PRACTICAL ASPECTS OF CORRESPONDENCE FACTOR ANALYSIS AND RELATED MULTIDIMENSIONAL METHODS

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THESIS

SUBMITTED IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN MATHEMATICAL STATISTICS

UNIVERSITY OF THE ORANGE FREE STATE

BLOEMFONTEIN

DEPARTMENT OF MATHEMATICAL STATISTICS

JUNE, 1977

A C K N O W L E D G E M E N T S

I would like to thank Prof. H.W. Browne for his valuable advise and literature in connection with this work. A great deal of the documentation for the Correspondence analysis was not available in a prescribed language. My sincerest thanks and appreciation go to Michael Greenacre for his effort in supplying me with a written translation of the necessary articles.

Furthermore my thanks are extended to Prof. D.J. de Waal for his guidance and support. Without his enthusiasm and experience this work would surely be unfounded.

Assistance was rended by Dr. Underhill who forwarded the programming with respect to Multidimensional Scaling.

A final thanks goes to the Computer Centre of the University of the Orange Free State. Members of Staff offered suggestions and assistance. A special thanks to Marie Strydom for her patience and effort in the presentations and editing of this thesis.

I N T R O D U C T I O N

INTRODUCTION

Correspondence Factor Analysis is a multivariate technique which general aim is to find associations and oppositions between subjects and variables, as in other multivariate methods. Its advantages compared to other methods are related to its more sophisticated mathematics and its simultaneous representation of subjects and variables on the same factorial axes. Theoretically the method is shown to be equivalent to a special case of Hotteling's canonical correlation analysis and also to a scale-free variant of Principal Components Analysis.

Correspondence Factor Analysis is developed by Prof. J.P. Bénzecri at the Mathematical Statistics Laboratory, Faculty of Science, Paris. The name "Correspondence Analysis" is a translation of J.P. Bénzecri's "Analyse Factorielle des Correspondances".

In Chapter 1 we discuss the domain of Correspondence Analysis and give a description of the data tables to which Correspondence Analysis has been applied.

Chapter 2 introduces the formulation of Correspondence Analysis. We present the theory how to compute the factors and explain the computation of the contributions. Furthermore we discuss the input data matrix and some of its restrictions.

Chapter 3 describe the results of the analysis which form the output of the programs in Chapter 4. We explain systematic methods of interpreting the listings produced by the computer, referring to Chapter 4's example on Israeli rainfall. Finally in Chapter 3 we gives a table which explain the differences between Principal Component Analysis and Correspondence Analysis. I would like to mention at this stage that most of the theory given in Chapter 1, Chapter 2 and Chapter 3 is take from Col. 2 of Prof. J.P. Bénzecri's

"L' Analyse des données" which was translated by Michael Greenacre. Useful ideas were also taken from an article "Correspondence Factor Analysis: An Outline of its Method by H. Teil in the Mathematical Geology, Vol 7, No. 1, 1975 page 3-12.

In Chapter 5 we discuss function plots of High-Dimensional Data. Furthermore we imply Andrew's method on the factors computed by Correspondence Analysis.

Finally in Chapter 6 we explain the theory of Multidimensional Scaling together with an example based on the factors as plotted in Chapter 5.

CHAPTER 1.

1.1 INTRODUCTION.

In this section we consider the different types of data sets to which correspondence analysis has been applied. Correspondence factor analysis may be applied to any tipe of data and to any number of data points.

The main types of data are

(1) Homogeneous data

(2) Heterogeneous data

(3) Exhausitive data

The best types of data are finite sets I and J with whole positive numbers, e.g., data from questionaires and surveys because these numbers are independent of any unitary system. Attention must be paid to the Unit of measure used in a study of a homogenous data set so that it has the same meaning throughout the matrix. The best method of handling heterogenous data is to use a logical code, i.e., divide each variable into classes of similar probability and consider each value as being present or absent in each class. Starting from the treatment of frequency tables, where one can check the validity of the results using a probability model, we shall gradually extend the argument to the treatment of several different kinds of data sets.

1.2 FREQUENCY TABLES.

This is the simplest type of data considered suited to correspondence analysis. Let us take two finite sets I and J; a probability distribution $p_{IJ} = \{p_{ij} | i \in I,\ j \in J\}$ on the Cartesian product $I \times J$ may also be considered as a system of nonnegative point masses with total sum equal to 1, each of these masses being assigned to a couple (i,j) comprising an element i of I and an element j of J.

The easiest case is met when one logs the occurencies of independent events (i,j) which are the conjunction of the realization of an element i of I and an element j of J. If $k(i,j)$ is the number of times one has observed the outcome of the event (i,j) and k is the total number of events observed, an estimate of the probability p_{ij} is given by the frequency $f_{ij} = k(i,j)/k$.

The simpliest mathematical structure of the data can be described as follows: first two finite sets I and J given "a priori", and secondly an integer-valued nonnegative function $k(i,j)$ that counts the independent or correlated events defined by simultaneous occurence of i and j.

1.3 CONTINGENCY TABLES.

If all the $k(i,j)$ are quantities of the same nature, for example all mass or all amounts of money, the choice of the units of measurement (e.g. kilogram or rand) does not affect the result of the analysis. This is due to the fact that a change in units is equivalent to the multiplication of all the $k(i,j)$ by a common coefficient; consequently the $f_{ij} = k(i,j)/k$ remain unchanged. In fact even if the quantities $k(i,j)$ are not expressed as rational fractions of some common unit, the particular choice of this unit is of no consequence to the results. We shall thus analyse as contingency tables those composed of homogeneous quantities (e.g. expenditures or income expressed in dollars) and those composed of integers counting population rather than events (e.g. number of persons practising a profession i in a given area j); here $k(i,j)$ is the weight of j (or number of j's) in i.

As regards the product set $I \times J$ used as framework for our observations it will often not appear at first glance as the product of two definite sets. Let us for example think of a study of the expenditures of the

R.S.A. citizens. The first set, I = the R.S.A. citizens, is too large for a detailed study; the second, J = the expenditures, is rather a continuum and its division into a finite number of classes presents a tricky problem. Specialists in economics of expenditure usually split the household's budget into a few dozen categories, say 50, and this practice should be accepted at first by the statistician. The advantage of correspondence analysis in such a situation is that, thanks to the principle of distributional equivalence (p. 11 Dis. χ^2 Corr.), the analysis is only slightly sensitive to the detail of the partition adopted.

For practical considerations it is clear that we cannot consider the complete set of R.S.A. citizens, but merely a sample the size of which does not exceed the capabilities of the available set of investigations. We see the choice of set J leads to the same kind of problems as those presented by the choice of I; in both cases the principal of distributional equivalence favours the stability of the results. Furthermore as in correspondence analysis the value of the computed factors for each individual i (or j) does not depend on the total mass but rather on the profile of each row (or column) describing this individual, elements of different magnitudes can be mixed in an analysis.

Let us discuss contingency table analysis as correspondence analysis is defined algebraically equivalent to Fisher's contingency table analysis. It was first published by Hirschfield(1935), but since that time it has suffered widespread neglect, and has been rediscovered by Guttman(1959). Hirschfield's treatment of the topic is clear and succinct, but was not cited by Fisher(1940). As a result, Fisher has frequently been regarded as the method's first inventor.

Thus, given on $m \times n$ contingency table $A = [a_{ij}]$, specifying the counts of joint occurences of two discrete variates, let $K = (I,J)$ be the bivariate random variable specifying the outcome of each individual observation from which the table was assembled. The count a_{ij} is then simply the number of times that the random variable K assumed the value (i,j) in the observed sample. Fisher's "contingency table analysis" consists of looking for functions f,g defined on the ranges of I,J, such that the correlation of the derived random variables f(I), g(J) is a maximum. Rephrasing the contingency table analysis amounts to looking for scores $X = (x_1, \ldots, x_m)^T$ and $y = (y_1, \ldots, y_n)^T$, such that when the functions f and g are defined by the relations $f(i) = x_i$ and $g(j) = y_j$, then the correlation of the random variables f(I) and g(J) is a maximum. Other expositions of this approach are given in Williams(1952), Kendall and Stuart(1961, p.569), Bénzecri(1969) and Lancaster(1969).

1.4 MEASUREMENT TABLES.

For contingency tables we were carefull to maintain two important properties regarding the data: homogeneity and exhaustivity. By homogeneity we mean that all the entities presented in the table are of the same nature. By exhaustivity we mean that the sets I and J represent a complete investigation of a natural phenomenon.

Suppose we are studying the distribution of commercial activity of Johannesburg. It is possible to follow the existing rule of subdivision of the city into 30 districts, thus defining I. As regard the commercial activity, the set J, it is feasible to subdivide it, say, into 10 classes. Now, by defining K(i,j) = number of shops of type j in district i, have we thus for satisfied the condition of homogeneity? Probably not, as with our numbering scheme we must want one individual for a large store as well as for a very small tobacconist. Even if one takes care to count the small tobacconists and the stores in different columns j, homogeneity is not satisfied because the unit of measurement chosen, the firm, does not have the same meaning in the two cases presented. We should thus replace our simple counting scheme for example, by a measurement $k(i,j)$ = surface occupied by the firm in the district i. It is necessary to choose a unit of measurement which bears the same meaning over the entire range of the table. Too large a difference in quantity between the large store and the tobacconist introduces here a qualitative heterogeneity.

We have gradually moved from the study of contingency tables to that of a larger set of measurement tables. The set I is now a set of individuals supposedly representative of a potentially interesting population which is generally very large and more or less well defined, for the human race. The set J is a set of measurements constructed in such a way that the vector $\{k(i,j) | j \in J\}$ (the j-th row of the table) is a satisfactory description of the individual i relative to the scope of the study. One may think of J as a sample of the set of all the variables that could be measured, nevertheless it is certainly true that the concepts of exhaustivity and equal weighting are now met in a very weak sense. The arbitrariness in the choice of the framework of the study (namely the set $I \times J$) has become rather large.

1.5 LOGICAL DESCRIPTION TABLES.

It has been noticed several times that one of the best ways of reducing a heterogeneous data matrix to a common unit is to use a logical coding scheme, i.e. where each measurement scale is replaced by a partition into classes of approximately equal probability, where instead of using a real-valued measurement one simply records whether the value falls into a certain class. For example, we can represent one measurement j by three columns j_1, j_2, j_3, so that for an individual i for

for which the value j is small and falls into the first class we shall record:

$$k(i,j_1) = 1; \quad k(i,j_2) = 0; \quad k(i,j_3) = 0$$

or where the value is fairly high, in the domain of class 2 and 3:

$$k(i,j_1) = 0; \quad k(i,j_2) = \tfrac{1}{2}; \quad k(i,j_3) = \tfrac{1}{2}$$

We can thus substitute a scheme of continuous measurements by a family of classes of logically coded variables.

We talk of a logical description table when the $k(i,j)$ assume the value 1 or 0 in a Boolean sense: $k(i,j) = 1$ means that individual i has property j and $k(i,j) = 0$ means that i does not have property j. It is possible to code various types of information in this way, for example plant j grows in area i, or student i gives the answer j.

We say that a logical description table is in complete disjunctive form if the following condition is satisfied: the set of columns (or properties) J is divided into a family Q of subsets (or questions) q such that:

$$\forall i \in I, \forall q \in Q, \; j \in Q : (k(i,j) = 1 \quad (j' \in q;\ j' \neq j) \Rightarrow k(i,j') = 0)$$

In other words each individual i has in each class q one and only one property j. One can think of Q as a "questionnaire"; to each question $q \in Q$ the subject i may answer by selecting a set of attitudes in which the abstention can be included; to each of these attitudes is assigned a column j of the matrix $k_{I \times J}$; if i gives the answer j to the question q then $k(i,j) = 1$ and for any other $j' \in q$, $k(i,j') = 0$. In the particular case where the abstention is not considered and where all the questions only allow answers yes or no for each question q, only two attitudes q^+ and q^- (yes-no) are possible, so that:

$$J = Q^+ \cup Q^- = \cup\{\{q^+, q^-\} | q \in Q\}$$

The person i who answers yes to q has $k(i,q^+) = 1$, $k(i,q^-) = 0$; and the one i' who answers no has $k(i',q^+) = 0$, $k(i',q^-) = 1$.

From a mathematical point of view a table k_{IJ} in complete disjunctive form presents the great advantage that the results of a correspondence analysis are equivalent to those obtained by analyzing a true contingency table. More precisely it can be shown that if t_{JJ} is the symmetric correspondence matrix with integer values on $J \times J$ such that,

$t(j,j") = \text{Card}\ \{i | i \in I;\ k(i,j) = k(i,j") = 1\}$ then the factors

ϕ^J (functions on J of mean 0 and variance 1) computed from t_{JJ} are the same as those computed from k_{IJ}; the eigenvalues computed from t_{JJ} are the square of those computed from k_{IJ}.

1.6 INTENSITY LEVEL TABLE.

Another type of data commonly considered is a table k_{IM} which gives the following information on a set of individuals I : for each $i \in I$, $m \in M$, $k(i,m)$ is an intensity level of each i between 0 and an upper bound Max_m which is usually the same for all the columns of matrix k_{IM}. A logical description table can be considered as a particular type of intensity level table in which all the intensity levels can take only the values 0 and 1. This analogy suggests a similar doubling of intensity level tables. To table k_{IM} is associated another table defined as follows: J is a set of all the couples m^+, m^-

in which each subject under study has been doubled, so that Card $J = 2 \times$ Card M:

$$J = \cup\{\{m^+, m^-\} m \in M\}$$

In column m^+ is recorded the initial value of the intensity level of subject m, and in column m^- is recorded the complement with respect to Max_m: We could for example chose Max_m as 20.

$k(i,m^+) = k(i,m)$; $k(i,m^-) = Max_m - k(i,m)$

We define $k(i,m^+)$ as the intensity level and $k(i,m^-)$ as the deficiency level.

The doubling concept also suggests another coding scheme which enables us to use correspondence analysis on tables with negative values. It is advisable to double each column with numbers of both signs into a possitive part column m^+ and a negative part column m^-. We have:

if $k(i,m) > 0$ then $k(i,m^+) = k(i,m)$ and $k(i,m^-) = 0$;
if $k(i,m) < 0$ then $k(i,m^+) = 0$ and $k(i,m^-) = -k(i,m)$

This method has been shown to be useful in practice. It can be very meaningful in certain cases; if for example $k(i,m^+)$ is an amount of exportation then $k(i,m^-)$ is the corresponding amount of importation.

1.7 MULTIDIMENSIONAL TABLES.

A rectangular table of numbers k_{IJ} can be considered as values of a two dimensional variable defined on the product $I \times J$ of two finite sets. More generally a multidimensional table has elements of multidimensional variable defined on the product $I_1 \times \ldots \times I_p$ of several finite sets. For example if I is a set of countries, J a set of districts, T a set of time intervals (e.g. set of months). A ternary table of rainfall $k_{I \times J \times T}$ may be defined where $k(i,j,t)$ is the rainfall in the country i of the district j during the month t.

Several methods have been developed for the analysis of multidimensional tables with a particular reference to those which are time dependant. Correspondence analysis of rectangular tables is applicable to the analysis of ternary tables and usually gives satisfactory results. In the rainfall example the product set $I \times J \times T$ may be considered in several ways as the product of two sets, one of them being itself the product of two other sets : e.g.

$$I \times J \times T = (I \times T) \times J$$

Consequently the ternary table $k_{I \times J \times T}$ may be presented as the rectangular table $k_{(I \times T) \times J}$ with margins $I \times T$ and J. A row (i,t) in this table will give the set of Card J elements which are rainfall $k(i,t),j) = k(i,j,t)$ for the country i during month t of district j. To the individual countries (I) we have substituted the individual countries at given month periods.

But we can equally consider that $I \times J \times T = I \times (J \times T)$. Now each line refers to a country i and gives the rainfall of all the months in the country i.e. the family indexed by $(j,t) \in J \times T$ of the rainfall by country i of the district t, at any month.

CHAPTER 2.

2. FORMULATION OF CORRESPONDENCE ANALYSIS.

2.1 Data matrix (input)

Correspondence analysis takes into account the probabilistic character of a data matrix $I \times J$ of positive numbers $\{k(i,j) | i \in I, j \in J\}$. This matrix could be obtained from the row data after different transformations as described above in section 1. The following notation is commonly used:

$$K = \Sigma\{k(i,j) | i \in I, j \in J\}$$

$k(i) = \Sigma\{k(i,j) | j \in J\}$; $k(j) = \Sigma\{k(i,j) | i \in I\}$ where k is the total of all values i and j in the matrix.

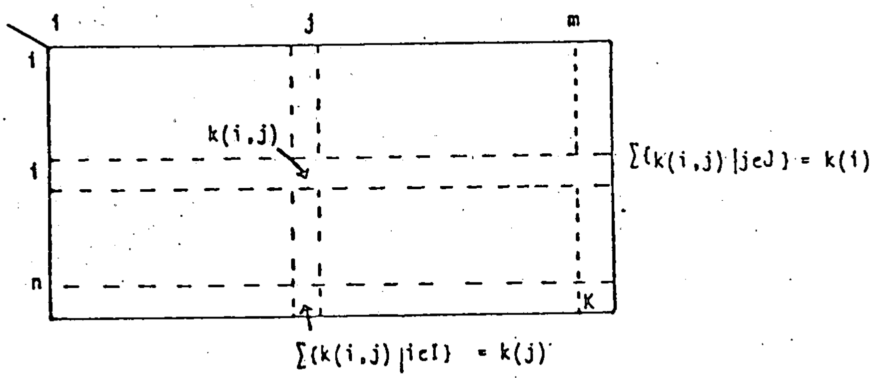


Figure 1. Data matrix $I \times J$.

We thus define a probability estimate matrix f_{IJ} of total sum 1:

$$f_{IJ} = \{f_{ij} | i \in I, j \in J\}; \; f_{ij} = k(i,j)/k$$

These formulae show that most of our computations will be performed as if f_{IJ} were a probability distribution on the finite set $I \times J$ (set of pairs (i,j). Nevertheless one can in any case compute a χ^2 distance which is a purely algebraic concept. We write:

$$f_I = \{k(i)/k | i \in I\}; \; f_J = \{k(j)/k | j \in J\}.$$

These distributions are called marginal distributions of the matrix.

Because $f_{iJ} = \{f_{ij} | j \varepsilon J\}$; $f_{iJ} \varepsilon K_J$, a probability law exists such that if we divide f_{iJ} by the total f_i of row i we obtain a distribution denoted by f_J :

$$f_J^i = f_{iJ} | f_i = \{f_{ij}/f_i | j \varepsilon J\}$$

f_J^i is called the conditional distribution on J for a given i, as f_j^i is in fact the relative weight of couple (i,j) amongst all the couples of the form (i,j') for $j' \varepsilon J$. If the actual data do not warrant a probabilistic interpretation, then f_J^i is known as the "profile" of the element i on J. The study of the "cloud" $N(I)$ of profiles f_J^i with masses f_i is the aim of correspondence analysis. The cloud is considered in the space R_J with the χ^2 distance from the center f_J.

The χ^2 distance $\| f_J^i - f_J^{i'} \| f_j$ is called the distributional distance between i and i'. Simultaneously we consider in R_I, structured by the χ^2 distance centred at f_I, the cloud $N(J)$ of the profiles f_I^j (or column profiles):

$f_I^j = \{f_{ij} | f_j | i \varepsilon I\}$ to which are respectively attached the masses f_j.

The χ^2 distance $\|f_j^i - f_j^{i'}\|$ is given by

$$D^2(i,i') = \sum_{j=1}^{p} \{(f_J^i - f_J^{i'})\}^2$$

In order to eliminate the influence of certain variables which may have large absolute values compared to the rest, and therefore would give unbalanced results, each difference is divided by f_J

(the sum of the column corresponding to the variables j). The new formula is thus:

$$D^2(i,i') = \sum_{j=1}^{m}\{f_J^i - f_J^{i'}\}^2/f_j$$

Also the cloud $N(J)$ of profiles f_J^i with masses f_j is considered in the space R_I which has the χ^2 distance from the center f_I.

The principle of distributional equivalence is one of the advantages existing in correspondence analysis because it gives stability to the results. It is explained as follows: if j' and j'' are two elements of J such that their corresponding column have the same profile i.e.,

$f_I^{j'} = f_I^{j''}$, or $f_I^{j'}$ is proportional to $f_I^{j''}$, then a column j^s equal to their sum can be substituted in their place without modifying the elements of I. The cloud $N(J)$ is evidently not modified because one unique point f_I^{js} with mass $f_j' + f_j''$ replace the two points.

2.2 Factorial axes and factors.

The cloud $N(I)$ of f_J^i in R_J has for its center of gravity G the marginal profile f_J itself. Around this center, there are principal axes of inertia.

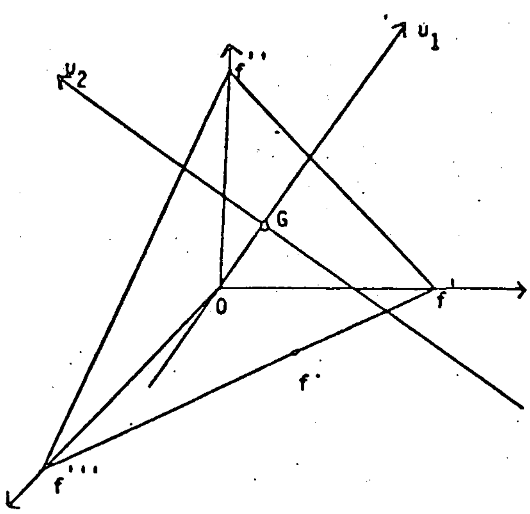


Figure 2. Set of factorial axes u_1 and u_2.

It may be compared to the situation met with in mechanics where the axes of inertian for a system of weighted points need to be found. In a similar manner, axes are extracted by decreasing inertia in correspondence analysis.

In R_J the factorial axis of order α is a unit vector $u_{\alpha J}$ in the sense of the χ^2 metric. We denote by $\phi_\alpha^J = (u_{\alpha J}/f_J)$ the density of this measure. The function ϕ_α^J has zero mean and owing to the unitary property has a variance of 1:

$$\Sigma\{\phi_\alpha^j \, f_j | j \in J\} = 0 \; : \; \Sigma\{(\phi_\sigma^j)^2 f_j | j \in J\} = 1$$

The successive factorial axes are mutually orthogonal in the sense of the χ^2 metric. For the functions ϕ_α^J this property is expressed by:

$$\Sigma\{\phi_\alpha^j \, \phi_\beta^j \, f_j | j \in J\} = \delta_{\alpha\beta} \, .$$

In this formula $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and 1 if $\alpha = \beta$; the two conditions of orthogonality and normalization (i.e. orthonormality) are thus expressed in a single formula. In the language of probability we would say that the functions ϕ_α^J are mutually independent on J with probability distribution f_J.

The cloud N(I) can be referenced relative to the system with origin at f_J and orthogonal axes $u_{\alpha J}$; we shall denote by A the index set, indexed by α, of the family of these axes. We have for each f_J^i coordinates denoted by $F_\alpha(i)$:

$$f_J^i = f_J + \Sigma\{F_\alpha(i) \, u_{\alpha J} | \alpha \in A\} \; ;$$

$$f_J^i = f_J(1 + \Sigma\{F_\alpha(i) \, \phi_\alpha^J | \alpha \in A\};$$

so that keeping in mind that $f_j^i = f_{ij}/f_i$ we have:

$$\forall i,j : f_{ij} = f_i f_j (1 + \Sigma\{F_\alpha(i)\ \phi_\alpha^j | \alpha \in A\})$$

Because the origin has been placed at the centre of gravity f_J of the cloud $N(I)$ the function $F_\alpha(i)$ of i is of zero mean on I relative to the system of masses f_I : $\Sigma\{F_\alpha(i) f_i | i \in I\} = 0$. The moment of inertia of the cloud $N(I)$ in the direction of the axis $U_{\alpha J}$ is by definition the sum of the $F_\alpha(i)^2$ weighted by f_i(i.e. the variance of F_α):

$$\lambda_\alpha = \Sigma\{(F_\alpha(i))^2 f_i | i \in I\}$$

It can be shown that in correspondence analysis the λ_α are necessarily positive numbers lying between 0 and 1. To the function F_α is related a function ϕ_α^I :

$$\phi_\alpha^I = \lambda_\alpha^{-\frac{1}{2}} F(i);\ F_\alpha(i) = \lambda_\alpha^{\frac{1}{2}} \phi_\alpha^i$$

From the definition of the principal axes of inertia we know that the F_α (or the ϕ_α^I) are mutually uncorrelated:

$$\Sigma\{\phi_\alpha\ \phi_\beta\ f_i | i \in I\} = \delta_{\alpha\beta}$$

The sum of the moments of inertia $\Sigma\{\lambda_\alpha | \alpha \in A\}$ is simply the sum $\Sigma\{f_i \| f_J^i - f_J \| f_J^2\ i \in I\}$, i.e. the total inertia of the cloud (the sum of the point masses f_i). To each moment of inertia λ_α corresponds a part of the total inertia:

$\lambda_\alpha / \Sigma\{\lambda_\alpha | \alpha \in A\}$, usually coded τ_α (proportion of inertia).

The cloud $N(J) = R_I$ is investigated in the same manner as for the cloud $N(I) \in R_J$, and it is noted that the functions ϕ_α and the values λ_α are the same for both clouds. Therefore, the function $\phi_\alpha^J = \lambda_\alpha^{-\frac{1}{2}} G_\alpha(j)$ has a complete symmetry with the function

ϕ_α^I. F_α and G_α are known as the factors and the λ_α as the characteristic values of the matrix as defined above. The sum of the λ_α gives the total inertia for N(I) and N(J) and is called the trace. Each factor F_α, G_α relative to the characteristic value λ_α extracts a part of inertia.

2.3 Computation of the eigenvalue λ_α with the aid of the transition formula.

When the factors are known on one of the sets it is possible to determine their value on the other set by using only the simple linear computations of the transition formula. If ϕ_α is one of the factors we have:

$$\Sigma\{f_j^i\ \phi_\alpha^j | j \ \varepsilon\ J\} = \Sigma\{f_{ij}/f_i)\phi_\alpha^j | j\ \varepsilon\ J\} = \lambda_\alpha^{\frac{1}{2}}\ \phi_\alpha^i$$

$$\Sigma\{f_i^j \phi_\alpha^i | i\ \varepsilon\ I\} = \Sigma\{f_{ij}/f_j)\phi_\alpha^i | i\ \varepsilon\ I\} = \lambda_\alpha^{\frac{1}{2}}\phi_\alpha^j$$

The transition formulae enables us to go from set I to set J and vice versa. In the above we can replace the ϕ_α^I, ϕ_α^J by the F_α, G_α because they are proportional.

The transition formula allows us to compute $F_\alpha(i) = \lambda_\alpha^{\frac{1}{2}}\phi_\alpha^i$, coordinate of the point f_J^i of the cloud, by orthogonal projection on the factorial axis $u_{\alpha J}$. The transition formula provide a new definition of the factors. If we start from a function ϕ^I on I we obtain by transition a function χ^J on J; and again by transition we return to a β^I on I:

$$\Sigma\{\phi^i f_i^j | i\ \varepsilon\ I\} = \chi^j;\ \Sigma\{\chi^j f_j^i | j\ \varepsilon\ J\} = \beta^i\ ;$$

ϕ^I is a factor relative to the eigenvalue λ if and only if we have $\beta^I = \lambda\phi^I$. The use of this formula gives not only the usual factors

ϕ_α^I with zero mean but also the constant function equal to 1 which is often called the trivial factor relative to the eigenvalue $\lambda = 1$.

Using the transition formula we can show why the eigenvalue λ_α lies between 0 and 1. Let us consider the set $I \times J$ with distribution f_{IJ} (system of point masses with total mass 1); ϕ_α^I and ϕ_α^J can be considered as functions on $I \times J$, each one being the function of only one of the two variables i or j thus: $\phi_\alpha^I(i,j) = \phi_\alpha^i$; $\phi_\alpha^J(i,j) = \phi_\alpha^j$. On $I \times J$ the ϕ_α^I and ϕ_α^J are functions with zero mean and variance 1 (as on I and J respectively). The correlation coefficient between these two functions can be expressed by:

$$\mathrm{cor}(\phi_\alpha^I,\phi_\alpha^J) = \Sigma\{\phi_\alpha^i\phi_\alpha^j\, f_{ij} | i \in I, j \in J\}$$

$$= \Sigma\{\phi_\alpha^i f_i \Sigma\{\phi_\alpha^j (f_{ij}/f_i) | j \in J\} i \in I\}$$

$$= \Sigma\{\phi_\alpha^i f_i \phi_\alpha^i\, \lambda_\alpha^{\frac{1}{2}} | i \in I\}$$

$$= \lambda_\alpha^{\frac{1}{2}}$$

where a double summation has been performed. We have thus shown, with the aid of the transition formula, that $\lambda_\alpha^{\frac{1}{2}}$ is a correlation coefficient, hence $\lambda_\alpha^{\frac{1}{2}} \in (0,1)$. In this way we have another interpretation of the factors computed from correspondence analysis, namely they are the couples $(\phi_\alpha^I, \phi_\alpha^J)$ of a function on I and a function on J which when considered as functions on $I \times J$ (with distribution f_{IJ}) are the most correlated.

2.4 Computation of the contributions.

In order to shorten the formulae in the following expressions we denote the χ^2 distance between an element of the cloud and the centre of gravity as a polar radius $\rho(i)$ (or $\rho(j)$):

$$\rho(i) = \|f_J^i - f_J\|f_J \quad ; \quad \rho(j) = \|f_I^i - f_I\|f_I$$

The total inertia of the cloud $N(I)$, as that of the cloud $N(J)$, is equal to the trace (or total sum of the eigenvalues), so that we have the formula

$$\Sigma\{\lambda_\alpha | \alpha \in A\} = \Sigma\{\rho(i)^2 f_i | i \in I\} = \Sigma\{\rho(j)^2 f_j | j \in J\} = \text{Trace}$$

We also know that each of the eigenvalues λ_α is a moment of inertia which can be expressed as a sum indexed by I or J:

$$\lambda_\alpha = \Sigma\{F_\alpha(i)^2 f_i | i \in I\} = \Sigma\{G_\alpha(j)^2 f_j | j \in J\}$$

As the square of a χ^2 distance is, according to the usual Euclidian formula, equal to the sum of the squares of the coordinate relative to an orthonormal system of axes (this condition is essential) we have:

$$\rho(i)^2 = \Sigma\{F_\alpha(i)^2 | \alpha \in A\}; \; \rho(j)^2 = \Sigma\{G_\alpha(j)^2 | \alpha \in A\}$$

This is based on the condition that a squared distance (in the sense of χ^2) is the sum of the squares of the coordinates in a system of orthogonal axes.

$\rho(i)^2 f_i$ is known as the absolute contribution of the element i to the trace; $F_\alpha(i)^2 f_i$ the absolute contribution of the element i to the moment of inertia λ_α; $F\,(i)^2$ the absolute contribution of the factor α to i. The relative contribution is the contribution of the factor to the element $- F_\alpha(i)^2/\rho(i)^2$, equal to the cosine squared.

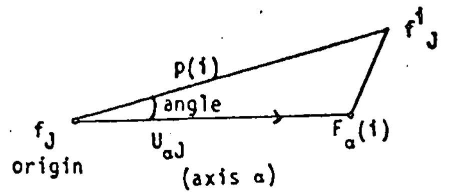


Figure 3. Relative contribution of factor to element is $F_\alpha(i)^2/p(i)^2$, equal to cosine squared.

$$\cos(u_{\alpha J},(f_J^i - f_J)) = F_\alpha(i)/\rho(i)$$

The usage of these terms will be discussed later.

2.5 Simultaneous Representation of I and J.

In order to represent the set J, the cloud $N(J)$ is projected in the space E_p with p axes of inertia. We have already seen that the cloud $N(J)$ is the set of points f_I^j of R_I, where the distances are those given by the χ^2 metric centred at f_I (f_I is the centre of gravity of the cloud $N(J)$).

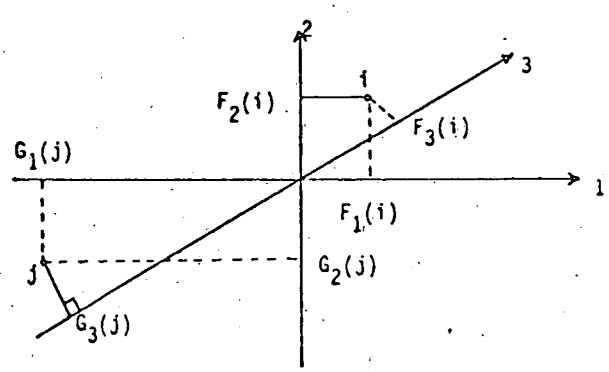
For a factor ϕ, there is an axis with vector $(\phi^I f_I)_I$ (measure with function ϕ^I as density relative to f_I).

A vector x_I of R_I has for its coordinate on the axis $\phi^I o X_I = \Sigma_i \phi^i x_i$. In particular, the vector δ_I^i (representing the unit mass placed at i) has a coordinate ϕ^i in the system of axes of $N(J)$ whereas the coordinates of the representative point i are $\lambda_i(\phi)^{\frac{1}{2}}\phi^i$. We can say that we pass from the system of the δ_I^i to the system of points representing the element i by means of a linear transformation r the eigenvalues of which are the principal axes of inertia of the cloud $N(J)$ relative to the eigenvalues $\lambda_\alpha^{\frac{1}{2}}$.

The point f_I^j, representing j in R, is exactly the barycenter of the δ_I^i weighted with the point masses f_i^j: $f_I^j = \Sigma_i f_i^j \delta_I^i$

In certain problems (especially when the cardinal of the set I, say, is small compared to the cardinal of J) it may be of interest to destroy the symmetric situation between I and J by presenting the cloud N(J) with the set $\{\delta_I^i\}$ representing the set I. Using the usual coding of the output listings this is equivalent to assigning to each i the coordinate $F(i)\lambda^{-\frac{1}{2}}$ instead of the usual F(i). In this way the point i is represented as a point j for which $\delta_I^i = f_I^i$ (the limiting case of a j associated with i only); hence j is exactly at the centre of gravity of the i (the δ_I^i) weighted by the masses f_i^j. It is then by a coefficient λ rather than $\lambda^{\frac{1}{2}}$ that i can be made to coincide with the barycentre of the j weighted by the masses f_j^i.

The clouds may be considered graphically on the same plan.



Graphs show the distribution of the points i and j with respect to the chosen axes.

2.6 Correspondence Analysis as a Method of Scaling.

Guttman(1941), Torgerson(1958, p.538) and Hill(1973) introduced correspondence analysis as a method of scaling. Let's consider a simpler method of scaling namely "gradient analysis" which was developed by R.H. Whittaker(1967).

The data (for example floristic data) of gradient analysis consists of a table of the incidences of a number of species at a number of sites. A particular species of grass indicate wet conditions, while another may indicate dry conditions. We can scale the species accordingly to their suspected preferences along a known physical gradient. For example, a grass with a score of 1 may be wet-loving, and a grass with a score of 10 may be dry-loving. It is clear that a grass of score 5 may be intermediate. The site scores are the averages of the scores of the species which occur in them. If one considers the various environmental gradients and obtain scores along the corresponding axes of variation, we could derive a multidimensional scaling as described in Chapter 6.

The problem arise that the user has to quess the important physical gradients in advance, and his results are therefore highly subjective. An experienced person may interpret the gradients correctly, but a novice is less trustworthy. We could take the data as basic, ignoring physical factors, and use standard multivariate methods to reveal the gradients. The revealed gradients are then related to such physical factors as are thought to be relevant.

Corresponding analysis can be regarded as a generalization of gradient analysis using the method of successive approximation. There are a few definitions and propositions related to Correspondence analysis and we would like to formulate them by looking at an example. Let A be an $m \times n$ data matrix of elements a_{ij} specifying the occurencies of m species (rows) at n sites (columns). If we apply the method of gradient analysis, we can calibrate the sites along a presumed physical gradient by assigning scores $y_j (j = 1, \ldots, n)$ to the sites so as to conform with the physical gradient. The species scores

$$x_i = \sum_j a_{ij} y_j / a_i$$

are the mean site scores of the sites at which they occur. The derived species scores can now be used to derive a new calibration

$$y'_j = \sum_i a_{ij} x_i / a_{.j}.$$

The scores y'_j are a gradient analysis of the sites. We could iterate the process with the new scores y'_j in place of the old ones y_j. Hill(1973) has called this process "reciprocal averaging".

The following two definitions and three propositions as mentioned above are taken from Applied Statist.(1974), 23, No.3, p.342, M.O. Hill.

DEFINITION 1.

Let A be an $m \times n$ table of non-negative numbers a_{ij}, and let $R \equiv \mathrm{diag}(\alpha_i)$ and $C \equiv \mathrm{diag}(\alpha_{.j})$ be the diagonal matrices of row and column totals. It is assumed that none of the totals is zero. The sequence of operations

$$y = C^{-1}A^T x;\ x' = R^{-1}Ay;\ y' = C^{-1}A^T x';\ \ldots$$

in which new sets of scores $y, x', y', \ldots$ are successively derived from an initial set of scores x is referred to here as the "two-way averaging algorithm" corresponding to the matrix A.

The two-way averaging algorithm is simply the process of cross-calibration outlined above; its eigenvectors are the solutions of the correspondence analysis problem defined by the matrix A.

DEFINITION 2.

Using the same notation as above, a triple (ρ,x,y) is a solution of the zero-order correspondence analysis of $A, C_0(A)$, if

$$\rho x = R^{-1}Ay; \quad \rho y = C^{-1}A^{T}x.$$

The elements of the vector x are called "row scores" and the elements of the vector y are called "column scores". The number ρ is, as explained below, the correlation of x and y with respect to the matrix A.

Before much can be said about the method, three simple propositions should be noted.

PROPOSITION 1. The correspondence analysis problem is equivalent to a singular value decomposition problem, and is therefore solved by extracting the eigenvectors of a positive semi-definite symmetric matrix.

PROOF: Defining $R^{\frac{1}{2}} \equiv \mathrm{diag}(\sqrt{a_{i.}})$, and defining $C^{\frac{1}{2}}$ similarly, then (ρ,x,y) is a solution of $C_0(A)$ if and only if

$$\rho(R^{\frac{1}{2}}x) = (R^{-\frac{1}{2}}AC^{-\frac{1}{2}})(C^{\frac{1}{2}}y)$$
$$\rho(C^{\frac{1}{2}}y) = (R^{-\frac{1}{2}}AC^{-\frac{1}{2}})^{T}(R^{\frac{1}{2}}x).$$

This establishes that the solutions are equivalent to a singular value decomposition. Looking at the matter from the point of view of x, we see that

$$\rho^{2}(R^{\frac{1}{2}}x) = (R^{-\frac{1}{2}}AC^{-\frac{1}{2}})(R^{-\frac{1}{2}}AC^{-\frac{1}{2}})^{T}(R^{\frac{1}{2}}x).$$

The matrix preceding $(R^{\frac{1}{2}}x)$ on the right-hand side of the equation is of the form BB^T and is therefore positive semi-definite; ρ^2 is the eigenvalue of the solution. The solutions or "axes" are deemed to be ordered by their eigenvalues.

PROPOSITION 2. The maximal solution of the correspondence analysis problem is $(1,\mathbf{1}_m,\mathbf{1}_n)$, where $\mathbf{1}_m$ is $(1, \ldots, 1)^T$, the m-vector of 1's and $\mathbf{1}_n$ is defined similarly.

PROOF: Recalling the two-way averaging algorithm of Definition 1, and the informal discussion which preceded it, it is clear that the range (i.e. maximum minus the minimum) of the column scores y_j - which are averages - cannot exceed that of the row scores x_i from which they were derived. Similarly the new row scores x_i' must have a range less that that of the column scores y_j. Therefore the range of the scores x_i' is less than that of the scores x_i, so that the eigenvalue ρ^2 of a solution cannot exceed 1. The triple $(1,\mathbf{1}_m,\mathbf{1}_n)$ is a solution, as the average of a set of 1's is 1. It must be maximal. as $\rho^2 = 1$.

PROPOSITION 3. Solutions other than the first satisfy the relation

$$\sum_i a_{i.} x_i = \sum_j a_{.j} y_j = 0.$$

PROOF: By Propositions 1 and 2 the condition of orthogonality to the trivial first axis is

$$(R^{\frac{1}{2}}\mathbf{1}_m)^T(R^{\frac{1}{2}}x) = 0.$$

An analogous result holds for the column scores y.

CHAPTER 3.

3. INTERPRETATION OF THE NUMERICAL AND GRAPHICAL OUTPUT OF THE ANALYSIS.

3.1 Number of interpreted factors, eigenvalues and proportions of inertia.

The computer program provides a listing of the values on I and J of the first extracted factors F , G , with associated eigenvalues λ_α and proportions of inertia $\tau_\alpha(\lambda_\alpha/\Sigma\lambda_\alpha)$. In the past it was the practice to compute only the fist five factors. The motive for this choice was that of covenience - printing the label of F(i) (generally three lettered), the numerical values relevant to the five factors and the total mass k(i) usually fills up the available space on a line of listing. In the early days the computation of the factors was a costly process and up to the fourth factor was sufficient, but today with highly development computers it allows us to go beyond the five factors. The question is now to know at which number to stop the examination.

In the case of a true frequency table the χ^2 test approximately indicates until which factor the explained part of the inertia dominates sampling fluctuations. The case of tables generated by independent events is rather difficult. The interpretation of the factors proceeds according to the meaning of associations and analogies which become opponent, according to typical shapes of the projections, governed by the computed contributions.

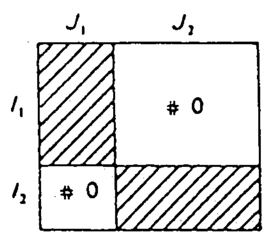
The set of eigenvalues and corresponding percentages must be examined even though they do not give strong indications. Usually we regard as highly meaningful a first factor which represents more than 50% of the total inertia. An eigenvalue greater than 0,6 indicates a possible dichotomy in the data itself, usually because a small number of elements oppose all the others. Therefore, the factor is not meaningful for the set as a whole. If the characteristic value is around 0, 2, the factor becomes interesting. A low value of λ_α indicates that the profile f_J of the individuals is similar to the mean profile f_J. The associated factor could be of significance.

3.2 Geometrical configurations of the planar representations.

To give a visual idea of the values of the factors for the different elements it is most convenient to use two dimensional diagrams considering firstly the one which represents simultaneously the clouds I and J with respect to factorial axes 1 and 2. The plot produced by the computer examines the results for a particular factor. The planar representation according to factorial axes 1 and 2 the projections i have $F_1(i)$ for abscissa and $F_2(i)$ for ordinate, the projections j have $G_1(j)$ for abscissa and $G_2(j)$ for ordinate. These sorted lists are particularly helpful when used in examining the contributions.

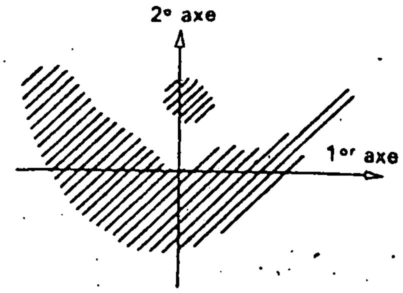
We know that if I is weighted by the system of masses f_i, the two factors $F_1(i)$ and $F_2(i)$ are functions of zero mean and respective variance λ_1 and λ_2, and they are also uncorrelated. A statistician acquainted with Gaussian variables instantly imagines in the plane 1-2, an elliptic cloud centred at the origin, with primary axis $\lambda_1^{\frac{1}{2}}$ (standard deviation) in the direction of the first and $\lambda_2^{\frac{1}{2}}$ in the direction of the second axis. Other shapes can also be observed and their occurrence is rather meaningful: in fact there exists no special reason for the factors to be Gaussian or even unimodal; and furthermore, two uncorrelated variables are not necessarily independent - there may indeed be a relationship between them which is compatible with orthoganality properties (e.g. one is a second degree polynomial of the other). Let us look at three typical shapes.

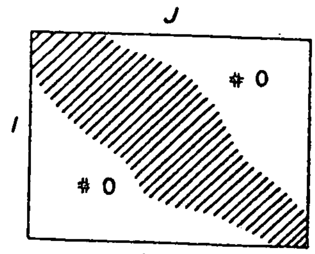
FIRST TYPICAL SHAPE: the cloud is divided into two seperate clusters: $I = I_1 \cup I_2$, $J = J_1 \cup J_2$, where I_1 is associated with J_1 and I_2 with J_2. If one reorganizes the data table by grouping together the rows (the columns) I_1 the I_2 (J_1 then J_2) we have approximately the following situation; outside of the diagonal blocks $I_1 \times J_1$ and $I_2 \times J_2$ the two blocks $I_1 \times J_2$ and $I_2 \times J_1$ are close to zero.



Sometimes one of the diagonial blocks ($I_2 \times J_2$ for example) is composed of a few elements only (e.g. Card $I_2 = 2$, Card $J_2 = 3$); in such a case it may be possible that the few isolated elements strongly disturb the analysis. Here it is advisable to repeat the analysis without these elements, that is analyse the table $k_{I_1 \times J_1}$. Correspondence analysis often reveals strong dichotomies on a few axes (division and subdivision of the cloud) and continuous spreadings on other axis.

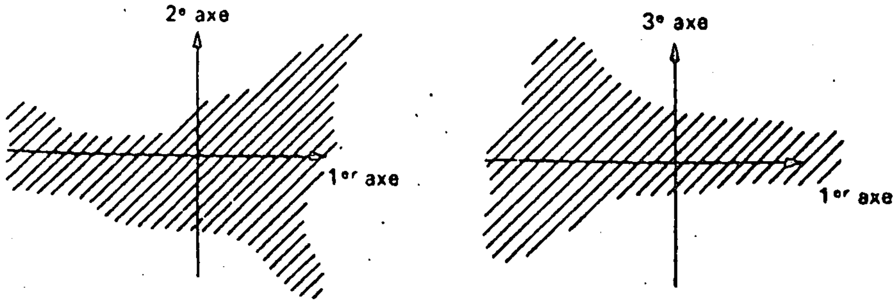
SECOND TYPICAL SHAPE: Cloud in the shape of a parabolic cresent:





Let us suppose (the common case) that the two extremisties of the crescent project onto the first axis on opposite sides of the origin. If the table is rearranged in such a way that the rows (set I) and the columns (set J) are in the same order as they project onto the first axis, there clearly appears a diagonal zone with rather heavy elements $(f_{i.} \times f_{.j} < f_{ij})$ between two rather light corner zones $(f_{ij} < f_{i.} \times f_{.j})$. The second axis contributes to the same general classification, but reveals some meaningful complementary nuances; to a point located on the graph inside the crescent (near the positive part of the second axis as seen in the above figure) corresponds in the table a row (or column) with a rather flat profile (i.e. with relatively high values outside of the diagonal of the table). Such a point does not have a well-defined rank in the general classification but is likely to be mixture of the two extremes (this is due to the barycentric principle, the geometrical expression of the transition formula).

<u>THIRD TYPICAL SHAPE:</u> Cloud in the shape of a triangle (or tetrahedron).



We see in the above illustration that if the first factor is negative the second shows little dispersion around zero, while for positive values of factor 1, the factor 2 is most dispersed (between its extreme negative and positive values). On the other hand it is possible that on a third axis we find dispersion if the factors 1 is negative and concentration around zero if factor 1 is positive. In this case the cloud will have the shape of a tetrahedron in the vector space generated by the three axes; this tetrahedron has two opposite vertices perpendicular to the axis 1, one parallel to axis 2 (on the positive axis 1), the other parallel to axis 3 (on the negative side of axis 1). We seldom obtain perfect triangular or tetrahedic shapes but even an approximate configuration is worth noting.

3.3 Guidelines to understand the meaning of the results.

If an interval of relative importance separates two factors x and y the most important one, say x, has a definite significance. If the percentage of y is low and the interval large, the factor y has relatively little importance. If x and y have similar percentages, they have equally significant, independent meanings.

To interpret an axis, the opposition between the negative and the positive sides of the axis needs to be explained. Its understanding may be clarified by considering the other axis which are not only different but uncorrelated.

Each factor is studied using the absolute contributions for all the elements, unless its characteristic value is high, in which case the relative contributions are used. If the value of an absolute contribution is greater than $\lambda_\alpha/4$ then the corresponding element should be placed as a supplementary element (i.e., it is not used in the calculation of the axes, but may be projected onto the factorial plan). This avoids the distortion of the axes resulting from an element with a high contribution. More detailed guidelines could be seen at the end of Chapter 4.

3.4 Difference between principal Component analysis and Correspondence Analysis.

The following table is given by H. Teil in the Mathematical Geology, Vol. 7, No. 1, 1975 page 11.

Main differences between Principal Component Analysis and Correspondence Analysis.

	Principal component analysis		Correspondence analysis
1.	Euclidean distance between two points	1.	x distance (eq 1)
2.	Individuals have equal weights	2.	Individuals have proportional weights $f_i = k_i/k$.
3.	The individual k(i,j) itself is considered	3.	The "profile" of the individual is described by the vector $\{k(i,j)/k_i \mid j \in J\}$
4.	Diagonolization of variance matrix (euclidean distance) or the correlation matrix (weighted euclidean distance)	4.	Diagonalization of the matrix of vector j profiles
5.	Characteristic values of I and J not equal, and so no representation of I and J on the same axes. Correlation coefficients calculated between a variable and the set of projections of individuals on the factorial axis	5.	Simultaneous representation of I and J because their matrices have the same characteristic values. Associations between I and J seen from the factorial plans

Principal component analysis	Correspondence analysis
6. Rotation of the principal axes of inertia	6. No rotation

CHAPTER 4.

4.1 ILLUSTRATIONS OF CORRESPONDENCE FACTOR ANALYSIS ON DATA.

Data of monthly rainfall averages in the rainy season were compiled by Katznelson(1968-69) for 55 stations in Israel. For each month the average rainfall (mm) was computed in the selected 55 stations (1921-50). The table taken from the Journal of Applied Meteorology Vol. 11. No. 7, October, 1972, pp. 1071-1077 is Table 1 as given on page 48.

A computer program (page 46) produces for each i and j their coordinates relative to m factors $F_\beta(\beta=1, \ldots, 55), G_\alpha(\alpha=1, \ldots, 9)$ together with their absolute and relative contributions (Table 2, page 51, and Table 3, page 52. For each factor, the characteristic value λ_α and the percentage variability explained by the factor is given.

A program (page 46) was written to produce two dimensional diagrams to give a visual idea of the values of the factors for the different elements. We considered the one which represents simultaneously the clouds I and J (3.1, in our case months and stations) with respect to factorial axes 1 and 2 (page 55).

4.2 PROGRAMS.

Before we look at the listings of the programs, it may be useful to summarize the method of Correspondence analysis and describe some of the variables used in the main program.

Input matrix: $K : p \times m$

All elements of K must be positive. Computation is reduced if $p \geq m$

Results are the same whether K or K' is input.

Calculations:

$k_{..} = \sum_{i=1}^{p} \sum_{j=1}^{m} k_{ij}$: Sum of all the elements of K.

$a_{ij} = \frac{k_{ij}}{k_{..}}$: $A : p \times m$ consists of elements of K scaled so that the sum of all the elements of A is 1.

$D_r : p \times p$ Diagonal matrix of row sums of $A . r_{ii} = \sum_{j=1}^{m} a_{ij}$.

$D_c : m \times m$ Diagonal matrix of columns sums of $A . c_{jj} = \sum_{i=1}^{p} a_{ij}$

$$B = D_r^{-\frac{1}{2}} A D_c^{-\frac{1}{2}}$$

The largest characteristic root of $B'B$ is equal to 1.

$V : m \times (m - 1)$ Matrix of unit length characteristic vectors corresponding to the $p - 1$ characteristic roots of $B'B$ which are less than 1.

$D_\gamma : (m - 1)\times(m - 1)$ Diagonal matrix of the p characteristic roots of $B'B$ which are ≤ 1.

($\therefore$ $B'B = VD_\gamma V' + \underline{r}^{\frac{1}{2}} 1 \underline{c}^{\frac{1}{2}}$ where $\underline{r}^{\frac{1}{2}}$ and $\underline{c}^{\frac{1}{2}}$ are vectors of diagonals of $D_r^{\frac{1}{2}}$ and $D_c^{\frac{1}{2}}$.

$V'V = I_{m-1}$).

$U = B\, VD_\gamma^{-\frac{1}{2}}$

$F = D_r^{-\frac{1}{2}} UD_\gamma^{\frac{1}{2}} = D_r^{-\frac{1}{2}} BV$ $\quad p \times (m - 1)$ matrix of row factor loadings.

$G = D_c^{-\frac{1}{2}} VD_\alpha^{\frac{1}{2}}$ $\quad m\times(m - 1)$ matrix of column factor loadings.

Check: $A = D_r(1_p 1'_m + F\, D_\gamma^{-\frac{1}{2}} G')D_c$ where 1_p, 1_m are unit vectors.

The program prints out the first few columns of F and G.

4.3 EXTENDED GUIDELINES TO UNDERSTAND THE MEANING OF THE RESULTS.

First we consider the graph on page 55 and look at the shape of the first two factors compared to the three typical shapes described on page 26. The two-dimensional diagrams are composed of labelled points. In order to go beyond an interpretation of the geometric shape of the clusters, one must take into account the significance of the labels. This significance is usually at hand for at least one of the two sets I or J, say J, which is called set of characteristics (or measurements). On the other hand the set of individuals I can be unknown or difficult to know in all its details.

The relations between the two sets I and J are ruled by the barycentric principle. However we should emphasize that two points i and j which are near one another on a plane graph $(\alpha\ \beta)$ do not necessarily have a high level of associatoin (i.e. a high $f_{ij}/(f_i \times f_j)$. This is due to the fact that the location of i is determined according to the barycentric principle.

The interpretation of an axis involves trying to express the analogies between the points on one side of the origin and similary between those on the other side of the origin, and then explaining as concisely and exactly as possible the opposition between the two extremes. Such a interpretation is usually difficult to find because one has to take into account not only the relative locations of the points the most distant to the right and to the left, but also the location of the points which bring large absolute contributions $(F_\alpha(i)^2 f_i$ or $G_\alpha(j)^2)$ to the factor of interest. It is also to be feared that one might stop, once an explanation more or less compatible with the geometric repartition of the points on the axis is found, without trying to discover the basic causes.

It is therefore necessary to handle each case carefully, if possible with a statistician,to balance the interpretation with the amount of information available. Very often the interpretation of a factor is improved by keeping in mind those which follow it. One has to remember that the successive factors are not only different but also mutually uncorrelated. If an interpretation given to axis 1 seems to be equally good for interpreting axis 3, for example, this is certainly a sign that one should review the analysis with more care.

Let us look at the table given on page 53. We could group the factor column according to their scores for the different subjects. Similarly for the objects. The importance in the interpretation of the factors has already become apparent and could be described the same as the factors of Principal Component analysis. We shall end this section by describing the format of the contributions and by giving some details of their use.

For an analysis involving, say 5 factors, the listing usually gives for each factor 3 contributions namely, the absolute, relative and cumulative contributions. The absolute contribution on page 53 is $F_\alpha(i)^2 f_i/\lambda_\alpha \times 100$. The relative contribution is $F\,(i)^2/\,(i) \times 100$. The cumulative contribution is the absolute of the relative contribution. It is important to point out that the absolute contributive elements are not necessarily those which have the most extreme position on the axis of interest. Another interesting factor is that the sum of the absolute contributions to λ_α is equal to $4\ \lambda_\alpha/5$. Further interpretation of the relative and absolute contribution has already become apparent if one only looks at their definition given in §2.4.

One could use the Multidimensional technique to plot the different 55 stations by using the χ^2 distance, $D^2(i,i')$ as given on page 11.

The result is shown on page 57. The virtual image of this graph is nearly the same as the two dimensional graph of the Correspondence factors on page 55.

4.4 Interpretation of Correspondence Analysis tables and graph.

H. Teil(1975) suggested a Correpondence Analysis table which look like the computer printout on page 53. Each i and j coordinates, relative to m factors F_α, G_α $(\alpha=1, \ldots, m)$ together with their absolute and relative contributions,are printed in this table. For each factor, the characteristic value λ_α and the percentage variability explained by the factor are given. Let us look at the example of Israeli rainfall at 55 stations and 9 months given on page 53.

On the top lefthand side of the table is printed 'Factor 1'. This was really the fifth factor as computed by Correspondence Analysis, but is now treated as Factor 1 because its characteristic value was the highest compared (ignoring the characteristic value 1). The characteristic value is .0153315 and the percentage of inertia is 54.567. One should add the percentage of inertia of Factor 1 and Factor 2 and this sum should be 60% or more to make the two dimensional graph on page 55 significant.

We could now divide this table into the objects and subjects. The nine different months and the 55 different stations are printed under the heading 'Object' and 'Subject' respectively. Consider the object table; under the heading 'Mass' for object 1 is printed; 29.5. This figure is the sum of all the actual rainfall values corresponding to the first month (i.e. object 1).

The distance .8302 (in the sense of χ^2) is the distance from the center of gravity. The next column gives the Correspondence Analysis factor loadings of the first factor. One could group these factors in the same manner as Principal Component analysis. It is clear that the first 4 objects 1,2,3 and 4 (i.e. -.3158, -.1666, -.1422, and -.1348 respectively) could be classified as group 1. It follows from the table that object 5.6.7.8 and 9 form the next group. The same interpretation on the subjects (i.e. stations) could be concluded.

The next three columns gives the contributions namely; absolute, relative and cumulative. The absolute contribution -.7834 is the contribution of object 1 to the λ_α - characteristic value .0153315. The sum of all the absolute contributions of the 9 objects must be equal to 100. The relative contribution -12.0154 is the contribution of the object 1 relative to the other factors to λ_α. The sum of all the relative contributions of object 1 must be equal to 100. The cumulative contribution is the absolute of the relative contribution.

Consider the two dimensional graph of factor 1 and factor 2 given on page 55. The objects and subjects are plotted on the same graph. M1 is associated to month 1 and 'S01' to station 1. One immediately sees that their exists a great variation in rainfall during month 1 and month 9 as these two points are extreme to the origin. The least variation is in months 5 and 6. One could also see that station 6 and 9 are positively correlated to month 1, while station 43 and 38 are negatively correlated to month 1. This discussion on correlation could be extended to all the months and stations. Furthermore one could group the stations and months together to form certain areas, page 56. The stations almost coincide with the factual graphical notation predetermined on a existing map of Israel (page 100).

REFERENCE TO PROGRAMS AND CORRESPONDING EXECUTION EXAMPLES.

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C ***********************************************************************
C      D.BESTER. CORRESPONDENCE FACTOR ANALYSIS
C
C      REFERENCE:MATHEMATICAL GEOLOGY,VOL.7,NO.1,1975. - H. TEIL.
C                HILL- APPL.STATIST.(1974),23,340-354.
C
C      EXECUTE CARDS NEEDED :
C      1.  TITLE CARD    (80 CHARACTERS ALFANUMERIC)
C
C      2.  PROBLEM CARD.
C               COLUMN 1-4      OBJECTS (MAXIMUM OF 80)
C               COLUMN 5-8      FACTORS TO BE TREATED BY CORRESPONDENCE
C                               ANALYSIS. (MAXIMUM OF 10)
C               COLUMN 9-12     SUBJECTS (MAXIMUM OF 330)
C               COLUMN13-16     0= NO PRINTING OF TABLES
C                               1= PRINT TABLES FOR SPECIFIED FACTORS AS
C                               GIVEN ON PAGE 8 OF H.TEIL.
C               COLUMN17-18     FACTORS TO BE PRINTED AS DESCRIBED IN
C                               COLUMN13-16 (NUMBER)
C               COLUMN19-20     FACTOR 1
C               COLUMN21-22     FACTOR 2
C               COLUMN23-24     FACTOR 3
C               .....................
C               .....................
C               .....................
C               COLUMN37-38     FACTOR 10
C      THE ANALYSER MUST BE CAREFUL THAT ALL THE DATA IS POSITIVE
C      AND THE OBJECTS MUST BE < OR = THAN THE SUBJECTS.
C ***********************************************************************
       DIMENSION KP(2),A(330,80),F(330,10),DC(330),DV(80),
      *          DG(80),OPS(20),FOR(60),V(80,80),G(80,10),B(80,80)
      *,SUBM(330),OBJM(80),NPF(10),V1(330,80)
       NWTR=6
C      -----------------------------------------------------------------
C      READ HEADING CARD FROM CARD-READER (TITLE CARD)
C      -----------------------------------------------------------------
       READ (5,2) OPS
     2 FORMAT (20A4)
C      -----------------------------------------------------------------
C      WRITE HEADING AND TITLE ON PRINTER
C      -----------------------------------------------------------------
       WRITE (6,10) OPS
    10 FORMAT(1H1,40X,39(1H*),/41X,'*   CORRESPONDENCE FACTOR ANALYSIS
      * *',/41X,39(1H*),//41X,20A4)
C      -----------------------------------------------------------------
C      READ THE PROBLEM CARD
C      -----------------------------------------------------------------
       READ (5,12)NUMFAC,NOB,NSUB,NT,NP,(NPF(I),I=1,NP)
    12 FORMAT(4I4,12I2)
       IND=1
C      -----------------------------------------------------------------
C      READ THE FORMAT CARD CARD WHICH MUST BE SMALLER THAN 60 CHARACTERS
C      AND WRITE THE FORMAT ON THE PRINTER
C      -----------------------------------------------------------------
       READ (5,14) FOR
    14 FORMAT  (60A1)
       WRITE (6,16) FOR
    16 FORMAT (/1X,'YOUR FORMAT IS : ',60A1)
C      -----------------------------------------------------------------
```

```
C       COMPUTE IF SUBJECTS,OBJECTS AND FACTORS ARE WITHIN LIMITS
C       ------------------------------------------------------------------
        IF (NOB.LE.80) GO TO 17
        WRITE (6,3) NOB
      3 FORMAT (//1X,'YOUR OBJECTS ',I3,' IS OUT OF RANGE')
        GO TO 99
     17 IF (NSUB.LT.NOB) GO TO 4
        IF (NSUB.LE.330) GO TO 7
      4 WRITE (6,5) NSUB,NOB
      5 FORMAT (1X,'YOUR SUBJECTS ',I3,' IS EITHER TOO BIG OR SMALLER
       *THAN YOUR OBJECTS',I3)
        GO TO 99
      7 WRITE (6,18) NOB,NSUB
     18 FORMAT (//1X,'YOUR OBJECTS = ',I3,//1X,
       *                   '     SUBJECTS = ',I3)
C       ------------------------------------------------------------------
C       READ THE INPUT DATA MATRIX FROM CARD-READER.
C       WRITE OBSERVATION MATRIX ON PRINTER
C       ------------------------------------------------------------------
        WRITE(6,66)
     66 FORMAT (///37X,20(1H-),/37X,'-   OBSERVATIONS   -',/37X,20(1H-))
         DO 77 I=1,NSUB
        READ (5,FOR) (A(I,J),J=1,NOB)
     77 WRITE (6,88) (A(I,J),J=1,NOB)
     88 FORMAT (1X,10F10.5)
        DO 41 I=1,NSUB
     41 SUBM(I)=0.0
        DO 42 J=1,NOB
     42 OBJM(J)=0.0
        DO 43 I=1,NSUB
        DO 43 J=1,NOB
        SUBM(I)=SUBM(I) + A(I,J)
     43 OBJM(J)=OBJM(J) + A(I,J)
        SK=0.
C       ------------------------------------------------------------------
C        COMPUTE THE DIAGONAL MATRIX OF ROW AND COLUMN SUMS
C       ------------------------------------------------------------------
        DO 121 I=1,NSUB
        DO 121 J=1,NOB
    121 SK = SK + A(I,J)
        DO 122 I=1,NOB
    122 DV(I) = 0.0
        DO 123I=1,NSUB
        DC(I)=0.0
        DO 123 J=1,NOB
        A(I,J)= A(I,J)/SK
        DV(J)=DV(J) + A(I,J)
    123 DC(I)=DC(I) + A(I,J)
C       ------------------------------------------------------------------
C       WRITE THE ROW- AND COLUMN TOTALS.
C       ------------------------------------------------------------------
        WRITE (NWTR,120)
    120 FORMAT (1H1,36X,21(1H-),/37X,'-      ROW-TOTALS    -'/37X,21(1H-),/
       *31X,'ROW NUMBER',15X,'TOTAL'/31X,10(1H-),15X,5(1H-))
        WRITE (NWTR,221)( I,DC(I) ,I=1,NSUB)
    221 FORMAT (31X,I3,18X,F10.5)
        WRITE (NWTR,230)
    230 FORMAT (1H1,36X,21(1H-),/37X,'-   COLUMN-TOTALS    -'/37X,21(1H-),/
       *31X,'COLUMN NUMBER',12X,'TOTAL',/31X,13(1H-),12X,5(1H-))
        WRITE (NWTR,231) ( J,DV(J) ,J=1,NOB)
```

```
  231 FORMAT (35X,I3,14X,F10.5)
      IF (IND.NE.0)GO TO 127
      DO 125 I=1,NSUB
      DO 125 J=1,NOB
  125 V1(I,J)= A(I,J)/DC(I)
      WRITE (NWTR,150) (( V1(I,J),J=1,NOB),I=1,NSUB)
  150 FORMAT ('1',19X,24(1H-),/20X,'THE COLUMN PROFILE MATIX',/20X,24(1H
     *-),//100(1X,12F10.5,/)   )
  127 DO 128 I=1,NOB
  128 DV(I)= 1./SQRT(DV(I))
      DO 129 J=1,NSUB
  129 DC(J)= 1./SQRT(DC(J))
      DO 211 I=1,NSUB
      DO 211 J=1,NOB
  211 A(I,J) = A(I,J)* DV(J)* DC(I)
      DO 212 I=1,NOB
      DO 212 J=1,I
      G(I,J)=0.
      DO 133 K=1,NSUB
  133 G(I,J)=G(I,J)+A(K,I)*A(K,J)
      G(J,I)=G(I,J)
      B(J,I)=G(J,I)
  212 B(I,J)=G(I,J)
C     ------------------------------------------------------------------
C     COMPUTE THE EIGENVALUES AND VECTORS OF THE MATRIX G BY MAKING USE
C     OF THE SUBROUTINE EIEW. THIS PROGRAM MUST BE CHANGED IF YOU HAVE
C     ANY STANDARD SUBROUTINES AVAILABLE.
C     ------------------------------------------------------------------
      EPS = 5.E-8
      CALL EIEW(B,V,80,NOB,EPS,888,132,1)
      GO TO 333
  132 WRITE (NWTR,134)
  134 FORMAT ('0','ERROR WITH COMPUTATION OF EIGENVALUES ---- IT MAY BE
     USEFUL TO ENLARGE THE VALUE OF 888 IN THE CALL STATEMENT ',/1X,'
     *OF THE SUBROUTINE EIEW.')
C     ------------------------------------------------------------------
C     COMPUTE THE FACTORS F AND G AS DESRIBED
C     IN THE THEORY OF CORRESPONDENCE ANALYSIS
C     ------------------------------------------------------------------
  333 DO 135 I=1,NOB
  135 DG(I)= B(I,I)
      DO 144 I=1,NSUB
      DO 144 J=1,NUMFAC
      F(I,J)=0.0
      DO 144 K=1,NOB
  144 F(I,J)= F(I,J) + V(K,J) * A(I,K)
      DO 155 I=1,NSUB
      DO 155 J=1,NUMFAC
  155 F(I,J)=F(I,J) * DC(I)
      DO 166 J=1,NUMFAC
      DGS = SQRT (ABS(DG(J)))
      DO 166 I=1,NOB
  166 G(I,J)= V(I,J) *DV(I) *DGS
C     ------------------------------------------------------------------
C     WRITE EIGENVALUES AND EIGENVECTORS.
C     ------------------------------------------------------------------
      WRITE (NWTR,20)
   20 FORMAT ('1',20X,'EIGENVALUES',/21X,11(1H-),//)
      WRITE (NWTR,21)(DG(I),I=1,NOB)
   21 FORMAT (1X,12F10.5)
```

```
      WRITE (NWTR,22)
   22 FORMAT (///,21X,'EIGENVECTORS',/21X,12(1H-),//)
      DO 23 I=1,NOB
   23 WRITE (NWTR,21) (V(I,J),J=1,NOB)
      TR= 0.0
      DO 11I=2,NOB
      TR= TR + DG(I)
   11 DV(I-1)= SQRT( ABS(DG(I)))
      WRITE (NWTR,30)
   30 FORMAT (///21X,'CORRELATIONS',/21X,12(1H-),//)
      WRITE (NWTR,31)(DV(I),I=1,NOB-1)
  31   FORMAT (1X,12F10.6)
      DO13 I=2,NOB
   13 DV(I-1)=DG(I)/TR
      WRITE (NWTR,40) TR
   40 FORMAT (//,21X,'TRACE = ',F12.6)
      WRITE (NWTR,50)
   50 FORMAT (//,21X,'PROPORTION OF TRACE',/21X,19(1H-),/ )
      WRITE (NWTR,55)(DV(I),I=1,NOB-1)
   55 FORMAT (1X,10F12.6)
      WRITE (NWTR,60) (I,I=1,NUMFAC)
C     ----------------------------------------------------------------
C       WRITE THE ROW AND COLUMN LOADINGS
C     ----------------------------------------------------------------
   60 FORMAT (1H1,20X,'ROW LOADINGS',/21X,12(1H-),/3X,'VARIABLE  ','FACT
     *OR', 8(I2,'    FACTOR'),I2)
      DO 61 I=1,NSUB
   61 WRITE(NWTR,62) I,(F(I,J),J=1,NUMFAC)
   62 FORMAT (4X,I3,3X,10F12.8)
      WRITE (NWTR,70) (I,I=1,NUMFAC)
   70 FORMAT (1H1,20X,'COLUMN LOADINGS',/21X,12(1H-),/3X,'VARIABLE  ','F
     *ACTOR', 8(I2,'    FACTOR'),I2)
      DO 71 I=1,NOB
   71 WRITE (NWTR,62) I,(G(I,J),J=1,NUMFAC)
      IF (NT.EQ.0) GO TO 99
      TOM=0.0
      TSM=0.0
C     ----------------------------------------------------------------
C      COMPUTE AND PRINT TABLES AS DESCRIBED ON PAGE 8 OF H.TEIL
C     ----------------------------------------------------------------
      DO 72 I=1,NOB
   72 TOM=TOM +OBJM(I)
      DO 73 J=1,NSUB
   73 TSM=TSM + SUBM(J)
      SOM=0.0
      DO 76 I=1,NP
      KK=NPF(I)
   76 SOM=SOM +DG(KK)
      DO 89 K=1,NP
      KK=NPF(K)
      PER = DG(KK)/ SOM  * 100.0
      WRITE (6,100)
      WRITE (6,101)
      WRITE (6,103)
      WRITE (6,102)
      WRITE (6,104) K,DG(KK),PER
      WRITE (6,103)
      WRITE (6,101)
      WRITE (6,103)
      WRITE (6,105)
```

```
      WRITE (6,103)
      WRITE (6,101)
      DO 74 I=1,NUMFAC
      RHO= 0.0
      DO 75 J =1,NP
      JJ=NPF(J)
   75 RHO=RHO + ((ABS(G(I,JJ))) ** 2.0)
      ACO =((ABS(G(I,KK))**2.0)*(OBJM(I)/TOM))/DG(KK)*100.0
      IF (G(I,KK).LT.0.0) ACO=ACO* (-1.0)
      RCO=((ABS(G(I,KK)))**2.0/RHO) * 100.
      IF (G(I,KK).LT.0.0) RCO=RCO * (-1.0)
      CCO =ABS(RCO)
   74 WRITE (6,106)I,OBJM(I),RHO,G(I,KK),ACO,RCO,CCO
      WRITE (6,101)
      WRITE (6,100)
      WRITE (6,101)
      WRITE (6,103)
      WRITE (6,102)
      WRITE (6,104) K,DG(KK),PER
      WRITE (6,103)
      WRITE (6,101)
      WRITE (6,103)
      WRITE (6,107)
      WRITE (6,103)
      WRITE (6,101)
      DO 78 I=1,NSUB
      RHS=0.0
      DO 79 J=1,NP
      JJ=NPF(J)
   79 RHS=RHS + ((ABS(F(I,JJ))) ** 2.0)
      ACS =((ABS(F(I,KK))**2.0)*(SUBM(I)/TSM))/DG(KK)*100.0
      IF (F(I,KK).LT.0.0) ACS=ACS* (-1.0)
      RCS=((ABS(F(I,KK)))**2.0/RHS) * 100.
      IF (F(I,KK).LT.0.0) RCS=RCS * (-1.0)
      CCS =ABS(RCS)
   78 WRITE(6,106)I,SUBM(I),RHS,F(I,KK),ACS,RCS,CCS
      WRITE (6,101)
   89 CONTINUE
  100 FORMAT(1H1)
  101 FORMAT( 1X,119(1H*))
  102 FORMAT(1X,'*',15X,'FACTOR',26X,'CHARACTERISTIC VALUE :',17X,'PERCE
     *NTAGE OF INERTIA :',8X,'*')
  103 FORMAT (1X,'*',117X,'*')
  104 FORMAT(1X,'*',17X,I2,34X,F9.7,33X,F6.3,16X,'*')
  105 FORMAT(1X,'*',92X,'CONTRIBUTIONS :',10X,'*'/ 1X,'*',5X,'OBJECT',
     *15X,'MASS',16X,'RHO',15X,'FACTOR', 47X,'*'/1X,'*',78X,'ABSOLUTE',
     *7X,'RELATIVE',6X,'CUMULATIVE*')
  107 FORMAT(1X,'*',92X,'CONTRIBUTIONS :',10X,'*'/ 1X,'*',4X,'SUBJECT',
     *15X,'MASS',16X,'RHO',15X,'FACTOR', 47X,'*'/1X,'*',78X,'ABSOLUTE',
     *7X,'RELATIVE',6X,'CUMULATIVE*')
  106 FORMAT(1X,'*',7X,I2,6X,'*',6X,  F10.4,5X,'*',5X, F10.4,
     *4X,'*',4X,F10.4,2X,'*',2(2X,F10.4,2X,'*'),2X,F10.4,'*')
C     ------------------------------------------------------------------
C      WRITE THE FIRST TWO FACTORS TO DISC.
C      THESE FACTORS ARE USED BY ANDREW'S METHOD.
C     ------------------------------------------------------------------
      WRITE (3,112) NOB,NSUB
  112 FORMAT (2I3)
      K= NPF(1)
      L= NPF(2)
```

```
      DO 113 I=1,NOB
  113 WRITE (3,114) G(I,K),G(I,L)
  114 FORMAT (2F10.7)
      DO 115 J=1,NSUB
  115 WRITE (3,114) F(J,K),F(J,L)
   99 STOP
      END
```

```
      SUBROUTINE EIEW(A,E,NM,N,ACC,IT,$,L)
      DIMENSION A(NM,NM),E(NM,NM)
      GO TO (1,2),L
1     DO 6 I=1,N
      DO 5 J=1,N
5     E(I,J)=0.
6     E(I,I)=1.
2     NI=0
      IND=0
      VF=ACC
      VI=0.
      DO 10J=2,N
      K=J-1
      DO 10 M=1,K
10    VI=VI+2.*A(M,J)*A(M,J)
      VI=SQRT(VI)
      IF(L.EQ.1) VF=VI*ACC/N
      IF(L.EQ.1) V=VI
14    V=V/N
      IF(V-VF) 110,15,15
15    DO 100 J=2,N
      K=J-1
      DO 90 M=1,K
      IF(ABS(A(M,J))-V) 90,20,20
20    IND=1
      NI=NI+1
      Y=-A(M,J)
      U=0.5*(A(M,M)-A(J,J))
      IF(A(M,M)-A(J,J)) 22,21,22
21    U=ABS(U)
22    W=Y*SIGN(1,U)/SQRT(Y*Y+U*U)
      S=W/SQRT(2.*(1.+SQRT(1.-W*W)))
      C=SQRT(1.-S*S)
      DO 50 I=1,N
      IF(I.EQ.M.OR.I.EQ.J) GO TO 45
      A(I,M)=A(I,M)*C-A(I,J)*S
      A(I,J)=A(M,I)*S+A(I,J)*C
      A(M,I)=A(I,M)
      A(J,I)=A(I,J)
45    SS=E(I,M)
      E(I,M)=E(I,M)*C - E(I,J)*S
50    E(I,J)=SS*S+E(I,J)*C
      SAM=A(M,M)
      A(M,M)=A(M,M)*C*C+A(J,J)*S*S-2.*A(M,J)*S*C
      A(J,J)=SAM*S*S+A(J,J)*C*C+2.*A(M,J)*S*C
      A(M,J)=0.
      A(J,M)=A(M,J)
      IF(NI.EQ.IT) GO TO 109
90    CONTINUE
100   CONTINUE
      IF(IND.EQ.0) GO TO 14
      IND=0
      GO TO 15
109   ACC=VF
      RETURN 1
110   IT=NI
      ACC=VF
      RETURN
      END
```

```
C     ----------------------------------------------------------------
C     THIS PROGRAM COULD BE INCLUDED IN THE MAIN PROGRAM AS A
C     SUBROUTINE. IT IS HOWEVER NOT POSSIBLE AT THE U.O.V.S.
C     AS THE PLOTTER PROGRAMS ARE WRITTEN IN A DIFFERENT LANGUAGE.
C     ----------------------------------------------------------------
      DIMENSION IBUF(1000),X(330),Y(330),IBYS(5),SIM(55),R(9)
      DATA IBYS /6HGRAPH ,6HOF 2 F,6HACTORI,6HAL AXI,6HS      /
      DATA SIM /'S01','S02','S03','S04','S05','S06','S07','S08','S09','S
     *10','S11','S12','S13','S14','S15','S16','S17','S18','S19','S20','S
     *21','S22','S23','S24','S25','S26','S27','S28','S29','S30','S31','S
     *32','S33','S34','S35','S36','S37','S38','S39','S40','S41','S42','S
     *43','S44','S45','S46','S47','S48','S49','S50','S51','S52','S53','S
     *54','S55'/
      DATA R /'M1','M2','M3','M4','M5','M6','M7','M8','M9'/
      CALL PLTIME (50)
      CALL PLOTS(IBUF(1),1000,9)
      CALL PLOT (0.0,0.5,-3)
      CALL FACTOR (0.6)
      CALL SYMBOL (7.0,20.5,0.3,IBYS(1),0.0,28)
C     ----------------------------------------------------------------
C     READ THE FACTORS AS COMPUTED IN THE MAIN PROGRAM OF
C     CORRESPONDENCE ANALYSIS
C     ----------------------------------------------------------------
      READ (5,3) K,L
    3 FORMAT (2I3)
      J=0
      DO 2 I=1,K
      J=J+1
    2 READ(5,4) X(J),Y(J)
    4 FORMAT(2F10.7)
      DO 6 I=1,L
      J=J+1
    6 READ(5,4) X(J),Y(J)
      J=K+L
C     ----------------------------------------------------------------
C     AMX,AMY IS THE MINIMUM OF THE TWO FACTORS
C     AX,XY IS THE MAXIMUM OF THE TWO FACTORS
C     ----------------------------------------------------------------
      AMY= 999.
      AMX= 999.
      AY =-999.
      AX =-999.
      DO 8 I=1,J
      IF (X(I).LT.AMX) AMX= X(I)
      IF (X(I).GT.AX) AX=X(I)
      IF (Y(I).LT.AMY) AMY= Y(I)
      IF (Y(I).GT.AY)  AY=Y(I)
    8 CONTINUE
C     ----------------------------------------------------------------
C     DX,DY INCREMENT ALONG THE X-AND Y-AXIS EXACTLY
C     ----------------------------------------------------------------
      DX = ((2* AX) - (1.0- ABS(AMX)))/20.0
      DY = ((2* AY) - (1.0 - ABS(AMY)))/20.0
      CALL AXIS (0.0,10.0,1H ,-1,20.0,0.0,-0.4,.04)
      CALL AXIS (10.0,0.0,1H ,1,20.0,90.0,-0.2,.02)
C     ----------------------------------------------------------------
C     COMPUTE ORDINATES AND PLOT FACTORS
C     ----------------------------------------------------------------
      DO 12 I=1,K
```

```
      IF (X(I)) 21,23,22
   21  X(I) = (0.4 - ABS(X(I)))
      GO TO 23
   22  X(I) = .4+ X(I)
   23 CONTINUE
      IF (Y(I)) 24,27,26
   24  Y(I) = (0.2 - ABS(Y(I)))
      GO TO 27
   26  Y (I) = 0.2 + Y(I)
   27 CONTINUE
      S = R(I)
      X(I) = X(I)/0.4 * 10.0
      Y(I) = Y(I)/.2 * 10.0
      CALL SYMBOL (X(I),Y(I),0.14,S,0.0,3)
   12 CONTINUE
      KK=K + 1
      DO 14 I =KK,J
      IF (X(I)) 31,33,32
   31  X(I) = (0.4 - ABS(X(I)))
      GO TO 33
   32 X(I)= .4 + X(I)
   33 CONTINUE
      IF (Y(I)) 34,36,35
   34 Y(I) = (0.2 -ABS(Y(I)))
      GO TO 36
   35 Y(I) = 0.2 + Y(I)
   36 CONTINUE
      II = I - K
      S = SIM(II)
      X(I) = X(I)/.4  * 10.0
      Y(I) = Y(I)/.2 * 10.0
   14 CALL SYMBOL (X(I),Y(I),0.14,S,0.0,3)
C     -------------------------------------------------------------
C     CLOSE OUTPUT PLOTTER
C     -------------------------------------------------------------
      CALL PLOT (32.0,0.0,999)
      STOP
      END
```

Area	Station	Altitude (m)	IX	X	XI	XII	I	II	III	IV	V
A. Coast											
1. North coast area	1. Rosh Hanikra	60	0.4	20.7	95.9	122.6	176.3	117.4	50.0	30.3	9.4
	2. Acre	10	1.4	21.7	81.7	126.0	154.2	122.2	44.7	19.7	5.4
2. Carmel area	3. Haifa, Carmel Blvd.	10	1.2	18.7	88.1	145.4	160.8	113.5	39.7	18.4	4.2
	4. Haifa, Mt. Carmel	300	0.4	22.8	94.9	161.3	182.5	123.1	44.5	23.5	8.0
	5. Yagur	30	0.3	18.8	99.8	165.7	220.3	155.1	58.5	23.0	7.5
	6. Atlit, Salt Co.	10	1.0	17.5	88.1	127.2	124.6	84.4	26.8	18.8	3.6
	7. Zichron Ya'akov	150	0.5	17.5	89.3	154.4	161.4	117.5	40.4	19.2	3.8
3. Central coastal plain	8. Hedera	20	0.9	15.3	83.5	165.9	153.0	105.2	37.9	14.1	4.2
	9. Tel-Aviv, Reading	3	3.3	17.8	83.1	150.4	124.0	89.9	34.4	13.8	2.3
	10. Lod Airport	40	1.9	17.4	61.9	120.7	121.8	112.0	42.4	15.8	2.1
	11. Mazkeret Batya	60	1.1	13.2	68.4	104.3	130.2	98.5	43.8	15.0	2.5
4. Southern coastal plain	12. Be'er Tuvia	55	0.6	11.4	72.0	114.1	123.2	89.2	43.0	12.0	1.5
	13. Gvar'am	100	0.4	7.9	61.1	99.2	93.7	69.4	37.4	8.5	1.4
	14. Gaza	45	0.3	14.4	60.7	90.5	90.9	70.2	32.9	12.6	3.5
	15. Han-Yunes	45	0.2	8.8	51.2	58.0	60.4	49.7	19.8	12.3	2.6
B. Mountains											
1. Upper Galilee	16. Kfar Giladi	340	0.2	18.7	79.4	137.3	207.9	192.8	88.4	48.4	13.9
	17. Rehanya	660	0.4	15.5	75.3	93.1	165.3	155.5	54.5	26.9	6.5
	18. Ma'alya	500	0.6	25.2	89.9	134.3	218.8	179.0	70.5	36.8	7.9
	19. Mt. Cna'an	935	2.1	15.5	80.4	139.0	195.3	172.2	72.0	35.6	15.9
2. Lower Galilee	20. Mitzpe	75	0.1	12.0	58.1	97.1	122.7	111.6	43.4	18.8	10.2
	21. Nazaret, Mushtelot Haye'ur	445	0.6	11.8	71.9	139.1	172.2	146.2	63.3	27.5	6.4
	22. Tavor, Agricultural School	145	0.2	8.7	54.7	102.6	130.6	120.7	49.0	22.4	4.1
3. Yezre'el Valley	23. Ramat David Airport	50	0.1	12.7	59.4	109.8	138.6	110.1	41.0	17.4	4.9
	24. Mishmar Haemek	100	0.3	16.1	74.0	137.4	172.7	137.0	48.9	19.4	5.2
	25. Afula South-West	65	0.3	13.3	59.5	108.4	143.7	112.7	41.4	21.5	5.2
4. Beisan Valley	26. Tel-Josef	5	0.4	12.4	50.7	83.5	114.0	91.7	39.6	16.7	5.0
	27. Hefziba, Gilboa	-80	0.4	10.8	53.5	86.5	118.8	93.8	40.3	17.7	4.2
	28. Beisan	-120	0.6	10.9	37.4	68.4	84.3	67.8	33.2	14.2	3.2
5. Samarian hills	29. Jenin	140	0.2	12.3	53.3	95.9	135.4	111.2	42.9	22.4	4.4
	30. Tul-Carem	80	0.3	12.2	75.5	127.1	149.9	128.2	37.4	19.7	3.7
	31. Nablus	490	0.5	14.9	63.5	127.0	176.6	152.4	66.3	31.9	4.9
	32. Bir-Zayit	780	1.2	11.2	58.4	132.4	194.1	174.6	88.9	38.1	3.1
6. Judean hills	33. Latrun Monastery	200	0.8	15.2	69.3	90.1	129.5	99.0	56.3	20.0	3.8
	34. Kiryat Anavim	700	1.2	15.6	75.1	112.5	177.3	158.0	82.2	28.6	2.5
	35. Jerusalem, Hanevi'im Street	810	0.6	9.2	59.0	91.7	143.2	142.9	69.6	24.8	4.0
	36. Bet-Jamal	360	0.1	13.4	63.4	88.3	126.1	108.6	54.3	19.1	2.7
	37. Hebron	930	0.3	7.3	53.5	76.1	127.2	109.9	68.4	22.6	2.7
	38. Zahariya	600	0.3	4.8	33.4	57.8	97.3	82.1	57.6	14.0	2.7
7. Negev	39. Ruchama	180	0.3	3.7	39.8	67.3	103.2	64.0	38.0	11.5	1.2
	40. Urim	90	0.2	5.0	38.9	42.2	51.5	43.0	21.0	5.8	1.4
	41. Be'er Sheva	270	0.3	4.0	25.2	39.6	47.9	40.8	30.5	7.4	4.3
	42. Mamshit	450	0.3	1.0	13.0	20.4	34.1	25.4	15.6	5.3	4.9
	43. Mash'avey Sade	350	0.2	1.7	12.2	18.3	22.4	22.9	19.3	5.0	3.0
	44. Nizana	250	0.1	2.1	10.9	12.4	19.8	18.1	15.1	7.7	3.8
C. Jordan Valley											
1. Hula region	45. Dafna	150	0.1	11.1	61.2	103.9	154.0	129.0	62.7	35.8	11.2
	46. Kfar Blum	75	0.2	5.6	56.3	93.3	127.5	115.1	53.9	25.4	11.7
	47. Ayelet Hashahar	175	0.8	8.5	57.4	93.1	132.6	109.8	46.7	24.4	8.7
2. Central Jordan valley	48. Dganya A	-200	0.4	8.3	42.4	76.1	100.9	88.9	38.3	22.8	5.9
	49. Naharayim	-220	0.3	13.1	46.8	80.7	108.7	90.8	42.8	22.3	5.5
	50. Tirat Zvi	-220	0.5	10.3	37.2	62.5	65.6	61.9	27.5	12.7	9.8
3. Dead Sea region	51. Jerico	-260	0.0	3.0	18.6	27.5	35.5	31.1	17.0	7.1	3.2
	52. Ashlag North	-385	0.0	2.6	14.1	17.7	19.5	16.2	11.1	4.5	1.3
	53. Ashlag South	-380	0.0	1.5	4.4	9.3	11.3	8.7	6.8	3.2	1.8
4. Arava	54. Ein Hazeva	-140	0.1	1.5	6.3	17.0	15.3	12.6	11.5	2.3	4.4
	55. Eilat	12	0.0	1.3	1.7	6.9	3.7	4.3	4.0	4.3	1.4

****
* CORRESPONDENCE FACTOR ANALYSIS *
****

TEST RUN ON ISRAELI RAINFALL

YOUR FORMAT IS : (9F5.1)

YOUR OBJECTS = 9

SUBJECTS = 55

- OBSERVATIONS -

.40000	20.70000	95.90000	122.60000	176.30000	117.40000	50.00000	30.30000	9.40000
1.40000	21.70000	81.70000	126.00000	154.20000	122.20000	44.70000	19.70000	5.40000
1.20000	18.70000	88.10000	145.40000	160.80000	113.50000	39.70000	18.40000	4.20000
.40000	22.80000	94.90000	161.30000	182.50000	123.10000	44.50000	23.50000	8.00000
.30000	18.80000	99.80000	165.70000	220.30000	155.10000	58.50000	23.00000	7.50000
1.00000	17.50000	88.10000	127.20000	124.60000	84.40000	26.80000	18.80000	3.60000
.50000	17.50000	89.30000	154.40000	161.40000	117.50000	40.40000	19.20000	3.80000
.90000	15.30000	83.50000	165.90000	153.00000	105.20000	37.90000	14.10000	4.20000
3.30000	17.80000	83.10000	150.40000	124.00000	89.90000	34.40000	13.80000	2.30000
1.90000	17.40000	61.90000	120.70000	121.80000	112.00000	42.40000	15.80000	2.10000
1.10000	13.20000	68.40000	104.30000	130.20000	98.50000	43.80000	15.00000	2.50000
.60000	11.40000	72.00000	114.10000	123.20000	89.20000	43.00000	12.00000	1.50000
.40000	7.90000	61.10000	99.20000	93.70000	69.40000	37.40000	8.50000	1.40000
.30000	14.40000	60.70000	90.50000	90.90000	70.20000	32.90000	12.60000	3.50000
.20000	8.80000	51.20000	58.00000	60.40000	49.70000	19.80000	12.30000	2.60000
.20000	18.70000	79.40000	137.30000	207.90000	192.80000	88.40000	48.40000	13.90000
.40000	15.50000	75.30000	93.10000	165.30000	155.50000	54.50000	26.90000	6.50000
.60000	25.50000	89.90000	134.30000	218.80000	179.00000	70.50000	36.80000	7.90000
2.10000	15.50000	80.40000	139.00000	195.30000	172.20000	72.00000	35.60000	15.90000
.10000	12.00000	58.10000	97.10000	122.70000	111.60000	43.40000	18.80000	10.20000
.60000	11.80000	71.90000	139.10000	172.20000	146.20000	63.30000	27.50000	6.40000
.20000	8.70000	54.70000	102.60000	130.60000	120.70000	49.00000	22.40000	4.10000
.10000	12.70000	59.40000	109.80000	138.60000	110.10000	41.00000	17.40000	4.90000
.30000	16.10000	74.00000	137.40000	172.70000	137.00000	48.90000	19.40000	5.20000
.30000	13.30000	59.50000	108.40000	143.70000	112.70000	41.40000	21.50000	5.20000
.40000	12.40000	50.70000	83.50000	114.00000	91.70000	39.60000	16.70000	5.00000
.40000	10.80000	53.50000	86.50000	118.80000	93.80000	40.30000	17.70000	4.20000
.60000	10.90000	37.40000	68.40000	84.30000	67.80000	33.20000	14.20000	3.20000
.20000	12.30000	53.30000	95.90000	135.40000	111.20000	42.90000	22.40000	4.40000
.30000	12.20000	75.50000	127.10000	149.90000	128.20000	37.40000	19.70000	3.70000
.50000	14.90000	63.50000	127.00000	176.60000	152.40000	66.30000	31.90000	4.90000
1.20000	11.20000	58.40000	132.40000	194.10000	174.60000	88.90000	38.10000	3.10000
.80000	15.20000	69.30000	90.10000	129.50000	99.00000	56.30000	20.00000	3.80000
1.20000	15.60000	75.10000	112.50000	177.30000	158.00000	82.20000	28.60000	2.50000
.60000	9.20000	59.00000	91.70000	143.20000	142.90000	69.60000	24.80000	4.00000
.10000	13.40000	63.40000	88.30000	126.10000	108.60000	54.30000	19.10000	2.70000
.30000	7.30000	53.50000	76.10000	127.20000	109.90000	68.40000	22.60000	2.70000
.30000	4.80000	33.40000	57.80000	97.30000	82.10000	57.60000	14.60000	2.70000
.30000	3.70000	39.80000	67.30000	103.20000	64.00000	38.00000	11.50000	1.20000
.20000	5.60000	38.90000	42.20000	51.50000	43.00000	21.00000	5.80000	1.20000
.30000	4.00000	25.20000	39.60000	47.90000	40.80000	30.50000	7.40000	4.30000
.30000	1.00000	13.00000	20.40000	34.10000	25.40000	15.60000	5.30000	4.90000
.20000	1.70000	12.20000	18.30000	22.40000	22.90000	19.30000	5.00000	3.00000
.10000	2.10000	10.90000	12.40000	19.80000	18.10000	15.10000	7.70000	3.80000
.10000	11.10000	61.20000	103.90000	154.00000	129.00000	62.70000	35.80000	11.20000
.20000	5.60000	56.30000	93.30000	127.50000	115.10000	53.90000	25.40000	11.70000
.80000	8.50000	57.40000	93.10000	132.60000	109.80000	46.70000	24.40000	8.70000
.40000	8.30000	42.40000	76.10000	100.90000	88.90000	38.30000	22.80000	5.90000
.30000	13.10000	46.80000	80.70000	108.70000	90.80000	42.80000	22.30000	5.50000
.50000	10.30000	37.20000	62.50000	65.60000	61.90000	27.50000	12.70000	9.80000
.00000	3.00000	18.60000	27.50000	35.50000	31.10000	17.00000	7.10000	3.20000
.00000	2.60000	14.10000	17.70000	19.50000	16.20000	11.10000	4.50000	1.30000
.00000	1.50000	4.40000	9.30000	11.30000	8.70000	6.80000	3.20000	1.80000
.10000	1.50000	6.30000	17.00000	15.30000	12.60000	11.50000	2.30000	4.40000
.00000	1.30000	1.70000	6.90000	3.70000	4.30000	4.00000	4.30000	1.40000

ROW-TOTALS

ROW NUMBER	TOTAL
1	.02543
2	.02355
3	.02408
4	.02698
5	.03057
6	.02008
7	.02465
8	.02367
9	.02118
10	.02024
11	.01947
12	.01906
13	.01547
14	.01535
15	.01073
16	.03212
17	.02420
18	.03115
19	.02971
20	.01935
21	.02608
22	.02012
23	.02016
24	.02494
25	.02065
26	.01690
27	.01739
28	.01306
29	.01951
30	.02261
31	.02604
32	.02865
33	.01975
34	.02665
35	.02224
36	.01943
37	.01910
38	.01431
39	.01343
40	.00855
41	.00816
42	.00490
43	.00429
44	.00367
45	.02322
46	.01996
47	.01967
48	.01567
49	.01677
50	.01175
51	.00584
52	.00355
53	.00192
54	.00290
55	.00113

COLUMN-TOTALS

COLUMN NUMBER	TOTAL
1	.00120
2	.02566
3	.12712
4	.20934
5	.26824
6	.21867
7	.09662
8	.04202
9	.01112

EIGENVALUES

.00088	.00065	.00190	1.00000	.01533	.00052	.00265	.00129	.00487

EIGENVECTORS

.47282	-.05425	-.13730	.03470	-.08851	-.68780	.18364	.48846	.01499
.22958	.53695	.45591	.16020	-.21553	.35486	-.13387	.46047	.15210
.02920	-.38954	.58071	.35654	-.40955	-.13419	.28785	-.30924	.13723
-.26410	-.05497	-.58702	.45754	-.49819	.21543	.12826	.21290	.13006
-.07532	.51066	-.04127	.51792	.02744	-.41536	-.30671	-.37618	-.23401
.45185	-.44796	-.05029	.46762	.30792	.29971	-.34471	.08069	-.25295
-.04311	.20417	.01596	.31084	.51401	.13317	.75268	.04738	-.09422
-.61944	-.21792	.24580	.20499	.32805	-.23728	-.24993	.44279	.20957
.24028	.05796	-.16305	.10544	.24979	-.01412	-.08464	-.24918	.87713

CORRELATIONS

.029729	.025438	.043624	1.000000	.123820	.022803	.051511	.035904	.069776

TRACE = 1.028096

PROPORTION OF TRACE

.000860	.000629	.001851	.972671	.014913	.000506	.002581	.001254	.004736

ROW LOADINGS

VARIABLE	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4	FACTOR 5	FACTOR 6	FACTOR 7	FACTOR 8	FACTOR 9
1	-.02740035	.02634675	.08654794	.99999935	-.05206452	-.03451231	-.02849357	-.02878722	.06813180
2	.04569522	.02167823	.03425666	.99999936	-.09867733	-.00143495	-.01898723	.02693910	.00729985
3	.00201498	.00969955	-.00338410	.99999935	-.16853558	-.01377761	-.01009500	.00093600	.00200566
4	-.02080829	.03539458	.00187166	.99999937	-.14442173	.00927708	-.03213638	-.01325390	.04877555
5	.00115489	.02714403	-.01732370	.99999937	-.07477388	-.00491431	-.03244638	-.05816613	-.01570936
6	-.03331551	-.02610697	.05180620	.99999934	-.24529887	-.02219513	.00108270	.01505290	.05326967
7	-.02452689	-.00570868	-.01837499	.99999934	-.17062590	.01388174	-.01230055	-.01087657	-.00396467
8	-.02348852	.00173816	-.08278171	.99999937	-.21874149	.01174543	.01543383	-.00904227	.01042400
9	.03462197	-.01739331	-.04779560	.99999936	-.27244527	-.05745125	.07083781	.09748186	.02162540
10	.05623873	-.00356901	-.03208013	.99999936	-.10122466	.01635170	.01280953	.09219011	-.03904779
11	.02063245	.00728776	.01300912	.99999938	-.08089668	-.01974935	.03207805	-.00370517	-.04249220
12	-.01281283	-.00591987	-.00895672	.99999937	-.13850987	.00573870	.07138524	-.02848265	-.04490856
13	-.02568232	-.02800975	-.03123637	.99999937	-.15941861	.02272375	.11894256	-.03647498	-.03030695
14	-.00689257	.00830405	.05090624	.99999937	-.14495706	.04481475	.05277654	.01364287	.04221377
15	-.03307375	-.07765152	.13238619	.99999936	-.15134590	.00333876	.03709257	-.00441167	.07096483
16	-.01779363	-.00804869	.01191306	.99999936	.16284770	.01149244	-.04235269	.02417658	.03497422
17	.05442393	-.02922860	.06886203	.99999938	.08428172	.00056626	-.07168404	-.02357873	-.03947603
18	.01667266	.03401582	.05654486	.99999937	.05019007	-.00027550	-.06540935	.01283488	-.02156895
19	.04648388	-.01384877	-.03278241	.99999937	.09275101	-.04238676	-.03074439	.00991911	.07078813
20	.03729240	-.01451000	-.01893094	.99999939	.03615809	.03594945	-.03443196	-.03386591	.07702962
21	-.01625205	-.01730622	-.05205773	.99999938	.02762888	.00423955	-.00748346	-.00186386	-.02510371
22	-.01431100	-.03949550	-.03838866	.99999940	.05318772	.02061437	-.02343973	.00051895	-.04782244
23	.00036288	.01343133	-.03212473	.99999937	-.03650775	.02362415	-.04140911	-.02415074	-.01999719
24	.01247030	.01560091	-.03581975	.99999935	-.05474237	.02235664	-.04302449	-.02368728	-.03422436
25	-.01069152	.01383247	-.01819675	.99999937	-.01586368	.00256873	-.05813938	-.00627666	-.01329396
26	.01604122	.02675345	.01095436	.99999937	.00873407	.00986292	-.01811391	-.00128247	.00181092
27	-.00179591	.00932268	.00600221	.99999939	.00419113	-.00553131	-.01486651	-.01204035	-.01875226
28	.00127859	.04111406	.00493622	.99999938	.00213606	.01120662	.01263843	.05565008	.00236833
29	-.01358033	.01029551	-.00523554	.99999938	.03117962	.00610429	-.05946372	.00404982	-.03198148
30	.00032750	-.04817463	-.02251876	.99999937	-.08628353	.01295777	-.05726387	-.02580354	-.03716337
31	-.02004440	.00781607	-.02425056	.99999937	.07751612	.00918563	-.03471378	.03032524	-.05336169
32	-.02728305	-.00047624	-.05186546	.99999938	.15840611	-.01254808	.00559456	.05372289	-.10410899
33	.00742720	.02853663	.07726767	.99999936	.01231001	-.01196899	.06603814	.00536842	-.02067491
34	.02248006	.00499928	.03240412	.99999938	.10450463	-.00563131	.04009387	.02424284	-.09837470
35	.02759019	-.03619895	.00477388	.99999935	.15488365	.01243080	.02526360	.00323291	-.08568140
36	.00481778	.00425427	.05777808	.99999937	.03716464	.03084079	.03480711	-.00771685	-.05865909
37	-.01907676	-.00229034	.02900110	.99999936	.17067183	-.00614179	.08456544	-.01356226	-.09145650
38	-.00124161	.03784591	-.02160918	.99999936	.21591833	.00379802	.12010814	-.01888692	-.09832208
39	-.05060955	.04316664	-.04072666	.99999936	.02113680	-.06073176	.05541477	-.07435112	-.10010222
40	.02833521	-.05367609	.10415881	.99999937	-.09925430	.00978334	.09662792	-.05865797	-.02201539
41	.02345758	.01661230	-.01989951	.99999940	.11828136	.01867156	.16314228	-.02494228	.07631354
42	.06151456	.02353780	-.07675063	.99999937	.20968696	-.08496026	.04522864	-.10686464	.20155675
43	.03350716	-.00936242	-.01391938	.99999937	.25287880	.02492971	.21619616	-.00028458	.12674409
44	-.04428009	-.00568279	.08925383	.99999935	.33510110	-.03139800	.11230681	.02728690	.28807543
45	-.04708588	-.00835015	.00044644	.99999938	.14409443	-.01478370	-.03043536	-.00104728	.06164847
46	-.00692129	-.04903393	-.04144035	.99999939	.13443205	-.01059126	-.00452015	-.04870818	.08320337
47	.00098285	-.02403071	-.01888670	.99999935	.07349809	-.04184462	-.02810420	-.01840259	.04321992
48	-.03292974	-.02548752	-.01030388	.99999938	.08835999	-.01142180	-.03732850	.02948561	.03413900
49	-.02128211	.02490442	.02624880	.99999940	.06088148	.01523392	-.02001276	.04197037	.02644802
50	.06582750	.00320120	-.00924015	.99999937	.02546406	.03875932	.00122464	.01722785	.22614392
51	-.01176894	-.02085787	.01269179	.99999940	.09838603	.01882065	.04342049	-.03332595	.09572067
52	-.05394648	-.02169568	.09557729	.99999939	.01574399	.03336888	.12678985	.00519907	.08510341
53	-.05544435	.08435359	-.01978063	.99999935	.20763331	.03249400	.06212822	.03854114	.25816723
54	.07954005	.09138618	-.19304527	.99999938	.20011729	.06273524	.17308152	-.06267912	.42907491
55	-.31600590	.00092800	.00199873	.99999938	.31250678	.06616535	.00647413	.36655699	.53196479

COLUMN LOADINGS

VARIABLE	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4	FACTOR 5	FACTOR 6	FACTOR 7	FACTOR 8	FACTOR 9
1	.40509492	-.03977143	-.17261725	.99999943	-.31584052	-.45200676	.27262220	.50541971	.03013511
2	.04260346	.08526091	.12414932	.99999933	-.16658890	.05051134	-.04304499	.10320140	.06625056
3	.00243497	-.02779215	.07105143	.99999940	-.14223161	-.00858258	.04158771	-.03114023	.02685651
4	-.01715967	-.00305588	-.05596912	.99999943	-.13482043	.01073684	.01443947	.01670673	.01983504
5	-.00432347	.02508117	-.00347581	.99999928	.00655977	-.01828758	-.03050468	-.02607786	-.03152653
6	.02872557	-.02436807	-.00469117	.99999939	.08153360	.01461472	-.03797163	.00619536	-.03774338
7	-.00412339	.01670788	.00224037	.99999936	.20475108	.00976912	.12473046	.00547291	-.02115051
8	-.06963376	-.02704225	.05230788	.99999937	.19815209	-.02639472	-.06230286	.07755389	.07133517
9	.06774794	.01398365	-.06745785	.99999940	.29333650	-.00305360	-.04135204	-.08484928	.58045103

FACTOR	CHARACTERISTIC VALUE :	PERCENTAGE OF INERTIA :
1	.0153315	54.567

OBJECT	MASS	RHO	FACTOR	CONTRIBUTIONS : ABSOLUTE	RELATIVE	CUMULATIVE
1	29.5000	.8302	-.3158	-.7834	-12.0154	12.0154
2	628.7999	.0717	-.1666	-4.6454	-38.7091	38.7091
3	3114.6996	.0296	-.1422	-16.7735	-68.4582	68.4582
4	5129.2993	.0226	-.1348	-24.8190	-80.3945	80.3945
5	6572.2991	.0036	.0066	.0753	1.1816	1.1816
6	5357.8992	.0112	.0815	9.4816	59.3172	59.3172
7	2367.3997	.0584	.2048	26.4204	71.8418	71.8418
8	1029.5999	.0665	.1982	10.7617	59.0032	59.0032
9	272.4000	.4412	.2933	6.2396	19.5017	19.5017

FACTOR	CHARACTERISTIC VALUE :	PERCENTAGE OF INERTIA :
1	.0153315	54.567

SUBJECT	MASS	RHO	FACTOR	CONTRIBUTIONS : ABSOLUTE	RELATIVE	CUMULATIVE
1	623.0000	.0191	-.0521	-.4496	-14.1775	14.1775
2	577.0000	.0146	-.0987	-1.4956	-66.6462	66.6462
3	590.0000	.0288	-.1685	-4.4612	-98.5900	98.5900
4	661.0000	.0262	-.1444	-3.6701	-79.5473	79.5473
5	749.0000	.0113	-.0748	-1.1148	-49.3203	49.3203
6	492.0000	.0682	-.2453	-7.8808	-88.2216	88.2216
7	604.0000	.0306	-.1706	-4.6810	-95.2563	95.2563
8	580.0000	.0558	-.2187	-7.3876	-85.7151	85.7151
9	519.0000	.0963	-.2724	-10.2551	-77.0664	77.0664
10	496.0000	.0249	-.1012	-1.3529	-41.1399	41.1399
11	477.0000	.0104	-.0809	-.8310	-62.7407	62.7407
12	467.0000	.0274	-.1385	-2.3850	-69.9639	69.9639
13	379.0000	.0447	-.1594	-2.5641	-56.7958	56.7958
14	376.0000	.0305	-.1450	-2.1032	-68.9336	68.9336
15	263.0000	.0540	-.1513	-1.6037	-42.4195	42.4195
16	787.0000	.0308	.1628	5.5559	86.1684	86.1684
17	593.0000	.0229	.0843	1.1213	30.9991	30.9991
18	763.3000	.0121	.0502	.5119	20.8879	20.8879
19	728.0000	.0199	.0928	1.6672	43.2708	43.2708
20	474.0000	.0128	.0362	.1650	10.1939	10.1939
21	639.0000	.0048	.0278	.1317	16.2845	16.2845
22	493.0000	.0093	.0532	.3713	30.3242	30.3242
23	494.0000	.0058	-.0365	-.1753	-22.9744	22.9744
24	611.0000	.0088	-.0547	-.4874	-34.2014	34.2014
25	506.0000	.0045	-.0159	-.0339	-5.6032	5.6032
26	414.0000	.0016	.0087	.0084	4.7688	4.7688
27	426.0000	.0009	.0042	.0020	1.9693	1.9693
28	320.0000	.0051	.0021	.0004	.0893	.0893
29	478.0000	.0059	.0312	.1237	16.4707	16.4707
30	554.0000	.0158	-.0863	-1.0979	-47.2184	47.2184
31	638.0000	.0121	.0775	1.0205	49.5926	49.5926
32	702.0000	.0424	.1584	4.6892	59.1237	59.1237
33	484.0000	.0120	.0123	.0195	1.2679	1.2679
34	653.0000	.0244	.1045	1.8985	44.7479	44.7479
35	545.0000	.0342	.1549	3.4804	70.0861	70.0861
36	476.0000	.0104	.0372	.1750	13.2503	13.2503
37	468.0000	.0461	.1707	3.6290	63.2187	63.2187
38	350.6000	.0730	.2159	4.3512	63.8764	63.8764
39	329.0000	.0288	.0211	.0391	1.5492	1.5492
40	209.4000	.0377	-.0993	-.5491	-26.1016	26.1016
41	200.0000	.0486	.1183	.7449	28.7736	28.7736
42	120.0000	.1155	.2097	1.4046	38.0660	38.0660
43	105.0000	.1288	.2529	1.7874	49.6572	49.6572
44	90.0000	.2196	.3351	2.6904	51.1392	51.1392
45	569.0000	.0280	.1441	3.1450	74.1630	74.1630
46	489.0000	.0317	.1344	2.3525	57.0645	57.0645
47	482.0000	.0111	.0735	.6931	48.7342	48.7342
48	384.0000	.0132	.0884	.7981	59.1190	59.1190
49	411.0000	.0086	.0609	.4055	43.2892	43.2892
50	288.0000	.0580	.0255	.0497	1.1176	1.1176
51	143.0000	.0229	.0984	.3685	42.2200	42.2200
52	87.0000	.0372	.0157	.0057	.6659	.6659
53	47.0000	.1267	.2076	.5394	34.0147	34.0147
54	71.0000	.3139	.2001	.7569	12.7571	12.7571
55	27.6000	.6193	.3125	.7175	15.7696	15.7696

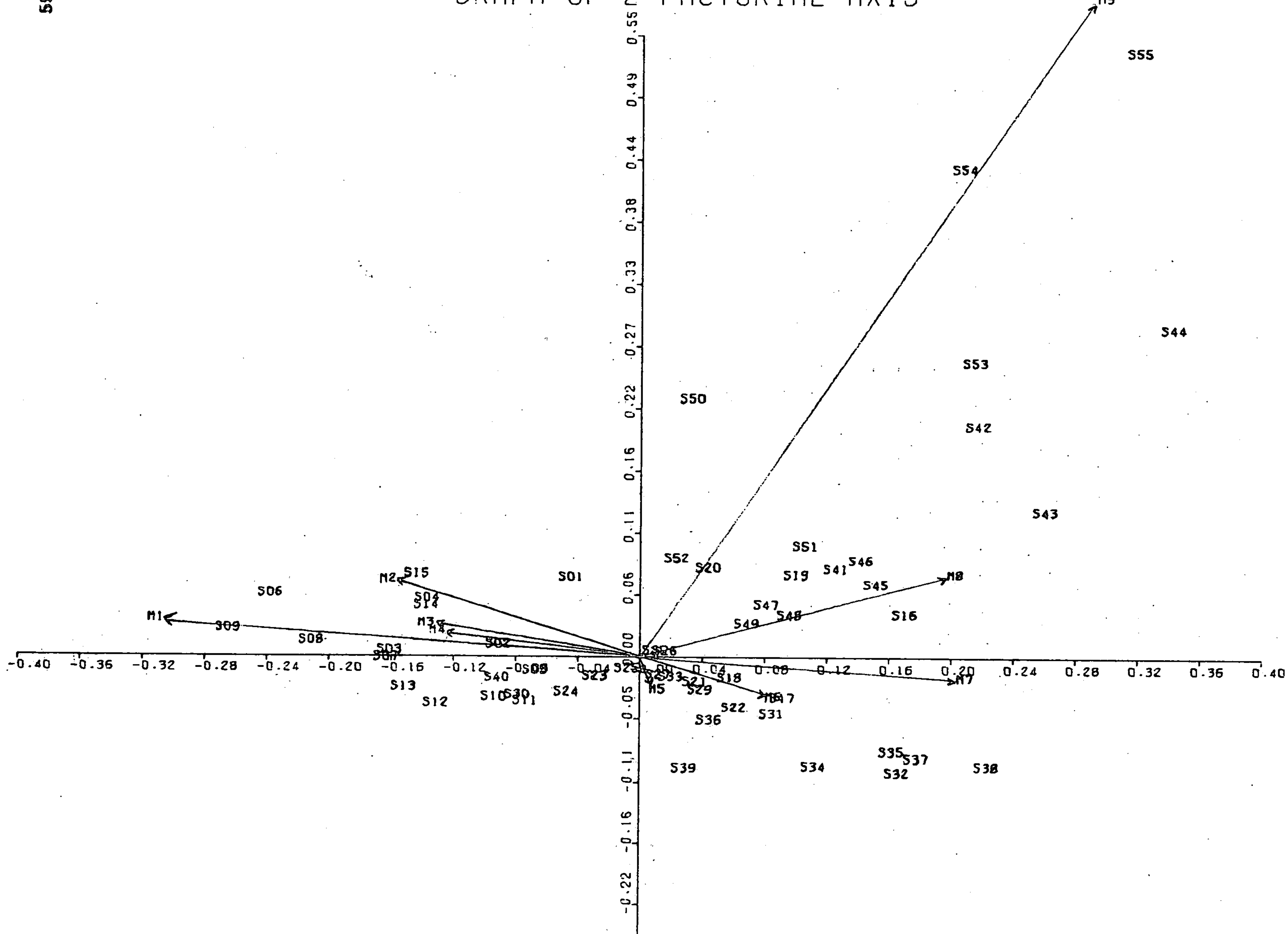
FACTOR 2 — CHARACTERISTIC VALUE : .0048687 — PERCENTAGE OF INERTIA : 17.328

OBJECT	MASS	RHO	FACTOR	CONTRIBUTIONS : ABSOLUTE	RELATIVE	CUMULATIVE
1	29.5000	.8302	.0301	.0225	.1094	.1094
2	628.7999	.0717	.0663	2.3135	6.1221	6.1221
3	3114.6996	.0296	.0269	1.8832	2.4408	2.4408
4	5129.2993	.0226	.0198	1.6916	1.7401	1.7401
5	6572.2991	.0036	-.0315	-5.4759	-27.2919	27.2919
6	5357.8992	.0112	-.0377	-6.3983	-12.7113	12.7113
7	2367.3997	.0584	-.0212	-.8878	-.7666	.7666
8	1029.5999	.0665	.0713	4.3920	7.6469	7.6469
9	272.4000	.4412	.5805	76.9351	76.3610	76.3610

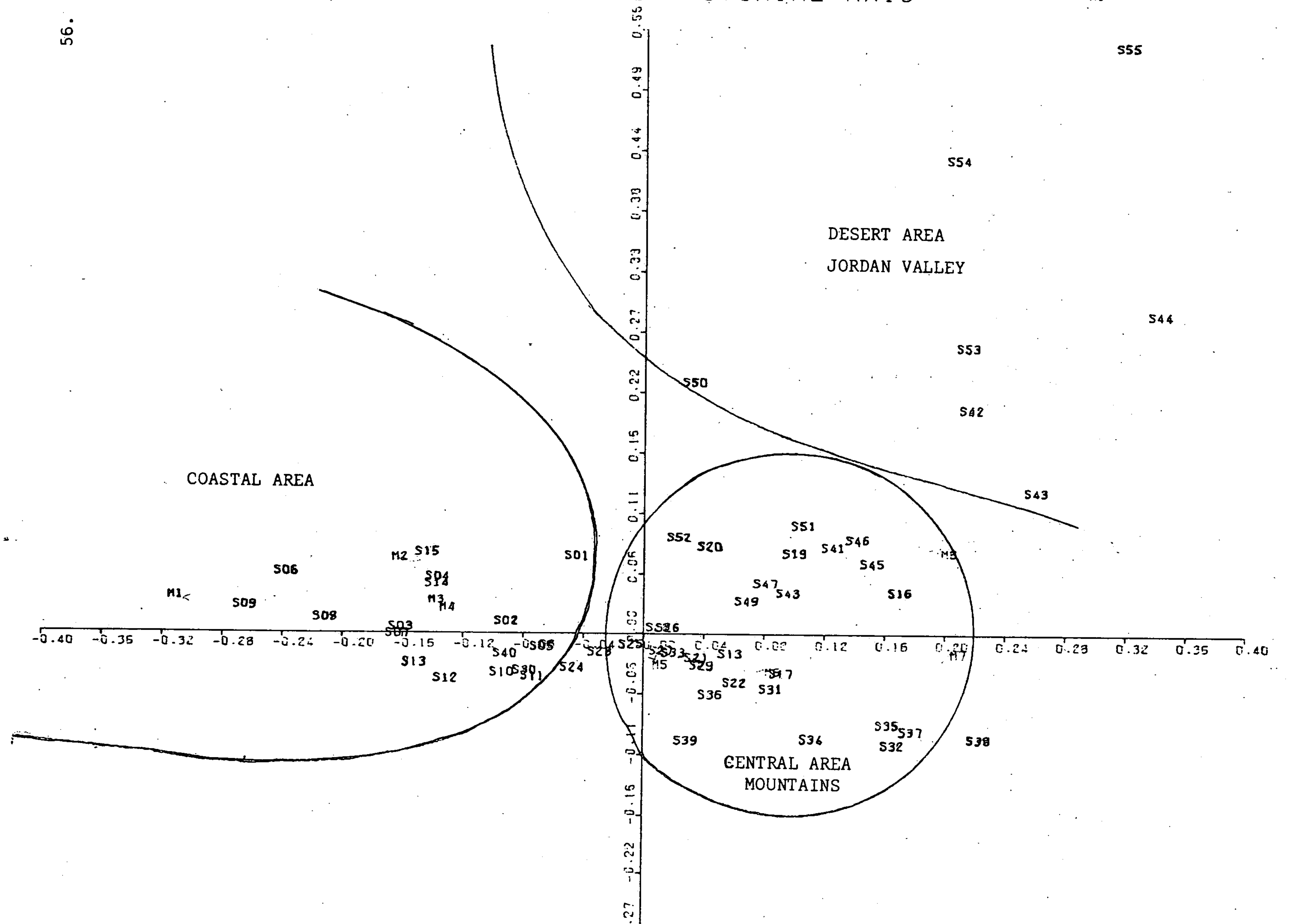
FACTOR 2 — CHARACTERISTIC VALUE : .0048687 — PERCENTAGE OF INERTIA : 17.328

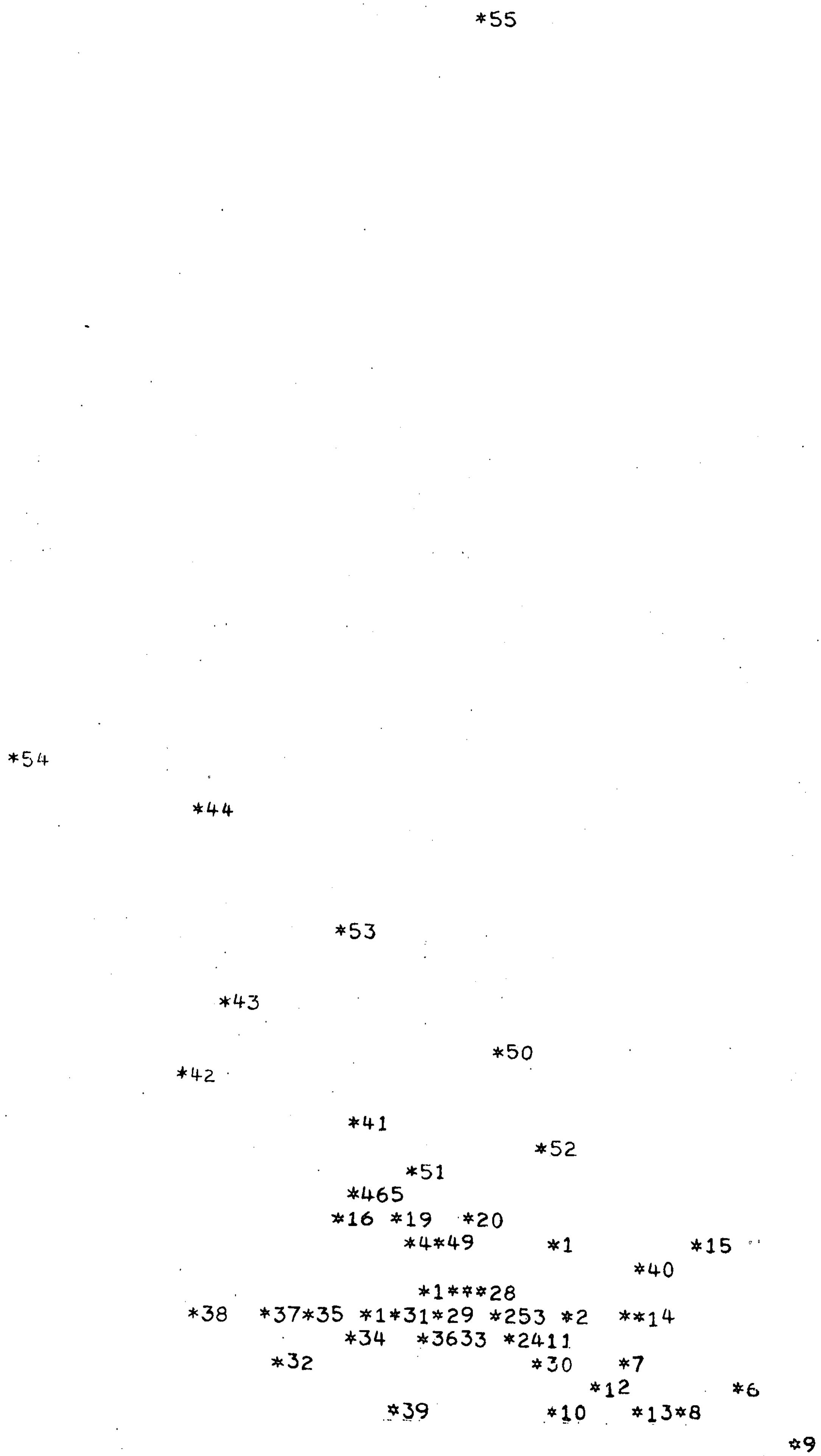
SUBJECT	MASS	RHO	FACTOR	CONTRIBUTIONS : ABSOLUTE	RELATIVE	CUMULATIVE
1	623.0000	.0191	.0681	2.4242	24.2782	24.2782
2	577.0000	.0146	.0073	.0258	.3647	.3647
3	590.0000	.0288	.0020	.0020	.0140	.0140
4	661.0000	.0262	.0488	1.3182	9.0733	9.0733
5	749.0000	.0113	-.0157	-.1549	-2.1769	2.1769
6	492.0000	.0682	.0533	1.1703	4.1605	4.1605
7	604.0000	.0306	-.0040	-.0080	-.0514	.0514
8	580.0000	.0558	.0104	.0528	.1947	.1947
9	519.0000	.0963	.0216	.2035	.4856	.4856
10	496.0000	.0249	-.0390	-.6340	-6.1219	6.1219
11	477.0000	.0104	-.0425	-.7220	-17.3104	17.3104
12	467.0000	.0274	-.0449	-.7895	-7.3548	7.3548
13	379.0000	.0447	-.0303	-.2918	-2.0527	2.0527
14	376.0000	.0305	.0422	.5617	5.8460	5.8460
15	263.0000	.0540	.0710	1.1103	9.3263	9.3263
16	787.0000	.0308	.0350	.8070	3.9745	3.9745
17	593.0000	.0229	-.0395	-.7747	-6.8006	6.8006
18	763.3000	.0121	-.0216	-.2977	-3.8576	3.8576
19	728.0000	.0199	.0708	3.0580	25.2045	25.2045
20	474.0000	.0128	.0770	2.3577	46.2640	46.2640
21	639.0000	.0048	-.0251	-.3376	-13.2513	13.2513
22	493.0000	.0093	-.0478	-.9451	-24.5149	24.5149
23	494.0000	.0058	-.0200	-.1656	-6.8931	6.8931
24	611.0000	.0088	-.0342	-.5999	-13.3680	13.3680
25	506.0000	.0045	-.0133	-.0750	-3.9349	3.9349
26	414.0000	.0016	.0018	.0011	.2050	.2050
27	426.0000	.0009	-.0188	-.1256	-39.4243	39.4243
28	320.0000	.0051	.0024	.0015	.1098	.1098
29	478.0000	.0059	-.0320	-.4098	-17.3288	17.3288
30	554.0000	.0158	-.0372	-.6414	-8.7596	8.7596
31	638.0000	.0121	-.0534	-1.5229	-23.5013	23.5013
32	702.0000	.0424	-.1041	-6.3782	-25.5384	25.5384
33	484.0000	.0120	-.0207	-.1734	-3.5764	3.5764
34	653.0000	.0244	-.0984	-5.2974	-39.6523	39.6523
35	545.0000	.0342	-.0857	-3.3539	-21.4483	21.4483
36	476.0000	.0104	-.0587	-1.3730	-33.0094	33.0094
37	468.0000	.0461	-.0915	-3.2814	-18.1531	18.1531
38	350.6000	.0730	-.0983	-2.8412	-13.2453	13.2453
39	329.0000	.0288	-.1001	-2.7636	-34.7476	34.7476
40	209.4000	.0377	-.0220	-.0851	-1.2842	1.2842
41	200.0000	.0486	.0763	.9764	11.9775	11.9775
42	120.0000	.1155	.2016	4.0866	35.1713	35.1713
43	105.0000	.1288	.1267	1.4139	12.4742	12.4742
44	90.0000	.2196	.2881	6.2610	37.7933	37.7933
45	569.0000	.0280	.0616	1.8128	13.5749	13.5749
46	489.0000	.0317	.0832	2.8378	21.8596	21.8596
47	482.0000	.0111	.0832	.7547	16.8519	16.8519
48	384.0000	.0132	.0341	.3752	8.8251	8.8251
49	411.0000	.0086	.0264	.2410	8.1695	8.1695
50	288.0000	.0580	.2261	12.3466	88.1454	88.1454
51	143.0000	.0229	.0957	1.0983	39.9634	39.9634
52	87.0000	.0372	.0851	.5282	19.4575	19.4575
53	47.0000	.1267	.2582	2.6259	52.5866	52.5866
54	71.0000	.3139	.4291	10.9575	58.6475	58.6475
55	27.6000	.6193	.5320	6.5473	45.6949	45.6949

GRAPH OF 2 FACTORIAL AXIS



GRAPH OF 2 FACTORIAL AXIS




*55
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*51
*465
*16 *19 *20
*4*49 *1 *15
*40
*1***28
*38 *37*35 *1*31*29 *253 *2 **14
*34 *3633 *2411
*32 *30 *7
*12 *6
*39 *10 *13*8
*9

CHAPTER 5.

GRAPHICAL TECHNIQUES FOR HIGH DIMENSIONAL DATA

(Andrew's method)

5.1 INTRODUCTION.

The method of Andrew's and Multidimensional Scaling should be followed when the sum of the percentage of inertia of the first two factors are less than 60. One could consider, say the first five factors, until a sufficient percentage of inertia is obtained.

Pictures of data may be important functions in statistical analysis and they could be useful in the early stage of the analysis. Pictures could also be useful in the subjective art of model formulation, where they assist in the selection of variables or effects to be included in a model, and could help in checking the assumptions of a model.

For multivariate data the need is greater than for univariate data as the amounts of multivariate data are larger and the relationships, though frequently geometrical cannot be readily assimulated from a listing of the data.

First we would like to describe in section 1 a method for mapping k dimensional points into functions which may then be plotted and briefly describe some of its properties. In Section 3 we show how the first 5 factors of Correspondence analysis could be plotted as described in Section 2.

5.2 FUNCTION PLOTS OF HIGH-DIMENSIONAL DATA.

The method is applicable to quantitive data in k dimensions, metric data. A data point may be represented as a vector $\underset{\sim}{x} = (x_1, \ldots, x_m)$ We could plot such a point by mapping it into a space of functions and plot the resulting functions. Many possible mappings could be done.

The following method has many useful geometrical and statistical properties and uses a family of functions which are widely understood.

For each k .variate observation x, define a function

$$f_{\underset{\sim}{x}}(t) = \frac{1}{\sqrt{2}} x_1 + x_2 \sin(t) + x_3 \cos(t) + x_4 \sin(2t) + x_5 \cos(2t) + \ldots..$$

This function may be plotted over the range $-\pi \leq t \leq \pi$. Each data point will thus produce a curve drawn across the page.

Some interesting properties of such plots are discussed in Andrews(1972) and we would like to summarize these properties as given in Discriminant Analysis and Applications edited by T. Cacoullos, page 38-40.

(1) The mapping is linear. If $\underset{\sim}{x},\underset{\sim}{y}$ and $\underset{\sim}{z}$ are points in k-dimensiona- space and if

$$\underset{\sim}{x} = a\underset{\sim}{y} + b\underset{\sim}{z}$$

then the functions share the same linear relation

$$f_{\underset{\sim}{x}}(t) = a\, f_{\underset{\sim}{y}}(t) + b\, f_{\underset{\sim}{z}}(t)$$

and so the functions of the average satisfies

$$f_{\underset{\sim}{x}}(t) = m^{-1} \sum_{i=1}^{m} f_{\underset{\sim}{x}_i}(t)$$

(2) The mapping preserves distances. If the distance between two functions f and g is defined by

$$\|f - g\|_{L_2} = \int_{-\pi}^{\pi} |f(t) - g(t)|^2 dt,$$

(The distance used in the program (page 63) is defined by

$$\delta(x,y) = \sum_i |f_{\underset{\sim}{x}}(t_i) - f_{\underset{\sim}{y}}(t_i)|)$$

then the distance between the functions corresponding to $\underset{\sim}{x}$ and y,

$$\|f_{\underset{\sim}{x}}(t) - f_{\underset{\sim}{y}}(t)\| L_2 = \pi\ \Sigma(x_i - y_i)^2$$

is proportional to the Euclidean distance between these points.

(3) The mapping yields one-dimensional projections. For a particular value of t say t_o, the function value $f_x(t_o)$ is proportional to the projection of $\underset{\sim}{x}$ on the vector $\underset{\sim}{f_1}$

$$\underset{\sim}{f_1}(t_o) = (1/\sqrt{2}, \sin(t_o), \cos(t_o), \sin(2t_o), \ldots).$$

As t_o changes the projections of the data on a continiun of directions are recorded on the plot.

(4) The mapping preserves variances. If the original data have been transformed so that the components are approximately independent with equal variance σ^2, then the variance of $f_{\underset{\sim}{x}}(t)$ is given by

$$\mathrm{var}(f_{\underset{\sim}{x}}(t)) = \sigma^2(\tfrac{1}{2} + \sin^2(t_o) + \cos^2(t_o) + \sin^2(2t_o) + \ldots)$$

$$= \sigma^2\begin{cases} k/2, & \text{if } k \text{ is odd,} \\ k/2 + \varepsilon, |\varepsilon| \le 1/2, & \text{if } k \text{ is even} \end{cases}$$

(5) The mapping generates over all tests. Since the function value $f_{\underset{\sim}{x}}(t)$ proportional to the projection of $\underset{\sim}{x}$ on any vector is less than the length of $\underset{\sim}{x}$, tests may be constructed for any value of t. Thus a test of the hypothesis $E(\underset{\sim}{x}) = \underset{\sim}{\mu}$ may be based on the inequality

$$\|f_{\underset{\sim}{x}}(t) - f_{\underset{\sim}{\mu}}(t)\|^2 \le \tfrac{1}{2}(k + 1)\sigma^2 \quad \chi_k^2(\alpha)$$

which is true with probability $1 - \alpha$ for <u>all</u> values of t.

The properties discussed above also hold with obvious slight changes for functions of the form

$$f_{\underset{\sim}{x}}(t) = x_1 \sin(t) + x_2 \cos(t) + x_3 \sin(2t) + x_4 \cos(2t) + \ldots$$

5.3 ANDREW'S METHOD IMPLIED ON THE FACTORS COMPUTED BY CORRESPONDENCE ANALYSIS.

A computer program (printed on page 63) takes the first n factors for its corresponding variable as computed by Correspondence analysis and plots the variable as described in section 2.

It is however necessary to decide how many factors present the vector $\underset{\sim}{x} = (x_1, \ldots, x_n)$ (§5.2) as described in chapter 3 to help with the decision.

If we look at the Israeli rainfall data and use for example the first 5 factors for the specific stations as computed in chapter 4, it is possible to plot for each variable (i.e. each station) a graph by using the Andrew's method. The specific graphs can be see on page 66 .

We are faced with another problem. If there are for example as in our Israeli rainfall analysis, say 55 variables, it is difficult to distinguish the difference or association between the variables on the 55 graphs. A program (page 64) was developed to solve this problem. During the plot of the different variables each x-axis is devided into n (say 1000) parts. For each value i on the x-axis the corresponding value on the y-axis is measured and subtracted from the value on the y-axis for the same i of every other graph. A total which specify the difference for all the y values is computed between each and every variable. Thus by just looking at the totals and then compare the graphs it may help us in finding the associations and oppositions between the different variables. This method will only save time because it could be very difficult to scan through all the graphs to find the associations. One could thus disregard this method especially in cases with not many variables.

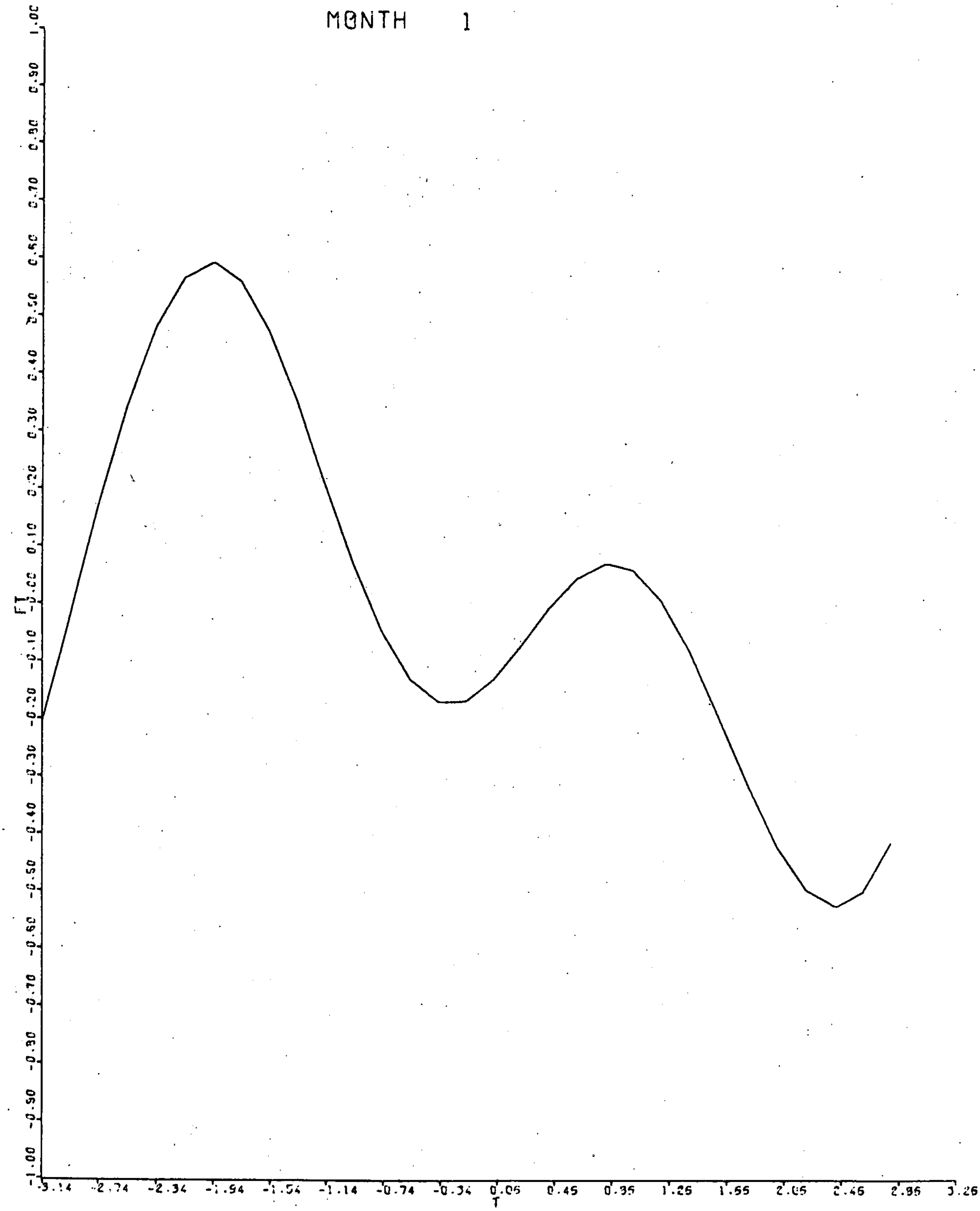
During the construction of these totals of differences we found it applicable to multidimensional scaling which leads us to the next chapter.

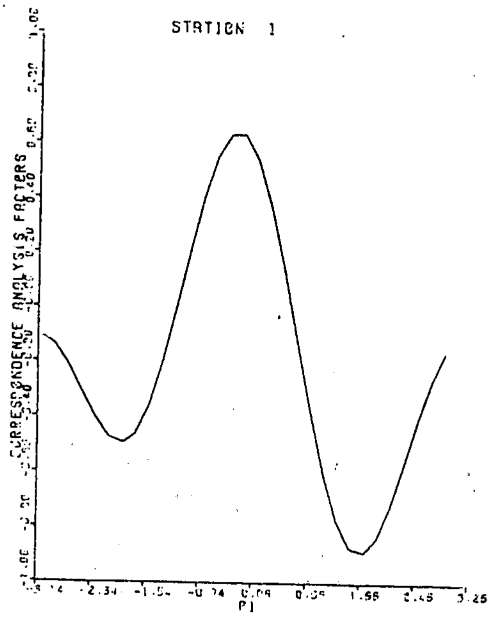
REFERENCE TO PROGRAMS AND CORRESPONDING EXECUTION EXAMPLES.

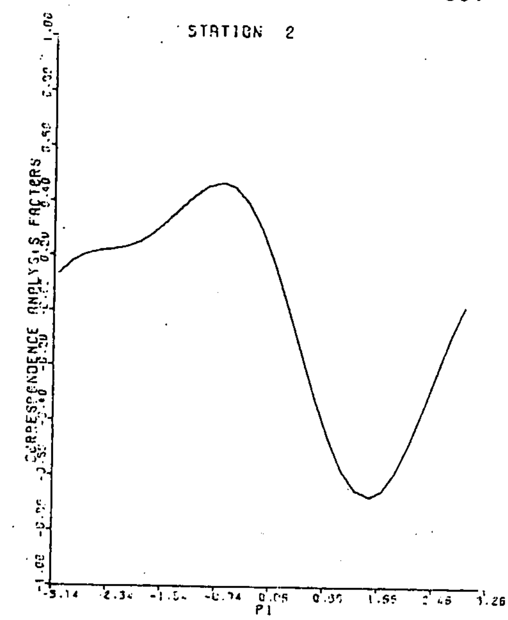
```
      SUBROUTINE ANDREW(X,K,M,IBYS,NK,N1,AMIN,AMAKS,Y,INT,IND)
C     ***************************************************************
C     X : MATRIX WITH FACTOR SCORES
C     K : FACTORS
C     M : GRAPHS
C     ***************************************************************
C     X : MATRIKS VAN HOOFKOMPONENTWAARDES
C     K : AANTAL HOOFKOMPONENTE WAT GEBRUIK WORD
C     M : AANTAL GRAFIEKE
      DIMENSION IBUF(1000),T(100),FT(100),IBYS(NK),X(K),Y(160,32)
      IF(IND.EQ.1)CALL PLOTS(IBUF(1),1000,9)
      CALL PLOT(0.0,-0.5,-3)
      CALL PLOT(0.0,0.5,-3)
      CALL FACTOR(0.400)
      T(1)=-3.1415926535
      INT=IFIX(6.2831853070/0.2)
      INT1=INT+1
      INT2=INT+2
      DO 2 J=2,INT
    2 T(J)=T(J-1)+0.2
      IF(K.GT.5) GO TO 100
C    -----------------------------------------------------------------
C    COMPUTE VALUES BY USING ANDREWS METHOD.
C    THE COMPUTATION IS DEPENDANT ON THESE FACTORS
C    -----------------------------------------------------------------
      GO TO (100,200,300,400,500),K
  100 WRITE(6,900)
  900 FORMAT(1H1,'ONGELDIGE AANTAL KOMPONENTE')
      GO TO 10
  200 DO 203J=1,INT
      Y(IND,J)=X(1)*SIN(T(J))+X(2)*COS(T(J))
  203 FT(J)=X(1)*SIN(T(J))+X(2)*COS(T(J))
      GO TO 1000
  300 DO 3 J=1,INT
      Y(IND,J)=X(1)*SIN(T(J))+X(2)*COS(T(J))+X(3)*SIN(2*T(J))
    3 FT(J)=X(1)*SIN(T(J))+X(2)*COS(T(J))+X(3)*SIN(2*T(J))
      GO TO 1000
  400 DO 403 J=1,INT
      Y(IND,J)=X(1)*SIN(T(J))+X(2)*COS(T(J))+X(3)*SIN(2*T(J))
     *+X(4)*COS(2*T(J))
  403 FT(J)=X(1)*SIN(T(J))+X(2)*COS(T(J))+X(3)*SIN(2*T(J))
     *+X(4)*COS(2*T(J))
      GO TO 1000
  500 DO 503J=1,INT
      Y(IND,J)=X(1)*SIN(T(J))+X(2)*COS(T(J))+X(3)*SIN(2*T(J))
     *+X(4)*COS(2*T(J))+X(5)*SIN(3*T(J))
  503 FT(J)=X(1)*SIN(T(J))+X(2)*COS(T(J))+X(3)*SIN(2*T(J))
     *+X(4)*COS(2*T(J))+X(5)*SIN(3*T(J))
 1000 FT(INT1)=AMIN
      FT(INT2)=AMAKS/5
      T(INT1)=-3.1415926535
      T(INT2)=0.8
C    -----------------------------------------------------------------
C    PLOT FACTORS
C    -----------------------------------------------------------------
      CALL SYMBOL(2.0,10.0,0.2,IBYS(1),0.0,N1)
      CALL AXIS(0.0,0.0,2HPI,-2,8.0,0.0,T(INT1),T(INT2))
      CALL AXIS(0.0,0.0,31HCORRESPONDENCE ANALYSIS FACTORS,31,
     *10.0,90.0,FT(INT1),FT(INT2))
```

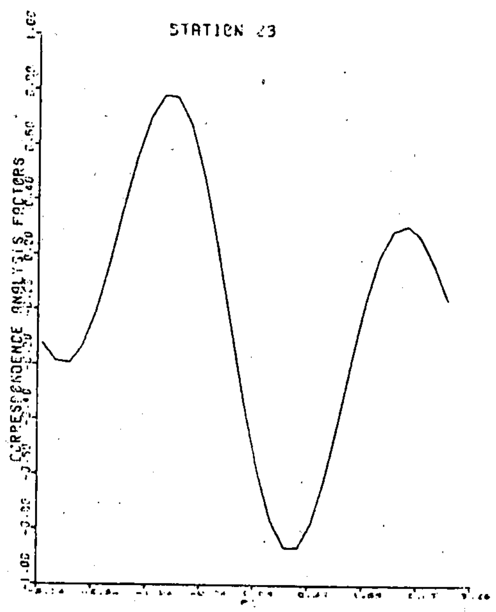
```
      CALL SCALE(T(1),8.0,INT,1)
      CALL SCALE(FT(1),10.0,INT,1)
      CALL LINE(T(1),FT(1),INT,1,0,4)
      CALL PLOT(10.0,0.0,-3)
      IF(IND.EQ.M)GO TO 999
      GO TO  10
  999 CALL PLOT(10.0,0.0,999)
  10  RETURN
      END
```

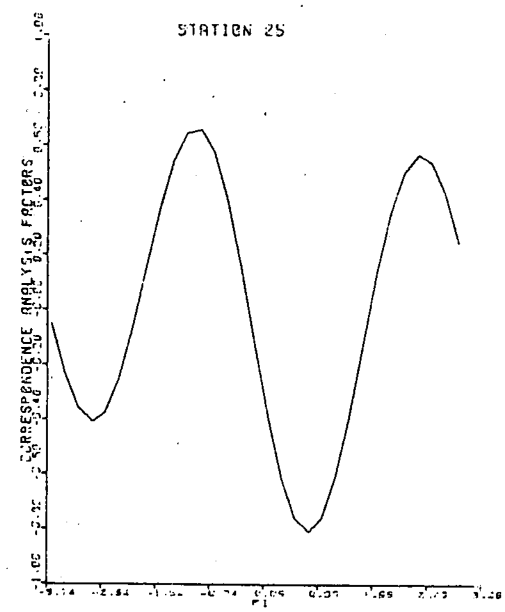
```
C,    ************************************************************************
C     ANDREWS METHOD: D.BESTER.
C     REFERENCE: CALCOULLOS,T. (1973) PAGE 37-59
C     PARAMETER CARDS NEEDED:
C     COLUMN  1-12   HEADING OF GRAPH (12 CHARACTERS ALFANUMERIC)
C     COLUMN 13-24   FACTOR1 (CORRESPONDENCE ANALYSIS)
C     COLUMN 25-37   FACTOR2
C        ::    ::        ::
C        ::    ::        ::
C        ::    ::        ::
C     ************************************************************************
      DIMENSION X(4),IBYS(2),Y(150,32)
      INT=0
C     ------------------------------------------------------------------------
C     REQUEST FOR 500 MINUTE MAXIMUM PLOTTER TIME
C     ------------------------------------------------------------------------
      CALL PLTIME(500)
      DO 10I=1,55
      READ(5,1)IBYS,X
    1 FORMAT(2A6,4F13.8)
   10 CALL ANDREW(X,4,55,IBYS,2,12,1.0,-1.0,Y,INT,I)
      DO 11K=1,54
      KK=K+1
      DO 11I=KK,55
      SOMVER       =0.0
      DO 12J=1,INT
      VERSK        =Y(K,J)-Y(I,J)
   12 SOMVER=SOMVER+ABS(VERSK)
   11 WRITE(3,13)K,I,SOMVER
   13 FORMAT(5X,2I3,F7.4)
      STOP
      END
```

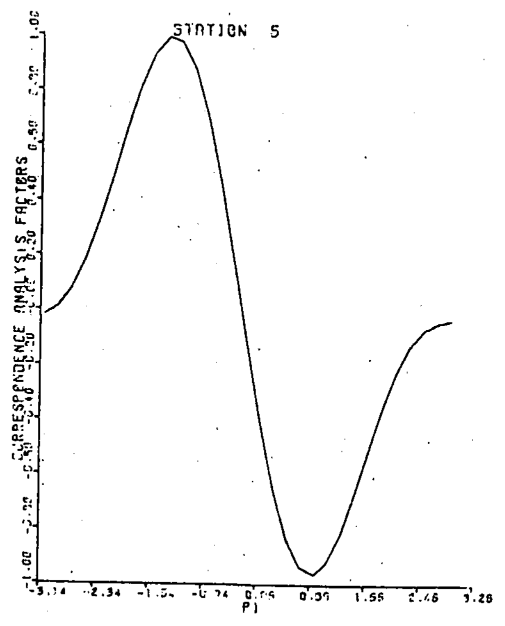

MONTH 1
FT
T
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
-0.70
-0.80
-0.90
-1.00
-3.14
-2.74
-2.34
-1.94
-1.54
-1.14
-0.74
-0.34
0.06
0.45
0.85
1.25
1.65
2.05
2.46
2.86
3.26

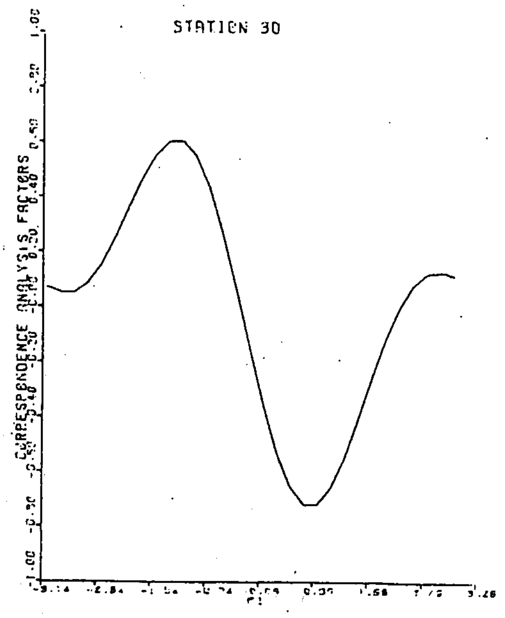

STATION 1
CORRESPONDENCE ANALYSIS FACTORS
PI

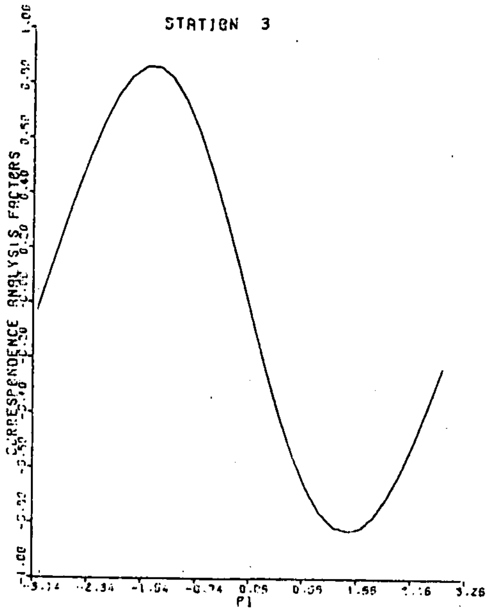

STATION 2
CORRESPONDENCE ANALYSIS FACTORS
PI

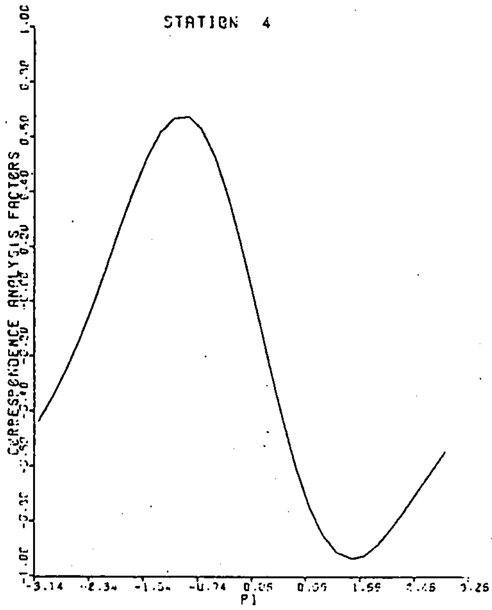

STATION 23
CORRESPONDENCE ANALYSIS FACTORS

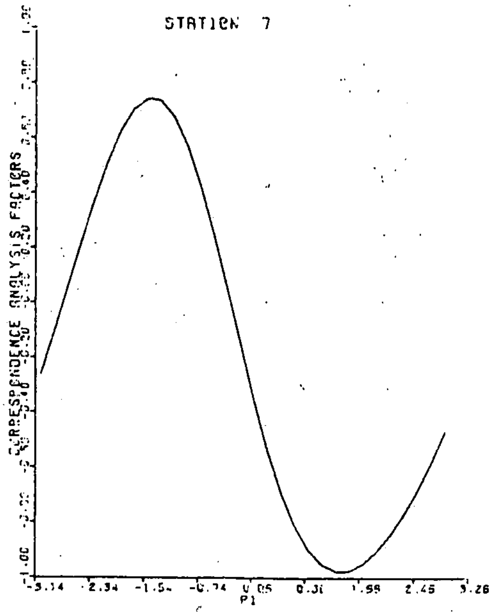

STATION 25
CORRESPONDENCE ANALYSIS FACTORS
PI

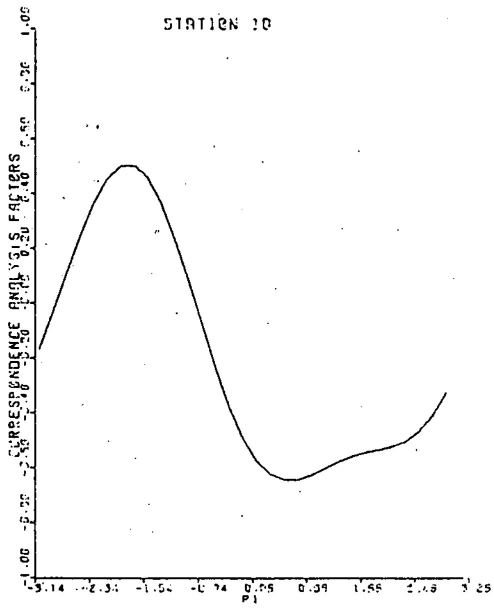

STATION 5
CORRESPONDENCE ANALYSIS FACTORS
PI

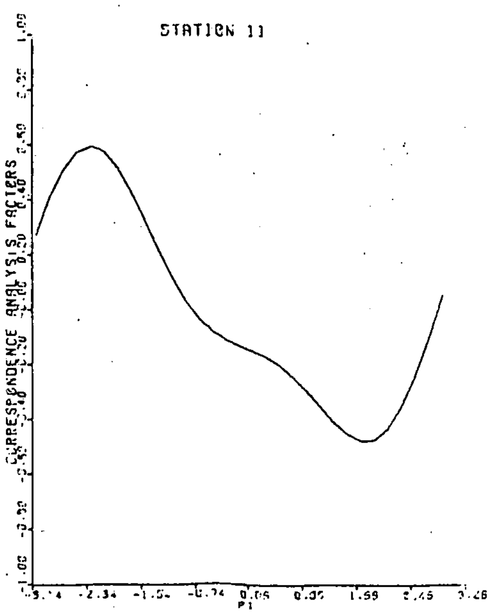

STATION 30
CORRESPONDENCE ANALYSIS FACTORS

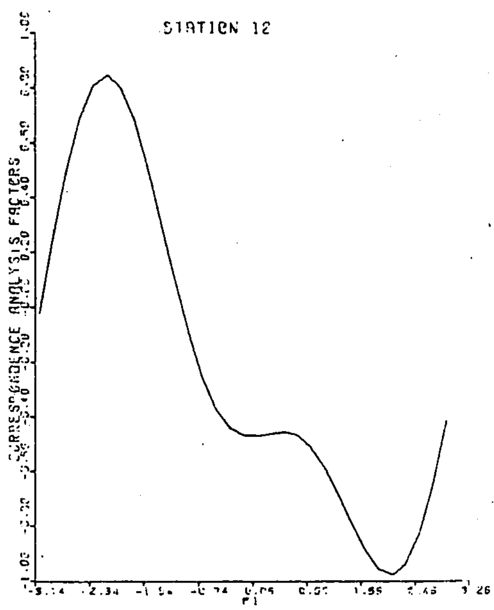

STATION 3
CORRESPONDENCE ANALYSIS FACTORS
PI

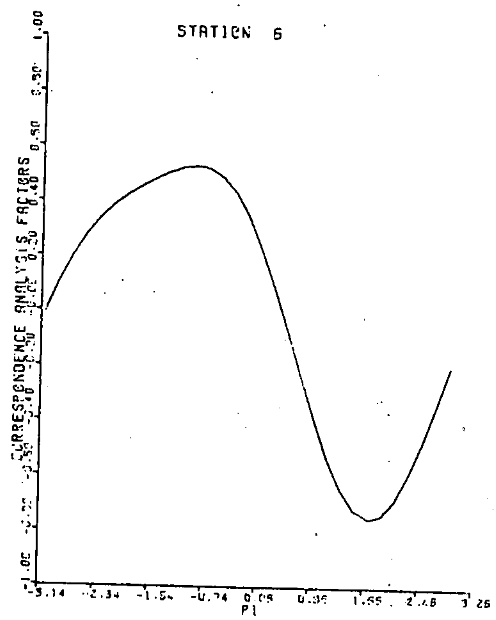

STATION 4
CORRESPONDENCE ANALYSIS FACTORS
PI

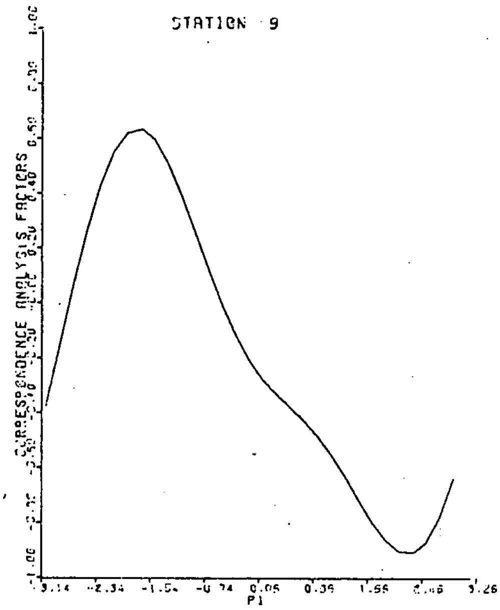

STATION 7
CORRESPONDENCE ANALYSIS FACTORS
PI

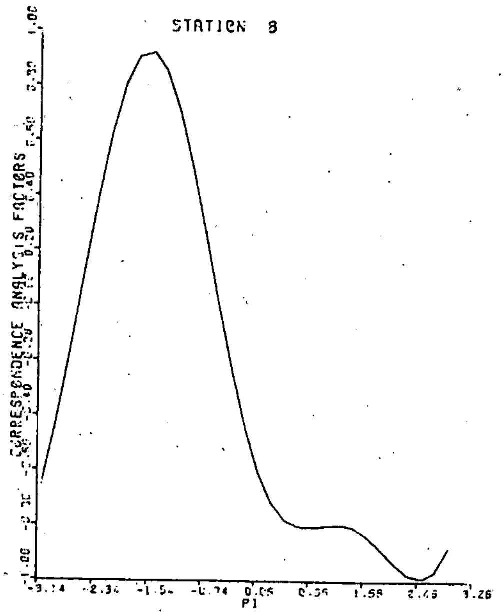

STATION 10
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 11
CORRESPONDENCE ANALYSIS FACTORS
PI

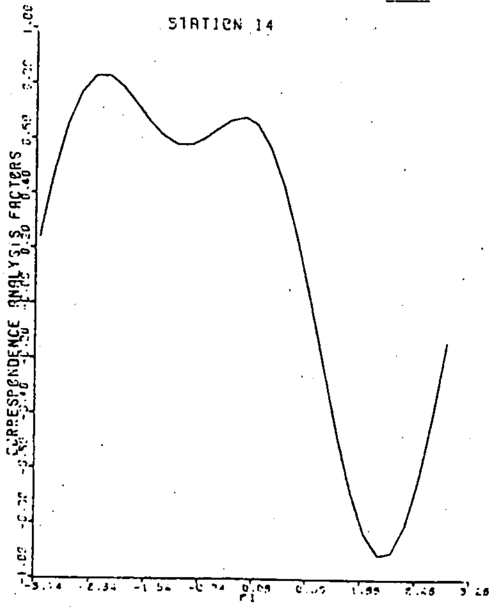

STATION 12
CORRESPONDENCE ANALYSIS FACTORS
PI

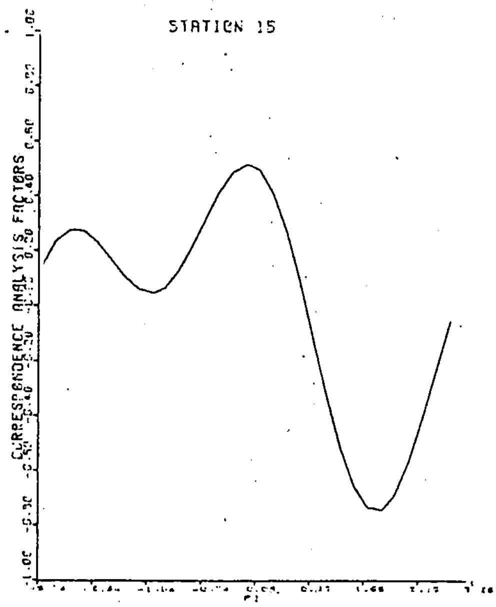

STATION 6
CORRESPONDENCE ANALYSIS FACTORS
PI

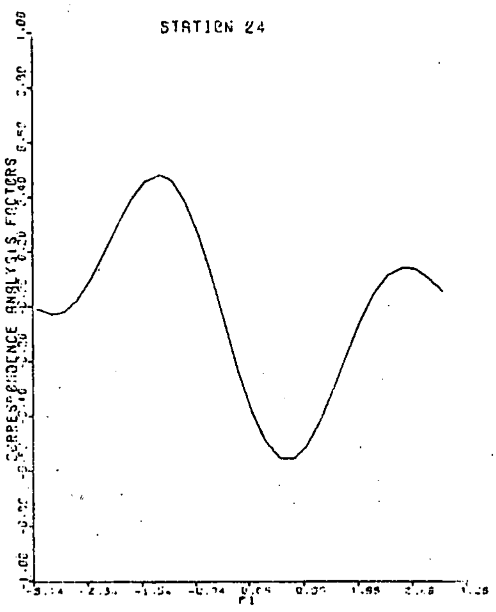

STATION 9
CORRESPONDENCE ANALYSIS FACTORS
PI

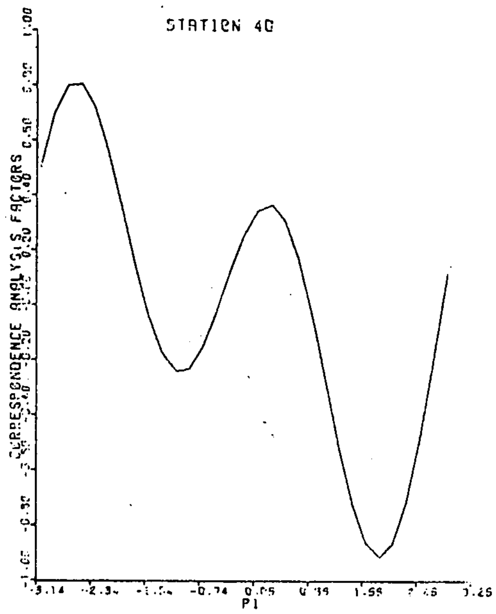

STATION 8
CORRESPONDENCE ANALYSIS FACTORS
PI

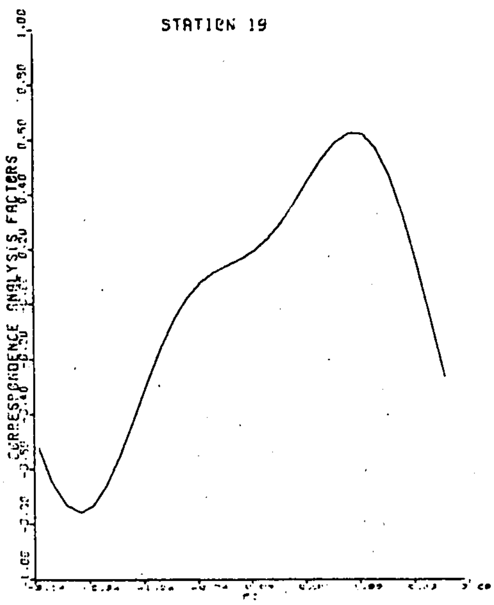

STATION 13
CORRESPONDENCE ANALYSIS FACTORS
PI

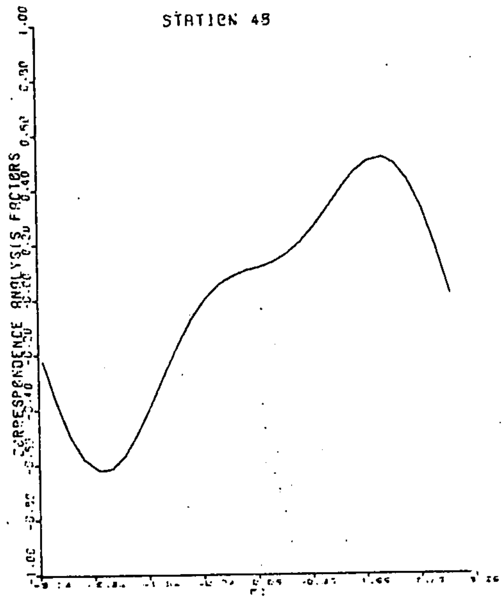

STATION 14
CORRESPONDENCE ANALYSIS FACTORS
PI

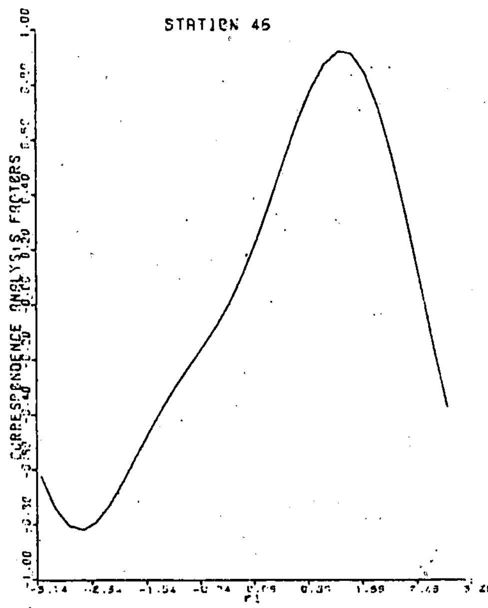

STATION 15
CORRESPONDENCE ANALYSIS FACTORS
PI

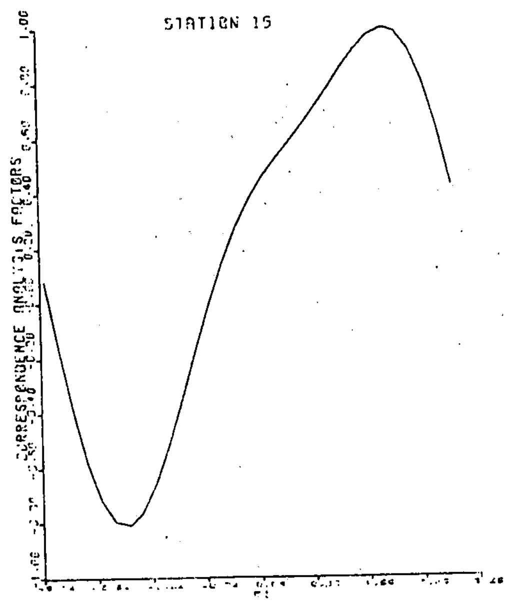

STATION 24
CORRESPONDENCE ANALYSIS FACTORS
PI

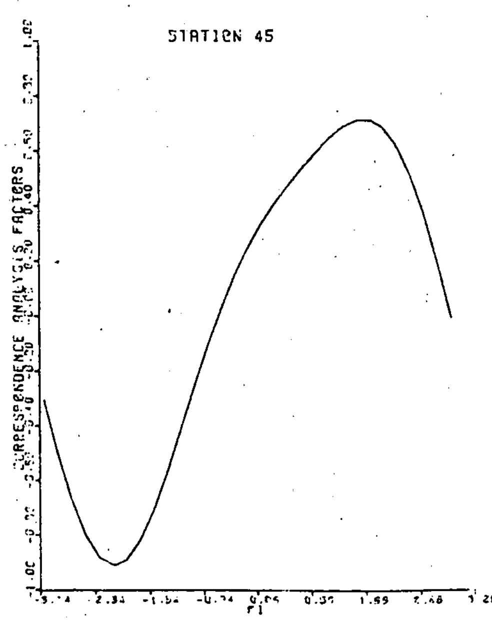

STATION 40
CORRESPONDENCE ANALYSIS FACTORS
PI

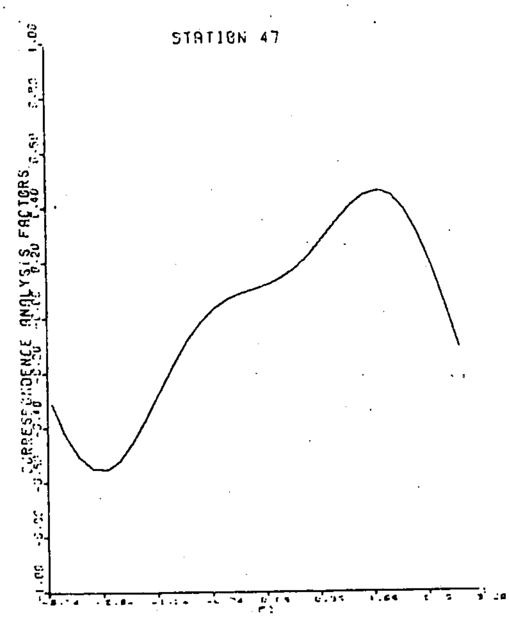

STATION 19
CORRESPONDENCE ANALYSIS FACTORS
PI

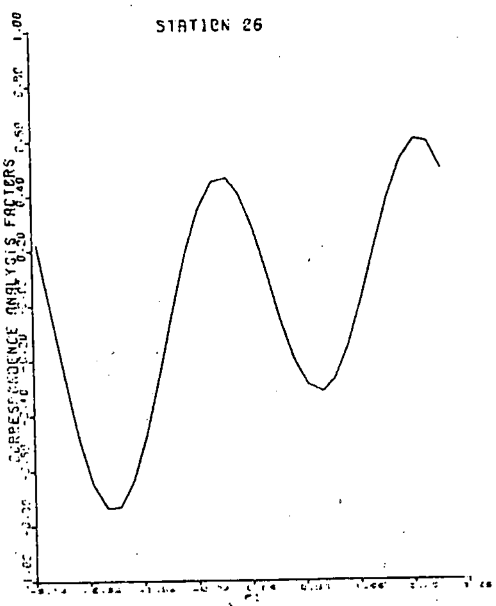

STATION 48
CORRESPONDENCE ANALYSIS FACTORS
PI

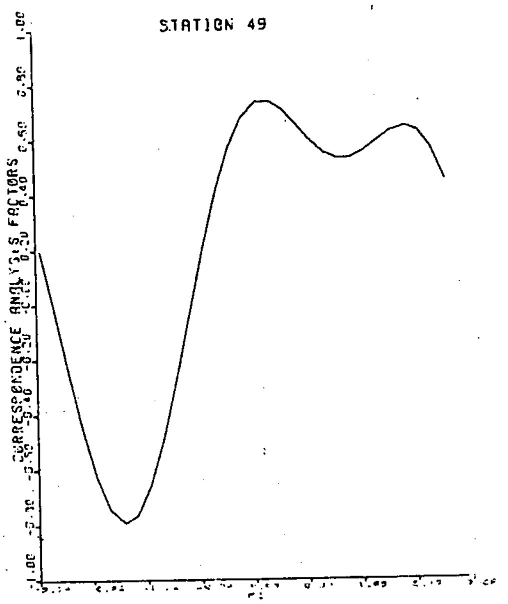

STATION 46
CORRESPONDENCE ANALYSIS FACTORS
PI

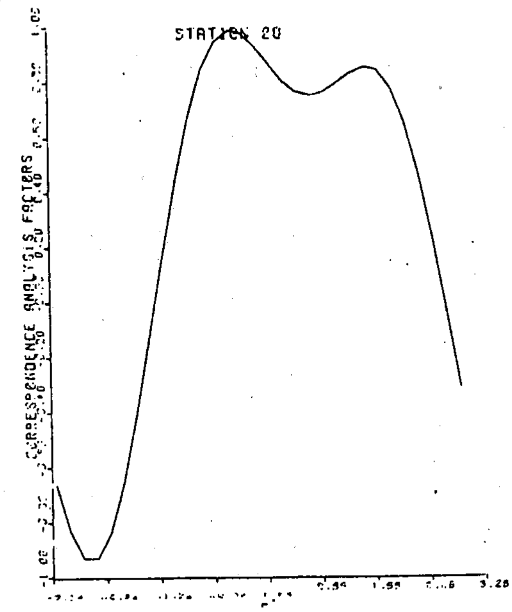

STATION 15
CORRESPONDENCE ANALYSIS FACTORS
PI

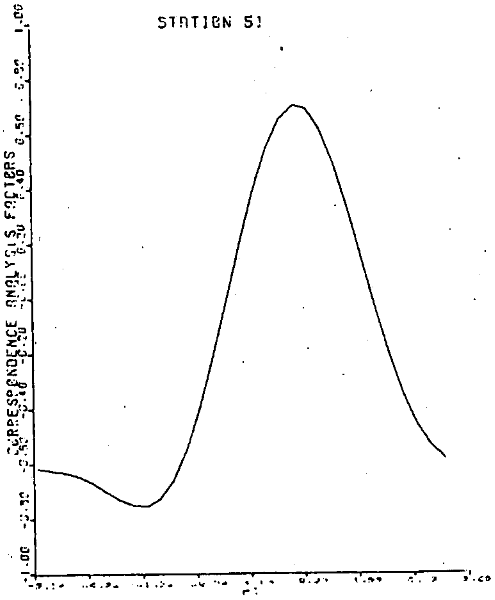

STATION 45
CORRESPONDENCE ANALYSIS FACTORS
PI

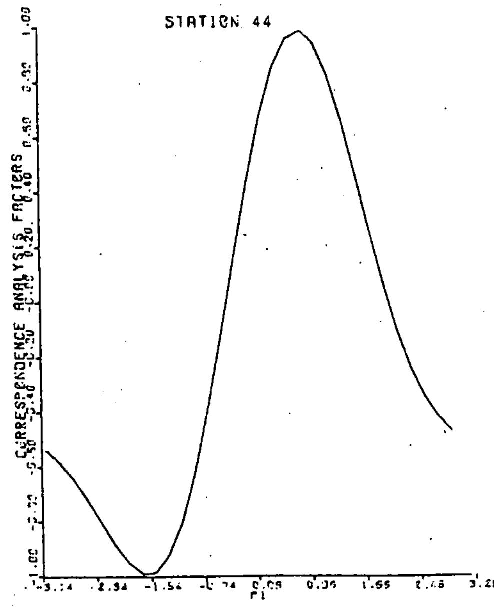

STATION 47
CORRESPONDENCE ANALYSIS FACTORS
PI

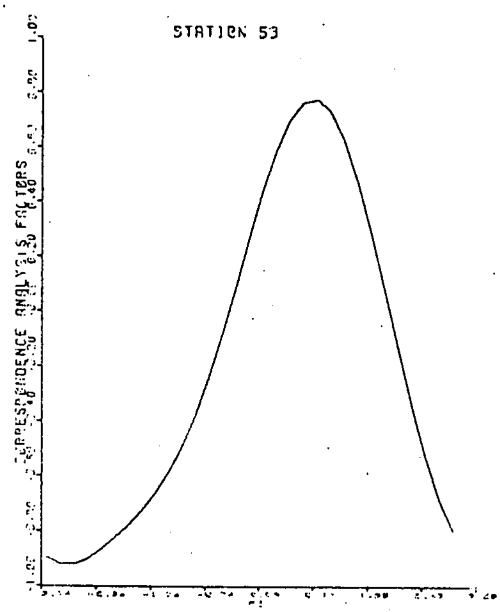

STATION 26
CORRESPONDENCE ANALYSIS FACTORS
PI

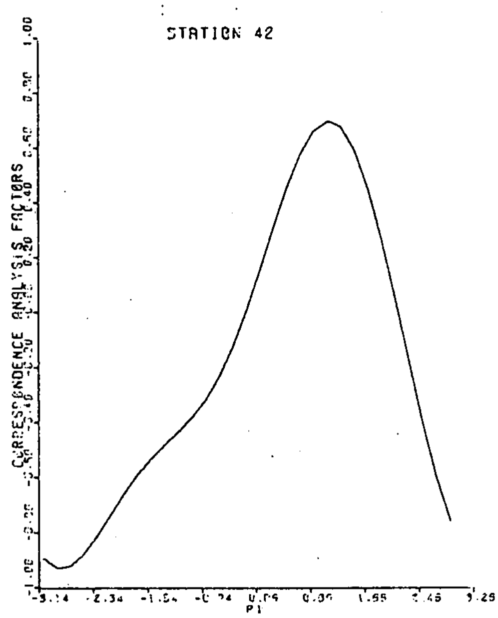

STATION 49
CORRESPONDENCE ANALYSIS FACTORS
PI

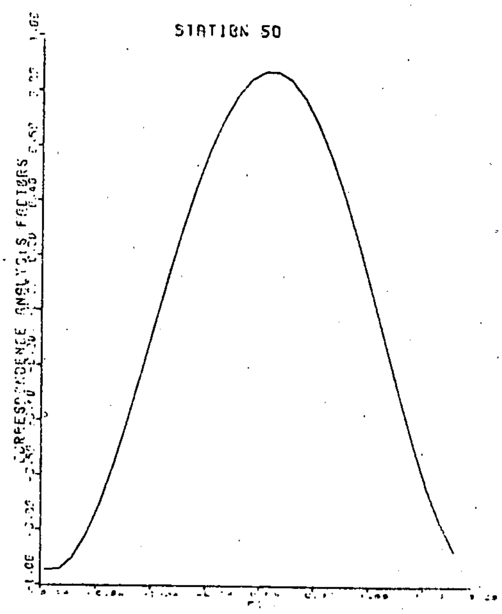

STATION 20
CORRESPONDENCE ANALYSIS FACTORS
3.26

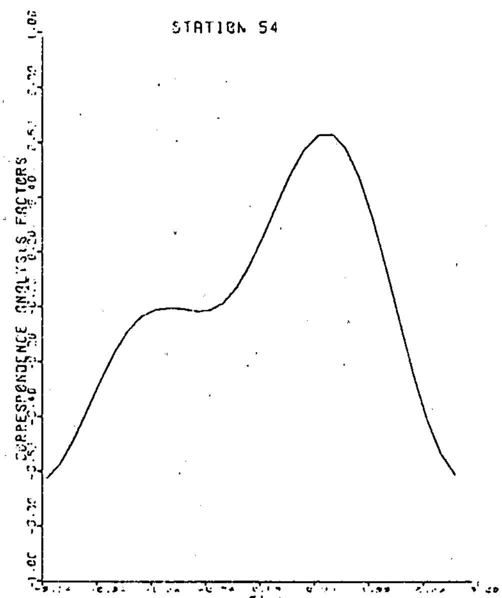

STATION 51
CORRESPONDENCE ANALYSIS FACTORS
PI

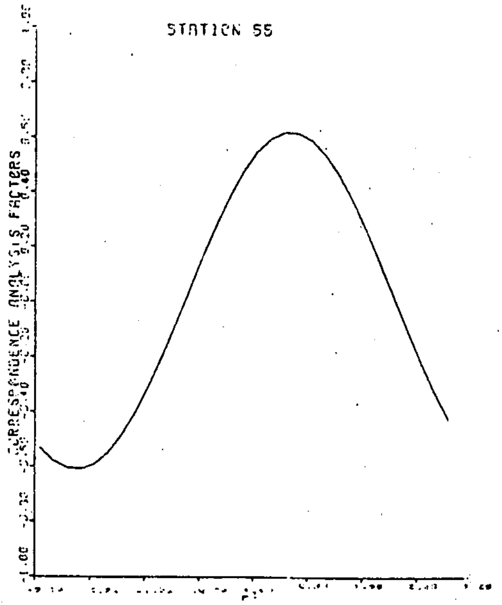

STATION 44
CORRESPONDENCE ANALYSIS FACTORS
1.00
0.20
-3.14
-2.34
-1.54
3.26
PI

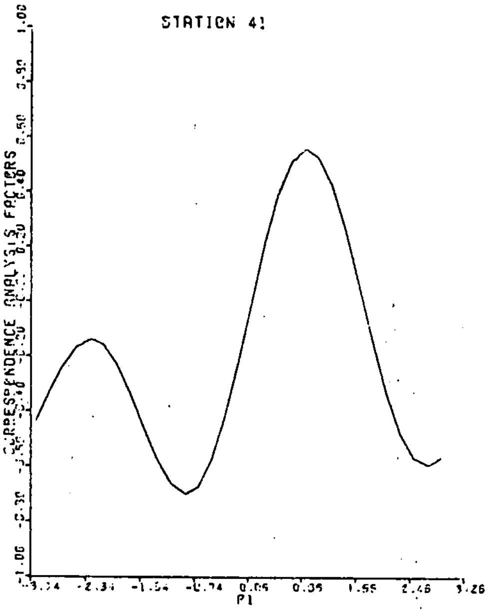

STATION 53
CORRESPONDENCE ANALYSIS FACTORS

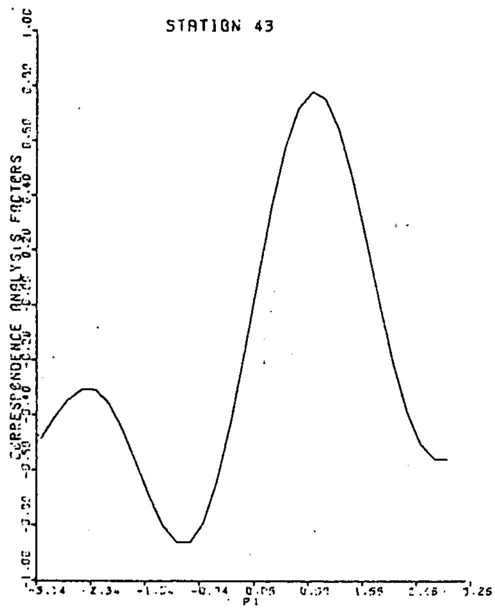

STATION 42
CORRESPONDENCE ANALYSIS FACTORS
PI

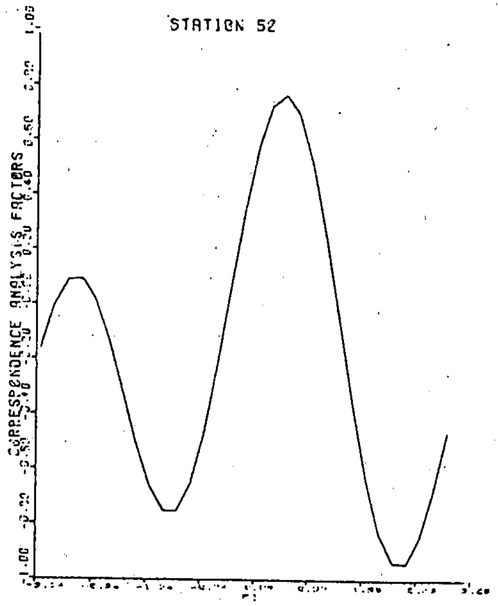

STATION 50
CORRESPONDENCE ANALYSIS FACTORS

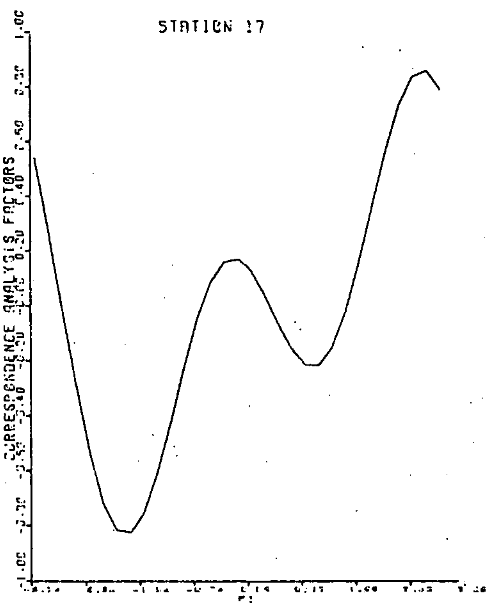

STATION 54
CORRESPONDENCE ANALYSIS FACTORS

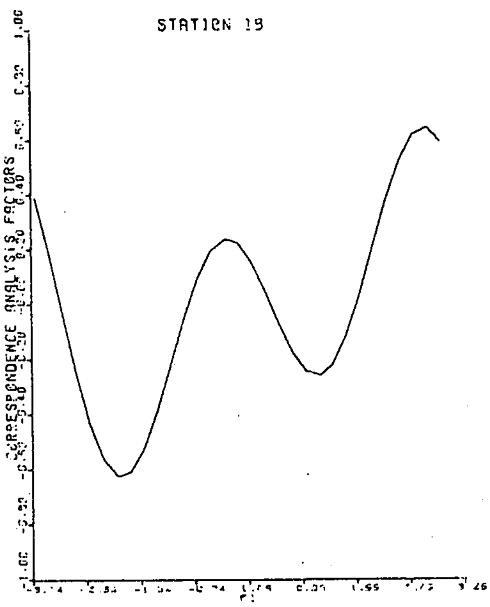

STATION 55
CORRESPONDENCE ANALYSIS FACTORS

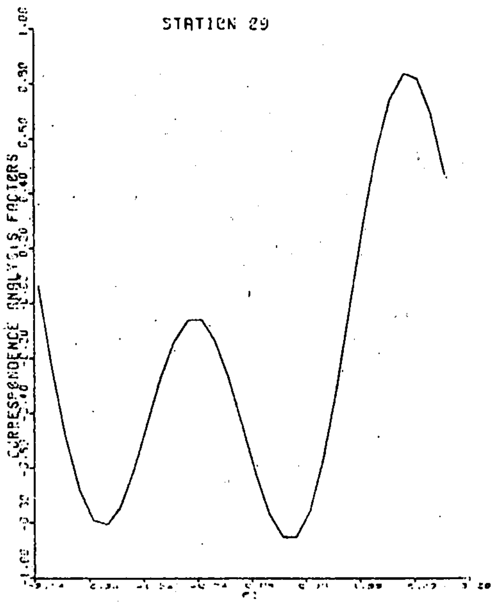

STATION 41
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 43
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 52
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 17
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 19
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 29
CORRESPONDENCE ANALYSIS FACTORS
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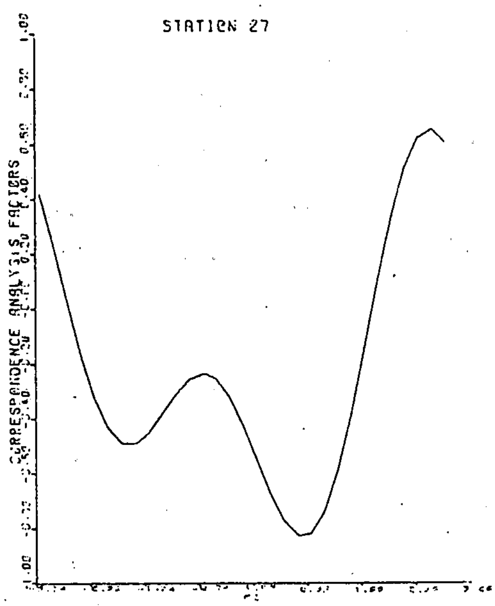
STATION 27
CORRESPONDENCE ANALYSIS FACTORS
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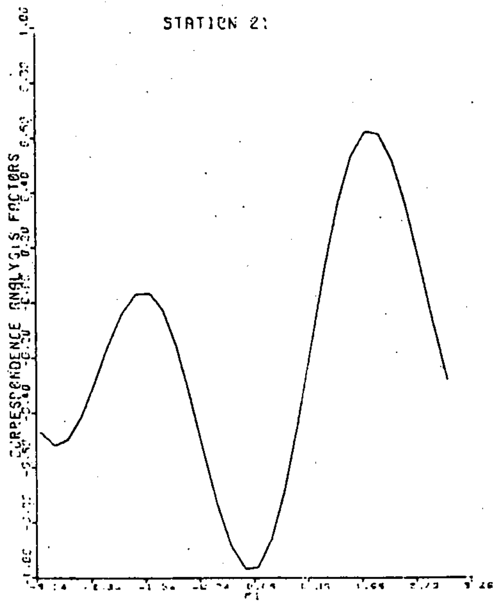
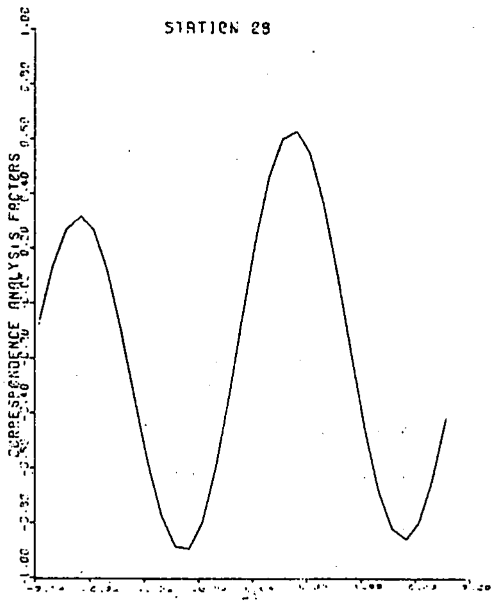
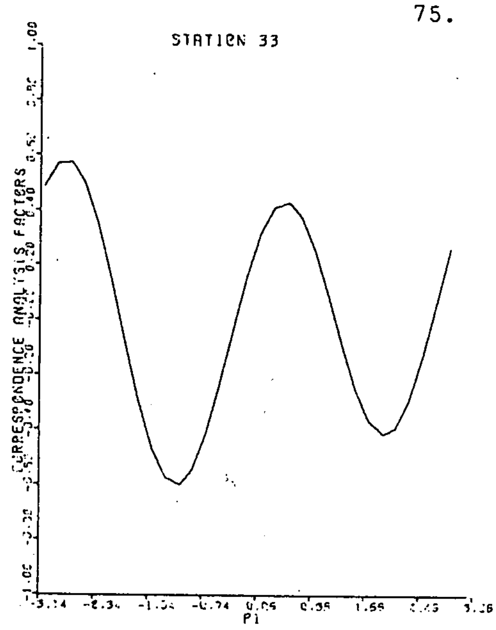
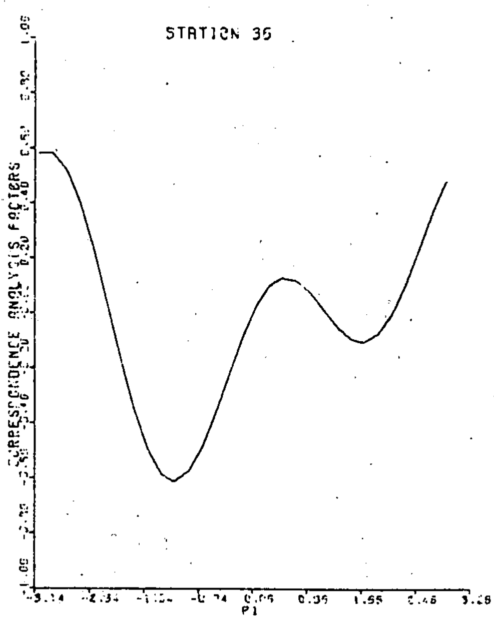

STATION 2
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 28
CORRESPONDENCE ANALYSIS FACTORS


STATION 33
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 35
CORRESPONDENCE ANALYSIS FACTORS
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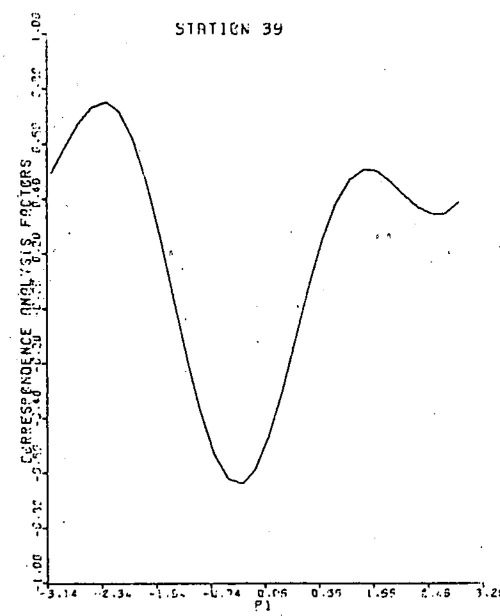
STATION 39
CORRESPONDENCE ANALYSIS FACTORS
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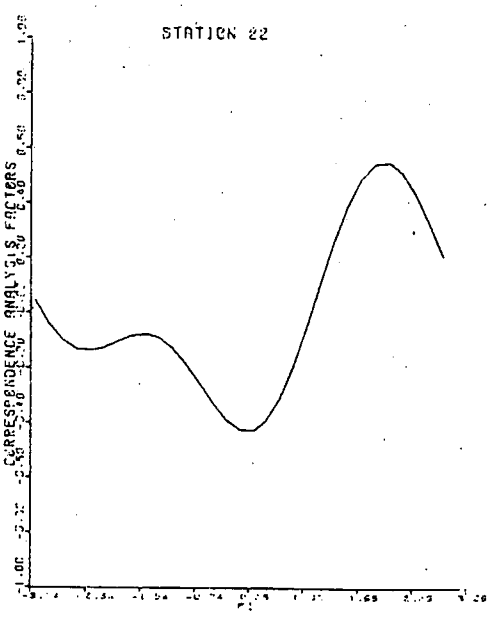
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CORRESPONDENCE ANALYSIS FACTORS
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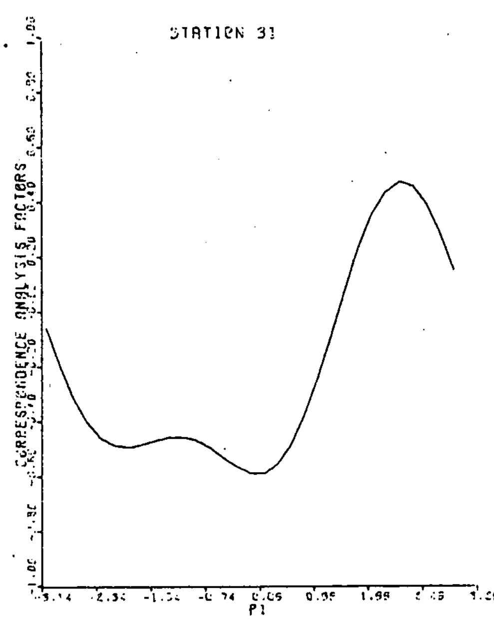
STATION 31
CORRESPONDENCE ANALYSIS FACTORS
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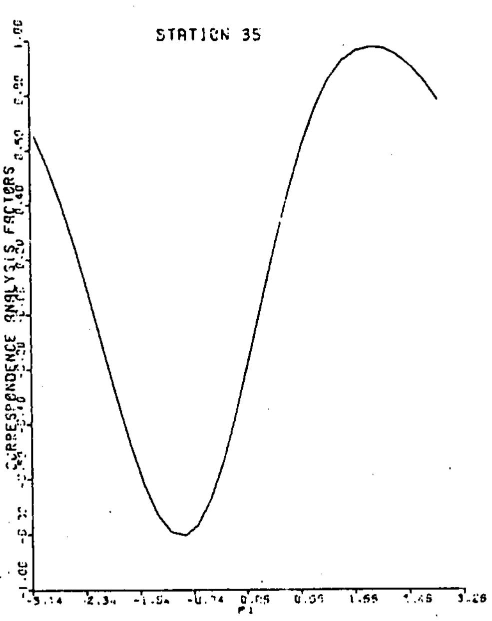
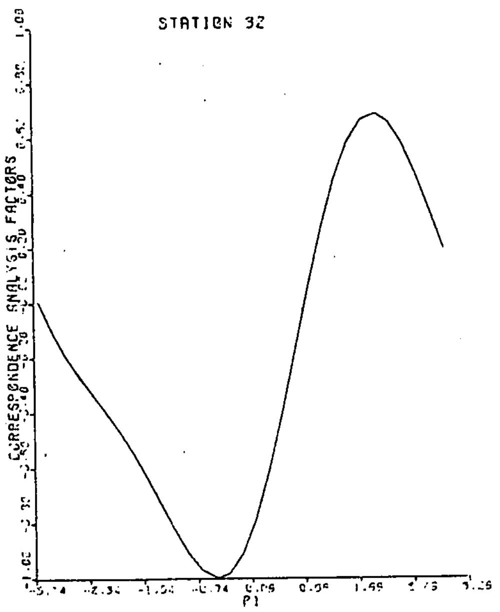

STATION 35
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 32
CORRESPONDENCE ANALYSIS FACTORS
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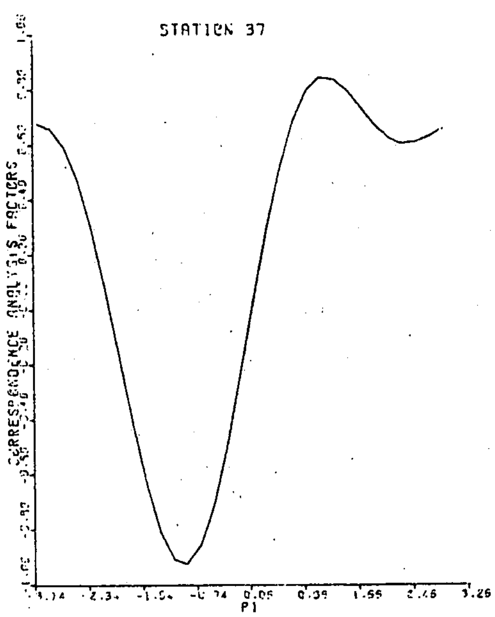
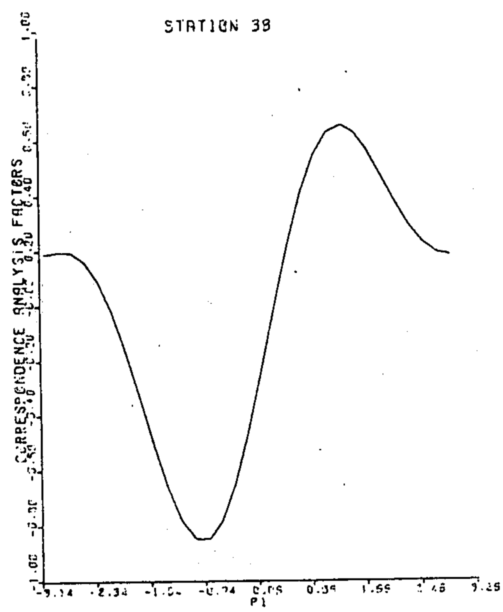

STATION 37
CORRESPONDENCE ANALYSIS FACTORS
PI


STATION 38
CORRESPONDENCE ANALYSIS FACTORS
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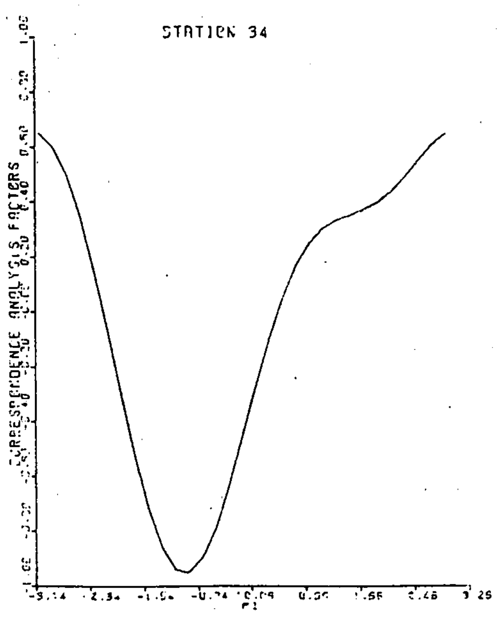
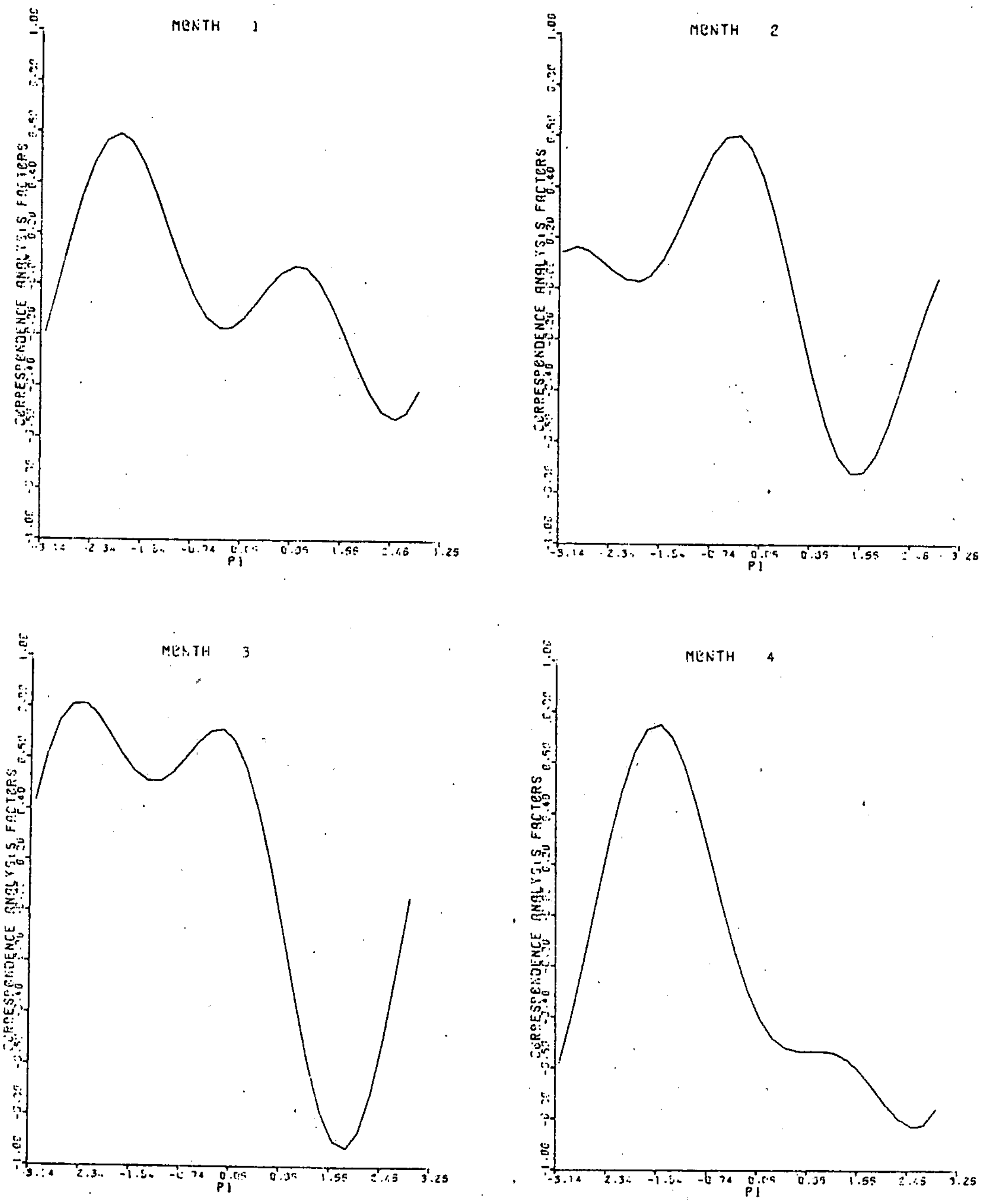
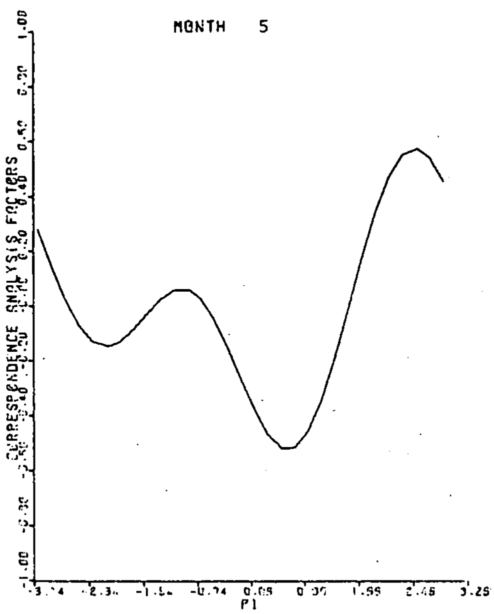
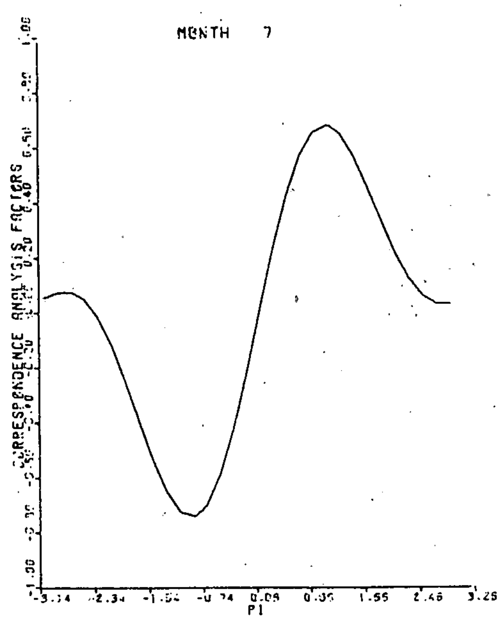

STATION 34
CORRESPONDENCE ANALYSIS FACTORS
PI


MONTH 1
MONTH 2
MONTH 3
MONTH 4
CORRESPONDENCE ANALYSIS FACTORS
PI


MONTH 5
CORRESPONDENCE ANALYSIS FACTORS
PI


MONTH 6
CORRESPONDENCE ANALYSIS FACTORS
PI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CORRESPONDENCE ANALYSIS FACTORS
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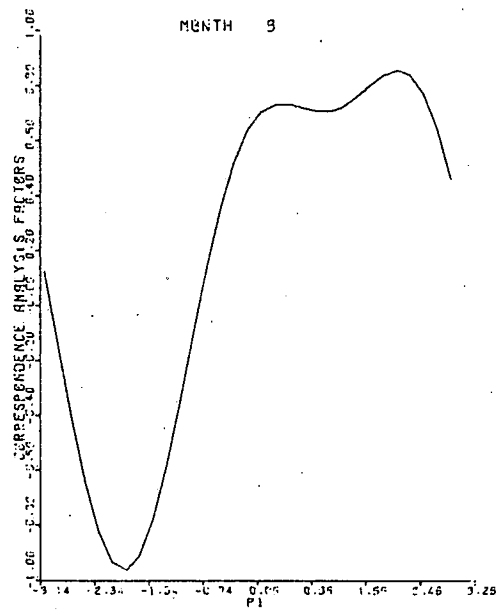
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CORRESPONDENCE ANALYSIS FACTORS
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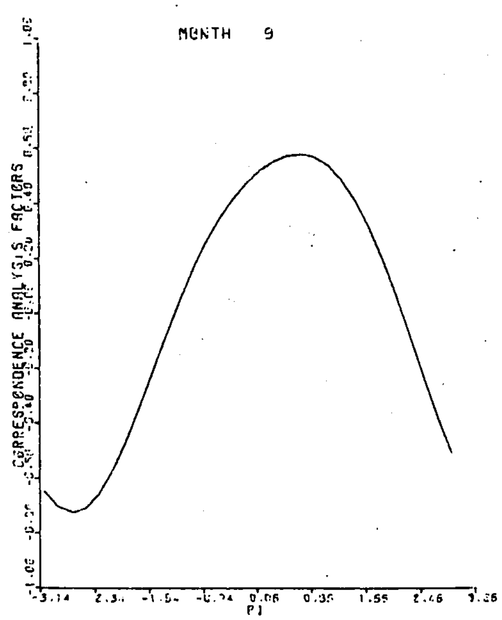

MONTH 9
CORRESPONDENCE ANALYSIS FACTORS
PI

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
1	2	1.6170	2	1	1.6170	3	1	2.9964	4	1	2.3077	5	1	2.4627	6	1	3.9334	7	1	3.1400	8	1	4.4724
1	3	2.9964	2	3	1.5520	3	2	1.5520	4	2	1.2829	5	2	1.1600	6	2	3.0905	7	2	1.7004	8	2	3.0701
1	4	2.3077	2	4	1.2829	3	4	1.0911	4	3	1.0911	5	3	1.9430	6	3	2.0051	7	3	.3138	8	3	1.7878
1	5	2.4627	2	5	1.1600	3	5	1.9430	4	5	1.8788	5	4	1.8788	6	4	2.2565	7	4	1.1738	8	4	2.3093
1	6	3.9334	2	6	3.0905	3	6	2.0051	4	6	2.2565	5	6	3.9040	6	5	3.9040	7	5	1.9229	8	5	3.0824
1	7	3.1400	2	7	1.7004	3	7	.3138	4	7	1.1738	5	7	1.9229	6	7	2.1082	7	6	2.1082	8	6	2.7742
1	8	4.4724	2	8	3.0701	3	8	1.7878	4	8	2.3093	5	8	3.0824	6	8	2.7742	7	8	1.5562	8	7	1.5562
1	9	5.2689	2	9	3.8284	3	9	2.4083	4	9	3.1419	5	9	4.0095	6	9	2.4696	7	9	2.3441	8	9	1.5602
1	10	3.0681	2	10	1.6099	3	10	1.5901	4	10	2.1901	5	10	1.0949	6	10	3.4943	7	10	1.5737	8	10	2.6081
1	11	2.6152	2	11	1.3802	3	11	2.1114	4	11	2.4420	5	11	1.4634	6	11	3.7773	7	11	2.1547	8	11	3.3040
1	12	3.5239	2	12	2.2249	3	12	1.8208	4	12	2.4782	5	12	2.3281	6	12	3.2234	7	12	1.8446	8	12	2.4004
1	13	4.3451	2	13	3.2064	3	13	2.6676	4	13	3.3101	5	13	3.2599	6	13	3.3981	7	13	2.6567	8	13	2.5223
1	14	2.3686	2	14	1.7054	3	14	1.8004	4	14	1.9518	5	14	2.5784	6	14	2.0460	7	14	2.0528	8	14	2.9746
1	15	2.3138	2	15	2.5661	3	15	3.0345	4	15	2.9240	5	15	3.7009	6	15	2.3178	7	15	3.3412	8	15	4.4260
1	16	4.5805	2	16	5.2764	3	16	6.6577	4	16	6.1301	5	16	4.8894	6	16	8.2039	7	16	6.7348	8	16	7.8457
1	17	3.5321	2	17	3.7337	3	17	5.1890	4	17	4.8516	5	17	3.4905	6	17	6.7911	7	17	5.3008	8	17	6.5729
1	18	2.8059	2	18	2.9923	3	18	4.4742	4	18	4.0955	5	18	2.8054	6	18	6.0614	7	18	4.5883	8	18	5.8877
1	19	3.4928	2	19	4.1076	3	19	5.3854	4	19	4.7981	5	19	3.7464	6	19	6.9424	7	19	5.4498	8	19	6.4500
1	20	2.4927	2	20	3.0336	3	20	4.3395	4	20	3.6681	5	20	2.8544	6	20	5.7799	7	20	4.4342	8	20	5.4528
1	21	3.3411	2	21	2.9608	3	21	4.0710	4	21	3.7528	5	21	2.1468	6	21	5.9694	7	21	4.0326	8	21	4.9538
1	22	3.4940	2	22	3.4515	3	22	4.5894	4	22	4.3848	5	22	2.6477	6	22	6.5070	7	22	4.5624	8	22	5.5261
1	23	2.7422	2	23	1.7616	3	23	2.7674	4	23	2.5828	5	23	.8320	6	23	4.7359	7	23	2.7196	8	23	3.7283
1	24	2.9113	2	24	1.7279	3	24	2.5209	4	24	2.4416	5	24	.6220	6	24	4.5165	7	24	2.4596	8	24	3.3732
1	25	2.5545	2	25	2.0506	3	25	3.0967	4	25	2.8930	5	25	1.1742	6	25	5.0508	7	25	3.0857	8	25	4.1964
1	26	2.0700	2	26	2.1967	3	26	3.5435	4	26	3.1837	5	26	1.7520	6	26	5.2411	7	26	3.6202	8	26	4.8301
1	27	2.3713	2	27	2.1677	3	27	3.4559	4	27	3.2481	5	27	1.6401	6	27	5.2479	7	27	3.5109	8	27	4.7044
1	28	2.2331	2	28	2.1005	3	28	3.4120	4	28	3.1110	5	28	1.6279	6	28	5.0964	7	28	3.4683	8	28	4.6681
1	29	2.8129	2	29	2.8838	3	29	4.0470	4	29	3.8610	5	29	2.1222	6	29	5.9724	7	29	4.0451	8	29	5.1531
1	30	2.8844	2	30	1.5302	3	30	2.0195	4	30	2.0666	5	30	.6304	6	30	4.0022	7	30	1.9234	8	30	2.8791
1	31	3.6548	2	31	3.8708	3	31	5.0546	4	31	4.8576	5	31	3.1129	6	31	6.9613	7	31	5.0276	8	31	6.0367
1	32	5.4297	2	32	5.6618	3	32	6.8576	4	32	6.7215	5	32	4.9615	6	32	8.7523	7	32	6.8473	8	32	7.8098
1	33	2.6929	2	33	2.7368	3	33	4.0558	4	33	4.0097	5	33	2.9980	6	33	5.4382	7	33	4.1818	8	33	5.4091
1	34	4.7439	2	34	4.6384	3	34	5.8746	4	34	5.8923	5	34	4.2084	6	34	7.5704	7	34	5.9167	8	34	7.0391
1	35	5.1825	2	35	5.3684	3	35	6.6704	4	35	6.5536	5	35	4.8705	6	35	8.4369	7	35	6.7010	8	35	7.7795
1	36	3.3234	2	36	3.1701	3	36	4.4642	4	36	4.4530	5	36	2.9036	6	36	6.0439	7	36	4.5488	8	36	5.7646
1	37	5.7642	2	37	5.8676	3	37	7.1548	4	37	7.1339	5	37	5.4828	6	37	8.7654	7	37	7.1968	8	37	8.3445
1	38	6.7001	2	38	6.7070	3	38	8.0012	4	38	7.9702	5	38	6.3223	6	38	9.6663	7	38	8.0433	8	38	9.1107
1	39	4.4897	2	39	3.5627	3	39	4.3535	4	39	4.7060	5	39	2.9760	6	39	6.1195	7	39	4.3541	8	39	5.3261
1	40	2.9524	2	40	2.7056	3	40	3.1751	4	40	3.4436	5	40	3.5422	6	40	3.7549	7	40	3.3980	8	40	4.3663
1	41	5.0190	2	41	5.4172	3	41	6.3632	4	41	5.7579	5	41	5.2100	6	41	7.5894	7	41	6.3889	8	41	6.8736
1	42	6.4944	2	42	7.4963	3	42	8.6288	4	42	7.9024	5	42	7.2694	6	42	9.8213	7	42	8.6884	8	42	9.3289
1	43	7.3097	2	43	8.2801	3	43	9.2638	4	43	8.6488	5	43	8.0506	6	43	10.4640	7	43	9.3096	8	43	9.7741
1	44	9.1291	2	44	10.4638	3	44	11.5992	4	44	10.7680	5	44	10.4082	6	44	12.4846	7	44	11.6940	8	44	12.3919
1	45	4.2257	2	45	4.9788	3	45	6.3761	4	45	5.7713	5	45	4.6319	6	45	7.8618	7	45	6.4532	8	45	7.5178
1	46	4.3484	2	46	5.0452	3	46	6.2777	4	46	5.6777	5	46	4.6494	6	46	7.7904	7	46	6.3412	8	46	7.2572
1	47	3.0918	2	47	3.6137	3	47	4.9073	4	47	4.3771	5	47	3.1740	6	47	6.5328	7	47	4.9780	8	47	6.0204
1	48	3.3238	2	48	3.8342	3	48	5.1764	4	48	4.6526	5	48	3.4028	6	48	6.8075	7	48	5.2465	8	48	6.3222
1	49	2.5761	2	49	3.2204	3	49	4.6391	4	49	4.1364	5	49	2.9036	6	49	6.1457	7	49	4.7342	8	49	5.9394
1	50	3.5916	2	50	4.9315	3	50	5.8435	4	50	4.8874	5	50	5.1709	6	50	6.3748	7	50	5.9753	8	50	6.4931
1	51	3.4968	2	51	4.4463	3	51	5.6545	4	51	4.9463	5	51	4.2699	6	51	6.9703	7	51	5.7189	8	51	6.7120
1	52	3.2644	2	52	3.7882	3	52	4.5899	4	52	4.4197	5	52	4.4071	6	52	5.2786	7	52	4.7947	8	52	5.6797
1	53	6.7397	2	53	7.9976	3	53	9.1330	4	53	8.3019	5	53	7.9691	6	53	10.0347	7	53	9.2278	8	53	9.8062
1	54	9.4914	2	54	10.5749	3	54	11.6058	4	54	10.8694	5	54	10.6282	6	54	12.2567	7	54	11.7151	8	54	12.0833
1	55	11.5990	2	55	13.1358	3	55	14.1290	4	55	13.1175	5	55	13.2524	6	55	14.5212	7	55	14.2463	8	55	14.7519

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
9	1	5.2689	10	1	3.0681	11	1	2.6152	12	1	3.5239	13	1	4.3451	14	1	2.3686	15	1	2.3138	16	1	4.5805
9	2	3.8284	10	2	1.6099	11	2	1.3802	12	2	2.2249	13	2	3.2064	14	2	1.7054	15	2	2.5661	16	2	5.2764
9	3	2.4083	10	3	1.5901	11	3	2.1114	12	3	1.8208	13	3	2.6676	14	3	1.8004	15	3	3.0345	16	3	6.6577
9	4	3.1419	10	4	2.1901	11	4	2.4420	12	4	2.4782	13	4	3.3101	14	4	1.9518	15	4	2.9240	16	4	6.1301
9	5	4.0095	10	5	1.0949	11	5	1.4634	12	5	2.3281	13	5	3.2599	14	5	2.5784	15	5	3.7009	16	5	4.8894
9	6	2.4696	10	6	3.4943	11	6	3.7773	12	6	3.2234	13	6	3.3981	14	6	2.0460	15	6	2.3178	16	6	8.2039
9	7	2.3441	10	7	1.5737	11	7	2.1547	12	7	1.8446	13	7	2.6567	14	7	2.0528	15	7	3.3412	16	7	6.7348
9	8	1.5602	10	8	2.6081	11	8	3.3040	12	8	2.4004	13	8	2.5223	14	8	2.9746	15	8	4.4260	16	8	7.8457
9	10	3.6330	10	9	3.6330	11	9	4.0113	12	9	2.9807	13	9	2.6225	14	9	3.0709	15	9	4.0739	16	9	8.7865
9	11	4.0113	10	11	1.0112	11	10	1.0112	12	10	1.3739	13	10	2.2688	14	10	2.2959	15	10	3.7720	16	10	5.6281
9	12	2.9807	10	12	1.3739	11	12	1.3065	12	11	1.3065	13	11	2.2282	14	11	2.1191	15	11	3.1615	16	11	5.1932
9	13	2.6225	10	13	2.2688	11	13	2.2282	12	13	1.0949	13	12	1.0949	14	12	2.0562	15	12	3.3966	16	12	6.4527
9	14	3.0709	10	14	2.2959	11	14	2.1191	12	14	2.0562	13	14	2.3694	14	13	2.3694	15	13	3.9172	16	13	6.9570
9	15	4.0739	10	15	3.7720	11	15	3.1615	12	15	3.3966	13	15	3.9172	14	15	1.6939	15	14	1.6939	16	14	6.2329
9	16	8.7865	10	16	5.6281	11	16	5.1932	12	16	6.4527	13	16	6.9570	14	16	6.2329	15	16	6.8434	16	15	6.8434
9	17	7.3758	10	17	4.2055	11	17	3.6330	12	17	4.9373	13	17	5.6626	14	17	5.0516	15	17	5.6493	16	17	2.2666
9	18	6.6906	10	18	3.5635	11	18	3.0860	12	18	4.3925	13	18	5.1558	14	18	4.3650	15	18	4.9558	16	18	2.6341
9	19	7.4079	10	19	4.4942	11	19	4.1893	12	19	5.3496	13	19	5.8792	14	19	5.0043	15	19	5.7177	16	19	1.6196
9	20	6.4102	10	20	3.6838	11	20	3.5485	12	20	4.6212	13	20	5.1604	14	20	3.8388	15	20	4.5957	16	20	2.6630
9	21	6.0749	10	21	2.6128	11	21	2.4619	12	21	3.4502	13	21	3.9311	14	21	4.2727	15	21	5.0020	16	21	3.2113
9	22	6.6365	10	22	3.0836	11	22	2.9377	12	22	3.9609	13	22	4.4621	14	22	4.8189	15	22	5.5371	16	22	2.8366
9	23	4.8186	10	23	1.5549	11	23	1.8539	12	23	2.7953	13	23	3.6576	14	23	3.2470	15	23	4.2514	16	23	4.2046
9	24	4.6220	10	24	1.3099	11	24	1.8180	12	24	2.6496	13	24	3.5455	14	24	3.1496	15	24	4.2870	16	24	4.6379
9	25	5.1480	10	25	2.0947	11	25	2.1762	12	25	3.2344	13	25	4.1036	14	25	3.5465	15	25	4.3520	16	25	3.7151
9	26	5.6892	10	26	2.5310	11	26	2.1464	12	26	3.4098	13	26	3.9839	14	26	3.4415	15	26	4.1602	16	26	3.1617
9	27	5.5555	10	27	2.3005	11	27	1.8491	12	27	3.1165	13	27	3.7573	14	27	3.5243	15	27	4.2119	16	27	3.3739
9	28	5.5320	10	28	2.2883	11	28	1.8920	12	28	3.1190	13	28	3.6783	14	28	3.2392	15	28	3.9428	16	28	3.3398
9	29	6.0940	10	29	2.8031	11	29	2.5454	12	29	3.7103	13	29	4.5680	14	29	4.3755	15	29	5.0322	16	29	2.9423
9	30	4.1639	10	30	1.4212	11	30	1.9185	12	30	2.6976	13	30	3.6455	14	30	3.0077	15	30	4.0629	16	30	5.1966
9	31	7.1017	10	31	3.5639	11	31	3.3526	12	31	4.3759	13	31	4.9128	14	31	5.2669	15	31	5.9458	16	31	2.4470
9	32	8.9206	10	32	5.3356	11	32	5.0974	12	32	6.1532	13	32	6.6607	14	32	6.9751	15	32	7.6076	16	32	3.0287
9	33	6.1032	10	33	2.9859	11	33	2.1720	12	33	3.3925	13	33	3.9711	14	33	3.3929	15	33	3.7345	16	33	3.5268
9	34	7.8559	10	34	4.4311	11	34	3.8507	12	34	4.9484	13	34	5.4012	14	34	5.6785	15	34	6.1397	16	34	3.2061
9	35	8.7458	10	35	5.2158	11	35	4.7701	12	35	5.8855	13	35	6.3112	14	35	6.5499	15	35	7.0907	16	35	2.3968
9	36	6.4830	10	36	3.1889	11	36	2.4927	12	36	3.6911	13	36	4.2351	14	36	4.1179	15	36	4.5883	16	36	3.3780
9	37	9.1413	10	37	5.7364	11	37	5.1361	12	37	6.2554	13	37	6.7118	14	37	6.7995	15	37	7.2025	16	37	3.1474
9	38	10.0298	10	38	6.5177	11	38	6.0283	12	38	7.1438	13	38	7.5722	14	38	7.7155	15	38	8.1446	16	38	3.9683
9	39	6.2814	10	39	2.7807	11	39	2.3674	12	39	3.4297	13	39	3.9544	14	39	4.3663	15	39	5.1783	16	39	4.3487
9	40	4.3090	10	40	3.1526	11	40	2.2492	12	40	2.3778	13	40	2.8138	14	40	1.8007	15	40	2.4145	16	40	5.5120
9	41	7.8727	10	41	5.4782	11	41	5.1668	12	41	5.8540	13	41	5.9310	14	41	5.6470	15	41	6.1769	16	41	4.2008
9	42	10.2344	10	42	7.8043	11	42	7.4982	12	42	8.4381	13	42	8.6671	14	42	7.8790	15	42	8.3047	16	42	3.6215
9	43	10.7011	10	43	8.3848	11	43	8.0735	12	43	8.7982	13	43	8.8561	14	43	8.5217	15	43	8.9270	16	43	5.4566
9	44	13.3120	10	44	10.9142	11	44	10.6155	12	44	11.5156	13	44	11.7861	14	44	10.7491	15	44	10.7141	16	44	6.8066
9	45	8.4286	10	45	5.4030	11	45	5.0032	12	45	6.2343	13	45	6.7673	14	45	5.8723	15	45	6.4312	16	45	.6964
9	46	8.2560	10	46	5.2700	11	46	4.9278	12	46	6.0563	13	46	6.5083	14	46	5.8179	15	46	6.4625	16	46	1.5254
9	47	6.9516	10	47	3.9264	11	47	3.5980	12	47	4.7978	13	47	5.3509	14	47	4.6409	15	47	5.3187	16	47	1.8612
9	48	7.2555	10	48	4.1863	11	48	3.7942	12	48	5.0441	13	48	5.5972	14	48	4.9229	15	48	5.5944	16	48	1.5368
9	49	6.8186	10	49	3.7031	11	49	3.2912	12	49	4.5586	13	49	5.1132	14	49	4.1938	15	49	4.8569	16	49	2.0607
9	50	7.3562	10	50	5.7879	11	50	5.7089	12	50	6.3672	13	50	6.5855	14	50	5.0514	15	50	4.7216	16	50	4.7044
9	51	7.6491	10	51	4.7846	11	51	4.4848	12	51	5.5299	13	51	5.9630	14	51	5.0012	15	51	5.4387	16	51	2.2381
9	52	6.3211	10	52	4.3366	11	52	3.8022	12	52	4.2261	13	52	4.4266	14	52	3.3374	15	52	3.5652	16	52	4.4437
9	53	10.6639	10	53	8.5162	11	53	8.2160	12	53	9.0749	13	53	9.3273	14	53	8.2224	15	53	8.3011	16	53	4.4730
9	54	12.5377	10	54	11.0521	11	54	10.7339	12	54	11.4449	13	54	11.4341	14	54	10.5054	15	54	10.6515	16	54	8.6857
9	55	15.4557	10	55	13.8908	11	55	13.7312	12	55	14.5023	13	55	14.6527	14	55	13.2592	15	55	12.9003	16	55	10.1822

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
17	1	3.5321	18	1	2.8059	19	1	3.4928	20	1	2.4927	21	1	3.3411	22	1	3.4940	23	1	2.7422	24	1	2.9113
17	2	3.7337	18	2	2.9923	19	2	4.1076	20	2	3.0336	21	2	2.9608	22	2	3.4515	23	2	1.7616	24	2	1.7279
17	3	5.1890	18	3	4.4742	19	3	5.3854	20	3	4.3395	21	3	4.0710	22	3	4.5894	23	3	2.7674	24	3	2.5209
17	4	4.8516	18	4	4.0955	19	4	4.7981	20	4	3.6681	21	4	3.7528	22	4	4.3848	23	4	2.5828	24	4	2.4416
17	5	3.4905	18	5	2.8054	19	5	3.7464	20	5	2.8544	21	5	2.1468	22	5	2.6477	23	5	.8320	24	5	.6220
17	6	6.7911	18	6	6.0614	19	6	6.9424	20	6	5.7799	21	6	5.9694	22	6	6.5070	23	6	4.7359	24	6	4.5165
17	7	5.3008	18	7	4.5883	19	7	5.4498	20	7	4.4342	21	7	4.0326	22	7	4.5624	23	7	2.7196	24	7	2.4596
17	8	6.5729	18	8	5.8877	19	8	6.4500	20	8	5.4528	21	8	4.9538	22	8	5.5261	23	8	3.7283	24	8	3.3732
17	9	7.3758	18	9	6.6906	19	9	7.4079	20	9	6.4102	21	9	6.0749	22	9	6.6365	23	9	4.8186	24	9	4.6220
17	10	4.2055	18	10	3.5635	19	10	4.4942	20	10	3.6838	21	10	2.6128	22	10	3.0836	23	10	1.5549	24	10	1.3099
17	11	3.6330	18	11	3.0860	19	11	4.1893	20	11	3.5485	21	11	2.4619	22	11	2.9377	23	11	1.8539	24	11	1.8180
17	12	4.9373	18	12	4.3925	19	12	5.3496	20	12	4.6212	21	12	3.4502	22	12	3.9609	23	12	2.7953	24	12	2.6498
17	13	5.6626	18	13	5.1558	19	13	5.8792	20	13	5.1604	21	13	3.9311	22	13	4.4621	23	13	3.6576	24	13	3.5455
17	14	5.0516	18	14	4.3650	19	14	5.0043	20	14	3.8388	21	14	4.2727	22	14	4.8189	23	14	3.2470	24	14	3.1496
17	15	5.6493	18	15	4.9558	19	15	5.7177	20	15	4.5957	21	15	5.0020	22	15	5.5371	23	15	4.2514	24	15	4.2870
17	16	2.2666	18	16	2.6341	19	16	1.6196	20	16	2.6630	21	16	3.2113	22	16	2.8366	23	16	4.2046	24	16	4.6379
17	18	.7580	18	17	.7580	19	17	2.7425	20	17	2.9041	21	17	2.8064	22	17	2.3449	23	17	2.9127	24	17	3.2650
17	19	2.7425	18	19	2.3982	19	18	2.3982	20	18	2.3586	21	18	2.4313	22	18	2.0750	23	18	2.2659	24	18	2.6133
17	20	2.9041	18	20	2.3586	19	20	1.1661	20	19	1.1661	21	19	2.3162	22	19	2.4590	23	19	3.1256	24	19	3.5845
17	21	2.8064	18	21	2.4313	19	21	2.3162	20	21	2.1134	21	20	2.1134	22	20	2.5056	23	20	2.4051	24	20	2.8530
17	22	2.3449	18	22	2.0750	19	22	2.4590	20	22	2.5056	21	22	.7359	22	21	.7359	23	21	1.3667	24	21	1.7375
17	23	2.9127	18	23	2.2659	19	23	3.1256	20	23	2.4051	21	23	1.3667	22	23	1.8641	23	22	1.8641	24	22	2.1682
17	24	3.2650	18	24	2.6133	19	24	3.5845	20	24	2.8530	21	24	1.7375	22	24	2.1682	23	24	.4596	24	23	.4596
17	25	2.4486	18	25	1.7897	19	25	2.7750	20	25	2.0900	21	25	1.3741	22	25	1.6039	23	25	.5570	24	25	.9754
17	26	1.9945	18	26	1.4501	19	26	2.1351	20	26	1.6817	21	26	1.3518	22	26	1.4324	23	26	1.2312	24	26	1.5559
17	27	2.0597	18	27	1.5558	19	27	2.5027	20	27	2.0643	21	27	1.1974	22	27	1.3252	23	27	1.1036	24	27	1.3902
17	28	2.4579	18	28	1.9872	19	28	2.3611	20	28	1.8510	21	28	1.3149	22	28	1.6552	23	28	1.4647	24	28	1.7241
17	29	1.6701	18	29	1.2540	19	29	2.3578	20	29	2.1421	21	29	1.3705	22	29	1.0460	23	29	1.4248	24	29	1.8066
17	30	3.7534	18	30	3.0682	19	30	4.1744	20	30	3.3354	21	30	2.4369	22	30	2.8092	23	30	1.0949	24	30	.7032
17	31	1.9807	18	31	1.8048	19	31	2.4529	20	31	2.6835	21	31	1.1844	22	31	.5411	23	31	2.3619	24	31	2.6635
17	32	3.1795	18	32	3.3117	19	32	3.7470	20	32	4.4123	21	32	3.0177	22	32	2.3965	23	32	4.2407	24	32	4.4624
17	33	2.9267	18	33	2.7131	19	33	3.4000	20	33	3.1071	21	33	2.9376	22	33	2.9774	23	33	3.0952	24	33	3.2552
17	34	2.5219	18	34	2.6206	19	34	3.5609	20	34	3.7101	21	34	2.4983	22	34	2.1423	23	34	3.6599	24	34	3.8834
17	35	2.6274	18	35	2.8805	19	35	3.3058	20	35	4.0853	21	35	3.0011	22	35	2.4287	23	35	4.2287	24	35	4.4751
17	36	2.2518	18	36	2.0789	19	36	3.3329	20	36	3.0789	21	36	2.3521	22	36	2.2335	23	36	2.6227	24	36	2.8092
17	37	3.5387	18	37	3.7294	19	37	3.8838	20	37	4.7269	21	37	3.7527	22	37	3.2230	23	37	4.9441	24	37	5.1675
17	38	4.7129	18	38	4.8453	19	38	4.8631	20	38	5.6676	21	38	4.5409	22	38	4.0130	23	38	5.7667	24	38	5.9519
17	39	3.5501	18	39	3.3164	19	39	3.8444	20	39	3.5301	21	39	1.7092	22	39	1.8179	23	39	2.5193	24	39	2.5660
17	40	4.4355	18	40	4.0531	19	40	4.8979	20	40	4.3148	21	40	4.1626	22	40	4.3840	23	40	3.9514	24	40	3.9842
17	41	5.2216	18	41	5.0165	19	41	3.8945	20	41	4.1004	21	41	4.0158	22	41	4.2228	23	41	4.9202	24	41	5.2430
17	42	5.5441	18	42	5.6978	19	42	3.7080	20	42	4.5571	21	42	5.7917	22	42	5.8467	23	42	6.7657	24	42	7.2136
17	43	6.8709	18	43	6.9056	19	43	5.5046	20	43	6.0037	21	43	6.5087	22	43	6.5755	23	43	7.6092	24	43	8.0267
17	44	8.7839	18	44	8.9560	19	44	7.0494	20	44	7.6925	21	44	9.1285	22	44	9.2318	23	44	9.9618	24	44	10.3976
17	45	2.3909	18	45	2.6256	19	45	1.1716	20	45	2.1988	21	45	3.0109	22	45	2.8228	23	45	3.9656	24	45	4.4116
17	46	3.1574	18	46	3.0363	19	46	.9676	20	46	2.0376	21	46	3.0061	22	46	3.0437	23	46	4.0197	24	46	4.4731
17	47	2.2739	18	47	1.9063	19	47	.6797	20	47	1.0002	21	47	1.6544	22	47	1.8677	23	47	2.5258	24	47	2.9785
17	48	2.0071	18	48	1.7197	19	48	.8139	20	48	1.3283	21	48	1.8438	22	48	1.7806	23	48	2.7256	24	48	3.1716
17	49	1.7342	18	49	1.2762	19	49	1.4232	20	49	1.2520	21	49	1.8111	22	49	1.8529	23	49	2.2457	24	49	2.6603
17	50	5.6218	18	50	5.0811	19	50	3.3318	20	50	2.9260	21	50	5.0116	22	50	5.4223	23	50	5.0387	24	50	5.4374
17	51	3.2098	18	51	2.9755	19	51	1.7805	20	51	1.9386	21	51	3.1157	22	51	3.3227	23	51	3.8615	24	51	4.2974
17	52	4.4687	18	52	4.1918	19	52	4.1797	20	52	3.9280	21	52	4.3099	22	52	4.4641	23	52	4.5542	24	52	4.7497
17	53	6.3913	18	53	6.5201	19	53	4.6340	20	53	5.2364	21	53	6.7809	22	53	6.9077	23	53	7.5593	24	53	8.0013
17	54	10.2634	18	54	9.9361	19	54	7.6550	20	54	8.0333	21	54	9.4904	22	54	9.7486	23	54	10.2621	24	54	10.7000
17	55	12.0603	18	55	12.0298	19	55	10.1120	20	55	10.4948	21	55	12.3697	22	55	12.5405	23	55	12.8888	24	55	13.3246

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
25	1	2.5545	26	1	2.0700	27	1	2.3713	28	1	2.2331	29	1	2.8129	30	1	2.8844	31	1	3.6548	32	1	5.4297
25	2	2.0506	26	2	2.1967	27	2	2.1677	28	2	2.1005	29	2	2.8838	30	2	1.5302	31	2	3.8708	32	2	5.6618
25	3	3.0967	26	3	3.5435	27	3	3.4559	28	3	3.4120	29	3	4.0470	30	3	2.0195	31	3	5.0546	32	3	6.8576
25	4	2.8930	26	4	3.1837	27	4	3.2481	28	4	3.1110	29	4	3.8610	30	4	2.0666	31	4	4.8576	32	4	6.7215
25	5	1.1742	26	5	1.7520	27	5	1.6401	28	5	1.6279	29	5	2.1222	30	5	.6304	31	5	3.1129	32	5	4.9615
25	6	5.0508	26	6	5.2411	27	6	5.2479	28	6	5.0964	29	6	5.9724	30	6	4.0022	31	6	6.9613	32	6	8.7523
25	7	3.0857	26	7	3.6202	27	7	3.5109	28	7	3.4683	29	7	4.0451	30	7	1.9234	31	7	5.0276	32	7	6.8473
25	8	4.1964	26	8	4.8301	27	8	4.7044	28	8	4.6681	29	8	5.1531	30	8	2.8791	31	8	6.0367	32	8	7.8098
25	9	5.1480	26	9	5.6892	27	9	5.5555	28	9	5.5320	29	9	6.0940	30	9	4.1639	31	9	7.1017	32	9	8.9206
25	10	2.0947	26	10	2.5310	27	10	2.3005	28	10	2.2883	29	10	2.8031	30	10	1.4212	31	10	3.5639	32	10	5.3356
25	11	2.1762	26	11	2.1464	27	11	1.8491	28	11	1.8920	29	11	2.5454	30	11	1.9185	31	11	3.3526	32	11	5.0974
25	12	3.2344	26	12	3.4098	27	12	3.1165	28	12	3.1190	29	12	3.7103	30	12	2.6976	31	12	4.3759	32	12	6.1532
25	13	4.1036	26	13	3.9839	27	13	3.7573	28	13	3.6783	29	13	4.5680	30	13	3.6455	31	13	4.9128	32	13	6.6607
25	14	3.5465	26	14	3.4415	27	14	3.5243	28	14	3.2392	29	14	4.3755	30	14	3.0077	31	14	5.2669	32	14	6.9751
25	15	4.3520	26	15	4.1602	27	15	4.2119	28	15	3.9426	29	15	5.0322	30	15	4.0629	31	15	5.9458	32	15	7.6076
25	16	3.7151	26	16	3.1617	27	16	3.3739	28	16	3.3398	29	16	2.9423	30	16	5.1966	31	16	2.4470	32	16	3.0287
25	17	2.4486	26	17	1.9945	27	17	2.0597	28	17	2.4579	29	17	1.6701	30	17	3.7534	31	17	1.9807	32	17	3.1795
25	18	1.7897	26	18	1.4501	27	18	1.5558	28	18	1.9872	29	18	1.2540	30	18	3.0682	31	18	1.8048	32	18	3.3117
25	19	2.7750	26	19	2.1351	27	19	2.5027	28	19	2.3611	29	19	2.3578	30	19	4.1744	31	19	2.4529	32	19	3.7470
25	20	2.0900	26	20	1.6817	27	20	2.0643	28	20	1.8510	29	20	2.1421	30	20	3.3354	31	20	2.6835	32	20	4.4123
25	21	1.3741	26	21	1.3518	27	21	1.1974	28	21	1.3149	29	21	1.3705	30	21	2.4369	31	21	1.1844	32	21	3.0177
25	22	1.6039	26	22	1.4324	27	22	1.3252	28	22	1.6552	29	22	1.0460	30	22	2.8092	31	22	.5411	32	22	2.3965
25	23	.5570	26	23	1.2312	27	23	1.1036	28	23	1.4647	29	23	1.4248	30	23	1.0949	31	23	2.3619	32	23	4.2407
25	24	.9754	26	24	1.5559	27	24	1.3902	28	24	1.7241	29	24	1.8066	30	24	.7032	31	24	2.6635	32	24	4.4624
25	26	1.0738	26	25	1.0738	27	25	1.0290	28	25	1.5311	29	25	1.0141	30	25	1.4835	31	25	2.0327	32	25	3.9858
25	27	1.0290	26	27	.4216	27	26	.4216	28	26	.6286	29	26	1.0672	30	26	2.0615	31	26	1.8285	32	26	3.6439
25	28	1.5311	26	28	.6286	27	28	.6207	28	27	.6207	29	27	1.0113	30	27	1.8917	31	27	1.7426	32	27	3.5303
25	29	1.0141	26	29	1.0672	27	29	1.0113	28	29	1.5866	29	28	1.5866	30	28	2.1782	31	28	2.1001	32	28	3.8007
25	30	1.4835	26	30	2.0615	27	30	1.8917	28	30	2.1782	29	30	2.3672	30	29	2.3672	31	29	1.0629	32	29	3.0619
25	31	2.0327	26	31	1.8285	27	31	1.7426	28	31	2.1001	29	31	1.0629	30	31	3.2791	31	30	3.2791	32	30	5.0450
25	32	3.9858	26	32	3.6439	27	32	3.5303	28	32	3.8007	29	32	3.0619	30	32	5.0450	31	32	1.9990	32	31	1.9990
25	33	3.1400	26	33	2.1460	27	33	2.1285	28	33	1.8025	29	33	2.9994	30	33	3.4749	31	33	3.0378	32	33	4.0715
25	34	3.4679	26	34	2.9408	27	34	2.7407	28	34	2.8792	29	34	2.6137	30	34	4.3969	31	34	1.9886	32	34	1.9727
25	35	3.9659	26	35	3.4428	27	35	3.3197	28	35	3.4803	29	35	3.0120	30	35	5.0274	31	35	1.9975	32	35	1.2132
25	36	2.5690	26	36	1.6836	27	36	1.5993	28	36	1.5308	29	36	2.2751	30	36	3.1501	31	36	2.2133	32	36	3.1693
25	37	4.7010	26	37	4.1147	27	37	3.9972	28	37	4.0575	29	37	3.8148	30	37	5.6935	31	37	3.0152	32	37	2.2620
25	38	5.5847	26	38	5.0154	27	38	4.8521	28	38	4.9378	29	38	4.7501	30	38	6.4789	31	38	3.8013	32	38	2.4979
25	39	2.7224	26	39	2.4792	27	39	2.1410	28	39	2.3327	29	39	2.6064	30	39	3.0517	31	39	2.1578	32	39	2.7594
25	40	4.0823	26	40	3.3338	27	40	3.3000	28	40	2.9951	29	40	4.2744	30	40	3.9665	31	40	4.6560	32	40	6.2926
25	41	4.9818	26	41	4.1265	27	41	4.1905	28	41	3.7016	29	41	4.8549	30	41	5.8006	31	41	4.3424	32	41	4.2061
25	42	6.4699	26	42	5.7342	27	42	6.0338	28	42	5.7035	29	42	6.0192	30	42	7.7882	31	42	5.6875	32	42	6.0752
25	43	7.3972	26	43	6.5988	27	43	6.8620	28	43	6.4437	29	43	7.0022	30	43	8.6112	31	43	6.4466	32	43	5.9301
25	44	9.7083	26	44	8.9076	27	44	9.2519	28	44	8.8904	29	44	9.3661	30	44	10.9616	31	44	9.0949	32	44	9.1079
25	45	3.5228	26	45	2.9452	27	45	3.2070	28	45	3.1154	29	45	2.9426	30	45	4.9961	31	45	2.6418	32	45	3.4756
25	46	3.6957	26	46	3.0377	27	46	3.2934	28	46	3.0673	29	46	3.2088	30	46	5.0870	31	46	2.9064	32	46	3.7182
25	47	2.1591	26	47	1.5195	27	47	1.8426	28	47	1.7395	29	47	1.7501	30	47	3.5774	31	47	1.8942	32	47	3.4680
25	48	2.2867	26	48	1.7063	27	48	1.9855	28	48	1.9284	29	48	1.7691	30	48	3.7560	31	48	1.7520	32	48	3.2582
25	49	1.8142	26	49	1.1765	27	49	1.4575	28	49	1.4404	29	49	1.5666	30	49	3.2139	31	49	1.8204	32	49	3.4286
25	50	4.7900	26	50	4.4124	27	50	4.8080	28	50	4.4134	29	50	5.0590	30	50	5.7016	31	50	5.5694	32	50	7.0327
25	51	3.6131	26	51	2.8027	27	51	3.1160	28	51	2.7165	29	51	3.3312	30	51	4.8277	31	51	3.2274	32	51	4.1847
25	52	4.5957	26	52	3.5443	27	52	3.6629	28	52	3.1540	29	52	4.5429	30	52	4.9896	31	52	4.5618	32	52	5.3176
25	53	7.2730	26	53	6.5234	27	53	6.8639	28	53	6.5092	29	53	6.9766	30	53	8.5137	31	53	6.7868	32	53	7.2378
25	54	10.0086	26	54	9.2546	27	54	9.5836	28	54	9.1843	29	54	9.7438	30	54	11.2282	31	54	9.8656	32	54	10.7544
25	55	12.5848	26	55	12.0087	27	55	12.4029	28	55	12.0906	29	55	12.4535	30	55	13.7592	31	55	12.4144	32	55	12.9111

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFEPENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
33	1	2.6929	34	1	4.7439	35	1	5.1825	36	1	3.3234	37	1	5.7642	38	1	6.7001	39	1	4.4897	40	1	2.9524
33	2	2.7368	34	2	4.6384	35	2	5.3684	36	2	3.1701	37	2	5.8676	38	2	6.7070	39	2	3.5627	40	2	2.7056
33	3	4.0558	34	3	5.8746	35	3	6.6704	36	3	4.4642	37	3	7.1548	38	3	8.0012	39	3	4.3535	40	3	3.1751
33	4	4.0097	34	4	5.8923	35	4	6.5536	36	4	4.4530	37	4	7.1339	38	4	7.9702	39	4	4.7060	40	4	3.4436
33	5	2.9980	34	5	4.2084	35	5	4.8705	36	5	2.9036	37	5	5.4828	38	5	6.3223	39	5	2.9760	40	5	3.5422
33	6	5.4382	34	6	7.5704	35	6	8.4369	36	6	6.0439	37	6	8.7654	38	6	9.6663	39	6	6.1195	40	6	3.7549
33	7	4.1818	34	7	5.9167	35	7	6.7010	36	7	4.5488	37	7	7.1968	38	7	8.0433	39	7	4.3541	40	7	3.3980
33	8	5.4091	34	8	7.0391	35	8	7.7795	36	8	5.7646	37	8	8.3445	38	8	9.1107	39	8	5.3261	40	8	4.3663
33	9	6.1032	34	9	7.8559	35	9	8.7458	36	9	6.4830	37	9	9.1413	38	9	10.0298	39	9	6.2814	40	9	4.3090
33	10	2.9859	34	10	4.4311	35	10	5.2158	36	10	3.1889	37	10	5.7364	38	10	6.5177	39	10	2.7807	40	10	3.1526
33	11	2.1720	34	11	3.8507	35	11	4.7701	36	11	2.4927	37	11	5.1361	38	11	6.0283	39	11	2.3674	40	11	2.2492
33	12	3.3925	34	12	4.9484	35	12	5.8855	36	12	3.6911	37	12	6.2554	38	12	7.1438	39	12	3.4297	40	12	2.3778
33	13	3.9711	34	13	5.4012	35	13	6.3112	36	13	4.2351	37	13	6.7118	38	13	7.5722	39	13	3.9544	40	13	2.8138
33	14	3.3929	34	14	5.6785	35	14	6.5499	36	14	4.1179	37	14	6.7995	38	14	7.7155	39	14	4.3663	40	14	1.8007
33	15	3.7345	34	15	6.1397	35	15	7.0907	36	15	4.5883	37	15	7.2025	38	15	8.1446	39	15	5.1783	40	15	2.4145
33	16	3.5268	34	16	3.2061	35	16	2.3968	36	16	3.3780	37	16	3.1474	38	16	3.9683	39	16	4.3487	40	16	5.5120
33	17	2.9267	34	17	2.5219	35	17	2.6274	36	17	2.2518	37	17	3.5387	38	17	4.7129	39	17	3.5501	40	17	4.4355
33	18	2.7131	34	18	2.6206	35	18	2.8805	36	18	2.0789	37	18	3.7294	38	18	4.8453	39	18	3.3164	40	18	4.0531
33	19	3.4000	34	19	3.5609	35	19	3.3058	36	19	3.3329	37	19	3.8838	38	19	4.8631	39	19	3.8444	40	19	4.8979
33	20	3.1071	34	20	3.7101	35	20	4.0853	36	20	3.0789	37	20	4.7269	38	20	5.6676	39	20	3.5301	40	20	4.3148
33	21	2.9376	34	21	2.4983	35	21	3.0011	36	21	2.3521	37	21	3.7527	38	21	4.5409	39	21	1.7092	40	21	4.1626
33	22	2.9774	34	22	2.1423	35	22	2.4287	36	22	2.2335	37	22	3.2230	38	22	4.0130	39	22	1.8179	40	22	4.3840
33	23	3.0952	34	23	3.6599	35	23	4.2287	36	23	2.6227	37	23	4.9441	38	23	5.7667	39	23	2.5193	40	23	3.9514
33	24	3.2552	34	24	3.8834	35	24	4.4751	36	24	2.8092	37	24	5.1675	38	24	5.9519	39	24	2.5660	40	24	3.9842
33	25	3.1400	34	25	3.4679	35	25	3.9659	36	25	2.5690	37	25	4.7010	38	25	5.5847	39	25	2.7224	40	25	4.0823
33	26	2.1460	34	26	2.9408	35	26	3.4428	36	26	1.6836	37	26	4.1147	38	26	5.0154	39	26	2.4792	40	26	3.3338
33	27	2.1285	34	27	2.7407	35	27	3.3197	36	27	1.5993	37	27	3.9972	38	27	4.8521	39	27	2.1410	40	27	3.3000
33	28	1.8025	34	28	2.8792	35	28	3.4803	36	28	1.5308	37	28	4.0575	38	28	4.9378	39	28	2.3327	40	28	2.9951
33	29	2.9994	34	29	2.6137	35	29	3.0120	36	29	2.2751	37	29	3.8148	38	29	4.7501	39	29	2.6064	40	29	4.2744
33	30	3.4749	34	30	4.3969	35	30	5.0274	36	30	3.1501	37	30	5.6935	38	30	6.4789	39	30	3.0517	40	30	3.9665
33	31	3.0378	34	31	1.9886	35	31	1.9975	36	31	2.2133	37	31	3.0152	38	31	3.8013	39	31	2.1578	40	31	4.6560
33	32	4.0715	34	32	1.9727	35	32	1.2132	36	32	3.1693	37	32	2.2620	38	32	2.4979	39	32	2.7594	40	32	6.2926
33	34	2.4945	34	33	2.4945	35	33	3.4566	36	33	1.0388	37	33	3.4740	38	33	4.5007	39	33	2.6462	40	33	2.3069
33	35	3.4566	34	35	1.1172	35	34	1.1172	36	34	1.5676	37	34	1.4159	38	34	2.6123	39	34	2.0511	40	34	4.6460
33	36	1.0388	34	36	1.5676	35	36	2.6106	36	35	2.6106	37	35	1.2928	38	35	2.1531	39	35	2.8427	40	35	5.6999
33	37	3.4740	34	37	1.4159	35	37	1.2928	36	37	2.7407	37	36	2.7407	38	36	3.8673	39	36	2.0791	40	36	3.1234
33	38	4.5007	34	38	2.6123	35	38	2.1531	36	38	3.8673	37	38	1.3968	38	37	1.3968	39	37	3.2273	40	37	5.7185
33	39	2.6462	34	39	2.0511	35	39	2.8427	36	39	2.0791	37	39	3.2273	38	39	3.9160	39	38	3.9160	40	38	6.8076
33	40	2.3069	34	40	4.6460	35	40	5.6999	36	40	3.1234	37	40	5.7185	38	40	6.8076	39	40	3.7244	40	39	3.7244
33	41	3.5129	34	41	3.7455	35	41	3.8020	36	41	3.7807	37	41	3.7689	38	41	3.9864	39	41	4.4039	40	41	5.2857
33	42	5.9733	34	42	6.4137	35	42	5.8695	36	42	6.2037	37	42	6.1109	38	42	5.9661	39	42	7.0117	40	42	7.5029
33	43	6.3629	34	43	6.0122	35	43	5.4246	36	43	6.5102	37	43	4.8273	38	43	4.5649	39	43	7.0644	40	43	8.1771
33	44	8.8682	34	44	9.0539	35	44	8.5148	36	44	9.1593	37	44	8.2440	38	44	8.1723	39	44	10.0519	40	44	10.5649
33	45	3.4657	34	45	3.4511	35	45	2.9047	36	45	3.4703	37	45	3.3273	38	45	4.2313	39	45	4.2956	40	45	5.2594
33	46	3.6561	34	46	3.9066	35	46	3.4168	36	46	3.7202	37	46	3.6682	38	46	4.3273	39	46	4.3832	40	46	5.3333
33	47	2.9363	34	47	2.9353	35	47	3.0999	36	47	2.6620	37	47	3.7590	38	47	4.7363	39	47	3.1831	40	47	4.4352
33	48	3.0042	34	48	2.7666	35	48	2.8627	36	48	2.6424	37	48	3.5483	38	48	4.5276	39	48	3.2821	40	48	4.5685
33	49	2.2215	34	49	2.7099	35	49	3.0277	36	49	2.0189	37	49	3.6783	38	49	4.6575	39	49	2.8834	40	49	3.7649
33	50	5.1332	34	50	6.5488	35	50	6.6082	36	50	5.7050	37	50	6.8839	38	50	7.7111	39	50	6.4373	40	50	6.0297
33	51	2.9872	34	51	3.8115	35	51	3.7215	36	51	3.2867	37	51	3.9439	38	51	4.7500	39	51	4.2111	40	51	4.5375
33	52	2.1942	34	52	4.1338	35	52	4.7551	36	52	3.0035	37	52	4.7091	38	52	5.4276	39	52	4.4346	40	52	3.1522
33	53	6.7027	34	53	7.2958	35	53	6.8200	36	53	7.0537	37	53	6.9492	38	53	6.9955	39	53	7.9659	40	53	8.1804
33	54	10.0373	34	54	11.0537	35	54	10.6579	36	54	10.5376	37	54	10.8811	38	54	10.7804	39	54	10.9809	40	54	10.7801
33	55	12.4905	34	55	13.0519	35	55	12.4767	36	55	12.8415	37	55	12.6032	38	55	12.5446	39	55	13.6788	40	55	13.7660

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
41	1	5.0190	42	1	6.4944	43	1	7.3097	44	1	9.1291	45	1	4.2257	46	1	4.3484	47	1	3.0918	48	1	3.3238
41	2	5.4172	42	2	7.4963	43	2	8.2801	44	2	10.4638	45	2	4.9788	46	2	5.0452	47	2	3.6137	48	2	3.8342
41	3	6.3632	42	3	8.6288	43	3	9.2638	44	3	11.5992	45	3	6.3761	46	3	6.2777	47	3	4.9073	48	3	5.1764
41	4	5.7579	42	4	7.9024	43	4	8.6488	44	4	10.7680	45	4	5.7713	46	4	5.6777	47	4	4.3771	48	4	4.6526
41	5	5.2100	42	5	7.2694	43	5	8.0506	44	5	10.4082	45	5	4.6319	46	5	4.6494	47	5	3.1740	48	5	3.4028
41	6	7.5894	42	6	9.8213	43	6	10.4640	44	6	12.4846	45	6	7.8618	46	6	7.7904	47	6	6.5328	48	6	6.8075
41	7	6.3889	42	7	8.6884	43	7	9.3096	44	7	11.6940	45	7	6.4532	46	7	6.3412	47	7	4.9780	48	7	5.2465
41	8	6.8738	42	8	9.3289	43	8	9.7741	44	8	12.3919	45	8	7.5178	46	8	7.2572	47	8	6.0204	48	8	6.3222
41	9	7.8727	42	9	10.2344	43	9	10.7011	44	9	13.3120	45	9	8.4286	46	9	8.2560	47	9	6.9516	48	9	7.2555
41	10	5.4782	42	10	7.8043	43	10	8.3848	44	10	10.9142	45	10	5.4030	46	10	5.2700	47	10	3.9264	48	10	4.1863
41	11	5.1668	42	11	7.4982	43	11	8.0735	44	11	10.6155	45	11	5.0032	46	11	4.9278	47	11	3.5980	48	11	3.7942
41	12	5.8540	42	12	8.4381	43	12	8.7982	44	12	11.5156	45	12	6.2343	46	12	6.0563	47	12	4.7978	48	12	5.0441
41	13	5.9310	42	13	8.6671	43	13	8.8561	44	13	11.7861	45	13	6.7673	46	13	6.5083	47	13	5.3509	48	13	5.5972
41	14	5.6470	42	14	7.8790	43	14	8.5217	44	14	10.7491	45	14	5.8723	46	14	5.8179	47	14	4.6409	48	14	4.9229
41	15	6.1769	42	15	8.3047	43	15	8.9270	44	15	10.7141	45	15	6.4312	46	15	6.4625	47	15	5.3187	48	15	5.5944
41	16	4.2008	42	16	3.6215	43	16	5.4566	44	16	6.8066	45	16	.6964	46	16	1.5254	47	16	1.8612	48	16	1.5368
41	17	5.2216	42	17	5.5441	43	17	6.8709	44	17	8.7839	45	17	2.3909	46	17	3.1574	47	17	2.2739	48	17	2.0071
41	18	5.0165	42	18	5.6978	43	18	6.9056	44	18	8.9560	45	18	2.6256	46	18	3.0363	47	18	1.9063	48	18	1.7197
41	19	3.8945	42	19	3.7080	43	19	5.5046	44	19	7.0494	45	19	1.1716	46	19	.9676	47	19	.6797	48	19	.8139
41	20	4.1004	42	20	4.5571	43	20	6.0037	44	20	7.6925	45	20	2.1988	46	20	2.0376	47	20	1.0002	48	20	1.3283
41	21	4.0158	42	21	5.7917	43	21	6.5087	44	21	9.1285	45	21	3.0109	46	21	3.0061	47	21	1.6544	48	21	1.8438
41	22	4.2228	42	22	5.8467	43	22	6.5755	44	22	9.2318	45	22	2.8228	46	22	3.0437	47	22	1.8677	48	22	1.7806
41	23	4.9202	42	23	6.7657	43	23	7.6092	44	23	9.9618	45	23	3.9656	46	23	4.0197	47	23	2.5258	48	23	2.7256
41	24	5.2430	42	24	7.2136	43	24	8.0267	44	24	10.3976	45	24	4.4116	46	24	4.4731	47	24	2.9785	48	24	3.1716
41	25	4.9818	42	25	6.4699	43	25	7.3972	44	25	9.7083	45	25	3.5228	46	25	3.6957	47	25	2.1591	48	25	2.2867
41	26	4.1265	42	26	5.7342	43	26	6.5988	44	26	8.9076	45	26	2.9452	46	26	3.0377	47	26	1.5195	48	26	1.7063
41	27	4.1905	42	27	6.0338	43	27	6.8620	44	27	9.2519	45	27	3.2070	46	27	3.2934	47	27	1.8426	48	27	1.9855
41	28	3.7016	42	28	5.7035	43	28	6.4437	44	28	8.8904	45	28	3.1154	46	28	3.0673	47	28	1.7395	48	28	1.9284
41	29	4.8549	42	29	6.0192	43	29	7.0022	44	29	9.3661	45	29	2.9426	46	29	3.2088	47	29	1.7501	48	29	1.7691
41	30	5.8006	42	30	7.7882	43	30	8.6112	44	30	10.9616	45	30	4.9961	46	30	5.0870	47	30	3.5774	48	30	3.7560
41	31	4.3424	42	31	5.6875	43	31	6.4466	44	31	9.0949	45	31	2.6418	46	31	2.9064	47	31	1.8942	48	31	1.7520
41	32	4.2061	42	32	6.0752	43	32	5.9301	44	32	9.1079	45	32	3.4756	46	32	3.7182	47	32	3.4680	48	32	3.2582
41	33	3.5129	42	33	5.9733	43	33	6.3629	44	33	8.8682	45	33	3.4657	46	33	3.6561	47	33	2.9363	48	33	3.0042
41	34	3.7455	42	34	6.4137	43	34	6.0122	44	34	9.0539	45	34	3.4511	46	34	3.9066	47	34	2.9353	48	34	2.7666
41	35	3.8020	42	35	5.8695	43	35	5.4246	44	35	8.5148	45	35	2.9047	46	35	3.4168	47	35	3.0999	48	35	2.8627
41	36	3.7807	42	36	6.2037	43	36	6.5102	44	36	9.1593	45	36	3.4703	46	36	3.7202	47	36	2.6620	48	36	2.6424
41	37	3.7689	42	37	6.1109	43	37	4.8273	44	37	8.2440	45	37	3.3273	46	37	3.6682	47	37	3.7590	48	37	3.5483
41	38	3.9864	42	38	5.9661	43	38	4.5649	44	38	8.1723	45	38	4.2313	46	38	4.3273	47	38	4.7363	48	38	4.5276
41	39	4.4039	42	39	7.0117	43	39	7.0644	44	39	10.0519	45	39	4.2956	46	39	4.3832	47	39	3.1831	48	39	3.2821
41	40	5.2857	42	40	7.5029	43	40	8.1771	44	40	10.5649	45	40	5.2594	46	40	5.3333	47	40	4.4352	48	40	4.5685
41	42	3.7591	42	41	3.7591	43	41	2.9443	44	41	6.1893	45	41	3.8966	46	41	3.3814	47	41	3.8785	48	41	4.0376
41	43	2.9443	42	43	3.8162	43	42	3.8162	44	42	4.2088	45	42	3.1795	46	42	2.8354	47	42	4.3185	48	42	4.2653
41	44	6.1893	42	44	4.2088	43	44	4.0376	44	43	4.0376	45	43	5.2241	46	43	4.7740	47	43	5.6664	48	43	5.7287
41	45	3.8966	42	45	3.1795	43	45	5.2241	44	45	6.5492	45	44	6.5492	46	44	6.2993	47	44	7.6160	48	44	7.6059
41	46	3.3814	42	46	2.8354	43	46	4.7740	44	46	6.2993	45	46	1.0159	46	45	1.0159	47	45	1.4974	48	45	1.2401
41	47	3.8785	42	47	4.3185	43	47	5.6664	44	47	7.6160	45	47	1.4974	46	47	1.5442	47	46	1.5442	48	46	1.4784
41	48	4.0376	42	48	4.2653	43	48	5.7287	44	48	7.6059	45	48	1.2401	46	48	1.4784	47	48	.3713	48	47	.3713
41	49	3.8911	42	49	4.6955	43	49	5.8250	44	49	7.9074	45	49	1.8009	46	49	2.0336	47	49	.9443	48	49	.8919
41	50	4.3683	42	50	3.8889	43	50	5.9530	44	50	6.4642	45	50	4.0213	46	50	3.5439	47	50	3.7120	48	50	3.9862
41	51	2.4974	42	51	3.0500	43	51	4.2029	44	51	6.1740	45	51	1.7124	46	51	1.5399	47	51	1.7827	48	51	1.8985
41	52	2.8489	42	52	5.0731	43	52	5.3635	44	52	7.5103	45	52	4.1651	46	52	4.1464	47	52	3.9905	48	52	4.1267
41	53	4.3075	42	53	1.4905	43	53	3.7082	44	53	3.1542	45	53	4.1657	46	53	3.9299	47	53	5.2266	48	53	5.2157
41	54	7.6074	42	54	5.1470	43	54	6.6806	44	54	6.2581	45	54	8.0330	46	54	7.2974	47	54	8.1911	48	54	8.3935
41	55	9.9820	42	55	6.8845	43	55	8.2390	44	55	5.0805	45	55	9.7999	46	55	9.5081	47	55	10.7458	48	55	10.7710

AN EXAMPLE OF THE DIFFERENCE TABLES

STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE	STATION		DIFFERENCE
49	1	2.5761	50	1	3.5916	51	1	3.4968	52	1	3.2644	53	1	6.7397	54	1	9.4914	55	1	11.5990
49	2	3.2204	50	2	4.9315	51	2	4.4463	52	2	3.7882	53	2	7.9976	54	2	10.5749	55	2	13.1358
49	3	4.6391	50	3	5.8435	51	3	5.6545	52	3	4.5899	53	3	9.1330	54	3	11.6058	55	3	14.1290
49	4	4.1364	50	4	4.8874	51	4	4.9463	52	4	4.4197	53	4	8.3019	54	4	10.8694	55	4	13.1175
49	5	2.9036	50	5	5.1709	51	5	4.2699	52	5	4.4071	53	5	7.9691	54	5	10.6282	55	5	13.2524
49	6	6.1457	50	6	6.3748	51	6	6.9703	52	6	5.2786	53	6	10.0347	54	6	12.2567	55	6	14.5212
49	7	4.7342	50	7	5.9753	51	7	5.7189	52	7	4.7947	53	7	9.2278	54	7	11.7151	55	7	14.2463
49	8	5.9394	50	8	6.4931	51	8	6.7120	52	8	5.6797	53	8	9.8062	54	8	12.0833	55	8	14.7519
49	9	6.8186	50	9	7.3562	51	9	7.6491	52	9	6.3211	53	9	10.6639	54	9	12.5377	55	9	15.4557
49	10	3.7031	50	10	5.7879	51	10	4.7846	52	10	4.3366	53	10	8.5162	54	10	11.0521	55	10	13.8908
49	11	3.2912	50	11	5.7089	51	11	4.4848	52	11	3.8022	53	11	8.2160	54	11	10.7339	55	11	13.7312
49	12	4.5586	50	12	6.3672	51	12	5.5299	52	12	4.2261	53	12	9.0749	54	12	11.4449	55	12	14.5023
49	13	5.1132	50	13	6.5855	51	13	5.9630	52	13	4.4266	53	13	9.3273	54	13	11.4341	55	13	14.6527
49	14	4.1938	50	14	5.0514	51	14	5.0012	52	14	3.3374	53	14	8.2224	54	14	10.5054	55	14	13.2592
49	15	4.8569	50	15	4.7216	51	15	5.4387	52	15	3.5652	53	15	8.3011	54	15	10.6515	55	15	12.9003
49	16	2.0607	50	16	4.7044	51	16	2.2381	52	16	4.4437	53	16	4.4730	54	16	8.6857	55	16	10.1822
49	17	1.7342	50	17	5.6218	51	17	3.2098	52	17	4.4687	53	17	6.3913	54	17	10.2634	55	17	12.0603
49	18	1.2762	50	18	5.0811	51	18	2.9755	52	18	4.1918	53	18	6.5201	54	18	9.9361	55	18	12.0298
49	19	1.4232	50	19	3.3318	51	19	1.7805	52	19	4.1797	53	19	4.6340	54	19	7.6550	55	19	10.1120
49	20	1.2520	50	20	2.9260	51	20	1.9386	52	20	3.9260	53	20	5.2364	54	20	8.0333	55	20	10.4948
49	21	1.8111	50	21	5.0116	51	21	3.1157	52	21	4.3099	53	21	6.7809	54	21	9.4904	55	21	12.3697
49	22	1.8529	50	22	5.4223	51	22	3.3227	52	22	4.4641	53	22	6.9077	54	22	9.7486	55	22	12.5405
49	23	2.2457	50	23	5.0387	51	23	3.8615	52	23	4.5542	53	23	7.5593	54	23	10.2621	55	23	12.8888
49	24	2.6603	50	24	5.4374	51	24	4.2974	52	24	4.7497	53	24	8.0013	54	24	10.7000	55	24	13.3246
49	25	1.8142	50	25	4.7900	51	25	3.6131	52	25	4.5957	53	25	7.2730	54	25	10.0086	55	25	12.5848
49	26	1.1765	50	26	4.4124	51	26	2.8027	52	26	3.5443	53	26	6.5234	54	26	9.2546	55	26	12.0087
49	27	1.4575	50	27	4.8080	51	27	3.1160	52	27	3.6629	53	27	6.8639	54	27	9.5836	55	27	12.4029
49	28	1.4404	50	28	4.4134	51	28	2.7165	52	28	3.1540	53	28	6.5092	54	28	9.1843	55	28	12.0906
49	29	1.5666	50	29	5.0590	51	29	3.3312	52	29	4.5429	53	29	6.9766	54	29	9.7438	55	29	12.4535
49	30	3.2139	50	30	5.7016	51	30	4.8277	52	30	4.9896	53	30	8.5137	54	30	11.2282	55	30	13.7592
49	31	1.8204	50	31	5.5694	51	31	3.2274	52	31	4.5618	53	31	6.7868	54	31	9.8656	55	31	12.4144
49	32	3.4286	50	32	7.0327	51	32	4.1847	52	32	5.3176	53	32	7.2378	54	32	10.7544	55	32	12.9111
49	33	2.2215	50	33	5.1332	51	33	2.9872	52	33	2.1942	53	33	6.7027	54	33	10.0373	55	33	12.4905
49	34	2.7099	50	34	6.5488	51	34	3.8115	52	34	4.1338	53	34	7.2958	54	34	11.0537	55	34	13.0519
49	35	3.0277	50	35	6.6082	51	35	3.7215	52	35	4.7551	53	35	6.8200	54	35	10.6579	55	35	12.4767
49	36	2.0189	50	36	5.7050	51	36	3.2867	52	36	3.0035	53	36	7.0537	54	36	10.5376	55	36	12.8415
49	37	3.6783	50	37	6.8839	51	37	3.9439	52	37	4.7091	53	37	6.9492	54	37	10.8811	55	37	12.6032
49	38	4.6575	50	38	7.7111	51	38	4.7500	52	38	5.4276	53	38	6.9955	54	38	10.7804	55	38	12.5446
49	39	2.8834	50	39	6.4373	51	39	4.2111	52	39	4.4346	53	39	7.9659	54	39	10.9809	55	39	13.6788
49	40	3.7649	50	40	6.0297	51	40	4.5375	52	40	3.1522	53	40	8.1804	54	40	10.7801	55	40	13.7660
49	41	3.8911	50	41	4.3683	51	41	2.4974	52	41	2.8489	53	41	4.3075	54	41	7.6074	55	41	9.9820
49	42	4.6955	50	42	3.8889	51	42	3.0500	52	42	5.0731	53	42	1.4905	54	42	5.1470	55	42	6.8845
49	43	5.8250	50	43	5.9530	51	43	4.2029	52	43	5.3635	53	43	3.7082	54	43	6.6806	55	43	8.2390
49	44	7.9074	50	44	6.4642	51	44	6.1740	52	44	7.5103	53	44	3.1542	54	44	6.2581	55	44	5.0805
49	45	1.8009	50	45	4.0213	51	45	1.7124	52	45	4.1651	53	45	4.1657	54	45	8.0330	55	45	9.7999
49	46	2.0336	50	46	3.5439	51	46	1.5399	52	46	4.1464	53	46	3.9299	54	46	7.2974	55	46	9.5081
49	47	.9443	50	47	3.7120	51	47	1.7827	52	47	3.9905	53	47	5.2266	54	47	8.1911	55	47	10.7458
49	48	.8919	50	48	3.9862	51	48	1.8985	52	48	4.1267	53	48	5.2157	54	48	8.3935	55	48	10.7710
49	50	4.0309	50	49	4.0309	51	49	1.8207	52	49	3.3990	53	49	5.5088	54	49	8.7589	55	49	11.0864
49	51	1.8207	50	51	2.9779	51	50	2.9779	52	50	3.8335	53	50	3.7520	54	50	6.3875	55	50	8.2901
49	52	3.3990	50	52	3.8335	51	52	2.6165	52	51	2.6165	53	51	3.8417	54	51	7.4050	55	51	9.5791
49	53	5.5088	50	53	3.7520	51	53	3.8417	52	53	5.2860	53	52	5.2860	54	52	8.4446	55	52	10.9355
49	54	8.7589	50	54	6.3875	51	54	7.4050	52	54	8.4446	53	54	4.7208	54	53	4.7208	55	53	5.7914
49	55	11.0864	50	55	8.2901	51	55	9.5791	52	55	10.9355	53	55	5.7914	54	55	5.5317	55	54	5.5317

CHAPTER 6.

6. MULTIDIMENSIONAL SCALING.

6.1 Introduction.

Multidimensional scaling refers to a collection of techniques which have been developed for inferring multidimensional metric structure from nonmetric ordinal data.

Suppose there are m objects (difference totals, stations) to be scaled. Define the symmetric input data matrix

$$\underset{(m \times m)}{\Delta} = \begin{pmatrix} \delta_{11} & \cdots & \delta_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \delta_{m1} & \cdots & \delta_{mm} \end{pmatrix},$$

where for $i \neq j$, δ_{ij} denotes the observed dissimilarity between objects i and j (δ_{ii} will never be used and therefore is not defined). Moreover, suppose δ_{ij} is not subject to error or stochastic variation but is specified exactly.

There are three ways of how the δ_{ij} may be obtained. First we could ask a subject to rank order all pairs of the m objects according to his perception of their degree of dissimilarity (there are $M \equiv m(m - 1)/2$ such pairs). The δ_{ij}'s could taken as the M ranks of the ordered pairs.

Secondly the subject could be asked to place each of the M pairs on a scale (ranging from 0 to 10) according to his perception of their degree of dissimilarity. The 10 will for instance denote most dissimilar, which a 0 will denote least dissimilar. The δ_{ij}'s would correspond to the values on the scale. Furthermore the N subjects might each be used to rank the M pairs of objects, ant the average rank could be used for the δ_{ij}'s.

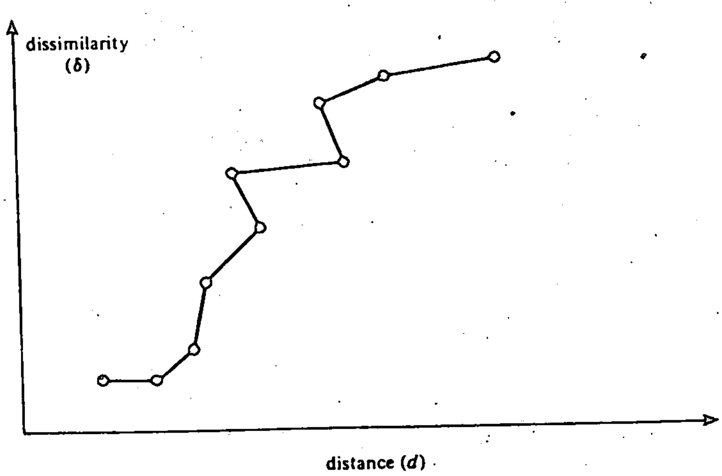
Third we could take the Δ as the rank correlation matrix of the m objects. The off-diagonal elements are the measures of degree of closeness of the objects.

Let $x_j : p \times 1$ denote the location vector of the j-th object in a space of dimension p, $j = 1, \ldots, m$. At the start of any scaling problem the x_j's are unknown. Assume a particular set of p-vectors $(x_1, \ldots, x_m)$ is selected arbitrarly, just see how well this set fits the observed data, i.e. the dissimilarities δ_{ij}. Furthermore the dimension p is generally unknown at the start, but lets assume it has been selected at some fixed value.

Define the distance between the i-th and j-th objects as

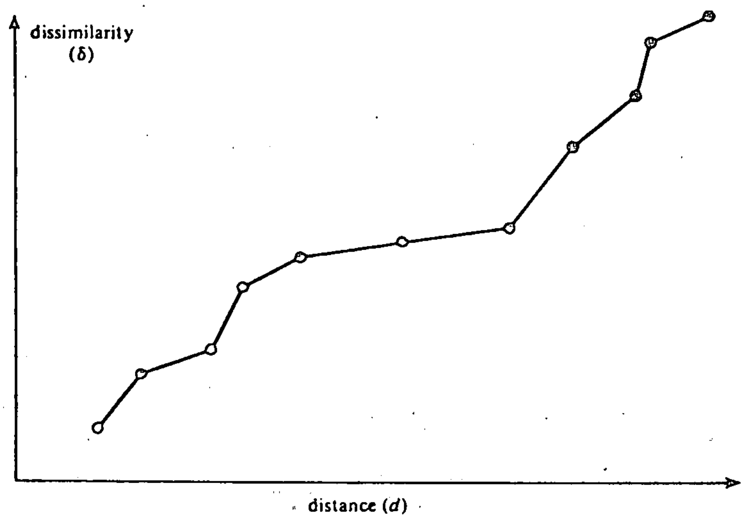
$$d_{ij} = |(x_i - x_j)'(x_i - x_j)|^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots(1)$$

A scatterdiagram of the points (d_{ij}, δ_{ij}) may now be plotted to compare how well the trial distances match the dissimilarities. The figure 15.2.1 below



Scattergram without Monotonicity

illustrates a hypothetical case in which monotonicity is not preserved. As the dissimilarities between objects increase, their distances do not necessarily increase, i.e. they sometimes decrease. A case in which there is a perfect match is shown in figure 15.2.2 below.



Scattergram Exhibiting Monotonicity

Lets discuss the goodness of Fit. Let $x_1, \ldots, x_m$ denote an arbitrary and known configuration of m points in p-space. The associated distances are defined in (1). Define the $n \times 1$ vector of distances between pairs of points

$$\underset{(n\times 1)}{D} = (d_{i_1 j_1}, \ldots, d_{i_n j_n})'.$$

Let $\hat{d}_{ij}$ denote a "fitted" value of d_{ij}. Kruskal(1964) defined the goodness-of-fit measure

$$\text{stress} = S = \left|\frac{(D-\hat{D})'(D-\hat{D})}{D'D}\right|^{\frac{1}{2}}$$

where $\hat{D} \equiv (\hat{d}_{ij})$. Thus the stress of the configuration of points $(x_1, \ldots, x_m)$ is defined as

$$S(x_1, \ldots, x_m) = \min_{\hat{D}} \left|\frac{(D-\hat{D})'(D-\hat{D})}{D'D}\right|^{\frac{1}{2}}$$

subject to the monotonity constraint

$$\hat{d}_{i_1j_1} \leq \hat{d}_{i_2j_2} \leq \dots \leq \hat{d}_{i_nj_n}$$

That is, $S(x_1, \dots, x_m)$ is a measure of how good a fit $(x_1, \dots, x_m)$ is to the given Δ.

Another problem arises, the choice of Dimension. Some problems dictate on appropriate value of p. For example there may be five fundamental factors which seems to characterize the essence of an object, and one could sought a solution in a space of dimension $p = 5$. as this is not always the case, the scaling problem should be solved for each of several values of p. If the p-stress drops below some tolerably low value such as 5 percent for some value of p, there is no need to go to a higher space of dimension since the fit is already quite good.

6.2 The use of multidimensional Scaling.

As already defined Correspondence analysis is one of a class of descriptive statistical methods, a class which also includes techniques such as multidimensional scaling and principal components analysis. The aim common to all these methods is to represent a data set by a number of points in multidimensional space, thus enabling a visual interpretation of the patterns existing in the data. The data observations can be imagined to occupy a space of high dimension.

Using the theory of multi-dimensional scaling, a program was developed by Dr. Underhill of the University of Cape Town to perform such a scaling. These programs have been changed to work at the University of the Orange Free State and is now available and working under

UNIVAC 1106 EXEC. These programs are listed on page 91.

The difference tables as described in Chapter 5 were being used as input for the Multidimensional Scaling program.

6.3 Interpretation of Andrew's and Multidimensional method on Correspondence analysis factors.

The main objective by using the method of Andrew's and Multidimensional scaling was to obtain the same graph as the two dimensional graph on page 55 when only the first two factors were considered. From the Multidimensional Scaling programs we obtained two printouts which reproduced firstly a diagram of the 55 Israeli stations and secondly a diagram of the 9 rainfall months.

These two diagrams were superimposed by using the centre points which were defined in the Multidimensional Scaling program. One must also note the scaling factor which is printed by this program and accordingly align the two graphs. The resultant graph can be seen on page 98. There is a definite simularity to the two dimensional graph on page 55.

One could also make the same interpretation as described on page 36. We again have the objects and subjects on the same graph. Note the variation in rainfall between month 1 and month 9. Station 6 and Station 9 are positively correlated to month 1, while Station 44 is negative.

We could in the same manner group the stations to form specific areas relating to page 99. On comparison to the actual map of Israel on page 100 there is again a definite resemblance.

```
      REAL  DISSIM(2000),CONF(5,400),DIST(2000)
      INTEGER T,IJ(2000),II(2000),JJ(2000),TIES(2000)
      READ (5,100)  M,T,L
  100 FORMAT(5X,I4,2I3)
      READ(5,101) (II(MM),JJ(MM),DISSIM(MM),MM=1,M)
  101 FORMAT(6X,2I3,F6.0)
      DO 1 MM=1,M
    1 IJ(MM)=1000*II(MM)+JJ(MM)
      CALL SORT(DISSIM,IJ,M, TIES)
      WRITE(6,102)     (DISSIM(I),I=1,M)
      WRITE(6,103) M, (IJ(I),I=1,M)
  102 FORMAT('1  DISSIMILARITIES MATRIX',/(12F10.3))
  103 FORMAT(' #DISSIM ',I10,/(12I10))
      WRITE(11)  M,T,L,IJ,DISSIM,DIST    , TIES
      DO  2  I = 1,12
      CALL     CONFIG  (M,T,L,IJ,TIES   ,CONF,DIST)
    2 CONTINUE
      END



      SUBROUTINE SORT (DISSIM,IJ,M,TIES)
      REAL DISSIM(1)
      INTEGER IJ(1)   , TIES(1)
      DO 1 J=2,M
      J1=J-1
      AMIN = DISSIM(J1)
      LEAST =J1
      DO 4 K=J,M
      IF (DISSIM(K)-AMIN) 3,3,4
    3 AMIN = DISSIM(K)
      LEAST = K
    4 CONTINUE
      SAVE = DISSIM(J1)
      ISAVE = IJ(J1)
      DISSIM(J1) = AMIN
      IJ(J1) = IJ(LEAST)
      DISSIM(LEAST) = SAVE
    1 IJ(LEAST) = ISAVE
      DISSIM(M+1)  = DISSIM(M)   +  1
      NOTIE  =  1
      I  =  0
      K  =  1
    2 I  =  I  +  1
    5 K  =  K  +  1
      IF(K.EQ.M+2)  GOTO 6
      IF  (DISSIM(I). EQ . DISSIM(K))  GOTO  5
      IF(K-I.EQ.1)  GOTO 2
      NOTIE = NOTIE  +  1
      TIES(NOTIE)  =  1000  *  I  +  K-1
      I  =  K
      GOTO  5
    6 IF  (NOTIE  .NE.1) TIES(1)  =  NOTIE
      RETURN
      END
```

```
      SUBROUTINE CONFIG(M,T,L,IJ,TIES  ,CONF,DIST)
      REAL CONF(5,1),DIST(1),DHAT(2000),GRAD(5,400),OLGRAD(5,400),
     * STRES(5)
      INTEGER  T,IJ(1),TIES(2000)
      DATA         ONE/1./
      IE  =  0
      SMALL  =  .01
      K = K+1
      DO 8 LL=1,L
      DO 8 IT = 1,T
    8 CONF(IT,LL) = 0.
      CALL CREATE(CONF,T,L)
      CALL NORMAL(CONF,T,L)
      GMAG  =  1.
      ITER  =  0
      GOTO  10
      ENTRY   FINCON( M,T,L,IJ,TIES  ,CONF,DIST)
  100 FORMAT(2I2)
      READ(5,100,END=40)   NUM  , IX
      DO 21  J = 1,NUM
   21 READ(11)  ITER,GMAG,STRES,ALPHA,GRAD,CONF
      IF(IX.EQ.0)  GOTO 22
      READ(5,103)  ( I ,(CONF(J,I),  J=1,3),  IT=1,IX)
  103 FORMAT  (I2,  3F6.2)
   22 CONTINUE
      WRITE(5,104)   NUM , ((CONF  (I,J),I=1,T),J = 1,L)
  104 FORMAT  ('0CONTINUATION OF ITERATION NO. ',I3/' STARTING FROM CONF
     *IGURATION'/ (12F10.3))
      IE  = 1
      SMALL  =  .000001
   10 CONTINUE
      ITER = ITER +1
      CALL NORMAL(CONF,T,L)
      CALL DISTAT(IJ,DIST,T,M,CONF,DHAT, TIES)
      CALL FITTER(DHAT,M)
      DO 2 LL=1,L
      DO 2 IT=1,T
      OLGRAD(IT,LL) = GRAD(IT,LL)
    2 GRAD(IT,LL) = 0.
      OLGMAG=GMAG
      OLNEW = 0.
      CALL GRADIT(DIST,DHAT,T,L,IJ,STRESS,GRAD,GMAG,CONF,M)
      CHANGE=ABS(STRESS - STRES(5) )
      IF(STRESS.LT.SMALL.OR. CHANGE.LT. 0.001)    GOTO 11
      DO 5 LL =1,L
      DO 5 IT = 1,T
    5  OLNEW = OLNEW + OLGRAD(IT,LL) *GRAD(IT,LL)
      COSTH = OLNEW/(L*OLGMAG*GMAG)
      DO 6 J=1,4
    6 STRES(J) = STRES(J+1)
      STRES(5) = STRESS
      IF(ITER.NE.1) GO TO 3
      OLGMAG = GMAG
      DO 4 J=1,5
    4 STRES (J) = STRESS
      COSTH = 1.
      ALPHA = 0.2
    3 ALPHA = ALPHA*4.**(COSTH**3.)*(1.3/(1.+AMIN1(ONE,
     * STRESS/STRES(1))))* AMIN1(ONE,STRESS/STRES (4))
```

```
      DO 7 LL=1,L
      DO 7 IT=1,T
    7 CONF(IT,LL) = CONF(IT,LL) + ALPHA*GRAD(IT,LL)/GMAG
      GOTO  10
   11 WRITE(6,105)K,ITER,CHANGE,STRESS,((CONF ( I,J),I=1,T),J=1,L)
  105 FORMAT(' ITERATION ',I3,'  NO OF ITERS ',I3,'  CHANGE ',F7.4,
     *'  STRESS ',F7.4/(12F10.3 )  )
      WRITE(6,101)  (IJ(I),I=1,M)
  101 FORMAT(' IJ MATRIX'/(12I10 ) )
      WRITE(6,102)(DIST(I),I=1,M)
      WRITE(6,120)(DHAT(I),I=1,M)
  102 FORMAT(' DIST MATRIX'/( 12F10.3 ))
  120 FORMAT(' DHAT MATRIX'/( 12F10.3 ))
      CALL TODPLT(CONF,L)
      IF(IE.NE.1) WRITE(11) ITER,GMAG,STRES,ALPHA,GRAD , CONF
      RETURN
   40 CALL EXIT
      END


      SUBROUTINE CREATE (CONF,T,L)
      INTEGER T
      REAL CONF(5,1)
      READ  100, N
  100 FORMAT  (I5)
      PRINT 101, N
  101 FORMAT ('1SEED:  ',I6)
      DO 1 LL=1,L
      DO 1 IT=1,T
    1 CONF(IT,LL) = RANDOM(N)
      RETURN
      END


      SUBROUTINE NORMAL (CONF,T,L)
      INTEGER T
      REAL CONF(5,1), AMEAN(100)
      SQUARE=0.
      DO 4 J=1,T
      AMEAN(J)=0.
      DO 3 K=1,L
    3 AMEAN(J)=AMEAN(J)+CONF(J,K)
    4 AMEAN(J)=AMEAN(J)/L
      DO 5 K=1,L
      DO 5 J=1,T
      CONF(J,K)=CONF(J,K)-AMEAN(J)
    5 SQUARE = SQUARE+ CONF(J,K)**2
      SQUARE = SQRT(L/SQUARE)
      DO 6 K=1,L
      DO 6 J=1,T
    6 CONF(J,K)=CONF(J,K)*SQUARE
      RETURN
      END
```

```
      SUBROUTINE DISTAT(IJ,DIST,T,M,CONF,DHAT,TIES)
      REAL DIST(1),DHAT(1),CONF(5,1)
      INTEGER IJ(1),T,TT     ,  TIES(1)
      DO 1 K=1,M
      I  =  IJ(K)/1000
      J = IJ(K)-1000*I
      SQUARE = 0.
      DO 2 TT=1,T
    2 SQUARE = SQUARE+(CONF(TT,I)-CONF(TT,J))**2
    1 DIST(K) = SQRT(SQUARE)
      IF(TIES(1) .NE.0)  CALL TIE(TIES,DIST,IJ)
      DO11  K = 1,M
   11 DHAT(K) = DIST(K)
      RETURN
      END


      SUBROUTINE FITTER(DHAT,M)
      REAL DHAT(1)
      INTEGER DOTHER(2000),SATIS
      DO 1 J=1,M
    1 DOTHER(J) = 0
      IACTIV = 1
    4 NUPDOW = 1
    5 CALL SATSFY(DHAT,DOTHER,M,NUPDOW,IACTIV,SATIS,NEXT,NEXT1,
     * IACTV1,NOBLOC)
      IF (SATIS.EQ.-1) GO TO 2
      NUPDOW = -NUPDOW
      CALL SATSFY(DHAT,DOTHER,M,NUPDOW,IACTIV,SATIS,NEXT,NEXT1,
     * IACTV1,NOBLOC)
      IF(SATIS.EQ.-1)  GO TO 2
      IF(DOTHER(IACTIV).NE.0) IACTIV = IACTIV + DOTHER(IACTIV)-1
      IACTIV = IACTIV+1
      IF(IACTIV.NE.M+1) GO TO 4
      J = 1
    3 J = J+1
      IF (J.EQ.M+2) RETURN
      IF (DOTHER(J-1).EQ.0) GO TO 3
      K = J+DOTHER(J-1)-2
      DO 6 KJ=J,K
    6  DHAT(KJ)   = DHAT(J-1)
      J=K+1
      GO TO 3
    2 CALL JOIN(DHAT,DOTHER,NEXT,IACTIV,NEXT1,IACTV1,NOBLOC,M)
      NUPDOW = -NUPDOW
      GO TO 5
      END


      FUNCTION RANDOM(N)
      N = N * 3125
      RANDOM = FLOAT(N)/ 34359738337.D0
        IF (RANDOM .LT.0.0)  RANDOM = -RANDOM
      RETURN
      END
```

```
      SUBROUTINE SATSFY(DHAT,DOTHER,M,NUPDOW,IACTIV,SATIS,NEXT,
     $ NEXT1,IACTV1,NOBLOC)
      INTEGERDOTHER(1),SATIS
      REAL DHAT(1)
      SATIS = 1
      IF (NUPDOW.EQ.-1. AND. IACTIV.EQ.1) RETURN
      IF (DOTHER(IACTIV).NE.0) GO TO 2
      IACTV1 = IACTIV
      NOACBL  = 1
      GO TO 1
    2 IACTV1 = IACTIV + 1
      NOACBL = DOTHER(IACTIV)
    1 IF (NUPDOW .EQ. -1) GO TO 4
      NEXT = IACTIV + NOACBL
      IF (NEXT.EQ.M+1) RETURN
      IF (DOTHER(NEXT) .NE.0) GO TO 3
    5 NEXT1 = NEXT
      NOBLOC = NOACBL+1
      GO TO 6
    4 NEXT = IACTIV-1
      IF(DOTHER(IACTIV-1).EQ.0) GO TO 5
      NEXT = DOTHER(IACTIV-1)
    3 NOBLOC = NOACBL + DOTHER(NEXT)
      NEXT1 = NEXT+1
    6 IF ((NUPDOW*DHAT(IACTIV)).GT.(NUPDOW*DHAT(NEXT))) SATIS=-1
      RETURN
      END
```

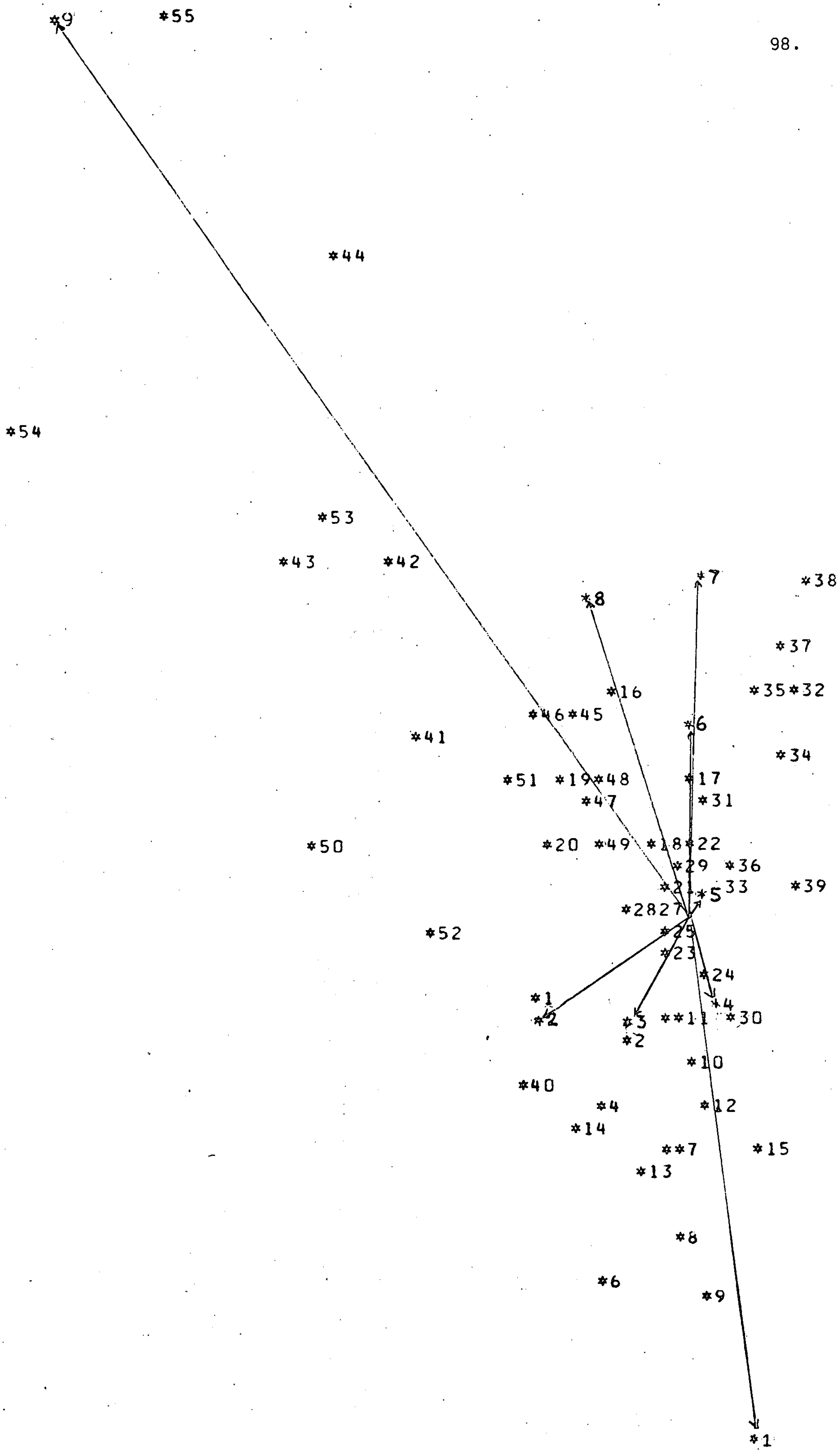
```
      SUBROUTINE GRADIT(DIST,DHAT,T,L,IJ,STRESS,GRAD,GMAG,CONF,M)
      REAL DIST(1),DHAT(1),GRAD(5,100),CONF(5,1)
      INTEGER IJ(1),T,TT
      SSTAR = 0.
      TSTAR = 0.
      GMAG = 0.
      DO 1 K=1,M
      SSTAR = SSTAR+(DIST(K)-DHAT(K))**2
    1 TSTAR = TSTAR + DIST(K) **2
      STRESS = SQRT(SSTAR/TSTAR)
      DO 4 K=1,M
      I = IJ(K)/1000
      J = IJ(K) -1000*I
      IF (I.EQ.J) GO TO 4
      TERM = STRESS*((DIST(K)-DHAT(K))/SSTAR-DIST(K)/TSTAR)
     * /DIST(K)
      DO 2 TT=1,T
      ATERM = -TERM*(CONF(TT,I)-CONF(TT,J))
      GRAD(TT,I) = GRAD(TT,I) + ATERM
      GRAD(TT,J) = GRAD(TT,J) - ATERM
    2 CONTINUE
    4 CONTINUE
      DO 3 LL=1,L
      DO 3 TT=1,T
    3 GMAG = GMAG+(GRAD(TT,LL)**2)
      GMAG = SQRT(GMAG/L)
      RETURN
      END
```

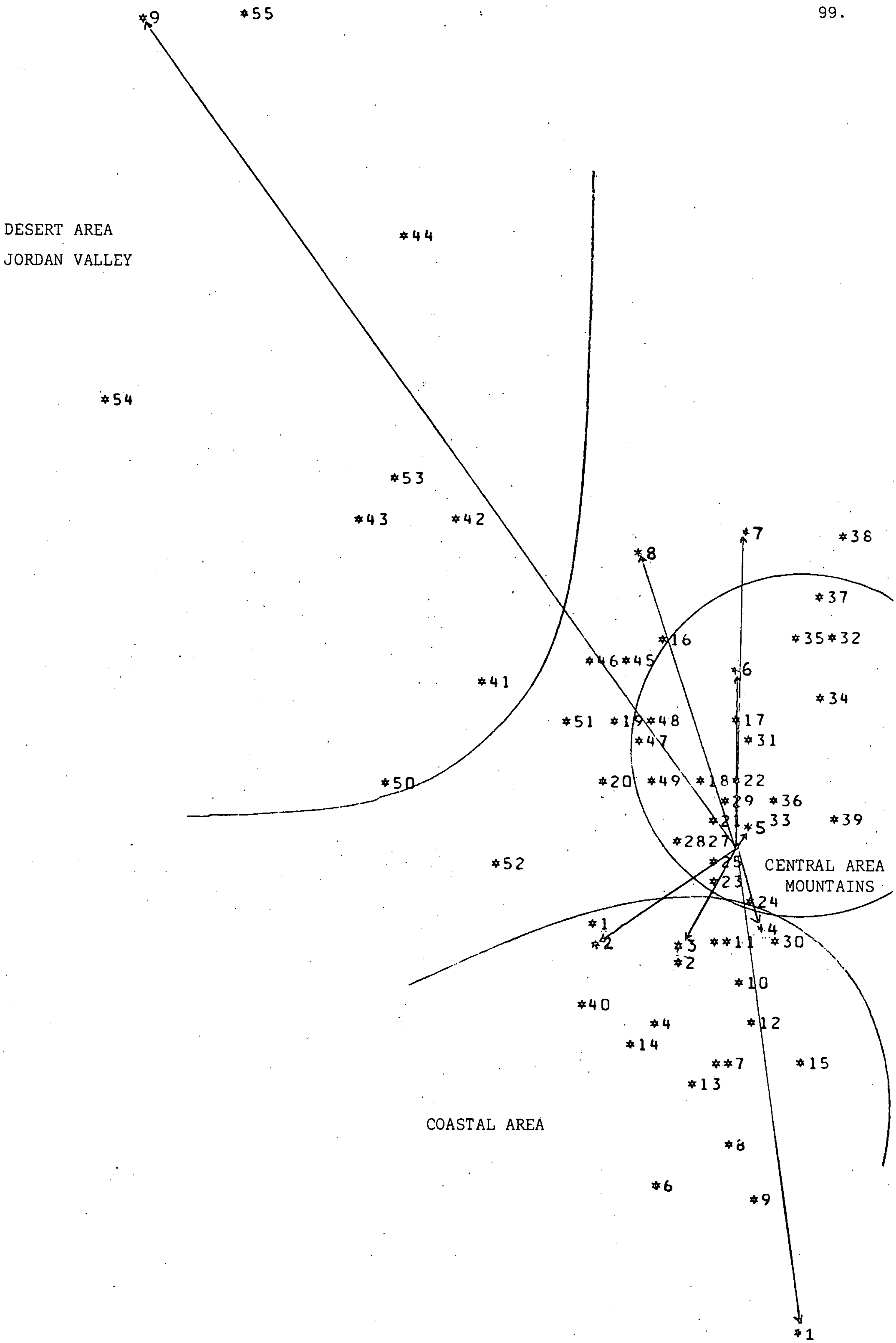
```
      SUBROUTINE JOIN (DHAT,DOTHER,I,J,I1,J1,NOBLOC,M)
      INTEGER DOTHER(1)
      REAL DHAT(1)
      I = MINO (I,J)
      J = I
      IEND = I+NOBLOC-1
      DHAT(I+1) = DHAT(J1)+DHAT(I1)
      DOTHER(I) = NOBLOC
      DHAT(I) = DHAT(I+1)/NOBLOC
      DOTHER(IEND) = I
      RETURN
      END
```

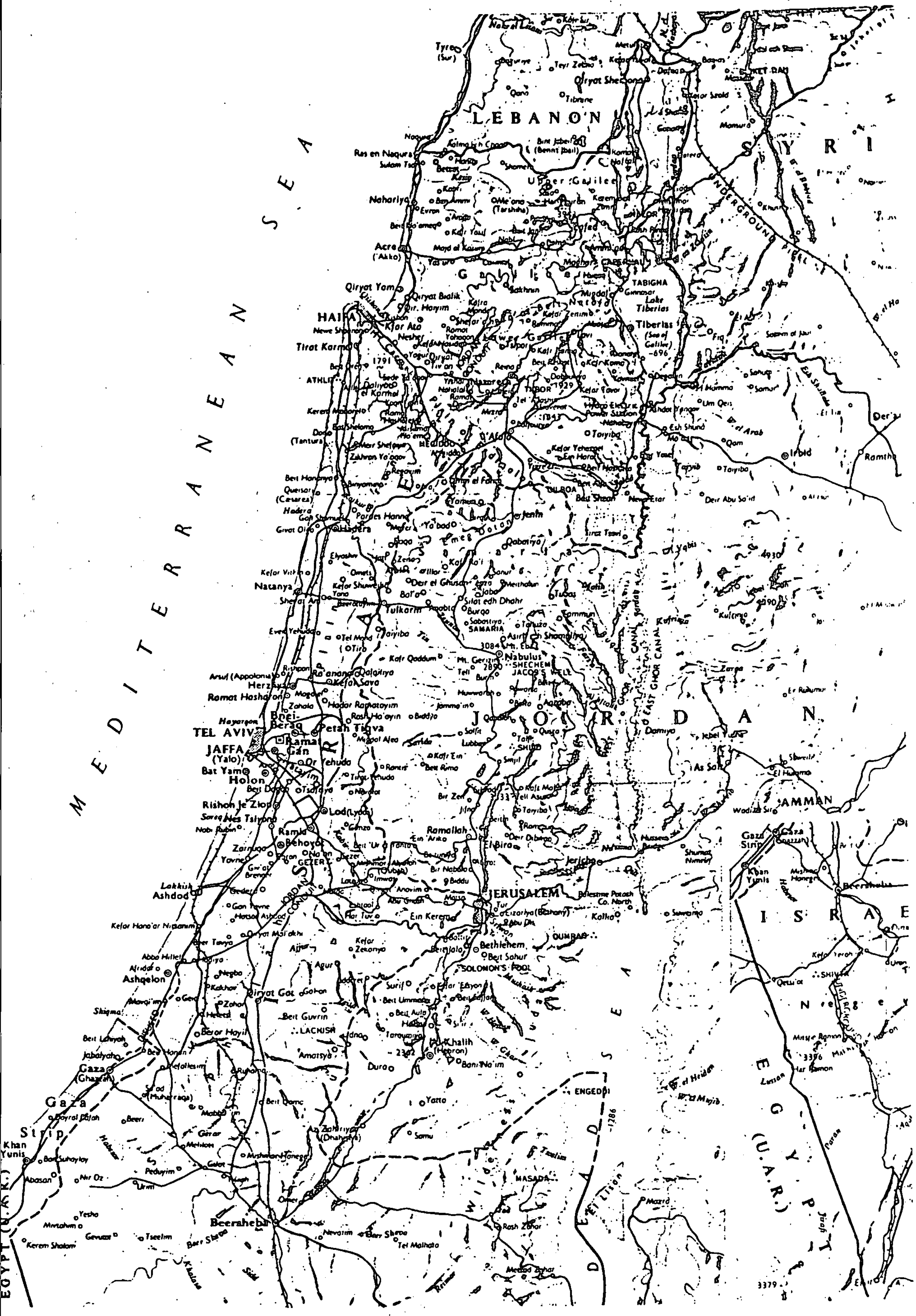
```
      SUBROUTINE  TIE(TIES,DIST,IJ)
      REAL DIST(1)
      INTEGER  TIES(1), IJ(1)
      ITIE  =  TIES(1)
      DO  1  KK = 2,ITIE
      I  =  TIES(KK)/1000
      J  = TIES(KK) - 1000 * I
      I  =  I+1
    3 MARK  =  0
      DO  2  K = I,J
      IF (DIST(K-1) . LE . DIST(K))  GO TO  2
      SAVE  = DIST (K)
      DIST(K)  =  DIST(K-1)
      DIST(K-1)  =  SAVE
      ISAVE  =  IJ(K)
      IJ(K)  =  IJ(K-1)
      IJ(K-1)  =  ISAVE
      MARK  =  1
    2 CONTINUE
      IF  (MARK .NE. 0) GOTO 3
    1 CONTINUE
      RETURN
      END
```

```
      FUNCTION  GAUSS  (AM,P,N)
    1 V1  =  2. * RANDOM(N) -1
      V2  =  2. * RANDOM(N) - 1
      S  =  V1 *V1  +  V2 *V2
      IF  (S.GE.1.)  GOTO 1
      VAR  =  V1 * SQRT(ALOG(S) *(-2.) / S)
      GAUSS  =  ( P * VAR + 1. ) * AM
      RETURN
      END
```

```
      SUBROUTINE TODPLT(CONFIG,M)
C     TO GENERATE MAP FROM CONFIGURATION
      REAL  CONFIG(5,1),MXC1,MXC2,MNC1,MNC2,MIDC,MIDR,SCALE
C     MAP SCALED TO 104*60 AS WIDTH:BREADTH::2.25:4.0
      CHARACTER*107 SPACE(61)
      CHARACTER*2 NUMBER(60)
      INTEGER  R,C,BLANK
      DATA NUMBER/'1','2','3','4','5','6','7','8','9','10',
     *'11','12','13','14','15','16','17','18','19','20',
     $'21','22','23','24','25','26','27','28','29','30'
     %,'31','32','33','34','35','36','37','38','39','40'
     &,'41','42','43','44','45','46','47','48','49','50'
     #,'51','52','53','54','55','56','57','58','59','60'/
C     INITIALIAZE SPACE TO BLANKS
      SPACE(1)='START'
      DO 1 R=2,61
        SPACE(R)=' '
    1   CONTINUE
      SUBSTR(SPACE(61),105,3)='END'
C
C     FIND MAX AND MIN VALUES OF CONFIG1, CONFIG2
      MXC1,MXC2,MNC1,MNC2=0
      DO 3 L = 1,M
        MXC1=AMAX1 (MXC1,CONFIG(1,L))
        MXC2=AMAX1 (MXC2,CONFIG(2,L))
        MNC2=AMIN1 (MNC2,CONFIG(2,L))
        MNC1=AMIN1 (MNC1,CONFIG(1,L))
    3   CONTINUE
C     DETR MIDPT CONFIGURATIONS (VERTICAL,HORIZONTAL)
C     AND SCALING FACTOR
      MIDC=(MXC1+MNC1)/2.
      MIDR=(MXC2+MNC2)/2.
      SCALE=AMAX1(MXC1-MNC1,MXC2-MNC2)/2.
      WRITE(6,100)MIDC,MIDR,SCALE
  100 FORMAT(T2,'CENTRE POINTS ARE ',2F10.3,'  SCALING FACTOR',
     *F8.4/T2,' COL FRAC  ROW FRAC TOWN  ROW  COL')
C     DETERMINE MAP FOR OUTPUT
      DO 4 L=1,M
        R=31+30./SCALE*(CONFIG(2,L)-MIDR)
        C=53+52./SCALE*(CONFIG(1,L)-MIDC)
        SUBSTR(SPACE(R),C,3)='*'&NUMBER(L)
        WRITE(6,101)CONFIG(1,L),CONFIG(2,L),L,R,C
  101   FORMAT(T2,2F10.2,3I5)
    4   CONTINUE
      I=I+1
      WRITE(6,102)I,SPACE
  102 FORMAT('1',I3,104('#')/(' [',A107, ']' ) )
      RETURN
      END
```


*9
*55
*44
*54
*53
*43
*42
*7
*38
*8
*37
*16
*35*32
*46*45
*41
*6
*34
*51
*19*48
*17
*47
*31
*50
*20
*49
*18*22
*29
*36
*21
*5
*33
*39
*2827
*52
*25
*23
*24
*1
*4
*2
*3
**11
*30
*2
*10
*40
*4
*12
*14
**7
*15
*13
*8
*6
*9
*1


DESERT AREA
JORDAN VALLEY
CENTRAL AREA
MOUNTAINS
COASTAL AREA


MEDITERRANEAN SEA
LEBANON
SYRIA
JORDAN
ISRAEL
EGYPT
(U.A.R.)
DEAD SEA
Upper Galilee
Lower Galilee
Tyre (Sur)
Acre ('Akko)
Nahariya
HAIFA
Qiryat Sherona
Safed
Tiberias
TABIGHA
Lake Tiberias
Nazareth
Jenin
Hadera
Natanya
Tulkarm
Nabulus
SHECHEM
JACOB'S WELL
SAMARIA
TEL AVIV
JAFFA (Yalo)
Bat Yam
Holon
Rishon le Zion
Petah Tiqva
Ramat Gan
Lod (Lydda)
Ramla
Rehovot
Ashdod
Ashqelon
Gaza
Gaza Strip
Khan Yunis
Ramallah
JERUSALEM
Jericho
Bethlehem
SOLOMON'S POOL
Hebron
Qiryat Gat
LACHISH
Beersheba
MASADA
ENGEDDI
AMMAN
Irbid
As Salt
EAST GHOR CANAL
TEL AVIV

SUMMARY

After the first trial run the Correspondence Analysis program prints all the factors as column and row loadings. Select two factors with the highest characteristic value (Ignore the characteristic value 1). If these factors contribute 60 or more to the percentage of inertia, the two dimensional graph will be obtained by plotting the subjects and objects on the same axis. If this is not the case the method of Andrew's and Multidimensional Scaling should be followed. The association between the subjects and objects is obtained by grouping the graphs as plotted by the Andrew's program with the aid of the difference tables. The representations of the objects and subjects on the same axis is obtained by the Multidimensional Scaling program.

Sometimes, when struggling with a particulary difficult interpretation and situation, we ask ourselves whether our efforts are worthwhile. If it is certain that Correspondence analysis reveal structural relationships between elements, can we trust in the individual axes which generate the planes in which we observe the projections of the clouds? Other statisticians do not use Correspondence analysis but other ways of constructing and analyzing clouds (e.g. classical factor analysis). Here it is common practise to rotate the axes, i.e. in the subspace spanned by the first principal axes of inertia of the cloud other axes (orthogonal or not) are chosen the interpretation of which appear easier. For these analysis the more a factor coincides with one of the variables of the table or a group of strongly correlated variables, the more it is interpretable. We are completely opposed to this practise for many reasons. Firstly, if the data is fairly homogenous and the sampling not too sparse the interpretation is often rather easy, thanks to the principle of distributional equivalence. Secondly, it has appeared in numerous applications that a modification of the original data usually does not modify the nature of the computed factors, causing perhaps only a permutation of their order. This kind of stability leads us to believe that one has to pay some respect to the individuality of the factors as computed in

Correspondence analysis.

Some practitioners often ask: among the set of characteristics (subjects) used to describe the objects, which are the least useful concerning the interpretation of the first computed factors. Another question may be measuring only what determines the important factors of a first study (the experimental basis of which is perhaps chosen to confirm some simple concepts) are we not running the risk of restricting ourselves. This is the sort of questions representative of an attitude which we are trying to avoid.

However, having established the conclusions of a study performed on a rather exhaustive table (k(I,J)) it is conceivable that one might wish to locate a set, say S, of supplementary elements on the factorial axes not by taking into account the whole table, but only a reduced table. One can compute the factors without repeating the analysis, with the usual formula:

$$G_{\alpha}^{r}(s) = (\lambda_{\alpha}^{r})^{-\frac{1}{2}}\Sigma\{F_{\alpha}(i)k(i,s)/k(s) | i \in I_{r}\}.$$

Here we are starting to tackle difficulties that can really only be evaluated by someone who has already practised Correspondence analysis.

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