta)$	$Q^*_{\eta_3}(Y,\theta)$	$Q^*_{\eta_{ML}}(Y,\theta)$
0.10	-1.2088	-0.2734	-0.1807	-5.5150	1.3803	0.3104	0.2062	-2.7967
0.05	-1.4357	-0.3301	-0.2155	-5.9351	1.7265	0.3926	0.2582	-2.6417
0.01	-1.9163	-0.4552	-0.2901	-6.8742	2.4915	0.5818	0.3756	-2.3685

Appendix B: LS and LLS families of distributions: Coverage probability and average length of confidence intervals for $\boldsymbol{\mu}, \boldsymbol{\sigma}$ and $\boldsymbol{\eta}$

B1: LS and LLS families of distributions: Coverage probability and average length of confidence intervals for $\boldsymbol{\mu}, \boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ $(\boldsymbol{n} = \mathbf{10})$

Table B1.1: Normal distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.9009	0.8989	0.9003	0.9153	0.9503	0.9491	0.9500	0.9501	0.9899	0.9896	0.9900	0.9898
Length	0.8955	1.3054	0.9170	0.8795	1.1132	1.6280	1.1383	1.1066	1.6211	2.4158	1.6596	1.6116

Table B1.2: Normal distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9014	0.9012	0.9017	0.9017	0.9501	0.9507	0.9502	0.9500	0.9901	0.9899	0.9898	0.9900
Length	1.1303	1.2168	1.1343	1.1296	1.3945	1.5414	1.4018	1.3928	2.0038	2.3333	2.0151	2.0004
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9011	0.9011	0.9014	0.9008	0.9502	0.9509	0.9503	0.9495	0.9901	0.9899	0.9898	0.9901
Length	1.1291	1.2163	1.1333	1.1263	1.3947	1.5427	1.4023	1.3900	2.0045	2.3322	2.0151	2.0019

Table B1.3: Normal distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.9009	0.9005	0.9011	0.9006	0.9496	0.9495	0.9495	0.9496	0.9899	0.9901	0.9900	0.9899
Length	2.0804	2.2406	2.0885	2.0726	2.5755	2.8455	2.5890	2.5647	3.7277	4.3848	3.7647	3.7174
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.8993	0.8975	0.8995	0.8999	0.9358	0.9394	0.9363	0.9493	0.9736	0.9793	0.9742	0.9899
Length	2.0220	2.1806	2.0321	2.0696	2.4951	2.7585	2.5061	2.5616	3.5849	4.1885	3.6075	3.7251

Table B1.4: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.8995	0.9002	0.8997	0.8994	0.9491	0.9499	0.9501	0.9484	0.9899	0.9901	0.9901	0.8999
Length	1.0303	1.4548	1.0731	1.0301	1.2824	1.8108	1.3349	1.2794	1.8554	2.6824	1.9290	1.8554

Table B1.5: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.8982	0.8981	0.8972	0.8978	0.9476	0.9479	0.9476	0.9474	0.9890	0.9893	0.9889	0.9890
Length	1.9522	2.1228	1.9663	1.9506	2.3864	2.6713	2.4139	2.3861	3.4092	4.0206	3.4510	3.4074
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9088	0.9070	0.9079	0.9002	0.9578	0.9559	0.9572	0.9494	0.9933	0.9923	0.9933	0.9895
Length	2.0232	2.2039	2.0392	1.9665	2.5229	2.8074	2.5497	2.4126	3.7220	4.3243	3.7667	3.4334

Table B1.6: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.8996	0.8993	0.8992	0.8994	0.9493	0.9494	0.9500	0.9495	0.9899	0.9899	0.9900	0.9898
Length	4.2622	4.5960	4.2902	4.2553	5.2692	5.8510	5.3440	5.2747	7.5800	8.9998	7.7426	7.5752
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.8892	0.8911	0.8903	0.8997	0.9263	0.9320	0.9275	0.9505	0.9697	0.9764	0.9708	0.9901
Length	4.1789	4.5505	4.2277	4.2594	5.2412	5.8103	5.2892	5.2960	7.7680	9.1812	7.8699	7.6828

Table B1.7: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.9017	0.8971	0.9002	0.9023	0.9505	0.9491	0.9495	0.9506	0.9901	0.9900	0.9903	0.9901
Length	0.5029	1.1001	0.5167	0.4423	0.6383	1.3767	0.6611	0.5571	0.9682	2.0218	1.0293	0.8346

Table B1.8: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.8988	0.9002	0.8984	0.8994	0.9494	0.9500	0.9528	0.9500	0.9893	0.9903	0.9910	0.9897
Length	0.3203	0.3547	0.3204	0.2742	0.4131	0.4706	0.4167	0.3474	0.6607	0.7925	0.6621	0.5285
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9294	0.9094	0.9293	0.8995	0.9546	0.9494	0.9547	0.9502	0.9826	0.9842	0.9824	0.9896
Length	0.2917	0.3244	0.2920	0.2745	0.3601	0.4135	0.3614	0.3484	0.5126	0.6257	0.5114	0.5208

Table B1.9: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.9015	0.8996	0.8999	0.9000	0.9495	0.9496	0.9508	0.9499	0.9898	0.9896	0.9903	0.9895
Length	0.3120	0.3441	0.3125	0.3093	0.4024	0.4590	0.4040	0.4038	0.6482	0.7757	0.6502	0.6536
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.9300	0.9078	0.9297	0.8986	0.9562	0.9499	0.9563	0.9483	0.9827	0.9837	0.9826	0.9892
Length	0.2823	0.3127	0.2825	0.3069	0.3483	0.4002	0.3488	0.3987	0.4997	0.6062	0.5011	0.6511

Table B1.10: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.8999	0.8986	0.8990	0.8999	0.9497	0.9496	0.9496	0.9498	0.9905	0.9901	0.9903	0.9905
Length	1.2920	1.7299	1.3614	1.2918	1.6126	2.1736	1.7028	1.6122	2.3853	3.2841	2.5259	2.3787

Table B1.11: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9000	0.9008	0.9006	0.9005	0.9496	0.9512	0.9501	0.9495	0.9899	0.9900	0.9901	0.9902
Length	0.3503	0.3869	0.3576	0.3506	0.4532	0.5189	0.4650	0.4535	0.7177	0.8681	0.7445	0.7173
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9397	0.9364	0.9399	0.8997	0.9679	0.9678	0.9684	0.9503	0.9930	0.9934	0.9934	0.9900
Length	0.3781	0.4136	0.3845	0.3486	0.4999	0.5583	0.5106	0.4534	0.8263	0.9657	0.8531	0.7218

Table B1.12: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.9000	0.8964	0.8988	0.9000	0.9495	0.9486	0.9492	0.9495	0.9902	0.9905	0.9906	0.9903
Length	4.6469	5.0522	4.7067	4.6454	5.7982	6.4986	5.9306	5.7975	8.5704	10.2677	8.8610	8.5666
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.8731	0.8802	0.8754	0.9010	0.9117	0.9203	0.9130	0.9508	0.9604	0.9677	0.9620	0.9904
Length	4.5244	4.9516	4.6037	4.6578	5.7840	6.4741	5.8869	5.8154	8.8901	10.4805	9.1614	8.5891

Table B1.13: Weibull distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.8993	0.9001	0.8998	0.8993	0.9491	0.9500	0.9498	0.9492	0.9899	0.9902	0.9897	0.9898
Length	0.9987	1.4189	1.0342	0.9954	1.2412	1.7711	1.2846	1.2368	1.8103	2.6233	1.8719	1.8049

Table B1.14: Weibull distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9000	0.8994	0.8996	0.8998	0.9492	0.9496	0.9492	0.9489	0.9900	0.9900	0.9899	0.9900
Length	1.2068	1.3100	1.2154	1.2054	1.4977	1.6684	1.5110	1.4957	2.1886	2.5626	2.2144	2.1851
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.8983	0.8984	0.8986	0.8997	0.9480	0.9477	0.9475	0.9489	0.9886	0.9890	0.9883	0.9897
Length	1.2142	1.3169	1.2224	1.2050	1.5137	1.6817	1.5253	1.4948	2.2095	2.5892	2.2343	2.1753

Table B1.15: Weibull distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 10$; $S = 100000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.8996	0.9000	0.8996	0.8993	0.9501	0.9500	0.9495	0.9499	0.9899	0.9900	0.9901	0.9900
Length	1.5574	1.6856	1.5669	1.5514	1.9459	2.1617	1.9619	1.9387	2.8701	3.3668	2.9054	2.8608
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.9067	0.9022	0.9058	0.8993	0.9439	0.9454	0.9446	0.9497	0.9804	0.9842	0.9806	0.9899
Length	1.5272	1.6543	1.5387	1.5512	1.9027	2.1136	1.9194	1.9356	2.7831	3.2552	2.8185	2.8550

B2: LS and LLS families of distributions: Coverage probability and average length of confidence intervals for μ, σ and η $(n = 25)$

Table B2.1: Normal distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.8999	0.9000	0.9001	0.8997	0.9493	0.9498	0.9505	0.9497	0.9899	0.9898	0.9897	0.9900
Length	0.5018	0.7602	0.5273	0.5001	0.6065	0.9207	0.6377	0.6048	0.8279	1.2598	0.8700	0.8253

Table B2.2: Normal distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9013	0.9011	0.9009	0.9010	0.9510	0.9504	0.9506	0.9508	0.9905	0.9904	0.9905	0.9904
Length	0.6773	0.7022	0.6794	0.6771	0.8171	0.8595	0.8214	0.8168	1.1078	1.1971	1.1164	1.1072
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9014	0.9009	0.9009	0.9009	0.9512	0.9499	0.9504	0.9508	0.9905	0.9904	0.9905	0.9904
Length	0.6776	0.7018	0.6794	0.6769	0.8180	0.8577	0.8206	0.8168	1.1081	1.1980	1.1156	1.1067

Table B2.3: Normal distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.9019	0.9015	0.9022	0.9020	0.9517	0.9514	0.9519	0.9518	0.9900	0.9897	0.9896	0.9898
Length	1.1936	1.2378	1.1983	1.1912	1.4430	1.5145	1.4490	1.4399	1.9543	2.1166	1.9688	1.9494
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.9016	0.9008	0.9022	0.9013	0.9452	0.9455	0.9454	0.9511	0.9808	0.9831	0.9813	0.9899
Length	1.1801	1.2233	1.1841	1.1883	1.4221	1.4953	1.4281	1.4345	1.9220	2.0786	1.9345	1.9521

Table B2.4: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.9008	0.9004	0.9008	0.9009	0.9502	0.9496	0.9499	0.9499	0.9901	0.9898	0.9903	0.9901
Length	0.5860	0.8405	0.6271	0.5855	0.7085	1.0192	0.7577	0.7076	0.9656	1.3869	1.0320	0.9647

Table B2.5: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9008	0.9000	0.9009	0.9011	0.9505	0.9495	0.9498	0.9502	0.9905	0.9903	0.9904	0.9905
Length	1.1784	1.2248	1.1854	1.1782	1.4177	1.4935	1.4278	1.4177	1.9118	2.0785	1.9349	1.9117
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9045	0.9030	0.9043	0.9007	0.9548	0.9536	0.9543	0.9502	0.9925	0.9920	0.9926	0.9906
Length	1.1919	1.2373	1.1982	1.1763	1.4466	1.5237	1.4578	1.4176	1.9867	2.1478	2.0085	1.9141

Table B2.6: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.9000	0.9004	0.9003	0.9004	0.9508	0.9498	0.9505	0.9507	0.9904	0.9900	0.9903	0.9904
Length	2.4500	2.5425	2.4616	2.4493	2.9570	3.1140	2.9786	2.9554	4.0196	4.3694	4.0664	4.0172
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.8991	0.8977	0.8985	0.9000	0.9397	0.9415	0.9404	0.9504	0.9766	0.9804	0.9775	0.9904
Length	2.4373	2.5318	2.4496	2.4466	2.9576	3.1172	2.9808	2.9510	4.06445	4.4025	4.1100	4.0168

Table B2.7: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.9000	0.8989	0.9007	0.9016	0.9497	0.9489	0.9500	0.9518	0.9899	0.9896	0.9900	0.9902
Length	0.1836	0.6261	0.1883	0.1208	0.2269	0.7552	0.2345	0.1522	0.3237	1.0216	0.3414	0.2276

Table B2.8: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9030	0.9030	0.8808	0.9011	0.9518	0.9500	0.9502	0.9507	0.9904	0.9898	0.9950	0.9907
Length	0.1207	0.1268	0.1200	0.1062	0.1524	0.1631	0.1518	0.1304	0.2291	0.2517	0.2306	0.1834
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9346	0.9089	0.9166	0.9014	0.9680	0.9529	0.9679	0.9508	0.9907	0.9894	0.9908	0.9905
Length	0.1158	0.1221	0.1143	0.1064	0.1430	0.1532	0.1425	0.1305	0.2041	0.2253	0.2043	0.1829

Table B2.9: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.8997	0.8971	0.9025	0.9014	0.9516	0.9499	0.9639	0.9503	0.9903	0.9904	0.9844	0.9903
Length	0.1178	0.1235	0.1181	0.1754	0.1487	0.1593	0.1500	0.2287	0.2231	0.2451	0.2231	0.3521
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.9264	0.9051	0.9324	0.9014	0.9637	0.9512	0.9657	0.9499	0.9906	0.9893	0.9905	0.9903
Length	0.1130	0.1188	0.1143	0.1753	0.1393	0.1494	0.1406	0.2280	0.1982	0.2196	0.1968	0.3525

Table B2.10: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for σ based on four CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}	σ_1	σ_2	σ_3	σ_{ML}
Coverage	0.8993	0.9002	0.8998	0.8992	0.9494	0.9497	0.9495	0.9492	0.9900	0.9899	0.9900	0.9899
Length	0.7150	0.9618	0.7745	0.7150	0.8661	1.1674	0.9381	0.8659	1.1859	1.6080	1.2861	1.1858

Table B2.11: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for μ based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples of data)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}	μ_1	μ_2	μ_3	μ_{ML}
Coverage	0.9023	0.9020	0.9061	0.9020	0.9517	0.9516	0.9487	0.9527	0.9901	0.9905	0.9916	0.9904
Length	0.1254	0.1320	0.1271	0.1254	0.1582	0.1695	0.1609	0.1582	0.2380	0.2626	0.2441	0.2376
	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*	μ_1^*	μ_2^*	μ_3^*	μ_{ML}^*
Coverage	0.9468	0.9384	0.9466	0.9024	0.9732	0.9722	0.9733	0.9505	0.9943	0.9945	0.9945	0.9902
Length	0.1307	0.1366	0.1319	0.1258	0.1675	0.1774	0.1697	0.1583	0.2557	0.2760	0.2603	0.2372

Table B2.12: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for η based on eight CPQs at specified confidence levels (sample size $n = 25$; $S = 1000000$ simulated samples)

Quantity	Nominal confidence level											
	0.90				0.95				0.99			
	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}	η_1	η_2	η_3	η_{ML}
Coverage	0.8994	0.9009	0.8997	0.8991	0.9494	0.9502	0.9501	0.9494	0.9899	0.9898	0.9900	0.9900
Length	2.6122	2.7282	2.6398	2.6120	3.1634	3.3553	3.2078	3.1627	4.3354	4.7442	4.4270	4.3363
	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*	η_1^*	η_2^*	η_3^*	η_{ML}^*
Coverage	0.8915	0.8909	0.8913	0.8991	0.9279	0.9318	0.9289	0.9494	0.9683	0.9729	0.9698	0.9899
Length	2.5897	2.7037	2.6162	2.6101	3.1630	3.3470	3.2031	3.1624	4.4089	4.8026	4.5014	4.3264

Appendix C: LS and LLS families of distributions: two-sample problem: lower and upper quantiles of the distribution of CPQs for $\rho = \sigma_1/\sigma_2$

C1: LS and LLS families of distributions: lower and upper quantiles of the distribution of CPQs for $\rho = \sigma_1/\sigma_2$; sample sizes $n_1 = n_2 = 10$

Table C1.1: Normal distribution: Lower and upper quantiles of two CPQs for $\rho = \sigma_1/\sigma_2$ at specified values of alpha (sample sizes $n_1 = n_2 = 10$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.5584	0.5603	1.7883	1.7832
0.05	0.4962	0.4978	2.0138	2.0069
0.01	0.3887	0.3901	2.5623	2.5565

Table C1.2: Logistic distribution: Lower and upper quantiles of two CPQs for $\rho = \sigma_1/\sigma_2$ at specified values of alpha (sample sizes $n_1 = n_2 = 10$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.5160	0.5161	1.9400	1.9385
0.05	0.4526	0.4530	2.2139	2.2115
0.01	0.3467	0.3466	2.8886	2.8867

Table C1.3: Uniform distribution: Lower and upper quantiles of two CPQs for $\rho = \sigma_1/\sigma_2$ at specified values of alpha (sample sizes $n_1 = n_2 = 10$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.7080	0.7962	1.4136	1.3592
0.05	0.6472	0.6806	1.5460	1.4705
0.01	0.5302	0.5728	1.8818	1.7448

Table C1.4: Pareto distribution: Lower and upper quantiles of two CPQs for $\rho = \sigma_1/\sigma_2$ at specified values of alpha (sample sizes $n_1 = n_2 = 10$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.4513	0.4513	2.2159	2.2154
0.05	0.3857	0.3857	2.5910	2.5905
0.01	0.2814	0.2815	3.5525	3.5529

Table C1.5: Weibull distribution: Lower and upper quantiles of two CPQs for $\rho = \sigma_1/\sigma_2$ at specified values of alpha (sample sizes $n_1 = n_2 = 10$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.5263	0.5273	1.9021	1.8978
0.05	0.4626	0.4635	2.1646	2.1588
0.01	0.3564	0.3572	2.8136	2.8059

C2: LS and LLS families of distributions: lower and upper quantiles of the distribution of CPQs for $\rho = \sigma_1/\sigma_2$; sample sizes $n_1 = n_2 = 25$

Table C2.1: Normal distribution: Lower and upper quantiles of two CPQs for ρ at specified values of alpha (sample sizes $n_1 = n_2 = 25$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.7092	0.7099	1.4093	1.4083
0.05	0.6631	0.6640	1.5084	1.5065
0.01	0.5807	0.5813	1.7261	1.7235

Table C2.2: Logistic distribution: Lower and upper quantiles of two CPQs for ρ at specified values of alpha (sample sizes $n_1 = n_2 = 25$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.6707	0.6710	1.4916	1.4916
0.05	0.6206	0.6209	1.6123	1.6116
0.01	0.5314	0.5317	1.8775	1.8766

Table C2.3: Uniform distribution: Lower and upper quantiles of two CPQs for ρ at specified values of alpha (sample sizes $n_1 = n_2 = 25$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.8749	0.9122	1.1434	1.0966
0.05	0.8455	0.8873	1.1837	1.1275
0.01	0.7823	0.8315	1.2777	1.2020

Table C2.4: Pareto distribution: Lower and upper quantiles of two CPQs for ρ at specified values of alpha (sample sizes $n_1 = n_2 = 25$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.7092	0.7099	1.4093	1.4083
0.05	0.6631	0.6640	1.5084	1.5065
0.01	0.5807	0.5813	1.7261	1.7235

Table C2.5: Weibull distribution: Lower and upper quantiles of two CPQs for ρ at specified values of alpha (sample sizes $n_1 = n_2 = 25$; $S = 1000000$ draws from the distribution of CPQs)

	Quantile			
	$\alpha/2$		$1-\alpha/2$	
α	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$	$\mathbb{Q}_{\rho_R}$	$\mathbb{Q}_{\rho_{ML}}$
0.10	0.6835	0.6842	1.4624	1.4610
0.05	0.6346	0.6353	1.5753	1.5731
0.01	0.5478	0.5486	1.8262	1.8229

Appendix D: LS and LLS families of distributions: two-sample problem: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$: equal and unequal σ_1 and σ_2

D1: LS and LLS families of distributions: coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$; samples sizes $n_1 = n_2 = 10$

Table D1.1: Normal distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.9012	0.9008	0.9506	0.9510	0.9901	0.9900
Average length	1.3070	1.2989	1.6127	1.6028	2.3098	2.3010
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.8999	0.9004	0.9499	0.9498	0.9900	0.9900
Average length	0.6546	0.6505	0.8078	0.8027	1.1569	1.1523

Table D1.2: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.9006	0.9003	0.9488	0.9490	0.9896	0.9897
Average length	1.5482	1.5460	1.9149	1.9113	2.7635	2.7608
$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9001	0.9004	0.9509	0.9507	0.9907	0.9909
Average length	0.7729	0.7720	0.9559	0.9544	1.3796	1.3786

Table D1.3: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.9006	0.8995	0.9505	0.9501	0.9899	0.9899
Average length	0.7221	0.6344	0.9198	0.8044	1.3832	1.1935
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9011	0.9006	0.9509	0.9510	0.9901	0.9897
Average length	0.3610	0.3172	0.4599	0.4022	0.6915	0.5968

Table D1.4: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.8982	0.8982	0.9494	0.9493	0.9900	0.9900
Average length	1.9802	1.9796	2.4748	2.4742	3.6708	3.6711
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9000	0.8999	0.9497	0.9496	0.9898	0.9897
Average length	3.9619	3.9609	4.9514	4.9504	7.3444	7.3452

Table D1.5: Weibull distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.9006	0.9001	0.9496	0.9495	0.9898	0.9897
Average length	1.4801	1.4739	1.8310	1.8238	2.6435	2.6335
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.8995	0.8993	0.9499	0.9500	0.9903	0.9904
Average length	2.9714	2.9585	3.6759	3.6596	5.3069	5.2860

D2: LS and LLS families of distributions: coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$; samples sizes $n_1 = n_2 = 25$

Table D2.1: Normal distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.8991	0.8990	0.9494	0.9496	0.9891	0.9892
Average length	0.7161	0.7142	0.8646	0.8616	1.1715	1.1680
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9009	0.9012	0.9510	0.9510	0.9904	0.9903
Average length	0.3575	0.3566	0.4316	0.4302	0.5849	0.5832

Table D2.2: Logistic distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.9001	0.8999	0.9496	0.9495	0.9900	0.9902
Average length	0.8454	0.8451	1.0213	1.0202	1.3863	1.3850
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.8977	0.8977	0.9484	0.9484	0.9900	0.9899
Average length	0.4233	0.4231	0.5114	0.5108	0.6941	0.6935

Table D2.3: Uniform distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.8984	0.8991	0.9499	0.9499	0.9901	0.9899
Average length	0.2693	0.1846	0.3393	0.2405	0.4969	0.3710
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9017	0.9006	0.9508	0.9497	0.9900	0.9900
Average length	0.1348	0.0924	0.1698	0.1203	0.2487	0.1886

Table D2.4: Pareto distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
`Coverage	0.9000	0.9018	0.9503	0.9503	0.9902	0.9903
Average length	1.0407	1.0437	1.2627	1.2626	1.7387	1.7374
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9011	0.9030	0.9500	0.9500	0.9900	0.9900
Average length	2.0807	2.0867	2.5246	2.5243	3.4761	3.4735

Table D2.5: Weibull distribution: Coverage probabilities and average lengths of confidence intervals for $\rho = \sigma_1/\sigma_2$ based on two CPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}	ρ_R	ρ_{ML}
Coverage	0.8958	0.8988	0.9488	0.9486	0.9901	0.9901
Average length	0.7895	0.7974	0.9659	0.9627	1.3127	1.3081
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.8983	0.9013	0.9509	0.9511	0.9904	0.9901
Average length	1.5772	1.5932	1.9296	1.9234	2.6223	2.6136

Appendix E: LS and LLS families of distributions: two-sample problem: coverage probabilities and average lengths of confidence intervals for $\boldsymbol{\delta = \mu_1 - \mu_2}$ and $\boldsymbol{d = \eta_1 - \eta_2}$: equal and unequal $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$

E1: LS and LLS families of distributions: coverage probabilities and average lengths of confidence intervals for $\boldsymbol{\delta = \mu_1 - \mu_2}$ and $\mathrm{d} = \boldsymbol{\eta_1 - \eta_2}$; samples sizes $\boldsymbol{n_1 = n_2 = 10}$

Table E1.1: Normal distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9172	0.9169	0.9619	0.9618	0.9936	0.9936
Length	1.6299	1.6285	1.9855	1.9834	2.7657	2.7617
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9117	0.9118	0.9568	0.9568	0.9918	0.9919
Length	2.5630	2.5608	3.1346	3.1311	4.4141	4.4073

Table E1.2: Normal distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9023	0.9024	0.9516	0.9513	0.9896	0.9897
Length	3.0515	3.0394	3.8333	3.8181	5.7480	5.7248
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9034	0.9035	0.9522	0.9518	0.9901	0.9903
Length	4.7803	4.7617	5.9805	5.9566	8.9277	8.8929

Table E1.3: Logistic distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1 \,/\, n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9197	0.9196	0.9638	0.9630	0.9927	0.9927
Length	2.8521	2.8512	3.4661	3.4653	4.8031	4.8024
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1 \,/\, n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9143	0.9145	0.9601	0.9599	0.9927	0.9926
Length	4.4593	4.4574	5.4378	5.4350	7.5990	7.5972

Table E1.4: Logistic distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9041	0.9036	0.9501	0.9506	0.9886	0.9883
Length	6.2858	6.2828	7.8824	7.8772	11.7886	11.7842
$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9025	0.9012	0.9518	0.9520	0.9901	0.9907
Length	9.7891	9.7828	12.2261	12.2178	18.1844	18.1745

Table E1.5: Uniform distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.8980	0.8984	0.9481	0.9488	0.9891	0.9896
Length	0.5070	0.4310	0.6871	0.5730	1.1681	0.9264
$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.8986	0.8985	0.9504	0.9494	0.9888	0.9889
Length	0.7755	0.6603	1.0443	0.8723	1.7534	1.3941

Table E1.6: Uniform distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9027	0.9028	0.9482	0.9490	0.9904	0.9905
Length	0.4929	0.4884	0.6686	0.6604	1.1357	1.1158
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9080	0.9107	0.9540	0.9552	0.9902	0.9910
Length	0.7568	0.7508	1.0203	1.0095	1.7350	1.7064

Table E1.7: Pareto distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9021	0.9018	0.9490	0.9492	0.9884	0.9884
Length	0.5620	0.5620	0.7602	0.7601	1.2906	1.2905
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9047	0.9048	0.9515	0.9515	0.9884	0.9882
Length	0.8594	0.8594	1.1549	1.1548	1.9297	1.9296

Table E1.8: Pareto distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9022	0.9020	0.9499	0.9496	0.9901	0.9904
Length	6.9827	6.9827	8.9203	8.9196	13.7998	13.7992
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9010	0.9016	0.9510	0.9514	0.9901	0.9902
Length	10.8615	10.8595	13.8002	13.7985	21.2816	21.2800

Table E1.9: Weibull distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9201	0.9206	0.9637	0.9637	0.9934	0.9937
Length	1.7510	1.7487	2.1400	2.1370	3.0069	3.0013
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9216	0.9216	0.9640	0.9641	0.9939	0.9938
Length	1.7532	1.7511	2.1424	2.1394	3.0103	3.0053

Table E1.10: Weibull distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9068	0.9072	0.9537	0.9539	0.9905	0.9907
Length	2.3020	2.2919	2.9169	2.9041	4.4561	4.4368
	$n_1 = 10;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 10;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9049	0.9052	0.9533	0.9532	0.9901	0.9899
Length	2.3048	2.2949	2.9211	2.9086	4.4619	4.4427

E2: LS and LLS families of distributions: coverage probabilities and average lengths of confidence intervals for $\boldsymbol{\delta = \mu_1 - \mu_2}$ and $\boldsymbol{d = \eta_1 - \eta_2}$; samples sizes $\boldsymbol{n_1 = n_2 = 25}$

Table E2.1: Normal distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9059	0.9059	0.9538	0.9538	0.9903	0.9902
Length	0.9644	0.9641	1.1570	1.1566	1.5467	1.5459
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9046	0.9046	0.9530	0.9531	0.9904	0.9903
Length	1.5208	1.5204	1.8276	1.8270	2.4550	2.4539

Table E2.2: Normal distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9004	0.9004	0.9502	0.9499	0.9893	0.9891
Length	1.7085	1.7045	2.0752	2.0705	2.8604	2.8536
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9005	0.9006	0.9497	0.9497	0.9882	0.9893
Length	2.6888	2.6828	3.2595	3.2524	4.4828	4.4730

Table E2.3: Logistic distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9050	0.9030	0.9480	0.9490	0.9900	0.9900
Length	1.6954	1.6956	2.0307	2.0309	2.790	2.7084
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9145	0.9134	0.9585	0.9586	0.9919	0.9918
Length	2.6477	2.6470	3.1792	3.1783	4.2604	4.2598

Table E2.4: Logistic distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9110	0.9080	0.9590	0.9590	0.9940	0.9940
Length	3.5523	3.5503	4.3109	4.3091	5.9319	5.9307
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9042	0.9033	0.9504	0.9499	0.9893	0.9890
Length	5.5432	5.5406	6.7189	6.7160	9.2329	9.2288

Table E2.5: Uniform distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9030	0.9021	0.9524	0.9527	0.9906	0.9906
Length	0.1902	0.1656	0.2511	0.2124	0.4002	0.3161
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9015	0.9016	0.9491	0.9486	0.9886	0.9883
Length	0.2912	0.2541	0.3819	0.3238	0.6021	0.4773

Table E2.6: Uniform distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1 \,/\, n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9065	0.9080	0.9547	0.9566	0.9906	0.9899
Length	0.1854	0.2490	0.2448	0.3137	0.3900	0.4618
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1 \,/\, n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.8989	0.9027	0.9493	0.9523	0.9890	0.9906
Length	0.2847	0.3915	0.3740	0.4996	0.5956	0.7530

Table E2.7: Pareto distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.8946	0.8947	0.9487	0.9487	0.9907	0.9907
Length	0.1986	0.1986	0.2620	0.2620	0.4171	0.4171
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.8961	0.8961	0.9469	0.9469	0.9867	0.9870
Length	0.3034	0.3034	0.3977	0.3976	0.6252	0.6252

Table E2.8: Pareto distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.8985	0.8988	0.9498	0.9503	0.9905	0.9906
Length	3.7802	3.7798	4.6307	4.6300	6.5049	6.5037
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.8964	0.8965	0.9476	0.9476	0.9907	0.9905
Length	5.9159	5.9152	7.2270	7.2260	10.1104	10.1090

Table E2.9: Weibull distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $\delta = \mu_1 - \mu_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	δ_R	δ_{ML}	δ_R	δ_{ML}	δ_R	δ_{ML}
Coverage	0.9120	0.9114	0.9609	0.9598	0.9917	0.9917
Length	1.0263	1.0257	1.2318	1.2311	1.6503	1.6492
$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$						
Coverage	0.9094	0.9097	0.9562	0.9565	0.9931	0.9928
Length	1.0230	1.0224	1.2276	1.2270	1.6443	1.6431

Table E2.10: Weibull distribution: Coverage probabilities and average lengths of fiducial generalized confidence intervals for $d = \eta_1 - \eta_2$ based on two FGPQs at specified nominal confidence levels ($S = 100000$ simulated samples of data; $I = 4000$ draws from the fiducial distribution)

$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 1$

Quantity	Nominal confidence level					
	0.90		0.95		0.99	
	d_R	d_{ML}	d_R	d_{ML}	d_R	d_{ML}
Coverage	0.9014	0.9001	0.9503	0.9503	0.9896	0.9895
Length	1.2501	1.2459	1.5244	1.5193	2.1204	2.1142
	$n_1 = 25;\ \mu_1 = 0;\ \sigma_1 = 1\ /\ n_2 = 25;\ \mu_2 = 0;\ \sigma_2 = 2$					
Coverage	0.9047	0.9045	0.9511	0.9523	0.9896	0.9897
Length	1.2457	1.2415	1.5186	1.5136	2.1130	2.1057

Appendix F: LSS families of distributions: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified confidence levels and ξ values, using theta star parametrization

F1: Generalized Extreme Value distribution

Table F1.1: GEV distribution: CPs and AVLs of FGCIs (θ^*parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.2$; $\eta_{0.975} = 2.0860$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.9648	0.9815	0.9948
		AVL	1.6816	1.9557	2.4990
	σ^*	CP	0.9263	0.9668	0.9932
		AVL	0.1395	0.1708	0.2416
	μ	CP	0.9660	0.9820	0.9952
		AVL	1.6369	1.9119	2.4489
	μ^*	CP	0.9140	0.9567	0.9895
		AVL	0.1511	0.1821	0.2465
	ξ	CP	0.9660	0.9845	0.9960
		AVL	0.4194	0.5031	0.6824
	$\eta_{0.975}$	CP	0.8888	0.9383	0.9808
		AVL	1.8862	2.6474	5.3264
50	σ	CP	0.9607	0.9822	0.9952
		AVL	1.4314	1.6588	2.0587
	σ^*	CP	0.9130	0.9593	0.9928
		AVL	0.0914	0.1105	0.1510
	μ	CP	0.9613	0.9832	0.9958
		AVL	1.3991	1.6196	2.0183
	μ^*	CP	0.9087	0.9543	0.9885
		AVL	0.1052	0.1259	0.1674
	ξ	CP	0.9648	0.9818	0.9960
		AVL	0.2993	0.3553	0.4766
	$\eta_{0.975}$	CP	0.8960	0.9410	0.9830
		AVL	0.9988	1.2938	2.1734

Table F1.2: GEV distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.25$; $\eta_{0.975} = 2.5069$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.9507	0.9760	0.9938
		AVL	2.0631	2.4260	3.1022
	σ^*	CP	0.9107	0.9575	0.9915
		AVL	0.1803	0.2210	0.3138
	μ	CP	0.9527	0.9765	0.9942
		AVL	2.0031	2.3551	3.0377
	μ^*	CP	0.9125	0.9543	0.9905
		AVL	0.1918	0.2309	0.3131
	ξ	CP	0.9600	0.9798	0.9932
		AVL	0.4609	0.5511	0.7392
	$\eta_{0.975}$	CP	0.8858	0.9355	0.9785
		AVL	2.9519	4.2046	8.8370
50	σ	CP	0.9373	0.9735	0.9960
		AVL	1.7052	1.9958	2.5091
	σ^*	CP	0.9033	0.9540	0.9888
		AVL	0.1173	0.1418	0.1939
	μ	CP	0.9365	0.9762	0.9962
		AVL	1.6627	1.9458	2.4595
	μ^*	CP	0.8982	0.9453	0.9902
		AVL	0.1331	0.1592	0.2112
	ξ	CP	0.9455	0.9795	0.9980
		AVL	0.3334	0.3947	0.5206
	$\eta_{0.975}$	CP	0.8910	0.9430	0.9858
		AVL	1.4694	1.9017	3.1616

Table F1.3: Generalized Extreme Value distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.33$; $\eta_{0.975} = 3.3641$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

			Nominal confidence level		
n	Parameter	Quantity	0.90	0.95	0.99
25	σ	CP	0.9065	0.9550	0.9908
		AVL	2.5406	3.0350	3.9584
	σ^*	CP	0.8995	0.9493	0.9928
		AVL	0.2508	0.3084	0.4411
	μ	CP	0.9077	0.9580	0.9918
		AVL	2.4542	2.9327	3.8712
	μ^*	CP	0.8998	0.9547	0.9905
		AVL	0.2573	0.3100	0.4200
	ξ	CP	0.9207	0.9648	0.9940
		AVL	0.5265	0.6283	0.8344
	$\eta_{0.975}$	CP	0.8685	0.9277	0.9782
		AVL	5.5922	8.2877	22.2975
50	σ	CP	0.8908	0.9490	0.9882
		AVL	1.9577	2.3477	3.0491
	σ^*	CP	0.9015	0.9490	0.9895
		AVL	0.1637	0.1982	0.2713
	μ	CP	0.8935	0.9483	0.9895
		AVL	1.8944	2.2739	2.9798
	μ^*	CP	0.9033	0.9493	0.9902
		AVL	0.1787	0.2138	0.2840
	ξ	CP	0.8990	0.9503	0.9922
		AVL	0.3889	0.4608	0.5998
	$\eta_{0.975}$	CP	0.8622	0.9263	0.9818
		AVL	2.6749	3.4631	5.7366

Table F1.4: Generalized Extreme Value distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels ($\mathbf{theta\ star\ parametrization}$; $\sigma = 1$; $\mu = 0$; $\xi = 0.5$; $\eta_{0.975} = 6.2847$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

			Nominal confidence level		
n	Parameter	Quantity	0.90	0.95	0.99
25	σ	CP	0.8595	0.9257	0.9870
		AVL	3.0575	3.7911	5.2567
	σ^*	CP	0.8968	0.9530	0.9915
		AVL	0.4356	0.5395	0.7905
	μ	CP	0.8605	0.9233	0.9860
		AVL	2.8903	3.5985	5.0860
	μ^*	CP	0.9002	0.9525	0.9905
		AVL	0.4052	0.4891	0.6663
	ξ	CP	0.8665	0.9325	0.9885
		AVL	0.6627	0.7927	1.0462
	$\eta_{0.975}$	CP	0.8177	0.8920	0.9715
		AVL	18.5573	28.4028	71.7104
50	σ	CP	0.8778	0.9343	0.9880
		AVL	1.8550	2.3289	3.2684
	σ^*	CP	0.9025	0.9537	0.9895
		AVL	0.2802	0.3402	0.4695
	μ	CP	0.8778	0.9357	0.9872
		AVL	1.7316	2.1883	3.1331
	μ^*	CP	0.8975	0.9440	0.9862
		AVL	0.2772	0.3318	0.4404
	ξ	CP	0.8798	0.9380	0.9880
		AVL	0.4760	0.5673	0.7400
	$\eta_{0.975}$	CP	0.8452	0.9077	0.9745
		AVL	7.9382	10.3809	17.4516

Table F1.5: Generalized Extreme Value distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels ($\mathbf{theta\ star\ parametrization}$; $\sigma = 1$; $\mu = 0$; $\xi = 1$; $\eta_{0.975} = 39.4979$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.8718	0.9305	0.9832
		AVL	2.7334	3.4431	5.1100
	σ^*	CP	0.8978	0.9535	0.9928
		AVL	1.3225	1.6814	2.6537
	μ	CP	0.8528	0.9135	0.9738
		AVL	2.1585	2.7821	4.3729
	μ^*	CP	0.8912	0.9525	0.9902
		AVL	0.8886	1.0851	1.5359
	ξ	CP	0.8592	0.9170	0.9785
		AVL	0.9557	1.1516	1.5509
	$\eta_{0.975}$	CP	0.8123	0.8832	0.9617
		AVL	495.6	908.1	3667.4
50	σ	CP	0.8815	0.9420	0.9870
		AVL	1.6224	1.8769	2.4400
	σ^*	CP	0.9048	0.9530	0.9928
		AVL	0.8574	1.0453	1.4792
	μ	CP	0.8790	0.9343	0.9812
		AVL	1.2153	1.4070	1.8621
	μ^*	CP	0.9035	0.9503	0.9890
		AVL	0.5984	0.7161	0.9578
	ξ	CP	0.8872	0.9393	0.9872
		AVL	0.6399	0.7681	1.0265
	$\eta_{0.975}$	CP	0.8428	0.9050	0.9685
		AVL	129.7349	180.5037	370.1645

F2: Generalized Pareto distribution

Table F2.1: Generalized Pareto distribution: CPs and AVLs of FGCIs for model parameters and $\boldsymbol{\eta_{0.975}}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\boldsymbol{\sigma = 1}$; $\boldsymbol{\mu = 0}$; $\boldsymbol{\xi = 0.05}$; $\boldsymbol{\eta_{0.975} = 1.2025}$; $\boldsymbol{S = 4000}$ simulated samples of data; $\boldsymbol{I = 1000}$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.9730	0.9848	0.9928
		AVL	4.5704	5.3757	6.8256
	σ^*	CP	0.9000	0.9530	0.9885
		AVL	0.0403	0.0489	0.0668
	μ	CP	0.9735	0.9855	0.9935
		AVL	4.5956	5.4172	6.9954
	μ^*	CP	0.8972	0.9535	0.9898
		AVL	0.0060	0.0076	0.0117
	ξ	CP	0.9742	0.9845	0.9932
		AVL	0.4375	0.5533	0.8248
	$\eta_{0.975}$	CP	0.8868	0.9403	0.9885
		AVL	0.2868	0.4234	1.1908
50	σ	CP	0.9620	0.9785	0.9902
		AVL	4.1649	4.8069	5.8421
	σ^*	CP	0.9000	0.9537	0.9915
		AVL	0.0280	0.0337	0.0453
	μ	CP	0.9625	0.9788	0.9910
		AVL	4.1831	4.8347	5.9455
	μ^*	CP	0.8980	0.9465	0.9862
		AVL	0.0029	0.0037	0.0055
	ξ	CP	0.9625	0.9788	0.9902
		AVL	0.3011	0.3756	0.5623
	$\eta_{0.975}$	CP	0.8910	0.9360	0.9860
		AVL	0.1527	0.1999	0.3582

Table F2.2: Generalized Pareto distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.1$; $\eta_{0.975} = 1.4461$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.9755	0.9865	0.9938
		AVL	9.2195	10.9364	14.0470
	σ^*	CP	0.9087	0.9543	0.9910
		AVL	0.0856	0.1037	0.1413
	μ	CP	0.9758	0.9868	0.9938
		AVL	9.2715	11.0247	14.3957
	μ^*	CP	0.8988	0.9513	0.9908
		AVL	0.0124	0.0157	0.0242
	ξ	CP	0.9772	0.9885	0.9945
		AVL	0.4854	0.6081	0.8901
	$\eta_{0.975}$	CP	0.8878	0.9377	0.9852
		AVL	0.7228	1.1102	3.4414
50	σ	CP	0.9545	0.9802	0.9938
		AVL	8.1255	9.4858	11.6804
	σ^*	CP	0.9157	0.9605	0.9940
		AVL	0.0601	0.0721	0.0966
	μ	CP	0.9555	0.9800	0.9938
		AVL	8.1650	9.5492	11.8951
	μ^*	CP	0.8952	0.9477	0.9868
		AVL	0.0061	0.0076	0.0114
	ξ	CP	0.9543	0.9798	0.9932
		AVL	0.3469	0.4274	0.6198
	$\eta_{0.975}$	CP	0.8970	0.9413	0.9855
		AVL	0.3643	0.4829	0.8758

Table F2.3: Generalized Pareto distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.2$; $\eta_{0.975} = 2.0913$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

			Nominal confidence level		
n	Parameter	Quantity	0.90	0.95	0.99
25	σ	CP	0.9245	0.9692	0.9958
		AVL	18.4111	22.2571	29.1453
	σ^*	CP	0.8975	0.9463	0.9880
		AVL	0.1934	0.2338	0.3168
	μ	CP	0.9240	0.9690	0.9955
		AVL	18.5317	22.4595	29.9631
	μ^*	CP	0.9080	0.9555	0.9912
		AVL	0.0265	0.0338	0.0522
	ξ	CP	0.9247	0.9752	0.9965
		AVL	0.5793	0.7162	1.0229
	$\eta_{0.975}$	CP	0.8718	0.9297	0.9818
		AVL	2.2764	3.7344	13.0599
50	σ	CP	0.8852	0.9545	0.9968
		AVL	14.8758	17.8083	22.5276
	σ^*	CP	0.9025	0.9533	0.9925
		AVL	0.1372	0.1642	0.2181
	μ	CP	0.8848	0.9525	0.9965
		AVL	14.9638	17.9501	22.9903
	μ^*	CP	0.8990	0.9505	0.9905
		AVL	0.0127	0.0161	0.0242
	ξ	CP	0.8858	0.9527	0.9952
		AVL	0.4366	0.5267	0.7258
	$\eta_{0.975}$	CP	0.8610	0.9240	0.9810
		AVL	1.0671	1.4356	2.6159

F3: Three-parameter Weibull distribution

Table F3.1: Three-parameter Weibull distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.2$; $\eta_{0.975} = 1.2983$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.9557	0.9748	0.9888
		AVL	1.3657	1.5948	2.0211
	σ^*	CP	0.9260	0.9673	0.9942
		AVL	0.1185	0.1479	0.2241
	μ	CP	0.9550	0.9760	0.9905
		AVL	1.3334	1.5547	1.9716
	μ^*	CP	0.9210	0.9625	0.9938
		AVL	0.1509	0.1829	0.2522
	ξ	CP	0.9590	0.9770	0.9905
		AVL	0.3402	0.4123	0.5767
	$\eta_{0.975}$	CP	0.9137	0.9550	0.9940
		AVL	0.2609	0.3292	0.5078
50	σ	CP	0.9435	0.9720	0.9872
		AVL	1.1477	1.3298	1.6472
	σ^*	CP	0.9220	0.9660	0.9952
		AVL	0.0779	0.0953	0.1356
	μ	CP	0.9435	0.9710	0.9882
		AVL	1.1214	1.2978	1.6047
	μ^*	CP	0.9107	0.9597	0.9920
		AVL	0.1041	0.1252	0.1694
	ξ	CP	0.9490	0.9712	0.9892
		AVL	0.2433	0.2881	0.3778
	$\eta_{0.975}$	CP	0.9080	0.9523	0.9905
		AVL	0.1652	0.2029	0.2912

Table F3.2: Three-parameter Weibull distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.25$; $\eta_{0.975} = 1.3859$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.9450	0.9750	0.9900
		AVL	1.6036	1.8872	2.4113
	σ^*	CP	0.9275	0.9708	0.9948
		AVL	0.1463	0.1822	0.2746
	μ	CP	0.9430	0.9740	0.9910
		AVL	1.5568	1.8303	2.3468
	μ^*	CP	0.9175	0.9613	0.9935
		AVL	0.1868	0.2263	0.3109
	ξ	CP	0.9525	0.9758	0.9920
		AVL	0.3732	0.4495	0.6197
	$\eta_{0.975}$	CP	0.9083	0.9525	0.9940
		AVL	0.3355	0.4226	0.6480
50	σ	CP	0.9130	0.9627	0.9885
		AVL	1.2921	1.5198	1.9162
	σ^*	CP	0.9190	0.9650	0.9948
		AVL	0.0965	0.1180	0.1679
	μ	CP	0.9135	0.9633	0.9885
		AVL	1.2553	1.4754	1.8627
	μ^*	CP	0.8990	0.9540	0.9922
		AVL	0.1294	0.1556	0.2100
	ξ	CP	0.9150	0.9648	0.9885
		AVL	0.2705	0.3203	0.4169
	$\eta_{0.975}$	CP	0.8932	0.9497	0.9920
		AVL	0.2138	0.2625	0.3753

Table F3.3: Three-parameter Weibull distribution: CPs and AVLs of FGCIs for model parameters and $\boldsymbol{\eta_{0.975}}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\boldsymbol{\sigma = 1}$; $\boldsymbol{\mu = 0}$; $\boldsymbol{\xi = 0.33}$; $\boldsymbol{\eta_{0.975} = 1.5384}$; $\boldsymbol{S = 4000}$ simulated samples of data; $\boldsymbol{I = 1000}$ draws from the fiducial distribution)

			Nominal confidence level		
n	Parameter	Quantity	0.90	0.95	0.99
25	σ	CP	0.9028	0.9593	0.9895
		AVL	1.8280	2.1933	2.8766
	σ^*	CP	0.9233	0.9688	0.9930
		AVL	0.1928	0.2403	0.3587
	μ	CP	0.9020	0.9563	0.9900
		AVL	1.7509	2.1012	2.7787
	μ^*	CP	0.8992	0.9575	0.9928
		AVL	0.2440	0.2953	0.4044
	ξ	CP	0.9075	0.9683	0.9920
		AVL	0.4301	0.5157	0.6967
	$\eta_{0.975}$	CP	0.8798	0.9435	0.9900
		AVL	0.4765	0.5993	0.9064
50	σ	CP	0.8860	0.9433	0.9868
		AVL	1.2842	1.5709	2.0936
	σ^*	CP	0.9133	0.9595	0.9958
		AVL	0.1284	0.1572	0.2228
	μ	CP	0.8872	0.9445	0.9870
		AVL	1.2244	1.5002	2.0112
	μ^*	CP	0.9067	0.9533	0.9902
		AVL	0.1708	0.2051	0.2758
	ξ	CP	0.8800	0.9397	0.9895
		AVL	0.3091	0.3686	0.4796
	$\eta_{0.975}$	CP	0.8835	0.9385	0.9882
		AVL	0.3085	0.3775	0.5351

Table F3.4: Three-parameter Weibull distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 0.5$; $\eta_{0.975} = 1.9206$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

			Nominal confidence level		
n	Parameter	Quantity	0.90	0.95	0.99
25	σ	CP	0.8772	0.9393	0.9912
		AVL	1.7184	2.1840	3.1241
	σ^*	CP	0.8995	0.9505	0.9938
		AVL	0.3041	0.3787	0.5596
	μ	CP	0.8752	0.9420	0.9902
		AVL	1.5490	1.9873	2.9184
	μ^*	CP	0.8972	0.9477	0.9900
		AVL	0.3682	0.4449	0.6058
	ξ	CP	0.8768	0.9407	0.9918
		AVL	0.5247	0.6320	0.8451
	$\eta_{0.975}$	CP	0.8650	0.9237	0.9852
		AVL	0.8747	1.0953	1.6268
50	σ	CP	0.8928	0.9470	0.9878
		AVL	0.7773	0.9940	1.4714
	σ^*	CP	0.9080	0.9533	0.9900
		AVL	0.2040	0.2491	0.3489
	μ	CP	0.8972	0.9485	0.9922
		AVL	0.6466	0.8392	1.2844
	μ^*	CP	0.8972	0.9517	0.9915
		AVL	0.2581	0.3098	0.4144
	ξ	CP	0.8968	0.9475	0.9885
		AVL	0.3401	0.4115	0.5501
	$\eta_{0.975}$	CP	0.8752	0.9367	0.9850
		AVL	0.5709	0.6972	0.9696

Table F3.5: Three-parameter Weibull distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 1$; $\eta_{0.975} = 3.6889$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

n	Parameter	Quantity	Nominal confidence level		
			0.90	0.95	0.99
25	σ	CP	0.8930	0.9450	0.9895
		AVL	0.8959	1.1048	1.5756
	σ^*	CP	0.8852	0.9420	0.9890
		AVL	0.8295	1.0413	1.5521
	μ	CP	0.8728	0.9395	0.9935
		AVL	0.2880	0.3928	0.6923
	μ^*	CP	0.8848	0.9433	0.9875
		AVL	0.7394	0.8964	1.2337
	ξ	CP	0.9010	0.9507	0.9920
		AVL	0.6838	0.8295	1.1291
	$\eta_{0.975}$	CP	0.8615	0.9257	0.9842
		AVL	3.5013	4.4031	6.5841
50	σ	CP	0.9023	0.9493	0.9905
		AVL	0.5392	0.6470	0.8639
	σ^*	CP	0.9002	0.9515	0.9902
		AVL	0.5506	0.6709	0.9294
	μ	CP	0.8875	0.9497	0.9948
		AVL	0.0899	0.1158	0.1824
	μ^*	CP	0.9002	0.9475	0.9915
		AVL	0.5098	0.6124	0.8200
	ξ	CP	0.8998	0.9483	0.9912
		AVL	0.4297	0.5165	0.6914
	$\eta_{0.975}$	CP	0.8752	0.9367	0.9840
		AVL	2.1907	2.6667	3.6815

Table F3.6: Three-parameter Weibull distribution: CPs and AVLs of FGCIs for model parameters and $\eta_{0.975}$ quantile based on six CFGPQs at specified nominal confidence levels (theta star parametrization; $\sigma = 1$; $\mu = 0$; $\xi = 2$; $\eta_{0.975} = 13.6078$; $S = 4000$ simulated samples of data; $I = 1000$ draws from the fiducial distribution)

			Nominal confidence level		
n	Parameter	Quantity	0.90	0.95	0.99
25	σ	CP	0.8932	0.9453	0.9885
		AVL	1.6675	2.0729	3.0432
	σ^*	CP	0.8900	0.9427	0.9895
		AVL	3.3412	4.3129	6.9342
	μ	CP	0.8988	0.9200	0.9330
		AVL	0.0289	0.0445	0.0979
	μ^*	CP	0.8930	0.9457	0.9885
		AVL	1.6607	2.0652	3.0368
	ξ	CP	0.9160	0.9587	0.9918
		AVL	1.1745	1.4213	1.9333
	$\eta_{0.975}$	CP	0.8972	0.9470	0.9885
		AVL	35.1922	46.8177	80.8819
50	σ	CP	0.8918	0.9487	0.9888
		AVL	1.0677	1.2991	1.7926
	σ^*	CP	0.8980	0.9483	0.9910
		AVL	2.0753	2.5632	3.6630
	μ	CP	0.8300	0.8610	0.8925
		AVL	0.0056	0.0087	0.0200
	μ^*	CP	0.8922	0.9485	0.9890
		AVL	1.0668	1.2979	1.7906
	ξ	CP	0.9065	0.9543	0.9920
		AVL	0.7723	0.9273	1.2390
	$\eta_{0.975}$	CP	0.9002	0.9545	0.9870
		AVL	18.8478	23.5094	34.4665

Appendix G: Proof for simultaneous confidence region for $\boldsymbol{\theta}$

The proof shown below is based on R. Schall (2012) research working paper.

As is shown in Section 4.4.12.1 above, an exact SCR for θ is given by

$$SCR(\theta) = \left\{\theta \middle| \left(\hat{\theta} - \theta\right)' X'V^{-1}X\left(\hat{\theta} - \theta\right) \leq \sigma^2 \cdot C_{1-\alpha}\right\} \qquad (G.1)$$

<u>Proof:</u>

Let $DD' = X'V^{-1}X$ be the Cholesky decomposition of $X'V^{-1}X$, such that

$$D = \begin{pmatrix} d_1 & 0 \\ d_2 & d_3 \end{pmatrix}$$

Then

$$\left(\hat{\theta} - \theta\right)' X'V^{-1}X\left(\hat{\theta} - \theta\right) \leq \sigma^2 \cdot C_{1-\alpha}$$

$$\Leftrightarrow [d_1(\hat{\mu} - \mu) + d_2(\hat{\sigma} - \sigma)]^2 + d_3^2(\hat{\sigma} - \sigma)^2 \leq \sigma^2 \cdot C_{1-\alpha}$$

$$\Leftrightarrow [d_1(\hat{\mu} - \mu) + d_2(\hat{\sigma} - \sigma)]^2 \leq \sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2$$

All solutions to the inequality must satisfy the inequality

$$\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2 \geq 0 \qquad (G.2)$$

Assuming this is the case, we distinguish the cases G.2.1 and G.2.2 below:

G.2.1. If $[d_1(\hat{\mu} - \mu) + d_2(\hat{\sigma} - \sigma)] \geq 0$:

$$\Leftrightarrow d_1(\hat{\mu} - \mu) + d_2(\hat{\sigma} - \sigma) \leq \sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2}$$

$$\Leftrightarrow \hat{\mu} - \mu \leq \left\{\sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2} - d_2(\hat{\sigma} - \sigma)\right\} \Big/ d_1$$

$$\Leftrightarrow \mu \geq \hat{\mu} - \left\{ \sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2} - d_2(\hat{\sigma} - \sigma) \right\} / d_1$$

G.2.2. If $[d_1(\hat{\mu} - \mu) + d_2(\hat{\sigma} - \sigma)] < 0$:

$$\Leftrightarrow d_1(\hat{\mu} - \mu) + d_2(\hat{\sigma} - \sigma) \geq -\sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2}$$

$$\Leftrightarrow \hat{\mu} - \mu \geq -\left\{ \sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2} + d_2(\hat{\sigma} - \sigma) \right\} / d_1$$

$$\Leftrightarrow \mu \leq \hat{\mu} + \left\{ \sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2} + d_2(\hat{\sigma} - \sigma) \right\} / d_1$$

Furthermore, returning to inequality (G.2) we have the following conditions on σ:

$$\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2 \geq 0$$

$$\Leftrightarrow \sigma^2(C_{1-\alpha} - d_3^2) + 2d_3^2\hat{\sigma}\,\sigma - d_3^2\hat{\sigma}^2 \geq 0$$

$$\Leftrightarrow \sigma^2(d_3^2 - C_{1-\alpha}) - 2d_3^2\hat{\sigma}\,\sigma + d_3^2\hat{\sigma}^2 \leq 0$$

$$\Leftrightarrow \left(\sigma\sqrt{d_3^2 - C_{1-\alpha}} - \frac{d_3^2\hat{\sigma}}{\sqrt{d_3^2 - C_{1-\alpha}}} \right)^2 \leq \frac{d_3^4\hat{\sigma}^2}{d_3^2 - C_{1-\alpha}} - d_3^2\hat{\sigma}^2$$

$$\Leftrightarrow \left(\sigma\sqrt{d_3^2 - C_{1-\alpha}} - \frac{d_3^2\hat{\sigma}}{\sqrt{d_3^2 - C_{1-\alpha}}} \right)^2 \leq d_3^2\hat{\sigma}^2 \cdot \left(\frac{d_3^2}{d_3^2 - C_{1-\alpha}} - 1 \right)$$

$$\Leftrightarrow \left(\sigma\sqrt{d_3^2 - C_{1-\alpha}} - \frac{d_3^2\hat{\sigma}}{\sqrt{d_3^2 - C_{1-\alpha}}} \right)^2 \leq d_3^2\hat{\sigma}^2 \cdot \frac{C_{1-\alpha}}{d_3^2 - C_{1-\alpha}}$$

$$\Leftrightarrow [\sigma(d_3^2 - C_{1-\alpha}) - d_3^2\hat{\sigma}]^2 \leq d_3^2\hat{\sigma}^2 \cdot C_{1-\alpha}$$

Again we distinguish two cases, G.2.3 and G.2.4, below:

G.2.3. If $[\sigma(d_3^2 - C_{1-\alpha}) - d_3^2\hat{\sigma}] \geq 0$:

$$\Leftrightarrow \sigma(d_3^2 - C_{1-\alpha}) - d_3^2\hat{\sigma} \leq d_3\hat{\sigma} \cdot \sqrt{C_{1-\alpha}}$$

$$\Leftrightarrow \sigma(d_3^2 - C_{1-\alpha}) \leq d_3\hat{\sigma} \cdot \sqrt{C_{1-\alpha}} + d_3^2\hat{\sigma}$$

$$\Leftrightarrow \sigma(d_3^2 - C_{1-\alpha}) \leq d_3\hat{\sigma} \cdot \left(\sqrt{C_{1-\alpha}} + d_3\right)$$

$$\Leftrightarrow \sigma \leq d_3\hat{\sigma} \cdot \frac{\sqrt{C_{1-\alpha}} + d_3}{d_3^2 - C_{1-\alpha}}$$

$$\Leftrightarrow \sigma \leq \hat{\sigma} \cdot \frac{d_3}{d_3 - \sqrt{C_{1-\alpha}}}$$

G.2.4. If $[\sigma(d_3^2 - C_{1-\alpha}) - d_3^2\hat{\sigma}] < 0$:

$$\Leftrightarrow \sigma(d_3^2 - C_{1-\alpha}) - d_3^2\hat{\sigma} \geq -d_3\hat{\sigma} \cdot \sqrt{C_{1-\alpha}}$$

$$\Leftrightarrow \sigma(d_3^2 - C_{1-\alpha}) \geq -d_3\hat{\sigma} \cdot \sqrt{C_{1-\alpha}} + d_3^2\hat{\sigma}$$

$$\Leftrightarrow \sigma(d_3^2 - C_{1-\alpha}) \geq d_3\hat{\sigma} \cdot \left(d_3 - \sqrt{C_{1-\alpha}}\right)$$

$$\Leftrightarrow \sigma \geq d_3\hat{\sigma} \cdot \frac{d_3 - \sqrt{C_{1-\alpha}}}{d_3^2 - C_{1-\alpha}}$$

$$\Leftrightarrow \sigma \geq \hat{\sigma} \cdot \frac{d_3}{d_3 + \sqrt{C_{1-\alpha}}}$$

In summary, the SCR for θ is given by all $\theta = [\mu, \sigma]'$ which satisfy the following two conditions:

$$\hat{\sigma} \cdot \frac{d_3}{d_3 + \sqrt{C_{1-\alpha}}} \leq \sigma \leq \hat{\sigma} \cdot \frac{d_3}{d_3 - \sqrt{C_{1-\alpha}}}$$

and

$$\hat{\mu} - \left\{\sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2} - d_2(\hat{\sigma} - \sigma)\right\}/d_1 \leq \mu \leq$$

$$\hat{\mu} + \left\{\sqrt{\sigma^2 \cdot C_{1-\alpha} - d_3^2(\hat{\sigma} - \sigma)^2} + d_2(\hat{\sigma} - \sigma)\right\}\Big/ d_1$$

Appendix H: MATLAB Programming Code

Please note that all code presented herein remains the intellectual property of the University of the Free State and may not be published without consent. The code for calculating the expected value and covariance matrice of Normal, Logistic, Uniform, Pareto, and Weibull distributions; and examples code for calculating the FGCIs for model parameters and quantile of Weibull (one and two sample problems) and Generalized Extreme Value distributions are solely presented in this thesis. However, other programs code are available electronically on request.

H1: LS and LLS families of distributions: Simulation of expected value and covariance matrix of Z for sample sizes $n = 10$ *and* $n = 25$

Code H1.1: Normal distribution $(n = 10;\ n = 25)$

```
clear
n = 10;
% n = 25;
mu = 0;
sig = 1;
I = 100000;
Z = zeros(I, n);
for i = 1 : I
    z = normrnd(mu, sig, [1, n]);
    s = sort(z);
    Z(i, :) = s;
end
EZ = mean(Z);
V = cov(Z);
```

```
v = ones(size(EZ));

X = [v; EZ]';

H = (X' / V * X) \ X' / V; % Linear predictor for mu and sigma
```

Code H1.2: Logistic distribution ($n = 10$; $n = 25$)

```
clear

n = 10;

% n = 25;

a = 0;

b = 1;

I = 100000;

Z = zeros(I, n);

for i = 1 : I

    u = unifrnd(a, b, [1, n]);

    z = - log((1 - u) ./ u);

    s = sort(z);

    Z(i, :) = s;

end

EZ = mean(Z);

V = cov(Z);

v = ones(size(EZ));

X = [v; EZ]';

H = (X' / V * X) \ X' / V;
```

Code H1.3: Uniform distribution $(n = 10;\ n = 25)$

```
clear
n = 10;
% n = 25;
a = 0;
b = 1;
I = 100000;
Z = zeros(I, n);
for i = 1 : I
    u = unifrnd(a, b, [1, n]);
    s = sort(u);
    Z(i, :) = s;
end
EZ = mean(Z);
V = cov(Z);
v = ones(size(EZ));
X = [v; EZ]';
H = (X' / V * X) \ X' / V;
```

Code H1.4: Pareto distribution $(n = 10;\ n = 25)$

```
clear
n = 10;
% n = 25;
a = 0;
b = 1;
I = 100000;
Z = zeros(I, n);
for i = 1 : I
    u = unifrnd(a, b, [1, n]);
    z = - log(1 - u);
    s = sort(z);
    Z(i, :) = s;
end
EZ = mean(Z);
V = cov(Z);
v = ones(size(EZ));
X = [v; EZ]';
H = (X' / V * X) \ X' / V;
```

Code H1.5: Weibull distribution $(n = 10;\ n = 25)$

```
clear
n = 10;
% n = 25;
mu_s = 1;
sig_s = 1;
I = 100000;
Z = zeros(I, n);
for i = 1 : I
    w = wblrnd(mu_s, sig_s, [1, n]);
    z = log(w);
    s = sort(z);
    Z(i, :) = s;
end
EZ = mean(Z);
V = cov(Z);
v = ones(size(EZ));
X = [v; EZ]';
H = (X' / V * X) \ X' / V;
```

H2: LS and LLS families of distributions: Calculation of point estimates and fiducial generalized confidence intervals for the model parameters of Log Weibull distributions using real data examples from Lawless (2003, p. 240)

Code H2.1: Illustrative examples for one-sample problem: Log Weibull distributions ($n = 20$)

```
clear
n = 20;
mu_s = 1;
sig_s = 1;
a1 = .10;
a2 = .05;
a3 = .01;
J = 10000; % Number of draws from fiducial distributions
LCP1 = round((J * a1 / 2) * 1) / 1;
    if LCP1 <= 0
        LCP1 = 1;
    end
UCP1 = round((J * (1 - a1 / 2)) * 1) / 1;
LCP2 = round((J * a2 / 2) * 1) / 1;
    if LCP2 <= 0
        LCP2 = 1;
    end
UCP2 = round((J * (1 - a2 / 2)) * 1) / 1;
LCP3 = round((J * a3 / 2) * 1) / 1;
    if LCP3 <= 0
        LCP3 = 1;
    end
UCP3 = round((J * (1 - a3 / 2)) * 1) / 1;
```

```
L1 = [1; 0];

    L2 = [0; 1];

    C1 = L1' * H; % Linear predictor for mu estimate

    C2 = L2' * H; % Linear predictor for sigma estimate

    w1s =
[32.0;35.4;36.2;39.8;41.2;43.3;45.5;46.0;46.2;46.4;46.5;46.8;
47.3;47.3;47.6;49.2;50.4;50.9;52.4;56.3]; % Order statistics of
sample 1

     phat1 = wblfit(w1s);

        muHAT1 = log(phat1(1));

            sigHAT1 = 1 / (phat1(2));

y1s = log(w1s);

    muhat1 = C1 * y1s;

        sighat1 = C2 * y1s;

        w2s =
[39.4;45.3;49.2;49.4;51.3;52.0;53.2;53.2;54.9;55.5;57.1;57.2;

57.5;59.2;61.0;62.4;63.8;64.3;67.3;67.7]; % Order statistics of
sample 2

    phat2 = wblfit(w2s);

        muHAT2 = log(phat2(1));

            sigHAT2 = 1 / (phat2(2));

y2s = log(w2s);

    muhat2 = C1 * y2s;

        sighat2 = C2 * y2s;

VR_sig_1 = zeros(1, J);

VML_sig_1 = zeros(1, J);

VR_mu_1 = zeros(1, J);

VML_mu_1 = zeros(1, J);
```

```
VR_sig_2 = zeros(1, J);

VML_sig_2 = zeros(1, J);

VR_mu_2 = zeros(1, J);

VML_mu_2 = zeros(1, J);

for j = 1 : J

    Z = wblrnd(mu_s, sig_s, [n, 1]);

    PHAT = wblfit(Z);

        MUHAT = log(PHAT(1));

            SIGHAT = 1 / (PHAT(2));

    z = log(Z);

        zs = sort(z);

            Muhat = C1 * zs;

                Sighat = C2 * zs;

    VR_sig_1(j) = sighat1 / Sighat; % Rank-based FGPQ for sigma 1

    VML_sig_1(j) = sigHAT1 / SIGHAT; % ML-based FGPQ for sigma 1

    VR_mu_1(j) = C1 * (y1s - (R_sig_1 * zs)); % Rank-based FGPQ
for mu 1

    VML_mu_1(j) = muHAT1 - (ML_sig_1 * MUHAT); % ML-based FGPQ
for mu 1

    VR_sig_2(j) = sighat2 / Sighat; % Rank-based FGPQ for sigma 2

    VML_sig_2(j) = sigHAT2 / SIGHAT; % ML-based FGPQ for sigma 2

    VR_mu_2(j) = C1 * (y2s - (R_sig_2 * zs)); % Rank-based FGPQ
for mu 2

    VML_mu_2(j) = muHAT2 - (ML_sig_2 * MUHAT); % ML-based FGPQ
for mu 2

end

% Rank-based point estimates
```

```
R_param_est = [muhat1, sighat1; muhat2, sighat2];

% ML-based point estimates

ML_param_est = [muHAT1, sigHAT1; muHAT2, sigHAT2];

% Rank and ML-based FGCIs

SVR_sig_1 = sort(VR_sig_1);

a1_RCI_Sig_1 = [SVR_sig_1(LCP1), SVR_sig_1(UCP1)];

SVML_sig_1 = sort(VML_sig_1);

a1_MLCI_Sig_1 = [SVML_sig_1(LCP1), SVML_sig_1(UCP1)];

a2_RCI_Sig_1 = [SVR_sig_1(LCP2), SVR_sig_1(UCP2)];

a2_MLCI_Sig_1 = [SVML_sig_1(LCP2), SVML_sig_1(UCP2)];

a3_RCI_Sig_1 = [SVR_sig_1(LCP3), SVR_sig_1(UCP3)];

a3_MLCI_Sig_1 = [SVML_sig_1(LCP3), SVML_sig_1(UCP3)];

SVR_sig_2 = sort(VR_sig_2);

a1_RCI_Sig_2 = [SVR_sig_2(LCP1), SVR_sig_2(UCP1)];

SVML_sig_2 = sort(VML_sig_2);

a1_MLCI_Sig_2 = [SVML_sig_2(LCP1), SVML_sig_2(UCP1)];

a2_RCI_Sig_2 = [SVR_sig_2(LCP2), SVR_sig_2(UCP2)];

a2_MLCI_Sig_2 = [SVML_sig_2(LCP2), SVML_sig_2(UCP2)];

a3_RCI_Sig_2 = [SVR_sig_2(LCP3), SVR_sig_2(UCP3)];

a3_MLCI_Sig_2 = [SVML_sig_2(LCP3), SVML_sig_2(UCP3)];

SVR_mu_1 = sort(VR_mu_1);

a1_RCI_mu_1 = [SVR_mu_1(LCP1), SVR_mu_1(UCP1)];

SVML_mu_1 = sort(VML_mu_1);

a1_MLCI_mu_1 = [SVML_mu_1(LCP1), SVML_mu_1(UCP1)];

a2_RCI_mu_1 = [SVR_mu_1(LCP2), SVR_mu_1(UCP2)];

a2_MLCI_mu_1 = [SVML_mu_1(LCP2), SVML_mu_1(UCP2)];
```

```
a3_RCI_mu_1 = [SVR_mu_1(LCP3), SVR_mu_1(UCP3)];

a3_MLCI_mu_1 = [SVML_mu_1(LCP3), SVML_mu_1(UCP3)];

SVR_mu_2 = sort(VR_mu_2);

a1_RCI_mu_2 = [SVR_mu_2(LCP1), SVR_mu_2(UCP1)];

SVML_mu_2 = sort(VML_mu_2);

a1_MLCI_mu_2 = [SVML_mu_2(LCP1), SVML_mu_2(UCP1)];

a2_RCI_mu_2 = [SVR_mu_2(LCP2), SVR_mu_2(UCP2)];

a2_MLCI_mu_2 = [SVML_mu_2(LCP2), SVML_mu_2(UCP2)];

a3_RCI_mu_2 = [SVR_mu_2(LCP3), SVR_mu_2(UCP3)];

a3_MLCI_mu_2 = [SVML_mu_2(LCP3), SVML_mu_2(UCP3)];
```

H3: LS and LLS families of distributions: Calculation of point estimates and fiducial generalized confidence intervals for the ratio of σ_1 and σ_2, difference of μ_1 and μ_2, and difference of η_1 and η_2 of Log Weibull distributions using real data example from Lawless (2003, p. 240)

Code H3.1: Illustrative example for two-sample problem: Log Weibull distributions ($n_1 = n_2 = 20$)

```
n1 = 20;

n2 = 20;

mu_s1 = 1;

sig_s1 = 1;

mu_s2 = 1;

sig_s2 = 1;

a1 = .10;

a2 = .05;

a3 = .01;

J = 10000; % Number of draws from fiducial distributions

LCP1 = round((J * a1 / 2) * 1) / 1;

    if LCP1 <= 0

        LCP1 = 1;

    end

UCP1 = round((J * (1 - a1 / 2)) * 1) / 1;

LCP2 = round((J * a2 / 2) * 1) / 1;

    if LCP2 <= 0

        LCP2 = 1;

    end

UCP2 = round((J * (1 - a2 / 2)) * 1) / 1;

LCP3 = round((J * a3 / 2) * 1) / 1;

    if LCP3 <= 0
```

```
        LCP3 = 1;

    end

UCP3 = round((J * (1 - a3 / 2)) * 1) / 1;

L1 = [1; 0];

    L2 = [0; 1];

    C1 = L1' * H;

    C2 = L2' * H;

    w1s =
[32.0;35.4;36.2;39.8;41.2;43.3;45.5;46.0;46.2;46.4;46.5;46.8;

47.3;47.3;47.6;49.2;50.4;50.9;52.4;56.3]; % Order statistics of
sample 1

       y1s = log(w1s);

   w2s =
[39.4;45.3;49.2;49.4;51.3;52.0;53.2;53.2;54.9;55.5;57.1;57.2;

57.5;59.2;61.0;62.4;63.8;64.3;67.3;67.7]; % Order statistics of
sample 2

        y2s = log(w2s);

% Rank-based poit estimates of parameters based on samples 1 & 2

muhat1 = 3.8688;

sighat1 = .1116;

muhat2 = 4.0824;

sighat2 = .1144;

% ML-based point estimates of parameters based on samples 1 & 2

muHAT1 = 3.8666;

sigHAT1 = .1066;

muHAT2 = 4.0796;

sigHAT2 = .1094;

p1 = .975; % failure probability for eta 1
```

```
p2 = .975; % failure probability for eta 2
z_p1 = log(- log(1 - p1));
    Lp1 = [1, z_p1]';
        eta_p1 = Lp1' * H * y1s;
z_p2 = log(- log(1 - p2));
    Lp2 = [1, z_p2]';
       eta_p2 = Lp2' * H * y2s;
  rho_R = sighat1 / sighat2; % rank-based rho
  rho_ML = sigHAT1 / sigHAT2; % ML-based rho
  delta_R = muhat1 - muhat2; % Rank-based delta
  delta_ML = muHAT1 - muHAT2; % ML-based delta
  dif_etas = eta_p1 - eta_p2; % ML-based diff
VR_rho = zeros(1, J);
VML_rho = zeros(1, J);
VR_delta = zeros(1, J);
VML_delta = zeros(1, J);
VR_eta_d = zeros(1, J);
VML_eta_d = zeros(1, J);
for j = 1 : J
W_1 = wblrnd(mu_s1, sig_s1, [n1, 1]);
    PHAT1 = wblfit(W_1);
        MUHAT1 = log(PHAT1(1));
            SIGHAT1 = 1 / (PHAT1(2));
    z1 = log(W_1);
        z1s = sort(z1);
```

```
            Muhat1 = C1 * z1s;

                Sighat1 = C2 * z1s;

W_2 = wblrnd(mu_s2, sig_s2, [n2, 1]);

    PHAT2 = wblfit(W_2);

        MUHAT2 = log(PHAT2(1));

            SIGHAT2 = 1 / (PHAT2(2));

    z2 = log(W_2);

        z2s = sort(z2);

            Muhat2 = C1 * z2s;

                Sighat2 = C2 * z2s;

    % Rank and ML based FGPQs

    VR_rho(j) = (sighat1 / Sighat1) * (Sighat2 / sighat2);

    VML_rho(j) = (sigHAT1 / SIGHAT1) * (SIGHAT2 / sigHAT2);

    VR_delta(j) = (C1 * (y1s - (sighat1 / Sighat1 * z1s))) - (C1
* (y2s -
(sighat2 / Sighat2 * z2s)));

    VML_delta(j) = (muHAT1 - (sigHAT1 / SIGHAT1 * MUHAT1)) -
(muHAT2 - (sigHAT2 / SIGHAT2 * MUHAT2));

    VR_eta_d(j) = (muhat1 - ((sighat1 / Sighat1) * (Muhat1 -
z_p1))) - (muhat2 - ((sighat2 / Sighat2) * (Muhat2 - z_p2)));

    VML_eta_d(j) = (muHAT1 - (sigHAT1 / SIGHAT1 * (MUHAT1 -
z_p1))) - (muHAT2 - (sigHAT2 / SIGHAT2 * (MUHAT2 - z_p2)));

end

% Rank-based point estimates of rho, delta, and eta_diff

R_param_est = [rho_R, delta_R, dif_etas]

% ML-based point estimates of rho, delta, and eta_diff

ML_param_est = [rho_ML, delta_ML, dif_etas]

% 90%, 95% and 99% FGCIs for rho_R, rho_ML, delta_R,
delta_ML,dif_etas
```

```
SVR_rho = sort(VR_rho);
a1_RCI_rho = [SVR_rho(LCP1), SVR_rho(UCP1)];
SVML_rho = sort(VML_rho);
a1_MLCI_rho = [SVML_rho(LCP1), SVML_rho(UCP1)];
a2_RCI_rho = [SVR_rho(LCP2), SVR_rho(UCP2)];
a2_MLCI_rho = [SVML_rho(LCP2), SVML_rho(UCP2)];
a3_RCI_rho = [SVR_rho(LCP3), SVR_rho(UCP3)];
a3_MLCI_rho = [SVML_rho(LCP3), SVML_rho(UCP3)];
SVR_delta = sort(VR_delta);
a1_RCI_delta = [SVR_delta(LCP1), SVR_delta(UCP1)];
SVML_delta = sort(VML_delta);
a1_MLCI_delta = [SVML_delta(LCP1), SVML_delta(UCP1)];
a2_RCI_delta = [SVR_delta(LCP2), SVR_delta(UCP2)];
a2_MLCI_delta = [SVML_delta(LCP2), SVML_delta(UCP2)];
a3_RCI_delta = [SVR_delta(LCP3), SVR_delta(UCP3)];
a3_MLCI_delta = [SVML_delta(LCP3), SVML_delta(UCP3)];
SVR_eta_d = sort(VR_eta_d);
a1_RCI_eta_d = [SVR_eta_d(LCP1), SVR_eta_d(UCP1)];
SVML_eta_d = sort(VML_eta_d);
a1_MLCI_eta_d = [SVML_eta_d(LCP1), SVML_eta_d(UCP1)];
a2_RCI_eta_d = [SVR_eta_d(LCP2), SVR_eta_d(UCP2)];
a2_MLCI_eta_d = [SVML_eta_d(LCP2), SVML_eta_d(UCP2)];
a3_RCI_eta_d = [SVR_eta_d(LCP3), SVR_eta_d(UCP3)];
a3_MLCI_eta_d = [SVML_eta_d(LCP3), SVML_eta_d(UCP3)];
```

H4: LSS family of distributions: Calculation of point estimates and fiducial generalized confidence intervals for the model parameters and quantile of Generalized Extreme Value distribution using real data examples, also analysed by Beirlant et al. (2004, pp. 452-459)

Code H4.1: Illustrative examples: Generalized Extreme Value distribution $(n = 70)$ *based on* θ^* *parametrization*

```
H_vec = H;

L2 = [0, 1]';

HM = H_X; % H matrix of demension n x 282 containing linear
predictors for
mu* and sigma*

DATA = ctwind; % 70 x 3 matrix of data: Harbour; Airport and
Robben Island

M = 10000; % Number of draws from the fiducial distribution of
FGPQs

n = 70;

c = 1; % c = 1 refers to (Cape Town Harbour data); c = 2 (Cape
Town Airport data); and c = 3 (Cape Town Robben Island data)

a = 0;

b = 1;

% Input THETA* parametrization

    mu_star = 35;

    sig_star = 4.5;

    xi = .14;

p = .975; % failure probability for p-quantile of GEV

lambda = 1;

z_p = - log(p) / lambda; % p_quantile of standard Exponential

 eta_p = mu_star + sig_star * ((z_p^(- xi) - 1) / xi); %
p_quantile of a GEV distribution based on THETA* parametrization

D = sort(ctwind(:, c)); % Order statistics of Harbour, Airport,
or Robben Island data
```

```
a1 = .10;

a2 = .05;

a3 = .01;

LCP1 = round((M * a1 / 2) * 1) / 1;

    if LCP1 <= 0

        LCP1 = 1;

    end

UCP1 = round((M * (1 - a1 / 2)) * 1) / 1;

LCP2 = round((M * a2 / 2) * 1) / 1;

    if LCP2 <= 0

        LCP2 = 1;
    end

UCP2 = round((M * (1 - a2 / 2)) * 1) / 1;

LCP3 = round((M * a3 / 2) * 1) / 1;

    if LCP3 <= 0

        LCP3 = 1;

    end

UCP3 = round((M * (1 - a3 / 2)) * 1) / 1;

xi_1 = .01; % Starting value of POSITIVE Xi's for the grid

xi_step = .01; % Increament steps of Xi

 X_i = zeros(1, M + 1);

     X_i(1) = 1;      % Filling the 1st position in vector X_i
with initial value of 1st element 1

sig_s = zeros(1, M + 1);

sig = zeros(1, M + 1);

mu_s = zeros(1, M + 1);

mu = zeros(1, M + 1);
```

```
ETA = zeros(1, M + 1);

ys = D';

for i = 1 : M

    k = round(X_i(i) * 100) / 100;

    c_mu_s = round(((2 * (k - xi_1) / xi_step) + 1) * 1) / 1;

       if c_mu_s <= 0

           c_mu_s = 1; % Keeping indices within the lower bound
(column 1)

       end

           if c_mu_s > 281 % last column number of mu* is 281

               c_mu_s = 281; % keeping indices within the upper
bound (column 281)

           end

    c_sig_s = c_mu_s + 1; % Columns numbers of linear predictors
of sigma*

    L1_xi = HM(:, c_mu_s); % Catching the linear predictors of
mu* from HM matrix

    L2_xi = HM(:, c_sig_s); % Catching the linear predictors of
sigma* from HM matrix

        u = unifrnd(a, b, [1, n]);

            z = - log(u);

        Zz_Xi = z .^ - X_i(i);

                Zz_s = sort(Zz_Xi);

                ZZ_Xi = (Zz_s - 1) ./ X_i(i);

% Drawing copies of parameters estimates from the distribution of
their respective CFGPQs

            sig_s(i+1) = L2_xi' * ys' / (L2_xi' * ZZ_Xi');

             sig(i+1) = sig_s(i+1) / X_i(i);
```

```
            mu_s(i+1) = L1_xi' * (ys' - (sig_s(i+1) * ZZ_Xi'));

            mu(i+1) = mu_s(i+1) - sig(i+1);

% since the support of the distribution y is lower bounded at mu,
we make % % sure that the estimate of mu is less than or equal to
the smallest %observation ys(1)

                   if mu(i+1) > ys(1)

                       mu(i+1) = ys(1) - .00001;

                   end

        u_2 = unifrnd(a, b, [1, n]); % Draw a new standard
Uniform

            z_2 = - log(u_2); % new standard Exponential

            z_2_s = sort(z_2 .^ - 1);

        X_i(i+1) = L2' * H * log(mu(i+1) - ys' ) / (L2' * H *
log(z_2_s'));

             if X_i(i+1) < .01  % Because we have simulated
linear %  %predictors with smallest value of xi = 0.01,we limit
all the estimates
% of X_i to >= 0.01

                    X_i(i+1) = .01;

             end

        ETA(i+1) = mu_s(i+1) + sig_s(i+1) * ((z_p ^ - X_i(i+1) -
1) / X_i(i+1));

end

% CFGCIs

% Sigma*:

    V1s = sort(sig_s(2 : length(sig_s)));

        V1s_L1 = V1s(LCP1);

        V1s_U1 = V1s(UCP1);

             V1s_L2 = V1s(LCP2);

             V1s_U2 = V1s(UCP2);
```

```
V1s_L3 = V1s(LCP3);

V1s_U3 = V1s(UCP3);

% Sigma:

V2s = sort(sig(2 : length(sig)));

V2s_L1 = V2s(LCP1);

V2s_U1 = V2s(UCP1);

V2s_L2 = V2s(LCP2);

V2s_U2 = V2s(UCP2);

V2s_L3 = V2s(LCP3);

V2s_U3 = V2s(UCP3);

% Mu*:

V3s = sort(mu_s(2 : length(mu_s)));

V3s_L1 = V3s(LCP1);

V3s_U1 = V3s(UCP1);

V3s_L2 = V3s(LCP2);

V3s_U2 = V3s(UCP2);

V3s_L3 = V3s(LCP3);

V3s_U3 = V3s(UCP3);

% Mu:

V4s = sort(mu(2 : length(mu)));

V4s_L1 = V4s(LCP1);

V4s_U1 = V4s(UCP1);

V4s_L2 = V4s(LCP2);

V4s_U2 = V4s(UCP2);

V4s_L3 = V4s(LCP3);

V4s_U3 = V4s(UCP3);
```

```
% Xi:
V5s = sort(X_i(2 : length(X_i)));
   V5s_L1 = V5s(LCP1);
   V5s_U1 = V5s(UCP1);
       V5s_L2 = V5s(LCP2);
       V5s_U2 = V5s(UCP2);
           V5s_L3 = V5s(LCP3);
           V5s_U3 = V5s(UCP3);
% Eta:
 V6s = sort(ETA(2 : length(ETA)));
       V6s_L1 = V6s(LCP1);
       V6s_U1 = V6s(UCP1);
           V6s_L2 = V6s(LCP2);
           V6s_U2 = V6s(UCP2);
               V6s_L3 = V6s(LCP3);
               V6s_U3 = V6s(UCP3);
Eta_Exp_p = z_p;
Eta_GEV_p_xi = eta_p;
% point estimates of model parameters and quantile
P_Est1 = sum(V1s) / M; % point estimate for sigma*
P_Est2 = sum(V2s) / M;  % point estimate for sigma
P_Est3 = sum(V3s) / M;  % point estimate for mu*
P_Est4 = sum(V4s) / M;  % point estimate for mu
P_Est5 = sum(V5s) / M;  % point estimate for xi
P_Est6 = sum(V6s) / M;  % point estimate for eta_.975
```

```
% FGCIs for the model parameters and quantile

FGCI1 = [V1s_L1, V1s_U1; V1s_L2, V1s_U2; V1s_L3, V1s_U3]; %
sigma*

FGCI2 = [V2s_L1, V2s_U1; V2s_L2, V2s_U2; V2s_L3, V2s_U3]; % sigma

FGCI3 = [V3s_L1, V3s_U1; V3s_L2, V3s_U2; V3s_L3, V3s_U3]; % mu*

FGCI4 = [V4s_L1, V4s_U1; V4s_L2, V4s_U2; V4s_L3, V4s_U3]; % mu

FGCI5 = [V5s_L1, V5s_U1; V5s_L2, V5s_U2; V5s_L3, V5s_U3]; % xi

FGCI6 = [V6s_L1, V6s_U1; V6s_L2, V6s_U2; V6s_L3, V6s_U3]; %
eta_.99
```