Bayesian Inference for the Lognormal Distribution

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Abstract

This thesis is concerned with objective Bayesian analysis (primarily estimation hypothesis testing and confidence statements) of data that are lognormally distributed. The lognormal distribution is currently used extensively to describe the distribution of positive random variables that are right-skewed. This is especially the case with data pertaining to occupational health and other biological data.

In Chapter 1 we begin with inference on the products of means and medians as discussed in Menzefricke (1991). Exposure risk modeling is a particular application of this setting. Exact posterior moments are derived and compared to the Monte Carlo simulation techniques.

Chapters 2 to 4 are concerned with inference on the mean of the lognormal distributtions in various settings. Other authors, namely Zou, Taleban and Huo (2009), have proposed procedures involving the so-called "method of variance estimates recovery" (MOVER), while an alternative approach based on simulation is the so-called generalized confidence interval, discussed by Krishnamoorthy and Mathew (2003). In this thesis we compare the performance of the MOVER-based confidence interval estimates and the generalized confidence interval procedure to coverage of credibility intervals obtained using Bayesian methodology using a variety of different prior distributions to estimate the appropriateness of each. An extensive simulation study is conducted to evaluate the coverage accuracy and interval width of the proposed methods. For the Bayesian approach both the equal-tailed and highest posterior density (HPD) credibility intervals are presented. Various prior distributions (independence Jeffreys' prior, the Jeffreys-rule prior, namely, the square root of the determinant of the Fisher Information matrix, Reference and Probability-Matching priors) are evaluated and compared to determine which give the best coverage with the most efficient interval width. The simulation studies show that the constructed Bayesian confidence intervals have satisfying coverage probabilities and in some cases outperform the MOVER and generalized confidence interval results. The Bayesian inference procedures (hypothesis tests and confidence

intervals) are also extended to the difference between two lognormal means as well as to the case of zero-valued observations and confidence intervals for the lognormal variance.

In Chapter 5, the variance of the lognormal distribution is the central focus. Similarly to previous chapters, various prior distributions are tested in different applications.

In the 6th chapter of this thesis the bivariate lognormal distribution is discussed and Bayesian confidence intervals are obtained for the difference between two correlated lognormal means as well as for the ratio of lognormal variances, using nine different priors.

Chapters 7 and 8 are an investigation into Bayesian methods for analysing the one-way random effects model. Chapter 7 presents the Bayesian framework and results for the balanced model and Chapter 8 is an extension of this setting for the unbalanced model. A new prior distribution, namely Gelman's prior (Gelman, 2006), is introduced.

Keywords: Bayesian procedure; Lognormal; Highest Posterior Density; MOVER; Credibility intervals; Coverage probabilities; Zero-valued observations; Bivariate Lognormal; Lognormal variance; One-way Random Effects Model

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Yes, everything else is worthless when compared with the infinite value of knowing Christ Jesus my Lord. For his sake I have discarded everything else, counting it all as garbage, so that I could gain Christ.

Phil 3:8

I would like to thank my Friend and Companion, Jesus Christ, for the privilege of knowing Him; compared to that there is nothing...

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May your future exploits far overshadow mine. You make your father's heart smile!

INTRODUCTION

As suggested by the title, this work is concerned with inference on the lognormal distribution. Lognormal data are found in many different settings and complexities unique to each of these settings are the focus. In each setting though, the aim is to present a Bayesian perspective or analysis method to deal with the setting. Various prior distributions are tested within each chapter and compared to each as well as to frequentist methods developed in the literature. The prior distributions primarily used throughout this work are the Independence Jeffreys prior, the Jeffreys Rule prior, the Reference prior and the Probability-Matching prior. In most chapters the effectiveness of the method is evaluated by performing simulation studies and assessing the coverage and interval lengths (primarily). Furthermore, these methods are then applied to practical examples used in literature.

In Chapter 1 we begin by developing the Bayesian framework for the analysis of the product of means and medians from a lognormal distribution. In this chapter the methodology for the derivation of the Probability-Matching prior in particular is presented in detail.

Chapter 2 is concerned with inference on the mean of a single sample from a lognormal population. The priors are developed and this chapter lays much of the foundation for the analyses following in Chapters 3 and 4. The Independence Jeffreys and Jeffreys Rule priors are developed and applied.

Where Chapter 2 was concerned with a single sample, Chapter 3 looks at the analysis of two samples, and in particular the ratio of two population means.

In Chapter 4, the situation in Chapter 3 is extended to include the situation where there are potential zero values, but the non-zero valued observations are lognormally distributed.

The next chapter deals with inference regarding the variance of the lognormal distribution. In previous chapters only the means or medians were considered. The difference as well as ratio between variances of two populations are analysed as well as the possibility of zero values.

Chapter 6 deals with the bivariate lognormal distribution.

Chapters 7 and 8 present the Bayesian perspective on the one way random effects model. In Chapter 7 we focus on a completely balanced model and in Chapter 8 the unbalanced case is considered. Various prior distributions are derived and tested in this section and the effectiveness of each is considered.

CHAPTER 1

Product of Means and Medians

Introduction

Consider k random variables $X_j, j = 1,2,\dots,k$ such that $Y_j = lnX_j$ is normally distributed with mean μ_j and variance σ_j^2. We are concerned with inferences (mainly estimation and confidence statements) about the product of their medians or means. As in Menzefricke (1991) this chapter addresses itself to obtain the posterior distribution of these parameters.

Mezenfricke succeeded in deriving the posterior distributions for the special case $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$ and σ^2 unknown. In this chapter, more general results will be given for σ_j^2 unknown, but not necessarily equal. Although it is difficult to derive the exact posterior distributions in the case of the Behrens-Fisher problem (variances not necessarily equal), the exact moments will be derived. By calculating the first four moments and by comparing Pearson curves to Monte Carlo simulation experiments it will be shown that good approximations of the true distributions can be obtained.

Berger and Bernardo (1989) addressed a similar problem. They gave an elegant analysis of the posterior distribution of the product of two means from a normal distribution. As mentioned by them, this problem arises, most obviously, in situations of determining area based on measurements of length and width. It also arises in other practical contexts, however, for instance in gypsy moth studies the hatching rate of larvae per unit area can be estimated as the product of the mean of egg masses per unit area times the mean number of larvae hatching per egg mass. Approximately independent samples can be obtained for each mean (see Southwood (1978)).

Another example application occurs in the assessment of risk due to exposure to radiation of various pollutants. It can be assumed that the dose per unit time, the units of time per day and the number of days during which an individual is exposed are three random variables each with an unknown mean and variance. If these variables are independently distributed then the total exposure is an estimate of the product of the means. For further details see Sun and Ye (1995) and Yfantis and Flatman (1991).

As mentioned the analysis in this chapter is restricted to variables X_j which are known to be positive. It was pointed out by Menzefricke (1991) that when the coefficients of variation are large it will be difficult to distinguish if they follow a normal or lognormal distribution. The latter approach was chosen since it appears to result in simpler analytical solutions.

In what follows both the terminology and layout of this chapter will be along the same lines as those given by Menzefricke (1991). Whereas he paid attention to exact distributional aspects, our exposition will place more emphasis on approximations.

Section 2 of this chapter deals with the easier case of the product of the medians, section 3 considers the case of the product of the means and in section 4 the problem of constructing a prior that is informative for a single parameter (the product of means parameter) is considered. The prior is constructed in such a way that the resulting one-sided credibility interval has accurate frequentist coverage. Section 5 contains numerical examples.

1.1 The Product of Medians: $\beta = exp\sum_{j=1}^{k}\mu_j$

Let $\{X_{ij}, i = 1,2,\ldots,n_j\}$ be a random sample of size n_j from the j-th log-normal population with parameters μ_j and σ_j^2 $(j = 1,2,\ldots,k)$. The first step in a Bayesian approach is to select some prior distribution. If one has little prior information, Jeffreys independence prior,

$$p(\mu_1, \ldots, \mu_k, \sigma_1^2, \ldots, \sigma_k^2) \propto \prod_{j=1}^{k} \sigma_j^{-2} \tag{1.1}$$

is appropriate. See Zellner (1971) and Box and Tiao (1973) for further discussion. Combining the prior density (1.1) with the likelihood function, the joint posterior density function is given by:

$$p(\mu_1, \ldots, \mu_k, \sigma_1^2, \ldots, \sigma_k^2|\boldsymbol{data}) = p(\mu_1, \ldots, \mu_k|\sigma_1^2, \ldots, \sigma_k^2, \boldsymbol{data}) \times p(\sigma_1^2, \ldots, \sigma_k^2|\boldsymbol{data})$$

$$= \prod_{j=1}^{k} \left(\frac{2\pi\sigma_j^2}{n_j}\right)^{-1/2} exp\left\{-\frac{n_j}{2}\frac{(\bar{y}_j - \mu_j)^2}{\sigma_j^2}\right\} \left\{\prod_{j=1}^{k} \left(\frac{\nu_j s_j^2}{2}\right)^{\frac{1}{2}\nu_j} \frac{(\sigma_j^2)^{-\frac{1}{2}\nu_j - 1} exp\left[-\frac{\nu_j s_j^2}{2\sigma_j^2}\right]}{\Gamma\left(\frac{\nu_j}{2}\right)}\right\} \tag{1.2}$$

where:

$$v_j = n_j - 1$$

$$y_{ij} = lnx_{ij}$$

$$v_j s_j^2 = \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_j)^2$$

$$\bar{y}_j = \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij}$$

The median of X_j is equal to e^{μ_j} and the product of these medians is denoted by β where β is such that:

$$ln\beta = \sum_{j=1}^{k} \mu_j$$

If $\sigma_j^2, j = 1,2, \dots, k$ is known it follows from (1.2) that the posterior distribution of β is lognormal, i.e.

$$ln\beta | \sigma_1^2, \dots, \sigma_k^2, \boldsymbol{data} \sim N\left(M, \sum_{j=1}^{k} \frac{\sigma_j^2}{n_j} \right) \tag{1.3}$$

where $M = \sum_{j=1}^{k} \bar{y}_j$.

For the special case $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$ and σ_j^2 unknown, the posterior distribution of $ln\beta$ is a t-distribution with $\nu = \sum_{j=1}^{k} n_j - k$ degrees of freedom, mean M and variance parameter $\frac{s^2}{N}$, where

$$s^2 = \frac{\sum_{j=1}^{k} \nu_j s_j^2}{\sum_{j=1}^{k} \nu_j}$$

$$N = \frac{1}{\sum_{j=1}^{k} \frac{1}{n_j}}$$

If the σ_j^2 $(j = 1, \dots, k)$ are unknown and not necessarily equal, the posterior distribution of the μ_j are t-distributions with ν_j degrees of freedom, mean $\bar{y}_j$, and variance parameter $\frac{s^2}{n_j}$. The distribution of $ln\beta$ is difficult to derive in general but when $k = 2$ approximations to the Behrens-Fisher distribution are discussed in Box and Tiao (1973, page 107).

As remarked in the introductory paragraph more general results will be given in this section as those derived by Menzefricke (1991). Our proposed solution is based on Monte Carlo simulations and approximations via Pearson curves. These approximations will be achieved by first deriving the exact posterior moments of $ln\beta$. The following results are of importance:

Theorem 1.1

The mean, variance and third and fourth central posterior moments of $ln\beta$ for unknown σ_j^2 $(j = 1, \dots, k)$ are given by:

$$\mu'_{1\beta} = M \tag{1.4}$$

$$\mu_{2\beta} = \sum_{j=1}^{k} \frac{v_j s_j^2}{(v_j - 2)(v_j + 1)} \tag{1.5}$$

$$\mu_{3\beta} = 0 \tag{1.6}$$

$$\mu_{4\beta} = 3\left\{\sum_{j=1}^{k} \frac{(v_j s_j^2)^2}{(v_j - 2)(v_j - 4)(v_j + 1)^2} + \sum_{j \neq l}^{k} \frac{(v_j s_j^2)\ (v_l s_l^2)}{(v_j - 2)(v_l - 1)(v_j + 1)(v_l + 1)}\right\} \tag{1.7}$$

Proof:

From (1.2) the unconditional posterior moments about zero can easily be obtained and using the relationships between moments about the origin and central moments, the proof follows.

Having calculated the moments it is possible to derive the Pearson curve density of $ln\beta$ from which an approximation of the posterior distribution of β can be obtained.

The Monte Carlo simulation procedure to estimate the posterior distribution of β can be performed in the following way:

a. From (1.2) it follows that $p\left(\sigma_j^2|\boldsymbol{data}\right) = \left(\frac{\nu_j s_j^2}{2}\right)^{\frac{1}{2}\nu_j} \frac{\left(\sigma_j^2\right)^{-\frac{1}{2}\nu_j - 1} exp\left[-\frac{\nu_j s_j^2}{2\sigma_j^2}\right]}{\Gamma\left(\frac{\nu_j}{2}\right)}, \sigma_j^2 > 0$

(1.8)

is an inverted Gamma Distribution and can therefore be generated in the following way:

i. Draw $u_j \sim \chi^2_{\nu_j}$ and

ii. Calculate $\sigma_j^2 = \frac{\nu_j s_j^2}{u_j}, j = 1,2, \ldots, k$

b. Given $\sigma_1^2, \ldots, \sigma_k^2$ the conditional posterior distribution of β is lognormal, i.e.

$$p(\beta|\sigma_1^2, \ldots, \sigma_k^2, \boldsymbol{data}) = \frac{1}{\beta\sqrt{\sum_{j=1}^k \frac{\sigma_j^2}{n_j} 2\pi}} exp\left\{ -\frac{1}{2}(ln\beta - M)^2 \Big/ \sum_{j=1}^k \frac{\sigma_j^2}{n_j} \right\}$$

c. Repeat steps (a) and (b) $l(= 1000 \; or \; 10000)$ times.

Using the Rao-Blackwell argument (see Gelfand and Smith (1991)) a density estimate of the unconditional posterior distribution of β can be obtained by averaging $p(\beta|\sigma_1^2, \ldots, \sigma_k^2, \boldsymbol{data})$ over the l repetitions.

1.2 The Product of Means: $\delta = exp\sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)$

Since X_j is lognormally distributed with parameters μ_j and σ_j^2, $j = 1,2,\ldots,k$, the mean of X_j is

$$E(X_j|\mu_j,\sigma_j^2) = exp\left(\mu_j + \frac{1}{2}\sigma_j^2\right) \qquad j = 1,2,\ldots,k \tag{1.9}$$

and so the product of the means is δ, where δ is such that

$$ln\delta = \sum\nolimits_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right) \tag{1.10}$$

If the σ_j^2 are known, it follows from (1.2) that the posterior distribution of δ is lognormal, i.e.

$$ln\delta|\sigma_1^2,\ldots,\sigma_k^2,\boldsymbol{data} \sim N\left(M + \frac{1}{2}\sum_{j=1}^{k}\sigma_j^2, \sum_{j=1}^{k}\frac{\sigma_j^2}{n_j}\right) \tag{1.11}$$

According to Menzefricke (1991) this distribution will be close to (1.3) if the values of the coefficients of variation are small.

For the special case $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$ and σ^2 unknown, Menzefricke (1991) derived for even v, the exact posterior distribution of $ln\delta$, which is:

$$p(ln\delta|\boldsymbol{data}) = \left(\pi^{\frac{1}{2}}\right)^{-1} exp\left\{\frac{Nk}{2}\left[ln\delta - M - \left(\frac{s_\delta}{N}\right)^{\frac{1}{2}}\right]\right\} \times$$

$$\left(\frac{N}{s_\delta}\right)^{\frac{(\nu+2)}{4}} \frac{\left(\frac{k}{4}\nu s^2\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \sum_{i=0}^{\frac{\nu}{2}} \frac{\Gamma\left(i+\frac{1}{2}\right)}{\left[k(Ns_\delta)^{\frac{1}{2}}\right]^i} \sum_{j=0}^{i} \binom{\nu+1}{2i-2j} \binom{\frac{\nu}{2}-i+j}{j}$$

(1.12)

where M, N, s^2, ν are defined as before and

$$s_\delta = \nu s^2 + N(ln\delta - M)^2$$

If ν is odd the posterior distribution for $ln\delta$ has a more complicated form. Menzefricke, however, did not treat the more general case, i.e. σ_i^2 unknown and not necessarily equal.

As was done in the previous section with the product of medians, more general results will be given here. This will again be achieved by using Monte Carlo simulation (similar to that described in the previous section) and Pearson curve approximations. In Theorem 1.2 the moments of $ln\delta = \sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)$ will be derived.

Theorem 1.2

The mean, variance, third and fourth central moments for the posterior distribution of $ln\delta = \sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)$ are given by:

$$\mu'_{1\delta} = M + \frac{1}{2}\sum_{j=1}^{k}\frac{\nu_j s_j^2}{(\nu_j - 2)}$$

(1.13)

$$\mu_{2\delta} = \sum_{j=1}^{k} \frac{\nu_j s_j^2}{(\nu_j - 2)(\nu_j + 1)} + \frac{1}{2}\sum_{j=1}^{k} \frac{(\nu_j s_j^2)^2}{(\nu_j - 2)^2(\nu_j - 4)} \tag{1.14}$$

$$\mu_{3\delta} = 3\sum_{j=1}^{k} \frac{(\nu_j s_j^2)^2}{(\nu_j - 2)^2(\nu_j + 1)(\nu_j - 4)} + 2\sum_{j=1}^{k} \frac{(\nu_j s_j^2)^3}{(\nu_j - 2)^3(\nu_j - 4)(\nu_j - 6)} \tag{1.15}$$

$$\begin{aligned}\mu_{4\delta} = {} & 3\sum_{j=1}^{k} \frac{(\nu_j s_j^2)^2}{(\nu_j + 1)^2(\nu_j - 2)(\nu_j - 4)} \\ & + 3\sum_{j\neq l}^{k} \frac{(\nu_j s_j^2)\ (\nu_l s_l^2)}{(\nu_j - 2)(\nu_l - 2)(\nu_j + 1)(\nu_l + 1)} \\ & + 3\sum_{j=1}^{k} \frac{(\nu_j s_j^2)^3(\nu_j + 2)}{(\nu_j + 1)\ (\nu_j - 2)^3(\nu_j - 4)(\nu_j - 6)} \\ & + 3\sum_{j\neq l}^{k} \frac{(\nu_j s_j^2)^2(\nu_l s_l^2)}{(\nu_j - 2)^2(\nu_j - 4)(\nu_l + 1)(\nu_l - 2)} \\ & + \frac{3}{4}\sum_{j=1}^{k} \frac{(\nu_j s_j^2)^4(\nu_j + 10)}{(\nu_j - 2)^4(\nu_j - 4)(\nu_j - 6)(\nu_j - 8)} \\ & - \frac{3}{8}\sum_{j\neq l}^{k} \frac{(\nu_j s_j^2)^2(\nu_l s_l^2)^2(\nu_j - \nu_l - 2)}{(\nu_j - 2)^2(\nu_l - 2)^2(\nu_j - 4)(\nu_l - 4)}\end{aligned} \tag{1.16}$$

Proof:

The proof is given in the appendix to this chapter.

Corollary to Theorem 1.2

For the special case $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$ the mean, variance, third and fourth central moments of the posterior distribution of $ln\delta^* = \sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma^2\right)$ are given by:

$$\mu'_{1\delta^*} = M + \frac{1}{2}k\frac{\nu s^2}{(\nu-2)}$$

(1.17)

$$\mu_{2\delta^*} = \frac{1}{N}\frac{\nu s^2}{(\nu-2)} + \frac{1}{2}k^2\frac{(\nu s^2)^2}{(\nu-2)^2(\nu-4)}$$

(1.18)

$$\mu_{3\delta^*} = 3k\frac{(\nu s^2)^2}{N(\nu-2)^2(\nu-4)} + 2k^3\frac{(\nu s^2)^3}{(\nu-2)^3(\nu-4)(\nu-6)}$$

(1.19)

$$\mu_{4\delta^*} = 3\frac{(\nu s^2)^2}{N^2(\nu-2)\ \ (\nu-4)} + 3k^2\frac{(\nu s^2)^3(\nu+2)}{N(\nu-2)^3(\nu-4)(\nu-6)} + \frac{3}{4}k^2\frac{(\nu s^2)^4(\nu+10)}{(\nu-2)^4(\nu-4)(\nu-6)(\nu-8)}$$

(1.20)

Having calculated the moments, the Pearson curve approximation of $ln\delta$ or δ can easily be obtained. For further details of how to determine the parameters of a Pearson curve, given the values of its moments, see for example Elderton (1953) or Elderton and Johnson (1969).

1.3 Non-informative Priors for δ

For making Bayesian inferences about μ_j and σ_j^2 in the case of the lognormal distribution the obvious prior is (1.1). In the previous section it was necessary to make inferences about a function of the parameters, i.e. $\delta = exp\left(\sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)\right)$. It was observed by Efron (1986) that the correct objective prior seems to depend on which parameters we want to estimate. Berger and Bernardo (1989) also mentioned that a good non-informative prior for the full parameter space need not be good for lower dimensional functions of it.

Datta and Ghosh (1995) derived the differential equation that a prior must satisfy if the posterior probability of a one-sided credibility interval for a parameter function and its frequentist probability agree up to $o(n^{-1})$ where n is the sample size. They showed that if $exp\left(\mu + \frac{1}{2}\sigma^2\right)$ is the parameter of interest for a lognormal distribution with parameters μ and σ^2, then the Probability-Matching prior is:

$$p_M(\mu\,,\sigma^2) \propto \frac{1}{\sigma^2}\left(1 + \frac{2}{\sigma^2}\right)^{\frac{1}{2}} \tag{1.21}$$

Probability-Matching priors often lead to procedures with good frequency validity while retaining the Bayesian flavor. The fact that the resulting Bayesian confidence intervals of level $1 - \alpha$ are also good frequentist confidence intervals at the same level is a very desirable situation. See also Bayarri and Berger (2004) and Severine, Mukerjee and Ghosh (2002) for a general discussion.

By extending the results of Datta and Ghosh (1995) the following theorem can be proved:

Theorem 1.3

Let $Y_j, j = 1,2,\dots,k$, be independently distributed as lognormal with parameter vector $[\mu_j, \sigma_j^2]'$. Suppose the parameter of interest is $t(\boldsymbol{\theta}) = \delta = exp\left(\sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)\right)$ where $\boldsymbol{\theta}' = [\mu_1,\dots,\mu_k,\sigma_1^2,\dots,\sigma_k^2] = [\theta_1,\dots,\theta_p]$ then the Probability-Matching prior is given by:

$$\pi(\boldsymbol{\theta}) = \pi(\mu_1,\dots,\mu_k,\sigma_1^2,\dots,\sigma_k^2) \propto \left\{\sum_{j=1}^{k}\sigma_j^2\left(1+\frac{1}{2}\sigma_j^2\right)\right\}^{\frac{1}{2}}\prod_{j=1}^{k}\sigma_j^{-4} \tag{1.22}$$

The proof is given in the appendix to this chapter.

For $k = 1$, (1.22) becomes equation (1.21).

Combining the prior distribution (1.22) and the likelihood function, the joint posterior distribution can be written as:

$$q(\mu_1,\dots,\mu_k,\sigma_1^2,\dots,\sigma_k^2|\boldsymbol{data}) = q(\mu_1,\dots,\mu_k|\sigma_1^2,\dots,\sigma_k^2,\boldsymbol{data})q(\sigma_1^2,\dots,\sigma_k^2|\boldsymbol{data}) \tag{1.23}$$

where

$$q(\mu_1,\dots,\mu_k|\sigma_1^2,\dots,\sigma_k^2,\boldsymbol{data}) = \prod_{j=1}^{k}q_j\left(\mu_j|\sigma_j^2,\boldsymbol{data}\right)$$

and

$$q_j\left(\mu_j|\sigma_j^2,\boldsymbol{data}\right) = \left(\frac{2\pi\sigma_j^2}{n_j}\right)^{-\frac{1}{2}} exp\left\{-\frac{n_j}{2}\frac{\left(\mu_j-\bar{y}_j\right)^2}{\sigma_j^2}\right\} \tag{1.24}$$

Also,

$$q(\sigma_1^2, \dots, \sigma_k^2 | \boldsymbol{data}) \propto \left\{ \sum_{j=1}^{k} \sigma_j^2 \left(1 + \frac{1}{2}\sigma_j^2\right) \right\}^{\frac{1}{2}} \prod_{j=1}^{k} \left(\sigma_j^2\right)^{-\frac{1}{2}(\nu_j+4)} exp\left(-\frac{\nu_j s_j^2}{2\sigma_j^2}\right)$$

$$\sigma_j^2 > 0$$

$$(1.25)$$

Equation (1.23) is quite complex. This has the implication that it is quite difficult to derive the exact posterior distribution for δ. It is an equally difficult task to derive the moments of the distribution. Significant advances in numerical integration techniques have however, assisted in simulating from such complex distributions, but there are still limitations associated with these techniques. Gelfand and Smith (1991), Gelfand et al (1990), Carlin and Polson (1991), Casella and George (1992), Carlin et al (1992), Gelfand et al (1992) and Wakefield et al (1994) used the Gibbs sampler quite effectively though in these kinds of situations and this has been shown to be helpful in simulations of Bayesian inference in a broad variety of statistical problems. The technique has also been used by Hastings (1970) and has also been applied by Geman and Geman (1984) in the field of image processing. The Gibbs sampler is a specific application of an adaptive Monte Carlo integration technique.

From the joint posterior distribution $q(\sigma_1^2, \dots, \sigma_k^2 | \boldsymbol{data})$ the conditional distributions $q_1(\sigma_1^2 | \sigma_2^2, \dots, \sigma_k^2 | \boldsymbol{data})$, $q_2(\sigma_2^2 | \sigma_1^2, \sigma_3^2, \dots, \sigma_k^2 | \boldsymbol{data})$,..., $q_k(\sigma_k^2 | \sigma_1^2, \sigma_2^2, \dots, \sigma_{k-1}^2 | \boldsymbol{data})$ can easily be obtained. These conditional distributions are not in closed form, hence we only have the kernels of their densities. Random numbers can however, be generated by using Gibbs sampling or the weighted bootstrap (Monte Carlo) method as, for example, discussed in Smith and Gelfand (1992), Stephens and Smith (1992), Guttman and Menzefricke (2003), Kim (2006) and Li (2007) or the rejection method as given in Rice (1995).

By repeating the process until m cycles have been performed and m pairs of parameter estimates have been obtained and using a Rao-Blackwell argument (see Gelfand and Smith (1991)) a density estimate of the unconditional posterior distribution of δ can be obtained by averaging the conditional distributions over the m repetitions., i.e.

$$p(\delta|\boldsymbol{data}) = \frac{1}{m}\sum_{l=1}^{m} p\left(\delta|\sigma_1^{2(l)}, \dots, \sigma_k^{2(l)}, \boldsymbol{data}\right) \tag{1.26}$$

where

$$p\left(\delta|\sigma_1^{2(l)}, \dots, \sigma_k^{2(l)}, \boldsymbol{data}\right) = \frac{1}{\delta\sqrt{\sigma_\delta^{2(l)} 2\pi}} exp\left\{-\frac{1}{2}\frac{\left(ln\delta - \mu_\delta^{(l)}\right)^2}{\sigma_\delta^{2(l)}}\right\} \tag{1.27}$$

$$\mu_\delta^{(l)} = M + \frac{1}{2}\sum_{j=1}^{k} \sigma_j^{2(l)} \tag{1.28}$$

and

$$\sigma_\delta^{2(l)} = \sum_{j=1}^{k} \frac{\sigma_j^{2(l)}}{n_j}$$

(1.29)

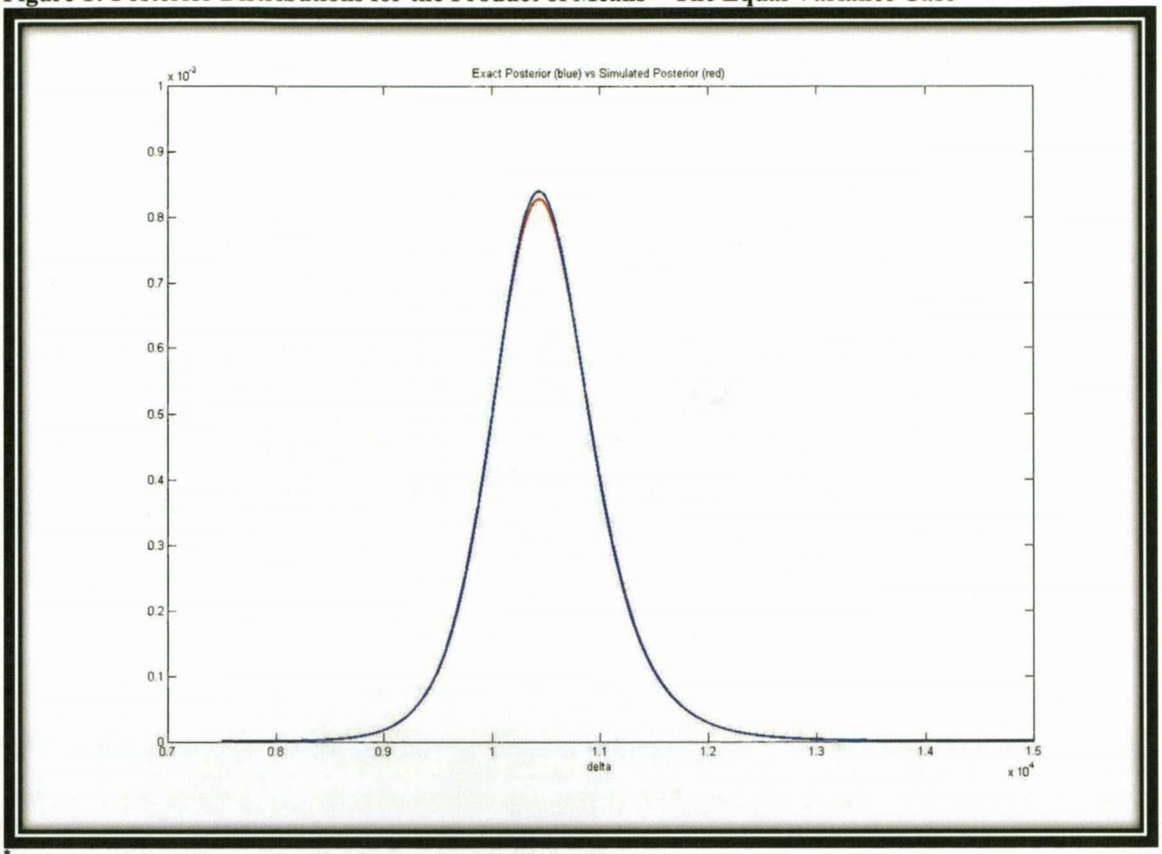
1.4 Numerical Examples

Our first problem in this section is the example given by Menzefricke (1991) and it involves the measurement of the area of a rectangle, with measurement uncertainty being similar for length and width.

The length measurements (in cm) are 106, 92, 100 and 106 and the width measurements are 97, 111, 102 and 104. If $X_1 = length$ and $X_2 = width$ then the means of the logged data are $\bar{y}_1 = 4.6135$ and $\bar{y}_2 = 4.6384$. Menzefricke assumed that $\sigma_1^2 = \sigma_2^2 = \sigma^2$. A pooled estimate of σ^2 is therefore, given by $s^2 = 0.0038$ and $\nu = 6$.

The exact posterior distribution of the product of means $\delta = exp\left(\sum_{j=1}^{2}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)\right)$ (for the equal variance case) can easily be derived from equation (1.12). By using a Monte Carlo simulation procedure similar to that described in section 2 earlier in this chapter, an estimate of the exact posterior density of δ can be obtained. These two densities are illustrated in Figure 1:

Figure 1: Posterior Distributions for the Product of Means – The Equal Variance Case



*Exact Posterior = Blue; Simulated Posterior = Red.

*Mean (simulated) = 10498.74; Mean (exact) = 10497.77.

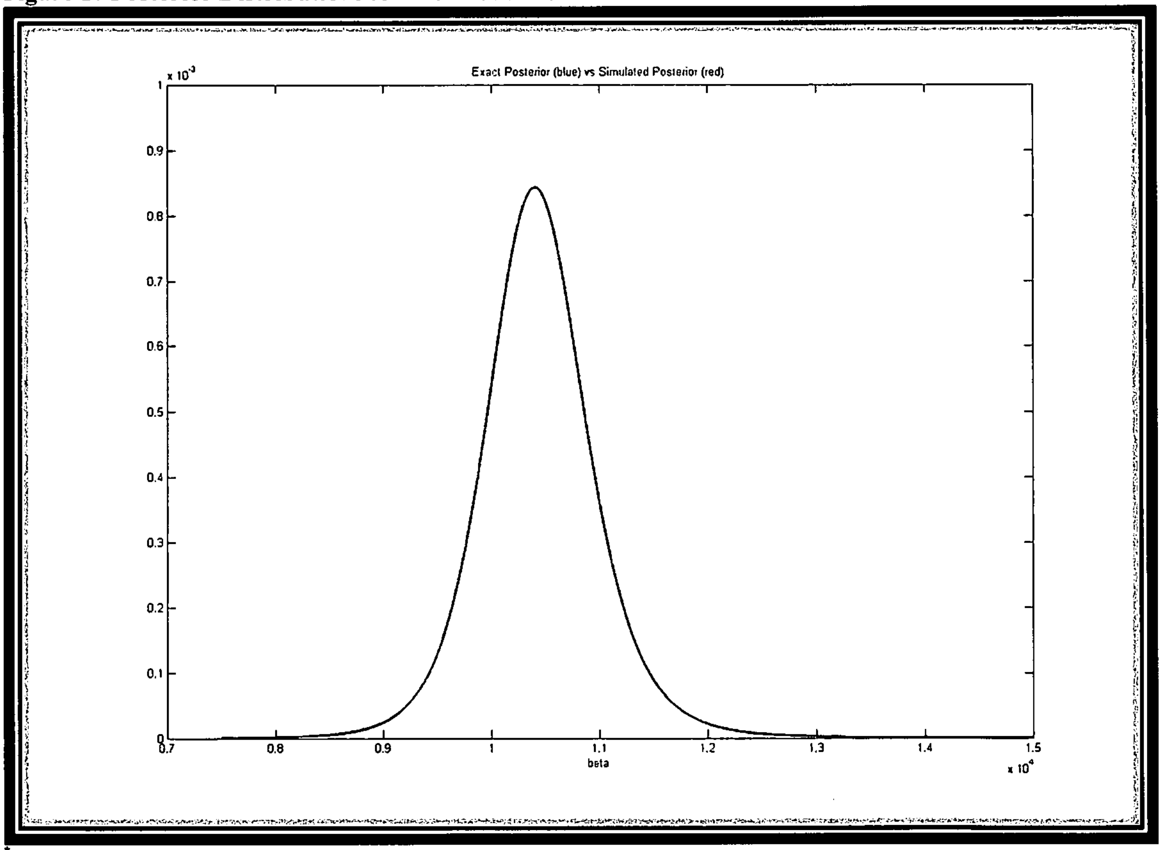
*Confidence Interval (simulated) = [9449; 11715]; Confidence Interval (exact) = [9453; 11703].

From the figure it can be seen that the simulated distribution is quite accurate and it fits the true density extremely well. The means, modes and credibility intervals are for all practical purposes the same. Since $\nu = 6$, the third and fourth central moments ($\mu_{3\delta^*}$ and $\mu_{4\delta^*}$ defined in (1.19) and (1.20)) do not exist, with the result that the Pearson curve approximations could not be obtained.

In Figure 2 the exact posterior distribution of the product of medians $\beta = exp \sum_{j=1}^{2} \mu_j$ for the same example is given. As mentioned in the section on the product of medians the exact distribution $ln\beta$ (for the equal variance case) is a t-distribution with $\nu = 6$

degrees of freedom. Also illustrated in Figure 2 is an estimation of the true density of β which is obtained by using Monte Carlo simulation. It is again clear that the simulated density is quite accurate because it fits the true density extremely well. A comparison of the means, modes and credibility intervals calculated from Figures 1 and 2 show that the posterior distributions of β and δ are quite similar. The reason for this is that the coefficient of variation is quite small. The posterior distributions of β and δ can however differ quite a lot in other examples.

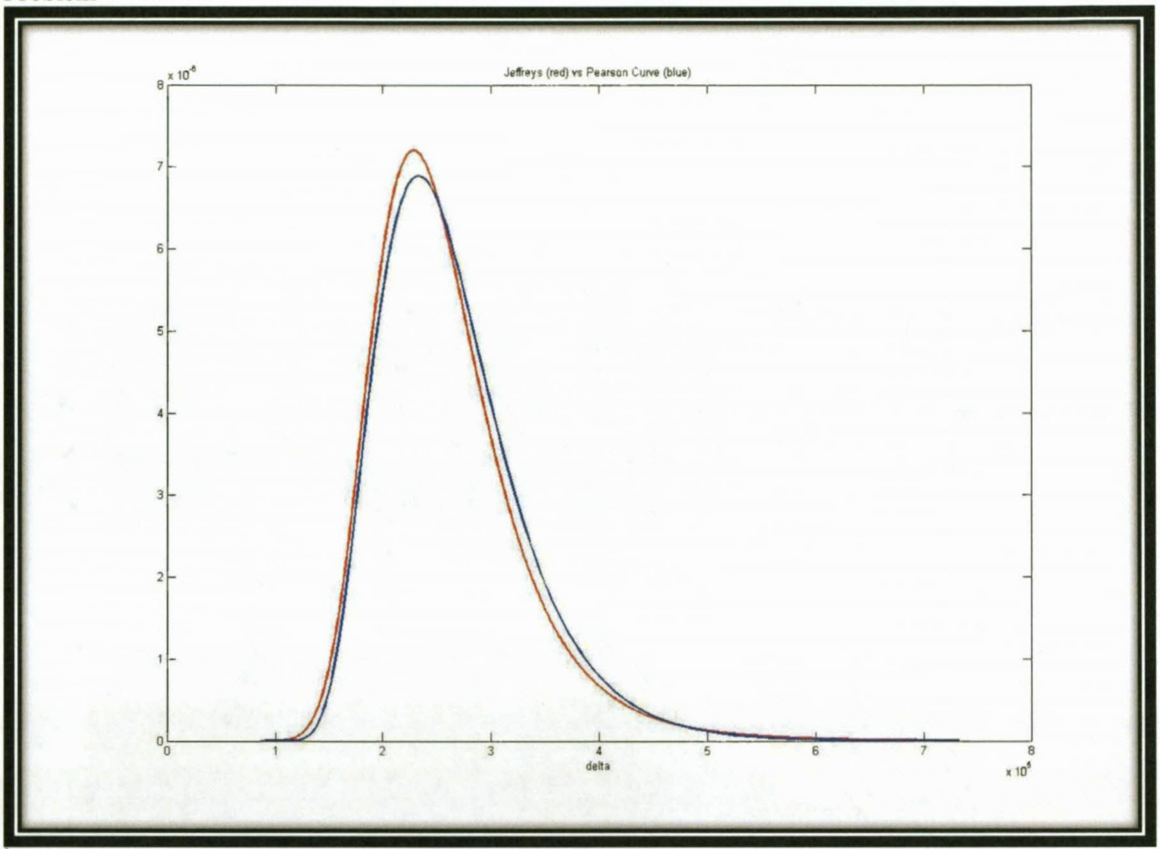
Figure 2: Posterior Distributions for the Product of Medians



*Exact Posterior = Blue; Simulated Posterior = Red.
*Mean (simulated) = 10438.26; Mean (exact) = 10438.55.
*Confidence Interval (simulated) = [9348; 11579]; Confidence Interval (exact) = [9372; 11594].

Our next example is an illustration of the Behrens-Fisher problem (unequal variances). To obtain sample data the parameters $\mu_1 = 2, \mu_2 = 4, \mu_3 = 6, \sigma_1^2 = 0.04, \sigma_2^2 = 0.16, \sigma_3^2 = 0.4$ were selected and sample sizes of $n_1 = 12, n_2 = 14, n_3 = 16$ were randomly drawn from the lognormal distribution. The sample statistics for the logged values were $\bar{y}_1 = 2.0404, \bar{y}_2 = 4.0307, \bar{y}_3 = 5.9967, s_1^2 = 0.0363, s_2^2 = 0.2154, s_3^2 = 0.3786$. By calculating $\mu'_{1\delta}, \mu_{2\delta}, \mu_{3\delta}$ and $\mu_{4\delta}$ (defined in (1.13) – (1.16)) a type I Pearson curve approximation for the posterior distribution of $ln\delta = \sum_{j=1}^{3}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)$ could be derived. The expression for the density function of δ is given by $\tilde{p}(\delta|\boldsymbol{data}) = \frac{1}{\delta} \times 165.6056\left(1 + \frac{ln\delta}{1.0440}\right)^{14.2520} \times \left(1 - \frac{ln\delta}{3.5630}\right)^{48.637}$ as illustrated in Figure 3. For details of how to determine the parameters of a Pearson curve given the values of the moments, see for example Elderton (1953) or Elderton and Johnson (1969). Also given in Figure 3 is the simulated posterior density. A comparison of the two densities shows that they are very much the same and therefore good approximations of the true density. Since σ_j^2 $(j = 1,2,3)$ are known to be unequal, the posterior distribution for δ for the equal variances case (equation (1.12)) could not be used for inferential purposes.

Figure 3: Posterior (Jeffreys) Distribution and Pearson Curve Approximation to the Behrens-Fisher Problem

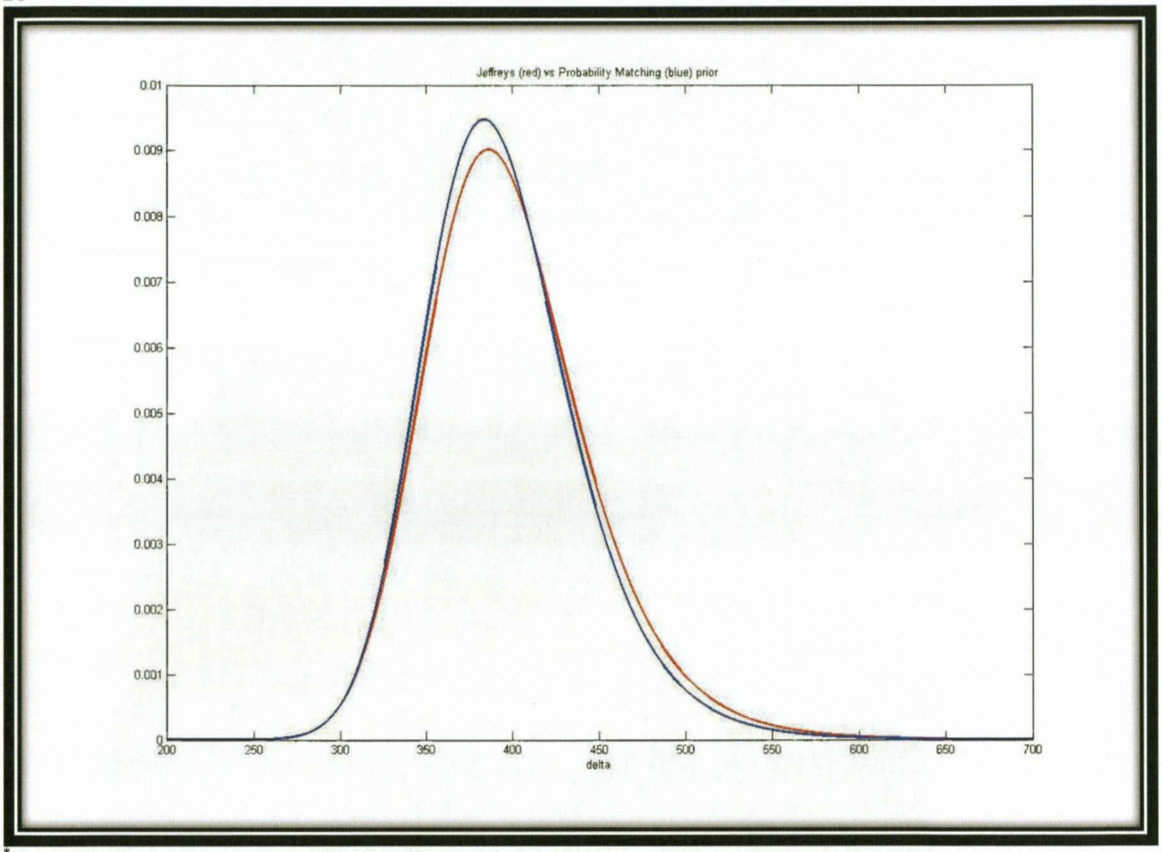


*Pearson Curve = Blue; Jeffreys Prior = Red.
*Mean (Jeffreys) = 258208.62; Mean (Pearson) = 249969.58.
*Confidence Interval (Jeffreys) = [161000; 425000]; Confidence Interval (Pearson) = [160256; 398127].

In our next example the posterior densities of δ for two different priors are compared. These are the Independence Jeffreys prior in equation (1.1) and the Probability-Matching prior derived in equation (1.22), which is an extension of the result obtained by Datta and Ghosh (1995). The same values for the parameters μ and σ^2 selected for the first two populations in the previous example were again used. Samples of sizes $n_j = 20$ ($j = 1,2$) were randomly drawn from the lognormal populations and the sample statistics were calculated as $\bar{y}_1 = 1.9810, \bar{y}_2 = 3.8745, s_1^2 = 0.0416, s_2^2 = 0.1838$.

In Figure 4 the Jeffreys and Probability-Matching posterior densities are displayed. These densities were obtained by conducting Monte Carlo simulation and Gibbs sampling procedures.

Figure 4: Posterior Distributions for the Jeffreys and Probability-Matching Priors – Sample Size n = 20

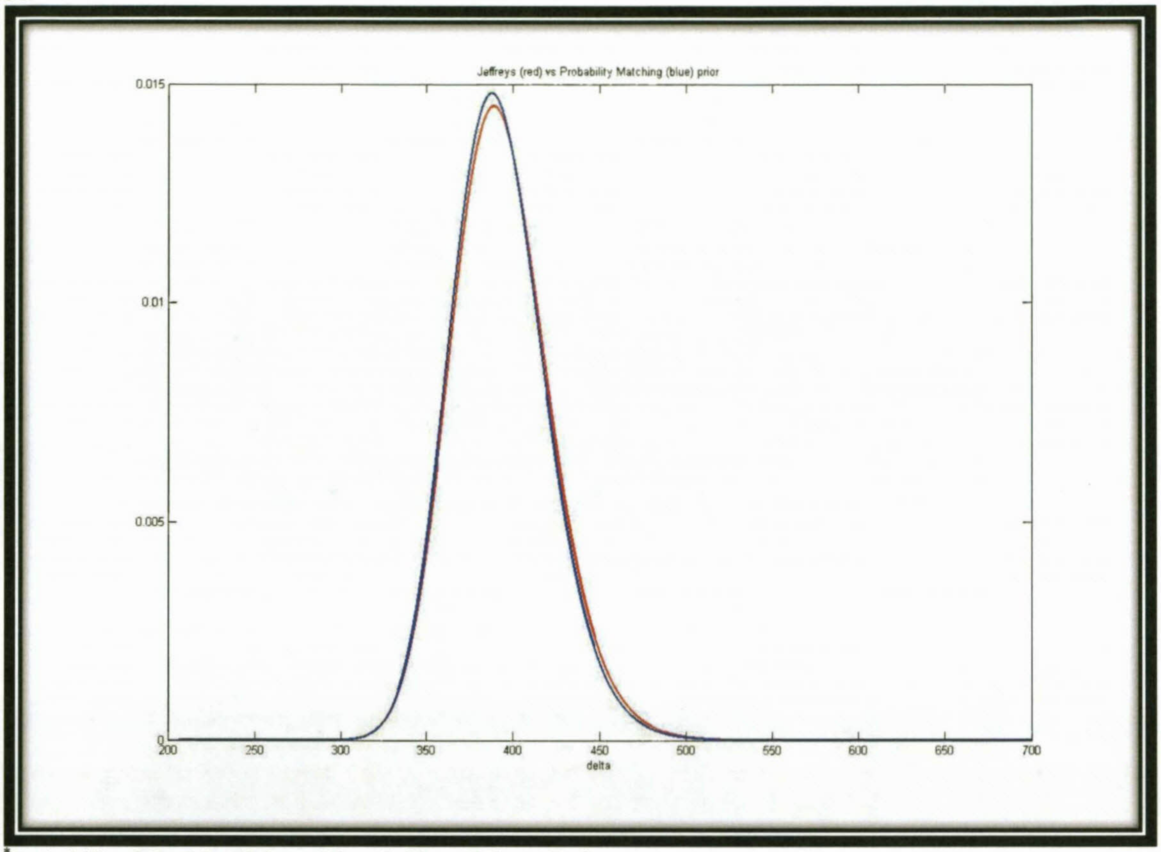


*Probability-Matching Prior = Blue; Jeffreys Prior = Red.
*Mean (Jeffreys) = 398.62; Mean (PMP) = 395.27.
*Confidence Interval (Jeffreys) = [318; 507]; Confidence Interval (PMP) = [318; 498.5].

It is clear from the figure that there is not much difference in the comparable posterior central values. The posterior using the Jeffreys prior is however slightly less peaked than the one obtained using the Probability-Matching prior with the consequence that the standard deviation of the former is larger and the 95% credibility interval will be wider. As mentioned before the Probability-Matching prior is derived such that the posterior probability of a one-sided credibility interval for δ and its frequentist probability agree up to $o(n^{-1})$ where n is the sample size.

In our last example a comparison is again made between the Jeffreys and Probability-Matching priors. The parameters selected are identical to those of the previous example, but in this case samples of size $n_j = 50$ $(j = 1,2)$ were randomly drawn from the lognormal distributions. The resulting posteriors are illustrated in Figure 5 and although the priors provide less information than in the previous example (the sample sizes are larger), the Jeffreys posterior is still somewhat less peaked with the consequence that the 95% credibility interval for the Probability-Matching posterior will be slightly narrower. For practical purposes the intervals are however equal.

Figure 5: Posterior Distributions for the Jeffreys and Probability-Matching Priors – Sample size n = 50



*Probability-Matching Prior = Blue; Jeffreys Prior = Red.

*Mean (Jeffreys) = 393.56; Mean (PMP) = 392.25.

*Confidence Interval (Jeffreys) = [343; 454]; Confidence Interval (PMP) = [342; 451].

Appendix to Chapter 1

Proof of Theorem 1.2

In order to prove equations (1.13) – (1.16) the following results will be used:

Lemma A.1

The $r-th$ moment about zero for the posterior distribution of σ_j^2 defined in equation (1.8) is given by:

$$E\left(\left(\sigma_j^2\right)^r |\boldsymbol{data}\right) = \frac{\left(\frac{1}{2}\nu_j s_j^2\right)^r \Gamma\left(\frac{1}{2}\nu_j - r\right)}{\Gamma\left(\frac{1}{2}\nu_j\right)} \tag{A.1}$$

Proof:

The proof follows easily; see for example Zellner (1971).

Lemma A.2

For the posterior distributions $p(\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\boldsymbol{data})$ defined in equation (1.2) it follows that

$$E\left(\sum_{j=1}^{k} \sigma_j^2 \;|\boldsymbol{data}\right) = \sum_{j=1}^{k} \frac{\nu_j s_j^2}{(\nu_j - 2)} \tag{A.2}$$

$$E\left(\left(\sum_{j=1}^{k} \sigma_j^2\right)^2 |\boldsymbol{data}\right) = \sum_{j=1}^{k} \frac{\left(\nu_j s_j^2\right)^2}{(\nu_j - 2)(\nu_j - 4)} + \sum_{j \neq l}^{k} \frac{(\nu_j s_j^2)(\nu_l s_l^2)}{(\nu_j - 2)(\nu_l - 2)} \tag{A.3}$$

$$E\left(\left(\sum_{j=1}^{k}\sigma_j^2\right)^3 |\boldsymbol{data}\right)$$

$$= \sum_{j=1}^{k}\frac{(v_j s_j^2)^3}{(v_j-2)(v_j-4)(v_j-6)} + 3\sum_{j\neq l}^{k}\frac{(v_j s_j^2)^2(v_l s_l^2)}{(v_j-2)(v_j-4)(v_l-2)}$$

$$+ \sum_{j\neq l\neq m}^{k}\frac{(v_j s_j^2)(v_l s_l^2)(v_m s_m^2)}{(v_j-2)(v_l-2)(v_m-2)}$$

(A.4)

$$E\left(\left(\sum_{j=1}^{k}\sigma_j^2\right)^4 |\boldsymbol{data}\right)$$

$$= \sum_{j=1}^{k}\frac{(v_j s_j^2)^4}{(v_j-2)(v_j-4)(v_j-6)(v_j-8)}$$

$$+ 4\sum_{j\neq l}^{k}\frac{(v_j s_j^2)^3(v_l s_l^2)}{(v_j-2)(v_j-4)(v_j-6)(v_l-2)}$$

$$+ 3\sum_{j\neq l}^{k}\frac{(v_j s_j^2)^2(v_l s_l^2)^2}{(v_j-2)(v_j-4)(v_l-2)(v_l-4)}$$

$$+ 6\sum_{j\neq l\neq m}^{k}\frac{(v_j s_j^2)^2(v_l s_l^2)(v_m s_m^2)}{(v_j-2)(v_j-4)(v_l-2)(v_m-2)}$$

$$+ \sum_{j\neq l\neq m\neq o}^{k}\frac{(v_j s_j^2)(v_l s_l^2)(v_m s_m^2)(v_o s_o^2)}{(v_j-2)(v_l-2)(v_m-2)(v_o-2)}$$

(A.5)

Proof:

It is well known (see for example Mood, Graybill and Boes (1974)) that

$$E\left(\sum_{j=1}^{k}\sigma_j^2\right)^r = E\left\{\sum\frac{r!}{\prod_{j=1}^{k} r_j!}\prod_{j=1}^{k}\left(\sigma_j^2\right)^{r_j}\right\}$$

(A.6)

where the summation is over all the nonnegative integers $r_1, r_2, \dots, r_k$ which sum to r. Substituting (A.1) for $r = 1,2,3,4$ the expected values (A.2) – (A.5) follow, from which the expressions for the first fours moments about zero can be derived. Using the relationship between moments about zero and central moments, equations (1.13) – (1.16) follow.

Proof of Theorem 1.3

Datta and Ghosh (1995) proved that the agreement between the posterior probability and frequentist probability holds if and only if

$$\sum_{\alpha=1}^{p}\frac{\partial}{\partial\theta_\alpha}\{\varphi_\alpha(\boldsymbol{\theta})\pi(\boldsymbol{\theta})\} = 0$$

(A.7)

where $\pi(\boldsymbol{\theta})$ is the Probability-Matching prior for $\boldsymbol{\theta}$, the vector of unknown parameters.

Also

$$\nabla_t(\boldsymbol{\theta}) = \left[\frac{\partial}{\partial\theta_1}t(\boldsymbol{\theta}), \dots, \frac{\partial}{\partial\theta_p}t(\boldsymbol{\theta})\right]'$$

(A.8)

where, as mentioned, $t(\boldsymbol{\theta}) = \delta = exp\left(\sum_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)\right)$ and

$$\varphi \ \ (\boldsymbol{\theta}) = \frac{I^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}{\sqrt{\nabla_t'(\boldsymbol{\theta})I^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}} = \left[\varphi_1(\boldsymbol{\theta}), \dots, \varphi_p(\boldsymbol{\theta})\right]'$$

(A.9)

It is clear that $\nabla_t'(\boldsymbol{\theta})I(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta}$ and $I^{-1}(\boldsymbol{\theta})$ is the inverse of the Fisher information matrix of $\boldsymbol{\theta}$ per unit observation.

For this example

$$I^{-1}(\boldsymbol{\theta}) = diag[\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2, 2\sigma_1^4, 2\sigma_2^4, \dots, 2\sigma_k^4]$$

(A.10)

$$\nabla_t(\boldsymbol{\theta}) = exp\sum\nolimits_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)\left[1,1,\dots,1,\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right]'$$

(A.11)

$$I^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta}) = exp\sum\nolimits_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)[\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2, \sigma_1^4, \sigma_2^4, \dots, \sigma_k^4]'$$

(A.12)

and

$$\{\nabla_t'(\boldsymbol{\theta})I^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})\}^{\frac{1}{2}} = exp\sum\nolimits_{j=1}^{k}\left(\mu_j + \frac{1}{2}\sigma_j^2\right)\left[\sum\nolimits_{j=1}^{k}\sigma_j^2\left(1 + \frac{1}{2}\sigma_j^2\right)\right]^{\frac{1}{2}}$$

(A.13)

Therefore,

$$\boldsymbol{\varphi}(\boldsymbol{\theta}) = \left\{\varphi_1(\boldsymbol{\theta}), \dots, \varphi_p(\boldsymbol{\theta})\right\}' = \frac{1}{\left[\sum_{j=1}^{k}\sigma_j^2\left(1 + \frac{1}{2}\sigma_j^2\right)\right]^{\frac{1}{2}}}[\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2, \sigma_1^4, \sigma_2^4, \dots, \sigma_k^4]'$$

(A.14)

The Probability-Matching equation (A.7) is therefore given by:

$$\sum_{j=1}^{k}\frac{\partial}{\partial\mu_j}\left\{\frac{\sigma_j^2}{\left[\sum_{j=1}^{k}\sigma_j^2\left(1+\frac{1}{2}\sigma_j^2\right)\right]^{\frac{1}{2}}}\pi(\boldsymbol{\theta})\right\}+\sum_{j=1}^{k}\frac{\partial}{\partial\sigma_j^2}\left\{\frac{\sigma_j^4}{\left[\sum_{j=1}^{k}\sigma_j^2\left(1+\frac{1}{2}\sigma_j^2\right)\right]^{\frac{1}{2}}}\right\}=0 \tag{A.15}$$

which has a solution given by

$$\pi(\boldsymbol{\theta}) = \pi(\mu_1,\dots,\mu_k,\sigma_1^2,\dots,\sigma_k^2) = \left[\sum_{j=1}^{k}\sigma_j^2\left(1+\frac{1}{2}\sigma_j^2\right)\right]^{\frac{1}{2}}\prod_{j=1}^{k}\sigma_j^{-4} \tag{A.16}$$

CHAPTER 2

Inference on the Mean: Single Sample

Introduction

As in Chapter 1, this chapter is primarily concerned with the analysis of lognormal data. The specification is that the populations of interest contain only non-zero, lognormally distributed observations. This is a simpler setting than the setting that will be proposed in Chapter 4, but it is convenient for highlighting certain aspects of the analysis of lognormal data.

We begin the chapter with a concise description of the setting as proposed by Krishnamoorthy and Mathew (2003). Given this background we proceed with a simulation study to compare the choice of various prior distributions for a single lognormal observation.

2.1 The Case of No Zero-Valued Observations

Inferences on the means of lognormal distributions using generalized p-values and generalized confidence intervals were proposed by Krishnamoorthy and Mathew (2003). The situation was as follows: suppose X follows a lognormal distribution, such that $Y = \ln(X) \sim N(\mu, \sigma^2)$, then the mean of X is defined by: $E(X) = E(\exp(Y)) = \exp(\eta)$, where $\eta = \mu + \frac{\sigma^2}{2}$. Now, let $X_1, X_2, \ldots, X_n$ be a random sample from this distribution. Many of the functions of interest with lognormal distributions are functions of both μ and σ^2. For example, the mean of the lognormal distribution is, as previously stated:

$$E(X) = E(\exp(Y)) = \exp(\eta)\text{, where } \eta = \mu + \frac{\sigma^2}{2}.$$

Thus, we can see that computations and confidence intervals for the mean of X are comparable to calculations and confidence intervals for the quantity η. Furthermore, we define the following sufficient statistics:

$$\hat{\mu} = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \text{ and } \hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2$$

where $\bar{y}$ and s^2 denote the observed values.

Krishnamoorthy and Mathew defined some of the following goals:

1. Obtain exact tests and confidence intervals for η using generalized p-values and generalized confidence intervals, with particular application to small samples. Generalized p-values and confidence intervals were developed by Tsui and Weerhandi (1989) and Weerhandi (1993) respectively. These references should be consulted for a detailed description of the methods involved.
2. Testing hypotheses and constructing confidence intervals for $\eta_1 - \eta_2$ were also discussed, where the two quantities are from independent lognormal populations with means $\exp(\eta_1)$ and $\exp(\eta_2)$. It is important to note that confidence intervals for $\eta_1 - \eta_2$ are essentially confidence intervals for the ratio of the means from two lognormally distributed populations.

The tests on the mean values and the confidence intervals were based on the so-called *generalized test variable* T_1. For the relevant conditions and criterion for this variable the interested reader is referred to the original text of Krishnamoorthy and Mathew (2003). This variable was defined as follows:

$$T_1 = \bar{y} - \frac{\bar{Y} - \mu}{S \big/ \sqrt{n}} s \big/ \sqrt{n} + \frac{\sigma^2}{2S^2} s^2 - \eta \tag{2.1}$$

where

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 .$$

This can further be written as:

$$T_1 = \bar{y} - \frac{Z}{U \big/ \sqrt{n-1}} \frac{s}{\sqrt{n}} + \frac{s^2}{2U^2 \big/ (n-1)} - \eta \tag{2.2}$$

where

$$Z = \sqrt{n}(\bar{Y} - \mu)/\sigma \sim N(0,1) \text{ independently of } U^2 = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1} .$$

Furthermore, the results obtained by Krishnamoorthy and Mathew (2003) were compared to those obtained by Angus (1994), where a parametric bootstrap method was used.

The first part of this chapter focuses on obtaining confidence intervals for η, except that a Bayesian perspective is considered, *i.e.* credibility intervals. Specifically, the construction of credibility intervals for the mean of a lognormally distributed population is considered and compared to the approach developed by Krishnamoorthy and Mathew (2003). It is interesting to note that due to the aforementioned specification the Bayesian procedure is "identical" to the method described by Krishnamoorthy and Mathew (2003) when the Independence Jeffreys Prior, $p(\mu, \sigma^2) \propto (\sigma^2)^{-1}$, is used. Therefore, we would expect this choice of prior distribution to provide similar results as those attained by Krishnamoorthy and Mathew (2003) and any potential differences can be considered to be "random". Various "*ad hoc*" priors were also assumed and confidence intervals and their corresponding coverage were analysed and compared to both the generalized test variable method and the parametric bootstrap method proposed in simulations by Angus (1994).

The following table represents the results obtained by Krishnamoorthy and Mathew (2003). The upper limits of the confidence intervals for η for the authors' method (as described briefly in the previous paragraphs and referred to as "KM") as well as results obtained using Land's (1971) formula and Angus' (1994) parametric bootstrap method are presented here for completeness.

Table 2: Upper Limits for η

n	σ	95% limits			99% limits		
		KM	**Land**	**Angus**	**KM**	**Land**	**Angus**
3	0.1	1.226	1.199	1.184	1.731	1.594	1.431
3	0.5	3.724	3.421	2.329	13.831	13.436	4.052
3	5	244.250	244.690	164.440	1242.41	1244.57	446.520
11	0.1	1.062	1.062	1.061	1.093	1.093	1.091
11	1	2.499	2.448	2.367	3.247	3.194	2.902
11	10	128.110	128.100	127.220	196.570	196.690	193.760
21	0.1	1.044	1.044	1.043	1.062	1.062	1.062
21	0.5	1.355	1.347	1.344	1.476	1.468	1.456
21	2	4.889	4.852	4.769	6.113	6.068	5.809
21	10	93.390	93.330	93.260	122.510	122.29	121.800
101	0.5	1.218	1.217	1.216	1.259	1.258	1.258
101	5	17.145	17.139	17.130	18.988	18.975	18.960
101	10	65.273	65.260	65.227	72.518	72.500	72.440
501	5	14.963	14.964	14.510	15.621	15.623	15.617
501	10	56.704	56.711	56.690	59.286	59.291	59.301
1001	5	14.510	14.510	14.510	14.951	14.954	14.953
1001	10	54.940	54.940	59.94	56.672	56.673	56.622

From the above table and as discussed by Krishnamoorthy and Mathew we can see that the method they proposed (generalized confidence intervals) gave very similar results to the confidence limits using Land's formula. The algorithm proposed by Angus (1994) had smaller limits than the other two methods for small sample sizes and large standard deviation values.

2.2 Intervals Based on a Bayesian Procedure

The following setting was introduced in Chapter 1. However, in this section it is adapted to the single sample case.

As in Chapter 1, let $y = \ln(X) \sim N(\mu, \sigma^2)$ then the likelihood function can be written as:

$$L(\mu, \sigma^2) \propto (\sigma^2)^{-\frac{n}{2}} exp\left\{-\frac{\sum_{i=1}^{n}(y_i - \mu)^2}{2\sigma^2}\right\} \quad (2.3)$$

2.3 The Independence Jeffreys Prior

Consider the first prior distribution:

$$p(\mu, \sigma^2) \propto (\sigma^2)^{-1}$$

Equation (2.3) is a location-scale model. Using the argument in Section 1.3.2 of Box and Tiao (1973) it follows that $p(\mu, \sigma^2) \propto (\sigma^2)^{-1}$.

Combining this prior distribution with the likelihood given by (2.3) results in the following posterior distribution:

$$p(\mu, \sigma^2 \mid data) = \left(\frac{2\pi\sigma^2}{n}\right)^{-\frac{1}{2}} \exp\left\{-\frac{n}{2\sigma^2}(\mu - \hat{\mu})^2\right\} \times \left\{\frac{\nu\hat{\sigma}^2}{2}\right\}^{\frac{1}{2}\nu} \left\{\frac{(\sigma^2)^{-\frac{1}{2}(\nu+2)} \exp\left(-\frac{\nu\hat{\sigma}^2}{2\sigma^2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right\} \quad (2.4)$$

where

$$\nu = n - 1$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$$

$$\hat{\sigma}^2 = \frac{1}{\nu}\sum_{i=1}^{n}(y_i - \hat{\mu})^2 = s^2$$

It follows from the posterior distribution (2.4) that μ has the following conditional distribution:

$$\mu \mid \sigma^2, data \sim N\left(\hat{\mu}, \frac{\sigma^2}{n}\right) \tag{2.5}$$

and the posterior density function for σ^2 is an Inverted Gamma density function. More specifically, the distribution can be written as:

$$p(\sigma^2 \mid data) = \left\{\frac{\nu\hat{\sigma}^2}{2}\right\}^{\frac{1}{2}\nu} \left\{\frac{(\sigma^2)^{-\frac{1}{2}(\nu+2)} \exp\left\{-\frac{\nu\hat{\sigma}^2}{2\sigma^2}\right\}}{\Gamma\left(\frac{\nu}{2}\right)}\right\}. \tag{2.6}$$

To obtain credibility intervals for this Bayesian procedure Monte Carlo simulation is applied.

2.4 Simulation Procedure

From the preceding derivations and similarly to the methodology introduced in Chapter 1, the following 'algorithm', or method, was applied using the MATLAB® package.

For given μ, σ^2 and n the procedure is as follows:

1. Since σ^2 is known, in accordance with Krishnamoorthy and Mathew (2003), set $\mu = -\frac{1}{2}\sigma^2$.

2. With these initial parameters we can simulate a sample of observations. From this the following can be calculated:

$\bar{Y}$ and $m = \nu\hat{\sigma}^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$, where $\nu = n-1$.

3. However, with respect to step 2, since we are only interested in the sufficient statistics these can be simulated directly, namely:

$\bar{Y} \sim N(\mu, \sigma^2/n)$ and $\frac{m}{\sigma^2} \sim \chi_\nu^2$

$\therefore m = \sigma^2(\chi_\nu^2)$

4. Given these calculated values we can simulate μ and σ^2 out of their posterior distributions (refer to previous section) as follows:

   a. $\sigma^2 = \frac{m}{\chi_\nu^2}$

   b. $\mu \mid \sigma^2 \sim N(\hat{\mu}, \frac{\sigma^2}{n})$

   c. $\eta = \mu + \frac{1}{2}\sigma^2$

   d. For this experiment/sample, simulate $l = 1000$ values of η.

5. Sort them in ascending order such that $\eta^*_{(1)} \le \eta^*_{(2)} \le \ldots \le \eta^*_{(l)}$.

6. Let $K_1 = \left[\frac{\alpha}{2}l\right]$ and $K_2 = \left[\left(1-\frac{\alpha}{2}\right)l\right]$ where $[a]$ denotes the largest integer not greater than a.

7. $\left\{\eta^*_{(k_1)}, \eta^*_{(k_2)}\right\}$ is then a $100(1-\alpha)\%$ Bayesian confidence interval for η.

8. Repeat the procedure for 1000 experiments.

Various values for n, σ^2 were considered, for $\mu = -\sigma^2/2$, as specified in the following table:

Table 1: Parameter Settings Used in the Simulation

n	σ^2
5	0.5
5	1
5	5
5	20
10	0.5
10	1
10	5
10	20
15	0.5
15	1
15	5
15	20
25	0.5
25	1
25	5
25	20
25	100

2.5 Other Prior Distributions

As previously mentioned, this procedure was repeated for three other prior distributions as well. Obviously the form of the posterior distribution, similar to (2.4), will change for each prior distribution. The simulation procedure is similar to that for posterior distribution (2.4) except that (given the choice of prior distributions) σ^2 is distributed from a central chi-squared distribution with the following degrees of freedom:

Table 2: Prior Distributions and Simulation Parameters

Name	Specification	Simulation
Prior 2	$p(\mu,\sigma^2) \propto \sigma^{-3}$	$\sigma^2 = \frac{m}{\chi^2_{v+1}}$
Prior 3	$p(\mu,\sigma^2) \propto Const$	$\sigma^2 = \frac{m}{\chi^2_{v-2}}$
Prior 4	$p(\mu,\sigma^2) \propto \sigma^{-1}$	$\sigma^2 = \frac{m}{\chi^2_{v-1}}$

Prior 2 is the Jeffreys Rule prior, which is the square root of the determinant of the Fisher Information matrix and Prior 3 is the uniform prior.

2.6 Results of Simulation Study – Single Variable

The following table compares the results obtained by Krishnamoorthy and Mathew (2003), indicated by "KM" in the below tables, to the Bayesian credibility coverage probabilities obtained for the various "*ad hoc*" priors (denoted by Prior 1 $[\sigma^{-2}]$, Prior 2 $[\sigma^{-3}]$, Prior 3 $[Const]$ and Prior 4 $[\sigma^{-1}]$).

Table 3: Results for a Single Mean

n	σ^2	Method	Coverage Probability $\alpha = 90\%$	Coverage Probability $\alpha = 95\%$	n	σ^2	Method	Coverage Probability $\alpha = 90\%$	Coverage Probability $\alpha = 95\%$
5	0.5	KM	0.895	0.947	15	0.5	KM	0.899	0.948
		Prior 1	0.903	0.959			Prior 1	0.892	0.950
		Prior 2	0.908	0.952			Prior 2	0.895	0.955
		Prior 3	0.897	0.954			Prior 3	0.905	0.949
		Prior 4	0.900	0.943			Prior 4	0.899	0.946
5	1	KM	0.880	0.949	15	1	KM	0.896	0.947
		Prior 1	0.895	0.956			Prior 1	0.904	0.942
		Prior 2	0.889	0.943			Prior 2	0.900	0.955
		Prior 3	0.892	0.955			Prior 3	0.913	0.949
		Prior 4	0.888	0.958			Prior 4	0.907	0.949
5	5	KM	0.893	0.948	15	5	KM	0.892	0.951
		Prior 1	0.903	0.947			Prior 1	0.917	0.946
		Prior 2	0.903	0.950			Prior 2	0.896	0.948
		Prior 3	0.888	0.949			Prior 3	0.902	0.961
		Prior 4	0.903	0.953			Prior 4	0.911	0.953
5	20	KM	0.897	0.948	15	20	KM	0.901	0.950
		Prior 1	0.889	0.954			Prior 1	0.904	0.947
		Prior 2	0.906	0.945			Prior 2	0.890	0.949
		Prior 3	0.894	0.947			Prior 3	0.900	0.944
		Prior 4	0.897	0.941			Prior 4	0.904	0.941
10	0.5	KM	0.895	0.952	25	0.5	KM	0.904	0.950
		Prior 1	0.919	0.951			Prior 1	0.898	0.951
		Prior 2	0.894	0.949			Prior 2	0.885	0.947
		Prior 3	0.889	0.945			Prior 3	0.900	0.944
		Prior 4	0.902	0.955			Prior 4	0.877	0.937
10	1	KM	0.897	0.950	25	1	KM	0.895	0.946
		Prior 1	0.901	0.946			Prior 1	0.908	0.957
		Prior 2	0.885	0.949			Prior 2	0.898	0.936
		Prior 3	0.910	0.950			Prior 3	0.893	0.955
		Prior 4	0.902	0.950			Prior 4	0.891	0.942
10	5	KM	0.899	0.950	25	5	KM	0.900	0.947
		Prior 1	0.891	0.949			Prior 1	0.904	0.950
		Prior 2	0.902	0.937			Prior 2	0.898	0.949
		Prior 3	0.899	0.953			Prior 3	0.888	0.934
		Prior 4	0.900	0.953			Prior 4	0.886	0.957
10	20	KM	0.901	0.949	25	20	KM	0.909	0.949
		Prior 1	0.891	0.965			Prior 1	0.898	0.957
		Prior 2	0.899	0.946			Prior 2	0.904	0.950
		Prior 3	0.884	0.954			Prior 3	0.906	0.948
		Prior 4	0.896	0.943			Prior 4	0.910	0.948
					25	100	KM	0.897	0.947
							Prior 1	0.913	0.947
							Prior 2	0.886	0.948
							Prior 3	0.901	0.948
							Prior 4	0.907	0.941

From these results we can see that the Bayesian procedures employed provide accurate results that are comparable to those obtained by Krishnamoorthy and Mathew (2003) using their technique of generalized p-values. In fact, the Bayesian results using the prior

$p(\mu, \sigma^2) \propto (\sigma^2)^{-1}$ should be exactly the same as those obtained by Krishnamoorthy and Mathew (2003) using the technique of generalized p-values. However, there is not much variation in the results for the choice of prior. The Bayesian methods provided no distinction in results between the priors. In general, the methods were comparable.

To accurately distinguish between the prior distributions used, additional measures are required. For example, the following characteristics were studied:

- coverage probabilities
- average interval lengths
- coverage error (target coverage – actual coverage),
- percentages of under-coverage on both sides ($\%BCI < \theta$ and $\%BCI > \theta$), where BCI is the Bayesian Confidence Interval or Credibility Interval.
- relative bias $\frac{|\%BCI < \theta - \%BCI > \theta|}{(\%BCI < \theta + \%BCI > \theta)}$.

Thus, the simulation results were run again and yielded the following results (for $\alpha = 0.05$):

Table 4: Aggregated Results for a Single Mean

n	σ^2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
5	0.5	Prior 1	0.951	-0.001	2.115	0.021	0.028	0.143
5	0.5	Prior 2	0.942	0.008	1.604	0.033	0.025	0.138
5	0.5	Prior 3	0.938	0.012	9.381	0.029	0.033	0.065
5	0.5	Prior 4	0.955	-0.005	3.464	0.020	0.025	0.111
5	1	Prior 1	0.947	0.003	4.366	0.023	0.030	0.132
5	1	Prior 2	0.956	-0.006	3.106	0.016	0.028	0.273
5	1	Prior 3	0.956	-0.006	20.913	0.022	0.022	0.000
5	1	Prior 4	0.947	0.003	7.334	0.027	0.026	0.019
5	5	Prior 1	0.951	-0.001	20.890	0.020	0.029	0.184
5	5	Prior 2	0.947	0.003	15.410	0.025	0.028	0.057
5	5	Prior 3	0.952	-0.002	97.644	0.027	0.021	0.125
5	5	Prior 4	0.962	-0.012	33.507	0.021	0.017	0.105
5	20	Prior 1	0.948	0.002	83.747	0.032	0.020	0.231
5	20	Prior 2	0.939	0.011	63.785	0.031	0.030	0.016
5	20	Prior 3	0.951	-0.001	385.677	0.027	0.022	0.102
5	20	Prior 4	0.942	0.008	147.256	0.031	0.027	0.069
10	0.5	Prior 1	0.957	-0.007	0.764	0.021	0.022	0.023
10	0.5	Prior 2	0.945	0.005	0.709	0.028	0.027	0.018
10	0.5	Prior 3	0.948	0.002	0.958	0.024	0.028	0.077
10	0.5	Prior 4	0.943	0.007	0.855	0.029	0.028	0.018
10	1	Prior 1	0.952	-0.002	1.525	0.028	0.020	0.167
10	1	Prior 2	0.945	0.005	1.404	0.032	0.023	0.164
10	1	Prior 3	0.952	-0.002	1.966	0.024	0.024	0.000
10	1	Prior 4	0.956	-0.006	1.692	0.018	0.026	0.182
10	5	Prior 1	0.946	0.004	7.643	0.028	0.026	0.037
10	5	Prior 2	0.941	0.009	7.072	0.029	0.030	0.017
10	5	Prior 3	0.943	0.007	9.650	0.025	0.032	0.123
10	5	Prior 4	0.952	-0.002	8.582	0.027	0.021	0.125
10	20	Prior 1	0.942	0.008	29.944	0.030	0.028	0.034
10	20	Prior 2	0.961	-0.011	27.576	0.019	0.020	0.026
10	20	Prior 3	0.944	0.006	40.287	0.024	0.032	0.143
10	20	Prior 4	0.959	-0.009	33.955	0.019	0.022	0.073
15	0.5	Prior 1	0.952	-0.002	0.510	0.023	0.025	0.042
15	0.5	Prior 2	0.948	0.002	0.479	0.026	0.026	0.000
15	0.5	Prior 3	0.946	0.004	0.576	0.030	0.024	0.111
15	0.5	Prior 4	0.954	-0.004	0.539	0.025	0.021	0.087
15	1	Prior 1	0.949	0.001	1.022	0.021	0.030	0.176
15	1	Prior 2	0.954	-0.004	0.971	0.020	0.026	0.130
15	1	Prior 3	0.949	0.001	1.154	0.025	0.026	0.020
15	1	Prior 4	0.948	0.002	1.089	0.032	0.020	0.231
15	5	Prior 1	0.956	-0.006	5.103	0.024	0.020	0.091
15	5	Prior 2	0.961	-0.011	4.841	0.017	0.022	0.128
15	5	Prior 3	0.936	0.014	5.773	0.030	0.034	0.063
15	5	Prior 4	0.948	0.002	5.496	0.028	0.024	0.077
15	20	Prior 1	0.957	-0.007	20.794	0.026	0.017	0.209
15	20	Prior 2	0.960	-0.010	19.204	0.019	0.021	0.050
15	20	Prior 3	0.947	0.003	22.817	0.026	0.027	0.019
15	20	Prior 4	0.955	-0.005	21.808	0.026	0.019	0.156

n	σ^2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
25	0.5	Prior 1	0.945	0.005	0.342	0.030	0.025	0.091
25	0.5	Prior 2	0.943	0.007	0.329	0.017	0.040	0.404
25	0.5	Prior 3	0.937	0.013	0.364	0.036	0.027	0.143
25	0.5	Prior 4	0.944	0.006	0.355	0.026	0.030	0.071
25	1	Prior 1	0.957	-0.007	0.687	0.022	0.021	0.023
25	1	Prior 2	0.958	-0.008	0.665	0.019	0.023	0.095
25	1	Prior 3	0.940	0.010	0.729	0.032	0.028	0.067
25	1	Prior 4	0.948	0.002	0.713	0.025	0.027	0.038
25	5	Prior 1	0.959	-0.009	3.405	0.019	0.022	0.073
25	5	Prior 2	0.948	0.002	3.341	0.022	0.030	0.154
25	5	Prior 3	0.942	0.008	3.589	0.020	0.038	0.310
25	5	Prior 4	0.945	0.005	3.479	0.022	0.033	0.200
25	20	Prior 1	0.952	-0.002	13.569	0.026	0.022	0.083
25	20	Prior 2	0.951	-0.001	13.126	0.020	0.029	0.184
25	20	Prior 3	0.957	-0.007	14.577	0.021	0.022	0.023
25	20	Prior 4	0.953	-0.003	13.902	0.024	0.023	0.021
25	100	Prior 1	0.968	-0.018	67.982	0.017	0.015	0.063
25	100	Prior 2	0.948	0.002	64.913	0.025	0.027	0.038
25	100	Prior 3	0.955	-0.005	71.396	0.014	0.031	0.378
25	100	Prior 4	0.947	0.003	70.425	0.030	0.023	0.132

At this preliminary stage, we examine the overall results of the different choices of prior distributions for a single mean from a lognormally distributed population. The table to follow presents the summative results of Table 45 in order to compare the results for the different prior distributions. Due to the results presented in Krishnamoorthy and Mathew (2003) their results can only be compared for the coverage probability and none of the other performance statistics.

Table 5: Summary Results for the Case of the Single Mean

Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
KM $\alpha = 0.95$	0.9487	N/A	N/A	N/A	N/A	N/A
Prior 1	0.9523	-0.0023	15.5534	0.0242	0.0235	0.1060
Prior 2	0.9498	0.0002	13.4432	0.0234	0.0268	0.1113
Prior 3	0.9466	0.0034	40.4383	0.0256	0.0277	0.1039
Prior 4	0.9505	-0.0005	20.8500	0.0253	0.0242	0.1009

The results for Prior 1 and Prior 2 both seem to compare well with the results obtained by Krishnamoorthy and Mathew (2003). For Prior 1 this is less surprising than what it is necessary, since as explained earlier this prior essentially results in the same procedure as used by Krishnamoorthy and Mathew (2003).

The coverage obtained by Prior 3 does not appear to be adequate. The poor coverage of this prior is even further highlighted when one considers the average length of the confidence intervals. We can see that Prior 3 is inefficient and results in wide intervals. For this prior we can also see a tendency to over-cover on the right hand side of the distribution. Also, by the same reasoning we note that even though Prior 4 results in good coverage the interval length is too wide and thus indicates inefficiency. These two priors will therefore not be discussed further in this setting.

Thus, we focus on Priors 1 and 2. We can see that Prior 1 (essentially the same as the KM approach) results in good coverage of the interval. The same is true for Prior 2 and despite "randomness" arising from the simulation procedure, we can regard these two priors as having similar performance characteristics. No discernable difference was observed. What is interesting, and perhaps surprising, is that Prior 2 (the Jeffreys Rule prior) achieves this coverage while at the same time resulting in a noticeably smaller interval width. The coverage probability and the Average Length can be regarded as the primary distinguishing characteristics between these priors distributions and thus we conclude that Prior 2 is a more appropriate distribution. The difference in Average

Length is more pronounced in situations of larger variance and even more so when a larger sample size is considered.

2.7 Comparison to the MOVER

Instead of adapting a simulation approach for making inferences on the lognormal mean, Zou, Taleban and Huo (2009a) proposed procedures involving the so-called "method of variance estimates recovery" (MOVER). The MOVER method was designed in order to apply to a general scenario and also to provide adequate coverage rates in estimation procedures relating to lognormally distributed data. The advantage of the MOVER is therefore that it is easily applicable to many different settings with little more than a basic knowledge of introductory statistical text.

The object of the paper by Zou et al (2009a, page 3760) was to demonstrate the MOVER in different scenarios, i.e. for a few different combinations of n and σ^2, where $\mu = -\frac{1}{2}\sigma^2$. The Bayesian confidence intervals and these will be compared to the results from the MOVER method and generalized confidence interval procedure to evaluate the performance of the Bayesian confidence intervals for both the equal-tailed and HPD (highest posterior density) intervals. The following characteristics are reported:

- coverage probabilities
- average interval lengths

A nominal significance level of $\alpha = 0.05$ will be used for each parameter setting.

The confidence limits for the MOVER, as given by Zou et al (2009a) on page 3758, are:

$$l = \hat{\mu} + \frac{\hat{\sigma}^2}{2} - \sqrt{Z^2_{\alpha/2}\frac{\hat{\sigma}^2}{n}\left\{\frac{\hat{\sigma}^2}{2}\left(1 - \frac{\nu}{\chi^2_{1-\alpha/2,\nu}}\right)\right\}^2}$$

$$u = \hat{\mu} + \frac{\hat{\sigma}^2}{2} + \sqrt{Z^2_{\alpha/2}\frac{\hat{\sigma}^2}{n}\left\{\frac{\hat{\sigma}^2}{2}\left(\frac{\nu}{\chi^2_{\alpha/2,\nu}} - 1\right)\right\}^2}$$

where $Z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution and $\chi^2_{\alpha/2,\nu}$ is the $\alpha/2$ percentile of the chi-squared distribution with ν degrees of freedom.

As mentioned in equation (2.2) the simulation procedure for the generalized confidence interval method can be summarized as follows:

$$\gamma^* = lnM^* = \hat{\mu} + \frac{Z^*}{\sqrt{\frac{\tau^*}{\nu}}}\frac{\hat{\sigma}}{n} + \frac{\hat{\sigma}^2}{\frac{2\tau^*}{\nu}}$$

where

$Z^* \sim N(0,1)$ and $\tau^* \sim \chi^2_\nu$, i.e. simulated values.

The following table presents the summary statistics of the results in both the Zou et al (2009a) simulation study and the Bayesian simulation study using Jeffreys' Independence prior.

The same designs are used as considered by Zou et al (2009a), where (< , >)% refers to the proportion of cases where the interval is below or above the true value respectively:

Table 6: Comparison of the MOVER and Independence Jeffreys' Prior for Constructing Two-sided 95% Confidence Intervals for $\mu+\frac{1}{2}\sigma^2$

		MOVER		GCI		Jeffreys' (Equal-tail)		Jeffreys' (HPD)	
n	σ^2	Cover (<, >)%	Width	Cover (<, >)%	Width	Cover (<, >)%	Width	Cover (<, >)%	Width
5	0.5	93.47 (3.19, 3.34)	2.54	93.99 (2.00, 4.01)	2.69	94.31 (1.87, 3.82)	2.68	95.67 (3.01, 1.32)	2.27
	1.0	94.65 (2.75, 2.60)	4.66	93.98 (1.92, 4.10)	4.78	93.81 (2.16, 4.03)	4.73	95.69 (3.27, 1.04)	3.80
	1.5	95.03 (2.92, 2.05)	6.67	94.15 (2.07, 3.78)	6.76	93.55 (2.09, 4.36)	6.75	95.83 (3.48, 0.69)	5.26
	2.0	95.10 (2.99, 1.91)	8.76	93.77 (2.36, 3.87)	8.82	94.13 (2.11, 3.76)	8.68	95.70 (3.66, 0.64)	6.64
	2.5	95.30 (2.89, 1.81)	10.55	94.08 (2.38, 3.54)	10.59	94.21 (2.14, 3.65)	10.82	95.84 (3.80, 0.36)	8.16
	3.0	95.35 (2.60, 2.05)	12.72	93.90 (2.09, 4.01)	12.74	94.35 (2.07, 3.58)	12.71	95.81 (3.68, 0.51)	9.50
20	0.5	94.24 (3.37, 2.39)	0.74	94.56 (2.35, 3.09)	0.76	94.56 (2.29, 3.15)	0.77	95.06 (3.00, 1.94)	0.75
	1.0	95.19 (2.87, 1.94)	1.20	94.90 (2.17, 2.93)	1.22	95.09 (1.88, 3.03)	1.22	95.76 (2.76, 1.48)	1.19
	1.5	94.76 (3.04, 2.20)	1.63	94.36 (2.40, 3.24)	1.64	94.77 (2.26, 2.97)	1.65	95.32 (3.28, 1.40)	1.59
	2.0	94.94 (2.89, 2.17)	2.04	94.39 (2.39, 3.22)	2.06	95.16 (2.30, 2.54)	2.05	95.77 (3.30, 0.93)	1.96
	2.5	95.24 (2.61, 2.15)	2.43	94.89 (2.14, 2.97)	2.44	94.98 (2.27, 2.75)	2.54	95.40 (3.48, 1.12)	2.39
	3.0	95.41 (2.82, 1.77)	2.86	94.97 (2.37, 2.66)	2.87	95.59 (2.36, 3.05)	2.86	95.06 (3.62, 1.32)	2.71

In comparison to the MOVER confidence intervals, the equal-tailed intervals also seem to compare reasonably well, with insignificant differences in both the proportion of confidence intervals above and below the true parameter, but the width of the intervals are larger than those of the MOVER. Naturally, as the sample size increases the width of the interval tends to decrease. On the other hand, if the variance increases the width of the intervals will also increase.

Thus, the equal-tailed intervals do not offer an improvement on the MOVER method. However, when considering the HPD (highest posterior density) intervals, which are only possible through the Bayesian framework in this setting, a large improvement on the MOVER can be gained, particularly when n is small. Of particular interest to note is that these HPD intervals result in considerable reductions in interval width. Also of note is that the proportion of intervals above the true parameter is considerably less than both the MOVER and the equal-tailed intervals.

So, the performance of the Independence Jeffreys prior is comparable (or improved for HPD intervals) to the MOVER. However, in terms of the literature, Box and Tiao

(1973), this would be the natural choice of prior distribution in this setting and thus, its accuracy is an expected result. In addition to the previously mentioned priors the following prior distributions will also be tested:

$$p_R\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma}\sqrt{1+\frac{2}{\sigma^2}} \tag{2.7}$$

$$p_P\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma^2}\sqrt{1+\frac{2}{\sigma^2}} \tag{2.8}$$

Prior distributions (2.7) and (2.8) are the Reference and Probability-Matching priors respectively, the derivations of which will be discussed and provided in the appendix to this chapter.

The table below represents the results of the Reference and Probability-Matching prior distributions, once again compared to the MOVER. However, these were only performed for the extreme values of σ^2 in Table 7. Also, only the coverage and interval widths are presented.

Table 7: Comparison of the MOVER and Other Prior Distributions for Constructing Two-sided 95% Confidence Intervals for $\mu+\frac{1}{2}\sigma^2$

n	σ^2	Prior / Method	Equal-Tailed / MOVER		HPD Intervals	
			Cover %	Width	Cover %	Width
5	0.5	MOVER	93.47	2.54	N/A	N/A
		$p(\theta) \propto \sigma^{-3}$ (Jeffreys Rule)	91.36	1.82	91.68	1.66
		Reference Prior	94.79	3.36	96.76	2.79
		Probability-Matching Prior	92.27	2.09	93.30	1.89
5	3	MOVER	95.35	12.72	N/A	N/A
		$p(\theta) \propto \sigma^{-3}$ (Jeffreys Rule)	91.54	7.68	91.23	6.27
		Reference Prior	93.57	11.89	96.41	10.09
		Probability-Matching Prior	92.76	8.68	93.35	7.37
20	0.5	MOVER	94.24	0.74	N/A	N/A
		$p(\theta) \propto \sigma^{-3}$ (Jeffreys Rule)	94.39	0.74	94.14	0.72
		Reference Prior	94.88	0.77	95.26	0.76
		Probability-Matching Prior	94.74	0.74	94.81	0.73
20	3	MOVER	95.41	2.86	N/A	N/A
		$p(\theta) \propto \sigma^{-3}$ (Jeffreys Rule)	94.59	2.66	94.21	2.54
		Reference Prior	94.71	2.99	95.66	2.84
		Probability-Matching Prior	94.78	2.78	94.95	2.65

It appears as though the coverage of the other priors is not as good as the Independence Jeffreys' prior, particularly for small sample sizes. However, as the sample size increases the effect of the prior distribution seems to decrease and the results are comparable.

From Table 8 it is also clear that the Reference prior seems to have better coverage than the Probability-Matching prior. It must be remembered that the Probability-Matching prior is derived for one-sided credibility intervals. This might be the reason for undercoverage if n is small.

Appendix to Chapter 2

As mentioned before Probability-Matching and Reference priors often lead to procedures with good frequency properties while retaining the Bayesian flavor. The fact that the resulting posterior intervals of level $1-\alpha$ are also good frequentist intervals at the same level is a very desirable situation.

The results of Datta and Ghosh (1995) will be briefly reviewed in the following theorem:

Theorem 1

For the mean, $M = e^{\mu+\frac{1}{2}\sigma^2}$, of the lognormal distribution, the Probability-Matching prior is given by:

$$p_P\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma^2}\sqrt{1+\frac{2}{\sigma^2}}. \tag{A.1}$$

Equation (A.1) is a special case of Theorem 1.3 (from Chapter 1). If $k=1$ is substituted in (1.22) the result follows.

Multiplying (A.1) by the likelihood function (2.3) it follows that:

$$p_P\left(\sigma_j^2|data\right) \propto \left(\sigma_j^2\right)^{-\frac{1}{2}(\nu_{j1}+2)}\left(1+\frac{2}{\sigma_j^2}\right)^{\frac{1}{2}} exp\left[-\frac{\nu_{j1}\hat{\sigma}_j^2}{2\sigma_j^2}\right]$$

for $j = 1,2$

(A.2)

Simulation from (A.2) can be obtained using the rejection method. Simulation of μ_j and δ_j are as before.

The Reference prior for $M \;=\; e^{\mu+\frac{1}{2}\sigma^2}$

The determination of reasonable, non-informative priors in multiparameter problems is not easy; common non-informative priors, such as Jeffreys' prior, can have features that have an unexpectedly dramatic effect on the posterior distribution. In recognition of this problem Berger and Bernardo (1992) proposed the *Reference prior* approach to the development of non-informative priors. As in the case of the Jeffreys and Probability-Matching priors, the Reference prior method is derived from the Fisher information matrix. Reference priors depend on the group ordering of the parameters. Berger and Bernardo (1992) suggested that multiple groups, ordered in terms of inferential importance, are allowed, with the Reference prior being determined through a succession of analyses for the implied conditional problems. They particularly recommend the Reference prior based on having each parameter in its own group, i.e. having each conditional Reference prior be only one dimensional.

As mentioned by Pearn and Wu (2005) the Reference prior maximises the difference in information (entropy) about the parameter provided by the prior and posterior distributions. In other words, the Reference prior is derived in such a way that it provides as little as possible information about the parameter.

The following theorem can now be stated.

Theorem 2

For the mean, $M = e^{\mu+\frac{1}{2}\sigma^2}$, of the lognormal distribution, the Reference prior relative to the ordered parameterisation (μ,σ^2) is given by:

$$p_R(\mu,\sigma^2) \propto \frac{1}{\sigma}\sqrt{1+\frac{2}{\sigma^2}}. \tag{A.3}$$

Proof:

The Fisher information matrix of $\boldsymbol{\theta} = [\mu,\sigma^2]$ per unit observation is given by:

$$F(\boldsymbol{\theta}) = F(\mu,\sigma^2) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{bmatrix}.$$

The parameter of interest is the mean of the lognormal distribution

$$t(\boldsymbol{\theta}) = e^{\mu+\frac{1}{2}\sigma^2} = M$$

Define $A = \dfrac{\partial(\mu,\sigma^2)}{\partial(t(\boldsymbol{\theta}),\sigma^2)} = \begin{bmatrix} 1/t(\boldsymbol{\theta}) & -1/2 \\ 0 & 1 \end{bmatrix}.$

Hence, the Fisher information matrix under the reparameterisation $(t(\boldsymbol{\theta}),\sigma^2)$ is given by

$$F(t(\boldsymbol{\theta}),\sigma^2) = A'F(\mu,\sigma^2)A = \begin{bmatrix} \dfrac{1}{t^2(\boldsymbol{\theta})\sigma^2} & \dfrac{-1}{2t(\boldsymbol{\theta})\sigma^2} \\ \dfrac{-1}{2t(\boldsymbol{\theta})\sigma^2} & \dfrac{1}{4\sigma^2}+\dfrac{1}{2\sigma^4} \end{bmatrix}.$$

Following the notation of Berger and Bernardo (1992), the functions $h_j, (j=1,2)$, which are needed to calculate the Reference prior for the group ordering $(t(\theta), \sigma^2)$, can be obtained from $F(t(\theta), \sigma^2)$ as follows:

$$h_1^{\frac{1}{2}} = \left| \frac{1}{t^2(\theta)\sigma^2} - \left(\frac{-1}{2t(\theta)\sigma^2} \right)^2 \left(\frac{1}{4\sigma^2} + \frac{1}{2\sigma^4} \right)^{-1} \right|^{\frac{1}{2}} = \frac{1}{t(\theta)} \left(\frac{1}{\sigma^2} - \frac{1}{2+\sigma^2} \right)^{\frac{1}{2}}$$

and

$$h_2^{\frac{1}{2}} = \left[\frac{1}{2\sigma^2} \left(\frac{1}{2} + \frac{1}{\sigma^2} \right) \right]^{\frac{1}{2}}.$$

Therefore, the Reference prior relative to the ordered parameterisation $(t(\theta), \sigma^2)$ is given by

$$p_R(t(\theta), \sigma^2) \propto \frac{1}{t(\theta)} \frac{1}{\sigma} \sqrt{1 + \frac{2}{\sigma^2}}.$$

In the (μ, σ^2) parameterisation this corresponds to

$$p_R(\mu, \sigma^2) \propto \frac{1}{t(\theta)} \frac{1}{\sigma} \sqrt{1 + \frac{2}{\sigma^2}} \left(t(\theta) \right) \propto \frac{1}{\sigma} \sqrt{1 + \frac{2}{\sigma^2}}.$$

This is the same result derived by Roman (2008).

The posterior distribution of σ_j^2 is now

$$p_P(\sigma_j^2 | data) \propto (\sigma_j^2)^{-\frac{1}{2}(v_{j1}+1)} \left(1 + \frac{2}{\sigma_j^2} \right)^{\frac{1}{2}} exp\left[-\frac{v_{j1}\hat{\sigma}_j^2}{2\sigma_j^2} \right]$$

for $j = 1,2$

(A.4)

CHAPTER 3

Inference on the Mean: Two Samples

Introduction

The problem setting for this chapter is similar to that of Chapter 2. We consider the same type of data (*i.e.* lognormally distributed observations without zero observations) as well as the same prior distributions. The point of departure in Chapter 3 from the previous chapter is that we consider lognormally distributed observations from two different populations. For analysis the ratio of the two means from the populations was the statistic of choice.

Similar to Chapter 2 though, a simulation study was undertaken to examine the effectiveness and efficiency of the various choices of prior distributions. This chapter begins with a description of the problem setting, but full details of all procedures are not necessarily supplied, since this setting is merely an extension of the setting found in the previous chapter.

Finally, we end the chapter with practical examples. There is however, a point of departure from the situation described by Krishnamoorthy and Mathew (2003) with regards to the analysis of the data. Research (Fernandez, C. and Steel F.J.M. [1999], Roman, L [2008] and Berger and Bernardo [1992]) suggests that when considering the prediction of a future value (specifically from a lognormal distribution) then the median and not the mean of the distributions is a more appropriate quantity of interest. This will be examined by means of an example.

3.1 The Case of No Zero-Valued Observations – Ratio of Means

As mentioned in the introduction to this chapter, the ratio of means from two different populations can also be examined. Due to the problem specification the ratio between these two means can be written as: $\eta_1 - \eta_2$. Thus, it is an easy matter to extend the simulation study applied earlier to this difference. Using the simulation methods mentioned before, first one population mean is simulated, then the next and finally they are subtracted from each other. The credibility intervals can then be calculated from this "differenced" data. Thus, the applicable steps and procedures are not re-stated here.

3.2 Ratio of Means from Two Different Populations - Results

Several combinations of differing population sizes were simulated to analyse the applicability of the method to both small and large samples of data. The following table outlines these population size specifications:

Table 8: Sample Size Settings Used in the Simulation Study

n_1	n_2
10	10
10	25
25	10
25	25
25	50
25	100
50	25
50	50
50	100
100	25
100	50
100	100

Furthermore, for each of the four prior distributions mentioned in Sections 2.3 and 2.5 and for the above sample sizes, populations with different designs, that is, different variances, σ_1^2 and σ_2^2, were simulated and analysed. The following table represents these designs chosen:

Table 9: Design Specification

Design	σ_1^2	σ_2^2
Design 1	3	1
Design 2	4	4
Design 3	2	0.5
Design 4	2	12
Design 5	30	4

Lastly, as was specified for the case of a single mean, the following was assumed for each population: $\mu = -\sigma^2/2$.

The results for each individual design are given in the appendix to this chapter as Tables 21 to 25.

As mentioned previously, one advantage of the Bayesian framework is the construction of HPD intervals. This is the shortest interval that gives the required coverage. The results for HPD intervals are again presented in the appendix as Tables 26 to 30.

In Chapter 2 another method for constructing confidence intervals was introduced, called the MOVER. In the case of the MOVER the $(1-\alpha)100\%$ confidence limits for $lnM_1 - lnM_2$ are given by

$$L = \hat{\theta}_1 - \hat{\theta}_2 - \sqrt{\left(\hat{\theta}_1 - l_1\right)^2 + \left(u_2 - \hat{\theta}_2\right)^2}$$

$$U = \hat{\theta}_1 - \hat{\theta}_2 + \sqrt{\left(\hat{\theta}_2 - l_2\right)^2 + \left(u_1 - \hat{\theta}_1\right)^2}$$

where

$$\hat{\theta}_i = \hat{\mu}_i + \frac{1}{2}\hat{\sigma}_i^2$$

and l_i and u_i $(i = 1,2)$ are defined in Section 2.6. The results for the various designs are given in the appendix to this chapter as Table 31. However, aggregated results are given later in this section for the MOVER to aid comparison with the Bayesian methods.

Furthermore, certain parameter settings for the simulation study were suggested by Krishnamoorthy and Mathew (2003). These settings were also simulated using the Bayesian procedures and are referred to as "KM Orig" in the summary table. The results for these designs are presented in the appendix to this chapter as Table 32, except that the MOVER and HPD intervals will not be supplied for these parameter settings.

In the following table, we examine the overall results of the different choices of prior distributions. It presents the summative results of Tables 21 to 32 (given in the appendix to this chapter) in order to compare the results for the different prior distributions.

Table 10: Summary Results for the Case of the Ratios of Two Means – Equal Tail Intervals

Method	Prior	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1	Prior 1	0.9506	-0.0006	2.1963	0.0227	0.0268	0.1294
	Prior 2	0.9473	0.0028	2.0908	0.0218	0.0310	0.2361
	Prior 3	0.9528	-0.0028	2.4964	0.0204	0.0268	0.1390
	Prior 4	0.9498	0.0003	2.3439	0.0199	0.0303	0.2307
Design 2	Prior 1	0.9513	-0.0013	4.1490	0.0223	0.0265	0.1244
	Prior 2	0.9492	0.0008	3.9165	0.0264	0.0244	0.1086
	Prior 3	0.9449	0.0051	4.7739	0.0257	0.0294	0.1637
	Prior 4	0.9503	-0.0003	4.4297	0.0238	0.0259	0.0750
Design 3	Prior 1	0.9516	-0.0016	1.4060	0.0200	0.0284	0.1864
	Prior 2	0.9509	-0.0009	1.3396	0.0209	0.0282	0.2181
	Prior 3	0.9493	0.0008	1.6032	0.0208	0.0300	0.2297
	Prior 4	0.9510	-0.0010	1.5023	0.0207	0.0283	0.1862
Design 4	Prior 1	0.9517	-0.0017	8.2989	0.0247	0.0237	0.1186
	Prior 2	0.9489	0.0011	7.8213	0.0297	0.0214	0.1772
	Prior 3	0.9455	0.0045	9.4078	0.0291	0.0254	0.1158
	Prior 4	0.9513	-0.0013	8.7498	0.0259	0.0228	0.1566
Design 5	Prior 1	0.9492	0.0008	20.2126	0.0225	0.0283	0.1322
	Prior 2	0.9503	-0.0003	19.3125	0.0222	0.0276	0.1675
	Prior 3	0.9513	-0.0013	22.8317	0.0207	0.0281	0.1605
	Prior 4	0.9483	0.0018	21.5613	0.0220	0.0298	0.1606
KM Orig	Prior 1	0.9502	-0.0002	14.2032	0.0248	0.0250	0.1536
	Prior 2	0.9525	-0.0025	10.9021	0.0242	0.0233	0.1442
	Prior 3	0.9519	-0.0019	553.3609	0.0238	0.0243	0.1750
	Prior 4	0.9518	-0.0018	27.6973	0.0244	0.0238	0.1674
Overall*	Prior 1	0.9507	-0.0007	8.4110	0.0228	0.0264	0.1408
	Prior 2	0.9498	0.0002	7.5638	0.0242	0.0260	0.1753
	Prior 3	0.9493	0.0007	99.0790	0.0234	0.0273	0.1640
	Prior 4	0.9504	-0.0004	11.0474	0.0228	0.0268	0.1628

*Average results across all designs

Table 11: Summary Results for the Case of the Ratios of Two Means – HPD Intervals

Method	Prior	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1	Prior 1	0.9571	-0.0071	2.0556	0.0329	0.0100	0.5383
	Prior 2	0.9443	0.0058	1.8675	0.0455	0.0103	0.6116
	Prior 3	0.9688	-0.0188	2.6828	0.0193	0.0119	0.2576
	Prior 4	0.9634	-0.0134	2.3312	0.0260	0.0106	0.4229
Design 2	Prior 1	0.9608	-0.0108	3.9526	0.0200	0.0192	0.3200
	Prior 2	0.9514	-0.0014	3.5770	0.0249	0.0237	0.3457
	Prior 3	0.9717	-0.0217	5.2558	0.0136	0.0148	0.1145
	Prior 4	0.9661	-0.0161	4.4906	0.0181	0.0158	0.2592
Design 3	Prior 1	0.9522	-0.0022	1.3183	0.0364	0.0114	0.5079
	Prior 2	0.9396	0.0104	1.1911	0.0514	0.0090	0.6907
	Prior 3	0.9631	-0.0131	1.7007	0.0238	0.0131	0.3324
	Prior 4	0.9603	-0.0103	1.4716	0.0279	0.0118	0.4396
Design 4	Prior 1	0.9481	0.0019	7.5209	0.0121	0.0398	0.5492
	Prior 2	0.9414	0.0086	6.8423	0.0090	0.0496	0.6774
	Prior 3	0.9634	-0.0134	9.7194	0.0140	0.0226	0.2428
	Prior 4	0.9540	-0.0040	8.4334	0.0129	0.0331	0.4619
Design 5	Prior 1	0.9452	0.0048	18.5868	0.0432	0.0117	0.5847
	Prior 2	0.9386	0.0114	16.8909	0.0512	0.0103	0.6474
	Prior 3	0.9603	-0.0103	23.7619	0.0241	0.0156	0.2370
	Prior 4	0.9563	-0.0063	20.8273	0.0308	0.0129	0.4453
Overall	Prior 1	0.9527	-0.0027	6.6868	0.0289	0.0184	0.5000
	Prior 2	0.9431	0.0070	6.0738	0.0364	0.0206	0.5945
	Prior 3	0.9655	-0.0155	8.6241	0.0190	0.0156	0.2368
	Prior 4	0.9600	-0.0100	7.5108	0.0232	0.0169	0.4058

Table 12: Summary Results for the Case of the Ratios of Two Means – MOVER Intervals

Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1	0.9497	0.0003	2.6343	0.0228	0.0275	0.1166
Design 2	0.9563	-0.0063	4.5983	0.0215	0.0222	0.1046
Design 3	0.9538	-0.0038	1.8260	0.0218	0.0244	0.1208
Design 4	0.9488	0.0012	8.4617	0.0287	0.0225	0.1454
Design 5	0.9552	-0.0052	20.1543	0.0216	0.0233	0.1374
Overall	0.9528	-0.0028	7.5349	0.0233	0.0240	0.1249

The above tables do little to enhance our understanding of the differences between the prior distributions. It does highlight certain issues, but detailed examination of Table 32 reveals the full extent of these differences. Enough variation in the designs was allowed for and thus it is possible to evaluate a few different scenarios.

Overall, when comparing the Bayesian results it appears that the HPD intervals results in somewhat better coverage and in some cases a substantial reduction in the average interval length. However, this additional coverage and reduction in interval length comes at the expense of increased relative bias. Furthermore, the HPD results follow the same pattern as the equal-tailed intervals which will be discussed in more detail further on.

The MOVER method results in adequate coverage for these designs and appears to be efficient as well. It is similar in performance to the equal-tailed Bayesian intervals, however, the HPD intervals discussed previously are still an improvement on the MOVER, particularly with regards to the average interval length. As will be discussed in further chapters, the MOVER is somewhat limited in the situations that it can be applied to, for example the comparison of variances, as discussed in Chapter 6.

Similar to the case for the single lognormal variable, we see that the overall coverages of Priors 1 $[\sigma^{-2}]$ and 2 $[\sigma^{-3}]$ are the best. Besides for a few lower decimals both give the desired coverage, *i.e.* 95%.

Furthermore, we note that Priors 3 (uniform) and 4 $[\sigma^{-1}]$ offer reasonable coverage, but again the effectiveness and efficiency of the prior distributions is called into question. Particularly with the uniform prior, the overall average interval length exceeds that of the other distributions by almost 10 times. It seems that the uniform prior is especially not suited to small sample sizes.

Prior 4 does not have the high average interval length of the uniform prior, but nevertheless it does seem excessive and not useful in this situation. Coverage probability and average interval length are the more important characteristics that we are interested in here.

Surprisingly though, between Priors 1 and 2 it seems once again that Prior 2 is the more appropriate choice. As mentioned earlier, both distributions resulted in similar and adequate coverage probabilities, but the additional measure of average interval length seems to indicate that Prior 2 is marginally more efficient than Prior 1. This is a slightly unexpected result, but nevertheless motivates the use of the Jeffreys Rule prior in applications.

From Table 32 we see the following tendencies in the results:

1. We see that the Bayesian methods are perhaps not particularly well adapted to situations of small sample sizes. The tendency of all four prior distributions is to over-cover, especially on the right hand side when small sample sizes and unequal population variances are considered. However, by "small sample sizes" we imply that n_1 and n_2 are both less than or equal to 10. Particularly Prior 3 results in excessive over-coverage. Thus, the Bayesian techniques may not be efficient for small sample sizes. However, this is by no means conclusive.
2. Once the sample sizes increase to at least 10 for each population then the stability and efficiency of all choices of prior distribution increase dramatically, but particularly so for Prior 3. The tendency to over-cover is still present, but not as prevalent as for small sample sizes. However, throughout all the designs examined it is apparent that this prior distribution (Prior 3) is less efficient than the others, particularly with regards to the interval length.

Thus, we would suggest the use of the Bayesian methods for larger sample situations, and even then we would suggest the use of only Prior distributions 1 and 2. The methods also seem to be more accurate when the population means are relatively close together.

3.3 Examples

As mentioned in the Introduction to this chapter, the examples in Krishnamoorthy and Mathew (2003) are analysed here according to the Bayesian methodology developed in the simulation study. Due to their performance in the simulation studies, it was decided to analyse the data from these real-life situations using only Priors 1 and 2. In addition the literature on the subject, in particular research by (Fernandez, C. and Steel F.J.M. [1999], Roman, L [2008] and Berger and Bernardo [1992]) suggests that when considering the prediction of a future value (specifically from a lognormal distribution) then the median and not the mean of the distributions is a more appropriate quantity of interest. It was decided thus to incorporate the analysis of the ratio of the medians from two different populations. The same prior distributions as used for the analysis of the means were also used for the analysis of the medians, namely Priors 1 and 2. According to the literature, the choice of the median as the parameter of interest is justified by the adequate coverage properties of the resulting posterior predictive distributions. The theory is particularly illustrated by Roman, L (2008).

The differences to the methods described previously are relatively minor. For the most part, the procedures are similar except for the following: instead of obtaining the means of the posterior distributions ($E(X) = E(\exp(Y)) = \exp(\eta)$, where $\eta = \mu + \frac{\sigma^2}{2}$) and taking the ratios thereof it is only necessary to obtain the ratio of the median, namely $Me(X) = \exp(\eta)$, where $\eta = \mu$.

3.3.1 Probability-Matching and Reference Priors – Lognormal Distribution

In addition to using the prior distributions (Priors 1 and 2) obtained from the simulation study in the previous sections the Probability-Matching prior and Reference prior distributions were applied for the analysis of both examples.

In this section we will only use the method described by Datta and Ghosh (1995) (as discussed in Chapter 2) to obtain Probability-Matching priors for the parameters of the lognormal distribution. The following theorems can be stated:

Theorem 3.1

For the mean, $M_e' = e^{\mu + \frac{1}{2}\sigma^2}$, of the lognormal distribution, the Reference prior relative to the ordered parameterisation (μ, σ^2) is given by:

$$p_R(\mu, \sigma^2) \propto \frac{1}{\sigma}\sqrt{1 + \frac{2}{\sigma^2}}.$$

Proof: The proof is given in Chapter 2.

Theorem 3.2

For the mean, $M_e' = e^{\mu + \frac{1}{2}\sigma^2}$, of the lognormal distribution, the Probability-Matching prior is given by:

$$p_P(\mu, \sigma^2) \propto \frac{1}{\sigma^2}\sqrt{1 + \frac{2}{\sigma^2}}.$$

Proof: Refer to Chapter 1 for more details regarding the derivation of this prior.

Corollary 1: The Reference prior is not a Probability-Matching prior for the mean.

Theorem 3.3

For the mode, $M_o = e^{\mu-\sigma^2}$, of the lognormal distribution, the Reference prior relative to the ordered parameterisation $\left(\mu,\sigma^2\right)$ is given by:

$$\pi_R\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma}\sqrt{1+\frac{1}{2\sigma^2}}.$$

Proof: The proof is given in the appendix to this chapter.

Theorem 3.4

For the mode, $M_o = e^{\mu-\sigma^2}$, of the lognormal distribution, the Probability-Matching prior is given by:

$$\pi_P\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma^2}\sqrt{1+\frac{1}{2\sigma^2}}.$$

Proof: The proof is given in the appendix to this chapter.

Corollary 2: The Reference prior is not a Probability-Matching prior for the mode of the lognormal distribution.

It is easy to prove that for the median, $M_e = e^{\mu}$, the Probability-Matching and Reference priors are the same, that is:

$$\phi_P\left(\mu,\sigma^2\right)=\phi_R\left(\mu,\sigma^2\right)\propto\frac{1}{\sigma^2}.$$

3.4 Procedure for Analysis of Data

The data obtained were in the form of readings (in different settings) from two populations. Let us denote the original readings as: X_{ij} where $i=1,2$. Denote the log-transformed data once again by $Y_{ij}=\log(X_{ij})$. From this log-transformed data the following statistics are calculated:

$$\bar{Y}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}Y_{ij}$$
$$s_i=\frac{1}{n_i-1}\sum_{j=1}^{n_i}\left(Y_{ij}-\bar{Y}_i\right)^2$$
$$v_i=n_i-1$$
$$m_i=v_i\times s_i$$

Using these calculated statistics, the simulation procedure was similar to the procedures described in Chapters 1 and 2. However, the simulation procedure is adjusted so as to simulate from two different populations and then the difference is calculated in each case. This difference results in the simulated values.

Probability-matching and Reference Prior Distributions

The aforementioned simulation procedure remains unchanged except for step 1a in Section 1.1. This step implicitly assumes the posterior distribution resulting from the prior distributions mentioned before (Priors 1 and 2). As will be shown here the form of the derived prior distributions only affects the posterior distribution of σ_i^2. The conditional distribution $\mu_i \mid \sigma_i^2$, as mentioned in step 1b in Section 1.1 remains unchanged.

Using the form of the likelihood derived in Chapter 2 the following posterior distributions are (for the Probability-Matching and Reference prior distributions respectively):

$$P(\mu,\sigma^2 \mid data) = \prod_{j=1}^{2}\left\{\left(\frac{1}{\sigma_j^2}\sqrt{1+\frac{2}{\sigma_j^2}}\right)\left(\frac{2\pi\sigma_j^2}{n_{j1}}\right)^{-\frac{1}{2}}\exp\left[-\frac{n_{j1}}{2\sigma_j^2}(\mu_j-\hat{\mu}_j)^2\right]\left(\frac{1}{\sigma_j^2}\right)^{\frac{1}{2}\nu_j}\exp\left(-\frac{\nu_j s_j^2}{2\sigma_j^2}\right)\right\}$$

(3.1)

where $\hat{\mu}_j = \frac{1}{n_{j1}}\sum_{i=1}^{n_{j1}} y_{ij}$,

$\nu_{j1} = n_{j1} - 1$ and

$\hat{\sigma}_j^2 = \frac{1}{\nu_{j1}}\sum_{i=1}^{n_{j1}}(y_{ij}-\hat{\mu}_j)^2$.

From (3.1) the following is evident:

$$\mu_j \mid \sigma^2, data \sim N\left(\hat{\mu}_j, \frac{\sigma_j^2}{n_{j1}}\right) \qquad (3.2)$$

and for σ_j^2, the posterior density function is as follows:

$$P(\sigma_j^2 \mid data) = \sqrt{1+\frac{2}{\sigma_j^2}}\left(\frac{1}{\sigma_j^2}\right)^{\frac{1}{2}(\nu_{j1}+2)}\exp\left[-\frac{\nu_{j1}\hat{\sigma}_j^2}{2\sigma_j^2}\right] \qquad (3.3)$$

for the Probability-Matching prior distribution and

$$P(\sigma_j^2 \mid data) = \sqrt{1+\frac{2}{\sigma_j^2}}\left(\frac{1}{\sigma_j^2}\right)^{\frac{1}{2}(\nu_{j1}+1)}\exp\left[-\frac{\nu_{j1}\hat{\sigma}_j^2}{2\sigma_j^2}\right] \qquad (3.4)$$

for the Reference prior distribution. Since both (3.3) and (3.4) are non-standard distributions the simulation procedure described in step 1a in Section 1.1 was adapted so as to use the rejection method to simulate σ_j^2 observations from these posterior distributions.

3.5 Results

Due to the definition of the ratio(s) above we expect ω to be centered around 0 (or the exponent of ω to be centered around 1) if the data from two different samples are equal. If the credibility interval (95%) does not contain 0 (or 1 alternatively) then it is unlikely that the data came from the same population.

3.5.1 Refinery Data

The example is the same as used in Krishnamoorthy and Mathew (2003). An oil refinery located at the northeast of San Francisco conducted a series of 31 daily measurements of the carbon monoxide levels arising from one of their stacks between April 16 and May 16 1993. The measurements were submitted as evidence for establishing a baseline to the Bay Area Air Quality Management District (BAAQMD). BAAQMD personnel also made 9 independent measurements of the carbon monoxide concentration from the same stack over the period September 11, 1990 to March 30, 1993. It appears that the refinery had an incentive to overestimate carbon monoxide emissions, and it was this assertion that was tested by Krishnamoorthy and Mathew (2003). The original measurements were as follows:

Table 13: Refinery Data

Source	Measurements Obtained
Refinery	45; 30; 38; 42; 63; 43; 102; 86; 99; 63; 58; 34; 37; 55; 58; 153; 75; 58; 36; 59; 43; 102; 52; 30; 21; 40; 141; 85; 161; 86; 71
BAAQMD	13; 20; 4; 20; 25; 170; 15; 20; 15

The log transformed measurements collected by the refinery can be summarised as follows:

$$n_1 = 31$$
$$\bar{y}_1 = 4.0743$$
$$s_1 = 0.5021$$

The log transformed measurements from the second sample can be summarised by:

$$n_2 = 9$$
$$\bar{y}_2 = 2.963$$
$$s_2 = 0.974$$

Krishnamoorthy and Mathew (2003) reached the conclusion that the data do not provide sufficient evidence to indicate that the mean measurement by the refinery is greater than that of BAAQMD, which was contrary to the speculation.

As mentioned earlier in this section we approached the problem from a Bayesian perspective and in addition to this, not only was the ratio of the means investigated, but also the ratio of the medians. It should be remembered that since the method proposed by Krishnamoorthy and Mathew (2003) is technically the same as the use of Prior 1, we expect Prior 1 once again to give the same results as obtained by these authors.

According to the methodology presented in Section 3.2, the following results were obtained:

Table 14: Summary Results for Refinery Data

Quantity	Prior	ω	$\exp(\omega)$
Mean	Prior 1	[-0.65853 : 1.07467]	[0.51761 : 2.92903]
	Prior 2	[-0.27764 : 1.08427]	[0.75757 : 2.95728]
Median	Prior 1	[0.78108 : 1.44468]	[2.18382 : 4.24049]
	Prior 2	[0.82579 : 1.39271]	[2.28369 : 4.02575]

These results indicate that for Priors 1 and 2 the same conclusions were obtained as were obtained by Krishnamoorthy and Mathew (2003) for the means. It does not appear that the means of the populations differ with any meaningful level of credibility. Thus, we cannot say that the means from the two populations are different, but we can also not say that they are not. It appears as though the ratio of the means is relatively skewed to the left. Due to the definitions (the mean of the refinery data is the numerator in the ratio) it appears as though there may be cause to suspect that the measurements do differ, but no conclusive results can be obtained. The histograms below indicate this situation (notice the skewed distribution). The first set of histograms represents the 100000 ω and $\exp(\omega)$ data points obtained for Prior 1. The second set of histograms represent this situation for Prior 2.

Figure 6: Histograms for Prior 1 (Means)

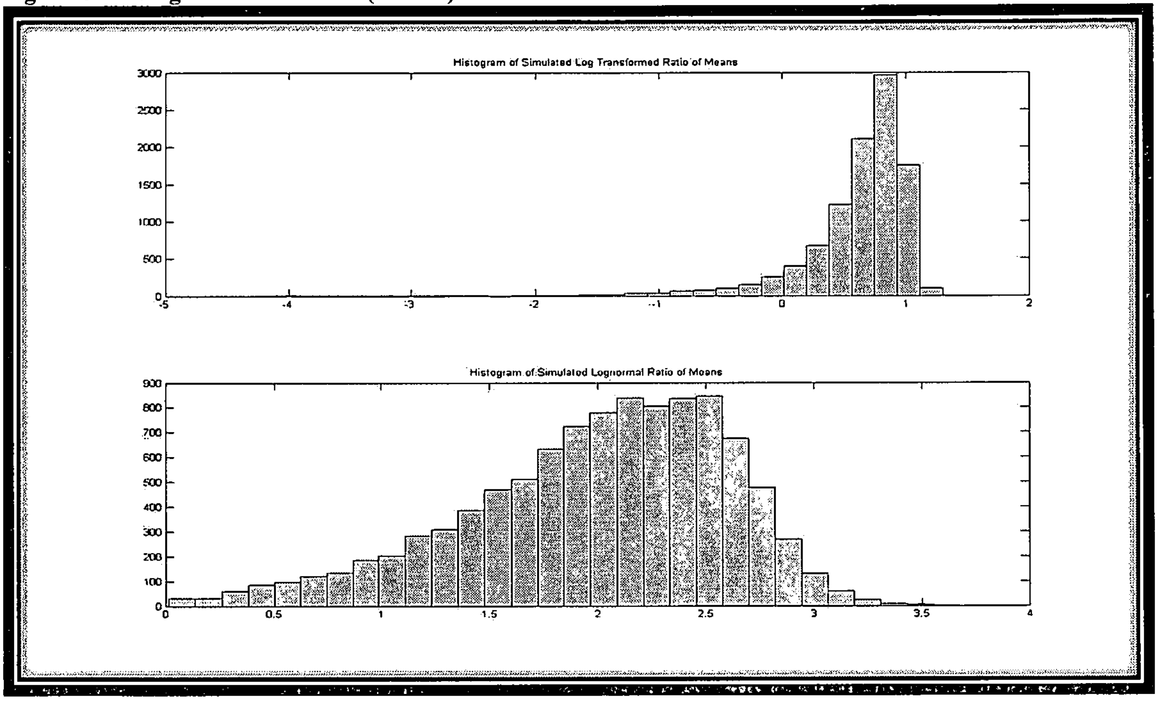
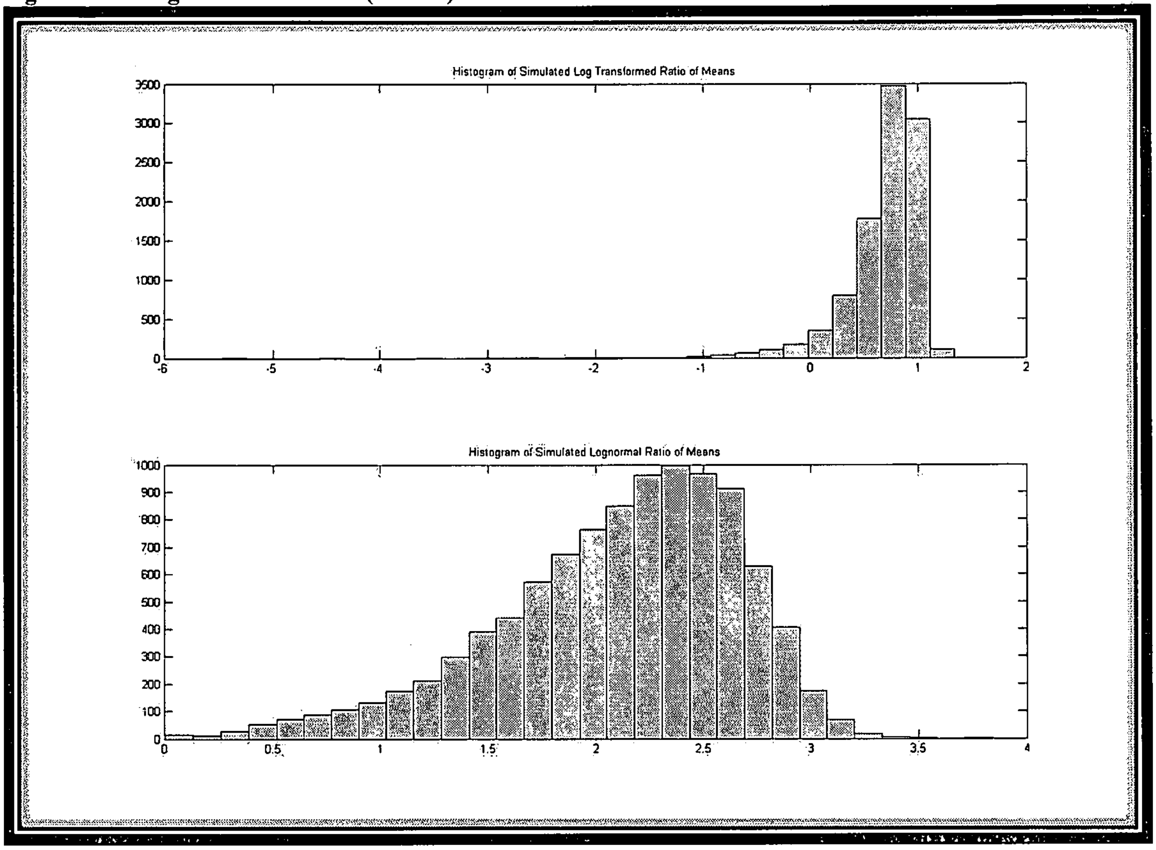


Figure 7: Histograms for Prior 2 (Means)



As suggested by the literature, different results were obtained when analysing the medians rather than the means of the data. Although different results are to be expected to those for the mean (since the lognormal distribution is not a symmetric distribution) the results indicate that the medians of the different samples are indeed not equal. This tends to support the supposition that the refinery measurements are excessive.

From the credibility intervals in the text it is evident that there is more than a 0.95 probability that the median refinery measurement is indeed larger than the median of the BAAQMD data. The following two sets of histograms indicate this situation for Priors 1 and 2.

Figure 8: Histograms for Prior 1 (Medians)

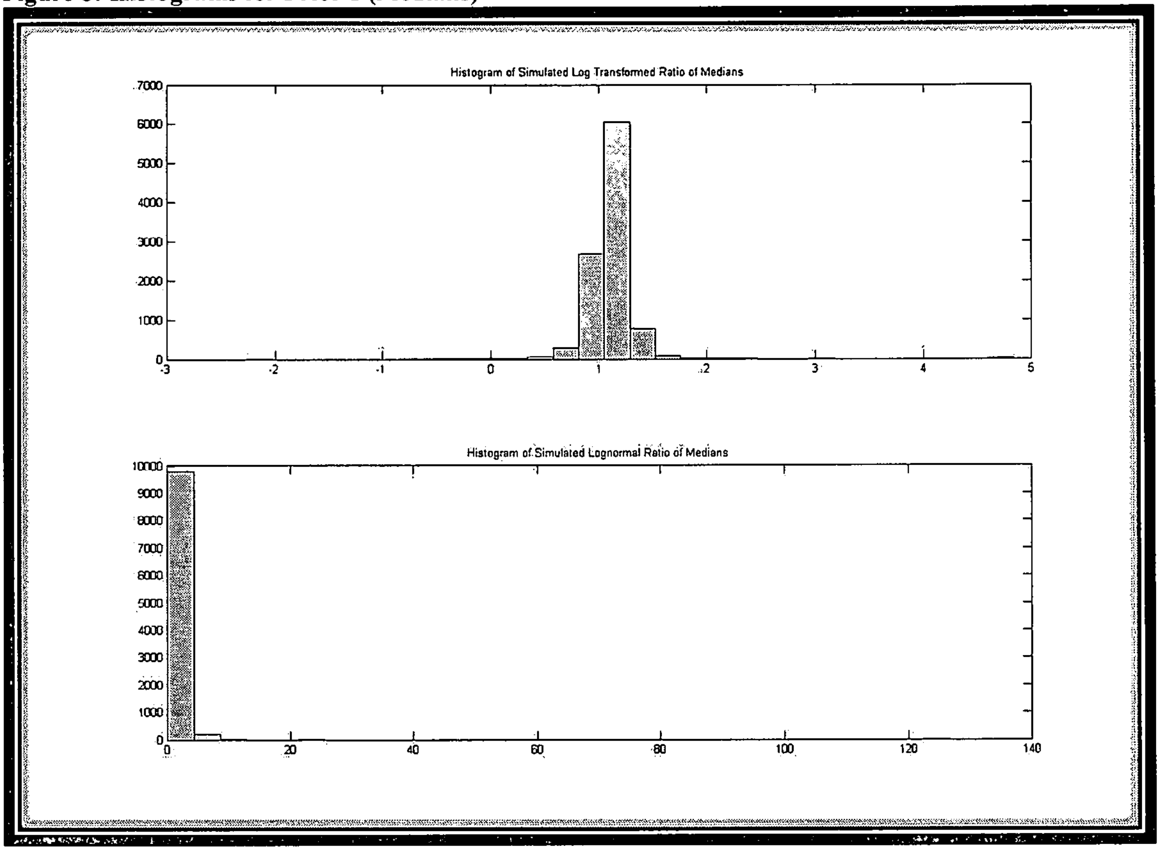
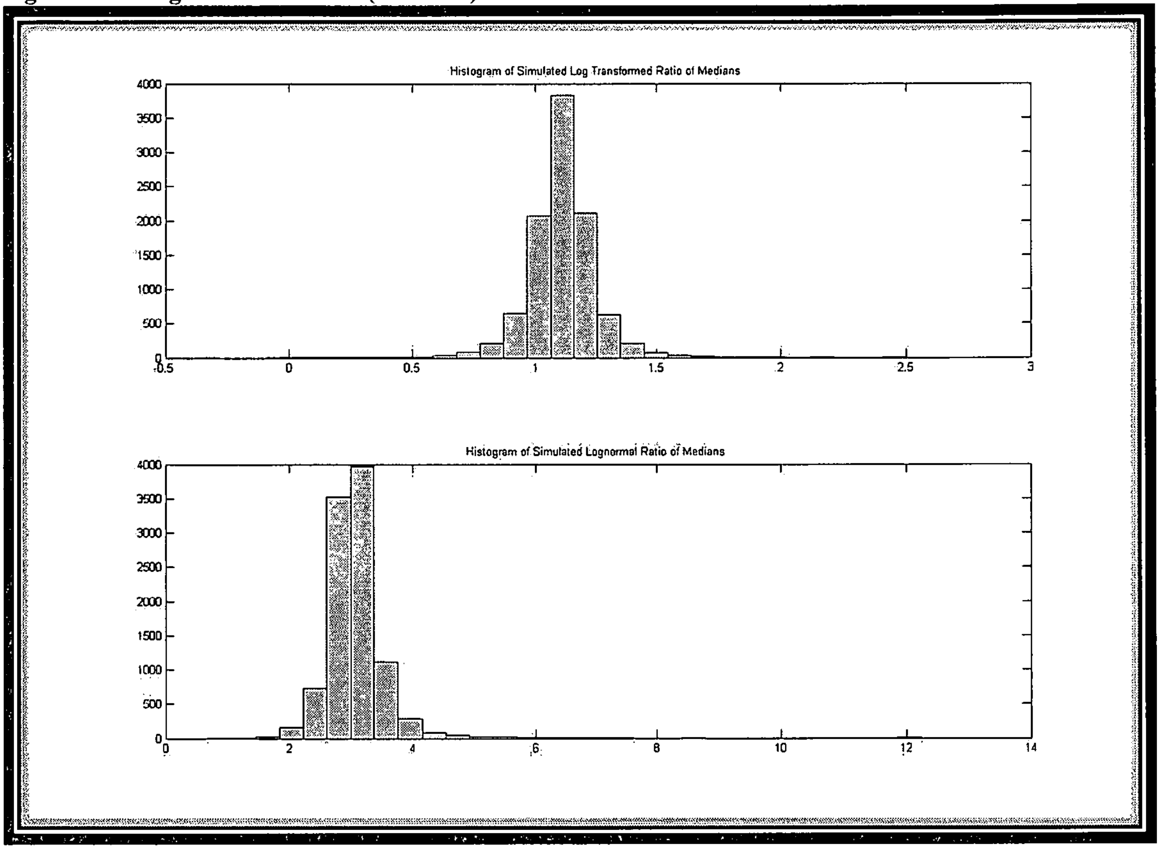


Figure 9: Histograms for Prior 2 (Medians)



Probability-matching and Reference Prior Distribution Results

As mentioned before, a similar analysis was conducted using the derived Probability-Matching and Reference prior distributions. The following results were obtained:

Table 15: Results for Refinery Data: Probability-Matching and Reference Priors

Quantity	Prior	ω	$\exp(\omega)$
Mean	PMP	[-0.37879 : 1.08436]	[0.68469 : 2.95754]
	Ref	[-0.79154 : 1.07592]	[0.45314 : 2.93268]
Median	PMP	[0.79618 : 1.41430]	[2.21704 : 4.11361]
	Ref	[0.74299 : 1.48256]	[2.10222 : 4.40422]

So we can see that by using these two prior distributions identical conclusions were reached as those obtained using Priors 1 and 2. The following histograms represent this situation:

Figure 10: Histograms for Probability-matching Prior (Means)

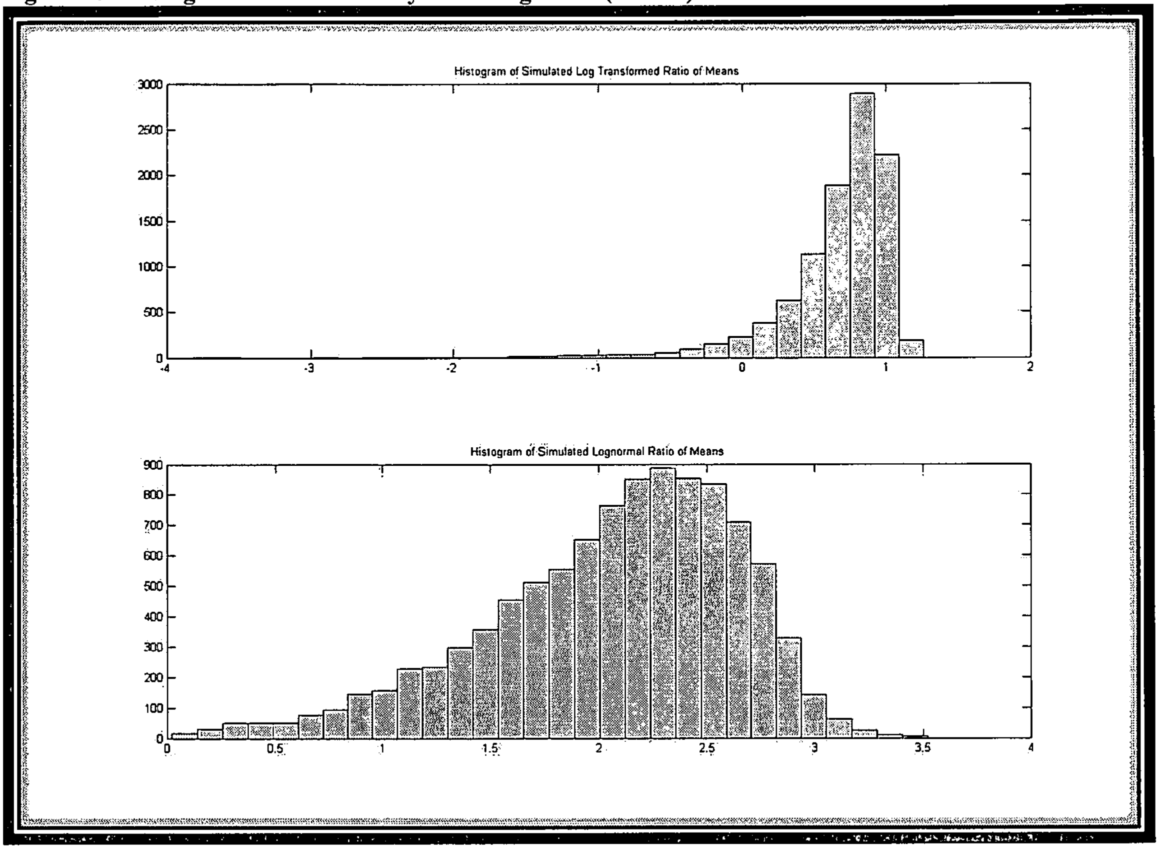


Figure 11: Histograms for Reference Prior (Means)

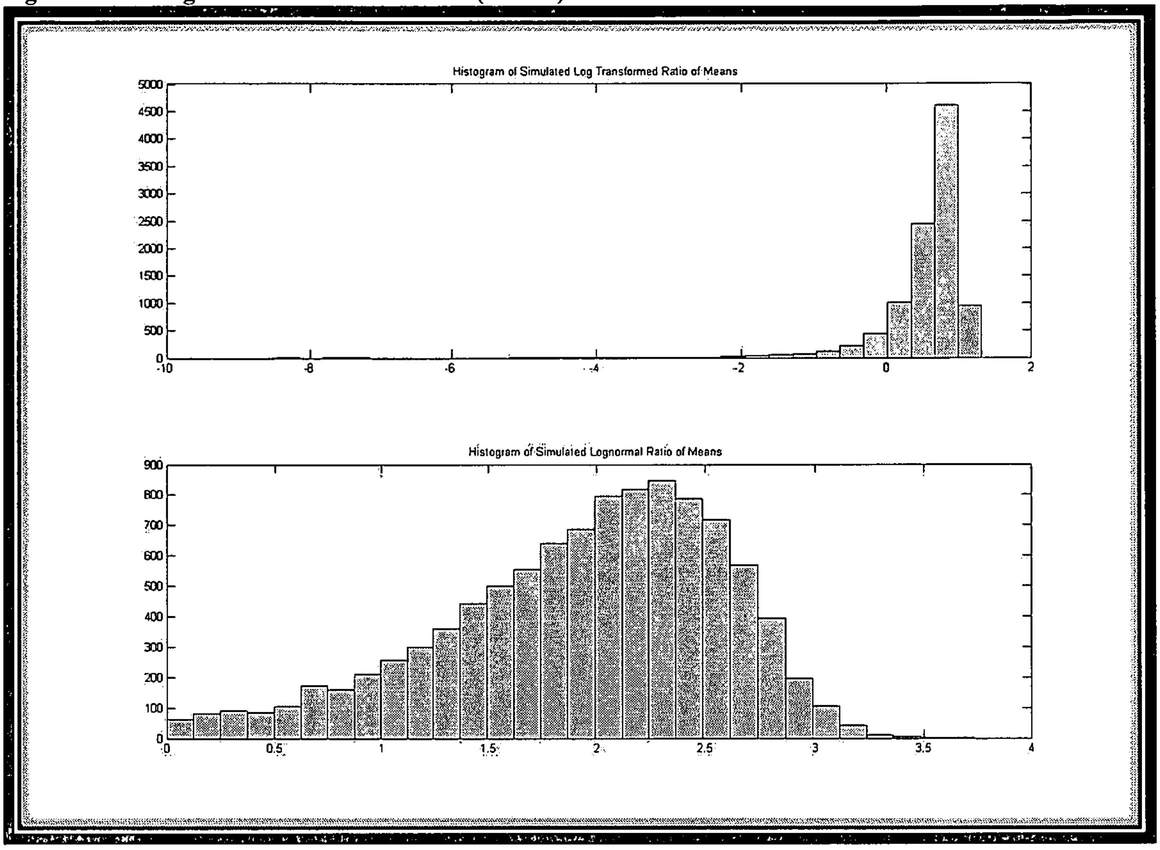


Figure 12: Histograms for Probability-Matching Prior (Medians)

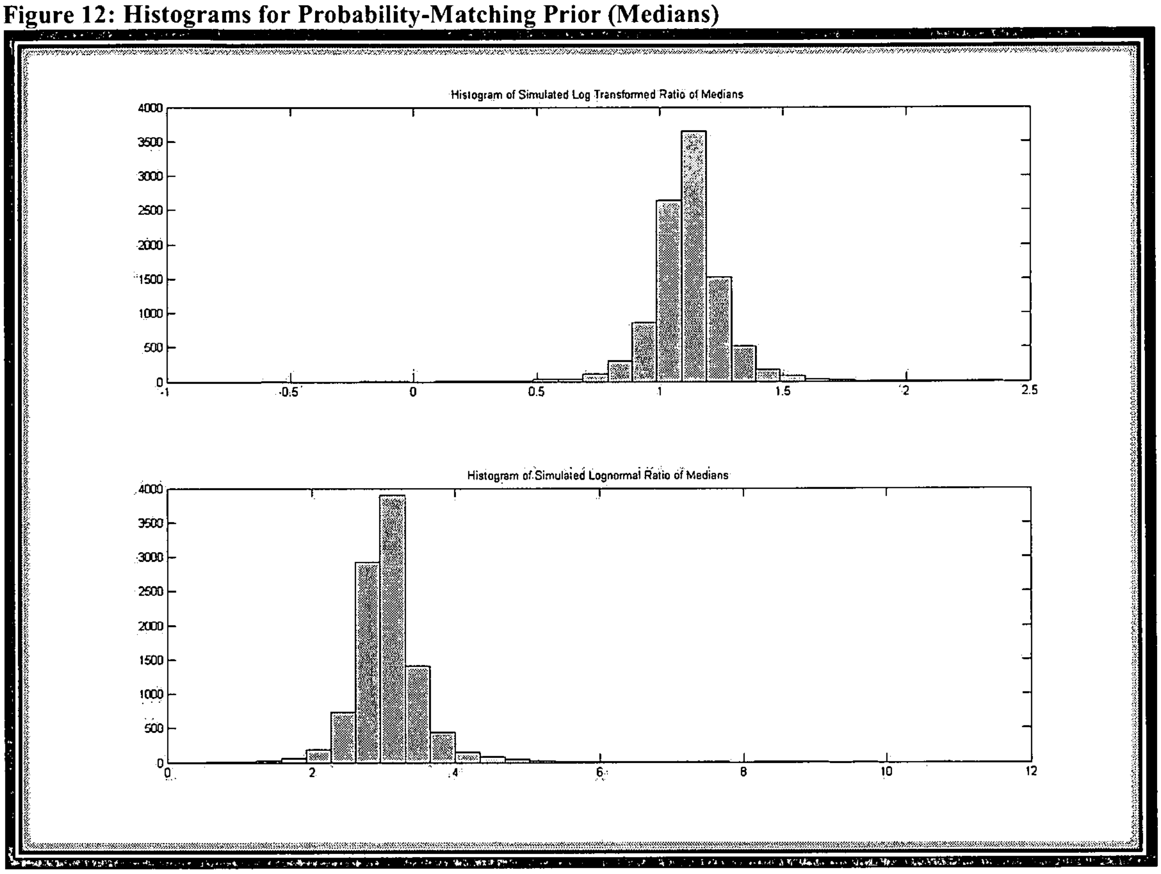
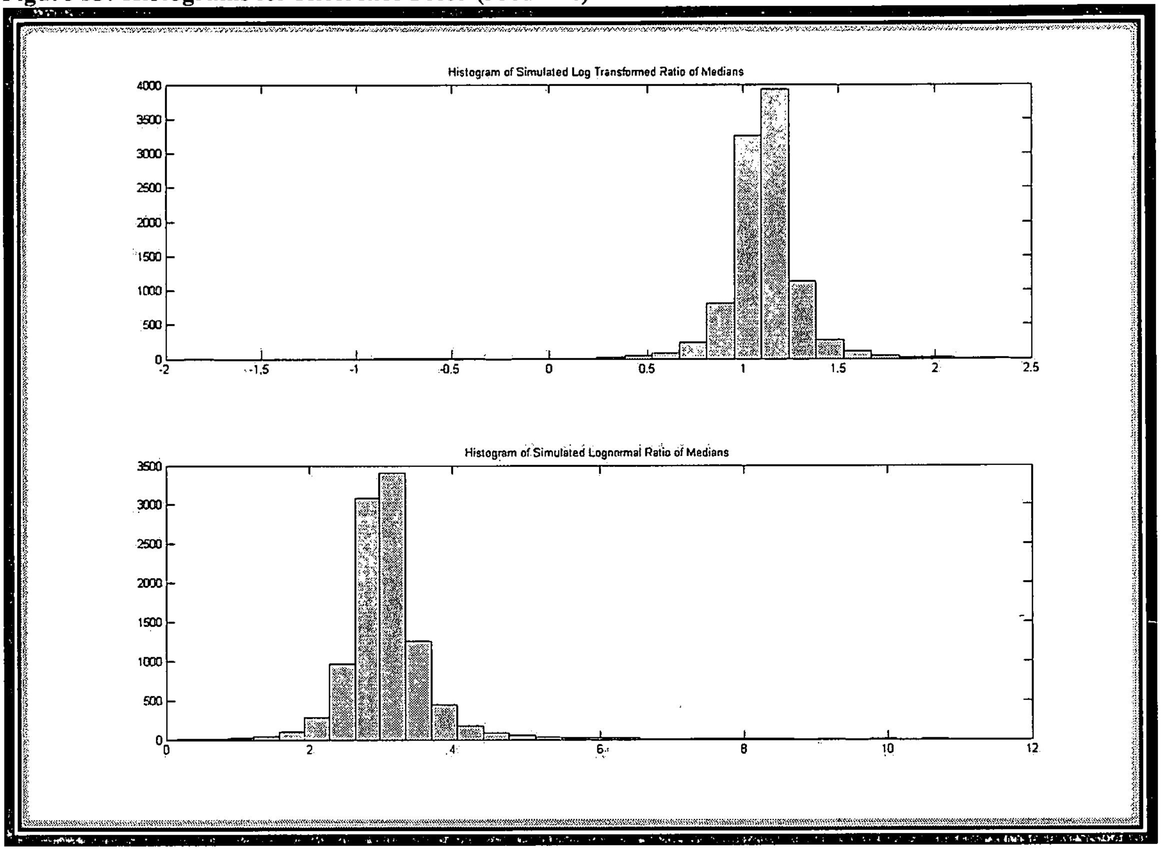


Figure 13: Histograms for Reference Prior (Medians)



We can see from the above results that the Reference prior tends to result in wider credibility intervals than the Probability-Matching prior in this setting for both the means and the medians of the population. Overall, Prior 2 results in the best results, in terms of interval length.

3.5.2 Cloud Seeding Data

The following example is conceptually similar to the previous example. It was also one of the sets of data analysed by Krishnamoorthy and Mathew (2003). The same procedure was used to determine the Bayesian credibility intervals. Thus, for the purposes of brevity the methodology and specification will not be presented again.

The data on the amount of rainfall (in acre-feet) from 52 clouds, 26 of which were chosen at random and seeded with silver nitrate, were obtained. Probability plots indicated that the lognormal distribution was a better fit than the normal distribution. The measurements were given as follows:

Table 16: Cloud Seeding Data

Source	Measurements Obtained
Unseeded	1202.6; 830.1; 372.4; 345.5; 321.2; 244.3; 163.0; 147.8; 95.0; 87.0; 81.2; 68.5; 47.3; 41.1; 36.6; 29.0; 28.6; 26.3; 26.1; 24.4; 21.7; 17.3; 11.5; 4.9; 4.9; 1.0
Seeded	2745.6; 1697.8; 1656.0; 978.0; 703.4; 489.1; 430.0; 334.1; 302.8; 274.7; 274.7; 255.0; 242.5; 200.7; 198.6; 129.6; 119.0; 118.3; 115.3; 92.4; 40.6; 32.7; 31.4; 17.5; 7.7; 4.1

The log transformed measurements collected from the unseeded clouds can be summarised as follows:

$$n_1 = 26$$
$$\bar{y}_1 = 3.990$$
$$s_1 = 1.642$$

The log transformed measurements for the seeded clouds can be summarised by:

$$n_2 = 26$$
$$\bar{y}_2 = 5.134$$
$$s_2 = 1.600$$

Naturally, we would like to test whether the seeding had a positive effect on the rainfall measured. Krishnamoorthy and Mathew's (2003) generalized confidence limit method indicated that there was insufficient data to be able to ascertain whether $\eta_2 > \eta_1$. The generalized p-value turned out to be 0.078. However, application of the two-sample t-test resulted in a p-value of 0.007, indicating that there is perhaps a difference between the rainfall from seeded and unseeded clouds. A third test, the Z-score test, yielded similar conclusions as the generalized p-value.

Using the Bayesian methodology, the following results were obtained:

Table 17: Results for Cloud Seeding Data

Quantity	Prior	ω	$\exp(\omega)$
Mean	Prior 1	[-2.36083 : 0.28328]	[0.09434 : 1.32748]
	Prior 2	[-2.28273 : 0.17100]	[0.10201 : 1.18650]
Median	Prior 1	[-1.46804 : -0.82284]	[0.23038 : 0.43918]
	Prior 2	[-1.44920 : -0.82870]	[0.23476 : 0.43662]

As with the previous example it appears as though the conclusions obtained from the means of the two populations agree with the conclusions reached by Krishnamoorthy and Mathew (2003). From the data we find that there is not enough evidence to be able to say with any certainty that the means of the distributions are different. However, once again we see that there is a high degree of probability that the medians of the two populations do indeed differ. Due to the definition of the variables under analysis there is more than a 0.95 probability that the medians differ for unseeded and seeded observations for both choices of prior distributions.

The following histograms once again illustrate the situation graphically for the different prior distributions as well as for the different quantities of interest:

Figure 14: Histograms for Prior 1 (Means)

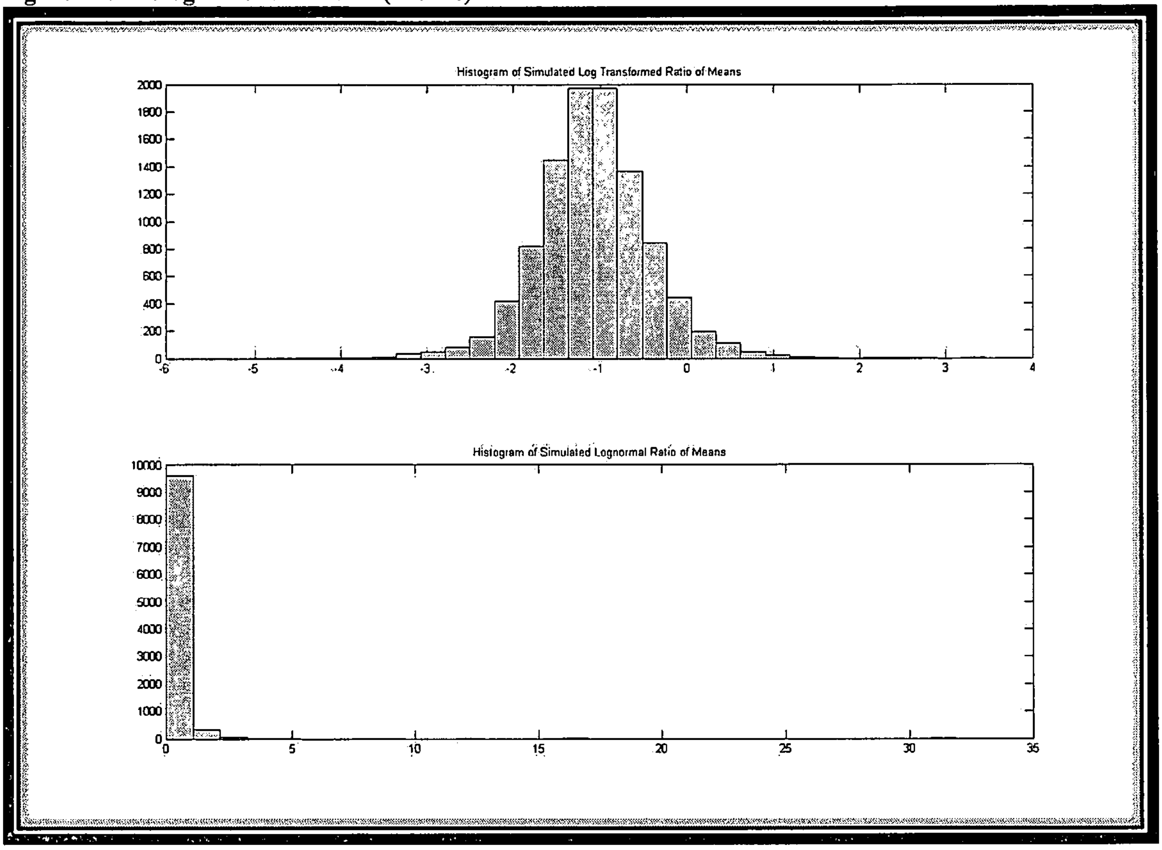


Figure 15: Histograms for Prior 2 (Means)

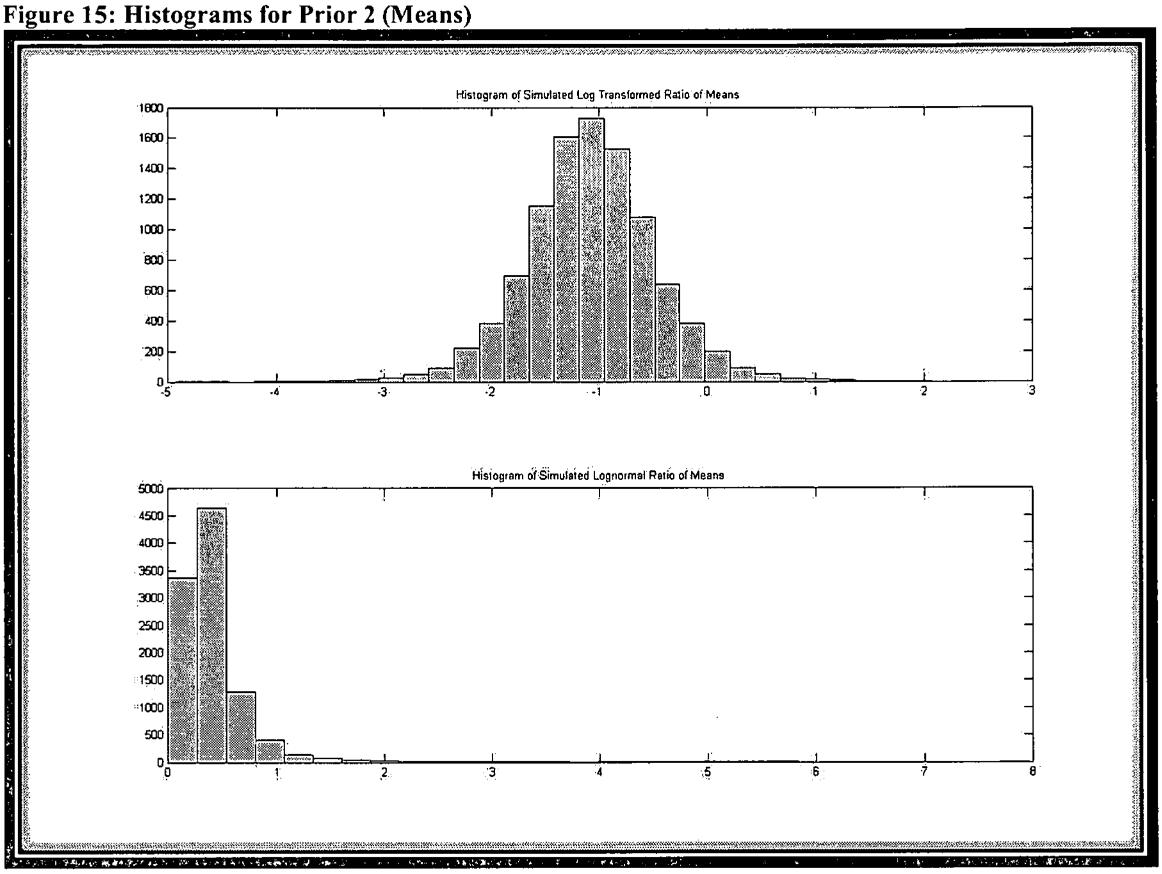


Figure 16: Histograms for Prior 1 (Medians)

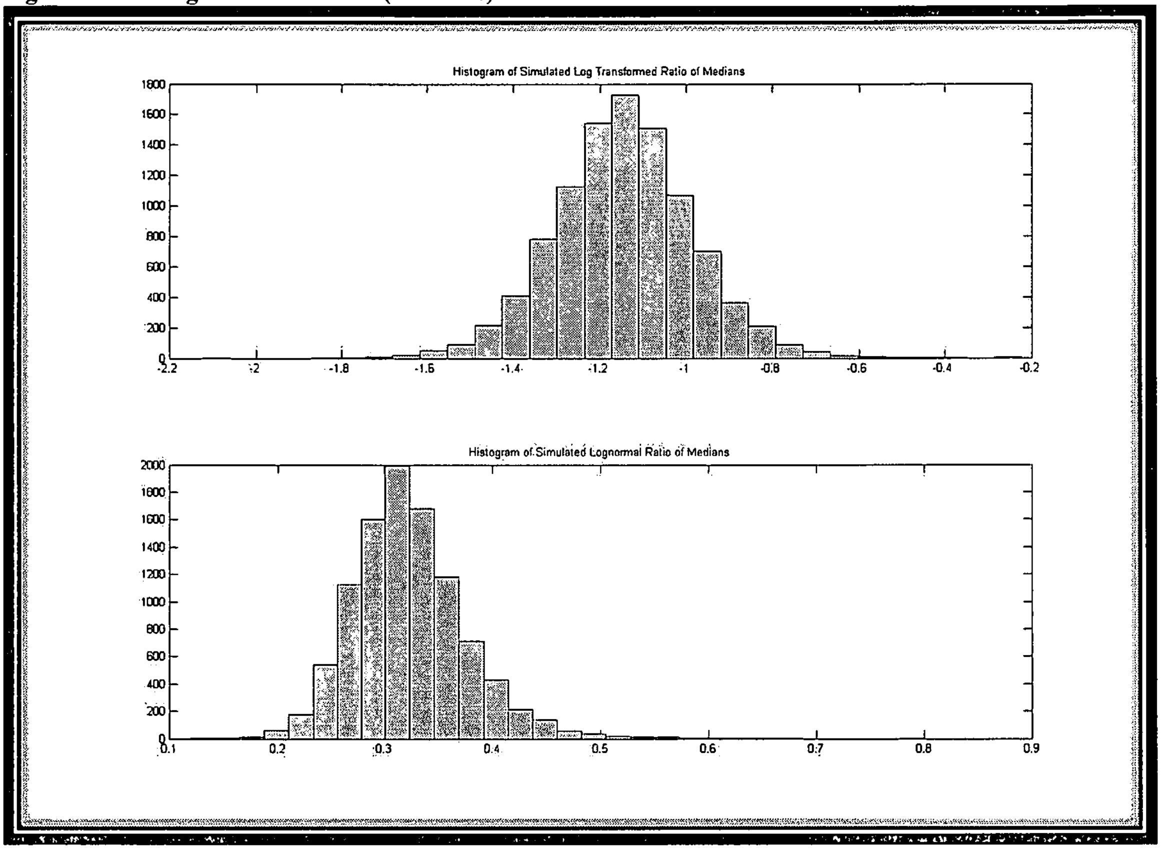
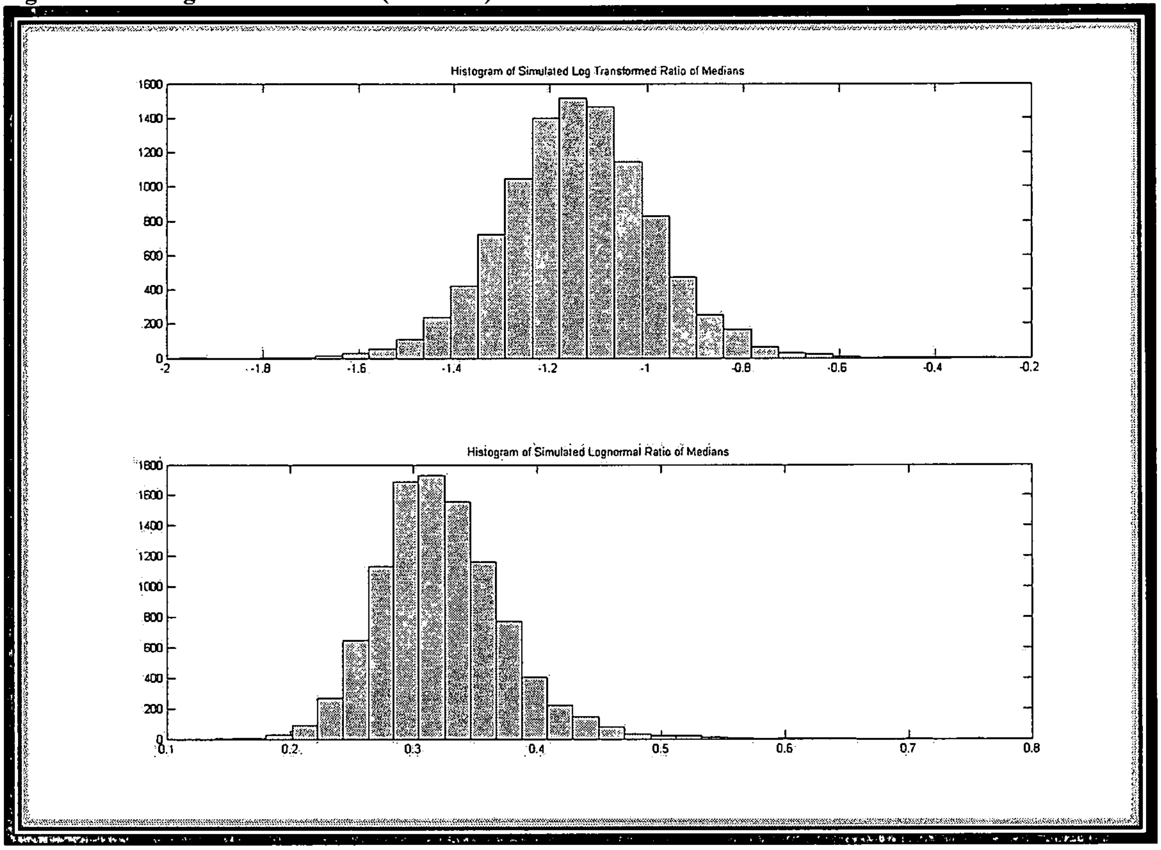


Figure 17: Histograms for Prior 2 (Medians)



Probability-matching and Reference Prior Distribution Results

As for the previous example, the Probability-Matching and Reference prior distributions were also used to analyse the data. This was done according to the same methodology described earlier. The results were as follows:

Table 18: Cloud Seeding Data Results - Probability-Matching and Reference Priors

Quantity	Prior	ω	$\exp(\omega)$
Mean	PMP	[-2.30325 : 0.23930]	[0.09993 : 1.27036]
	Ref	[-2.39517 : 0.31731]	[0.09116 : 1.37343]
Median	PMP	[-1.45412 : -0.81964]	[0.23361 : 0.44059]
	Ref	[-1.47673 : -0.81758]	[0.22838 : 0.44150]

Once again, the same conclusions that were reached by using the Priors 1 and 2 were also obtained using the Probability-Matching and Reference prior distributions. The following histograms graphically represent these results:

Figure 18: Histograms for Probability-Matching Prior (Means)

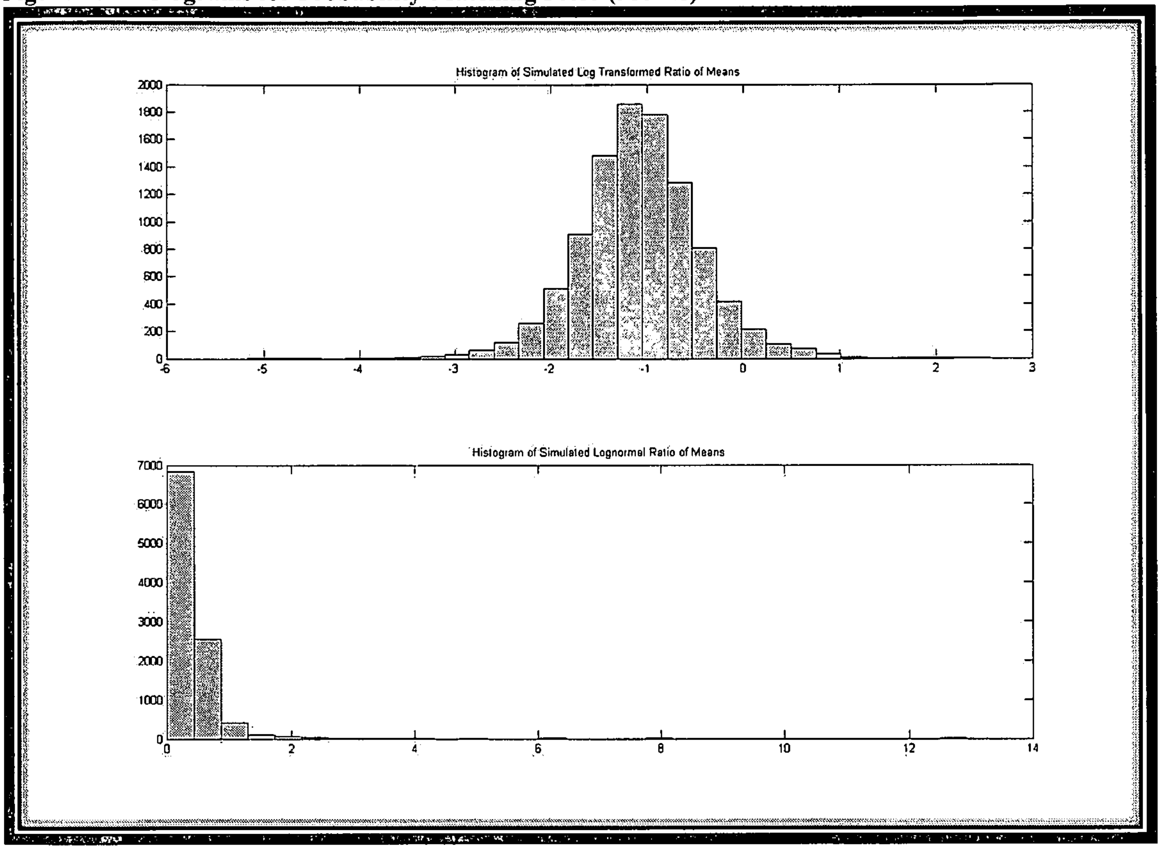


Figure 19: Histograms for Reference Prior (Means)

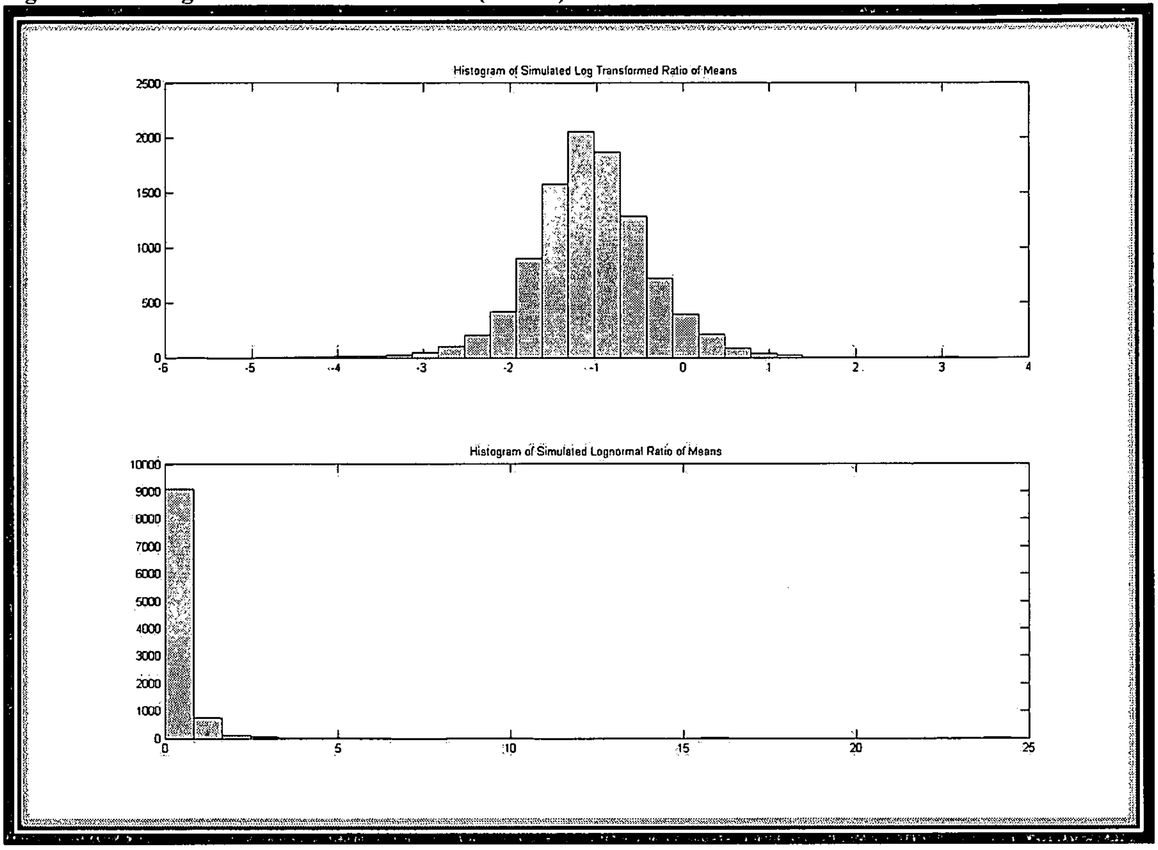


Figure 20: Histograms for Probability-Matching Prior (Medians)

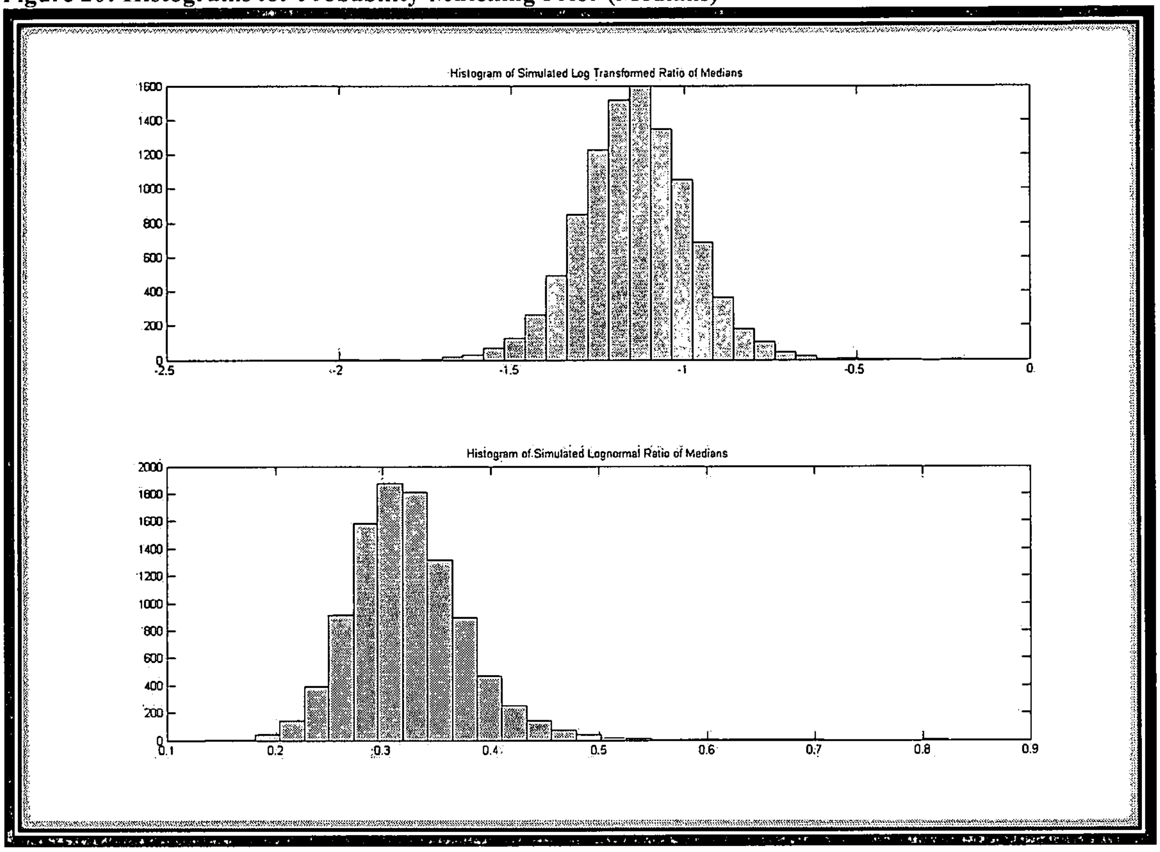
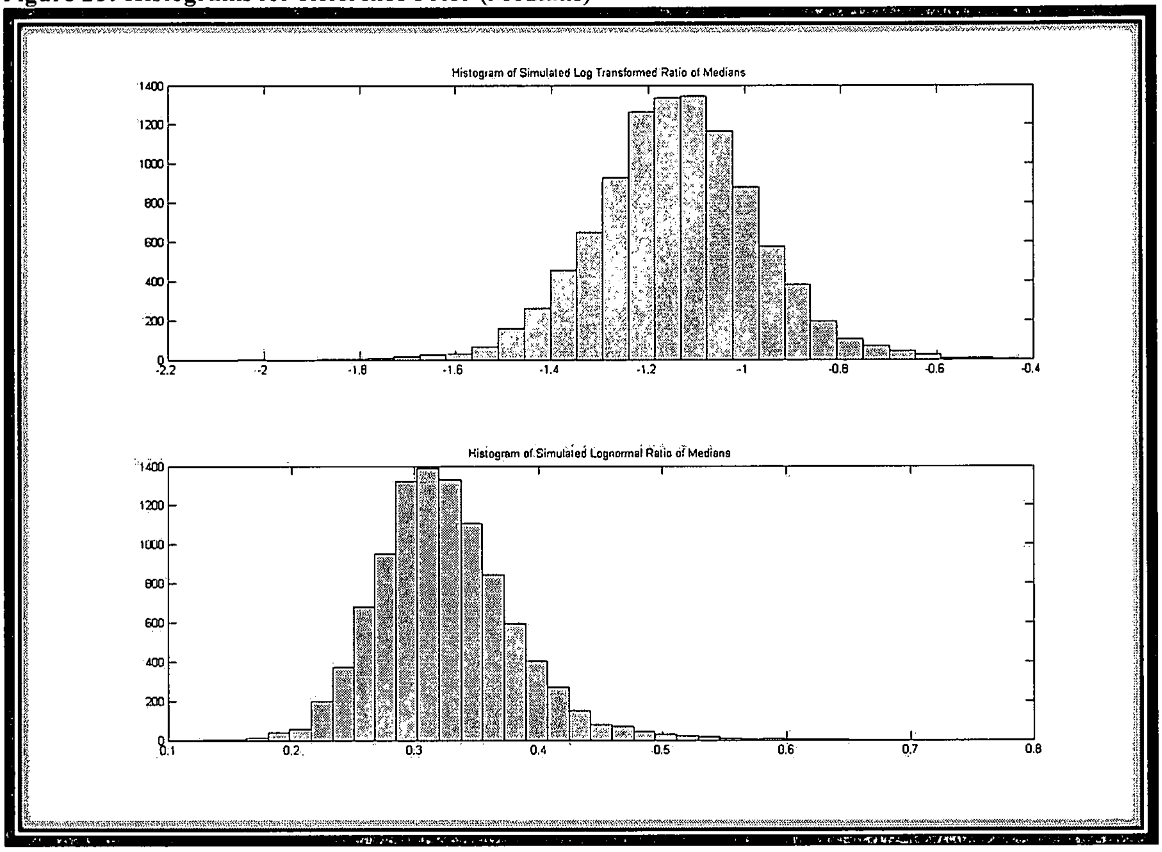


Figure 21: Histograms for Reference Prior (Medians)



It is therefore clear that the results obtained from the different prior distributions are for all practical purposes the same.

3.5.3 Diabetes Data Example

On page 3761 of Zou et al (2009a) the following example was given (as referenced in Zhou et al (1997b)): the effect of race on the cost of medical care for type I diabetes was investigated using MOVER. Log transformed cost data for 119 black patients and 106 white patients. For the black patients the following log-transformed data was available: $\hat{\mu}_1 = \bar{y}_1 = 9.06694$ and $\hat{\sigma}_1^2 = s_1^2 = 1.82426$. For the white patients this was $\hat{\mu}_2 =$

$\bar{y}_2 = 8.69306$ and $\hat{\sigma}_2^2 = s_2^2 = 2.69186$. For the MOVER the following results were obtained and this is compared to results obtained from the Bayes methods:

Table 19: Diabetes Results

	MOVER	Jeffreys Rule	Independence Jeffreys	Reference Prior	Probability-Matching Prior
Black Patients					
Lower Limit (equal-tailed)	15806.00	15970.19	15882.58	16021.01	15906.92
Upper Limit (equal-tailed)	31388.77	31759.23	31752.94	32025.42	31557.68
Lower Limit (HPD)		15708.48	15626.14	15753.04	15671.29
Upper Limit (HPD)		31063.21	31121.86	31351.21	30982.03
White Patients					
Lower Limit (equal-tailed)	14842.03	15097.15	15039.52	15081.46	14962.31
Upper Limit (equal-tailed)	39722.09	40079.56	40156.06	40483.85	40019.03
Lower Limit (HPD)		14744.94	14668.91	14765.88	14461.04
Upper Limit (HPD)		38799.26	38847.41	39404.13	38289.58
Difference					
Lower Limit (equal-tailed)	-19112.18	-19156.98	-19241.18	-19812.57	-19169.17
Upper Limit (equal-tailed)	11371.14	11229.07	11381.24	11457.97	11361.90
Lower Limit (HPD)		-19161.99	-19254.62	-19820.21	-19170.15
Upper Limit (HPD)		11225.65	11367.43	11446.41	11360.83

Since the sample sizes are large the interval lengths for the different procedures are more or less the same. It is however, clear that the HPD intervals for the Independence Jeffreys' prior are somewhat shorter for black and white patients than those of the MOVER.

Derivation of the Central Moments of $\delta = \gamma_1 - \gamma_2$

As an alternative to the simulation study discussed in previous sections, another method is available to determine a confidence interval for the ratio of the means from two lognormal populations. This method requires the derivation of the first four central moments from the original problem specification.

The next section describes the derivation of the required central moments for the ratio of means from two lognormally distributed populations. This is performed for the setting described in previous sections of this chapter, *i.e.* populations without zero observations. Only the main results are given in the next section and the interested reader is referred to the appendices to this chapter for a more complete discussion of the derivation.

3.6 First Four Central Moments:

First let us restate and re-define the problem setting as follows: consider a random variable X_j that has a lognormal distribution and let μ_j and $\sigma^2{}_j$ denote the mean and variance respectively of $\ln(X_j)$ such that $Y = \ln(X_j) \sim N(\mu_j, \sigma_j{}^2)$, for $j = 1,2$. The mean of the particular lognormal distribution is defined as:

$$M_j = \exp(\mu_j + \frac{1}{2}\sigma_j^2).$$

Furthermore, define:

$$\delta = \log\left(\frac{M_1}{M_2}\right) = \left(\mu_1 + \frac{1}{2}\sigma_1^2\right) - \left(\mu_2 + \frac{1}{2}\sigma_2^2\right)$$

$$= (\mu_1 - \mu_2) + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right) = \gamma_1 - \gamma_2$$

We are interested in the mean and variance, as well as the third and fourth central moments of the posterior distribution of δ. If the same prior distributions as in Section 2.2 are used, then:

$$\mu_j \mid \sigma_j^2, data \sim N(\hat{\mu}_j, \sigma_j^2 / n_j) \text{ for } (j = 1,2)$$

where

$$\hat{\mu}_j = \overline{y}_{j.} = \frac{1}{n_j}\sum_{i=1}^{n_j} y_{ij} \, .$$

Furthermore, the posterior distribution of ${\sigma^2}_j$ is an Inverse Gamma distribution and can be written as:

$$p\left(\sigma_j^2 \mid data\right) = K\left(\sigma_j^2\right)^{-\frac{1}{2}(\nu_j+2)} \exp\left\{-\frac{1}{2}\frac{\nu_j s_j^2}{\sigma_j^2}\right\}; \quad \sigma_j^2 > 0$$

where

$$K = \left(\frac{\nu_j s_j^2}{2}\right)^{\frac{1}{2}\nu_j} \frac{1}{\Gamma\left(\frac{1}{2}\nu_j\right)},$$

$\nu_j = n_j - 1$ and

$$s_j^2 = \frac{1}{\nu_j}\sum_{i=1}^{n_j}(y_{ij} - \overline{y}_j)^2 = \hat{\sigma}_j^2 \, .$$

We are now ready to state the following theorem:

<u>Theorem 3.5</u>

The mean and variance and the third and fourth central moments of the posterior distribution of δ are respectively given by:

$$E\left(\delta \mid data\right) = \left(\overline{y}_{1\bullet} - \overline{y}_{2\bullet}\right) + \frac{1}{2}\left(\frac{\nu_1 s_1^2}{\nu_1 - 2} - \frac{\nu_2 s_2^2}{\nu_2 - 2}\right), \tag{3.8}$$

$$Var(\delta \mid data) = \sum_{j=1}^{2}\frac{\nu_j s_j^2}{(\nu_j+1)(\nu_j-2)} + \frac{1}{2}\sum_{j=1}^{2}\frac{\left(\nu_j s_j^2\right)^2}{(\nu_j-2)^2(\nu_j-4)}, \tag{3.9}$$

$$E\left\{\left[\delta - E\left(\delta\right)\right]^3 \mid data\right\} = 3\left\{\frac{\left(\nu_1 s_1\right)^2}{n_1(\nu_1-2)^2(\nu_1-4)} - \frac{\left(\nu_2 s_2\right)^2}{n_2(\nu_2-2)^2(\nu_2-4)}\right\}$$

$$+2\left\{\frac{\left(\nu_1 s_1\right)^3}{(\nu_1-2)^3(\nu_1-4)(\nu_1-6)}-\frac{\left(\nu_2 s_2\right)^3}{(\nu_2-2)^3(\nu_2-4)(\nu_2-6)}\right\} \tag{3.10}$$

and

$$E\left\{\left[\delta-E(\delta)\right]^4 \mid data\right\}=3\sum_{j=1}^{2}\frac{\left(\nu_j s_j^2\right)}{n_j^2(\nu_j-2)(\nu_j-4)}+3\sum_{j\neq i}^{2}\frac{\left(\nu_j s_j^2\right)\left(\nu_i s_i^2\right)}{n_j n_i(\nu_j-2)(\nu_i-2)}$$

$$+3\sum_{j=1}^{2}\frac{\left(\nu_j+2\right)\left(\nu_j s_j^2\right)^3}{n_j(\nu_j-2)^3(\nu_j-4)(\nu_j-6)}+3\sum_{j\neq i}^{2}\frac{\left(\nu_j s_j^2\right)^2\left(\nu_i s_i^2\right)}{n_i(\nu_j-2)(\nu_j-4)(\nu_i-2)}$$

$$+\frac{3}{4}\sum_{j=1}^{2}\frac{\left(\nu_j+10\right)\left(\nu_j s_j^2\right)^4}{n_j(\nu_j-2)^4(\nu_j-4)(\nu_j-6)(\nu_j-8)}$$

$$+\frac{3}{4}\sum_{j\neq i}^{2}\frac{\left(\nu_j s_j^2\right)^2\left(\nu_i s_i^2\right)^2}{(\nu_j-2)^2(\nu_i-2)^2(\nu_j-4)(\nu_i-4)}. \tag{3.11}$$

<u>Proof:</u> The proof is given in the appendix to this chapter.

Appendix to Chapter 3
Proof of Theorem 3.3

The parameter of interest in the mode of the lognormal distribution is

$$M_o = t(\theta) = e^{\mu-\sigma^2}.$$

Define

$$A = \frac{\partial(\mu,\sigma^2)}{\partial(t(\theta),\sigma^2)} = \begin{bmatrix} 1/t(\theta) & 1 \\ 0 & 1 \end{bmatrix}.$$

Hence, the Fisher information matrix under the reparameterisation $(t(\theta),\sigma^2)$ is given by

$$F(t(\theta),\sigma^2) = A'F(\mu,\sigma^2)A = \begin{bmatrix} \dfrac{1}{t^2(\theta)\sigma^2} & \dfrac{1}{t(\theta)\sigma^2} \\ \dfrac{1}{t(\theta)\sigma^2} & \dfrac{1}{\sigma^2}+\dfrac{1}{2\sigma^4} \end{bmatrix}.$$

In this case

$$h_1^{\frac{1}{2}} = \left|\frac{1}{t^2(\theta)\sigma^2} - \left(\frac{1}{t(\theta)\sigma^2}\right)^2\left(\frac{1}{\sigma^2}+\frac{1}{2\sigma^4}\right)^{-1}\right|^{\frac{1}{2}} = \frac{1}{t(\theta)}\left(\frac{1}{\sigma^2}-\frac{2}{2\sigma^2+1}\right)^{\frac{1}{2}}$$

$$h_2^{\frac{1}{2}} = \left[\frac{1}{\sigma^2}\left(1+\frac{1}{2\sigma^2}\right)\right]^{\frac{1}{2}}.$$

Therefore, the Reference prior relative to the ordered parameterisation $(t(\theta),\sigma^2)$ is given by

$$\pi_R(t(\theta),\sigma^2) \propto \frac{1}{t(\theta)}\frac{1}{\sigma}\sqrt{1+\frac{1}{2\sigma^2}}.$$

In the (μ,σ^2) parameterisation this corresponds to

$$\pi_R\left(\mu,\sigma^2\right)\propto\frac{1}{t(\boldsymbol{\theta})}\frac{1}{\sigma}\sqrt{1+\frac{1}{2\sigma^2}}\left(t(\boldsymbol{\theta})\right)\propto\frac{1}{\sigma}\sqrt{1+\frac{1}{2\sigma^2}}.$$

This is the same result derived by Roman (2008).

Proof of Theorem 3.4

$$M_o=t(\boldsymbol{\theta})=e^{\mu-\sigma^2}$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial\mu}=e^{\mu-\sigma^2}\text{ and }\frac{\partial t(\boldsymbol{\theta})}{\partial\sigma^2}=-e^{\mu-\sigma^2}.$$

Also

$$\nabla_t'(\boldsymbol{\theta})=\begin{bmatrix}\frac{\partial t(\boldsymbol{\theta})}{\partial\mu} & \frac{\partial t(\boldsymbol{\theta})}{\partial\sigma^2}\end{bmatrix}=e^{\mu-\sigma^2}\begin{bmatrix}1 & -1\end{bmatrix}$$

and

$$\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})=e^{\mu-\sigma^2}\begin{bmatrix}\sigma^2 & -2\sigma^4\end{bmatrix}$$

$$\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})=e^{2\mu-2\sigma^2}\left(\sigma^2+2\sigma^4\right)$$

and

Therefore

$$\gamma'(\boldsymbol{\theta})=\frac{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})}{\sqrt{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}}=\begin{bmatrix}\gamma_1(\boldsymbol{\theta}) & \gamma_2(\boldsymbol{\theta})\end{bmatrix}=\frac{1}{\sqrt{\sigma^2+2\sigma^4}}\begin{bmatrix}\sigma^2 & -2\sigma^4\end{bmatrix}.$$

For a prior $p_P(\boldsymbol{\theta})=p_P\left(\mu,\sigma^2\right)$ to be a Probability-Matching prior, the differential equation

$$\frac{\partial}{\partial\mu}\left[\gamma_1(\boldsymbol{\theta})p_P(\boldsymbol{\theta})\right]+\frac{\partial}{\partial\sigma^2}\left[\gamma_2(\boldsymbol{\theta})p_P(\boldsymbol{\theta})\right]=0$$

must be satisfied. If we take

$$\pi_P\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma^2}\sqrt{1+\frac{1}{2\sigma^2}}$$

then the differential equation will be satisfied.

Proof of Theorem 3.5

From the posterior distribution of μ_j (conditionally normally distributed) and $\sigma^2{}_j$ we know that

$$E\left\{\mu_j \mid \sigma_j^2, data\right\} = \bar{y}_j, \tag{A.1}$$

$$E\left\{\left(\sigma_j^2\right)^r \mid data\right\} = \frac{\left(\frac{1}{2}\nu_j s_j^2\right)^r \Gamma\left(\frac{1}{2}\nu_j - r\right)}{\Gamma\left(\frac{1}{2}\nu_j\right)}, \tag{A.2}$$

$$Var\left\{\mu_j \mid \sigma_j^2, data\right\} = E\left\{\left(\mu_j - \bar{y}_j\right)^2 \mid \sigma_j^2, data\right\} = \frac{\sigma_j^2}{n_j} \tag{A.3}$$

$$E\left\{\left(\mu_j - \bar{y}_j\right)^3 \mid \sigma_j^2, data\right\} = 0 \text{ and} \tag{A.4}$$

$$E\left\{\left(\mu_j - \bar{y}_j\right)^4 \mid \sigma_j^2, data\right\} = 3\left(\frac{\sigma_j^2}{n_j}\right)^2. \tag{A.5}$$

Now, the following equations represent the relationships between the moments around the origin, μ_1', μ_2', μ_3' and μ_4' and the central moments, μ_1, μ_2, μ_3 and μ_4, where $\mu_1' = \mu_1$, are given by:

$$\mu_2' = \mu_2 + \left(\mu_1'\right)^2, \tag{A.6}$$

$$\mu_3' = \mu_3 + 3\mu_2\mu_1' + \left(\mu_1'\right)^3 \text{ and} \tag{A.7}$$

$$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\left(\mu_1'\right)^2 \mu_2 + \left(\mu_1'\right)^4. \tag{A.8}$$

The above relationships were derived by expanding the definitions of the central moments, *i.e.* expanding $\mu_r = E\left\{\left[X-\mu_1'\right]^r\right\}$ and $\mu_r' = E\left\{X^r\right\}$.

Proof of (3.8)

From the definition and posterior distribution of δ we know the following:

$$E\left(\delta \mid \sigma_1^2, \sigma_2^2, data\right) = \left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right) + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)$$

and using (A.2) above and taking the expectation with respect to each σ_j^2, we find that the unconditional expectation can be written as

$$E\left(\delta \mid data\right) = \left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right) + \frac{1}{2}\left(\frac{\nu_1 s_1^2}{\nu_1 - 2} - \frac{\nu_2 s_2^2}{\nu_2 - 2}\right) \qquad \text{(A.9)}$$

Proof of (3.9)

To find the unconditional variance, note that

$$\delta^2 = \left\{\left(\mu_1 - \mu_2\right) + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)\right\}^2$$

$$= \mu_1^2 - 2\mu_1\mu_2 + \mu_2^2 + \left(\mu_1 - \mu_2\right)\left(\sigma_1^2 - \sigma_2^2\right) + \frac{1}{4}\left(\sigma_1^2 - \sigma_2^2\right)^2.$$

Taking the expectation

$$E\left(\delta^2 \mid \sigma_1^2, \sigma_2^2, data\right) = \bar{y}_{1\bullet}^2 + \frac{1}{n_1}\sigma_1^2 - 2\bar{y}_{1\bullet}\bar{y}_{2\bullet} + \bar{y}_{2\bullet}^2 + \frac{1}{n_2}\sigma_2^2$$

$$+\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)\left(\sigma_1^2 - \sigma_2^2\right) + \frac{1}{4}\left(\sigma_1^2 - \sigma_2^2\right)^2 \qquad \text{(A.10)}$$

Once again, taking the expectation of (A.10) with respect to each σ_j^2 using relationship (A.2) we can calculate the following:

$$E\left(\delta^2 \mid data\right) = \bar{y}_{1\bullet}^2 + \frac{1}{n_1}\frac{\nu_1 s_1^2}{\nu_1 - 2} - 2\bar{y}_{1\bullet}\bar{y}_{2\bullet} + \bar{y}_{2\bullet}^2 + \frac{1}{n_2}\frac{\nu_2 s_2^2}{\nu_2 - 2}$$

$$+\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)\left\{\frac{\nu_1 s_1^2}{\nu_1 - 2} - \frac{\nu_2 s_2^2}{\nu_2 - 2}\right\}$$

$$+\frac{1}{4}\left\{\frac{\left(\nu_1 s_1^2\right)^2}{\left(\nu_1 - 2\right)\left(\nu_1 - 4\right)} - 2\frac{\nu_1 s_1^2}{\nu_1 - 2}\frac{\nu_2 s_2^2}{\nu_2 - 2} - \frac{\left(\nu_2 s_2^2\right)^2}{\left(\nu_2 - 2\right)\left(\nu_2 - 4\right)}\right\} \quad \text{(A.11)}$$

Lastly, combining (A.11) and squaring (A.9) we can find the unconditional variance of δ with respect to the following relationship:

$$Var\left(\delta \mid data\right) = E\left(\delta^2 \mid data\right) - \left\{E\left(\delta \mid data\right)\right\}^2$$

$$= \sum_{j=1}^{2}\frac{\nu_j s_j^2}{\left(\nu_j + 1\right)\left(\nu_j - 2\right)} + \frac{1}{2}\sum_{j=1}^{2}\frac{\left(\nu_j s_j^2\right)^2}{\left(\nu_j - 2\right)^2\left(\nu_j - 4\right)} \quad \text{(A.12)}$$

<u>Proof of (3.10)</u>

Since $\delta \mid \sigma_1^2, \sigma_2^2, data \sim N\left\{\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right) + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right); \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right\}$, or equivalently in a condensed notation $\delta \mid \sigma_1^2, \sigma_2^2, data \sim N\left\{\mu_{1\delta}'; \mu_{2\delta}\right\}$, and similarly to (A.4) and (A.5)

$$\mu_{3\delta} = 0 \text{ and } \mu_{4\delta} = 3\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2$$

then relationship (A.7) can be written as:

$$E\left(\delta^3 \mid \sigma_1^2, \sigma_2^2, data\right) = \mu_{3\delta} + 3\mu_{2\delta}\mu'_{1\delta} + \left(\mu'_{1\delta}\right)^3 \tag{A.13}$$

$$= 3\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right) + \frac{3}{2}\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)\left(\sigma_1^2 - \sigma_2^2\right)$$

$$+\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)^3 + \frac{3}{2}\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)^2\left(\sigma_1^2 - \sigma_2^2\right)$$

$$+\frac{3}{4}\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)\left(\sigma_1^2 - \sigma_2^2\right)^2 + \frac{1}{8}\left(\sigma_1^2 - \sigma_2^2\right)^3 \tag{A.14}$$

Taking the expectation of (A.14) with respect to σ_j^2 we obtain

$$E\left(\delta^3 \mid data\right) = 3\left(\frac{\nu_1 s_1^2}{(\nu_1 - 2)n_1} + \frac{\nu_2 s_2^2}{(\nu_2 - 2)n_2}\right)\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)$$

$$+\frac{3}{2}\left\{\frac{\left(\nu_1 s_1^2\right)^2}{n_1(\nu_1 - 2)(\nu_1 - 4)} + \frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{n_2(\nu_1 - 2)(\nu_2 - 2)} - \frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{n_1(\nu_1 - 2)(\nu_2 - 2)} - \frac{\left(\nu_2 s_2^2\right)^2}{n_2(\nu_2 - 2)(\nu_2 - 4)}\right\}$$

$$+\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)^3 + \frac{3}{2}\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)^2\left(\frac{\nu_1 s_1^2}{(\nu_1 - 2)} - \frac{\nu_2 s_2^2}{(\nu_2 - 2)}\right)$$

$$+\frac{3}{4}\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right)\left\{\frac{\left(\nu_1 s_1^2\right)^2}{(\nu_1 - 2)(\nu_1 - 4)} - \frac{2\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{(\nu_1 - 2)(\nu_2 - 2)} + \frac{\left(\nu_2 s_2^2\right)^2}{(\nu_2 - 2)(\nu_2 - 4)}\right\}$$

$$+\frac{1}{8}\left\{\frac{\left(\nu_1 s_1^2\right)^3}{(\nu_1 - 2)(\nu_1 - 4)(\nu_1 - 6)} - \frac{3\left(\nu_1 s_1^2\right)^2\left(\nu_2 s_2^2\right)}{(\nu_1 - 2)(\nu_1 - 4)(\nu_2 - 2)} + \frac{3\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)^2}{(\nu_1 - 2)(\nu_2 - 4)(\nu_2 - 2)} - \frac{\left(\nu_2 s_2^2\right)^3}{(\nu_2 - 2)(\nu_2 - 4)(\nu_2 - 6)}\right\}$$

$$\tag{A.15}$$

Once again, using a relationship similar to (A.7) we know that

$$E\left\{\left[\delta - E(\delta)\right]^3 \mid data\right\} = E\left(\delta^3 \mid data\right) - 3Var\left(\delta \mid data\right)E\left(\delta \mid data\right) - \left\{E\left(\delta \mid data\right)\right\}^3$$

and using (A.9), (A.12) and (A.15) this simplifies to:

$$E\left\{\left[\delta - E(\delta)\right]^3 \mid data\right\} = \left\{\frac{\left(\nu_1 s_1^2\right)^2}{n_1\left(\nu_1 - 2\right)^2\left(\nu_1 - 4\right)} - \frac{\left(\nu_2 s_2^2\right)^2}{n_2\left(\nu_2 - 2\right)^2\left(\nu_2 - 4\right)}\right\}$$

$$+2\left\{\frac{\left(\nu_1 s_1^2\right)^3}{\left(\nu_1 - 2\right)^3\left(\nu_1 - 4\right)\left(\nu_1 - 6\right)} - \frac{\left(\nu_2 s_2^2\right)^3}{\left(\nu_2 - 2\right)^3\left(\nu_2 - 4\right)\left(\nu_2 - 6\right)}\right\}$$

(A.16)

<u>Proof of (3.11)</u>

Since $\mu_{4\delta} = 3\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2$ we can re-write this relationship as

$$\mu_{4\delta} = 3\left(\frac{\sigma_1^4}{n_1^2} + \frac{\sigma_2^4}{n_2^2}\right) + 6\left(\frac{\sigma_1^2}{n_1}\right)\left(\frac{\sigma_2^2}{n_2}\right) \tag{A.17}$$

From relationship (A.8):

$$E\left(\delta^4 \mid \sigma_1^2, \sigma_2^2, data\right) = \mu_{4\delta} + 4\mu'_{1\delta}\mu_{3\delta} + 6\left(\mu'_{1\delta}\right)^2 \mu_{2\delta} + \left(\mu'_{1\delta}\right)^4$$

$$= 3\left(\frac{\sigma_1^4}{n_1^2} + \frac{\sigma_2^4}{n_2^2}\right) + 6\left(\frac{\sigma_1^2}{n_1}\right)\left(\frac{\sigma_2^2}{n_2}\right) + 0 + 6\left\{\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right) + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)\right\}^2\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$+\left\{\left(\bar{y}_{1\bullet} - \bar{y}_{2\bullet}\right) + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)\right\}^4 \tag{A.18}$$

Now, re-writing and taking the expectation of (A.18) with respect to σ_j^2 it is possible to derive the following using (A.9):

$$E\left(\delta^4 \mid data\right) = 3\left\{\frac{\left(\nu_1 s_1^2\right)^2}{n_1^2\left(\nu_1 - 2\right)\left(\nu_1 - 4\right)} + \frac{\left(\nu_2 s_2^2\right)^2}{n_2^2\left(\nu_2 - 2\right)\left(\nu_2 - 4\right)}\right\}$$

$$+6\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{n_1 n_2\left(\nu_1-2\right)\left(\nu_2-2\right)}+6\left(\bar{y}_{1\bullet}-\bar{y}_{2\bullet}\right)^2\left(\frac{\left(\nu_1 s_1^2\right)}{n_1\left(\nu_1-2\right)}+\frac{\left(\nu_2 s_2^2\right)}{n_2\left(\nu_2-2\right)}\right)$$

$$+6\left(\bar{y}_{1\bullet}-\bar{y}_{2\bullet}\right)\left\{\frac{\left(\nu_1 s_1^2\right)^2}{n_1\left(\nu_1-2\right)\left(\nu_1-4\right)}-\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{n_1\left(\nu_1-2\right)\left(\nu_2-2\right)}+\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{n_2\left(\nu_1-2\right)\left(\nu_2-2\right)}-\frac{\left(\nu_2 s_2^2\right)^2}{n_2\left(\nu_2-2\right)\left(\nu_2-4\right)}\right\}$$

$$+\frac{3}{2}\left\{\begin{array}{l}\frac{\left(\nu_1 s_1^2\right)^3}{n_1\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_1-6\right)}-\frac{2\left(\nu_1 s_1^2\right)^2\left(\nu_2 s_2^2\right)}{n_1\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_2-2\right)}+\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)^2}{n_1\left(\nu_1-2\right)\left(\nu_2-2\right)\left(\nu_2-4\right)}\\+\frac{\left(\nu_1 s_1^2\right)^2\left(\nu_2 s_2^2\right)}{n_2\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_2-2\right)}-\frac{2\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)^2}{n_2\left(\nu_1-2\right)\left(\nu_2-2\right)\left(\nu_2-4\right)}+\frac{\left(\nu_2 s_2^2\right)^3}{n_2\left(\nu_2-2\right)\left(\nu_2-4\right)\left(\nu_2-6\right)}\end{array}\right\}$$

$$+\left(\bar{y}_{1\bullet}-\bar{y}_{2\bullet}\right)^4+2\left(\bar{y}_{1\bullet}-\bar{y}_{2\bullet}\right)^3\left\{\frac{\nu_1 s_1^2}{\nu_1-2}-\frac{\nu_2 s_2^2}{\nu_2-2}\right\}$$

$$+\frac{3}{2}\left(\bar{y}_{1\bullet}-\bar{y}_{2\bullet}\right)^2\left\{\frac{\left(\nu_1 s_1^2\right)^2}{\left(\nu_1-2\right)\left(\nu_1-4\right)}-2\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)}{\left(\nu_1-2\right)\left(\nu_2-2\right)}+\frac{\left(\nu_2 s_2^2\right)^2}{\left(\nu_2-2\right)\left(\nu_2-2\right)}\right\}$$

$$+\frac{1}{2}\left(\bar{y}_{1\bullet}-\bar{y}_{2\bullet}\right)\left\{\begin{array}{l}\frac{\left(\nu_1 s_1^2\right)^3}{\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_1-6\right)}-3\frac{\left(\nu_1 s_1^2\right)^2\left(\nu_2 s_2^2\right)}{\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_2-2\right)}\\+3\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)^2}{\left(\nu_1-2\right)\left(\nu_2-2\right)\left(\nu_2-4\right)}-\frac{\left(\nu_2 s_2^2\right)^3}{\left(\nu_2-2\right)\left(\nu_2-4\right)\left(\nu_2-6\right)}\end{array}\right\}$$

$$+\frac{1}{16}\left\{\begin{array}{l}\frac{\left(\nu_1 s_1^2\right)^4}{\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_1-6\right)\left(\nu_1-8\right)}-4\frac{\left(\nu_1 s_1^2\right)^3\left(\nu_2 s_2^2\right)}{\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_1-6\right)\left(\nu_2-2\right)}\\+6\frac{\left(\nu_1 s_1^2\right)^2\left(\nu_2 s_2^2\right)^2}{\left(\nu_1-2\right)\left(\nu_1-4\right)\left(\nu_2-2\right)\left(\nu_2-4\right)}-4\frac{\left(\nu_1 s_1^2\right)\left(\nu_2 s_2^2\right)^3}{\left(\nu_1-2\right)\left(\nu_2-2\right)\left(\nu_2-4\right)\left(\nu_2-6\right)}\\+\frac{\left(\nu_2 s_2^2\right)^4}{\left(\nu_2-2\right)\left(\nu_2-4\right)\left(\nu_2-6\right)\left(\nu_2-8\right)}\end{array}\right\}$$

(A.19)

Using a relationship similar to (A.8) we know that

$$E\left\{\left[\delta-E\left(\delta\right)\right]^4 \mid data\right\}=E\left(\delta^4 \mid data\right)-4E\left(\delta \mid data\right)E\left\{\left[\delta-E\left(\delta\right)\right]^3 \mid data\right\}$$

$$-6\left\{E\left(\delta \mid data\right)\right\}^2 Var\left\{\delta \mid data\right\} - \left[E\left\{\delta \mid data\right\}\right]^4$$

(A.20)

Therefore, using the previous results obtained, namely (A.9), (A.12), (A.16) and (A.19), substituting these into (A.20) and simplifying, we arrive at the desired result:

$$E\left\{\left[\delta - E(\delta)\right]^4 \mid data\right\} = 3\sum_{i=1}^{2} \frac{\left(v_i s_i^2\right)^2}{n_i^2\left(v_i-2\right)\left(v_i-4\right)} + 3\sum_{i\neq j}^{2} \frac{\left(v_i s_i^2\right)\left(v_j s_j^2\right)}{n_i n_j\left(v_i-2\right)\left(v_j-2\right)}$$

$$+3\sum_{i=1}^{2} \frac{\left(v_{i+2}\right)\left(v_i s_i^2\right)^3}{n_i\left(v_i-2\right)^3\left(v_i-4\right)\left(v_i-6\right)} + 3\sum_{i\neq j}^{2} \frac{\left(v_i s_i^2\right)^2\left(v_j s_j^2\right)}{n_j\left(v_i-2\right)^2\left(v_i-4\right)\left(v_j-2\right)}$$

$$+\frac{3}{4}\sum_{i=1}^{2} \frac{\left(v_{i+10}\right)\left(v_i s_i^2\right)^4}{\left(v_i-2\right)^4\left(v_i-4\right)\left(v_i-6\right)\left(v_i-8\right)} + \frac{3}{4}\sum_{i\neq j}^{2} \frac{\left(v_i s_i^2\right)^2\left(v_j s_j^2\right)^2}{\left(v_i-2\right)^2\left(v_j-2\right)^2\left(v_i-4\right)\left(v_j-4\right)}$$

(A.21)

Individual Results of Simulation Studies

Table 20: Results for Design 1 – Equal-Tailed Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.942	0.008	5.050	0.025	0.033	0.138
		Prior 2	0.944	0.006	4.731	0.021	0.035	0.250
		Prior 3	0.951	-0.001	6.661	0.022	0.027	0.102
		Prior 4	0.955	-0.005	5.861	0.020	0.025	0.111
10	25	Prior 1	0.950	0.000	4.753	0.024	0.026	0.040
		Prior 2	0.955	-0.005	4.198	0.016	0.029	0.289
		Prior 3	0.938	0.012	5.848	0.031	0.031	0.000
		Prior 4	0.946	0.004	5.189	0.020	0.034	0.259
25	10	Prior 1	0.964	-0.014	2.809	0.014	0.022	0.222
		Prior 2	0.949	0.001	2.666	0.018	0.033	0.294
		Prior 3	0.953	-0.003	3.196	0.019	0.028	0.191
		Prior 4	0.949	0.001	3.000	0.014	0.037	0.451
25	25	Prior 1	0.947	0.003	2.213	0.018	0.035	0.321
		Prior 2	0.948	0.002	2.135	0.017	0.035	0.346
		Prior 3	0.966	-0.016	2.352	0.017	0.017	0.000
		Prior 4	0.957	-0.007	2.304	0.022	0.021	0.023
25	50	Prior 1	0.955	-0.005	2.100	0.018	0.027	0.200
		Prior 2	0.954	-0.004	2.089	0.031	0.015	0.348
		Prior 3	0.948	0.002	2.239	0.024	0.028	0.077
		Prior 4	0.948	0.002	2.189	0.017	0.035	0.346
25	100	Prior 1	0.951	-0.001	2.107	0.029	0.020	0.184
		Prior 2	0.942	0.008	2.032	0.027	0.031	0.069
		Prior 3	0.954	-0.004	2.181	0.021	0.025	0.087
		Prior 4	0.954	-0.004	2.147	0.023	0.023	0.000
50	25	Prior 1	0.948	0.002	1.521	0.028	0.024	0.077
		Prior 2	0.945	0.005	1.495	0.019	0.036	0.309
		Prior 3	0.950	0.000	1.569	0.018	0.032	0.280
		Prior 4	0.951	-0.001	1.554	0.017	0.032	0.306
50	50	Prior 1	0.958	-0.008	1.387	0.019	0.023	0.095
		Prior 2	0.948	0.002	1.378	0.027	0.025	0.038
		Prior 3	0.960	-0.010	1.414	0.021	0.019	0.050
		Prior 4	0.951	-0.001	1.400	0.014	0.035	0.429
50	100	Prior 1	0.947	0.003	1.344	0.023	0.030	0.132
		Prior 2	0.947	0.003	1.329	0.019	0.034	0.283
		Prior 3	0.947	0.003	1.370	0.015	0.038	0.434
		Prior 4	0.941	0.009	1.373	0.028	0.031	0.051
100	25	Prior 1	0.949	0.001	1.147	0.025	0.026	0.020
		Prior 2	0.946	0.004	1.134	0.019	0.035	0.296
		Prior 3	0.958	-0.008	1.183	0.017	0.025	0.190
		Prior 4	0.943	0.007	1.176	0.011	0.046	0.614
100	50	Prior 1	0.953	-0.003	0.992	0.021	0.026	0.106
		Prior 2	0.943	0.007	0.984	0.026	0.031	0.088
		Prior 3	0.942	0.008	1.007	0.025	0.033	0.138
		Prior 4	0.958	-0.008	1.004	0.024	0.018	0.143
100	100	Prior 1	0.943	0.007	0.933	0.028	0.029	0.018
		Prior 2	0.946	0.004	0.918	0.021	0.033	0.222
		Prior 3	0.966	-0.016	0.937	0.015	0.019	0.118
		Prior 4	0.944	0.006	0.931	0.029	0.027	0.036

Table 21: Results for Design 2 – Equal-Tailed Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.955	-0.005	9.840	0.020	0.025	0.111
		Prior 2	0.947	0.003	8.944	0.024	0.029	0.094
		Prior 3	0.937	0.013	12.758	0.026	0.037	0.175
		Prior 4	0.960	-0.010	11.378	0.018	0.022	0.100
10	25	Prior 1	0.949	0.001	7.121	0.017	0.034	0.333
		Prior 2	0.941	0.009	6.540	0.031	0.028	0.051
		Prior 3	0.944	0.006	8.981	0.032	0.024	0.143
		Prior 4	0.945	0.005	7.852	0.029	0.026	0.055
25	10	Prior 1	0.962	-0.012	7.329	0.020	0.018	0.053
		Prior 2	0.956	-0.006	6.479	0.021	0.023	0.045
		Prior 3	0.952	-0.002	8.936	0.025	0.023	0.042
		Prior 4	0.955	-0.005	7.867	0.021	0.024	0.067
25	25	Prior 1	0.953	-0.003	4.119	0.021	0.026	0.106
		Prior 2	0.953	-0.003	3.997	0.023	0.024	0.021
		Prior 3	0.949	0.001	4.396	0.016	0.035	0.373
		Prior 4	0.951	-0.001	4.316	0.026	0.023	0.061
25	50	Prior 1	0.950	0.000	3.333	0.020	0.030	0.200
		Prior 2	0.952	-0.002	3.299	0.022	0.026	0.083
		Prior 3	0.944	0.006	3.576	0.025	0.031	0.107
		Prior 4	0.961	-0.011	3.421	0.020	0.019	0.026
25	100	Prior 1	0.950	0.000	3.062	0.025	0.025	0.000
		Prior 2	0.950	0.000	2.984	0.022	0.028	0.120
		Prior 3	0.955	-0.005	3.167	0.017	0.028	0.244
		Prior 4	0.960	-0.010	3.119	0.020	0.020	0.000
50	25	Prior 1	0.937	0.013	3.401	0.030	0.033	0.048
		Prior 2	0.947	0.003	3.335	0.028	0.025	0.057
		Prior 3	0.947	0.003	3.557	0.023	0.030	0.132
		Prior 4	0.944	0.006	3.463	0.029	0.027	0.036
50	50	Prior 1	0.957	-0.007	2.546	0.019	0.024	0.116
		Prior 2	0.953	-0.003	2.523	0.022	0.025	0.064
		Prior 3	0.927	0.023	2.614	0.035	0.038	0.041
		Prior 4	0.953	-0.003	2.579	0.022	0.025	0.064
50	100	Prior 1	0.946	0.004	2.130	0.024	0.030	0.111
		Prior 2	0.955	-0.005	2.123	0.029	0.016	0.289
		Prior 3	0.942	0.008	2.179	0.033	0.025	0.138
		Prior 4	0.944	0.006	2.148	0.020	0.036	0.286
100	25	Prior 1	0.954	-0.004	3.090	0.019	0.027	0.174
		Prior 2	0.943	0.007	2.961	0.034	0.023	0.193
		Prior 3	0.959	-0.009	3.227	0.019	0.022	0.073
		Prior 4	0.944	0.006	3.163	0.029	0.027	0.036
100	50	Prior 1	0.953	-0.003	2.137	0.022	0.025	0.064
		Prior 2	0.945	0.005	2.131	0.029	0.026	0.055
		Prior 3	0.934	0.016	2.190	0.025	0.041	0.242
		Prior 4	0.946	0.004	2.170	0.026	0.028	0.037
100	100	Prior 1	0.949	0.001	1.680	0.030	0.021	0.176
		Prior 2	0.948	0.002	1.680	0.032	0.020	0.231
		Prior 3	0.949	0.001	1.707	0.032	0.019	0.255
		Prior 4	0.940	0.010	1.682	0.026	0.034	0.133

Table 22: Results for Design 3 – Equal-Tailed Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.952	-0.002	3.273	0.014	0.034	0.417
		Prior 2	0.957	-0.007	2.992	0.015	0.028	0.302
		Prior 3	0.956	-0.006	4.142	0.011	0.033	0.500
		Prior 4	0.955	-0.005	3.764	0.023	0.022	0.022
10	25	Prior 1	0.950	0.000	3.065	0.020	0.030	0.200
		Prior 2	0.949	0.001	2.791	0.022	0.029	0.137
		Prior 3	0.948	0.002	3.989	0.024	0.028	0.077
		Prior 4	0.946	0.004	3.439	0.030	0.024	0.111
25	10	Prior 1	0.957	-0.007	1.718	0.018	0.025	0.163
		Prior 2	0.944	0.006	1.580	0.013	0.043	0.536
		Prior 3	0.962	-0.012	1.922	0.015	0.023	0.211
		Prior 4	0.959	-0.009	1.789	0.013	0.028	0.366
25	25	Prior 1	0.948	0.002	1.413	0.021	0.031	0.192
		Prior 2	0.967	-0.017	1.391	0.009	0.024	0.455
		Prior 3	0.961	-0.011	1.509	0.017	0.022	0.128
		Prior 4	0.952	-0.002	1.501	0.022	0.026	0.083
25	50	Prior 1	0.967	-0.017	1.361	0.011	0.022	0.333
		Prior 2	0.952	-0.002	1.353	0.018	0.030	0.250
		Prior 3	0.951	-0.001	1.478	0.020	0.029	0.184
		Prior 4	0.955	-0.005	1.411	0.023	0.022	0.022
25	100	Prior 1	0.952	-0.002	1.387	0.025	0.023	0.042
		Prior 2	0.959	-0.009	1.344	0.014	0.027	0.317
		Prior 3	0.949	0.001	1.433	0.019	0.032	0.255
		Prior 4	0.957	-0.007	1.416	0.020	0.023	0.070
50	25	Prior 1	0.939	0.011	0.952	0.025	0.036	0.180
		Prior 2	0.939	0.011	0.943	0.023	0.038	0.246
		Prior 3	0.942	0.008	0.978	0.020	0.038	0.310
		Prior 4	0.943	0.007	0.966	0.025	0.032	0.123
50	50	Prior 1	0.964	-0.014	0.898	0.014	0.022	0.222
		Prior 2	0.949	0.001	0.896	0.025	0.026	0.020
		Prior 3	0.934	0.016	0.925	0.029	0.037	0.121
		Prior 4	0.952	-0.002	0.915	0.018	0.030	0.250
50	100	Prior 1	0.945	0.005	0.876	0.021	0.034	0.236
		Prior 2	0.946	0.004	0.873	0.032	0.022	0.185
		Prior 3	0.946	0.004	0.901	0.031	0.023	0.148
		Prior 4	0.950	0.000	0.885	0.016	0.034	0.360
100	25	Prior 1	0.951	-0.001	0.698	0.021	0.028	0.143
		Prior 2	0.950	0.000	0.685	0.026	0.024	0.040
		Prior 3	0.955	-0.005	0.715	0.014	0.031	0.378
		Prior 4	0.948	0.002	0.706	0.018	0.034	0.308
100	50	Prior 1	0.943	0.007	0.626	0.026	0.031	0.088
		Prior 2	0.944	0.006	0.625	0.031	0.025	0.107
		Prior 3	0.944	0.006	0.631	0.018	0.038	0.357
		Prior 4	0.947	0.003	0.634	0.027	0.026	0.019
100	100	Prior 1	0.951	-0.001	0.604	0.024	0.025	0.020
		Prior 2	0.955	-0.005	0.603	0.023	0.022	0.022
		Prior 3	0.943	0.007	0.615	0.031	0.026	0.088
		Prior 4	0.948	0.002	0.602	0.013	0.039	0.500

Table 23: Results for Design 4 – Equal-Tailed Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.942	0.008	5.050	0.025	0.033	0.138
		Prior 2	0.944	0.006	4.731	0.021	0.035	0.250
		Prior 3	0.951	-0.001	6.661	0.022	0.027	0.102
		Prior 4	0.955	-0.005	5.861	0.020	0.025	0.111
10	25	Prior 1	0.950	0.000	4.753	0.024	0.026	0.040
		Prior 2	0.955	-0.005	4.198	0.016	0.029	0.289
		Prior 3	0.938	0.012	5.848	0.031	0.031	0.000
		Prior 4	0.946	0.004	5.189	0.020	0.034	0.259
25	10	Prior 1	0.964	-0.014	2.809	0.014	0.022	0.222
		Prior 2	0.949	0.001	2.666	0.018	0.033	0.294
		Prior 3	0.953	-0.003	3.196	0.019	0.028	0.191
		Prior 4	0.949	0.001	3.000	0.014	0.037	0.451
25	25	Prior 1	0.947	0.003	2.213	0.018	0.035	0.321
		Prior 2	0.948	0.002	2.135	0.017	0.035	0.346
		Prior 3	0.966	-0.016	2.352	0.017	0.017	0.000
		Prior 4	0.957	-0.007	2.304	0.022	0.021	0.023
25	50	Prior 1	0.955	-0.005	2.100	0.018	0.027	0.200
		Prior 2	0.954	-0.004	2.089	0.031	0.015	0.348
		Prior 3	0.948	0.002	2.239	0.024	0.028	0.077
		Prior 4	0.948	0.002	2.189	0.017	0.035	0.346
25	100	Prior 1	0.951	-0.001	2.107	0.029	0.020	0.184
		Prior 2	0.942	0.008	2.032	0.027	0.031	0.069
		Prior 3	0.954	-0.004	2.181	0.021	0.025	0.087
		Prior 4	0.954	-0.004	2.147	0.023	0.023	0.000
50	25	Prior 1	0.948	0.002	1.521	0.028	0.024	0.077
		Prior 2	0.945	0.005	1.495	0.019	0.036	0.309
		Prior 3	0.950	0.000	1.569	0.018	0.032	0.280
		Prior 4	0.951	-0.001	1.554	0.017	0.032	0.306
50	50	Prior 1	0.958	-0.008	1.387	0.019	0.023	0.095
		Prior 2	0.948	0.002	1.378	0.027	0.025	0.038
		Prior 3	0.960	-0.010	1.414	0.021	0.019	0.050
		Prior 4	0.951	-0.001	1.400	0.014	0.035	0.429
50	100	Prior 1	0.947	0.003	1.344	0.023	0.030	0.132
		Prior 2	0.947	0.003	1.329	0.019	0.034	0.283
		Prior 3	0.947	0.003	1.370	0.015	0.038	0.434
		Prior 4	0.941	0.009	1.373	0.028	0.031	0.051
100	25	Prior 1	0.949	0.001	1.147	0.025	0.026	0.020
		Prior 2	0.946	0.004	1.134	0.019	0.035	0.296
		Prior 3	0.958	-0.008	1.183	0.017	0.025	0.190
		Prior 4	0.943	0.007	1.176	0.011	0.046	0.614
100	50	Prior 1	0.953	-0.003	0.992	0.021	0.026	0.106
		Prior 2	0.943	0.007	0.984	0.026	0.031	0.088
		Prior 3	0.942	0.008	1.007	0.025	0.033	0.138
		Prior 4	0.958	-0.008	1.004	0.024	0.018	0.143
100	100	Prior 1	0.943	0.007	0.933	0.028	0.029	0.018
		Prior 2	0.946	0.004	0.918	0.021	0.033	0.222
		Prior 3	0.966	-0.016	0.937	0.015	0.019	0.118
		Prior 4	0.944	0.006	0.931	0.029	0.027	0.036

Table 24: Results for Design 5 – Equal-Tailed Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.954	-0.004	46.710	0.015	0.031	0.348
		Prior 2	0.953	-0.003	42.795	0.018	0.029	0.234
		Prior 3	0.947	0.003	59.779	0.024	0.029	0.094
		Prior 4	0.952	-0.002	53.149	0.023	0.025	0.042
10	25	Prior 1	0.950	0.000	45.454	0.018	0.032	0.280
		Prior 2	0.953	-0.003	41.351	0.021	0.026	0.106
		Prior 3	0.935	0.015	57.378	0.029	0.036	0.108
		Prior 4	0.945	0.005	50.807	0.025	0.030	0.091
25	10	Prior 1	0.954	-0.004	22.581	0.022	0.024	0.043
		Prior 2	0.944	0.006	21.121	0.016	0.040	0.429
		Prior 3	0.962	-0.012	23.718	0.012	0.026	0.368
		Prior 4	0.953	-0.003	23.047	0.015	0.032	0.362
25	25	Prior 1	0.947	0.003	20.485	0.026	0.027	0.019
		Prior 2	0.964	-0.014	20.193	0.013	0.023	0.278
		Prior 3	0.962	-0.012	21.958	0.018	0.020	0.053
		Prior 4	0.960	-0.010	21.614	0.021	0.019	0.050
25	50	Prior 1	0.966	-0.016	20.147	0.012	0.022	0.294
		Prior 2	0.952	-0.002	20.031	0.018	0.030	0.250
		Prior 3	0.947	0.003	21.801	0.028	0.025	0.057
		Prior 4	0.938	0.012	21.308	0.025	0.037	0.194
25	100	Prior 1	0.953	-0.003	20.689	0.024	0.023	0.021
		Prior 2	0.956	-0.006	20.043	0.017	0.027	0.227
		Prior 3	0.954	-0.004	21.562	0.023	0.023	0.000
		Prior 4	0.949	0.001	21.240	0.026	0.025	0.020
50	25	Prior 1	0.936	0.014	13.351	0.027	0.037	0.156
		Prior 2	0.934	0.016	13.275	0.030	0.036	0.091
		Prior 3	0.951	-0.001	13.692	0.020	0.029	0.184
		Prior 4	0.946	0.004	13.699	0.023	0.031	0.148
50	50	Prior 1	0.958	-0.008	13.107	0.018	0.024	0.143
		Prior 2	0.948	0.002	13.081	0.025	0.027	0.038
		Prior 3	0.958	-0.008	13.373	0.021	0.021	0.000
		Prior 4	0.952	-0.002	13.251	0.016	0.032	0.333
50	100	Prior 1	0.944	0.006	12.984	0.022	0.034	0.214
		Prior 2	0.945	0.005	12.936	0.035	0.020	0.273
		Prior 3	0.940	0.010	13.361	0.022	0.038	0.267
		Prior 4	0.938	0.012	13.398	0.029	0.033	0.065
100	25	Prior 1	0.947	0.003	9.276	0.026	0.027	0.019
		Prior 2	0.950	0.000	9.191	0.025	0.025	0.000
		Prior 3	0.949	0.001	9.402	0.017	0.034	0.333
		Prior 4	0.945	0.005	9.325	0.014	0.041	0.491
100	50	Prior 1	0.934	0.016	8.911	0.034	0.032	0.030
		Prior 2	0.948	0.002	8.910	0.025	0.027	0.038
		Prior 3	0.946	0.004	9.031	0.019	0.035	0.296
		Prior 4	0.955	-0.005	9.052	0.023	0.022	0.022
100	100	Prior 1	0.947	0.003	8.856	0.026	0.027	0.019
		Prior 2	0.956	-0.006	8.822	0.023	0.021	0.045
		Prior 3	0.964	-0.014	8.925	0.015	0.021	0.167
		Prior 4	0.946	0.004	8.845	0.024	0.030	0.111

Table 25: Results for Design 1 – HPD Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9760	-0.0260	4.6714	0.0220	0.0020	0.8333
		Prior 2	0.9330	0.0170	3.9112	0.0610	0.0060	0.8209
		Prior 3	0.9900	-0.0400	7.8038	0.0050	0.0050	0.0000
		Prior 4	0.9840	-0.0340	6.1121	0.0100	0.0060	0.2500
10	25	Prior 1	0.9560	-0.0060	4.0985	0.0380	0.0060	0.7273
		Prior 2	0.9300	0.0200	3.4780	0.0670	0.0030	0.9143
		Prior 3	0.9880	-0.0380	6.2292	0.0080	0.0040	0.3333
		Prior 4	0.9720	-0.0220	5.0025	0.0260	0.0020	0.8571
25	10	Prior 1	0.9770	-0.0270	2.7080	0.0180	0.0050	0.5652
		Prior 2	0.9630	-0.0130	2.4075	0.0300	0.0070	0.6216
		Prior 3	0.9870	-0.0370	3.6598	0.0090	0.0040	0.3846
		Prior 4	0.9750	-0.0250	3.0990	0.0130	0.0120	0.0400
25	25	Prior 1	0.9600	-0.0100	2.1201	0.0340	0.0060	0.7000
		Prior 2	0.9380	0.0120	1.9625	0.0580	0.0040	0.8710
		Prior 3	0.9710	-0.0210	2.4125	0.0190	0.0100	0.3103
		Prior 4	0.9740	-0.0240	2.2689	0.0210	0.0050	0.6154
25	50	Prior 1	0.9530	-0.0030	1.9881	0.0380	0.0090	0.6170
		Prior 2	0.9440	0.0060	1.8703	0.0460	0.0100	0.6429
		Prior 3	0.9690	-0.0190	2.2726	0.0190	0.0120	0.2258
		Prior 4	0.9610	-0.0110	2.1003	0.0320	0.0070	0.6410
25	100	Prior 1	0.9380	0.0120	1.9518	0.0530	0.0090	0.7097
		Prior 2	0.9480	0.0020	1.8397	0.0470	0.0050	0.8077
		Prior 3	0.9600	-0.0100	2.2142	0.0280	0.0120	0.4000
		Prior 4	0.9580	-0.0080	2.0520	0.0330	0.0090	0.5714
50	25	Prior 1	0.9700	-0.0200	1.4832	0.0190	0.0110	0.2667
		Prior 2	0.9550	-0.0050	1.4320	0.0310	0.0140	0.3778
		Prior 3	0.9690	-0.0190	1.6328	0.0190	0.0120	0.2258
		Prior 4	0.9650	-0.0150	1.5487	0.0250	0.0100	0.4286
50	50	Prior 1	0.9460	0.0040	1.3362	0.0440	0.0100	0.6296
		Prior 2	0.9480	0.0020	1.3092	0.0390	0.0130	0.5000
		Prior 3	0.9550	-0.0050	1.4186	0.0340	0.0110	0.5111
		Prior 4	0.9590	-0.0090	1.3943	0.0290	0.0120	0.4146
50	100	Prior 1	0.9550	-0.0050	1.2993	0.0340	0.0110	0.5111
		Prior 2	0.9410	0.0090	1.2539	0.0520	0.0070	0.7627
		Prior 3	0.9560	-0.0060	1.3795	0.0280	0.0160	0.2727
		Prior 4	0.9520	-0.0020	1.3239	0.0300	0.0180	0.2500
100	25	Prior 1	0.9590	-0.0090	1.1295	0.0290	0.0120	0.4146
		Prior 2	0.9530	-0.0030	1.1080	0.0260	0.0210	0.1064
		Prior 3	0.9640	-0.0140	1.2220	0.0160	0.0200	0.1111
		Prior 4	0.9620	-0.0120	1.1678	0.0250	0.0130	0.3158
100	50	Prior 1	0.9510	-0.0010	0.9769	0.0250	0.0240	0.0204
		Prior 2	0.9420	0.0080	0.9508	0.0410	0.0170	0.4138
		Prior 3	0.9580	-0.0080	1.0176	0.0200	0.0220	0.0476
		Prior 4	0.9500	0.0000	0.9920	0.0310	0.0190	0.2400
100	100	Prior 1	0.9440	0.0060	0.9040	0.0410	0.0150	0.4643
		Prior 2	0.9360	0.0140	0.8862	0.0480	0.0160	0.5000
		Prior 3	0.9590	-0.0090	0.9314	0.0260	0.0150	0.2683
		Prior 4	0.9490	0.0010	0.9134	0.0370	0.0140	0.4510

Table 26: Results for Design 2 – HPD Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9820	-0.0320	9.3278	0.0130	0.0050	0.4444
		Prior 2	0.9610	-0.0110	7.9338	0.0260	0.0130	0.3333
		Prior 3	0.9930	-0.0430	15.7816	0.0030	0.0040	0.1429
		Prior 4	0.9870	-0.0370	11.8379	0.0040	0.0090	0.3846
10	25	Prior 1	0.9690	-0.0190	6.7261	0.0260	0.0050	0.6774
		Prior 2	0.9400	0.0100	5.6835	0.0490	0.0110	0.6333
		Prior 3	0.9820	-0.0320	9.9831	0.0100	0.0080	0.1111
		Prior 4	0.9780	-0.0280	7.9336	0.0200	0.0020	0.8182
25	10	Prior 1	0.9760	-0.0260	6.5254	0.0040	0.0200	0.6667
		Prior 2	0.9580	-0.0080	5.7126	0.0080	0.0340	0.6190
		Prior 3	0.9880	-0.0380	9.8738	0.0060	0.0060	0.0000
		Prior 4	0.9780	-0.0280	7.9985	0.0050	0.0170	0.5455
25	25	Prior 1	0.9630	-0.0130	4.0543	0.0200	0.0170	0.0811
		Prior 2	0.9640	-0.0140	3.7452	0.0170	0.0190	0.0556
		Prior 3	0.9710	-0.0210	4.6650	0.0130	0.0160	0.1034
		Prior 4	0.9650	-0.0150	4.3494	0.0180	0.0170	0.0286
25	50	Prior 1	0.9650	-0.0150	3.3085	0.0190	0.0160	0.0857
		Prior 2	0.9540	-0.0040	3.1009	0.0290	0.0170	0.2609
		Prior 3	0.9730	-0.0230	3.7214	0.0130	0.0140	0.0370
		Prior 4	0.9600	-0.0100	3.4990	0.0240	0.0160	0.2000
25	100	Prior 1	0.9500	0.0000	2.9260	0.0340	0.0160	0.3600
		Prior 2	0.9270	0.0230	2.7509	0.0580	0.0150	0.5890
		Prior 3	0.9680	-0.0180	3.2681	0.0180	0.0140	0.1250
		Prior 4	0.9680	-0.0180	3.0667	0.0230	0.0090	0.4375
50	25	Prior 1	0.9610	-0.0110	3.3002	0.0120	0.0270	0.3846
		Prior 2	0.9540	-0.0040	3.0951	0.0190	0.0270	0.1739
		Prior 3	0.9670	-0.0170	3.6903	0.0180	0.0150	0.0909
		Prior 4	0.9650	-0.0150	3.4972	0.0140	0.0210	0.2000
50	50	Prior 1	0.9560	-0.0060	2.4863	0.0220	0.0220	0.0000
		Prior 2	0.9550	-0.0050	2.4199	0.0240	0.0210	0.0667
		Prior 3	0.9620	-0.0120	2.6901	0.0190	0.0190	0.0000
		Prior 4	0.9660	-0.0160	2.5985	0.0180	0.0160	0.0588
50	100	Prior 1	0.9520	-0.0020	2.1122	0.0290	0.0190	0.2083
		Prior 2	0.9540	-0.0040	2.0420	0.0280	0.0180	0.2174
		Prior 3	0.9670	-0.0170	2.2120	0.0150	0.0180	0.0909
		Prior 4	0.9490	0.0010	2.1598	0.0330	0.0180	0.2941
100	25	Prior 1	0.9540	-0.0040	2.9148	0.0080	0.0380	0.6522
		Prior 2	0.9390	0.0110	2.7753	0.0100	0.0510	0.6721
		Prior 3	0.9730	-0.0230	3.2536	0.0070	0.0200	0.4815
		Prior 4	0.9700	-0.0200	3.0908	0.0150	0.0150	0.0000
100	50	Prior 1	0.9530	-0.0030	2.1013	0.0220	0.0250	0.0638
		Prior 2	0.9480	0.0020	2.0358	0.0130	0.0390	0.5000
		Prior 3	0.9590	-0.0090	2.2198	0.0180	0.0230	0.1220
		Prior 4	0.9500	0.0000	2.1721	0.0220	0.0280	0.1200
100	100	Prior 1	0.9490	0.0010	1.6486	0.0310	0.0200	0.2157
		Prior 2	0.9630	-0.0130	1.6293	0.0180	0.0190	0.0270
		Prior 3	0.9570	-0.0070	1.7112	0.0230	0.0200	0.0698
		Prior 4	0.9570	-0.0070	1.6839	0.0210	0.0220	0.0233

Table 27: Results for Design 3 – HPD Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9680	-0.0180	3.0344	0.0250	0.0070	0.5625
		Prior 2	0.9290	0.0210	2.4629	0.0690	0.0020	0.9437
		Prior 3	0.9900	-0.0400	4.8140	0.0080	0.0020	0.6000
		Prior 4	0.9880	-0.0380	3.7254	0.0110	0.0010	0.8333
10	25	Prior 1	0.9440	0.0060	2.6485	0.0530	0.0030	0.8929
		Prior 2	0.9240	0.0260	2.2143	0.0720	0.0040	0.8947
		Prior 3	0.9870	-0.0370	4.0992	0.0100	0.0030	0.5385
		Prior 4	0.9720	-0.0220	3.1898	0.0240	0.0040	0.7143
25	10	Prior 1	0.9660	-0.0160	1.6454	0.0240	0.0100	0.4118
		Prior 2	0.9520	-0.0020	1.4892	0.0430	0.0050	0.7917
		Prior 3	0.9790	-0.0290	2.1353	0.0160	0.0050	0.5238
		Prior 4	0.9770	-0.0270	1.8380	0.0150	0.0080	0.3043
25	25	Prior 1	0.9550	-0.0050	1.3565	0.0340	0.0110	0.5111
		Prior 2	0.9370	0.0130	1.2499	0.0600	0.0030	0.9048
		Prior 3	0.9680	-0.0180	1.5686	0.0220	0.0100	0.3750
		Prior 4	0.9580	-0.0080	1.4458	0.0350	0.0070	0.6667
25	50	Prior 1	0.9460	0.0040	1.3121	0.0460	0.0080	0.7037
		Prior 2	0.9490	0.0010	1.2373	0.0420	0.0090	0.6471
		Prior 3	0.9550	-0.0050	1.4862	0.0280	0.0170	0.2444
		Prior 4	0.9600	-0.0100	1.4164	0.0260	0.0140	0.3000
25	100	Prior 1	0.9430	0.0070	1.2835	0.0510	0.0060	0.7895
		Prior 2	0.9420	0.0080	1.2297	0.0500	0.0080	0.7241
		Prior 3	0.9560	-0.0060	1.4801	0.0260	0.0180	0.1818
		Prior 4	0.9520	-0.0020	1.3784	0.0340	0.0140	0.4167
50	25	Prior 1	0.9580	-0.0080	0.9239	0.0250	0.0170	0.1905
		Prior 2	0.9490	0.0010	0.8973	0.0420	0.0090	0.6471
		Prior 3	0.9500	0.0000	1.0177	0.0260	0.0240	0.0400
		Prior 4	0.9590	-0.0090	0.9647	0.0320	0.0090	0.5610
50	50	Prior 1	0.9510	-0.0010	0.8672	0.0370	0.0120	0.5102
		Prior 2	0.9280	0.0220	0.8470	0.0570	0.0150	0.5833
		Prior 3	0.9530	-0.0030	0.9257	0.0320	0.0150	0.3617
		Prior 4	0.9430	0.0070	0.8991	0.0390	0.0180	0.3684
50	100	Prior 1	0.9430	0.0070	0.8465	0.0430	0.0140	0.5088
		Prior 2	0.9310	0.0190	0.8198	0.0630	0.0060	0.8261
		Prior 3	0.9570	-0.0070	0.9048	0.0270	0.0160	0.2558
		Prior 4	0.9410	0.0090	0.8749	0.0360	0.0230	0.2203
100	25	Prior 1	0.9490	0.0010	0.6865	0.0330	0.0180	0.2941
		Prior 2	0.9480	0.0020	0.6613	0.0380	0.0140	0.4615
		Prior 3	0.9640	-0.0140	0.7319	0.0240	0.0120	0.3333
		Prior 4	0.9630	-0.0130	0.7002	0.0210	0.0160	0.1351
100	50	Prior 1	0.9540	-0.0040	0.6172	0.0310	0.0150	0.3478
		Prior 2	0.9510	-0.0010	0.6056	0.0370	0.0120	0.5102
		Prior 3	0.9380	0.0120	0.6363	0.0460	0.0160	0.4839
		Prior 4	0.9550	-0.0050	0.6325	0.0290	0.0160	0.2889
100	100	Prior 1	0.9490	0.0010	0.5973	0.0350	0.0160	0.3725
		Prior 2	0.9350	0.0150	0.5785	0.0440	0.0210	0.3538
		Prior 3	0.9600	-0.0100	0.6086	0.0210	0.0190	0.0500
		Prior 4	0.9550	-0.0050	0.5945	0.0330	0.0120	0.4667

Table 28: Results for Design 4 – HPD Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9520	-0.0020	16.6503	0.0080	0.0400	0.6667
10	10	Prior 2	0.9270	0.0230	13.7130	0.0060	0.0670	0.8356
10	10	Prior 3	0.9870	-0.0370	26.8994	0.0050	0.0080	0.2308
10	10	Prior 4	0.9760	-0.0260	21.0004	0.0040	0.0200	0.6667
10	25	Prior 1	0.9640	-0.0140	8.7842	0.0060	0.0300	0.6667
10	25	Prior 2	0.9530	-0.0030	8.0792	0.0080	0.0390	0.6596
10	25	Prior 3	0.9820	-0.0320	11.1447	0.0060	0.0120	0.3333
10	25	Prior 4	0.9700	-0.0200	9.7959	0.0070	0.0230	0.5333
25	10	Prior 1	0.9560	-0.0060	15.5453	0.0010	0.0430	0.9545
25	10	Prior 2	0.9420	0.0080	13.1742	0.0020	0.0560	0.9310
25	10	Prior 3	0.9790	-0.0290	24.3382	0.0080	0.0130	0.2381
25	10	Prior 4	0.9580	-0.0080	18.6804	0.0050	0.0370	0.7619
25	25	Prior 1	0.9590	-0.0090	7.9776	0.0090	0.0320	0.5610
25	25	Prior 2	0.9380	0.0120	7.3406	0.0050	0.0570	0.8387
25	25	Prior 3	0.9650	-0.0150	9.0611	0.0150	0.0200	0.1429
25	25	Prior 4	0.9540	-0.0040	8.4699	0.0150	0.0310	0.3478
25	50	Prior 1	0.9450	0.0050	5.3021	0.0100	0.0450	0.6364
25	50	Prior 2	0.9400	0.0100	5.0620	0.0120	0.0480	0.6000
25	50	Prior 3	0.9700	-0.0200	5.7817	0.0120	0.0180	0.2000
25	50	Prior 4	0.9510	-0.0010	5.4223	0.0100	0.0390	0.5918
25	100	Prior 1	0.9430	0.0070	3.7434	0.0250	0.0320	0.1228
25	100	Prior 2	0.9470	0.0030	3.6532	0.0150	0.0380	0.4340
25	100	Prior 3	0.9570	-0.0070	3.9511	0.0130	0.0300	0.3953
25	100	Prior 4	0.9470	0.0030	3.8299	0.0190	0.0340	0.2830
50	25	Prior 1	0.9420	0.0080	7.5669	0.0090	0.0490	0.6897
50	25	Prior 2	0.9230	0.0270	7.1903	0.0050	0.0720	0.8701
50	25	Prior 3	0.9550	-0.0050	8.7226	0.0210	0.0240	0.0667
50	25	Prior 4	0.9440	0.0060	8.3323	0.0200	0.0360	0.2857
50	50	Prior 1	0.9460	0.0040	5.1445	0.0160	0.0380	0.4074
50	50	Prior 2	0.9500	0.0000	4.9229	0.0070	0.0430	0.7200
50	50	Prior 3	0.9490	0.0010	5.4288	0.0180	0.0330	0.2941
50	50	Prior 4	0.9530	-0.0030	5.2266	0.0110	0.0360	0.5319
50	100	Prior 1	0.9490	0.0010	3.5357	0.0200	0.0310	0.2157
50	100	Prior 2	0.9460	0.0040	3.4618	0.0150	0.0390	0.4444
50	100	Prior 3	0.9620	-0.0120	3.6634	0.0120	0.0260	0.3684
50	100	Prior 4	0.9510	-0.0010	3.5936	0.0190	0.0300	0.2245
100	25	Prior 1	0.9440	0.0060	7.5310	0.0120	0.0440	0.5714
100	25	Prior 2	0.9360	0.0140	7.2025	0.0080	0.0560	0.7500
100	25	Prior 3	0.9570	-0.0070	8.7510	0.0180	0.0250	0.1628
100	25	Prior 4	0.9660	-0.0160	8.1663	0.0080	0.0260	0.5294
100	50	Prior 1	0.9470	0.0030	5.0141	0.0080	0.0450	0.6981
100	50	Prior 2	0.9470	0.0030	4.9041	0.0090	0.0440	0.6604
100	50	Prior 3	0.9590	-0.0090	5.3207	0.0130	0.0280	0.3659
100	50	Prior 4	0.9390	0.0110	5.1859	0.0220	0.0390	0.2787
100	100	Prior 1	0.9300	0.0200	3.4553	0.0210	0.0490	0.4000
100	100	Prior 2	0.9480	0.0020	3.4038	0.0160	0.0360	0.3846
100	100	Prior 3	0.9390	0.0110	3.5704	0.0270	0.0340	0.1148
100	100	Prior 4	0.9390	0.0110	3.4976	0.0150	0.0460	0.5082

Table 29: Results for Design 5 – HPD Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9610	-0.0110	42.0286	0.0370	0.0020	0.8974
		Prior 2	0.9170	0.0330	33.1307	0.0800	0.0030	0.9277
		Prior 3	0.9840	-0.0340	65.0471	0.0110	0.0050	0.3750
		Prior 4	0.9780	-0.0280	51.3259	0.0190	0.0030	0.7273
10	25	Prior 1	0.9400	0.0100	37.9984	0.0530	0.0070	0.7667
		Prior 2	0.9320	0.0180	32.4749	0.0650	0.0030	0.9118
		Prior 3	0.9800	-0.0300	59.9525	0.0140	0.0060	0.4000
		Prior 4	0.9640	-0.0140	47.7541	0.0290	0.0070	0.6111
25	10	Prior 1	0.9500	0.0000	21.0905	0.0400	0.0100	0.6000
		Prior 2	0.9430	0.0070	19.6784	0.0530	0.0040	0.8596
		Prior 3	0.9730	-0.0230	26.4142	0.0170	0.0100	0.2593
		Prior 4	0.9630	-0.0130	23.2377	0.0350	0.0020	0.8919
25	25	Prior 1	0.9480	0.0020	19.7136	0.0440	0.0080	0.6923
		Prior 2	0.9410	0.0090	18.4556	0.0520	0.0070	0.7627
		Prior 3	0.9600	-0.0100	22.3421	0.0270	0.0130	0.3500
		Prior 4	0.9580	-0.0080	20.5554	0.0310	0.0110	0.4762
25	50	Prior 1	0.9340	0.0160	18.9412	0.0530	0.0130	0.6061
		Prior 2	0.9340	0.0160	18.1917	0.0560	0.0100	0.6970
		Prior 3	0.9470	0.0030	21.6797	0.0330	0.0200	0.2453
		Prior 4	0.9620	-0.0120	20.5223	0.0270	0.0110	0.4211
25	100	Prior 1	0.9400	0.0100	18.8515	0.0510	0.0090	0.7000
		Prior 2	0.9570	-0.0070	17.8733	0.0400	0.0030	0.8605
		Prior 3	0.9580	-0.0080	21.5753	0.0220	0.0200	0.0476
		Prior 4	0.9670	-0.0170	20.1917	0.0250	0.0080	0.5152
50	25	Prior 1	0.9450	0.0050	12.9297	0.0380	0.0170	0.3818
		Prior 2	0.9330	0.0170	12.4521	0.0550	0.0120	0.6418
		Prior 3	0.9630	-0.0130	13.9230	0.0230	0.0140	0.2432
		Prior 4	0.9550	-0.0050	13.4322	0.0270	0.0180	0.2000
50	50	Prior 1	0.9410	0.0090	12.5595	0.0450	0.0140	0.5254
		Prior 2	0.9310	0.0190	12.3670	0.0560	0.0130	0.6232
		Prior 3	0.9650	-0.0150	13.5397	0.0210	0.0140	0.2000
		Prior 4	0.9430	0.0070	12.9614	0.0380	0.0190	0.3333
50	100	Prior 1	0.9460	0.0040	12.5178	0.0410	0.0130	0.5185
		Prior 2	0.9310	0.0190	12.0836	0.0580	0.0110	0.6812
		Prior 3	0.9510	-0.0010	13.2897	0.0240	0.0250	0.0204
		Prior 4	0.9520	-0.0020	12.9590	0.0310	0.0170	0.2917
100	25	Prior 1	0.9570	-0.0070	9.0946	0.0350	0.0080	0.6279
		Prior 2	0.9470	0.0030	8.9296	0.0280	0.0250	0.0566
		Prior 3	0.9580	-0.0080	9.5105	0.0270	0.0150	0.2857
		Prior 4	0.9490	0.0010	9.3470	0.0350	0.0160	0.3725
100	50	Prior 1	0.9400	0.0100	8.7045	0.0400	0.0200	0.3333
		Prior 2	0.9420	0.0080	8.5888	0.0410	0.0170	0.4138
		Prior 3	0.9380	0.0120	9.0115	0.0410	0.0210	0.3226
		Prior 4	0.9380	0.0120	8.8689	0.0420	0.0200	0.3548
100	100	Prior 1	0.9400	0.0100	8.6111	0.0410	0.0190	0.3667
		Prior 2	0.9550	-0.0050	8.4654	0.0300	0.0150	0.3333
		Prior 3	0.9470	0.0030	8.8573	0.0290	0.0240	0.0943
		Prior 4	0.9460	0.0040	8.7723	0.0310	0.0230	0.1481

Table 30: Results for MOVER

Design	n_1	n_2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
1	10	10	0.9570	-0.0070	5.7067	0.0190	0.0240	0.1163
	10	25	0.9620	-0.0120	5.1626	0.0150	0.0230	0.2105
	25	10	0.9490	0.0010	3.3341	0.0240	0.0270	0.0588
	25	25	0.9430	0.0070	2.7284	0.0250	0.0320	0.1228
	25	50	0.9430	0.0070	2.6016	0.0250	0.0320	0.1228
	25	100	0.9420	0.0080	2.5145	0.0250	0.0330	0.1379
	50	25	0.9500	0.0000	1.9672	0.0240	0.0260	0.0400
	50	50	0.9490	0.0010	1.7973	0.0280	0.0230	0.0980
	50	100	0.9420	0.0080	1.6969	0.0220	0.0360	0.2414
	100	25	0.9550	-0.0050	1.5629	0.0240	0.0210	0.0667
	100	50	0.9430	0.0070	1.3263	0.0240	0.0330	0.1579
	100	100	0.9610	-0.0110	1.2132	0.0190	0.0200	0.0256
2	10	10	0.9620	-0.0120	10.3330	0.0230	0.0150	0.2105
	10	25	0.9580	-0.0080	7.6838	0.0200	0.0220	0.0476
	25	10	0.9660	-0.0160	7.6931	0.0190	0.0150	0.1176
	25	25	0.9530	-0.0030	4.6819	0.0240	0.0230	0.0213
	25	50	0.9570	-0.0070	3.8461	0.0230	0.0200	0.0698
	25	100	0.9550	-0.0050	3.5033	0.0250	0.0200	0.1111
	50	25	0.9480	0.0020	3.8768	0.0220	0.0300	0.1538
	50	50	0.9480	0.0020	2.9944	0.0250	0.0270	0.0385
	50	100	0.9600	-0.0100	2.5252	0.0190	0.0210	0.0500
	100	25	0.9580	-0.0080	3.5003	0.0220	0.0200	0.0476
	100	50	0.9580	-0.0080	2.5334	0.0160	0.0260	0.2381
	100	100	0.9530	-0.0030	2.0085	0.0200	0.0270	0.1489
3	10	10	0.9550	-0.0050	3.8553	0.0210	0.0240	0.0667
	10	25	0.9620	-0.0120	3.6153	0.0140	0.0240	0.2632
	25	10	0.9630	-0.0130	2.2257	0.0210	0.0160	0.1351
	25	25	0.9460	0.0040	1.9074	0.0200	0.0340	0.2593
	25	50	0.9560	-0.0060	1.8269	0.0250	0.0190	0.1364
	25	100	0.9490	0.0010	1.8019	0.0210	0.0300	0.1765
	50	25	0.9580	-0.0080	1.3582	0.0200	0.0220	0.0476
	50	50	0.9580	-0.0080	1.2582	0.0240	0.0180	0.1429
	50	100	0.9470	0.0030	1.2180	0.0260	0.0270	0.0189
	100	25	0.9520	-0.0020	1.0491	0.0220	0.0260	0.0833
	100	50	0.9500	0.0000	0.9279	0.0220	0.0280	0.1200
	100	100	0.9500	0.0000	0.8681	0.0250	0.0250	0.0000
4	10	10	0.9580	-0.0080	18.6512	0.0290	0.0130	0.3810
	10	25	0.9680	-0.0180	9.7632	0.0140	0.0180	0.1250
	25	10	0.9480	0.0020	18.3130	0.0310	0.0210	0.1923
	25	25	0.9500	0.0000	8.6657	0.0360	0.0140	0.4400
	25	50	0.9470	0.0030	5.8718	0.0310	0.0220	0.1698
	25	100	0.9390	0.0110	4.2069	0.0300	0.0310	0.0164
	50	25	0.9500	0.0000	8.5461	0.0260	0.0240	0.0400
	50	50	0.9390	0.0110	5.6246	0.0340	0.0270	0.1148
	50	100	0.9510	-0.0010	3.9296	0.0260	0.0230	0.0612
	100	25	0.9430	0.0070	8.5884	0.0300	0.0270	0.0526
	100	50	0.9550	-0.0050	5.5608	0.0220	0.0230	0.0222
	100	100	0.9380	0.0120	3.8197	0.0350	0.0270	0.1290
5	10	10	0.9590	-0.0090	45.2531	0.0180	0.0230	0.1220
	10	25	0.9580	-0.0080	43.8887	0.0220	0.0200	0.0476
	25	10	0.9530	-0.0030	22.2846	0.0150	0.0320	0.3617
	25	25	0.9640	-0.0140	20.7190	0.0150	0.0210	0.1667
	25	50	0.9570	-0.0070	20.7546	0.0230	0.0200	0.0698
	25	100	0.9480	0.0020	20.6499	0.0300	0.0220	0.1538
	50	25	0.9500	0.0000	13.7947	0.0240	0.0260	0.0400
	50	50	0.9530	-0.0030	13.3402	0.0190	0.0280	0.1915
	50	100	0.9550	-0.0050	13.2087	0.0210	0.0240	0.0667
	100	25	0.9610	-0.0110	9.6378	0.0260	0.0130	0.3333
	100	50	0.9580	-0.0080	9.2326	0.0190	0.0230	0.0952
	100	100	0.9460	0.0040	9.0882	0.0270	0.0270	0.0000

Table 31: Results for Parameter Settings used by Krishnamoorthy & Mathew(2003)

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
4	4	0	0	Prior 1	3	3	0.960	-0.010	39.906	0.018	0.022	0.100
4	4	0	0	Prior 2	3	3	0.968	-0.018	23.592	0.013	0.019	0.188
4	4	0	0	Prior 3	3	3	0.980	-0.030	3092.565	0.006	0.014	0.400
4	4	0	0	Prior 4	3	3	0.972	-0.022	115.391	0.008	0.020	0.429
4	4	0	0	Prior 1	12	4	0.969	-0.019	114.652	0.004	0.027	0.742
4	4	0	0	Prior 2	12	4	0.964	-0.014	63.259	0.008	0.028	0.556
4	4	0	0	Prior 3	12	4	0.971	-0.021	8016.133	0.003	0.026	0.793
4	4	0	0	Prior 4	12	4	0.964	-0.014	296.915	0.004	0.032	0.778
4	4	0	0	Prior 1	12	12	0.972	-0.022	159.340	0.016	0.012	0.143
4	4	0	0	Prior 2	12	12	0.970	-0.020	95.730	0.014	0.016	0.067
4	4	0	0	Prior 3	12	12	0.976	-0.026	11611.268	0.010	0.014	0.167
4	4	0	0	Prior 4	12	12	0.973	-0.023	468.521	0.014	0.013	0.037
4	4	0	0	Prior 1	20	4	0.961	-0.011	162.943	0.004	0.035	0.795
4	4	0	0	Prior 2	20	4	0.973	-0.023	97.742	0.005	0.022	0.630
4	4	0	0	Prior 3	20	4	0.969	-0.019	11687.968	0.002	0.029	0.871
4	4	0	0	Prior 4	20	4	0.962	-0.012	428.705	0.006	0.032	0.684
4	4	1	0	Prior 1	2	4	0.975	-0.025	41.581	0.017	0.008	0.360
4	4	1	0	Prior 2	2	4	0.970	-0.020	24.411	0.025	0.005	0.667
4	4	1	0	Prior 3	2	4	0.967	-0.017	3021.916	0.023	0.010	0.394
4	4	1	0	Prior 4	2	4	0.972	-0.022	113.749	0.022	0.006	0.571
4	4	3	0	Prior 1	2	4	0.970	-0.020	41.782	0.022	0.008	0.467
4	4	3	0	Prior 2	2	4	0.973	-0.023	23.527	0.021	0.006	0.556
4	4	3	0	Prior 3	2	4	0.981	-0.031	3245.080	0.012	0.007	0.263
4	4	3	0	Prior 4	2	4	0.976	-0.026	117.672	0.023	0.001	0.917
4	4	4	0	Prior 1	1	1	0.967	-0.017	13.012	0.014	0.019	0.152
4	4	4	0	Prior 2	1	1	0.973	-0.023	7.841	0.014	0.013	0.037
4	4	4	0	Prior 3	1	1	0.985	-0.035	981.994	0.006	0.009	0.200
4	4	4	0	Prior 4	1	1	0.970	-0.020	38.312	0.012	0.018	0.200

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
4	4	5	0	Prior 1	2	12	0.965	-0.015	99.475	0.030	0.005	0.714
4	4	5	0	Prior 2	2	12	0.963	-0.013	58.480	0.024	0.013	0.297
4	4	5	0	Prior 3	2	12	0.972	-0.022	7478.827	0.026	0.002	0.857
4	4	5	0	Prior 4	2	12	0.964	-0.014	281.549	0.034	0.002	0.889
10	10	0	0	Prior 1	3	3	0.945	0.005	7.381	0.032	0.023	0.164
10	10	0	0	Prior 2	3	3	0.952	-0.002	6.760	0.026	0.022	0.083
10	10	0	0	Prior 3	3	3	0.959	-0.009	9.737	0.025	0.016	0.220
10	10	0	0	Prior 4	3	3	0.955	-0.005	8.249	0.030	0.015	0.333
10	10	0	0	Prior 1	12	4	0.960	-0.010	20.426	0.014	0.026	0.300
10	10	0	0	Prior 2	12	4	0.963	-0.013	18.692	0.010	0.027	0.459
10	10	0	0	Prior 3	12	4	0.959	-0.009	26.399	0.010	0.031	0.512
10	10	0	0	Prior 4	12	4	0.962	-0.012	23.277	0.012	0.026	0.368
10	10	0	0	Prior 1	12	12	0.958	-0.008	29.669	0.020	0.022	0.048
10	10	0	0	Prior 2	12	12	0.950	0.000	26.858	0.022	0.028	0.120
10	10	0	0	Prior 3	12	12	0.952	-0.002	38.851	0.018	0.030	0.250
10	10	0	0	Prior 4	12	12	0.944	0.006	33.006	0.025	0.031	0.107
10	10	0	0	Prior 1	20	4	0.944	0.006	32.668	0.027	0.029	0.036
10	10	0	0	Prior 2	20	4	0.963	-0.013	29.969	0.019	0.018	0.027
10	10	0	0	Prior 3	20	4	0.955	-0.005	40.383	0.013	0.032	0.422
10	10	0	0	Prior 4	20	4	0.947	0.003	35.945	0.020	0.033	0.245
10	10	1	0	Prior 1	2	4	0.955	-0.005	7.574	0.029	0.016	0.289
10	10	1	0	Prior 2	2	4	0.943	0.007	6.853	0.035	0.022	0.228
10	10	1	0	Prior 3	2	4	0.953	-0.003	10.180	0.026	0.021	0.106
10	10	1	0	Prior 4	2	4	0.951	-0.001	8.522	0.029	0.020	0.184
10	10	3	0	Prior 1	2	4	0.945	0.005	7.562	0.028	0.027	0.018
10	10	3	0	Prior 2	2	4	0.957	-0.007	7.005	0.027	0.016	0.256
10	10	3	0	Prior 3	2	4	0.960	-0.010	10.004	0.028	0.012	0.400
10	10	3	0	Prior 4	2	4	0.957	-0.007	8.409	0.027	0.016	0.256

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	4	0	Prior 1	1	1	0.954	-0.004	2.504	0.027	0.019	0.174
10	10	4	0	Prior 2	1	1	0.966	-0.016	2.229	0.012	0.022	0.294
10	10	4	0	Prior 3	1	1	0.933	0.017	3.283	0.023	0.044	0.313
10	10	4	0	Prior 4	1	1	0.952	-0.002	2.772	0.025	0.023	0.042
10	10	5	0	Prior 1	2	12	0.948	0.002	19.054	0.036	0.016	0.385
10	10	5	0	Prior 2	2	12	0.953	-0.003	17.305	0.026	0.021	0.106
10	10	5	0	Prior 3	2	12	0.945	0.005	25.308	0.032	0.023	0.164
10	10	5	0	Prior 4	2	12	0.956	-0.006	21.584	0.023	0.021	0.045
25	25	0	0	Prior 1	1	1	0.942	0.008	1.029	0.027	0.031	0.069
25	25	0	0	Prior 2	1	1	0.955	-0.005	1.003	0.018	0.027	0.200
25	25	0	0	Prior 3	1	1	0.955	-0.005	1.100	0.022	0.023	0.022
25	25	0	0	Prior 4	1	1	0.949	0.001	1.073	0.025	0.026	0.020
25	25	0	0	Prior 1	4	1	0.953	-0.003	2.876	0.027	0.020	0.149
25	25	0	0	Prior 2	4	1	0.946	0.004	2.826	0.030	0.024	0.111
25	25	0	0	Prior 3	4	1	0.955	-0.005	3.046	0.019	0.026	0.156
25	25	0	0	Prior 4	4	1	0.949	0.001	2.919	0.018	0.033	0.294
25	25	0	0	Prior 1	4	2	0.944	0.006	3.171	0.027	0.029	0.036
25	25	0	0	Prior 2	4	2	0.958	-0.008	3.084	0.013	0.029	0.381
25	25	0	0	Prior 3	4	2	0.961	-0.011	3.418	0.014	0.025	0.282
25	25	0	0	Prior 4	4	2	0.950	0.000	3.328	0.023	0.027	0.080
25	25	0	0	Prior 1	5	5	0.947	0.003	5.183	0.021	0.032	0.208
25	25	0	0	Prior 2	5	5	0.955	-0.005	5.042	0.024	0.021	0.067
25	25	0	0	Prior 3	5	5	0.953	-0.003	5.525	0.028	0.019	0.191
25	25	0	0	Prior 4	5	5	0.946	0.004	5.333	0.027	0.027	0.000
25	25	0	0	Prior 1	9	7	0.953	-0.003	8.307	0.020	0.027	0.149
25	25	0	0	Prior 2	9	7	0.954	-0.004	8.023	0.022	0.024	0.043
25	25	0	0	Prior 3	9	7	0.947	0.003	8.709	0.019	0.034	0.283
25	25	0	0	Prior 4	9	7	0.937	0.013	8.590	0.029	0.034	0.079

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
25	25	0	0	Prior 1	10	10	0.946	0.004	10.382	0.031	0.023	0.148
25	25	0	0	Prior 2	10	10	0.957	-0.007	10.088	0.025	0.018	0.163
25	25	0	0	Prior 3	10	10	0.959	-0.009	10.874	0.019	0.022	0.073
25	25	0	0	Prior 4	10	10	0.951	-0.001	10.634	0.024	0.025	0.020
25	25	0	0	Prior 1	100	100	0.954	-0.004	102.226	0.023	0.023	0.000
25	25	0	0	Prior 2	100	100	0.948	0.002	101.288	0.029	0.023	0.115
25	25	0	0	Prior 3	100	100	0.961	-0.011	109.839	0.023	0.016	0.179
25	25	0	0	Prior 4	100	100	0.951	-0.001	105.927	0.026	0.023	0.061
25	25	1	0	Prior 1	1	1	0.942	0.008	1.035	0.029	0.029	0.000
25	25	1	0	Prior 2	1	1	0.949	0.001	1.004	0.027	0.024	0.059
25	25	1	0	Prior 3	1	1	0.955	-0.005	1.095	0.027	0.018	0.200
25	25	1	0	Prior 4	1	1	0.943	0.007	1.049	0.031	0.026	0.088
25	25	1	0	Prior 1	5	5	0.943	0.007	5.225	0.026	0.031	0.088
25	25	1	0	Prior 2	5	5	0.947	0.003	5.009	0.024	0.029	0.094
25	25	1	0	Prior 3	5	5	0.949	0.001	5.436	0.027	0.024	0.059
25	25	1	0	Prior 4	5	5	0.959	-0.009	5.306	0.023	0.018	0.122
25	25	1	0	Prior 1	10	10	0.944	0.006	10.410	0.037	0.019	0.321
25	25	1	0	Prior 2	10	10	0.951	-0.001	10.103	0.031	0.018	0.265
25	25	1	0	Prior 3	10	10	0.950	0.000	11.014	0.029	0.021	0.160
25	25	1	0	Prior 4	10	10	0.941	0.009	10.654	0.031	0.028	0.051
25	25	2	0	Prior 1	4	8	0.953	-0.003	6.348	0.029	0.018	0.234
25	25	2	0	Prior 2	4	8	0.962	-0.012	6.242	0.022	0.016	0.158
25	25	2	0	Prior 3	4	8	0.939	0.011	6.824	0.042	0.019	0.377
25	25	2	0	Prior 4	4	8	0.949	0.001	6.554	0.031	0.020	0.216
25	25	4	0	Prior 1	8	16	0.957	-0.007	12.899	0.026	0.017	0.209
25	25	4	0	Prior 2	8	16	0.957	-0.007	12.433	0.022	0.021	0.023
25	25	4	0	Prior 3	8	16	0.954	-0.004	13.769	0.030	0.016	0.304
25	25	4	0	Prior 4	8	16	0.944	0.006	13.328	0.028	0.028	0.000

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
25	40	0	0	Prior 1	1	1	0.946	0.004	0.877	0.024	0.030	0.111
25	40	0	0	Prior 2	1	1	0.952	-0.002	0.870	0.020	0.028	0.167
25	40	0	0	Prior 3	1	1	0.956	-0.006	0.932	0.021	0.023	0.045
25	40	0	0	Prior 4	1	1	0.945	0.005	0.911	0.024	0.031	0.127
25	40	0	0	Prior 1	5	5	0.964	-0.014	4.449	0.017	0.019	0.056
25	40	0	0	Prior 2	5	5	0.947	0.003	4.389	0.031	0.022	0.170
25	40	0	0	Prior 3	5	5	0.954	-0.004	4.697	0.022	0.024	0.043
25	40	0	0	Prior 4	5	5	0.951	-0.001	4.612	0.026	0.023	0.061
25	40	0	0	Prior 1	10	10	0.938	0.012	8.940	0.032	0.030	0.032
25	40	0	0	Prior 2	10	10	0.950	0.000	8.639	0.028	0.022	0.120
25	40	0	0	Prior 3	10	10	0.951	-0.001	9.335	0.029	0.020	0.184
25	40	0	0	Prior 4	10	10	0.939	0.011	9.119	0.028	0.033	0.082
25	40	1	0	Prior 1	1	1	0.952	-0.002	0.894	0.027	0.021	0.125
25	40	1	0	Prior 2	1	1	0.946	0.004	0.868	0.023	0.031	0.148
25	40	1	0	Prior 3	1	1	0.951	-0.001	0.931	0.033	0.016	0.347
25	40	1	0	Prior 4	1	1	0.941	0.009	0.905	0.029	0.030	0.017
25	40	1	0	Prior 1	5	4	0.948	0.002	4.166	0.024	0.028	0.077
25	40	1	0	Prior 2	5	4	0.956	-0.006	4.098	0.025	0.019	0.136
25	40	1	0	Prior 3	5	4	0.938	0.012	4.384	0.030	0.032	0.032
25	40	1	0	Prior 4	5	4	0.954	-0.004	4.276	0.031	0.015	0.348
25	40	1	0	Prior 1	5	5	0.939	0.011	4.455	0.027	0.034	0.115
25	40	1	0	Prior 2	5	5	0.955	-0.005	4.337	0.022	0.023	0.022
25	40	1	0	Prior 3	5	5	0.932	0.018	4.617	0.030	0.038	0.118
25	40	1	0	Prior 4	5	5	0.948	0.002	4.546	0.030	0.022	0.154
25	40	1	0	Prior 1	10	9	0.940	0.010	8.596	0.029	0.031	0.033
25	40	1	0	Prior 2	10	9	0.957	-0.007	8.471	0.022	0.021	0.023
25	40	1	0	Prior 3	10	9	0.942	0.008	8.914	0.025	0.033	0.138
25	40	1	0	Prior 4	10	9	0.947	0.003	8.746	0.028	0.025	0.057

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
25	40	1	0	Prior 1	10	10	0.946	0.004	8.968	0.026	0.028	0.037
25	40	1	0	Prior 2	10	10	0.945	0.005	8.722	0.024	0.031	0.127
25	40	1	0	Prior 3	10	10	0.960	-0.010	9.302	0.015	0.025	0.250
25	40	1	0	Prior 4	10	10	0.946	0.004	9.181	0.026	0.028	0.037
25	40	5	0	Prior 1	2	12	0.956	-0.006	6.236	0.027	0.017	0.227
25	40	5	0	Prior 2	2	12	0.943	0.007	6.016	0.029	0.028	0.018
25	40	5	0	Prior 3	2	12	0.953	-0.003	6.470	0.026	0.021	0.106
25	40	5	0	Prior 4	2	12	0.952	-0.002	6.320	0.031	0.017	0.292
25	100	0	0	Prior 1	1	1	0.946	0.004	0.759	0.031	0.023	0.148
25	100	0	0	Prior 2	1	1	0.946	0.004	0.743	0.031	0.023	0.148
25	100	0	0	Prior 3	1	1	0.934	0.016	0.800	0.030	0.036	0.091
25	100	0	0	Prior 4	1	1	0.954	-0.004	0.788	0.024	0.022	0.043
25	100	0	0	Prior 1	2	1	0.946	0.004	1.407	0.024	0.030	0.111
25	100	0	0	Prior 2	2	1	0.945	0.005	1.361	0.020	0.035	0.273
25	100	0	0	Prior 3	2	1	0.946	0.004	1.476	0.026	0.028	0.037
25	100	0	0	Prior 4	2	1	0.953	-0.003	1.426	0.024	0.023	0.021
25	100	0	0	Prior 1	3	1	0.966	-0.016	2.055	0.010	0.024	0.412
25	100	0	0	Prior 2	3	1	0.959	-0.009	2.051	0.023	0.018	0.122
25	100	0	0	Prior 3	3	1	0.951	-0.001	2.193	0.028	0.021	0.143
25	100	0	0	Prior 4	3	1	0.955	-0.005	2.170	0.022	0.023	0.022
25	100	0	0	Prior 1	5	5	0.945	0.005	3.832	0.030	0.025	0.091
25	100	0	0	Prior 2	5	5	0.942	0.008	3.743	0.034	0.024	0.172
25	100	0	0	Prior 3	5	5	0.945	0.005	4.068	0.029	0.026	0.055
25	100	0	0	Prior 4	5	5	0.941	0.009	3.944	0.034	0.025	0.153
25	100	0	0	Prior 1	10	10	0.957	-0.007	7.550	0.024	0.019	0.116
25	100	0	0	Prior 2	10	10	0.948	0.002	7.488	0.024	0.028	0.077
25	100	0	0	Prior 3	10	10	0.952	-0.002	8.010	0.017	0.031	0.292
25	100	0	0	Prior 4	10	10	0.961	-0.011	7.919	0.019	0.020	0.026

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
25	100	1	0	Prior 1	1	1	0.951	-0.001	0.759	0.024	0.025	0.020
25	100	1	0	Prior 2	1	1	0.954	-0.004	0.748	0.025	0.021	0.087
25	100	1	0	Prior 3	1	1	0.944	0.006	0.798	0.029	0.027	0.036
25	100	1	0	Prior 4	1	1	0.955	-0.005	0.782	0.021	0.024	0.067
25	100	1	0	Prior 1	5	4	0.954	-0.004	3.662	0.024	0.022	0.043
25	100	1	0	Prior 2	5	4	0.952	-0.002	3.640	0.026	0.022	0.083
25	100	1	0	Prior 3	5	4	0.953	-0.003	3.903	0.024	0.023	0.021
25	100	1	0	Prior 4	5	4	0.956	-0.006	3.818	0.026	0.018	0.182
25	100	1	0	Prior 1	5	5	0.941	0.009	3.769	0.026	0.033	0.119
25	100	1	0	Prior 2	5	5	0.942	0.008	3.693	0.032	0.026	0.103
25	100	1	0	Prior 3	5	5	0.954	-0.004	3.968	0.027	0.019	0.174
25	100	1	0	Prior 4	5	5	0.951	-0.001	3.945	0.030	0.019	0.224
25	100	1	0	Prior 1	10	9	0.943	0.007	7.573	0.032	0.025	0.123
25	100	1	0	Prior 2	10	9	0.948	0.002	7.391	0.034	0.018	0.308
25	100	1	0	Prior 3	10	9	0.949	0.001	7.995	0.028	0.023	0.098
25	100	1	0	Prior 4	10	9	0.945	0.005	7.665	0.023	0.032	0.164
25	100	1	0	Prior 1	10	10	0.948	0.002	7.656	0.029	0.023	0.115
25	100	1	0	Prior 2	10	10	0.948	0.002	7.523	0.032	0.020	0.231
25	100	1	0	Prior 3	10	10	0.949	0.001	7.995	0.028	0.023	0.098
25	100	1	0	Prior 4	10	10	0.953	-0.003	7.877	0.023	0.024	0.021
25	100	13	0	Prior 1	4	30	0.951	-0.001	9.325	0.020	0.029	0.184
25	100	13	0	Prior 2	4	30	0.963	-0.013	9.260	0.018	0.019	0.027
25	100	13	0	Prior 3	4	30	0.943	0.007	9.424	0.033	0.024	0.158
25	100	13	0	Prior 4	4	30	0.945	0.005	9.369	0.034	0.021	0.236
25	300	0.8	0	Prior 1	12	12	0.947	0.003	8.563	0.022	0.031	0.170
25	300	0.8	0	Prior 2	12	12	0.940	0.010	8.269	0.038	0.022	0.267
25	300	0.8	0	Prior 3	12	12	0.942	0.008	8.922	0.024	0.034	0.172
25	300	0.8	0	Prior 4	12	12	0.947	0.003	8.719	0.027	0.026	0.019

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
40	25	0	0	Prior 1	1	1	0.952	-0.002	0.892	0.028	0.020	0.167
40	25	0	0	Prior 2	1	1	0.941	0.009	0.870	0.027	0.032	0.085
40	25	0	0	Prior 3	1	1	0.959	-0.009	0.946	0.022	0.019	0.073
40	25	0	0	Prior 4	1	1	0.953	-0.003	0.922	0.027	0.020	0.149
40	25	0	0	Prior 1	5	5	0.946	0.004	4.419	0.032	0.022	0.185
40	25	0	0	Prior 2	5	5	0.948	0.002	4.366	0.027	0.025	0.038
40	25	0	0	Prior 3	5	5	0.955	-0.005	4.662	0.025	0.020	0.111
40	25	0	0	Prior 4	5	5	0.959	-0.009	4.578	0.022	0.019	0.073
40	25	0	0	Prior 1	10	10	0.941	0.009	8.849	0.025	0.034	0.153
40	25	0	0	Prior 2	10	10	0.958	-0.008	8.740	0.022	0.020	0.048
40	25	0	0	Prior 3	10	10	0.957	-0.007	9.349	0.022	0.021	0.023
40	25	0	0	Prior 4	10	10	0.949	0.001	9.161	0.026	0.025	0.020
40	25	1	0	Prior 1	1	1	0.950	0.000	0.896	0.035	0.015	0.400
40	25	1	0	Prior 2	1	1	0.953	-0.003	0.874	0.021	0.026	0.106
40	25	1	0	Prior 3	1	1	0.944	0.006	0.942	0.025	0.031	0.107
40	25	1	0	Prior 4	1	1	0.941	0.009	0.919	0.025	0.034	0.153
40	25	1	0	Prior 1	5	4	0.956	-0.006	3.941	0.024	0.020	0.091
40	25	1	0	Prior 2	5	4	0.947	0.003	3.796	0.026	0.027	0.019
40	25	1	0	Prior 3	5	4	0.941	0.009	4.135	0.029	0.030	0.017
40	25	1	0	Prior 4	5	4	0.950	0.000	4.009	0.024	0.026	0.040
40	25	1	0	Prior 1	5	5	0.946	0.004	4.411	0.026	0.028	0.037
40	25	1	0	Prior 2	5	5	0.952	-0.002	4.343	0.030	0.018	0.250
40	25	1	0	Prior 3	5	5	0.943	0.007	4.670	0.026	0.031	0.088
40	25	1	0	Prior 4	5	5	0.950	0.000	4.593	0.023	0.027	0.080
40	25	1	0	Prior 1	10	9	0.945	0.005	8.378	0.023	0.032	0.164
40	25	1	0	Prior 2	10	9	0.948	0.002	8.224	0.022	0.030	0.154
40	25	1	0	Prior 3	10	9	0.952	-0.002	8.861	0.023	0.025	0.042
40	25	1	0	Prior 4	10	9	0.938	0.012	8.494	0.034	0.028	0.097

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
40	25	1	0	Prior 1	10	10	0.947	0.003	8.816	0.021	0.032	0.208
40	25	1	0	Prior 2	10	10	0.960	-0.010	8.615	0.020	0.020	0.000
40	25	1	0	Prior 3	10	10	0.948	0.002	9.468	0.026	0.026	0.000
40	25	1	0	Prior 4	10	10	0.948	0.002	9.142	0.026	0.026	0.000
40	25	5	0	Prior 1	2	12	0.952	-0.002	8.344	0.028	0.020	0.167
40	25	5	0	Prior 2	2	12	0.960	-0.010	8.163	0.022	0.018	0.100
40	25	5	0	Prior 3	2	12	0.947	0.003	8.840	0.027	0.026	0.019
40	25	5	0	Prior 4	2	12	0.949	0.001	8.602	0.027	0.024	0.059
40	40	8	0	Prior 1	4	20	0.958	-0.008	10.218	0.022	0.020	0.048
40	40	8	0	Prior 2	4	20	0.950	0.000	10.161	0.025	0.025	0.000
40	40	8	0	Prior 3	4	20	0.959	-0.009	10.579	0.018	0.023	0.122
40	40	8	0	Prior 4	4	20	0.948	0.002	10.287	0.034	0.018	0.308
40	40	14	0	Prior 1	4	32	0.947	0.003	16.126	0.028	0.025	0.057
40	40	14	0	Prior 2	4	32	0.958	-0.008	16.144	0.020	0.022	0.048
40	40	14	0	Prior 3	4	32	0.949	0.001	16.764	0.019	0.032	0.255
40	40	14	0	Prior 4	4	32	0.948	0.002	16.278	0.032	0.020	0.231
50	200	0	0	Prior 1	1	1	0.950	0.000	0.484	0.025	0.025	0.000
50	200	0	0	Prior 2	1	1	0.958	-0.008	0.483	0.021	0.021	0.000
50	200	0	0	Prior 3	1	1	0.948	0.002	0.493	0.025	0.027	0.038
50	200	0	0	Prior 4	1	1	0.951	-0.001	0.491	0.030	0.019	0.224
50	200	0	0	Prior 1	2	1	0.943	0.007	0.887	0.022	0.035	0.228
50	200	0	0	Prior 2	2	1	0.952	-0.002	0.885	0.025	0.023	0.042
50	200	0	0	Prior 3	2	1	0.948	0.002	0.927	0.028	0.024	0.077
50	200	0	0	Prior 4	2	1	0.956	-0.006	0.900	0.017	0.027	0.227
50	200	0	0	Prior 1	3	1	0.934	0.016	1.323	0.033	0.033	0.000
50	200	0	0	Prior 2	3	1	0.957	-0.007	1.317	0.027	0.016	0.256
50	200	0	0	Prior 3	3	1	0.934	0.016	1.329	0.027	0.039	0.182
50	200	0	0	Prior 4	3	1	0.954	-0.004	1.333	0.021	0.025	0.087

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
50	200	0	0	Prior 1	5	5	0.954	-0.004	2.436	0.024	0.022	0.043
50	200	0	0	Prior 2	5	5	0.961	-0.011	2.413	0.020	0.019	0.026
50	200	0	0	Prior 3	5	5	0.938	0.012	2.476	0.025	0.037	0.194
50	200	0	0	Prior 4	5	5	0.950	0.000	2.460	0.029	0.021	0.160
50	200	0	0	Prior 1	10	10	0.953	-0.003	4.837	0.018	0.029	0.234
50	200	0	0	Prior 2	10	10	0.954	-0.004	4.854	0.028	0.018	0.217
50	200	0	0	Prior 3	10	10	0.964	-0.014	4.968	0.016	0.020	0.111
50	200	0	0	Prior 4	10	10	0.954	-0.004	4.904	0.019	0.027	0.174
50	200	1	0	Prior 1	1	1	0.957	-0.007	0.485	0.017	0.026	0.209
50	200	1	0	Prior 2	1	1	0.933	0.017	0.478	0.030	0.037	0.104
50	200	1	0	Prior 3	1	1	0.953	-0.003	0.496	0.026	0.021	0.106
50	200	1	0	Prior 4	1	1	0.953	-0.003	0.491	0.017	0.030	0.277
50	200	1	0	Prior 1	5	5	0.941	0.009	2.407	0.029	0.030	0.017
50	200	1	0	Prior 2	5	5	0.939	0.011	2.413	0.032	0.029	0.049
50	200	1	0	Prior 3	5	5	0.952	-0.002	2.473	0.024	0.024	0.000
50	200	1	0	Prior 4	5	5	0.945	0.005	2.448	0.030	0.025	0.091
50	200	1	0	Prior 1	10	10	0.947	0.003	4.811	0.022	0.031	0.170
50	200	1	0	Prior 2	10	10	0.952	-0.002	4.828	0.023	0.025	0.042
50	200	1	0	Prior 3	10	10	0.955	-0.005	4.994	0.024	0.021	0.067
50	200	1	0	Prior 4	10	10	0.959	-0.009	4.927	0.017	0.024	0.171
100	25	0	0	Prior 1	1	1	0.954	-0.004	0.759	0.024	0.022	0.043
100	25	0	0	Prior 2	1	1	0.940	0.010	0.742	0.031	0.029	0.033
100	25	0	0	Prior 3	1	1	0.944	0.006	0.790	0.032	0.024	0.143
100	25	0	0	Prior 4	1	1	0.950	0.000	0.794	0.025	0.025	0.000
100	25	0	0	Prior 1	2	1	0.945	0.005	0.943	0.025	0.030	0.091
100	25	0	0	Prior 2	2	1	0.944	0.006	0.929	0.022	0.034	0.214
100	25	0	0	Prior 3	2	1	0.950	0.000	0.967	0.024	0.026	0.040
100	25	0	0	Prior 4	2	1	0.944	0.006	0.952	0.021	0.035	0.250

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
100	25	0	0	Prior 1	3	1	0.942	0.008	1.155	0.019	0.039	0.345
100	25	0	0	Prior 2	3	1	0.961	-0.011	1.146	0.018	0.021	0.077
100	25	0	0	Prior 3	3	1	0.935	0.015	1.186	0.026	0.039	0.200
100	25	0	0	Prior 4	3	1	0.953	-0.003	1.173	0.018	0.029	0.234
100	25	0	0	Prior 1	5	5	0.951	-0.001	3.880	0.021	0.028	0.143
100	25	0	0	Prior 2	5	5	0.957	-0.007	3.678	0.029	0.014	0.349
100	25	0	0	Prior 3	5	5	0.952	-0.002	3.996	0.027	0.021	0.125
100	25	0	0	Prior 4	5	5	0.953	-0.003	3.887	0.021	0.026	0.106
100	25	0	0	Prior 1	10	10	0.950	0.000	7.802	0.021	0.029	0.160
100	25	0	0	Prior 2	10	10	0.948	0.002	7.522	0.022	0.030	0.154
100	25	0	0	Prior 3	10	10	0.955	-0.005	7.987	0.025	0.020	0.111
100	25	0	0	Prior 4	10	10	0.942	0.008	7.817	0.027	0.031	0.069
100	25	1	0	Prior 1	1	1	0.942	0.008	0.767	0.027	0.031	0.069
100	25	1	0	Prior 2	1	1	0.938	0.012	0.750	0.034	0.028	0.097
100	25	1	0	Prior 3	1	1	0.950	0.000	0.803	0.022	0.028	0.120
100	25	1	0	Prior 4	1	1	0.954	-0.004	0.783	0.024	0.022	0.043
100	25	1	0	Prior 1	5	4	0.946	0.004	3.239	0.021	0.033	0.222
100	25	1	0	Prior 2	5	4	0.939	0.011	3.147	0.031	0.030	0.016
100	25	1	0	Prior 3	5	4	0.942	0.008	3.363	0.032	0.026	0.103
100	25	1	0	Prior 4	5	4	0.941	0.009	3.302	0.019	0.040	0.356
100	25	1	0	Prior 1	5	5	0.949	0.001	3.837	0.028	0.023	0.098
100	25	1	0	Prior 2	5	5	0.953	-0.003	3.743	0.022	0.025	0.064
100	25	1	0	Prior 3	5	5	0.949	0.001	3.995	0.025	0.026	0.020
100	25	1	0	Prior 4	5	5	0.957	-0.007	3.889	0.023	0.020	0.070
100	25	1	0	Prior 1	10	9	0.958	-0.008	7.006	0.022	0.020	0.048
100	25	1	0	Prior 2	10	9	0.950	0.000	6.857	0.024	0.026	0.040
100	25	1	0	Prior 3	10	9	0.961	-0.011	7.338	0.017	0.022	0.128
100	25	1	0	Prior 4	10	9	0.952	-0.002	7.127	0.025	0.023	0.042

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
100	25	1	0	Prior 1	10	10	0.938	0.012	7.607	0.035	0.027	0.129
100	25	1	0	Prior 2	10	10	0.942	0.008	7.583	0.027	0.031	0.069
100	25	1	0	Prior 3	10	10	0.945	0.005	8.037	0.030	0.025	0.091
100	25	1	0	Prior 4	10	10	0.961	-0.011	7.756	0.023	0.016	0.179
100	25	13	0	Prior 1	4	30	0.945	0.005	20.665	0.026	0.029	0.055
100	25	13	0	Prior 2	4	30	0.949	0.001	20.114	0.025	0.026	0.020
100	25	13	0	Prior 3	4	30	0.948	0.002	21.831	0.031	0.021	0.192
100	25	13	0	Prior 4	4	30	0.946	0.004	21.078	0.026	0.028	0.037
100	100	13	0	Prior 1	4	30	0.936	0.014	8.899	0.033	0.031	0.031
100	100	13	0	Prior 2	4	30	0.952	-0.002	8.806	0.028	0.020	0.167
100	100	13	0	Prior 3	4	30	0.948	0.002	8.952	0.025	0.027	0.038
100	100	13	0	Prior 4	4	30	0.948	0.002	8.914	0.026	0.026	0.000
200	50	0	0	Prior 1	1	1	0.948	0.002	0.491	0.025	0.027	0.038
200	50	0	0	Prior 2	1	1	0.944	0.006	0.484	0.030	0.026	0.071
200	50	0	0	Prior 3	1	1	0.956	-0.006	0.495	0.025	0.019	0.136
200	50	0	0	Prior 4	1	1	0.956	-0.006	0.488	0.030	0.014	0.364
200	50	0	0	Prior 1	2	1	0.944	0.006	0.603	0.022	0.034	0.214
200	50	0	0	Prior 2	2	1	0.944	0.006	0.600	0.024	0.032	0.143
200	50	0	0	Prior 3	2	1	0.938	0.012	0.615	0.018	0.044	0.419
200	50	0	0	Prior 4	2	1	0.941	0.009	0.610	0.026	0.033	0.119
200	50	0	0	Prior 1	3	1	0.960	-0.010	0.758	0.019	0.021	0.050
200	50	0	0	Prior 2	3	1	0.957	-0.007	0.752	0.018	0.025	0.163
200	50	0	0	Prior 3	3	1	0.957	-0.007	0.765	0.016	0.027	0.256
200	50	0	0	Prior 4	3	1	0.939	0.011	0.755	0.023	0.038	0.246
200	50	0	0	Prior 1	5	5	0.940	0.010	2.415	0.034	0.026	0.133
200	50	0	0	Prior 2	5	5	0.949	0.001	2.397	0.026	0.025	0.020
200	50	0	0	Prior 3	5	5	0.961	-0.011	2.479	0.017	0.022	0.128
200	50	0	0	Prior 4	5	5	0.954	-0.004	2.481	0.025	0.021	0.087

n_1	n_2	μ_1	μ_2	Method	σ_1^2	σ_2^2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
200	50	0	0	Prior 1	10	10	0.940	0.010	4.857	0.029	0.031	0.033
200	50	0	0	Prior 2	10	10	0.959	-0.009	4.800	0.021	0.020	0.024
200	50	0	0	Prior 3	10	10	0.953	-0.003	4.929	0.027	0.020	0.149
200	50	0	0	Prior 4	10	10	0.963	-0.013	4.936	0.014	0.023	0.243
200	50	1	0	Prior 1	1	1	0.948	0.002	0.487	0.027	0.025	0.038
200	50	1	0	Prior 2	1	1	0.948	0.002	0.482	0.022	0.030	0.154
200	50	1	0	Prior 3	1	1	0.939	0.011	0.493	0.036	0.025	0.180
200	50	1	0	Prior 4	1	1	0.954	-0.004	0.492	0.024	0.022	0.043
200	50	1	0	Prior 1	5	5	0.954	-0.004	2.419	0.024	0.022	0.043
200	50	1	0	Prior 2	5	5	0.942	0.008	2.397	0.025	0.033	0.138
200	50	1	0	Prior 3	5	5	0.953	-0.003	2.484	0.023	0.024	0.021
200	50	1	0	Prior 4	5	5	0.953	-0.003	2.447	0.026	0.021	0.106
200	50	1	0	Prior 1	10	10	0.943	0.007	4.869	0.035	0.022	0.228
200	50	1	0	Prior 2	10	10	0.958	-0.008	4.819	0.022	0.020	0.048
200	50	1	0	Prior 3	10	10	0.939	0.011	4.955	0.033	0.028	0.082
200	50	1	0	Prior 4	10	10	0.956	-0.006	4.913	0.025	0.019	0.136
200	200	24	0	Prior 1	12	60	0.956	-0.006	12.313	0.022	0.022	0.000
200	200	24	0	Prior 2	12	60	0.946	0.004	12.353	0.031	0.023	0.148
200	200	24	0	Prior 3	12	60	0.944	0.006	12.388	0.031	0.025	0.107
200	200	24	0	Prior 4	12	60	0.964	-0.014	12.378	0.019	0.017	0.056
200	200	40	0	Prior 1	2	82	0.959	-0.009	16.441	0.022	0.019	0.073
200	200	40	0	Prior 2	2	82	0.944	0.006	16.404	0.031	0.025	0.107
200	200	40	0	Prior 3	2	82	0.955	-0.005	16.580	0.024	0.021	0.067
200	200	40	0	Prior 4	2	82	0.943	0.007	16.535	0.034	0.023	0.193

CHAPTER 4

Inference on the Mean: Two Samples and Zero Valued Observations

Introduction

In this chapter we shift our focus from the previous two chapters. Until now we have just been concerned with strictly lognormally distributed data. However, the lognormal distribution in itself does not allow for zero values to be included in the data. This suggests an interesting setting, namely the analysis of data that contains both zero and non-zero values, with the non-zero values being lognormally distributed.

The situation was analysed by Zhou and Tu (2000) with specific application to medical diagnostic charges. These authors proposed both a maximum likelihood method and a bootstrap method to analyse this form of data. This chapter may be viewed as the Bayesian version of this work. Various prior distributions were specifically chosen and a simulation study was performed to examine the effectiveness of these Bayesian methods.

In addition to this, it was noted in previous chapters that perhaps the Bayesian methods are not particularly well suited to the situation of small sample sizes. This was not considered by Zhou and Tu (2000) either, but will be presented for additional insight.

The Case of Zero-Valued Observations

4.1 Model Formulation

From the specification of the problem in the Introduction we can assume that the populations of interest contain both zero and non-zero (positive observations) and we furthermore assume that the probability of obtaining a zero observation from the j-th population (j = 1,2) is δ_j where $0 \le \delta_j \le 1$. Furthermore, we assume that the non-zero observations are distributed lognormally with mean μ_j and variance σ_j^2. Now, let $X_{1j}, X_{2j}, \ldots, X_{nj}$ be a random sample from the j^{th} population and let $M_j = E(X_{ij})$. From this preliminary setting specification we wish to construct credibility intervals for the ratio of the means, M_1 and M_2, of the two populations. As in Zhou and Tu (2000) we assume that in the j^{th} sample the non-zero observations come first: $X_{ij} > 0$, and $\ln(X_{ij}) \mid n_{j1} \sim N(\mu_j, \sigma_j)$, for $i = 1, \ldots, n_{j1}$. In addition, $X_{ij} = 0$, for $i = n_{j1} + 1, \ldots, n_j$ and $n_{j0} = n_j - n_{j1} \sim Bin(n_j, \delta_j)$. From this it follows that the mean of the j^{th} population, which is a function of μ_j, σ_j^2 and δ_j, is given by:

$$M_j = (1 - \delta_j)\exp(\mu_j + \frac{1}{2}\sigma_j^2).$$

To compare the two population means we will construct credibility intervals for the ratio of the means:

$$\frac{M_1}{M_2} = \frac{(1 - \delta_1)\exp(\mu_1 + \frac{1}{2}\sigma_1^2)}{(1 - \delta_2)\exp(\mu_2 + \frac{1}{2}\sigma_2^2)}.$$

4.2 Intervals Based on a Bayesian Procedure

Denote $y_{ij} = \ln X_{ij}$ and $\boldsymbol{\theta} = [\delta_1 \quad \mu_1 \quad \sigma_1^2 \; \delta_2 \quad \mu_2 \quad \sigma_2^2]'$ then the likelihood function is given by:

$$L(\boldsymbol{\theta} \mid data) \propto \prod_{j=1}^{2} \{\delta_j^{n_{j0}} (1-\delta_j)^{n_{j1}} \prod_{i=1}^{n_{j1}} (\frac{1}{\sigma_j^2})^{\frac{1}{2}} \exp[-\frac{(y_{ij}-\mu_j)^2}{2\sigma_j^2}]\} \tag{4.1}$$

The choice of prior to be used in this setting will be discussed in further sections. Given the previous specification of the likelihood, the Fisher Information Matrix in our case can be written as:

$$I(\boldsymbol{\theta}) = -E\left\{\left[\frac{\partial^2}{\partial^2\boldsymbol{\theta}} \ln L\left(\boldsymbol{\theta} \mid data\right)\right]\right\}$$

$$\therefore I(\boldsymbol{\theta}) = diag\begin{bmatrix} \frac{n_1}{\delta_1(1-\delta_1)} & \frac{n_1(1-\delta_1)}{\sigma_1^2} & \frac{n_1(1-\delta_1)}{2\sigma_1^4} & \frac{n_2}{\delta_2(1-\delta_2)} & \frac{n_2(1-\delta_2)}{\sigma_2^2} & \frac{n_2(1-\delta_2)}{2\sigma_2^4} \end{bmatrix} \tag{4.2}$$

4.2.1 Independence Jeffreys Prior:

Since $\boldsymbol{\theta}$ is unknown the prior

$$p(\boldsymbol{\theta}) \propto \prod_{j=1}^{2} \sigma_j^{-2} \delta_j^{-1/2} (1-\delta_j)^{-1/2} \tag{4.3}$$

will be specified for the unknown parameters. This is known as the independence Jeffreys prior. In (4.3) we have assumed μ_j and σ_j^2, for $j = 1,2$ to be independently distributed, *a priori*, with μ_j and $\log\sigma_j^2$ each uniformly distributed. See Zellner (1971) and Box and Tiao (1973) for further discussion. The prior $p(\delta_j) \propto \delta_j^{-\frac{1}{2}} (1-\delta_j)^{-\frac{1}{2}}$ is the one proposed by Jeffreys (1967) for the binomial parameter. Combining the likelihood function (4.1) and the prior density function (4.3) the joint posterior density function can be written as:

$$P(\theta \mid data) = \prod_{j=1}^{2} \left\{ \begin{array}{l} \dfrac{1}{B(n_{j0}+0.5; n_{j1}+0.5)} \delta_j^{n_{j0}-\frac{1}{2}} (1-\delta_j)^{n_{j1}-\frac{1}{2}} \\ \times \left(\dfrac{2\pi\sigma_j^2}{n_{j1}} \right)^{-\frac{1}{2}} \exp\left[-\dfrac{n_{j1}}{2\sigma_j^2} (\mu_j - \hat{\mu}_j)^2 \right] \left(\dfrac{\nu_{j1}\hat{\sigma}_j^2}{2} \right)^{\frac{1}{2}\nu_{j1}} \left(\dfrac{(\sigma_j^2)^{-\frac{1}{2}(\nu_{j1}+2)} \exp\left[-\dfrac{\nu_{j1}\hat{\sigma}_j^2}{2\sigma_j^2} \right]}{\Gamma\left(\dfrac{\nu_{j1}}{2} \right)} \right) \end{array} \right\} \tag{4.4}$$

where $\hat{\mu}_j = \dfrac{1}{n_{j1}} \sum_{i=1}^{n_{j1}} y_{ij}$,

$\nu_{j1} = n_{j1} - 1$,

$\hat{\sigma}_j^2 = \dfrac{1}{\nu_{j1}} \sum_{i=1}^{n_{j1}} (y_{ij} - \hat{\mu}_j)^2$ and

$B(m,n) = \dfrac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$.

From (4.4) it follows that the posterior distribution of δ_j is a Beta distribution (specifically $B\left(n_{j0}+\dfrac{1}{2}; n_{j1}+\dfrac{1}{2}\right)$) and δ_j is independently distributed of μ_j and σ_j^2, where the conditional posterior distribution of μ_j is normal:

$$\mu_j \mid \sigma^2, data \sim N\left(\hat{\mu}_j, \frac{\sigma_j^2}{n_{j1}} \right) \tag{4.5}$$

and for σ_j^2, the posterior density function is an Inverted Gamma density, specifically:

$$P(\sigma_j^2 \mid data) = \left(\frac{\nu_{j1}\hat{\sigma}_j^2}{2}\right)^{\frac{1}{2}\nu_{j1}} \left(\frac{(\sigma_j^2)^{-\frac{1}{2}(\nu_{j1}+2)} \exp\left[-\frac{\nu_{j1}\hat{\sigma}_j^2}{2\sigma_j^2}\right]}{\Gamma\left(\frac{\nu_{j1}}{2}\right)} \right). \tag{4.6}$$

The method proposed here to find the Bayesian credibility intervals for $D = \ln M_1 - \ln M_2$, the log of the ratio of the population means, is through Monte Carlo simulation. Since $\ln M_j = \ln(1-\delta_j) + \mu_j + \frac{\sigma_j^2}{2}$ (j = 1,2), standard routines can be used in the simulation procedure.

4.2.2 Simulation Procedure:

The following simulation was obtained from the preceding theory using the MATLAB® package:

1. Simulation of σ_j^2 can be obtained from (4.6) in the following way:

   a. Simulate τ_j^* from a $\chi^2_{\nu_{j1}}$ distribution, as the sum of ν_{j1} squared independent normal random variables.

   b. Calculate $\sigma_j^{2^*} = \frac{\nu_{j1}\hat{\sigma}_j^2}{\tau_j^{*2}}$

2. Given σ_j^{2*}, simulate μ_j^* from (4.5).

3. A simulated value of δ_j (a Beta random variable), namely δ_j^*, was obtained using a MATLAB function.

4. Substitute the simulated values σ_j^{2*}, μ_j^* and δ_j^* into the expression for D to obtain D^*, a simulated value for the log-ratio of the population means.

Once these values have been obtained, the simulation procedure is similar to those discussed in previous chapters.

4.2.3 Alternate Prior Distributions – Jeffreys Rule Prior:

As mentioned in the Introduction to this thesis, one of the objectives was to compare the Bayesian procedure for different choices of prior distributions for $\boldsymbol{\theta}$, the unknown parameters. In the previous two sections we discussed the analysis methods using Jeffreys' non-informative prior and the resulting simulation technique. In this and subsequent sections different choices of prior distributions will be discussed in an effort to eventually compare the results. The choice of density applied in this section is the square root of the determinant of the Fisher Information Matrix, which is an adaptation of the Jeffreys' rule used in the previous section.

Since $\boldsymbol{\theta}$ is unknown this prior becomes

$$p(\boldsymbol{\theta}) \propto \prod_{j=1}^{2} \sigma_j^{-3} \delta_j^{-1/2} (1-\delta_j)^{1/2} \tag{4.7}$$

This was derived from $|I(\boldsymbol{\theta})|^{1/2}$, which was defined in (4.2). In (4.6) we have assumed μ_j and σ_j^2, for $j = 1,2$ to be independently distributed, *a priori*, with μ_j and $\log \sigma_j^2$ each uniformly distributed. Combining the likelihood function (4.1) and the prior density function (4.7) the joint posterior density function can be written as:

$$P(\theta \mid data) = \prod_{j=1}^{2} \left\{ \begin{array}{l} \dfrac{1}{B(n_{j0}+0.5; n_{j1}+1.5)} \delta_j^{n_{j0}-\frac{1}{2}} (1-\delta_j)^{n_{j1}+\frac{1}{2}} \\ \times \left(\dfrac{2\pi\sigma_j^2}{n_{j1}} \right)^{-\frac{1}{2}} \exp\left[-\dfrac{n_{j1}}{2\sigma_j^2} (\mu_j - \hat{\mu}_j)^2 \right] \left(\dfrac{(\nu_{j1})\hat{\sigma}_j^2}{2} \right)^{\frac{1}{2}(\nu_{j1}+1)} \left(\dfrac{(\sigma_j^2)^{-\frac{1}{2}(\nu_{j1}+3)} \exp\left[-\dfrac{(\nu_{j1})\hat{\sigma}_j^2}{2\sigma_j^2} \right]}{\Gamma\left(\dfrac{(\nu_{j1}+1)}{2} \right)} \right) \end{array} \right\} \tag{4.8}$$

where $\hat{\mu}_j$, ν_{j1}, $\hat{\sigma}_j^2$ and $B(m,n)$ are defined as before.

From (4.7) it follows that the posterior distribution of δ_j is a Beta distribution (specifically $B\left(n_{j0}+\frac{1}{2}; n_{j1}+\frac{3}{2}\right)$) and δ_j is independently distributed of μ_j and σ_j^2, where the conditional posterior distribution of μ_j is normal:

$$\mu_j \mid \sigma^2, data \sim N\left(\hat{\mu}_j, \frac{\sigma_j^2}{n_{j1}} \right) \tag{4.9}$$

and for σ_j^2, the posterior density function (as before) is an Inverted Gamma density, specifically:

$$P(\sigma_j^2 \mid data) = \left(\frac{(\nu_{j1})\hat{\sigma}_j^2}{2} \right)^{\frac{1}{2}(\nu_{j1}+1)} \left(\frac{(\sigma_j^2)^{-\frac{1}{2}(\nu_{j1}+3)} \exp\left[-\dfrac{(\nu_{j1})\hat{\sigma}_j^2}{2\sigma_j^2} \right]}{\Gamma\left(\dfrac{(\nu_{j1}+1)}{2} \right)} \right). \tag{4.10}$$

A similar simulation procedure to the one previously described can be used except that we simulate τ_j^* from a $\chi^2_{\nu_{j1+1}}$ distribution, as the sum of $\nu_{j1}+1$ squared independent normal random variables.

4.2.4 Alternate Prior Distributions – Constant (Uniform) Prior:

Since θ is unknown this prior becomes

$$p(\theta) \propto const \tag{4.11}$$

In (4.11) we have assumed μ_j and σ_j^2, for $j = 1,2$ to be independently distributed, *a priori*, with μ_j and $\log \sigma_j^2$ each uniformly distributed. Combining the likelihood function (4.1) and the prior density function (4.11) the joint posterior density function can be written as:

$$P(\theta \mid data) = \prod_{j=1}^{2} \left\{ \begin{array}{l} \dfrac{1}{B(n_{j0}+1; n_{j1}+1)} \delta_j^{n_{j0}} (1-\delta_j)^{n_{j1}} \\ \times \left(\dfrac{2\pi\sigma_j^2}{n_{j1}} \right)^{-\frac{1}{2}} \exp\left[-\dfrac{n_{j1}}{2\sigma_j^2} (\mu_j - \hat{\mu}_j)^2 \right] \left(\dfrac{(v_{j1})\hat{\sigma}_j^2}{2} \right)^{\frac{1}{2}(v_{j1}-2)} \left(\dfrac{(\sigma_j^2)^{-\frac{1}{2}(v_{j1})} \exp\left[-\dfrac{(v_{j1})\hat{\sigma}_j^2}{2\sigma_j^2} \right]}{\Gamma\left(\dfrac{(v_{j1}-2)}{2} \right)} \right) \end{array} \right\} \tag{4.12}$$

with the symbols defined as before.

From (4.12) it follows that the posterior distribution of δ_j is a Beta distribution (specifically $B\left(n_{j0}+1; n_{j1}+1\right)$) and δ_j is independently distributed of μ_j and σ_j^2, where the conditional posterior distribution of μ_j is normal:

$$\mu_j \mid \sigma^2, data \sim N\left(\hat{\mu}_j, \frac{\sigma_j^2}{n_{j1}} \right) \tag{4.13}$$

and for σ_j^2, the posterior density function is an Inverted Gamma density, specifically:

$$P(\sigma_j^2 \mid data) = \left(\frac{(\nu_{j1})\hat{\sigma}_j^2}{2}\right)^{\frac{1}{2}(\nu_{j1}-2)} \left(\frac{(\sigma_j^2)^{-\frac{1}{2}(\nu_{j1})} \exp\left[-\frac{(\nu_{j1})\hat{\sigma}_j^2}{2\sigma_j^2}\right]}{\Gamma\left(\frac{(\nu_{j1}-2)}{2}\right)}\right). \tag{4.14}$$

A similar simulation procedure to the one previously described except that we simulate τ_j^* from a $\chi^2_{\nu_{j1}-2}$ distribution, as the sum of $\nu_{j1}-2$ squared independent normal random variables.

4.3 Method of Variance Estimates Recovery (MOVER)

In addition to the Bayesian priors considered, the performance of the Bayesian prior distributions was also compared to the MOVER in the case where zero values are possible. Zou, Taleban and Huo (2009a) proposed procedures involving the so-called "method of variance estimates recovery" (MOVER). The MOVER method was designed in order to apply to a general scenario and also to provide adequate coverage rates in estimation procedures relating to lognormally distributed data. The advantage of the MOVER is therefore that it is easily applicable to many different settings with little more than a basic knowledge of introductory statistical texts.

The $(1-\alpha)100\%$ confidence limits for $\tilde{\theta}_i = \mu_i + \frac{1}{2}\sigma_i^2$, $i = 1,2$, using the MOVER as given by Zou et al (2009) on page 3758 are:

$$\tilde{L}_i = \hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2} - \sqrt{Z_{\alpha/2}^2 \frac{\hat{\sigma}_i^2}{n_i}\left\{\frac{\hat{\sigma}_i^2}{2}\left(1 - \frac{\nu_i}{\chi^2_{1-\alpha/2,\nu_i}}\right)\right\}^2}$$

$$\tilde{U}_i = \hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2} + \sqrt{Z_{\alpha/2}^2 \frac{\hat{\sigma}_i^2}{n_i}\left\{\frac{\hat{\sigma}_i^2}{2}\left(\frac{\nu_i}{\chi^2_{\alpha/2,\nu_i}} - 1\right)\right\}^2}$$

Furthermore, due to the presence of zero values the confidence limits for δ_i are given by:

$$\left[\hat{\delta}_i + \frac{Z_{\alpha/2}^2}{2n_i} \pm \sqrt{\frac{\left\{\hat{\delta}_i\left(1-\hat{\delta}_i\right) + \frac{Z_{\alpha/2}^2}{4n_i}\right\}}{n_i}} \middle/ \left(1 + \frac{Z_{\alpha/2}^2}{n_i}\right)\right]$$

where

$$\hat{\delta}_i = \frac{n_{i0}}{n_i}$$

It is then a simple matter to find the confidence limits for $ln(1 - \delta_i)$, which will be denoted by $\tilde{l}_i$ and $\tilde{u}_i$, respectively.

Underlying these limits is the well known result that the $(1 - \alpha)100\%$ confidence interval for σ_i^2 is given by $\left[{\nu_i s_i^2}/{\chi^2_{1-\alpha/2,\nu_i}} ; {\nu_i s_i^2}/{\chi^2_{\alpha/2,\nu_i}}\right]$ where $\chi^2_{\alpha/2,\nu}$ is the ${\alpha}/{2}$th percentile from the chi-squared distribution with ν degrees of freedom where $\nu_i = n_i - 1$. $Z_{\alpha/2}$ is the upper ${\alpha}/{2}\, th$ quantile of the standard normal distribution and $\hat{\mu}_i = \bar{x}_i$ and $\hat{\sigma}_i^2 = \frac{1}{\nu} s_{ii}$.

The $(1-\alpha)100\%$ confidence interval for $ln(1-\delta_i)+\mu_i+\frac{1}{2}\sigma_i^2$ is therefore given by:

$$L = ln\left(1-\hat{\delta}_i\right)+\hat{\mu}_i+\frac{1}{2}\hat{\sigma}_i^2-\sqrt{\left(ln\left(1-\hat{\delta}_i\right)-\tilde{l}_i\right)^2+\left(\tilde{\theta}_i-\tilde{L}_i\right)^2}$$

$$U = ln\left(1-\hat{\delta}_i\right)+\hat{\mu}_i+\frac{1}{2}\hat{\sigma}_i^2-\sqrt{\left(\tilde{u}_i-ln\left(1-\hat{\delta}_i\right)\right)^2+\left(\tilde{U}_i-\tilde{\theta}_i\right)^2}$$

4.4 Simulation Study

4.4.1 Results for a Single Sample with Zero Values

Zou et al (2009a) used the MOVER for what they termed the one sample Δ-distribution, which is the same situation as the previously described setting except that only a single population is considered. The following results were obtained:

Table 32: Comparison of the MOVER and GCI against Independence Jeffreys' Prior for Zero Values Included for Constructing Two-sided 95% Confidence Intervals for $ln\left(1-\delta\right)+\left(\mu+\frac{1}{2}\sigma^2\right)$

			MOVER		GCI		Equal-Tailed		HPD Intervals	
δ	n	σ^2	Cover %	Width	Cover %	Width	Cover %	Width	Cover %	Width
0.1	15	1	95.03 (3.60, 1.37)	1.65	95.53 (2.34, 2.13)	1.72	95.03 (2.59, 2.38)	1.66	95.36 (1.26, 3.38)	1.59
		2	95.22 (3.13, 1.65)	2.78	95.50 (2.26, 2.24)	2.85	95.14 (2.62, 2.24)	2.79	95.56 (0.91, 3.53)	2.62
		3	94.87 (2.90, 2.23)	3.88	94.94 (2.35, 2.71)	3.94	94.86 (2.73, 2.41)	3.89	95.42 (0.80, 3.78)	3.59
	25	1	95.21 (3.09, 1.70)	1.17	95.95 (2.03, 2.02)	1.22	95.75 (2.30, 2.45)	1.18	95.34 (1.34, 3.32)	1.15
		2	94.94 (2.87, 2.19)	1.93	95.21 (2.19, 2.60)	1.97	95.00 (2.52, 2.48)	1.93	95.43 (1.22, 3.35)	1.86
		3	95.09 (2.80, 2.11)	2.67	95.31 (2.26, 2.43)	2.71	94.85 (2.60, 2.55)	2.66	95.22 (1.04, 3.74)	2.54
	50	1	95.10 (3.02, 1.88)	0.78	95.79 (2.26, 1.95)	0.80	94.93 (2.47, 2.60)	0.78	94.98 (1.78, 3.24)	0.77
		2	95.16 (2.87, 1.97)	1.26	95.41 (2.37, 2.22)	1.29	95.28 (2.47, 2.25)	1.27	95.56 (1.47, 2.97)	1.24
		3	94.86 (2.69, 2.45)	1.73	94.87 (2.43, 2.70)	1.76	94.93 (2.68, 2.39)	1.73	95.13 (1.53, 3.34)	1.69
0.2	15	1	95.17 (3.30, 1.53)	1.87	95.99 (2.14, 1.87)	1.98	95.48 (2.44, 2.08)	1.89	96.18 (1.08, 2.74)	1.80
		2	95.41 (2.78, 1.81)	3.13	95.70 (2.02, 2.28)	3.23	94.98 (2.76, 2.26)	3.13	95.78 (0.88, 3.34)	2.91
		3	94.97 (3.11, 1.92)	4.38	94.93 (2.54, 2.53)	4.47	94.60 (2.83, 2.57)	4.37	95.10 (0.81, 4.09)	3.99
	25	1	95.17 (3.20, 1.63)	1.30	96.01 (2.14, 1.85)	1.36	95.32 (2.49, 2.19)	1.30	95.81 (1.45, 2.74)	1.27
		2	95.34 (2.79, 1.87)	2.10	95.70 (2.11, 2.19)	2.16	95.34 (2.47, 2.19)	2.11	95.83 (1.08, 3.09)	2.03
		3	95.27 (2.74, 1.99)	2.91	95.57 (2.17, 2.26)	2.96	94.92 (2.48, 2.60)	2.90	95.26 (0.89, 3.85)	2.76
	50	1	95.00 (2.99, 2.01)	0.85	95.56 (2.26, 2.18)	0.88	94.90 (2.62, 2.48)	0.86	95.20 (1.84, 2.96)	0.85
		2	94.95 (2.92, 2.13)	1.37	95.28 (2.36, 2.36)	1.39	95.09 (2.74, 2.17)	1.37	95.20 (1.72, 3.08)	1.34
		3	95.30 (2.58, 2.12)	1.87	95.39 (2.24, 2.37)	1.90	95.13 (2.77, 2.10)	1.87	95.46 (1.45, 3.09)	1.83

From the above it is evident that the interval lengths and coverage of the equal-tailed intervals are very similar to those of MOVER, with the lengths being almost identical. It is interesting to note however, that proportion of intervals above and below the true value differ substantially. The Bayesian HPD intervals are therefore a large improvement on the MOVER and generalized confidence intervals.

In simulations based on samples without zero values the equal-tailed Bayesian intervals using the Jeffreys' prior $p(\mu,\sigma^2) \propto \sigma^{-2}$ are identical to the generalized confidence intervals. This will not be the case in Table 33. The reason for this is the simulation of δ. In the Bayesian case (using the Independence Jeffreys prior) the posterior distribution of δ is the Beta, $B\left(n_0 + \frac{1}{2}, n_1 + \frac{1}{2}\right)$, distribution while Zou et al (2009a) (see also Tian (2005)) used two pivotal quantities, $B(n_0 + 1, n_1)$ and $B(n_0, n_1 + 1)$ for δ, which are combined with the pivotal quantity of the lognormal mean to simulate $ln\widetilde{M} = ln(1-\delta\ ) + \left(\mu\ \ + \frac{1}{2}\sigma^2\right)$. From Table 33 it is also clear that the Bayesian equal-tailed intervals are shorter than those of the generalized confidence interval procedure with just as good or better coverage probabilities.

Tian and Wu (2006) also considered an approach based on the adjusted log-likelihood ratio statistics for constructing a confidence interval for the mean of lognormally distributed data with excess zeros. Because of different parameter values only a few results could be compared. It does seem, however, that the procedures described in Table 33 result in better results than the adjusted log likelihood method.

Once again, other prior distributions were also evaluated, but only for the case of $\delta = 0.1$. In addition, the proportion of intervals above and below the true parameter values was also not recorded.

Table 33: Comparison of the MOVER and Other Prior Distributions for Constructing Two-sided 95% Confidence Intervals for $ln(1-\delta)+\left(\mu+\frac{1}{2}\sigma^2\right)$

n	σ^2	Prior / Method	Equal-Tailed / MOVER Cover %	Equal-Tailed / MOVER Width	HPD Intervals Cover %	HPD Intervals Width
15	1	MOVER	95.03	1.65		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.35	1.53	94.10	1.48
		Reference Prior	95.45	1.75	96.35	1.67
		Probability-Matching Prior	96.25	1.67	94.41	1.59
	2	MOVER	95.22	2.78		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.06	2.52	93.93	2.39
		Reference Prior	95.01	3.02	96.10	2.81
		Probability-Matching Prior	94.76	2.68	94.79	1.59
	3	MOVER	94.87	3.88		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.47	3.45	94.02	3.23
		Reference Prior	95.26	4.27	96.48	3.90
		Probability-Matching Prior	94.22	3.76	94.45	3.49
25	1	MOVER	95.21	1.17		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.12	1.12	94.18	1.10
		Reference Prior	95.03	1.19	95.46	1.17
		Probability-Matching Prior	94.82	1.15	94.81	1.12
	2	MOVER	94.94	1.93		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.65	1.82	94.47	1.76
		Reference Prior	94.96	1.99	95.89	1.92
		Probability-Matching Prior	95.14	1.88	95.20	1.82
	3	MOVER	95.09	2.67		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.82	2.51	94.33	2.41
		Reference Prior	95.01	2.78	95.82	2.64
		Probability-Matching Prior	94.44	2.60	94.35	2.49
50	1	MOVER	95.10	0.78		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.79	0.76	94.72	0.76
		Reference Prior	94.93	0.79	94.95	0.79
		Probability-Matching Prior	94.80	0.77	94.85	0.76
	2	MOVER	95.16	1.26		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	94.37	1.23	94.25	1.21
		Reference Prior	95.34	1.28	95.78	1.26
		Probability-Matching Prior	94.79	1.25	95.00	1.23
	3	MOVER	94.86	1.73		
		$p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$	95.03	1.68	94.51	1.65
		Reference Prior	95.11	1.77	95.53	1.72
		Probability-Matching Prior	94.65	1.72	94.66	1.68

In this instance it appears as though the Probability-Matching prior, $p(\tilde{\boldsymbol{\theta}}) \propto \delta^{-1/2}(1-\delta)^{-1/2}\sigma^{-2}\left(1+\frac{2}{\sigma^2}\right)^{1/2}$, and the Jeffreys' Rule prior ($p(\tilde{\boldsymbol{\theta}}) \propto \sigma^{-3}\delta^{-1/2}(1-\delta)^{1/2}$) tend to give coverage probabilities slightly less than 0.95. The Reference prior, $p(\tilde{\boldsymbol{\theta}}) \propto \delta^{-1/2}(1-\delta)^{-1/2}\sigma^{-1}\left(1+\frac{2}{\sigma^2}\right)^{1/2}$ on the other hand gives the correct coverage probabilities, but the intervals are wider than that of the MOVER.

Example 2: Two-sided 95% Confidence Intervals for $(1-\delta)exp\left(\mu+\frac{1}{2}\sigma^2\right)$

On page 3761 of Zou et al (2009a) the following example was given (as referenced in Zhou and Tu (2000)): diagnostic test charges on 40 patients were investigated. Among them, 10 patients had no diagnostic test charges and the charges for the remaining patients were approximated using a lognormal distribution. On the log scale the following values were observed: $\bar{y} = 6.8535$ and $s^2 = 1.8696$.

For the MOVER the following interval was obtained and this is compared to results obtained from the Bayes and GCI methods:

Table 34: Results of Diagnostic Test Charge Data

	MOVER	GCI	Jeffreys Rule	Independence Jeffreys	Reference Prior	Probability-Matching Prior
Lower Limit (equal-tailed)	955.50	970.81	1002.18	975.03	996.10	982.41
Upper Limit (equal-tailed)	4491.55	4687.37	4690.34	4310.66	4802.44	4519.89
Lower Limit (HPD)			958.05	906.87	931.60	926.10
Upper Limit (HPD)			4345.73	3932.04	4356.93	4111.21

From the table it is clear that the intervals do not differ much. The intervals for the MOVER and the Probability-Matching prior are for all practical purposes the same. The shortest intervals are obtained from the Independence Jeffreys' and Probability-Matching prior. As mentioned before, the equal-tailed Independence Jeffreys interval and the GCI interval will not be the same because of the difference in the simulation of δ.

4.4.2 Results for the Ratio Between Two Samples with Zero Values

As was done in Zhou and Tu (2000) we will use computer simulations to study the operating characteristics of the proposed Bayesian confidence interval procedure in finite sample sizes. Random sample sizes containing both zero and lognormal observations are generated using the following different sample sizes:

Table 35: Sample Sizes Analysed by Monte Carlo Simulation Techniques

n_1	n_2
10	10
25	25
50	50
100	100
10	25
25	10
25	50

Zero proportions with different skewness coefficients are also considered. Based on these generated samples the credibility intervals (or Bayesian confidence intervals [BCI's]) are constructed. The following additional characteristics are reported:

- coverage probabilities
- average interval lengths
- coverage error (target coverage – actual coverage),
- percentages of under-coverage on both sides ($\%BCI < \theta$ and $\%BCI > \theta$)
- relative bias $\dfrac{|\%BCI < \theta - \%BCI > \theta|}{(\%BCI < \theta + \%BCI > \theta)}$.

As was in Zhou and Tu (2000) the nominal significance level of $\alpha = 0.05$ will be used and for each parameter setting, $C = 10000$ random samples are simulated to ensure that the margin of error is less than 0.005 with 95% confidence. l is taken to be 1000.

In the following table the parameter settings used in the simulation study are presented:

Table 36: Parameter Settings used in the Simulation Study

Design	σ_1^2	σ_2^2	δ_1	δ_2	γ_1	γ_2
1	3.0	1.0	0.0	0.0	96.4851	6.1849
2	4.0	4.0	0.0	0.0	414.3593	414.3593
3	3.0	1.0	0.1	0.1	100.9809	6.1763
4	2.0	0.5	0.0	0.1	23.7323	2.6848
5	2.0	0.5	0.1	0.2	24.5572	2.5806

Tables 49 to 53 represent the results of the simulation study for designs one to five, respectively, for equal-tailed Bayesian confidence intervals only. The results from the simulation study performed by Zhou and Tu (2000) for the Maximum Likelihood and Bootstrap methods have been supplied as well for the purposes of comparison. The results for individual designs are presented in the appendix to this chapter as Tables 49 to 53. Only summary results will be presented in the discussion in the main body of text in this chapter.

As described in previous chapters, an advantage of the Bayesian approach is the construction of HPD intervals. Tables 54 to 58 represent the HPD interval results from these simulation studies. Table 59 is given for the same designs for the MOVER and is also given in the appendix to this chapter. Only summarised results will be presented and discussed here.

4.5 Discussion of Results for the Simulation Studies

As mentioned, Zhou and Tu (2000) presented the results for the ML and Bootstrap methods and compared only these. They ascertained that when the two population skewness coefficients are the same the ML-based method results in better coverage probabilities in comparison with the stated nominal level. However, it is found that the ML-based method is more biased than the bootstrap method, as evidenced by a larger relative bias. This was particularly evident when the sample sizes were not the same. The ML method tends to cover too many observations on the left and too few on the right.

When the two skewness coefficients are not the same the results indicate better coverage accuracy for the bootstrap method. This method is also less biased than the ML-based method.

However, the objective of this chapter was to compare these results against results obtained from a Bayesian-based simulation study using a specifically chosen set of prior distributions and to evaluate the performance of each prior distribution against both the other distributions and the results obtained by Zhou and Tu (2000), overall.

The following table presents the summary statistics of the results in both the Zhou and Tu (2000) simulation study and the Bayesian simulation study. Both HPD and equal-tailed confidence intervals are presented for the Bayesian confidence intervals. The MOVER results have been added for reference.

Table 37: Summary Results for Simulation Studies

Design	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
1	Bootstrap	0.9266	0.0234	2.6252	0.0393	0.0341	0.0857
	ML	0.9285	0.0215	2.3625	0.0636	0.0080	0.6678
	MOVER	0.9584	-0.0084	3.2313	0.0161	0.0254	0.2171
	Prior 1	0.9550	-0.0050	3.2020	0.0234	0.0216	0.1126
	Prior 1 - HPD	0.9596	-0.0096	2.5737	0.0304	0.0100	0.4851
	Prior 2	0.9473	0.0027	2.8889	0.0226	0.0301	0.2028
	Prior 2 - HPD	0.9484	0.0016	2.2595	0.0430	0.0086	0.5974
	Prior 3	0.9603	-0.0103	4.3057	0.0280	0.0117	0.4019
	Prior 3 - HPD	0.9716	-0.0216	3.5341	0.0183	0.0101	0.3478
2	Bootstrap	0.9369	0.0131	3.9506	0.0324	0.0307	0.0287
	ML	0.9476	0.0024	3.8491	0.0278	0.0246	0.3417
	MOVER	0.9580	-0.0080	5.6237	0.0227	0.0193	0.1166
	Prior 1	0.9494	0.0006	5.4961	0.0249	0.0257	0.1409
	Prior 1 - HPD	0.9644	-0.0144	4.9120	0.0183	0.0173	0.2659
	Prior 2	0.9429	0.0071	4.8849	0.0271	0.0300	0.1506
	Prior 2 - HPD	0.9593	-0.0093	4.3118	0.0234	0.0173	0.3452
	Prior 3	0.9590	-0.0090	7.6650	0.0207	0.0460	0.2535
	Prior 3 - HPD	0.9766	-0.0266	6.9142	0.0133	0.0101	0.3087
3	Bootstrap	0.9298	0.0202	2.7003	0.0393	0.0308	0.1269
	ML	0.9237	0.0263	2.5127	0.0675	0.0088	0.7266
	MOVER	0.9536	-0.0036	3.5978	0.0227	0.0237	0.1983
	Prior 1	0.9486	0.0014	3.5759	0.0257	0.0514	0.1422
	Prior 1 - HPD	0.9569	-0.0069	2.8775	0.0341	0.0090	0.5919
	Prior 2	0.9446	0.0093	3.1424	0.0227	0.0327	0.2508
	Prior 2 - HPD	0.9461	0.0039	2.4856	0.0440	0.0099	0.6056
	Prior 3	0.9569	-0.0069	7.3283	0.0289	0.0143	0.3953
	Prior 3 - HPD	0.9721	-0.0221	4.4808	0.0166	0.0113	0.1725
4	Bootstrap	0.9294	0.0206	1.8569	0.0380	0.0326	0.0716
	ML	0.9274	0.0226	1.7269	0.0624	0.0102	0.6828
	MOVER	0.9637	-0.0137	2.2705	0.0164	0.0199	0.1357
	Prior 1	0.9564	-0.0064	2.2691	0.0211	0.0224	0.0608
	Prior 1 - HPD	0.9513	-0.0013	1.6996	0.0391	0.0096	0.6104
	Prior 2	0.9513	-0.0013	2.0566	0.0206	0.0268	0.1699
	Prior 2 - HPD	0.9446	0.0054	1.5012	0.0463	0.0091	0.6292
	Prior 3	0.9619	-0.0157	3.0409	0.0224	0.0119	0.3122
	Prior 3 - HPD	0.9724	-0.0224	2.4051	0.0174	0.0101	0.3272
5	Bootstrap	0.9346	0.0154	1.9795	0.0349	0.0305	0.0596
	ML	0.9274	0.0226	1.8666	0.0619	0.0108	0.6650
	MOVER	0.9570	-0.0070	2.5871	0.0199	0.0231	0.1718
	Prior 1	0.9524	-0.0024	2.5810	0.0241	0.0234	0.1119
	Prior 1 - HPD	0.9549	-0.0049	1.9568	0.0346	0.0106	0.5139
	Prior 2	0.9450	0.0050	2.2786	0.0243	0.0307	0.1356
	Prior 2 - HPD	0.9437	0.0063	1.6839	0.0459	0.0104	0.6040
	Prior 3	0.9627	-0.0127	4.6573	0.0250	0.0123	0.4981
	Prior 3 - HPD	0.9740	-0.0240	3.4718	0.0169	0.0091	0.3673
Overall	Bootstrap	0.9315	0.0185	2.6225	0.0368	0.0317	0.0745
	ML	0.9309	0.0191	2.4636	0.0566	0.0125	0.6168
	MOVER	0.9581	-0.0081	3.4621	0.0196	0.0223	0.1679
	Prior 1	0.9524	-0.0024	3.4248	0.0239	0.0289	0.1137
	Prior 1 - HPD	0.9574	-0.0074	2.8039	0.0313	0.0113	0.4934
	Prior 2	0.9462	0.0046	3.0503	0.0235	0.0301	0.1819
	Prior 2 - HPD	0.9484	0.0016	2.4484	0.0405	0.0111	0.5563
	Prior 3	0.9601	-0.0109	5.3994	0.0250	0.0192	0.3722
	Prior 3 - HPD	0.9733	-0.0233	4.1612	0.0165	0.0102	0.3047

From the overall summary statistics we see that the choices of prior distributions have better coverage than both the ML-based and bootstrap methods. However, this does not provide the full picture.

Coverage Probabilities

As evident from the above summary table, it is apparent that the ML and Bootstrap methods are comparable in terms of the coverage probability. The ML method, as noted by Zhou and Tu (2000), gives better coverage for designs 1, 2 and 3, *i.e.*, when the skewness coefficients of the two populations are the same. Otherwise the Bootstrap method offers superior coverage. However, the coverage probabilities overall for the ML and Bootstrap methods were 0.9309 and 0.9315 respectively.

The three Bayesian methods considered here all provide better coverage than the ML and Bootstrap methods proposed. However, at least one of the Bayesian methods, the method of the constant or uniform prior distribution, results in over-coverage, with an overall coverage ratio of 0.9601. Naturally, this will imply a larger coverage error when compared with the other prior distributions used, which is ultimately due to a larger average interval length. But this will be discussed further in later sections.

Overall, the best prior distribution to be used in terms of coverage probability was the independence Jeffreys prior. In terms of the literature, Box and Tiao (1973), this would be the natural choice of prior distribution in this setting and thus, its accuracy compared to the other prior distributions should be expected. The overall coverage probability was 0.9524 compared to 0.9462 for the Jeffreys Rule prior described previously.

However, the Jeffreys Rule appears to be nearly as good as the Independence Jeffreys prior. The prior tends to undercover, but not by much at all. What is particularly positive is that even though there is slight under coverage, the average interval length is shorter for the Jeffreys Rule prior.

A point of interest is that the coverage of the Bayesian methods does not appear to be affected by the skewness coefficients of the different designs.

The better coverage probabilities are as a direct result of the increased average interval lengths for the Bayesian methods. However, this is discussed in more comprehensively in subsequent sections.

The MOVER appears to provide adequate coverage. The performance of the MOVER is matched quite well to the coverage of Prior 1 (i.e. the independence Jeffreys' prior). However, the advantage of the Bayesian method is evident when regarding the HPD intervals, which result in slightly better coverage, but increased efficiency in terms of the average interval length.

Coverage Error

Overall, the coverage error for the Bootstrap method is better than if compared to the ML method. The only possible exception to this overall figure is perhaps the case when the population skewness coefficients are similar. However, this is by no means concrete.

For the Bayesian methods the overall coverage error was better for all choices of prior distributions, as opposed to the ML and Bootstrap methods. For the independence Jeffreys prior the coverage error appears smallest, thereby reinforcing the observation of the better coverage probability. This error for the uniform prior appears to increase when the population skewness coefficients are different.

Average Length

Firstly, results by Zhou and Tu (2000) indicate that the Bootstrap method results in intervals with longer interval lengths. Overall, the interval length for the Bootstrap method was 2.6225 compared to the 2.4636 of the ML method. As previously mentioned, coverage probability and average interval length are related. Thus, we would expect the average interval length for the Bootstrap method to be greater since it provides better probability of coverage. However, the average interval length for both these methods appears to be related to the population skewness coefficients in the following way: when the coefficients are the same (designs 1, 2 and 3) the average interval lengths are distinctly larger for the Bootstrap method, particularly when sample sizes are small.

Overall, when analyzing the results from the Bayesian methods it is apparent that the interval lengths are larger. Once again this would be expected due to the previously mentioned relationship between the coverage probabilities and the interval length. As with the methods proposed by Zhou and Tu (2000), the average interval length decreases when the population skewness coefficients are different. The Independence Jeffreys prior and the Jeffreys Rule prior produced average interval lengths of 3.4248 and 3.0503 respectively.

Lastly, as was mentioned previously, the constant or uniform prior tends to over-cover. This inefficiency is accurately portrayed by the average interval length, namely: 5.3994.

The HPD intervals are an improvement on both the equal-tailed intervals as well as the MOVER in terms of interval length.

Coverage on Left and Right and Bias

As mentioned in previous sections, the results obtained by Zhou and Tu (2000) indicate that the ML method covers too many observation on the left and too many on the right. The only exception to this is Design 2. Overall, the Bootstrap method had a better spread.

The Bayesian methods employed indicate a much more equal spread of observations above and below. The uniform prior is the only possible exception. Thus, although the average interval lengths are greater for the Bayesian case the spread of the interval appears better, *i.e.*, the Bayesian methods overall tend to cover as many observations on the right as on the left.

The MOVER is comparable to independence Jeffreys prior. In particular, the HPD intervals result in increased relative bias compared to the equal-tailed intervals and the MOVER.

4.6 Example – Rainfall Data

For the purposes of comparison of the different methods, an example was chosen using raw data obtained from the South African Weather Service. The data consisted of the monthly rainfall totals for the cities of Bloemfontein and Kimberley, two South African cities, over a period of 69 to 70 years of measurement. However, these two cities are both located in relatively arid regions and are characterised by mainly summer rainfall. For that reason, the winter months do contain some rainfall data, but also contain many years where the total monthly rainfall data were zero. Based on past studies it is evident that rainfall data can be modeled according to a lognormal distribution. The data can be summarised as follows:

Table 38: Summary of the Rainfall Data

City	Parameter	Value
Bloemfontein	Number of Years of Available Data	70
	Number of Zero Valued Observations	18
	Mean of Log-Transformed Data	1.9578
	Variance of Log-Transformed Data	2.1265
Kimberley	Number of Years of Available Data	69
	Number of Zero Valued Observations	10
	Mean of Log-Transformed Data	1.0526
	Variance of Log-Transformed Data	3.1589

In order to compare the results, both the Maximum Likelihood and Bootstrap methods of Zhou and Tu (2000) were applied to the data. The Maximum Likelihood method is sufficient in its simplicity and can be referred to the text of Zhou and Tu (2000). The bootstrap method was applied as follows:

1. From the data delta was calculated as follows: $\hat{\delta}_j = \frac{n_{j0}}{n_j}$
2. The mean ($\hat{\mu}_j$) and variance ($\hat{\sigma}_j^2$) of the log-transformed observations were calculated (i.e. the non-zero log-transformed observations).
3. The following steps were then repeated 100000 times:
    a. Binomial samples were generated using the estimates obtained in 1) above, which will for simplicity be called $\hat{n}_{j0}$. Using these we calculate $\hat{n}_{j1} = n_j - \hat{n}_{j0}$.
    b. We then generate $\hat{n}_{j1}$ $N\left(\hat{\mu}_j, \hat{\sigma}_j^2\right)$ observations (since the log-transformed data is normally distributed).

c. Calculate $\hat{\delta}_j^* = \dfrac{\hat{n}_{j0}}{n_j}$ from these data as well as $\hat{\mu}_j^*$ as the mean of the $\hat{n}_{j1}$ observations and $\hat{\sigma}_j^{2*}$ as the standard deviation.

d. Calculate: $\log M_j^* = \log(1-\hat{\delta}_j^*) + \hat{\mu}_j^* + \hat{\sigma}_j^{2*}/2$.

e. Calculate se^* according to the formula given by Zhou and Tu (2000).

f. Calculate the $T^* = \dfrac{\log M_1^* - \log M_2^* - \log M_1 + \log M_2}{se^*}$

4. From these simulated T^* observations calculate the following:

a. If $\alpha = 0.05$, then $k_1^* = \left(1 - \alpha/2\right) \times 100000$ and $k_2^* = \left(\alpha/2\right) \times 100000$

b. Then the upper and lower limits (respectively) of the bootstrap confidence intervals for the log transformed data are $Upper = \log M_1 - \log M_2 + T_{k_1}^* \times se^*$ and $Lower = \log M_1 - \log M_2 + T_{k_2}^* \times se^*$.

c. To obtain the confidence intervals of the original log-normal data take the anti-log.

In addition to the maximum likelihood and bootstrap confidence intervals the confidence intervals were obtained using the Bayesian methods described in the preceding text and for the following priors: Independence Jeffreys Prior (Prior 1 in the table), the Jeffreys Rule Prior (Prior 2), the constant prior (Prior 3) , the Reference Prior (Prior 4) and the Probability-Matching Prior (Prior 5). The results are presented in the following table:

Table 39: Summary of Results for the Rainfall Data

	Maximum Likelihood	Bootstrap	Prior 1	Prior 2	Prior 3	Reference Prior	Probability-Matching Prior
Lower Limit of Logged Data	-0.6914	-0.6404	-0.8019	-0.7942	-0.8624	-0.8188	-0.7794
Upper Limit of Logged Data	1.1880	1.2444	1.2013	1.1954	1.2248	1.2104	1.1982
Lower Limit	0.5009	0.5271	0.4485	0.4520	0.4221	0.4410	0.4587
Upper Limit	3.2805	3.4710	3.3245	3.3048	3.4035	3.3548	3.3141

In terms of interval width it is worth noting that Prior 2 and the Probability-Matching prior were efficient in this situation. However, the above results do not indicate coverage and this is where the Jeffreys-based priors are superior to the other methods (as observed from the previous simulation study results).

The following graphs also illustrate these results:

Figure 22: Results for Prior 1

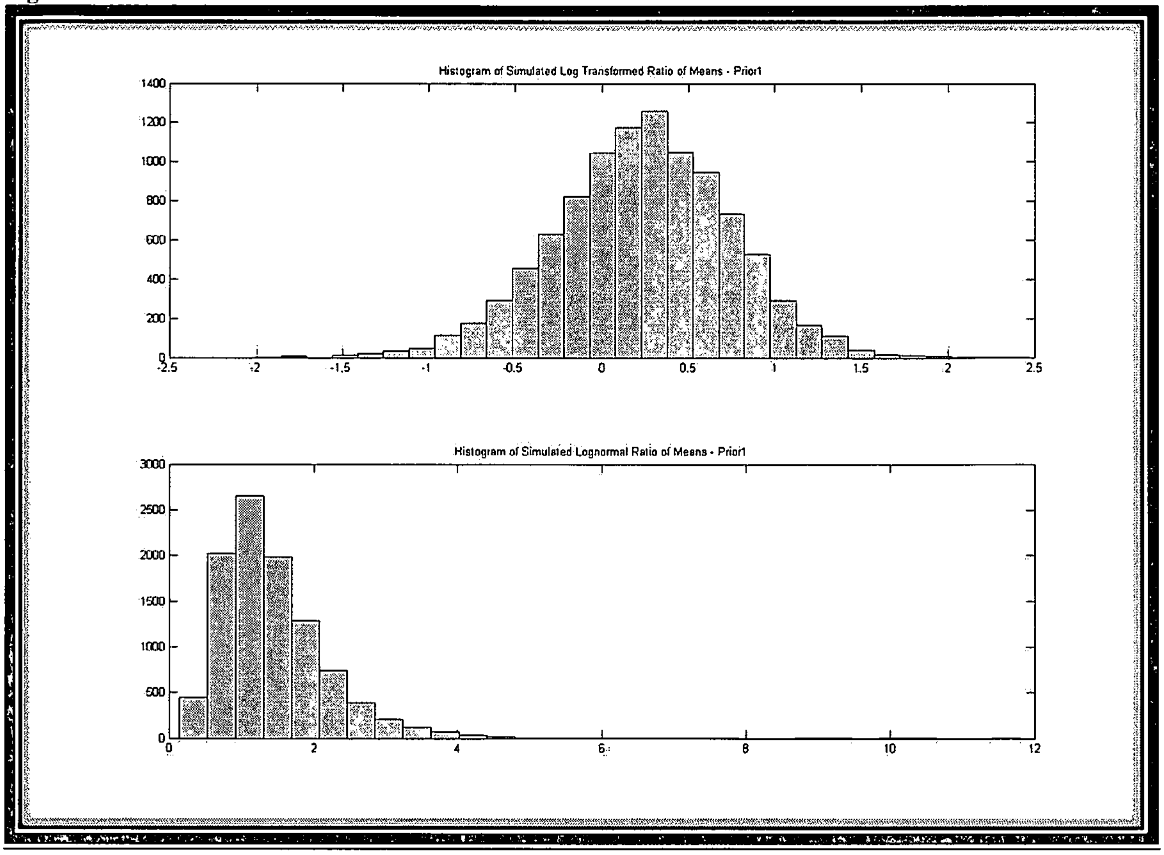


Figure 23: Results for Prior 2

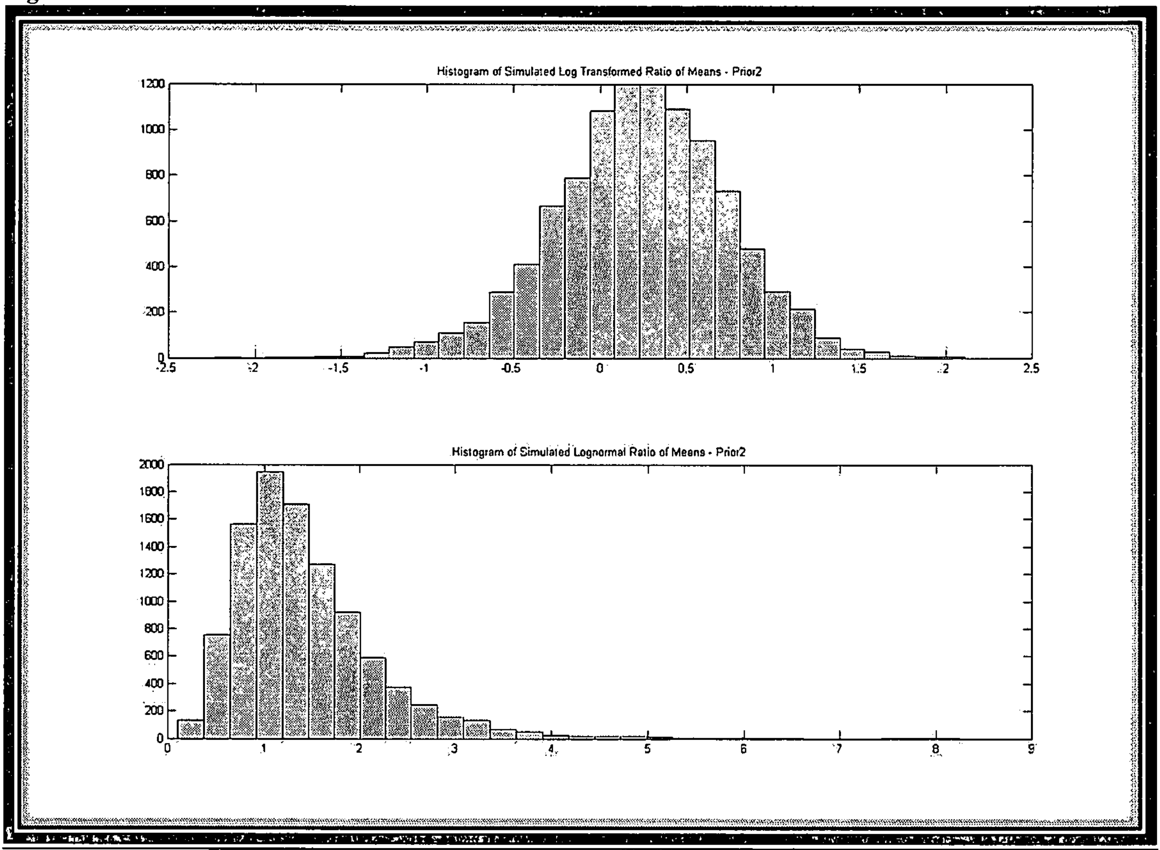


Figure 24: Results for Prior 3

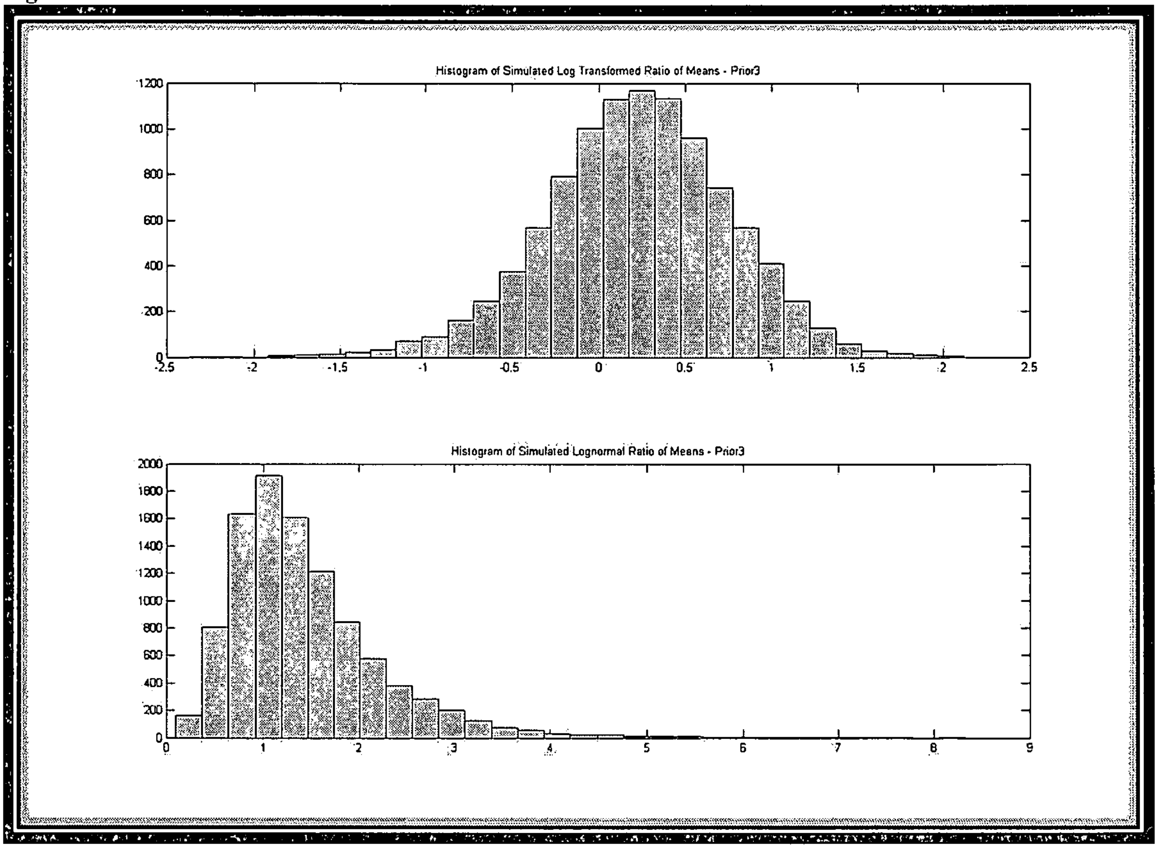


Figure 25: Results for the Reference Prior

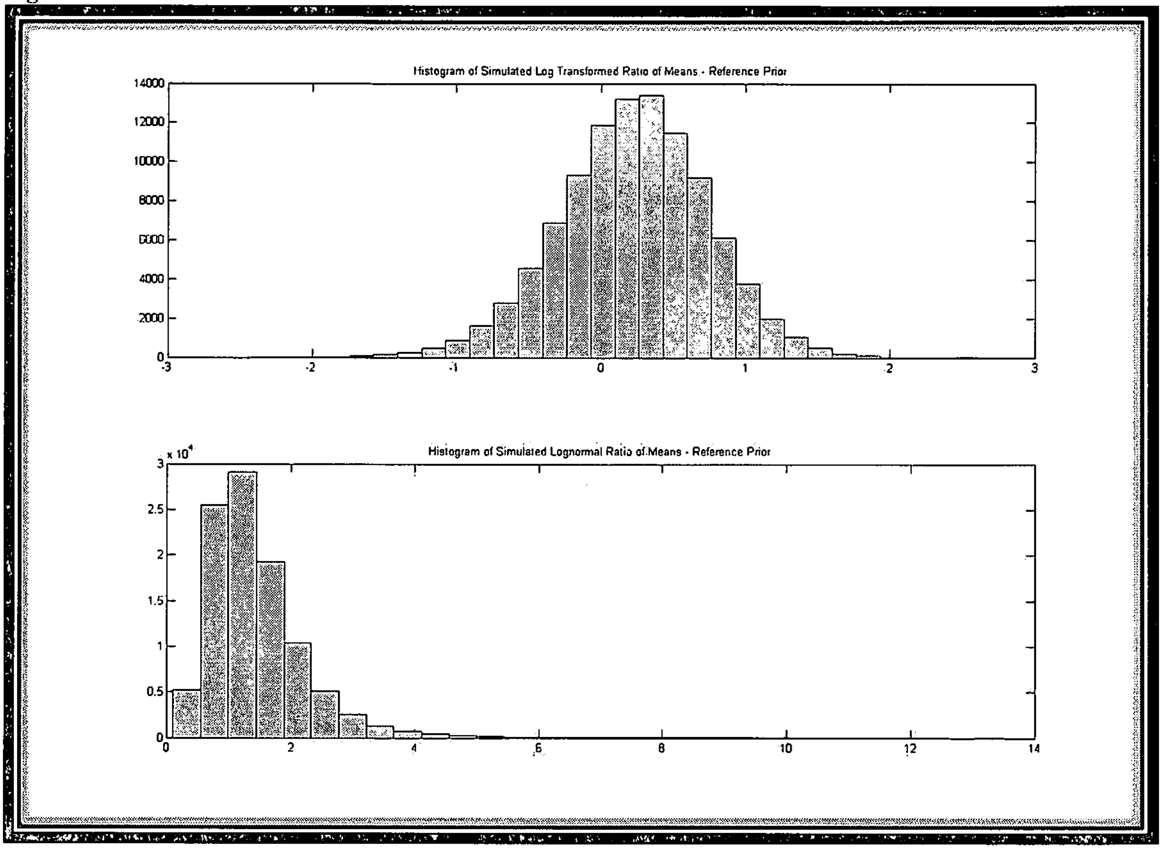


Figure 26: Results for the Probability-Matching Prior

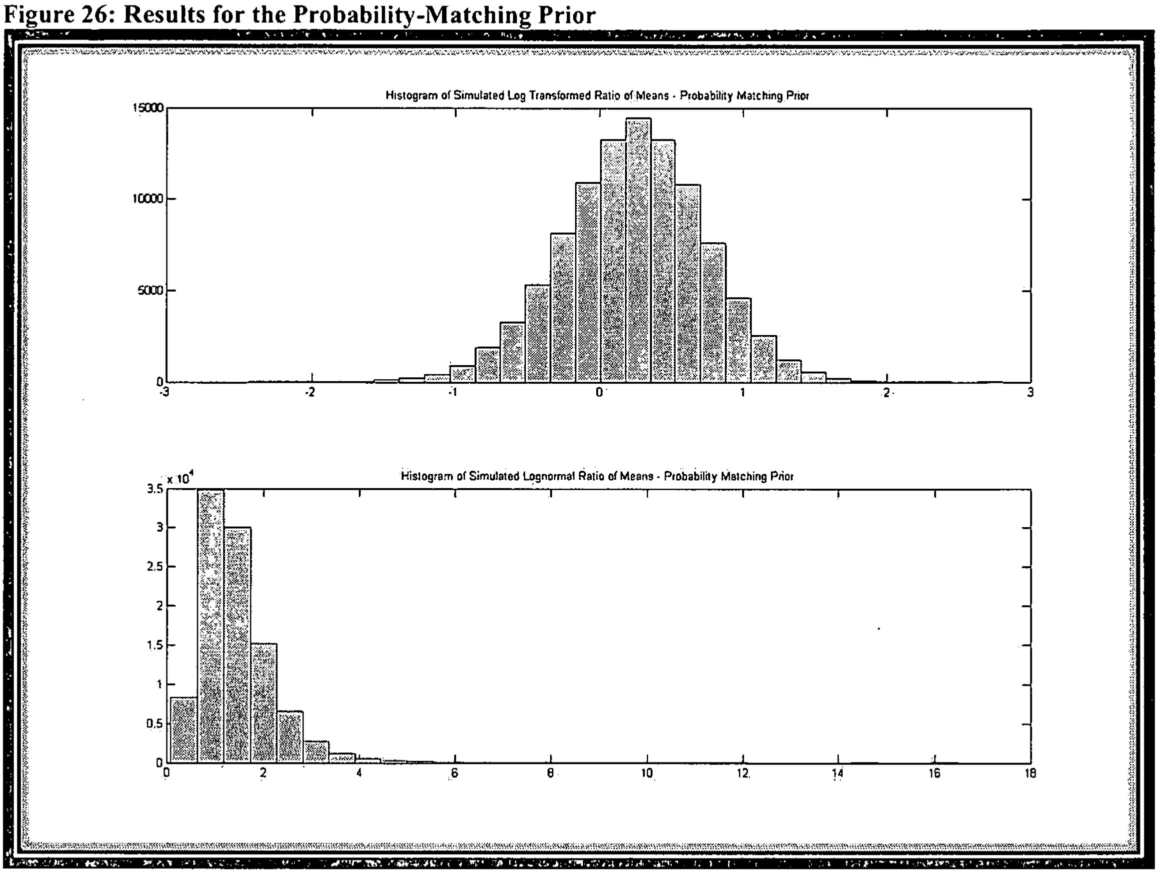
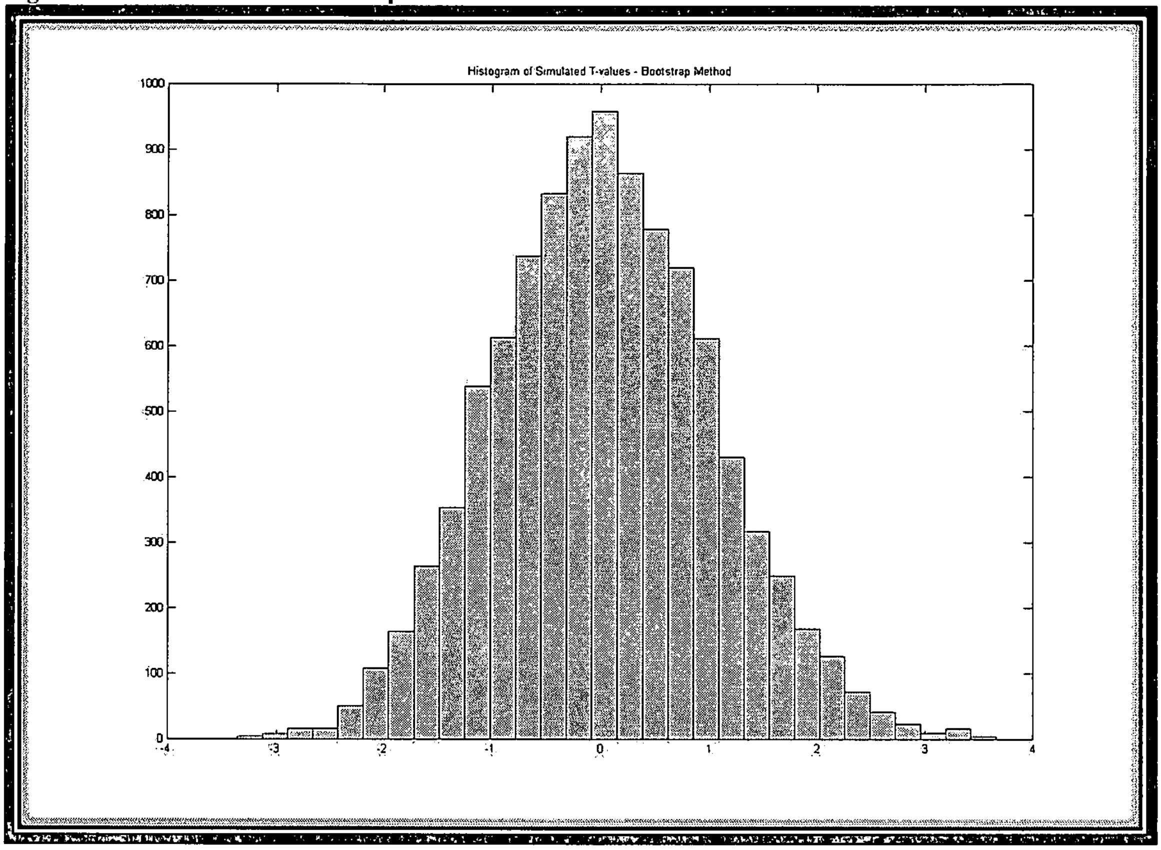


Figure 27: Results for the Bootstrap Simulation



4.7 Example: Shock Research Unit

The situation of lognormally distributed observations where there is also the presence of zero-value observations has also been examined by numerous other authors. A specific example was presented in Tian and Wu (2006) where these authors were analyzing a situation presented by Afifi and Azen (1979). A study was performed at the Shock Research Unit at the University of Southern California, Los Angeles, California. In a group of 113 critically ill patients a number of different physiological variables were measured at different time-points. Among these data there was a subset of patients that can be described as follows: 59 males of which 17 were dead at the end of the study and 42 were alive at the end of the study. For each of these two groups ("Dead" or "Alive") the urinary output (UO) at Baseline was measured. It was furthermore found that many zero-valued observations were included in each group (15 for the "Alive" group and 8 for the "Dead" group). The remaining non-zero observations could be described by a lognormal distribution and Tian and Wu (2006) calculated the *p*-values from Kolmogorov-Smirnov tests for normality of the logarithm of the non-zero observations as 0.454 and 0.574 for the "Alive" and "Dead" groups respectively. So it is evident that this practical setting mirrors the proposed theoretical setting that has been described in earlier sections of this Chapter. The following table represents the key features of the sample (all measures of location and dispersion are given for the logarithm of the non-zero observations for each group respectively):

Table 40: Summary of the Urinary Output Data

Group	Parameter	Value
Alive	Number of Patients	42
	Number of Zero Valued Observations	15
	Mean of Log-Transformed Data	3.43
	Variance of Log-Transformed Data	3.4586
Dead	Number of Patients	17
	Number of Zero Valued Observations	8
	Mean of Log-Transformed Data	2.42
	Variance of Log-Transformed Data	4.3292

The analysis objective as studied by Tian and Wu (2006) was somewhat different to the situation considered here. They considered confidence intervals for each group separately, whereas the objective of this Chapter is to examine the ratio of the means, which essentially tests whether the mean UO of the "Alive" group is the same as the mean for the "Dead" group. As in the previous example, the situation will be analysed from a Bayesian perspective using the methodology developed in previous sections.

The following table represents the results of the analysis for a comparison of the different methods (ML, Bootstrap, Independence Jeffreys Prior (Prior 1 in the table), the Jeffreys Rule Prior (Prior 2), the constant prior (Prior 3) , the Reference Prior (Prior 4) and the Probability-Matching Prior (Prior 5). The results are presented in the following table:

Table 41: Summary of Results for the Urinary Output Data

	Maximum Likelihood	Bootstrap	Prior 1	Prior 2	Prior 3	Reference Prior	Probability-Matching Prior
Lower Limit of Logged Data	-1.9592	-1.3160	-5.2459	-3.8440	-10.4773	-7.0417	-4.8931
Upper Limit of Logged Data	3.4969	4.7800	3.2195	3.1501	3.1654	3.1725	3.1984
Lower Limit	0.1410	0.2682	0.0053	0.0214	0.0000	0.0009	0.0075
Upper Limit	33.0136	119.0985	25.0156	23.3395	23.6972	23.8667	24.4935

The above table again indicates that in terms of interval width alone Prior 2 and the Probability-Matching prior are most efficient. In this particular setting the maximum likelihood method falls short and the Bootstrap method results in intervals widths that are too wide.

The following graphs also illustrate these results:

Figure 28: Results for Prior 1

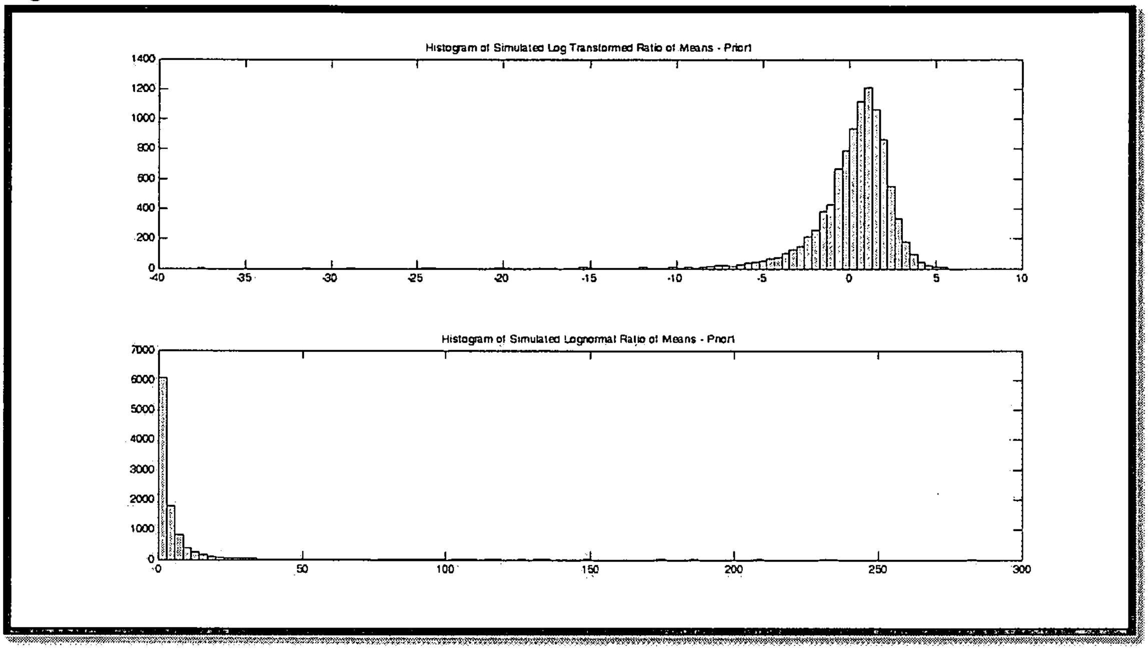


Figure 29: Results for Prior 2

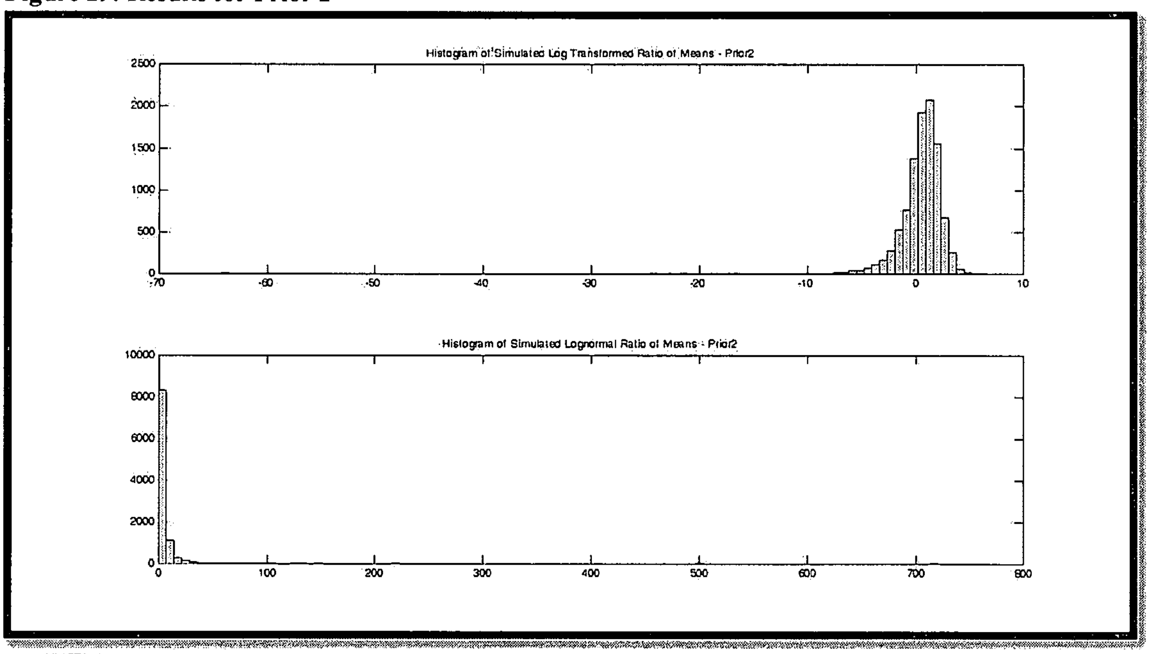


Figure 30: Results for Prior 3

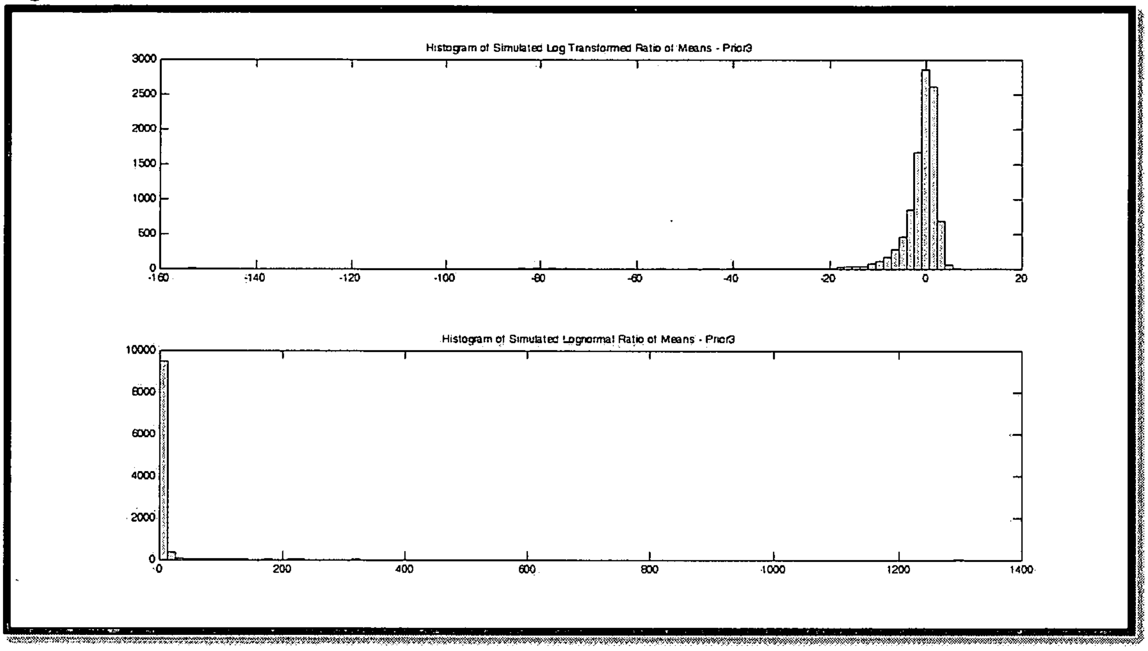


Figure 31: Results for the Reference Prior

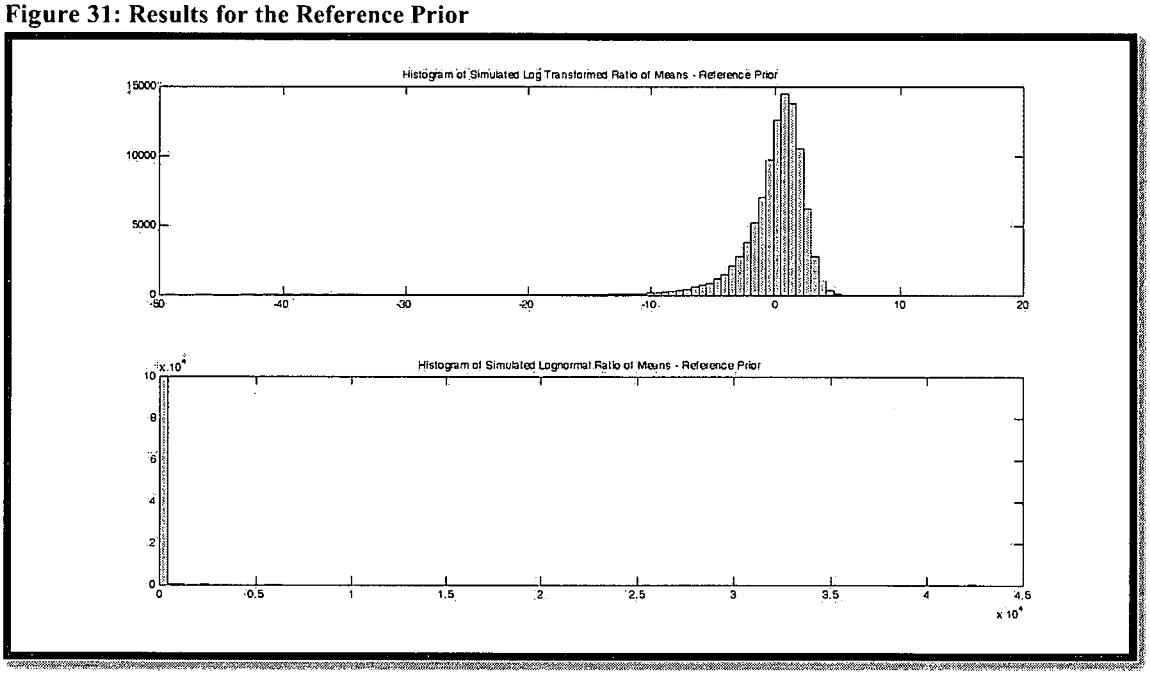


Figure 32: Results for the Probability-Matching Prior

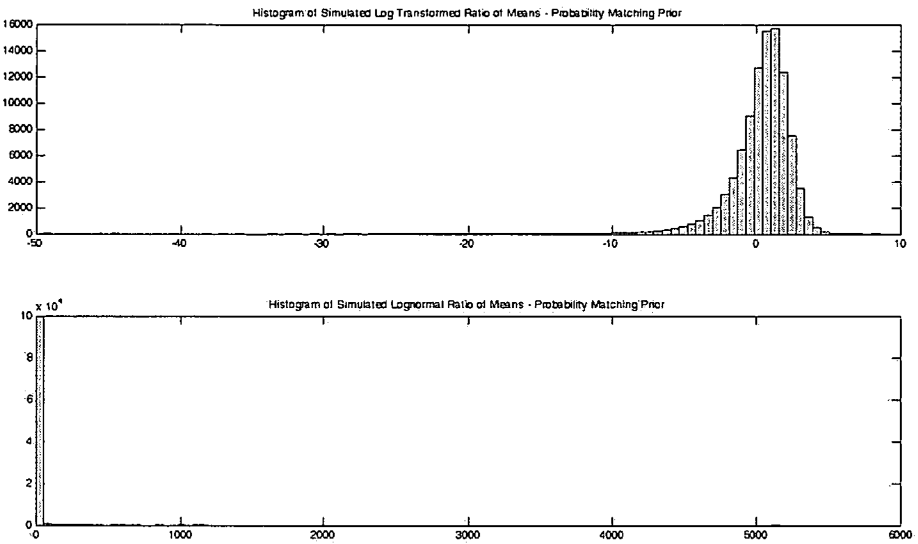
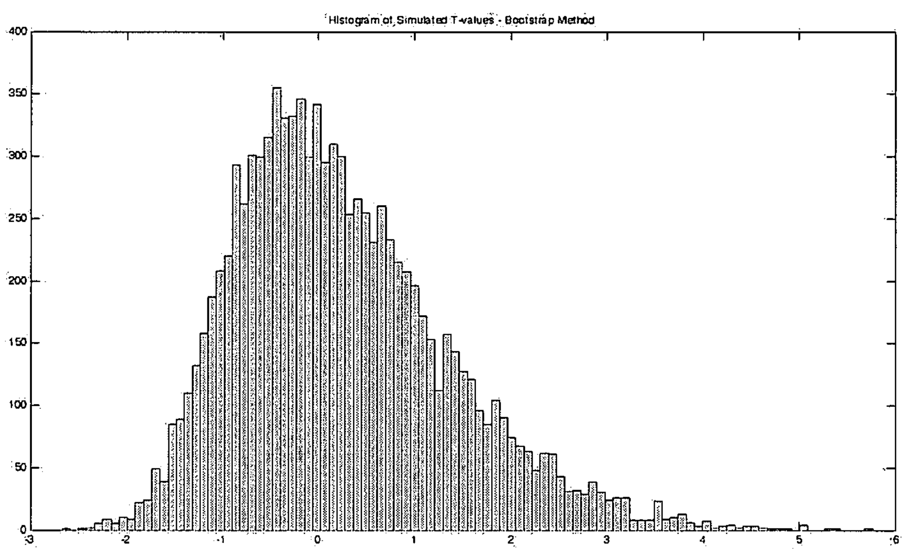


Figure 33: Results for the Bootstrap Simulation



4.8 Small Sample Application

As mentioned in the Introduction to this chapter, the situation as described in the previous chapters was applied to smaller samples. Although not specifically tested by Zhou and Tu (2000), when analysing data for the case where there are no zero observations, as described by Krishnamoorthy and Mathew (2003), smaller sample sizes (less than 10 observations per sample) were investigated by the authors. It was found that the coverage was still acceptable under these smaller sample constraints, but that the average interval length required to offer the required degree of credibility was much larger than for the situation of the sample sizes above 10 observations per sample.

Unfortunately, the situation was not specifically analysed by Zhou and Tu (2000), so we do not have a clear picture as to whether the average interval length is "excessive" or not. Also, although Krishnamoorthy and Mathew (2003) did analyse small samples the average interval length was not one of the factors that were considered in determining the efficacy of the particular method. Thus, in previous chapters we found that the Bayesian methods are perhaps not very well suited to the small sample situation, but without comparison to other methods this is by no means a definitive statement.

It was therefore decided to analyse the situation where there are potentially zero values included in the data from a Bayesian perspective. However, from previous sections it was found that the constant or uniform prior distributions (as well as the "ad hoc" prior proposed in Chapter 2) are definitely not efficient, particularly when analysing the average length in addition to the coverage. Thus, we suspect that the same would follow for the case of the inclusion of zero-valued observations. For that reason only the two Jeffreys prior distributions (Independence Jeffreys Prior and the Jeffreys Rule Prior) were used in the below analysis. The same techniques as described in earlier sections of this chapter were followed, the only exception being that smaller sample sizes were used.

The following table represents that sample sizes that were used in the simulation study:

Table 42: Small Sample Sizes Analysed by Monte Carlo Simulation Techniques

n_1	n_2
6	6
7	7
8	8
6	8

These sample sizes were analysed for the same designs as specified in Table 36.

4.8.1 Simulation Study: Results for Jeffreys Prior Distributions

As mentioned earlier, the same Monte Carlo simulation technique, as mentioned in Section 4.2.2 was applied to these smaller sample sizes. The same designs and parameter settings were applied.

The following table represents the results of this simulation study.

Table 43: Results for Simulation Study – Small Sample

Design (n_1 & n_2)		Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1								
6	6	Prior 1	0.9740	-0.0240	10.9766	0.0080	0.0180	0.3846
7	7		0.9610	-0.0110	8.4701	0.0140	0.0250	0.2821
8	8		0.9470	0.0030	7.1725	0.0240	0.0290	0.0943
6	8		0.9620	-0.0120	10.2292	0.0200	0.0180	0.0526
6	6	Prior 2	0.9480	0.0020	7.5417	0.0090	0.0430	0.6538
7	7		0.9400	0.0100	6.3774	0.0160	0.0440	0.4667
8	8		0.9390	0.0110	5.6805	0.0210	0.0400	0.3115
6	8		0.9450	0.0050	7.1669	0.0160	0.0390	0.4182
Design 2								
6	6	Prior 1	0.9550	-0.0050	20.3295	0.0240	0.0210	0.0667
7	7		0.9630	-0.0130	15.7294	0.0160	0.0210	0.1351
8	8		0.9610	-0.0110	13.1721	0.0240	0.0150	0.2308
6	8		0.9630	-0.0130	16.7904	0.0200	0.0170	0.0811
6	6	Prior 2	0.9320	0.0180	13.4673	0.0350	0.0330	0.0294
7	7		0.9440	0.0060	11.4623	0.0240	0.0320	0.1429
8	8		0.9490	0.0010	10.1460	0.0350	0.0160	0.3725
6	8		0.9390	0.0110	11.9156	0.0290	0.0320	0.0492
Design 3								
6	6	Prior 1	0.9690	-0.0190	15.1784	0.0140	0.0170	0.0968
7	7		0.9630	-0.0130	10.9404	0.0160	0.0210	0.1351
8	8		0.9540	-0.0040	8.8139	0.0230	0.0230	0.0000
6	8		0.9670	-0.0170	13.6443	0.0120	0.0210	0.2727
6	6	Prior 2	0.9410	0.0090	8.8024	0.0150	0.0440	0.4915
7	7		0.9480	0.0020	7.3555	0.0190	0.0330	0.2692
8	8		0.9390	0.0110	6.4876	0.0250	0.0360	0.1803
6	8		0.9380	0.0120	8.1097	0.0140	0.0480	0.5484
Design 4								
6	6	Prior 1	0.9680	-0.0180	7.7355	0.0170	0.0150	0.0625
7	7		0.9600	-0.0100	6.0932	0.0160	0.0240	0.2000
8	8		0.9500	0.0000	5.1013	0.0240	0.0260	0.0400
6	8		0.9590	-0.0090	7.0047	0.0180	0.0230	0.1220
6	6	Prior 2	0.9480	0.0020	5.2435	0.0180	0.0340	0.3077
7	7		0.9490	0.0010	4.5593	0.0170	0.0340	0.3333
8	8		0.9330	0.0170	4.0483	0.0240	0.0430	0.2836
6	8		0.9340	0.0160	4.9760	0.0190	0.0470	0.4242
Design 5								
6	6	Prior 1	0.9720	-0.0220	18.3263	0.0150	0.0130	0.0714
7	7		0.9650	-0.0150	9.5105	0.0170	0.0180	0.0286
8	8		0.9570	-0.0070	9.0032	0.0170	0.0260	0.2093
6	8		0.9740	-0.0240	10.3328	0.0140	0.0120	0.0769
6	6	Prior 2	0.9570	-0.0070	5.9971	0.0120	0.0310	0.4419
7	7		0.9400	0.0100	5.2233	0.0190	0.0410	0.3667
8	8		0.9420	0.0080	4.6606	0.0170	0.0410	0.4138
6	8		0.9480	0.0020	6.0476	0.0130	0.0390	0.5000

4.8.2 Discussion of Results– Jeffreys Prior Distributions

The results of the simulation study highlight certain interesting differences between the two types of Jeffreys prior distributions, the mostly obvious are the following:

1. The independence Jeffreys prior seems to over cover and the Jeffreys Rule prior seems to undercover. From the specification of the situation and of the analysis and due to the methods implemented, we expect each interval above to contain the true population parameter with 0.95 probability (Bayesian credibility intervals). However, the probability of covering the true parameter for the independence Jeffreys prior interval exceeds 0.95, while being below 0.95 for the Jeffreys Rule Prior.
2. The following may be related to the previous point to some degree: the average interval length for the independence Jeffreys prior is higher than for the Jeffreys Rule prior.

Since these are the two characteristics of primary importance it is difficult to rank these two choices of prior distributions based solely on these two criterion, as they tend to "cancel out."

The following table is a summary of the situation described above:

Table 44: Summary Results – Small Sample

Design	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1	Prior 1	0.9610	-0.0110	9.2121	0.0165	0.0225	0.2034
	Prior 2	0.9430	0.0070	6.6916	0.0155	0.0415	0.4625
Design 2	Prior 1	0.9605	-0.0105	16.5054	0.0210	0.0185	0.1284
	Prior 2	0.9410	0.0090	11.7478	0.0308	0.0283	0.1485
Design 3	Prior 1	0.9633	-0.0133	12.1442	0.0163	0.0205	0.1262
	Prior 2	0.9415	0.0085	7.6888	0.0183	0.0403	0.3724
Design 4	Prior 1	0.9593	-0.0093	6.4837	0.0188	0.0220	0.1061
	Prior 2	0.9410	0.0090	4.7068	0.0195	0.0395	0.3372
Design 5	Prior 1	0.9670	-0.0170	11.7932	0.0158	0.0173	0.0966
	Prior 2	0.9468	0.0033	5.4821	0.0153	0.0380	0.4306
Overall	Prior 1	0.9622	-0.0122	11.2277	0.0177	0.0202	0.1321
	Prior 2	0.9427	0.0074	7.2634	0.0199	0.0375	0.3502

We can see from the above the situation that was described earlier. Overall, the independence Jeffreys prior resulted in credibility intervals that had a 0.9622 probability of containing the true parameter, whereas the same probability for the Jeffreys rule prior was 0.9407. The difference is particularly evident in Design 5. This design was characterised by small (yet distinctly unequal) population variances and non-negative (yet differing probabilities) of obtaining zero-valued observations in the sample.

Regarding the average interval length, Design 4 resulted in the least disparity, while Design 2 resulted in the largest difference. Design 4 is similar to Design 4 (described above) whereas Design 2 is characterised by large(r) population variances and matching (zero) probabilities of obtaining a zero-valued observation in the sample.

Thus, the previous situation describes some of the (potential) areas where the Bayesian credibility interval method may not be particularly effective for small samples [however, as mentioned before, the comparison with methods proposed by Krishnamoorthy and Mathew (2003) and Zhou and Tu (2000) was not possible to provide as reference].

In conclusion, both prior distributions have their positive and negative characteristics. The analysis of the Relative Bias may indicate then that perhaps the independence Jeffreys prior is better suited to the analysis of small samples than the Jeffreys Rule prior.

4.8.3 Other Prior Distributions

In addition to the Jeffreys prior distributions mentioned earlier, it was decided to analyse the case of small sample sizes using two other prior distributions. These were the Probability-Matching Prior distribution and the Reference Prior distribution. These have been derived and discussed in previous chapters and thus derivations of the priors are not repeated here.

Probability-matching Prior Distribution

As mentioned earlier, the analysis of small sample data was extended so as to include Probability-Matching and Reference prior distributions. Define:

$$\gamma = e^{\mu + \frac{1}{2}\sigma^2},$$

given the situation described in earlier parts of this chapter. Then, as in Chapter 1, the following prior distribution was found:

$$p_p\left(\mu, \sigma^2\right) \propto \frac{1}{\sigma^2}\sqrt{1 + \frac{2}{\sigma^2}} \tag{4.15}$$

This resulted in the following posterior distribution:

$$P(\boldsymbol{\theta} \mid data) = \prod_{j=1}^{2} \left\{ \begin{array}{l} \left(\frac{1}{\sigma_j^2} \sqrt{1 + \frac{2}{\sigma_j^2}} \right) \frac{1}{B(n_{j0} + 0.5; n_{j1} + 0.5)} \delta_j^{n_{j0} - \frac{1}{2}} (1 - \delta_j)^{n_{j1} - \frac{1}{2}} \\ \times \left(\frac{2\pi\sigma_j^2}{n_{j1}} \right)^{-\frac{1}{2}} \exp\left[-\frac{n_{j1}}{2\sigma_j^2} (\mu_j - \hat{\mu}_j)^2 \right] \left(\frac{1}{\sigma_j^2} \right)^{\frac{1}{2}\nu_j} \exp\left(-\frac{\nu_j s_j^2}{2\sigma_j^2} \right) \end{array} \right\} \tag{4.16}$$

with the parameters as previously defined.

From (4.16) it follows that the posterior distribution of δ_j is a Beta distribution (specifically $B\left(n_{j0} + \frac{1}{2}; n_{j1} + \frac{1}{2}\right)$) and δ_j is independently distributed of μ_j and σ_j^2, where the conditional posterior distribution of μ_j is normal:

$$\mu_j \mid \sigma^2, data \sim N\left(\hat{\mu}_j, \frac{\sigma_j^2}{n_{j1}} \right) \tag{4.17}$$

and for σ_j^2, the posterior density function is as follows:

$$P(\sigma_j^2 \mid data) = \sqrt{1 + \frac{2}{\sigma_j^2}} \left(\frac{1}{\sigma_j^2} \right)^{\frac{1}{2}(\nu_{j1} + 2)} \exp\left[-\frac{\nu_{j1} \hat{\sigma}_j^2}{2\sigma_j^2} \right]. \tag{4.18}$$

Since 4.18 is a non-standard distribution the simulation procedure described in Section 4.2.2 was adapted so as to use the rejection method to simulate σ_j^2 observations from the posterior distribution.

The following table gives the simulation results for the small sample situation described in previous sections (using the same designs).

Table 45: Results for Simulation Study – Small Sample using the Probability-Matching Prior

Design (n_1 & n_2)		Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1								
6	6	PMP	0.8860	0.0640	8.3088	0.0730	0.0410	0.2807
7	7		0.8970	0.0530	6.8487	0.0560	0.0470	0.0874
8	8		0.8920	0.0580	6.0430	0.0700	0.0380	0.2963
6	8		0.8700	0.0800	7.7802	0.0830	0.0470	0.2769
8	6		0.9080	0.0420	6.5609	0.0630	0.0290	0.3696
Design 2								
6	6	PMP	0.9180	0.0320	13.6107	0.0370	0.0450	0.0976
7	7		0.9150	0.0350	11.4752	0.0350	0.0500	0.1765
8	8		0.9040	0.0460	9.9376	0.0520	0.0440	0.0833
6	8		0.9140	0.0360	12.0036	0.0460	0.0400	0.0698
8	6		0.9210	0.0290	11.9525	0.0410	0.0380	0.0380
Design 3								
6	6	PMP	0.8740	0.0760	9.9885	0.0830	0.0430	0.3175
7	7		0.8790	0.0710	8.0840	0.0790	0.0420	0.3058
8	8		0.8980	0.0520	6.8948	0.0630	0.0390	0.2353
6	8		0.8890	0.0610	9.0930	0.0680	0.0430	0.2252
8	6		0.9030	0.0470	7.7391	0.0550	0.0420	0.1340
Design 4								
6	6	PMP	0.8680	0.0820	5.9284	0.0820	0.0500	0.2424
7	7		0.8890	0.0610	4.8517	0.0770	0.0340	0.3874
8	8		0.8830	0.0670	4.1719	0.0740	0.0430	0.2650
6	8		0.8630	0.0870	5.6938	0.1050	0.0320	0.5328
8	6		0.8840	0.0660	4.6066	0.0720	0.0440	0.2414
Design 5								
6	6	PMP	0.8970	0.0530	7.7622	0.0630	0.0400	0.2233
7	7		0.8800	0.0700	6.0044	0.0810	0.0390	0.3500
8	8		0.8770	0.0730	4.9045	0.0860	0.0370	0.3984
6	8		0.8740	0.0760	6.8774	0.0940	0.0320	0.4921
8	6		0.9110	0.0390	5.8825	0.0610	0.0280	0.3708

Reference Prior Distribution

In addition to the Probability-Matching prior, the Reference prior distribution was also determined and analysed in a similar way for small samples. The following derivations were made:

$$p_R\left(\mu,\sigma^2\right) \propto \frac{1}{\sigma}\sqrt{1+\frac{2}{\sigma^2}} \tag{4.19}$$

This Reference prior distribution resulted in the following posterior distribution:

$$P(\boldsymbol{\theta} \mid data) = \prod_{j=1}^{2} \left\{ \begin{array}{l} \left(\dfrac{1}{\sigma_j} \sqrt{1 + \dfrac{2}{\sigma_j^2}} \right) \dfrac{1}{B(n_{j0} + 0.5; n_{j1} + 0.5)} \delta_j^{n_{j0} - \frac{1}{2}} (1 - \delta_j)^{n_{j1} - \frac{1}{2}} \\ \times \left(\dfrac{2\pi\sigma_j^2}{n_{j1}} \right)^{-\frac{1}{2}} \exp\left[-\dfrac{n_{j1}}{2\sigma_j^2} (\mu_j - \hat{\mu}_j)^2 \right] \left(\dfrac{1}{\sigma_j^2} \right)^{\frac{1}{2}\nu_j} \exp\left(-\dfrac{\nu_j s_j^2}{2\sigma_j^2} \right) \end{array} \right\}$$

(4.20)

and for σ_j^2, the posterior density function is as follows:

$$P(\sigma_j^2 \mid data) = \sqrt{1 + \frac{2}{\sigma_j^2}} \left(\frac{1}{\sigma_j^2} \right)^{\frac{1}{2}(\nu_{j1} + 1)} \exp\left[-\frac{\nu_{j1} \hat{\sigma}_j^2}{2\sigma_j^2} \right]. \tag{4.21}$$

Thus, the simulation procedure for the case of small sample was much the same, and the following results were obtained:

Table 46: Results for Simulation Study – Small Sample using the Reference Prior

Design (n_1 & n_2)		Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1								
6	6	Reference	0.9430	0.0070	11.5800	0.0450	0.0120	0.5789
7	7		0.9210	0.0290	9.0672	0.0580	0.0210	0.4684
8	8		0.9100	0.0400	7.3777	0.0630	0.0270	0.4000
6	8		0.9020	0.0480	10.2335	0.0820	0.0160	0.6735
8	6		0.9390	0.0110	8.6540	0.0420	0.0190	0.3770
Design 2								
6	6	Reference	0.9520	-0.0020	17.0831	0.0240	0.0240	0.0000
7	7		0.9490	0.0010	13.9435	0.0160	0.0350	0.3725
8	8		0.9440	0.0060	11.7032	0.0290	0.0270	0.0357
6	8		0.9520	-0.0020	14.6619	0.0290	0.0190	0.2083
8	6		0.9450	0.0050	14.5962	0.0230	0.0320	0.1636
Design 3								
6	6	Reference	0.9330	0.0170	14.5905	0.0470	0.0200	0.4030
7	7		0.9290	0.0210	11.2982	0.0450	0.0260	0.2676
8	8		0.9250	0.0250	9.0506	0.0530	0.0220	0.4133
6	8		0.9140	0.0360	12.6436	0.0630	0.0230	0.4651
8	6		0.9490	0.0010	11.0166	0.0300	0.0210	0.1765
Design 4								
6	6	Reference	0.9300	0.0200	9.1304	0.0490	0.0210	0.4000
7	7		0.8950	0.0550	6.9607	0.0840	0.0210	0.6000
8	8		0.9090	0.0410	5.3835	0.0800	0.0110	0.7582
6	8		0.9050	0.0450	8.0477	0.0830	0.0120	0.7474
8	6		0.9120	0.0380	6.7466	0.0510	0.0370	0.1591
Design 5								
6	6	Reference	0.9480	0.0020	12.8853	0.0410	0.0110	0.5769
7	7		0.9230	0.0270	9.1517	0.0620	0.0150	0.6104
8	8		0.9170	0.0330	7.0684	0.0640	0.0190	0.5422
6	8		0.9110	0.0390	10.5350	0.0750	0.0140	0.6854
8	6		0.9390	0.0110	9.3918	0.0390	0.0220	0.2787

The following is a summary of both the Probability-Matching and Reference priors:

Table 47: Results for Simulation Study – Other Prior Distributions

Design	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
Design 1							
	PMP	0.8906	0.0594	7.1083	0.0690	0.0404	0.2622
	Reference	0.9230	0.0270	9.3825	0.0580	0.0190	0.4996
Design 2							
	PMP	0.9144	0.0356	11.7959	0.0422	0.0434	0.0930
	Reference	0.9484	0.0016	14.3976	0.0242	0.0274	0.1560
Design 3							
	PMP	0.8886	0.0614	8.3599	0.0696	0.0418	0.2436
	Reference	0.9300	0.0200	11.7199	0.0476	0.0224	0.3451
Design 4							
	PMP	0.8774	0.0726	5.0505	0.0820	0.0406	0.3338
	Reference	0.9102	0.0398	7.2538	0.0694	0.0204	0.5329
Design 5							
	PMP	0.8878	0.0622	6.2862	0.0770	0.0352	0.3669
	Reference	0.9276	0.0224	9.8064	0.0562	0.0162	0.5387
Overall							
	PMP	0.8918	0.0582	7.7202	0.0680	0.0403	0.2599
	Reference	0.9278	0.0222	10.5120	0.0511	0.0211	0.4145

<u>Results and Conclusions</u>

Tables 46 to 48 represent 95% credibility intervals. The coverage is not adequate as is evident from the results. Given the aforementioned results it appears evident that neither the Probability-Matching or the Reference prior distributions offer significant benefit over the Jeffreys prior distributions.

The Probability-Matching prior was particular not suited to the small sample situation offering an average coverage probability of only 0.8918. One possible explanation for the performance of this prior distribution is that it was derived for one-sided Bayesian confidence intervals and may not be particular well-suited to two-sided intervals. The Reference prior distribution performed only slightly better. What is surprising though is that the Reference prior performed best in Design 2. Design 2 is characterised by large equal variances between the population. This could offer some insight into the potential usefulness of this prior distribution in applications. However, the necessity for equal variances is a significant constraint.

No other advantages in terms of the average interval length was observed. The Jeffreys prior distributions offer similar average interval lengths to those given in Table 48, but without the desired benefit of more accurate coverage probability.

In conclusion, given the setting, the Jeffreys prior distributions appear more applicable to the setting, particularly under small sample considerations.

Appendix to Chapter 4

Derivation of the Probability-Matching Prior for $t(\theta) = \frac{M_1}{M_2} = \frac{(1-\delta_1)\exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)}{(1-\delta_2)\exp\left(\mu_2 + \frac{1}{2}\sigma_2^2\right)}$

In this section the derivation of the Probability-Matching prior for the situation described by Zhou and Tu (2000) is presented. Summarizing the previous sections, we are interested in interval estimation for the ratio of means from two different lognormally distributed populations, with the added complication that zero values could possibly be included. The Probability-Matching prior was derived for the above-mentioned quantity in contrast to deriving the prior for simply the mean of the lognormal distribution (see Chapter 1), but simulation results were not included since these did not improve on the performance of the simple priors. For completeness the derivation is stated here.

As previously, we are interested in:

$$M_j = (1-\delta_j)\exp(\mu_j + \frac{1}{2}\sigma_j^2)$$

and in particular, in a credibility interval for the following relationship:

$$\frac{M_1}{M_2} = \frac{(1-\delta_1)\exp(\mu_1 + \frac{1}{2}\sigma_1^2)}{(1-\delta_2)\exp(\mu_2 + \frac{1}{2}\sigma_2^2)}.$$

As previously, let $y_{ij} = \ln W_{ij}$ and $\boldsymbol{\theta} = [\delta_1 \quad \mu_1 \quad \sigma_1^2\, \delta_2 \quad \mu_2 \quad \sigma_2^2]'$. Then the likelihood function is given by:

$$L(\theta \mid data) \propto \prod_{j=1}^{2}\{\delta_j^{n_{j0}}(1-\delta_j)^{n_{j1}}\prod_{i=1}^{n_{j1}}(\frac{1}{\sigma_j^2})^{\frac{1}{2}}\exp[-\frac{(y_{ij}-\mu_j)^2}{2\sigma_j^2}]\}$$

Given the previous specification of the likelihood, the Fisher Information Matrix is required. Although the derivation of this matrix is not repeated here in detail the information matrix is defined as the variance of the score. Therefore, the matrix can be written as the second moment of the score, which is the derivative of the log of the likelihood function with respect to $\boldsymbol{\theta}$. Equivalently, the Fisher Information Matrix can be defined as:

$$I(\boldsymbol{\theta}) = -E\left\{\left[\frac{\partial^2}{\partial^2\theta}\ln L\left(\theta \mid data\right)\right]\right\}$$

$$\therefore I(\boldsymbol{\theta}) = diag\left[\frac{n_1}{\delta_1(1-\delta_1)} \quad \frac{n_1(1-\delta_1)}{\sigma_1^2} \quad \frac{n_1(1-\delta_1)}{2\sigma_1^4} \quad \frac{n_2}{\delta_2(1-\delta_2)} \quad \frac{n_2(1-\delta_2)}{\sigma_2^2} \quad \frac{n_2(1-\delta_2)}{2\sigma_2^4}\right]$$

The inverse of the Fisher Information Matrix can then be calculated as:

$$I^{-1}(\boldsymbol{\theta}) = diag[\frac{\delta_1(1-\delta_1)}{n_1} \quad \frac{\sigma_1^2}{n_1(1-\delta_1)} \quad \frac{2\sigma_1^4}{n_1(1-\delta_1)} \quad \frac{\delta_2(1-\delta_2)}{n_2} \quad \frac{\sigma_2^2}{n_2(1-\delta_2)} \quad \frac{2\sigma_2^4}{n_2(1-\delta_2)}]$$

Now, following from the definitions of the quantity of interest defined above

$$t(\boldsymbol{\theta}) = \frac{M_1}{M_2} = \frac{(1-\delta_1)\exp(\mu_1 + \frac{1}{2}\sigma_1^2)}{(1-\delta_2)\exp(\mu_2 + \frac{1}{2}\sigma_2^2)}$$

$$= \frac{(1-\delta_1)}{(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$

The Probability-Matching prior is derived from the inverse of the Fisher information matrix. Now

$$I^{-1}(\boldsymbol{\theta}) = I^{-1}(\mu,\sigma^2) = diag\left[\frac{n_1}{\delta_1(1-\delta_1)} \quad \frac{n_1(1-\delta_1)}{\sigma_1^2} \quad \frac{n_1(1-\delta_1)}{2\sigma_1^4} \quad \frac{n_2}{\delta_2(1-\delta_2)} \quad \frac{n_2(1-\delta_2)}{\sigma_2^2} \quad \frac{n_2(1-\delta_2)}{2\sigma_2^4}\right]$$

Now, $t(\boldsymbol{\theta})$ is defined as above from which it follows that

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \delta_1} = \frac{-1}{(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$
$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu_1} = \frac{(1-\delta_1)}{(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$
$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_1^2} = \frac{(1-\delta_1)}{2(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$
$$\frac{\partial t(\boldsymbol{\theta})}{\partial \delta_2} = \frac{(1-\delta_1)}{(1-\delta_2)^2}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$
$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu_2} = \frac{-(1-\delta_1)}{(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$
$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_2^2} = \frac{-(1-\delta_1)}{2(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$

Also, define

$$\nabla_t'(\boldsymbol{\theta}) = \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \delta_1} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \mu_1} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_1^2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \delta_2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \mu_2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_2^2}\right].$$

which gives

$$\nabla_t'(\boldsymbol{\theta}) = \frac{(1-\delta_1)}{(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\} \times [\frac{-1}{(1-\delta_1)} \quad 1 \quad \frac{1}{2} \quad \frac{1}{(1-\delta_2)} \quad -1 \quad -\frac{1}{2}]$$

Furthermore,

$$\nabla_t'(\boldsymbol{\theta}) I^{-1}(\boldsymbol{\theta}) = \frac{(1-\delta_1)}{(1-\delta_2)}\exp\{\mu_1 - \mu_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\}$$

$$\times [\frac{-\delta_1}{n_1} \quad \frac{\sigma_1^2}{n_1(1-\delta_1)} \quad \frac{\sigma_1^4}{n_1(1-\delta_1)} \quad \frac{\delta_2}{n_2} \quad \frac{-\sigma_2^2}{n_2(1-\delta_2)} \quad \frac{-\sigma_2^4}{n_2(1-\delta_2)}]$$

and

$$\sqrt{\nabla_t'(\theta)I^{-1}(\theta)\nabla_t(\theta)} = \frac{(1-\delta_1)}{(1-\delta_2)}\exp\{\mu_1-\mu_2+\frac{1}{2}(\sigma_1^2-\sigma_2^2)\}\times$$

$$\sqrt{\frac{\delta_1}{n_1(1-\delta_1)}+\frac{\sigma_1^2}{n_1(1-\delta_1)}+\frac{\sigma_1^4}{2n_1(1-\delta_1)}+\frac{\delta_2}{n_2(1-\delta_2)}+\frac{\sigma_2^2}{n_2(1-\delta_2)}+\frac{\sigma_2^4}{2n_2(1-\delta_2)}}$$

Now, agreement between the posterior probability and the frequentist probability holds if and only if the differential equation

$$\sum_{\alpha=1}^{m}\frac{\partial}{\partial\theta_\alpha}\{\gamma_\alpha(\theta)p(\theta)\}=0$$

is satisfied, where $p(\theta)$ is the Probability-Matching prior distribution for θ, the vector of unknown parameters.

By definition,

$$\gamma(\theta)=\frac{I^{-1}(\theta)\nabla_t(\theta)}{\sqrt{\nabla_t'(\theta)I^{-1}(\theta)\nabla_t(\theta)}}=\left[\gamma_1(\theta),\ldots,\gamma_m(\theta)\right]'.$$

It is clear that $\gamma'(\theta)I(\theta)\gamma(\theta)=1$ for all θ where $I^{-1}(\theta)$ is the inverse of $I(\theta)$. $I(\theta)$ is the Fisher information matrix of θ and $t(\theta)$ is the parameter of interest.

Now,

$$\gamma(\theta)=\frac{I^{-1}(\theta)\nabla_t(\theta)}{\sqrt{\nabla_t'(\theta)I^{-1}(\theta)\nabla_t(\theta)}}=\begin{bmatrix}\gamma_1(\theta)\\ \gamma_2(\theta)\\ \gamma_3(\theta)\\ \gamma_4(\theta)\\ \gamma_5(\theta)\\ \gamma_6(\theta)\end{bmatrix}$$

$$
= \frac{1}{\sqrt{\frac{\delta_1}{n_1(1-\delta_1)} + \frac{\sigma_1^2}{n_1(1-\delta_1)} + \frac{\sigma_1^4}{2n_1(1-\delta_1)} + \frac{\delta_2}{n_2(1-\delta_2)} + \frac{\sigma_2^2}{n_2(1-\delta_2)} + \frac{\sigma_2^4}{2n_2(1-\delta_2)}}} \begin{bmatrix} \frac{-\delta_1}{n_1} \\ \frac{\sigma_1^2}{n_1(1-\delta_1)} \\ \frac{\sigma_1^4}{n_1(1-\delta_1)} \\ \frac{\delta_2}{n_2} \\ \frac{-\sigma_2^2}{n_2(1-\delta_2)} \\ \frac{-\sigma_2^4}{n_2(1-\delta_2)} \end{bmatrix}
$$

This differential equation mentioned earlier will be satisfied if:

$$
p(\theta) = p(\delta_1 \quad \mu_1 \quad \sigma_1^2\, \delta_2 \quad \mu_2 \quad \sigma_2^2)
$$

$$
\propto \frac{\sqrt{\frac{\delta_1}{n_1(1-\delta_1)} + \frac{\sigma_1^2}{n_1(1-\delta_1)} + \frac{\sigma_1^4}{2n_1(1-\delta_1)} + \frac{\delta_2}{n_2(1-\delta_2)} + \frac{\sigma_2^2}{n_2(1-\delta_2)} + \frac{\sigma_2^4}{2n_2(1-\delta_2)}}}{\delta_1\sigma_1^4\delta_2\sigma_2^4}
$$

This is known as the Probability-Matching Prior.

Results of Simulation Studies for Individual Designs

Table 48: Results for Design 1 – Equal-Tailed Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	ML	0.9059	0.0441	3.4669	0.0883	0.0058	0.8767
		Bootstrap	0.9069	0.0431	4.0458	0.0509	0.0422	0.0934
		Prior 1	0.9590	-0.0090	5.6082	0.0230	0.0180	0.1220
		Prior 2	0.9470	0.0030	4.7554	0.0210	0.0320	0.2076
		Prior 3	0.9710	-0.0210	8.7990	0.0220	0.0070	0.5172
25	25	ML	0.9340	0.0160	2.2798	0.0589	0.0071	0.7848
		Bootstrap	0.9261	0.0239	2.4284	0.0393	0.0346	0.0636
		Prior 1	0.9550	-0.0050	2.7427	0.0210	0.0240	0.0667
		Prior 2	0.9490	0.0010	2.6049	0.0200	0.0310	0.2157
		Prior 3	0.9600	-0.0100	3.0862	0.0270	0.0130	0.3500
50	50	ML	0.9411	0.0089	1.6375	0.0489	0.0100	0.6604
		Bootstrap	0.9374	0.0126	1.6877	0.0330	0.0296	0.0543
		Prior 1	0.9520	-0.0020	1.7693	0.0240	0.0240	0.0000
		Prior 2	0.9480	0.0020	1.7282	0.0210	0.0310	0.1923
		Prior 3	0.9550	-0.0050	1.8613	0.0290	0.0160	0.2889
100	100	ML	0.9468	0.0032	1.1678	0.0411	0.0121	0.0545
		Bootstrap	0.9451	0.0049	1.1735	0.0279	0.0270	0.0164
		Prior 1	0.9550	-0.0050	1.2215	0.0290	0.0160	0.2889
		Prior 2	0.9530	-0.0030	1.2105	0.0300	0.0170	0.2766
		Prior 3	0.9550	-0.0050	1.2492	0.0310	0.0140	0.3778
10	25	ML	0.8939	0.0561	3.2666	0.1042	0.0019	0.9642
		Bootstrap	0.9097	0.0403	4.0089	0.0577	0.0326	0.2780
		Prior 1	0.9500	0.0000	5.1595	0.0230	0.0270	0.0800
		Prior 2	0.9350	0.0150	4.4142	0.0160	0.0490	0.5077
		Prior 3	0.9560	-0.0060	7.9412	0.0360	0.0080	0.6364
25	10	ML	0.9463	0.0037	2.5361	0.0394	0.0143	0.4674
		Bootstrap	0.9232	0.0268	2.6510	0.0373	0.0395	0.0286
		Prior 1	0.9620	-0.0120	3.3218	0.0170	0.0210	0.1053
		Prior 2	0.9490	0.0010	3.0425	0.0250	0.0260	0.0196
		Prior 3	0.9710	-0.0210	4.3101	0.0140	0.0150	0.0345
25	50	ML	0.9312	0.0188	2.1828	0.0642	0.0046	0.8663
		Bootstrap	0.9377	0.0123	2.3810	0.0291	0.0332	0.0658
		Prior 1	0.9520	-0.0020	2.5907	0.0270	0.0210	0.1250
		Prior 2	0.9500	0.0000	2.4663	0.0250	0.0250	0.0000
		Prior 3	0.9540	-0.0040	2.8929	0.0370	0.0090	0.6087

Table 49: Results for Design 2 – Equal-Tailed Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	ML	0.9521	-0.0021	5.7248	0.0226	0.0253	0.0564
		Bootstrap	0.9205	0.0295	6.0589	0.0396	0.0399	0.0004
		Prior 1	0.9470	0.0030	10.0370	0.0230	0.0300	0.1321
		Prior 2	0.9310	0.0190	8.3373	0.0290	0.0400	0.1594
		Prior 3	0.9700	-0.0200	16.4340	0.0150	0.0150	0.0000
25	25	ML	0.9563	-0.0063	3.7488	0.0219	0.0218	0.0023
		Bootstrap	0.9376	0.0124	3.7540	0.0313	0.0311	0.0032
		Prior 1	0.9560	-0.0060	4.6371	0.0220	0.0220	0.0000
		Prior 2	0.9490	0.0010	4.3947	0.0290	0.0220	0.1373
		Prior 3	0.9620	-0.0120	5.2733	0.0220	0.0160	0.1579
50	50	ML	0.9538	-0.0038	2.6800	0.0234	0.0228	0.0130
		Bootstrap	0.9385	0.0115	2.6727	0.0309	0.0306	0.0049
		Prior 1	0.9500	0.0000	2.9760	0.0240	0.0260	0.0400
		Prior 2	0.9530	-0.0030	2.9023	0.0210	0.0260	0.1064
		Prior 3	0.9600	-0.0100	3.1470	0.0200	0.0200	0.0000
100	100	ML	0.9524	-0.0024	1.9096	0.0240	0.0236	0.0083
		Bootstrap	0.9501	-0.0001	1.9038	0.0251	0.0248	0.0060
		Prior 1	0.9540	-0.0040	2.0100	0.0290	0.0170	0.2609
		Prior 2	0.9530	-0.0030	1.9789	0.0300	0.0170	0.2766
		Prior 3	0.9650	-0.0150	2.0618	0.0200	0.0150	0.1429
10	25	ML	0.9368	0.0132	4.8149	0.0581	0.0051	0.8368
		Bootstrap	0.9359	0.0141	4.9238	0.0363	0.0278	0.1326
		Prior 1	0.9560	-0.0060	7.4354	0.0270	0.0170	0.2273
		Prior 2	0.9450	0.0050	6.4141	0.0270	0.0280	0.0182
		Prior 3	0.9550	-0.0050	11.1520	0.0350	0.0100	0.5556
25	10	ML	0.9320	0.0180	4.8104	0.0043	0.0637	0.8735
		Bootstrap	0.9344	0.0156	4.9631	0.0344	0.0312	0.0488
		Prior 1	0.9420	0.0080	7.5502	0.0220	0.0360	0.2414
		Prior 2	0.9310	0.0190	6.5120	0.0300	0.0390	0.1304
		Prior 3	0.9480	0.0020	11.3410	0.0060	0.0460	0.7692
25	50	ML	0.9498	0.0002	3.2550	0.0402	0.0100	0.6016
		Bootstrap	0.9413	0.0087	3.3780	0.0292	0.0295	0.0051
		Prior 1	0.9410	0.0090	3.8268	0.0270	0.0320	0.0847
		Prior 2	0.9380	0.0120	3.6553	0.0240	0.0380	0.2258
		Prior 3	0.9530	-0.0030	4.2456	0.0270	0.2000	0.1489

Table 50: Results for Design 3 – Equal-Tailed Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	ML	0.9046	0.0454	3.6808	0.0906	0.0048	0.8993
		Bootstrap	0.9210	0.0290	4.2114	0.0465	0.0325	0.1772
		Prior 1	0.9500	0.0000	6.3804	0.0220	0.0280	0.1200
		Prior 2	0.9370	0.0130	5.1994	0.0210	0.0420	0.3333
		Prior 3	0.9790	-0.0290	14.7330	0.0170	0.0040	0.6191
25	25	ML	0.9275	0.0225	2.4315	0.0652	0.0073	0.7986
		Bootstrap	0.9300	0.0200	2.5564	0.0413	0.0287	0.1800
		Prior 1	0.9580	-0.0080	2.9558	0.0160	0.0260	0.2381
		Prior 2	0.9530	-0.0030	2.7899	0.0130	0.0340	0.4468
		Prior 3	0.9590	-0.0090	3.3615	0.0210	0.0200	0.0244
50	50	ML	0.9400	0.0100	1.7453	0.0496	0.0104	0.6533
		Bootstrap	0.9412	0.0088	1.7499	0.0347	0.0241	0.1803
		Prior 1	0.9520	-0.0020	1.9132	0.0220	0.0260	0.0833
		Prior 2	0.9480	0.0020	1.8667	0.0230	0.0290	0.1154
		Prior 3	0.9530	-0.0030	2.0207	0.0270	0.0200	0.1489
100	100	ML	0.9460	0.0040	1.2447	0.0391	0.0149	0.4481
		Bootstrap	0.9492	0.0008	1.2244	0.0278	0.0230	0.0945
		Prior 1	0.9370	0.0130	1.3059	0.0380	0.0250	0.2064
		Prior 2	0.9420	0.0080	1.2895	0.0340	0.0240	0.1724
		Prior 3	0.9420	0.0080	1.3385	0.0390	0.0190	0.3448
10	25	ML	0.8845	0.0655	3.4497	0.1134	0.0021	0.9636
		Bootstrap	0.9011	0.0489	3.9343	0.0513	0.0476	0.0374
		Prior 1	0.9370	0.0130	5.9428	0.0320	0.0310	0.0159
		Prior 2	0.9320	0.0180	4.8762	0.0200	0.0480	0.4118
		Prior 3	0.9460	0.0040	21.2630	0.0450	0.0090	0.6667
25	10	ML	0.9377	0.0123	2.7183	0.0450	0.0173	0.4446
		Bootstrap	0.9370	0.0130	2.7412	0.0375	0.0255	0.1905
		Prior 1	0.9570	-0.0070	3.7529	0.0190	0.0240	0.1163
		Prior 2	0.9470	0.0300	3.3444	0.0220	0.0310	0.1698
		Prior 3	0.9730	-0.0230	5.4339	0.0090	0.0180	0.3333
25	50	ML	0.9258	0.0242	2.3188	0.0697	0.0045	0.8787
		Bootstrap	0.9294	0.0206	2.4843	0.0363	0.0343	0.0283
		Prior 1	0.9490	0.0010	2.7803	0.0310	0.0200	0.2157
		Prior 2	0.9530	-0.0030	2.6305	0.0260	0.0210	0.1064
		Prior 3	0.9460	0.0040	3.1472	0.0440	0.0100	0.6296

Table 51: Results for Design 4 – Equal-Tailed Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	ML	0.9079	0.0421	2.5265	0.0843	0.0078	0.8306
		Bootstrap	0.9108	0.0392	3.0611	0.0493	0.0399	0.1046
		Prior 1	0.9620	-0.0120	3.9119	0.0180	0.0200	0.0526
		Prior 2	0.9490	0.0010	3.3242	0.0150	0.0.60	0.4118
		Prior 3	0.9580	-0.0350	6.1603	0.0100	0.0050	0.3333
25	25	ML	0.9329	0.0171	1.6702	0.0563	0.0108	0.7065
		Bootstrap	0.9331	0.0169	1.7039	0.0371	0.0298	0.1091
		Prior 1	0.9570	-0.0070	1.9227	0.0210	0.0220	0.0233
		Prior 2	0.9530	-0.0030	1.8370	0.0220	0.0250	0.0638
		Prior 3	0.9600	-0.0100	2.1406	0.0250	0.0150	0.2500
50	50	ML	0.9422	0.0078	1.1973	0.0460	0.0118	0.5917
		Bootstrap	0.9397	0.0103	1.2556	0.0321	0.0282	0.0647
		Prior 1	0.9390	0.0110	1.2980	0.0330	0.0280	0.0820
		Prior 2	0.9400	0.0100	1.2739	0.0310	0.0290	0.0333
		Prior 3	0.9490	0.0010	1.3610	0.0350	0.0160	0.3726
100	100	ML	0.9499	0.0001	0.8538	0.0362	0.0139	0.4451
		Bootstrap	0.9470	0.0030	0.8820	0.0268	0.0262	0.0113
		Prior 1	0.9640	-0.0140	0.8823	0.0170	0.0190	0.0556
		Prior 2	0.9640	-0.0140	0.8732	0.0150	0.0210	0.1667
		Prior 3	0.9660	-0.0160	0.9014	0.0210	0.0130	0.2353
10	25	ML	0.8898	0.0602	2.3841	0.1055	0.0047	0.9147
		Bootstrap	0.9041	0.0459	2.5504	0.0521	0.0438	0.0865
		Prior 1	0.9530	-0.0030	3.6296	0.0230	0.0240	0.0213
		Prior 2	0.9450	0.0050	3.1407	0.0250	0.0300	0.0909
		Prior 3	0.9560	-0.0060	5.4701	0.0350	0.0090	0.5909
25	10	ML	0.9385	0.0115	1.8614	0.0445	0.0170	0.4472
		Bootstrap	0.9298	0.0202	1.9356	0.0387	0.0315	0.1026
		Prior 1	0.9620	-0.0120	2.3899	0.0190	0.0190	0.0000
		Prior 2	0.9580	-0.0080	2.1812	0.0180	0.0240	0.1429
		Prior 3	0.9770	-0.0270	3.2064	0.0100	0.0130	0.1304
25	50	ML	0.9308	0.0192	1.5949	0.0638	0.0054	0.8439
		Bootstrap	0.9411	0.0089	1.6100	0.0301	0.0288	0.0221
		Prior 1	0.9580	-0.0080	1.8496	0.0170	0.0250	0.1905
		Prior 2	0.9500	0.0000	1.7658	0.0180	0.0320	0.2800
		Prior 3	0.9670	-0.0170	2.0465	0.0210	0.0120	0.2727

Table 52: Results for Design 5 – Equal-Tailed Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	ML	0.9062	0.0438	2.7395	0.0852	0.0086	0.8166
		Bootstrap	0.9211	0.0289	2.9173	0.0423	0.0366	0.0640
		Prior 1	0.9660	-0.0160	4.6583	0.0160	0.0180	0.0588
		Prior 2	0.9510	-0.0010	3.8112	0.0220	0.0270	0.1020
		Prior 3	0.9870	-0.0370	11.3360	0.0120	0.0010	0.8462
25	25	ML	0.9348	0.0152	1.8030	0.0561	0.0091	0.7209
		Bootstrap	0.9331	0.0169	2.0132	0.0350	0.0319	0.0463
		Prior 1	0.9520	-0.0020	2.1272	0.0300	0.0180	0.2500
		Prior 2	0.9530	-0.0030	2.0187	0.0260	0.0210	0.1064
		Prior 3	0.9580	-0.0080	2.3997	0.0310	0.0110	0.4762
50	50	ML	0.9451	0.0049	1.2941	0.0429	0.0120	0.5628
		Bootstrap	0.9434	0.0066	1.2911	0.0297	0.0269	0.0495
		Prior 1	0.9510	-0.0010	1.3950	0.0270	0.0220	0.1020
		Prior 2	0.9500	0.0000	1.3654	0.0240	0.0260	0.0400
		Prior 3	0.9540	-0.0040	1.4658	0.0310	0.0150	0.3478
100	100	ML	0.9465	0.0035	0.9196	0.0401	0.0134	0.4991
		Bootstrap	0.9477	0.0023	0.9503	0.0266	0.0257	0.0173
		Prior 1	0.9510	-0.0010	0.9530	0.0210	0.0280	0.1429
		Prior 2	0.9470	0.0030	0.9437	0.0210	0.0320	0.2076
		Prior 3	0.9550	-0.0050	0.9775	0.0240	0.0210	0.0667
10	25	ML	0.8911	0.0589	2.5576	0.1043	0.0046	0.9155
		Bootstrap	0.9154	0.0346	2.7803	0.0501	0.0345	0.1844
		Prior 1	0.9480	0.0020	4.2069	0.0270	0.0250	0.0385
		Prior 2	0.9320	0.0180	3.4904	0.0230	0.0450	0.3235
		Prior 3	0.9600	-0.0100	7.3017	0.0360	0.0040	0.8000
25	10	ML	0.9376	0.0124	2.0337	0.0421	0.0203	0.3494
		Bootstrap	0.9393	0.0107	2.2042	0.0311	0.0296	0.0247
		Prior 1	0.9550	-0.0050	2.6866	0.0190	0.0260	0.1556
		Prior 2	0.9410	0.0090	2.3820	0.0260	0.0330	0.1186
		Prior 3	0.9770	-0.0270	6.8346	0.0050	0.0180	0.5652
25	50	ML	0.9303	0.0197	1.7188	0.0624	0.0073	0.7905
		Bootstrap	0.9422	0.0078	1.6998	0.0298	0.0280	0.0311
		Prior 1	0.9440	0.0060	2.0401	0.0290	0.0270	0.0357
		Prior 2	0.9410	0.0090	1.9391	0.0280	0.0310	0.0508
		Prior 3	0.9480	0.0020	2.2861	0.0360	0.0160	0.3846

Table 53: Results for Design 1 – HPD Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9570	-0.0070	4.8347	0.0320	0.0110	0.4884
		Prior 2	0.9530	-0.0030	3.9066	0.0450	0.0020	0.9149
		Prior 3	0.9920	-0.0420	7.7611	0.0060	0.0020	0.5000
10	25	Prior 1	0.9660	-0.0160	4.1246	0.0310	0.0030	0.8235
		Prior 2	0.9330	0.0170	3.4356	0.0660	0.0010	0.9701
		Prior 3	0.9840	-0.0340	6.2847	0.0140	0.0020	0.7500
25	10	Prior 1	0.9720	-0.0220	2.7055	0.0150	0.0130	0.0714
		Prior 2	0.9840	-0.0340	2.4298	0.0080	0.0080	0.0000
		Prior 3	0.9860	-0.0360	3.6555	0.0080	0.0060	0.1429
25	25	Prior 1	0.9650	-0.0150	2.1047	0.0260	0.0090	0.4857
		Prior 2	0.9450	0.0050	1.9682	0.0460	0.0090	0.6727
		Prior 3	0.9720	-0.0220	2.4207	0.0190	0.0090	0.3571
25	50	Prior 1	0.9480	0.0020	1.9860	0.0420	0.0100	0.6154
		Prior 2	0.9390	0.0110	1.8776	0.0530	0.0080	0.7377
		Prior 3	0.9630	-0.0130	2.2322	0.0250	0.0120	0.3514
50	50	Prior 1	0.9580	-0.0080	1.3599	0.0260	0.0160	0.2381
		Prior 2	0.9410	0.0090	1.3029	0.0430	0.0160	0.4576
		Prior 3	0.9520	-0.0020	1.4535	0.0280	0.0200	0.1667
100	100	Prior 1	0.9510	-0.0010	0.9005	0.0410	0.0080	0.6735
		Prior 2	0.9440	0.0060	0.8960	0.0400	0.0160	0.4286
		Prior 3	0.9520	-0.0020	0.9312	0.0280	0.0200	0.1667

Table 54: Results for Design 2 – HPD Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9810	-0.0310	9.4602	0.0050	0.0140	0.4737
		Prior 2	0.9730	-0.0230	7.8847	0.0130	0.0140	0.0370
		Prior 3	0.9930	-0.0430	15.6870	0.0020	0.0050	0.4286
10	25	Prior 1	0.9630	-0.0130	6.6368	0.0300	0.0070	0.6216
		Prior 2	0.9430	0.0070	5.7021	0.0530	0.0040	0.8596
		Prior 3	0.9860	-0.0360	9.9837	0.0130	0.0010	0.8571
25	10	Prior 1	0.9650	-0.0150	6.7691	0.0090	0.0260	0.4857
		Prior 2	0.9640	-0.0140	5.6596	0.0050	0.0310	0.7222
		Prior 3	0.9910	-0.0410	9.9329	0.0050	0.0040	0.1111
25	25	Prior 1	0.9670	-0.0170	4.0537	0.0160	0.0170	0.0303
		Prior 2	0.9640	-0.0140	3.7854	0.0160	0.0200	0.1111
		Prior 3	0.9770	-0.0270	4.7098	0.0090	0.0140	0.2174
25	50	Prior 1	0.9650	-0.0150	3.2799	0.0180	0.0170	0.0286
		Prior 2	0.9520	-0.0020	3.1131	0.0360	0.0120	0.5000
		Prior 3	0.9660	-0.0160	3.6776	0.0250	0.0090	0.4706
50	50	Prior 1	0.9540	-0.0040	2.5321	0.0260	0.0200	0.1304
		Prior 2	0.9510	-0.0010	2.4161	0.0230	0.0260	0.0612
		Prior 3	0.9650	-0.0150	2.6956	0.0170	0.0180	0.0286
100	100	Prior 1	0.9560	-0.0060	1.6519	0.0240	0.0200	0.0909
		Prior 2	0.9680	-0.0180	1.6220	0.0180	0.0140	0.1250
		Prior 3	0.9580	-0.0080	1.7129	0.0220	0.0200	0.0476

Table 55: Results for Design 3 – HPD Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9670	-0.0170	5.5532	0.0290	0.0040	0.7576
		Prior 2	0.9400	0.0100	4.3116	0.0530	0.0070	0.7667
		Prior 3	0.9930	-0.0430	10.4133	0.0040	0.0030	0.1429
10	25	Prior 1	0.9550	-0.0050	4.6276	0.0390	0.0060	0.7333
		Prior 2	0.9380	0.0120	3.8159	0.0580	0.0040	0.8710
		Prior 3	0.9820	-0.0320	8.5058	0.0100	0.0080	0.1111
25	10	Prior 1	0.9710	-0.0210	3.0150	0.0200	0.0090	0.3793
		Prior 2	0.9560	-0.0060	2.6912	0.0260	0.0180	0.1818
		Prior 3	0.9880	-0.0380	4.5997	0.0070	0.0050	0.1667
25	25	Prior 1	0.9570	-0.0070	2.3320	0.0360	0.0070	0.6744
		Prior 2	0.9430	0.0070	2.1450	0.0480	0.0090	0.6842
		Prior 3	0.9610	-0.0110	2.7197	0.0240	0.0150	0.2308
25	50	Prior 1	0.9550	-0.0050	2.1745	0.0380	0.0070	0.6889
		Prior 2	0.9380	0.0120	2.0464	0.0570	0.0050	0.8387
		Prior 3	0.9610	-0.0110	2.5241	0.0200	0.0190	0.0256
50	50	Prior 1	0.9540	-0.0040	1.4611	0.0360	0.0100	0.5652
		Prior 2	0.9620	-0.0120	1.4252	0.0290	0.0090	0.5263
		Prior 3	0.9630	-0.0130	1.5931	0.0210	0.0160	0.1351
100	100	Prior 1	0.9390	0.0110	0.9791	0.0410	0.0200	0.3443
		Prior 2	0.9460	0.0040	0.9636	0.0370	0.0170	0.3704
		Prior 3	0.9570	-0.0070	1.0100	0.0300	0.0130	0.3953

Table 56: Results for Design 4 – HPD Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9570	-0.0070	3.1202	0.0390	0.0040	0.8140
		Prior 2	0.9460	0.0040	2.5807	0.0490	0.0050	0.8148
		Prior 3	0.9930	-0.0430	5.3104	0.0060	0.0010	0.7143
10	25	Prior 1	0.9560	-0.0060	2.7157	0.0430	0.0010	0.9545
		Prior 2	0.9350	0.0150	2.2919	0.0650	0.0000	1.0000
		Prior 3	0.9850	-0.0350	4.2682	0.0100	0.0050	0.3333
25	10	Prior 1	0.9650	-0.0150	1.8114	0.0220	0.0130	0.2571
		Prior 2	0.9670	-0.0170	1.5772	0.0210	0.0120	0.2727
		Prior 3	0.9890	-0.0390	2.5082	0.0070	0.0040	0.2727
25	25	Prior 1	0.9480	0.0020	1.4107	0.0460	0.0060	0.7692
		Prior 2	0.9290	0.0210	1.3249	0.0610	0.0100	0.7183
		Prior 3	0.9720	-0.0220	1.6280	0.0180	0.0100	0.2857
25	50	Prior 1	0.9460	0.0040	1.3228	0.0490	0.0050	0.8148
		Prior 2	0.9380	0.0120	1.2580	0.0580	0.0040	0.8710
		Prior 3	0.9580	-0.0080	1.5289	0.0250	0.0170	0.1905
50	50	Prior 1	0.9490	0.0010	0.9066	0.0350	0.0160	0.3725
		Prior 2	0.9500	0.0000	0.8767	0.0380	0.0120	0.5200
		Prior 3	0.9580	-0.0080	0.9652	0.0270	0.0150	0.2857
100	100	Prior 1	0.9380	0.0120	0.6097	0.0400	0.0220	0.2903
		Prior 2	0.9470	0.0030	0.5989	0.0320	0.0210	0.2075
		Prior 3	0.9520	-0.0020	0.6270	0.0290	0.0190	0.2083

Table 57: Results for Design 5 – HPD Bayesian Credibility Intervals

n_1	n_2	Method	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
10	10	Prior 1	0.9700	-0.0200	3.6830	0.0260	0.0040	0.7333
10	10	Prior 2	0.9570	-0.0070	2.8799	0.0390	0.0040	0.8140
10	10	Prior 3	0.9930	-0.0430	8.6341	0.0040	0.0030	0.1429
10	25	Prior 1	0.9560	-0.0060	3.2310	0.0360	0.0080	0.6364
10	25	Prior 2	0.9250	0.0250	2.5733	0.0700	0.0050	0.8667
10	25	Prior 3	0.9820	-0.0320	5.3696	0.0120	0.0060	0.3333
25	10	Prior 1	0.9670	-0.0170	2.0684	0.0220	0.0110	0.3333
25	10	Prior 2	0.9530	-0.0030	1.8358	0.0270	0.0200	0.1489
25	10	Prior 3	0.9910	-0.0410	4.9927	0.0080	0.0010	0.7778
25	25	Prior 1	0.9500	0.0000	1.5694	0.0380	0.0120	0.5200
25	25	Prior 2	0.9470	0.0030	1.4907	0.0400	0.0130	0.5094
25	25	Prior 3	0.9770	-0.0270	1.8212	0.0190	0.0040	0.6522
25	50	Prior 1	0.9340	0.0160	1.4741	0.0570	0.0090	0.7273
25	50	Prior 2	0.9320	0.0180	1.3773	0.0600	0.0080	0.7647
25	50	Prior 3	0.9700	-0.0200	1.7286	0.0200	0.0100	0.3333
50	50	Prior 1	0.9610	-0.0110	1.0050	0.0220	0.0170	0.1282
50	50	Prior 2	0.9420	0.0080	0.9634	0.0500	0.0080	0.7241
50	50	Prior 3	0.9560	-0.0060	1.0625	0.0280	0.0160	0.2727
100	100	Prior 1	0.9460	0.0040	0.6669	0.0410	0.0130	0.5185
100	100	Prior 2	0.9500	0.0000	0.6671	0.0350	0.0150	0.4000
100	100	Prior 3	0.9490	0.0010	0.6942	0.0270	0.0240	0.0588

Table 58: Results for MOVER

Design	n_1	n_2	Coverage Probability	Coverage Error	Average Length	$\%CI < \theta$	$\%CI > \theta$	Relative Bias
1	10	10	0.9620	-0.0120	5.7026	0.0170	0.0210	0.1053
	10	25	0.9550	-0.0050	5.2727	0.0190	0.0260	0.1556
	25	10	0.9610	-0.0110	3.3565	0.0160	0.0230	0.1795
	25	25	0.9580	-0.0080	2.6984	0.0180	0.0240	0.1429
	25	50	0.9560	-0.0060	2.5970	0.0160	0.0280	0.2727
	50	50	0.9510	-0.0010	1.7782	0.0140	0.0350	0.4286
	100	100	0.9660	-0.0160	1.2138	0.0130	0.0210	0.2353
2	10	10	0.9700	-0.0200	10.4234	0.0180	0.0120	0.2000
	10	25	0.9650	-0.0150	7.6625	0.0200	0.0150	0.1429
	25	10	0.9460	0.0040	7.6824	0.0290	0.0250	0.0741
	25	25	0.9610	-0.0110	4.6886	0.0180	0.0210	0.0769
	25	50	0.9490	0.0010	3.9023	0.0250	0.0260	0.0196
	50	50	0.9650	-0.0150	2.9862	0.0200	0.0150	0.1429
	100	100	0.9500	0.0000	2.0205	0.0290	0.0210	0.1600
3	10	10	0.9770	-0.0270	6.5483	0.0040	0.0190	0.6522
	10	25	0.9460	0.0040	5.9409	0.0250	0.0290	0.0741
	25	10	0.9530	-0.0030	3.7494	0.0230	0.0240	0.0213
	25	25	0.9510	-0.0010	2.9558	0.0320	0.0170	0.3061
	25	50	0.9460	0.0040	2.7663	0.0230	0.0310	0.1481
	50	50	0.9400	0.0100	1.9191	0.0340	0.0260	0.1333
	100	100	0.9620	-0.0120	1.3048	0.0180	0.0200	0.0526
4	10	10	0.9700	-0.0200	3.9909	0.0140	0.0160	0.0667
	10	25	0.9670	-0.0170	3.5649	0.0090	0.0240	0.4545
	25	10	0.9640	-0.0140	2.3665	0.0200	0.0160	0.1111
	25	25	0.9690	-0.0190	1.9358	0.0130	0.0180	0.1613
	25	50	0.9670	-0.0170	1.8579	0.0150	0.0180	0.0909
	50	50	0.9550	-0.0050	1.2905	0.0220	0.0230	0.0222
	100	100	0.9540	-0.0040	0.8871	0.0220	0.0240	0.0435
5	10	10	0.9640	-0.0140	4.7565	0.0190	0.0170	0.0556
	10	25	0.9430	0.0070	4.1594	0.0230	0.0340	0.1930
	25	10	0.9660	-0.0160	2.7002	0.0120	0.0220	0.2941
	25	25	0.9660	-0.0160	2.1226	0.0130	0.0210	0.2353
	25	50	0.9480	0.0020	2.0177	0.0220	0.0300	0.1538
	50	50	0.9530	-0.0030	1.4002	0.0270	0.0200	0.1489
	100	100	0.9590	-0.0090	0.9532	0.0230	0.0180	0.1220

CHAPTER 5

Inference on the Variance: One and Two Sample Approach

Introduction

From Chapter 2 a Bayesian methodology was developed and introduced for lognormally distributed data. Credibility intervals were calculated for the mean of the distribution where no zero values were included in the data and the performance of these credibility intervals was compared for different choices of prior distributions, including the Independence Jeffreys prior, the Jeffreys Rule prior as well as a constant prior, Reference prior and Probability-Matching prior.

In this next chapter we look at a similar situation in terms of the distribution of the data. In this case we are primarily concerned with Bayesian inference for the variance of the distribution as opposed to the mean of the distribution. Similarly to previous chapters, credibility intervals (Bayesian confidence intervals) will be developed based on different choices of prior distributions. Again, the Probability-Matching prior and Reference prior for the variance will be derived. The relative performance of these various prior distributions will be compared similarly to previous chapters.

The comparison of the variances together with appropriate confidence intervals becomes necessary when we need to compare variability amongst measurements. According to Krishnamoorthy, Mathew and Ramachandran (2006) there are no readily available procedures available for computing confidence intervals for the variance. They proposed methods based on generalized p-values and generalized confidence intervals for addressing this problem. The authors in particular applied the procedures they developed to address situations involving patient and worker exposure data. This chapter presents an alternative method based on Bayesian confidence intervals to address the same problem. As mentioned previously, the effectiveness of a variety of non-informative prior distributions is of primary importance.

5.1 Intervals Based on a Bayesian Procedure

Let $y = \ln(X) \sim N(\mu, \sigma^2)$ then the likelihood function can be written as:

$$L(\mu, \sigma^2) \propto (\sigma^2)^{-\frac{1}{2}v} exp\left\{-\frac{vs^2}{2\sigma^2}\right\}\left(\frac{n}{2\sigma^2\pi}\right)^{\frac{1}{2}} exp\left\{-\frac{n(\mu-\bar{y})^2}{2\sigma^2}\right\}$$

$$L(\mu, \sigma^2) \propto L(\sigma^2)L(\mu|\sigma^2) \qquad (5.1)$$

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$vs^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2$ and

$v = n - 1$

We are interested in credibility intervals for the variance of the distribution, namely:

$$Var(X) = exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\} = t(\mu, \sigma^2)$$

5.2 The Independence Jeffreys Prior

Consider the first prior distribution:

$$p(\mu, \sigma^2) \propto (\sigma^2)^{-1}$$

As mentioned and according to Box and Tiao (1973) it is usually appropriate to take location parameters to be distributed independently of scale parameters. Using the argument in Section 1.3.2 of Box and Tiao (1973) it follows that $p(\mu,\sigma^2) \propto (\sigma^2)^{-1}$.

Combining this prior distribution with the likelihood given by (5.1) results in the following posterior distribution:

$$p(\mu,\sigma^2|data) = p_I(\sigma^2|data)p(\mu|\sigma^2,data)$$

where

$$p_I(\sigma^2|data) = \left(\frac{\upsilon s^2}{2}\right)^{\frac{1}{2}\upsilon} \frac{1}{\Gamma\left(\frac{\upsilon}{2}\right)} (\sigma^2)^{-\frac{1}{2}(\upsilon+2)} \, exp\left\{-\frac{\upsilon s^2}{2\sigma^2}\right\} \tag{5.2}$$

for $\sigma^2 > 0$, which is an Inverse Gamma distribution, and

$$\mu|\sigma^2, data \sim N\left(\bar{y}, \frac{1}{n}\sigma^2\right)$$

From (5.2) it follows that

$$\frac{\upsilon s^2}{\sigma} \sim \chi^2_\upsilon$$

To obtain credibility intervals for this Bayesian procedure Monte Carlo simulation is applied.

5.3 Simulation Procedure

The simulation of the required data can be performed in a similar way to that described in previous chapters, particularly in Chapters 1 and 2. Using these methods we can simulate μ and σ^2 out of their posterior distribution. Using these values we can simulate a

10000 values of $t(\mu, \sigma^2) = exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}$. These are sorted and credibility intervals are obtained in a similar manner as in the previously referred to chapters.

5.4 Other Prior Distributions.

As previously mentioned, this procedure was repeated for other prior distributions as well. Obviously the form of the posterior distribution will change for each prior distribution. The simulation procedure is similar to that for posterior distribution (5.2) except that (given the choice of prior distributions) σ^2 is distributed from a central chi-squared distribution with the degrees of freedom changing based on the form of (5.2).

The Jeffreys Rule prior, which is the square root of the determinant of the Fisher Information matrix namely, $p(\sigma^2) \propto \sigma^{-3}$.

5.4.1 Probability-Matching Prior Distribution

In addition to the two previously mentioned priors by Jeffreys, both the Reference and Probability-Matching prior distributions were derived. The Probability-Matching prior for $t(\mu, \sigma^2) = Var(X)$ is given by:

$$p_M(\mu, \sigma^2) \propto \sigma^{-3}\sqrt{\frac{2\{exp(\sigma^2) - 1\}^2}{\{2exp(\sigma^2) - 1\}^2} + \sigma^2} \tag{5.3}$$

The derivation is given in the appendix to this chapter.

From (5.1) it follows that if we multiply (5.1) by (5.3)

$$p_M(\mu,\sigma^2|data) \propto p_M(\sigma^2|data) \times p_M(\mu|\sigma^2,data)$$

where

$$\mu|\sigma^2, data \sim N\left(\bar{y},\frac{1}{n}\sigma^2\right)$$

and

$$p_M(\sigma^2|data) \propto L(\sigma^2) \times p_M(\mu,\sigma^2) \tag{5.4}$$

Using these results one is able to simulate from the posterior distribution by first simulating σ^2 from (5.4). A similar procedure to that described in section 5.3 can be used to simulate values of $t(\mu,\sigma^2)$.

5.4.2 Reference Prior Distribution

In the same way the Reference prior for $t(\mu,\sigma^2) = Var(X)$ is given by:

$$p_R(\mu,\sigma^2) \propto \sigma^{-1}\sqrt{\left\{\frac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right\}^2 + \frac{2}{\sigma^2}} \tag{5.5}$$

The derivation is available in the appendix to this chapter.

A similar procedure to that described in Section 5.4.1 can be used to simulate observations from $t(\mu,\sigma^2) = Var(X)$.

5.5 Simulation Study – Example 1 - Single Sample

In this example the following was done:

1. Take the following initial values: $n = 10, \nu s^2 = 6, \bar{y} = 1$

2. Simulate the posterior distributions of:

   a. $p_J(\mu, \sigma^2|data)$

   b. $p_M(\mu, \sigma^2|data)$

   c. $p_R(\mu, \sigma^2|data)$

   The above posterior distributions are obtained by multiplying the prior with the likelihood.

3. Normalise the posterior distributions so that the area under the curve is equal to 1

4. Simulate $t(\mu, \sigma^2) = Var(X)$ for a), b) and c) above.

5. Calculate 95% credibility intervals.

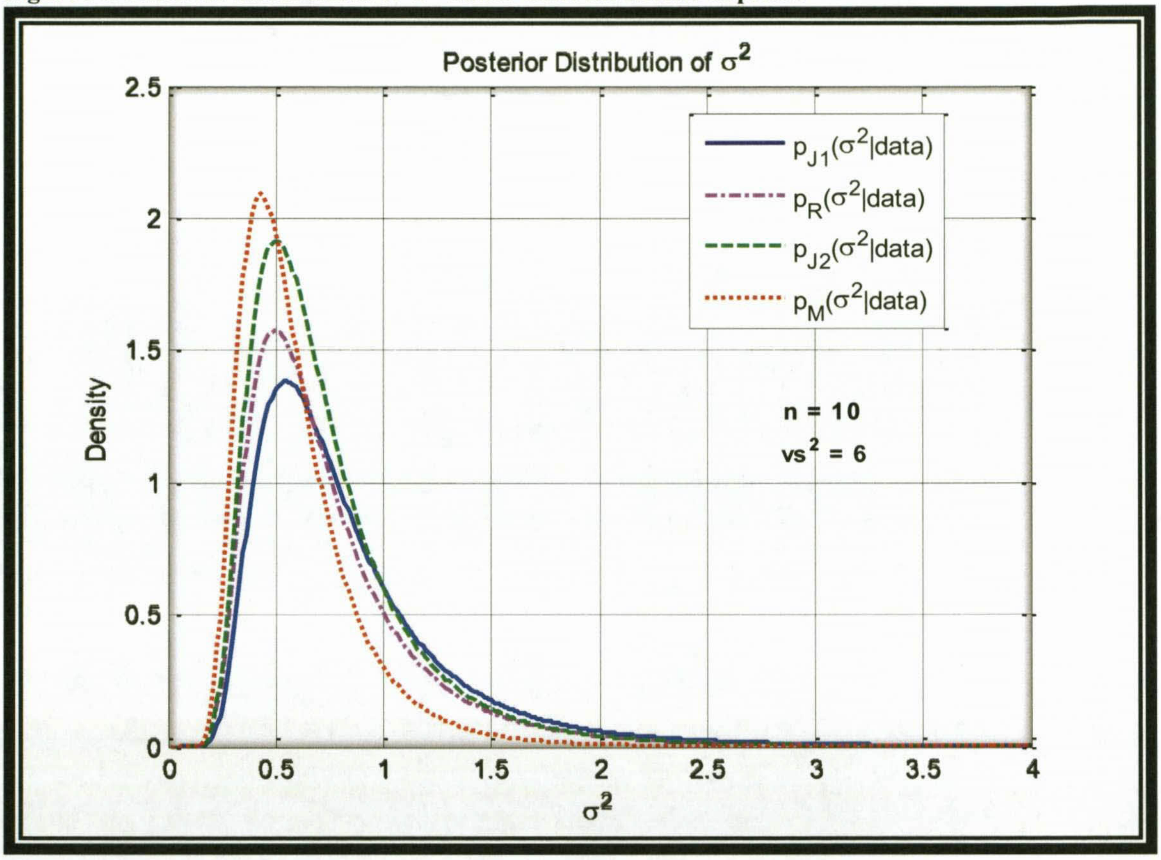
The results were as follows:

Table 59: Results for Various Priors

Prior	Equal -tail Intervals			HPD Intervals		
	Lower	Upper	Length	Lower	Upper	Length
Ind Jeffreys	3.7246	554.497	550.772	1.5494	229.647	228.097
Jeffreys Rule	3.3337	253.727	250.394	1.5496	120.675	119.125
Probability-Matching	2.8121	85.472	82.660	1.4988	50.461	48.962
Reference	3.2913	322.017	318.725	1.5394	146.464	144.925

In terms of interval length, we can see from the above results that the prior distribution that performs the best is the Jeffrey's Rule prior. The Probability-Matching prior appears to have the shortest interval length, however, as will be shown in later results this prior distribution suffers from insufficient coverage.

Figure 34: Posterior distributions of σ^2 under different choices of priors



In the above figure, p_{J1} and p_{J2} refer to the Independence Jeffreys prior and Jeffreys Rule prior respectively, whereas p_R and p_M refer to the Reference and Probability-matching priors respectively. The form of these posteriors is given by the following:

$$p_{J1} \propto \frac{\left(\frac{\upsilon s}{2}\right)^{\frac{\upsilon}{2}} \sigma^{-\frac{1}{2}(\upsilon+2)} exp\left(-\frac{\frac{\upsilon s}{2}}{\sigma}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)}$$

$$p_{J2} \propto \frac{\left(\frac{\upsilon s}{2}\right)^{\frac{\upsilon}{2}} \sigma^{-\frac{1}{2}(\upsilon+3)} exp\left(-\frac{\frac{\upsilon s}{2}}{\sigma}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)}$$

$$p_R \propto \sigma^{-\frac{1}{2}\upsilon} exp\left(-\frac{\frac{\upsilon s}{2}}{\sigma}\right) \sigma^{-1} \sqrt{\left\{\frac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right\}^2 + \frac{2}{\sigma^2}}$$

$$p_M \propto \sigma^{-\frac{1}{2}\upsilon} exp\left(-\frac{\frac{\upsilon s}{2}}{\sigma}\right)\sigma^{-3}\sqrt{\frac{2\{exp(\sigma^2)-1\}^2}{\{2exp(\sigma^2)-1\}^2}+\sigma^2}$$

Figure 35: Posterior Distributions of $Var(X)=exp(2\mu+\sigma^2)\{exp(\sigma^2)-1\}$ using the Jeffreys prior distributions

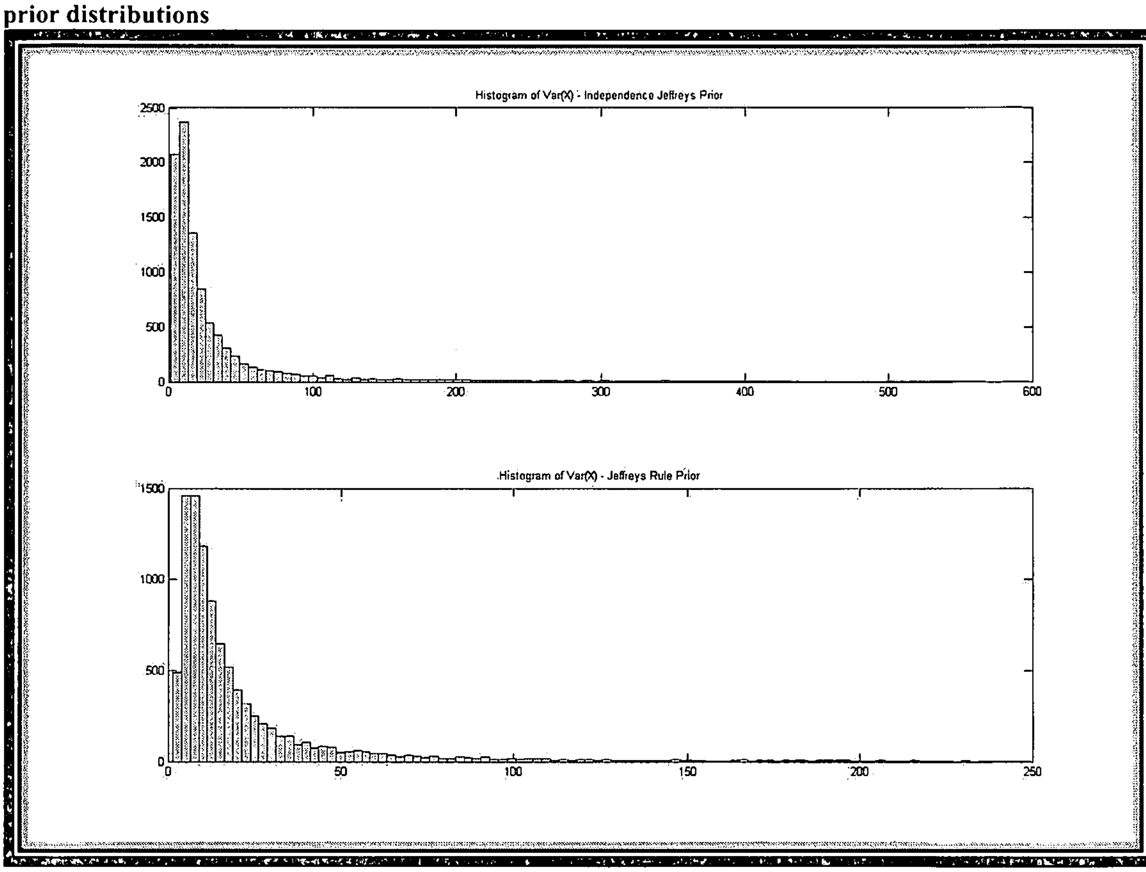
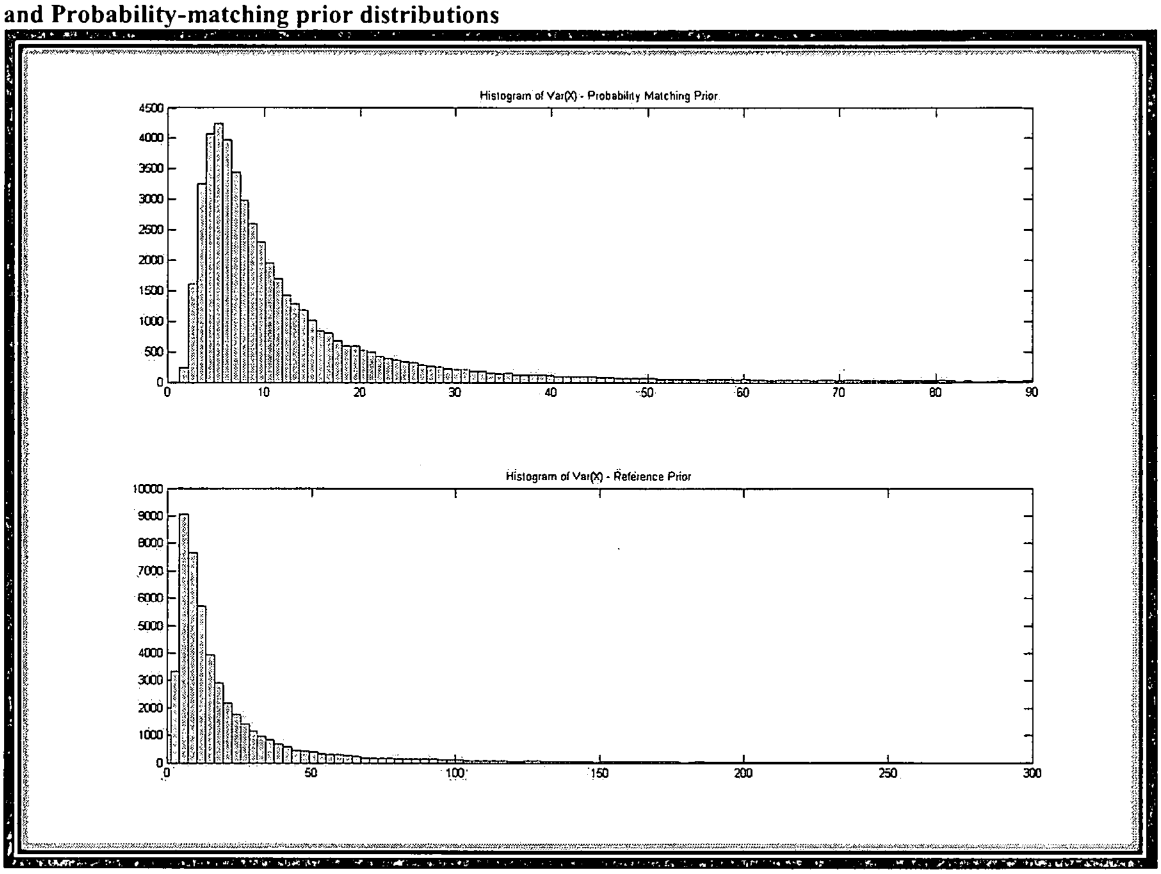


Figure 36: Posterior Distributions of $Var(X) = exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}$ using the Reference and Probability-matching prior distributions



In addition to the above analyses the simulations were performed for a wider range of parameter settings:

Table 60: Parameter Values

Parameters	Parameter Values Chosen
μ	0; 0.3; 0.7; 1; 1.3; 1.5; 1.7; 2
σ^2	0.25; 1; 2.25
n	10, 15, 20

The following results were obtained:

Table 61: Results for the Independence Jeffreys prior

$\sigma^2 = 0.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	94.70	94.92	94.76	94.87	95.05	94.94
	Length	5.4988	1.8618	1.1972	2.8917	1.3380	0.9517
$\mu = 0.3$	Coverage	94.93	95.27	95.35	94.76	95.53	95.30
	Length	10.055	3.3917	2.1720	5.2938	2.4396	1.7286
$\mu = 0.7$	Coverage	94.90	94.65	95.37	94.95	95.19	95.67
	Length	21.834	7.5889	4.8576	11.539	5.4555	3.8654
$\mu = 1$	Coverage	94.96	94.56	94.90	95.06	94.86	95.24
	Length	43.266	13.684	8.8105	22.056	9.8541	7.0135
$\mu = 1.3$	Coverage	94.76	95.08	95.01	94.62	94.93	95.19
	Length	71.389	24.710	16.024	38.059	17.838	12.751
$\mu = 1.5$	Coverage	94.50	95.14	95.04	94.83	95.36	95.00
	Length	110.80	37.235	23.766	58.258	26.807	18.923
$\mu = 1.7$	Coverage	94.94	95.29	95.25	94.66	95.33	95.48
	Length	164.73	55.104	35.903	85.752	39.733	28.561
$\mu = 2$	Coverage	95.32	94.92	94.55	95.03	95.12	95.06
	Length	296.02	101.04	64.801	156.59	72.773	51.526

$\sigma^2 = 1$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	94.86	94.64	95.01	95.09	95.00	95.13
	Length	2.51e7	2965.7	277.45	2.56e5	571.27	112.98
$\mu = 0.3$	Coverage	95.24	94.94	95.42	95.04	95.35	95.07
	Length	4.32e9	8408.2	475.52	1.22e7	1259.8	194.57
$\mu = 0.7$	Coverage	94.88	95.01	95.63	94.98	95.02	95.18
	Length	3.15e7	1.26e4	1134.5	5.65e5	2181.2	457.82
$\mu = 1$	Coverage	94.98	94.76	95.01	95.01	94.97	95.05
	Length	1.16e9	2.89e4	2412.7	8.77e6	5054.2	904.33
$\mu = 1.3$	Coverage	94.87	95.02	95.39	94.87	95.08	95.30
	Length	4.41e8	2.94e4	4078.3	4.46e6	6402.3	1588.6
$\mu = 1.5$	Coverage	94.97	95.14	94.74	95.07	95.17	95.15
	Length	3.19e8	6.56e4	6047.4	3.03e6	1.25e4	2425.7
$\mu = 1.7$	Coverage	94.73	95.02	94.65	94.62	95.20	94.85
	Length	9.99e8	2.34e5	8392.4	1.09e7	3.27e4	3390.2
$\mu = 2$	Coverage	94.91	95.21	95.08	94.87	95.08	95.30
	Length	3.94e8	2.70e5	1.44e4	6.91e6	4.22e4	6036.8

$\sigma^2 = 2.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	95.07	94.68	95.26	95.17	94.61	95.33
	Length	5.5e22	1.3e11	3.5e8	1.0e18	1.26e9	8.91e6
$\mu = 0.3$	Coverage	94.78	95.04	94.94	94.50	95.13	95.08
	Length	7.0e19	8.2e11	5.89e7	3.4e15	5.24e9	3.85e6
$\mu = 0.7$	Coverage	94.61	95.38	94.80	94.70	95.14	94.50
	Length	1.8e23	5.3e12	2.31e8	3.8e18	2.2e10	1.19e7
$\mu = 1$	Coverage	95.08	95.01	94.91	95.01	95.16	95.10
	Length	3.5e20	9.9e10	3.73e8	2.4e16	1.54e9	2.05e7
$\mu = 1.3$	Coverage	94.95	95.14	95.38	94.86	94.88	95.43
	Length	7.9e23	1.4e13	3.98e8	4.1e18	7.8e10	2.69e7
$\mu = 1.5$	Coverage	95.30	95.13	94.92	95.28	94.98	95.03
	Length	1.2e21	3.3e11	1.87e9	6.1e16	4.07e9	7.85e7
$\mu = 1.7$	Coverage	94.85	94.94	95.13	94.55	95.18	94.97
	Length	4.5e22	1.1e13	6.31e8	6.7e17	8.6e10	4.14e7
$\mu = 2$	Coverage	94.87	94.97	95.15	95.10	95.32	94.90
	Length	5.3e24	6.1e12	2.89e9	3.0e19	3.5e10	1.26e8

Table 62: Results for the Jeffreys Rule prior

$\sigma^2 = 0.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	93.78	94.45	94.49	91.27	92.83	93.50
	Length	3.0271	1.4520	1.0168	1.8300	1.0822	0.8239
$\mu = 0.3$	Coverage	93.56	94.60	94.11	91.07	92.68	93.10
	Length	5.3989	2.5880	1.8660	3.2736	1.9388	1.5113
$\mu = 0.7$	Coverage	94.09	94.06	94.11	91.87	92.77	92.63
	Length	12.272	5.8462	4.0922	7.4596	4.3715	3.3182
$\mu = 1$	Coverage	93.64	93.93	94.17	91.72	92.18	93.04
	Length	22.975	10.779	7.4761	13.846	8.0435	6.0590
$\mu = 1.3$	Coverage	93.83	94.41	94.18	91.66	92.27	93.16
	Length	40.136	18.946	13.711	24.438	14.203	11.106
$\mu = 1.5$	Coverage	93.50	94.57	94.05	91.79	92.88	93.13
	Length	63.707	28.462	20.689	38.185	21.325	16.757
$\mu = 1.7$	Coverage	93.56	94.53	94.32	91.43	92.88	93.04
	Length	93.302	43.324	30.054	55.946	32.391	24.381
$\mu = 2$	Coverage	94.13	94.21	94.78	91.77	92.66	93.31
	Length	162.94	79.336	55.543	99.071	59.270	45.029

$\sigma^2 = 1$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	94.04	94.30	94.32	91.77	92.75	92.61
	Length	4.29e6	1061.9	181.91	5.56e4	266.77	78.534
$\mu = 0.3$	Coverage	93.92	94.26	94.36	91.56	92.56	93.23
	Length	5.69e5	1688.9	563.75	1.92e4	444.97	186.26
$\mu = 0.7$	Coverage	93.65	94.31	94.81	91.10	92.25	92.96
	Length	1.60e6	3938.4	647.31	4.66e4	1006.5	289.00
$\mu = 1$	Coverage	93.66	94.27	94.62	91.53	92.40	93.39
	Length	5.45e6	6318.9	1374.3	1.68e5	1749.8	590.46
$\mu = 1.3$	Coverage	93.70	94.25	94.55	91.66	92.62	93.54
	Length	5.83e6	1.23e4	2288.2	1.91e5	3191.2	1024.6
$\mu = 1.5$	Coverage	93.57	94.55	94.47	91.38	93.07	92.96
	Length	2.44e6	2.96e4	3349.7	1.25e5	6300.4	1516.4
$\mu = 1.7$	Coverage	93.86	94.45	94.74	91.43	92.69	93.19
	Length	3.95e7	2.99e4	4591.0	9.04e5	7647.0	2130.1
$\mu = 2$	Coverage	94.18	94.48	94.31	92.16	92.51	92.84
	Length	1.74e8	7.64e4	8795.8	2.16e6	1.67e4	4008.6

$\sigma^2 = 2.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	93.47	94.67	94.19	90.93	92.53	92.86
	Length	1.3e16	5.38e8	1.24e7	3.1e12	1.64e7	9.66e5
$\mu = 0.3$	Coverage	93.91	94.06	93.98	91.78	92.53	92.48
	Length	5.9e16	5.97e9	8.89e6	2.0e13	1.19e8	7.38e5
$\mu = 0.7$	Coverage	93.76	94.07	94.24	91.61	92.02	92.79
	Length	7.4e15	1.2e10	2.15e8	3.9e12	2.93e8	1.37e7
$\mu = 1$	Coverage	93.94	94.45	94.29	91.74	92.13	92.78
	Length	1.1e17	3.2e11	2.35e9	4.9e13	3.53e9	7.33e7
$\mu = 1.3$	Coverage	93.55	94.46	94.12	91.44	92.69	92.81
	Length	3.8e18	9.56e9	2.15e8	8.3e14	2.30e8	1.64e7
$\mu = 1.5$	Coverage	93.57	94.69	94.42	91.33	92.61	92.80
	Length	3.8e18	4.0e10	2.09e8	7.2e14	8.32e8	1.46e7
$\mu = 1.7$	Coverage	94.05	94.23	94.81	91.76	92.21	93.03
	Length	1.3e19	8.8e11	2.04e8	1.4e15	7.69e9	1.49e7
$\mu = 2$	Coverage	94.08	94.29	94.34	91.74	92.75	92.57
	Length	2.6e21	2.8e11	5.48e9	9.6e16	5.51e9	2.94e8

Table 63: Results for the Reference prior

$\sigma^2 = 0.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	92.48	93.50	93.84	89.20	91.40	92.10
	Length	3.0992	1.3826	1.0063	1.7988	1.0293	0.8107
$\mu = 0.3$	Coverage	92.72	93.60	93.55	89.40	91.02	91.67
	Length	5.9863	2.6106	1.8098	3.4336	1.9355	1.4591
$\mu = 0.7$	Coverage	92.40	93.62	93.87	89.32	91.37	92.30
	Length	12.742	5.8852	4.0598	7.4066	4.3492	3.2737
$\mu = 1$	Coverage	92.32	93.47	94.00	89.70	91.55	92.00
	Length	25.025	10.659	7.2417	14.029	7.8707	5.8474
$\mu = 1.3$	Coverage	92.60	93.60	93.55	89.50	91.02	91.67
	Length	44.237	19.289	13.372	25.379	14.301	10.781
$\mu = 1.5$	Coverage	92.40	93.62	93.87	89.32	91.37	92.30
	Length	63.113	29.149	20.108	36.685	21.541	16.214
$\mu = 1.7$	Coverage	92.32	93.50	94.00	89.70	91.40	92.00
	Length	101.48	41.429	29.366	56.892	30.841	23.712
$\mu = 2$	Coverage	92.60	93.60	93.55	89.50	91.02	91.67
	Length	179.39	78.223	54.228	102.91	57.994	43.720

$\sigma^2 = 1$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	93.17	94.15	94.55	91.45	92.67	93.37
	Length	1.92e4	1551.8	221.92	4033.5	418.66	95.296
$\mu = 0.3$	Coverage	93.07	94.07	94.77	91.30	92.92	93.02
	Length	3.95e4	3425.6	333.48	8818.3	901.26	148.03
$\mu = 0.7$	Coverage	93.50	94.00	93.77	91.70	92.57	92.72
	Length	8.59e4	4782.7	877.24	1.91e4	1368.3	375.18
$\mu = 1$	Coverage	93.17	94.15	94.55	91.45	92.67	93.37
	Length	1.42e5	1.15e4	1639.8	2.98e4	3093.2	704.14
$\mu = 1.3$	Coverage	93.07	94.07	94.77	91.30	92.92	93.02
	Length	2.92e5	2.53e4	2464.2	6.52e4	6659.4	1093.8
$\mu = 1.5$	Coverage	93.50	94.00	93.77	91.70	92.57	92.72
	Length	4.26e5	2.37e4	4345.0	9.46e4	6777.9	1858.3
$\mu = 1.7$	Coverage	93.17	94.15	94.55	91.45	92.67	93.37
	Length	5.77e5	4.65e4	6649.9	1.21e5	1.25e4	2855.5
$\mu = 2$	Coverage	93.07	94.20	94.67	91.30	92.40	93.65
	Length	1.18e6	5.29e4	1.25e4	2.64e5	1.51e4	5090.2

$\sigma^2 = 2.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	94.65	93.52	94.12	92.95	92.80	93.72
	Length	1.42e6	4.89e5	1.88e5	4.90e5	1.84e5	7.04e4
$\mu = 0.3$	Coverage	94.65	94.52	95.17	92.65	93.75	94.37
	Length	2.39e6	8.44e5	3.35e5	8.13e5	3.07e5	1.24e5
$\mu = 0.7$	Coverage	94.02	94.62	94.80	92.95	93.52	94.30
	Length	6.04e6	1.74e6	7.07e5	2.14e6	6.11e5	2.56e5
$\mu = 1$	Coverage	94.30	94.72	94.82	92.62	94.05	94.12
	Length	9.77e6	3.28e6	1.39e6	3.28e6	1.19e6	5.15e5
$\mu = 1.3$	Coverage	94.40	94.22	94.97	92.77	93.77	93.97
	Length	1.80e7	6.03e6	2.26e6	6.13e6	2.18e6	7.88e5
$\mu = 1.5$	Coverage	94.42	94.55	94.17	93.17	93.50	93.57
	Length	2.73e7	9.24e6	3.62e6	9.47e6	3.36e6	1.32e6
$\mu = 1.7$	Coverage	94.37	94.75	94.90	92.77	93.75	94.00
	Length	4.07e7	1.43e7	5.55e6	1.40e7	5.25e6	2.04e6
$\mu = 2$	Coverage	94.22	94.62	94.80	92.67	93.52	94.30
	Length	7.75e7	2.34e7	9.52e6	2.69e7	8.23e6	3.44e6

Table 64: Results for the Probability-Matching prior

$\sigma^2 = 0.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	87.76	90.62	91.58	81.56	86.02	87.98
	Length	1.3341	0.9336	0.7757	0.9462	0.7381	0.6447
$\mu = 0.3$	Coverage	87.82	90.32	91.32	81.65	85.57	87.52
	Length	2.5094	1.7519	1.3944	1.7656	1.3809	1.1610
$\mu = 0.7$	Coverage	87.30	90.30	91.75	81.80	85.82	87.92
	Length	5.4664	3.9323	3.1360	3.8705	3.0938	2.6040
$\mu = 1$	Coverage	87.87	90.70	91.92	81.60	86.15	87.97
	Length	10.228	7.1017	5.5984	7.1983	5.5991	4.6635
$\mu = 1.3$	Coverage	87.88	90.32	91.32	81.72	85.57	87.52
	Length	18.552	12.945	10.303	13.053	10.203	8.5784
$\mu = 1.5$	Coverage	87.30	90.30	91.75	81.80	85.82	87.92
	Length	27.075	19.476	15.532	19.170	15.323	12.897
$\mu = 1.7$	Coverage	87.87	90.62	91.92	81.60	86.02	87.97
	Length	41.478	27.973	22.702	29.190	22.115	18.911
$\mu = 2$	Coverage	87.87	90.32	91.32	81.72	85.57	87.52
	Length	75.232	52.494	41.782	52.936	41.377	34.787

$\sigma^2 = 1$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	86.72	90.07	91.55	80.72	85.27	87.12
	Length	840.11	173.72	66.074	209.73	70.038	35.665
$\mu = 0.3$	Coverage	87.20	90.40	91.45	80.95	85.52	86.72
	Length	2566.0	369.34	105.71	691.37	145.04	58.835
$\mu = 0.7$	Coverage	87.55	90.10	91.30	80.92	85.07	86.80
	Length	4898.3	610.57	258.24	1156.2	256.62	141.15
$\mu = 1$	Coverage	86.72	90.07	91.55	80.72	85.27	87.12
	Length	6211.2	1284.3	488.26	1542.7	517.37	263.58
$\mu = 1.3$	Coverage	87.20	90.40	91.45	80.95	85.52	86.72
	Length	1.90e4	2729.1	781.16	5115.6	1071.7	434.72
$\mu = 1.5$	Coverage	87.55	90.10	91.30	80.92	85.07	86.80
	Length	2.42e4	3024.6	1279.1	5735.0	1271.2	699.23
$\mu = 1.7$	Coverage	86.72	90.07	91.55	80.72	85.27	87.12
	Length	2.52e4	5206.3	1979.8	6255.3	2098.5	1069.7
$\mu = 2$	Coverage	87.20	90.32	92.07	80.95	85.05	87.20
	Length	7.89e4	6935.4	3503.4	2.07e4	2012.6	1886.6

$\sigma^2 = 2.25$		Equal-tail Intervals			HPD Intervals		
	n	10	15	20	10	15	20
$\mu = 0$	Coverage	87.62	89.60	91.00	81.55	84.07	86.77
	Length	2.94e5	1.24e5	5.12e4	8.98e4	4.29e4	1.87e4
$\mu = 0.3$	Coverage	87.95	89.80	91.40	81.07	84.55	87.02
	Length	4.71e5	2.05e5	8.97e4	1.44e5	6.90e4	3.24e4
$\mu = 0.7$	Coverage	87.45	90.15	91.87	80.27	84.65	87.02
	Length	1.39e6	3.79e5	1.80e5	4.57e5	1.21e5	6.17e4
$\mu = 1$	Coverage	87.22	90.65	91.77	80.85	84.95	87.50
	Length	1.90e6	7.77e5	3.66e5	5.72e5	2.52e5	1.33e5
$\mu = 1.3$	Coverage	87.45	90.00	91.27	81.37	84.90	86.22
	Length	3.62e6	1.42e6	5.29e5	1.12e6	4.61e5	1.76e5
$\mu = 1.5$	Coverage	87.22	89.82	91.02	80.60	84.72	86.60
	Length	5.91e6	2.19e6	9.17e5	1.87e6	7.18e5	3.19e5
$\mu = 1.7$	Coverage	86.37	90.02	91.52	80.22	85.17	86.70
	Length	8.41e6	3.52e6	1.47e6	2.61e6	1.19e6	5.05e5
$\mu = 2$	Coverage	86.97	90.15	91.87	80.57	84.65	87.02
	Length	1.67e7	5.10e6	2.42e6	5.24e6	1.63e6	8.31e5

Furthermore, all four prior distributions were compared for a larger sample size of 70.

Table 65: Comparison of the 4 prior distributions

$n = 70, \mu = 0$		Equal-tail Interval			HPD Interval		
Prior		$\sigma2 = 0.25$	$\sigma2 = 1$	$\sigma2 = 2.25$	$\sigma2 = 0.25$	$\sigma2 = 1$	$\sigma2 = 2.25$
Ind. Jeffreys	Coverage	94.99	95.37	94.78	95.38	95.85	95.00
	Length	0.3899	13.650	1280.0	0.3668	11.352	808.10
Jeffreys Rule	Coverage	95.01	94.70	95.01	94.59	94.24	93.99
	Length	0.3747	12.709	1065.0	0.3530	10.620	682.84
Matching	Coverage	93.95	93.37	93.97	92.10	91.50	92.20
	Length	0.3527	10.843	862.03	0.3279	9.1612	564.27
Reference	Coverage	94.42	94.72	94.80	93.67	93.95	94.65
	Length	0.3729	12.717	1274.5	0.3463	10.601	804.05

Based on the above results we can see that the two Jeffreys priors perform the best. The Reference prior results in slight undercoverage, while the Probability-Matching prior results in poor coverage. The Independence Jeffreys prior distribution achieves the best results although the interval lengths are somewhat longer than the Jeffreys rule prior. HPD intervals result in a reduced interval lengths. The results of the Independence Jeffreys prior agree with the results obtained by Krishnamoorthy, Mathew and Ramachandran (2006).

5.6 Example - Two Samples

A similar methodology was repeated for two samples from the described distribution. The objective is to calculate credibility intervals for the difference between the sample variances or the ratio between the sample variances. In so doing one is actually testing the hypothesis that the variances are equal. In this example the following was done:

1. Take the following initial values: $n_1 = 10$, $v_1 s_1^2 = 6$, $\bar{y}_1 = 1$ and for the second sample $n_2 = 30, v_2 s_2^2 = 8, \bar{y}_2 = 1$
2. Using the prior distributions described previously simulate the posterior distribution of:

a. $\delta_1 = \frac{Var(X_1)}{Var(X_2)}$

b. $\delta_2 = Var(X_1) - Var(X_2)$

The results were as follows:

Table 66: 95% Credibility Intervals for the Ratio: $\delta_1 = \dfrac{Var(X_1)}{Var(X_2)}$

Prior	Equal -tail Intervals			HPD Intervals		
	Lower	Upper	Length	Lower	Upper	Length
Ind Jeffreys	0.8834	181.918	181.035	0.1828	74.491	74.308
Jeffreys Rule	0.8313	87.565	86.733	0.1694	42.670	42.500
Probability-Matching	0.7599	32.773	32.013	0.2277	19.478	19.250
Reference	0.8412	113.374	112.553	0.1748	49.976	49.801

Table 67: 95% Credibility Interval for the difference: $\delta_2 = Var(X_1) - Var(X_2)$

Prior	Equal -tail Intervals			HPD Intervals		
	Lower	Upper	Length	Lower	Upper	Length
Ind Jeffreys	-0.5419	568.080	568.622	-9.1488	225.357	234.506
Jeffreys Rule	-0.8138	256.634	257.448	-6.2640	123.744	130.008
Probability-Matching	-1.0809	82.061	83.142	-4.2851	49.650	53.936
Reference	-0.7407	322.310	323.051	-7.8606	141.330	149.191

The following histograms graphically show the posterior distributions (note: "Jeffreys 1" refers to the Independence Jeffreys Prior and "Jeffreys 2" refer to the Jeffreys Rule Prior"):

Figure 37: Posterior Distribution of $Var(X_1)/Var(X_2)$ using the Jeffreys Priors

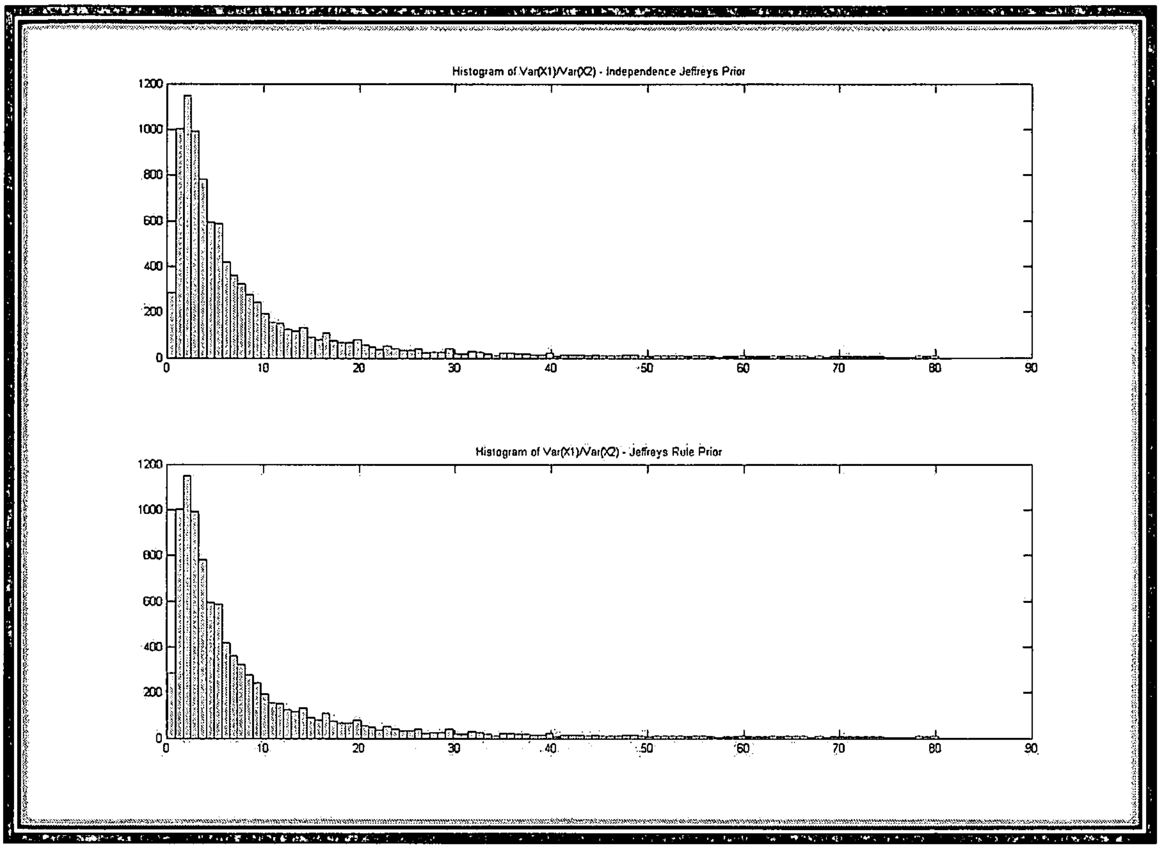


Figure 38: Posterior Distribution of $Var(X_1)/Var(X_2)$ using the Probability-Matching and Reference Priors

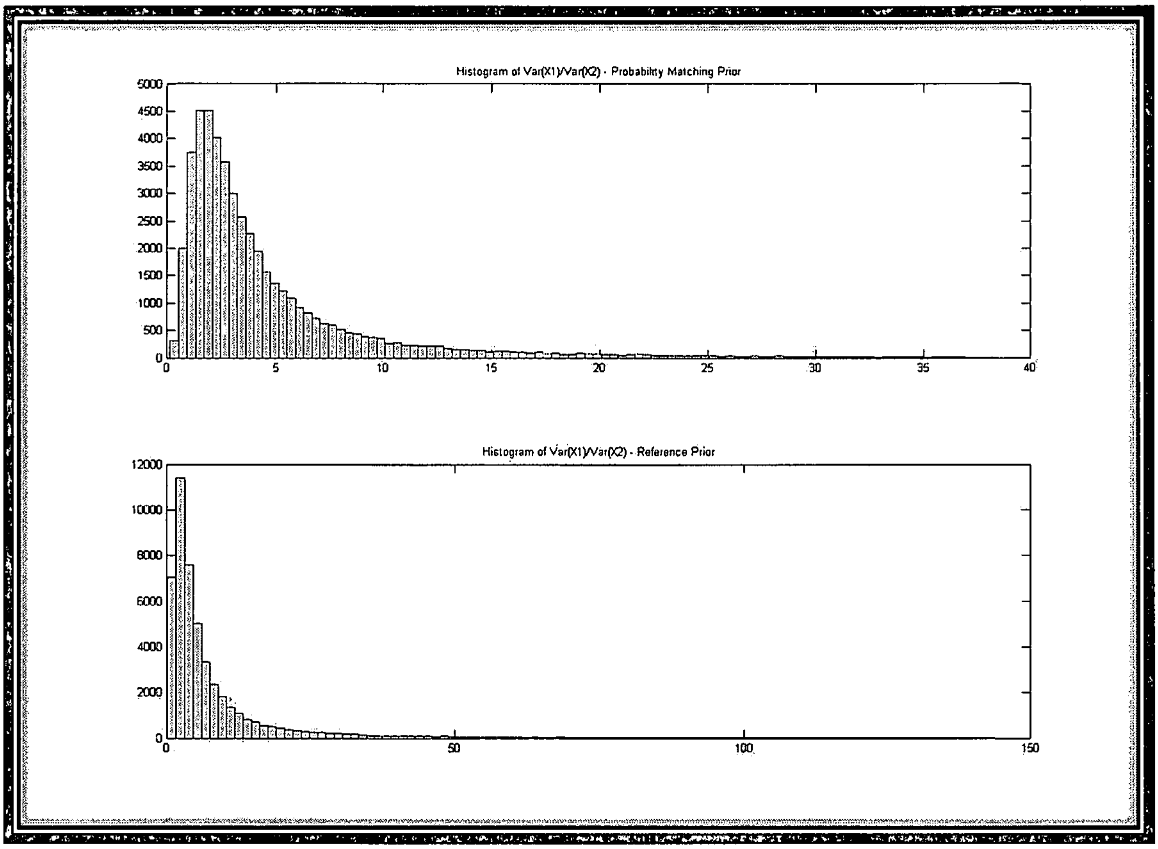


Figure 39: Posterior Distribution of $Var(X_1) - Var(X_2)$ using the Jeffreys Priors

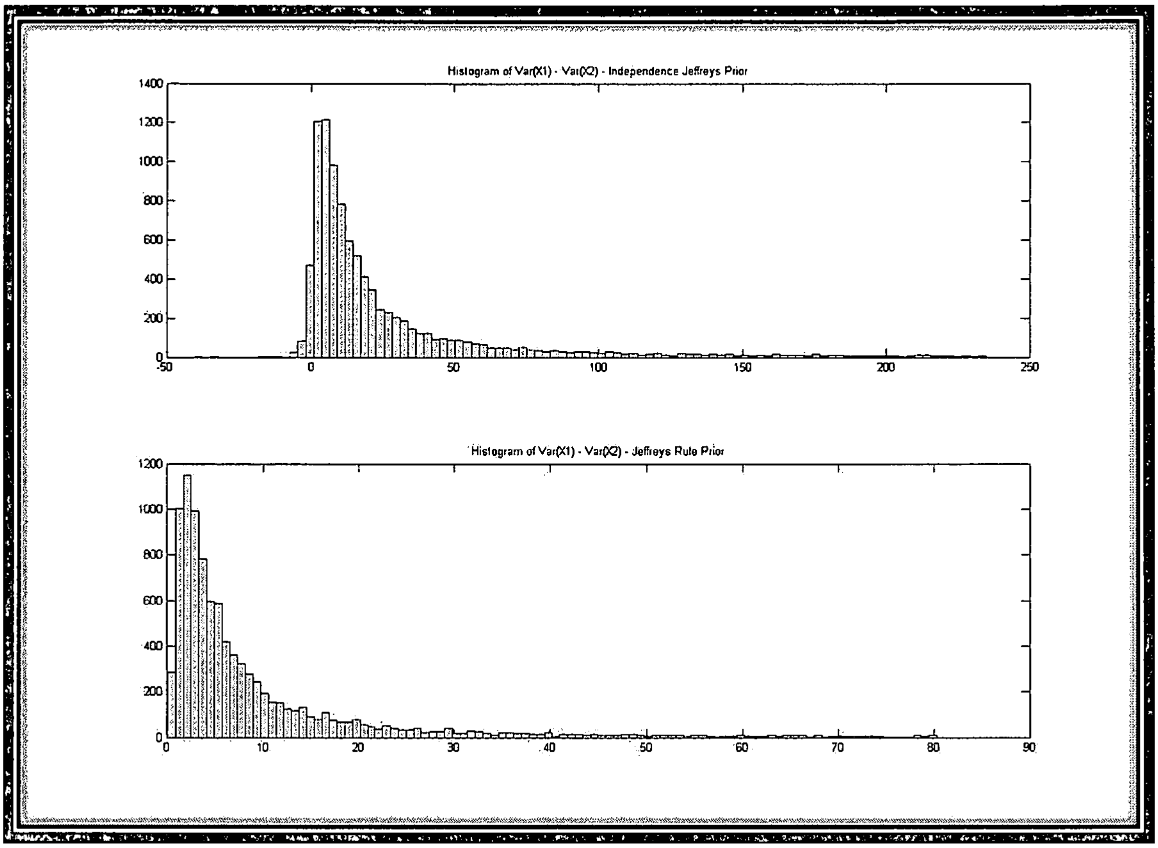
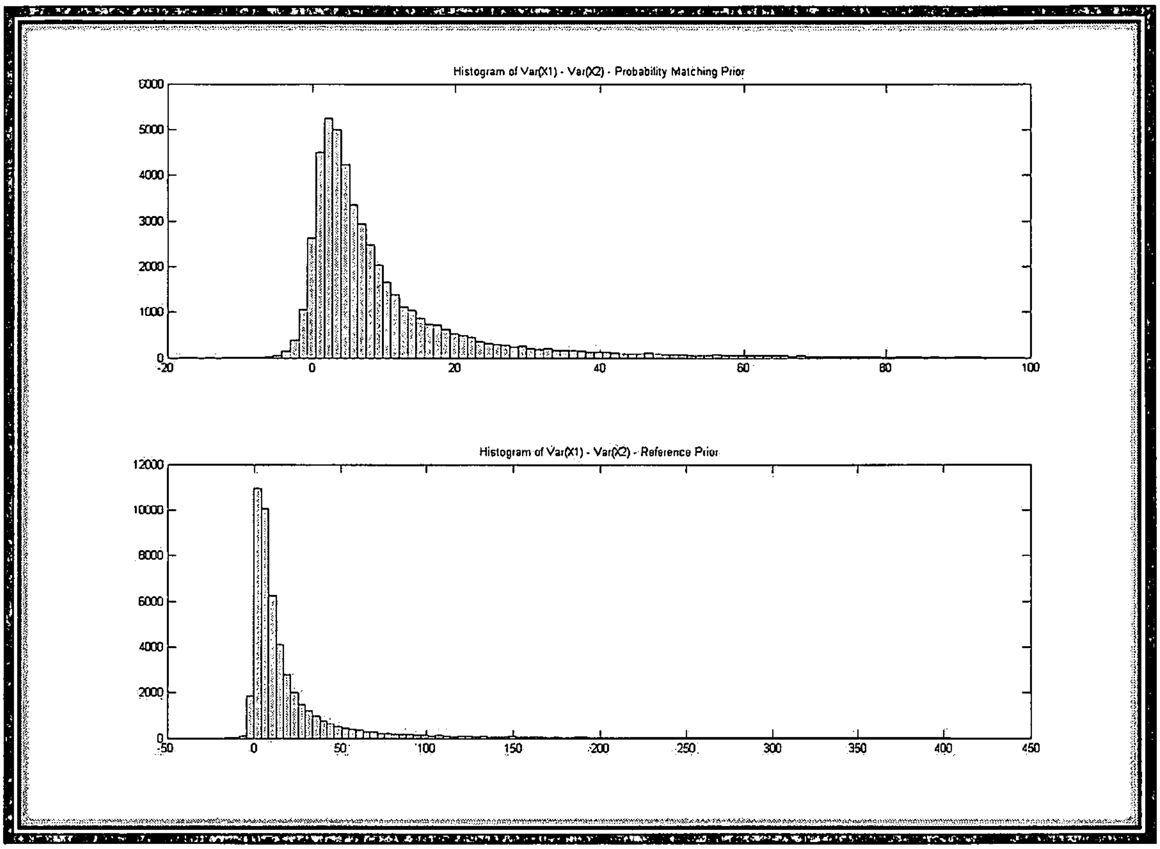


Figure 40: Posterior Distribution of $Var(X_1)-Var(X_2)$ using the Probability-Matching and Reference Priors



From the above results we can observe that both for the ratio and difference of two variances the Independence Jeffreys prior (Jeffreys 1) results in the narrowest credibility intervals. The Reference and Probability-Matching priors results in posterior distributions with a higher degree of variance, which is certainly evident for the Probability-Matching prior. Once again, this may be due to the fact that the Probability-Matching prior was developed for use in one-sided credibility intervals.

5.7 Zero Values

For the assessment of the extent of variability among health care costs or among exposure measurements, confidence intervals or tests concerning the variance $\tilde{\sigma}^2$ of lognormally distributed data with zero-valued observations becomes necessary. Krishnamoorthy, Mathew and Ramachandran (2006) made inference about the lognormal variance, while Bebu and Mathew (2008) obtained confidence intervals for the ratio of variances in the case of the bivariate lognormal distribution. However, as far as we know no procedures are known for computing confidence intervals for $\tilde{\sigma}^2$.

Non-zero observations are assumed to follow a lognormal distribution, but the inclusion of the possibility of zero values necessitates the addition of a binomial parameter to model this possibility. The Jeffreys Independence and Jeffreys Rule prior distributions were once again analysed in this situation. What follows is a brief description of the setting and then the corresponding results.

Let

$$f(x) = \begin{cases} \delta & for\, x = 0 \\ (1-\delta)\,\dfrac{1}{\sigma\sqrt{2\pi x}}\,exp\left(-\dfrac{(lnx - \mu)^2}{2\sigma^2}\right) & for\, x > 0 \end{cases}$$

Then,

$$E(x) = (1-\delta)exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$E(x^2) = (1-\delta)exp\big(2(\mu + \sigma^2)\big)$$

and

$$Var(x) = (1-\delta)exp\big(2(\mu + \sigma^2)\big)\{exp(\sigma^2) - (1-\delta)\}$$

The following prior distributions were taken:

Jeffreys Independence Prior:

$$p_I(\delta)p_I(\mu\,,\sigma^2)\ \propto\ \delta^{-\frac{1}{2}}(1-\delta)^{-\frac{1}{2}}\sigma^{-2}$$

Jeffreys Rule Prior:

$$p_D(\delta)p_D(\mu\,,\sigma^2)\ \propto\ \delta^{-\frac{1}{2}}(1-\delta)^{\frac{1}{2}}\sigma^{-3}$$

Using the simulation methodology described in this and previous chapters (with particular reference to Chapter 4 concerning the modeling of the possibility of zero values using a Beta distribution) credibility intervals were calculated for a single sample. Both equal-tailed and highest posterior density (HPD) intervals were calculated. These were calculated for various combinations of the four parameters, as follows:

Table 68: Parameter Settings for the Simulation Study

Parameters	Parameter Values Chosen
δ	0.1 ; 0.2 ; 0.3
μ	0; 1; 2
σ^2	0.25; 1; 2.25
n	10, 15, 20, 50

The following results were obtained:

Table 69: 95% Credibility Intervals for $Var\left(X\right)$ with Zero values included using the Independence Jeffreys Prior – Setting 1

$\delta = 0.1$			Equal-tail Interval			HPD Interval		
n	μ		$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$	$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$
10	0	Coverage	95.07	95.05	95.34	95.43	95.05	95.45
		Length	82.874	1.4e16	3.3e35	7.0863	3.3e10	2.4e24
	1	Coverage	95.41	95.11	94.95	95.66	94.94	94.99
		Length	4033.4	1.7e14	5.5e53	411.00	3.18e9	4.5e34
	2	Coverage	95.06	94.86	94.93	95.47	94.65	95.29
		Length	1371.9	2.7e16	2e180	303.44	1.9e11	1.6e88
15	0	Coverage	95.46	94.90	95.20	95.41	94.81	95.45
		Length	2.7576	5.79e4	8.8e13	1.8573	3317.7	9.9e10
	1	Coverage	95.14	95.04	94.68	95.44	95.04	95.16
		Length	20.724	4.53e5	2.8e14	13.870	2.47e4	7.7e11
	2	Coverage	95.00	94.83	95.08	95.27	94.38	94.86
		Length	155.78	2.90e8	4.4e17	104.12	3.71e6	1.5e13
20	0	Coverage	95.12	94.53	95.05	95.59	95.12	95.50
		Length	1.6494	1465.4	1.64e9	1.2664	330.75	4.26e7
	1	Coverage	95.21	94.92	95.32	95.80	95.16	95.08
		Length	12.069	6765.9	2.1e10	9.2813	1897.6	4.18e8
	2	Coverage	95.30	95.23	95.52	95.80	94.95	95.63
		Length	90.337	2.02e5	2.5e11	69.396	2.70e4	2.79e9
50	0	Coverage	94.99	95.00	95.33	95.37	94.92	95.45
		Length	0.6237	27.362	6647.4	0.5691	20.024	2964.5
	1	Coverage	95.28	95.07	94.95	95.47	95.32	94.88
		Length	4.6485	203.27	4.98e4	4.2399	148.77	2.21e4
	2	Coverage	95.03	95.05	94.51	95.49	94.85	94.71
		Length	34.284	1460.6	4.16e5	31.267	1070.3	1.82e5

Table 70: 95% Credibility Intervals for $Var(X)$ with Zero values included using the Independence Jeffreys Prior – Setting 2

$\delta = 0.2$			Equal-tail Interval			HPD Interval		
n	μ		$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$	$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$
10	0	Coverage	95.12	95.07	95.06	95.04	94.75	95.32
		Length	9.9e15	6.4e45	1e105	6.83e5	6.9e26	6.8e64
	1	Coverage	95.46	95.05	95.41	95.05	94.65	95.36
		Length	3.3e7	1e134	6e137	268.72	5.2e65	1.2e60
	2	Coverage	95.60	95.29	94.77	94.81	95.06	95.36
		Length	1e182	9.9e52	1e291	1.3e49	6.1e30	6e129
15	0	Coverage	94.89	94.91	95.16	95.07	94.73	95.13
		Length	4.0079	3.27e6	6.6e23	2.3347	4.08e4	3.0e17
	1	Coverage	95.10	95.19	94.86	95.21	95.29	95.19
		Length	32.146	1.53e7	8.6e25	17.190	1.57e5	1.1e20
	2	Coverage	95.39	95.08	95.13	95.29	94.94	95.10
		Length	238.12	1.3e11	8.5e26	129.84	2.17e8	1.0e20
20	0	Coverage	95.57	94.91	95.10	95.67	94.64	95.35
		Length	1.9881	2729.8	1.5e14	1.4438	469.51	1.6e11
	1	Coverage	95.24	94.91	95.13	94.92	95.17	94.86
		Length	14.248	3.87e4	6.4e13	10.462	5093.6	1.4e11
	2	Coverage	94.80	94.67	95.05	95.00	94.79	95.10
		Length	104.79	3.03e5	1.3e18	77.277	3.61e4	1.7e14
50	0	Coverage	95.22	94.74	95.12	95.50	95.09	95.08
		Length	0.6706	30.764	1.15e4	0.6082	21.597	4463.1
	1	Coverage	95.17	94.97	95.06	95.47	95.24	95.28
		Length	4.9141	227.52	8.18e4	4.4581	159.78	3.22e4
	2	Coverage	95.15	95.39	95.32	95.25	95.25	95.45
		Length	36.603	1640.7	6.32e5	33.195	1152.3	2.46e5

Table 71: 95% Credibility Intervals for $Var(X)$ with Zero values included using the Independence Jeffreys Prior – Setting 3

$\delta = 0.3$			Equal-tail Interval			HPD Interval		
n	μ		$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$	$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$
15	0	**Coverage**	95.30	94.94	94.73	95.05	94.67	95.04
		Length	6.7e14	Inf	1e132	2.64e4	1.6e223	1.4e63
	1	**Coverage**	95.24	94.70	94.91	94.86	95.22	95.31
		Length	4.01e4	2e191	1.0e75	54.324	1.1e47	4.6e45
	2	**Coverage**	95.24	95.02	95.09	94.49	95.37	94.96
		Length	3.88e6	3.3e25	3.4e39	926.96	2.8e16	1.2e29
20	0	**Coverage**	95.22	95.04	94.87	94.56	95.08	95.19
		Length	2.4084	6.41e4	2.7e21	1.6433	3010.5	5.4e14
	1	**Coverage**	95.52	95.11	94.95	94.73	95.09	95.22
		Length	17.271	1.15e7	1.5e19	11.875	1.05e5	1.4e14
	2	**Coverage**	95.39	95.18	94.79	95.14	95.13	95.11
		Length	133.20	2.21e7	1.8e27	89.537	3.27e5	1.2e19
50	0	**Coverage**	95.36	95.24	95.09	95.50	95.19	95.04
		Length	0.7194	36.287	3.76e4	0.6443	24.004	1.03e4
	1	**Coverage**	95.38	95.22	94.88	95.19	94.99	95.12
		Length	5.2838	269.33	2.89e5	4.7321	177.32	7.74e4
	2	**Coverage**	95.22	94.69	95.03	95.31	94.88	95.13
		Length	39.303	2029.4	2.39e6	35.201	1337.2	6.36e5

Table 72: 95% Credibility Intervals for $Var(X)$ with Zero values included using the Jeffreys Rule Prior – Setting 1

$\delta = 0.1$			Equal-tail Interval			HPD Interval		
n	μ		$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$	$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$
10	0	Coverage	94.35	93.53	93.50	92.20	91.60	91.20
		Length	10.157	3.6e10	5.1e39	3.1912	7.95e7	7.5e27
	1	Coverage	94.18	93.93	93.84	92.47	91.98	91.75
		Length	44.682	2.5e14	8.2e22	21.453	4.22e9	1.2e17
	2	Coverage	93.97	93.98	93.47	92.43	91.97	90.84
		Length	296.05	4.12e10	5.1e29	152.70	6.78e7	3.8e22
15	0	Coverage	94.26	94.43	94.26	93.46	92.72	92.63
		Length	2.1081	3246.5	1.2e17	1.4935	572.16	4.7e13
	1	Coverage	94.62	94.27	94.46	93.46	92.31	93.04
		Length	15.047	3.72e4	4.3e14	10.742	4778.1	9.7e10
	2	Coverage	94.50	94.23	94.11	93.39	92.44	92.38
		Length	113.04	2.52e5	6.3e14	80.677	3.78e4	1.7e12
20	0	Coverage	94.72	94.52	94.32	93.82	93.00	92.92
		Length	1.3685	358.68	1.57e8	1.0811	128.53	6.89e6
	1	Coverage	94.93	94.65	94.17	93.81	92.92	92.84
		Length	10.131	2965.3	1.56e8	7.9939	998.52	9.69e6
	2	Coverage	94.50	94.38	94.70	93.45	93.13	93.10
		Length	75.407	1.95e4	9.28e9	59.404	7074.4	3.74e8
50	0	Coverage	95.28	94.93	94.74	94.92	94.20	93.77
		Length	0.5968	24.152	4975.8	0.5461	17.910	2335.9
	1	Coverage	94.76	94.42	94.65	94.36	93.65	93.68
		Length	4.4163	182.89	4.98e4	4.0421	135.16	1.87e4
	2	Coverage	95.10	94.83	94.70	94.69	93.99	93.75
		Length	32.456	1337.3	2.67e5	29.715	990.61	1.25e5

Table 73: 95% Credibility Intervals for $Var(X)$ with Zero values included using the Jeffreys Rule Prior – Setting 2

$\delta = 0.2$			Equal-tail Interval			HPD Interval		
n	μ		$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$	$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$
10	0	**Coverage**	93.48	93.11	93.40	91.16	90.76	91.28
		Length	50.229	1.5e18	1.1e43	4.6232	1.5e11	4.4e28
	1	**Coverage**	93.67	93.69	93.42	91.09	91.53	91.47
		Length	7.48e4	3.8e21	3.7e46	33.528	4.6e11	2.2e29
	2	**Coverage**	94.02	93.47	93.50	91.85	91.67	91.52
		Length	7.40e4	1.4e25	4.1e41	367.61	5.2e15	3.4e28
15	0	**Coverage**	94.46	93.94	93.97	92.83	92.42	91.98
		Length	2.5312	2.10e4	6.0e15	1.7007	1787.8	3.8e12
	1	**Coverage**	93.96	93.83	94.09	92.48	92.20	92.05
		Length	19.235	2.7e10	1.7e17	12.524	4.02e7	2.2e13
	2	**Coverage**	94.37	94.25	94.11	92.45	92.33	92.02
		Length	133.85	2.52e7	1.7e20	90.152	3.46e5	2.1e14
20	0	**Coverage**	94.72	94.53	94.54	93.00	92.88	92.98
		Length	1.5595	1328.8	1.0e11	1.1967	277.71	8.13e8
	1	**Coverage**	94.24	94.22	94.34	93.27	92.86	92.99
		Length	11.415	8321.6	1.9e11	8.7722	1949.0	2.39e9
	2	**Coverage**	94.15	94.24	94.09	92.76	92.97	92.47
		Length	83.826	8.69e4	7.4e11	64.478	1.61e4	6.24e9
50	0	**Coverage**	95.05	94.88	95.12	94.63	93.95	94.06
		Length	0.6314	26.433	8758.9	0.5754	18.887	3418.2
	1	**Coverage**	94.57	95.00	94.51	94.40	93.93	93.59
		Length	4.7305	196.88	2.50e5	4.3058	140.68	1.25e5
	2	**Coverage**	94.65	94.90	94.61	94.07	93.93	93.68
		Length	34.668	1428.0	4.46e5	31.576	1021.5	1.84e5

Table 74: 95% Credibility Intervals for $Var(X)$ with Zero values included using the Jeffreys Rule Prior – Setting 3

$\delta = 0.3$			Equal-tail Interval			HPD Interval		
n	μ		$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$	$\sigma^2 = 0.25$	$\sigma^2 = 1$	$\sigma^2 = 2.25$
15	0	Coverage	93.82	93.84	93.84	91.72	92.37	92.28
		Length	3.7986	6.2e12	3.8e27	2.0463	1.75e9	3.4e20
	1	Coverage	93.99	93.37	93.68	91.90	91.94	91.87
		Length	49.165	8.90e8	1.9e23	15.620	1.57e6	1.1e16
	2	Coverage	94.11	93.93	93.47	92.24	92.31	91.66
		Length	184.47	2.6e53	8.4e26	108.84	6.2e34	1.0e20
20	0	Coverage	94.15	94.29	94.12	92.94	92.78	92.69
		Length	1.7678	5833.8	2.7e13	1.3041	688.14	4.1e10
	1	Coverage	94.43	94.80	94.10	92.85	93.26	92.67
		Length	13.168	1.23e5	2.6e13	9.6657	4652.3	3.5e10
	2	Coverage	94.52	93.86	94.34	92.83	92.83	92.74
		Length	100.49	5.00e6	3.6e17	72.478	1.80e5	1.2e13
50	0	Coverage	94.53	94.67	94.76	94.07	93.88	94.03
		Length	0.6673	30.933	1.97e4	0.6015	20.863	6224.7
	1	Coverage	94.18	94.79	94.60	93.40	94.18	93.63
		Length	4.9536	229.04	1.54e5	4.4631	154.89	5.03e4
	2	Coverage	94.75	94.68	94.84	94.49	94.35	94.01
		Length	36.775	1738.8	9.99e5	33.132	1171.6	3.28e5

From the above results it is clear that with both priors the HPD intervals are a considerable improvement, particularly in terms of interval length, on the standard equal-tailed intervals. Thus, the flexibility of a Bayesian approach to handling these situations is evident.

The results do not clearly define which prior distribution is better suited. In small sample size situations with small σ^2 values the Independence Jeffreys prior, while offering better coverage, does so at the expense of wide interval widths. As the values of σ^2 increases this tendency reverses with regards to the interval width, with the Independence Jeffreys prior still offering the better coverage. As the proportion of zero-valued observations increases the width of the credibility intervals also increase. Thus, with respect to coverage, the Independence Jeffreys prior seems better suited to the situation and with regards to interval width the prominence is evident, except in small sample size settings.

Appendix to Chapter 5

Derivation of the Probability-Matching Prior Distribution for the Variance of a Lognormal Distribution

Define:

$$t(\boldsymbol{\theta}) = exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}$$

Now,

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} = 2exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma^2} = exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\} + exp(2\mu + \sigma^2)exp(\sigma^2)$$

$$\nabla_t'(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} & \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma^2} \end{bmatrix} = exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}\begin{bmatrix} 2 & \frac{2exp(\sigma^2) - 1}{exp(\sigma^2) - 1} \end{bmatrix}$$

Using the results for the Fisher Information Matrix obtained in the appendix to Chapter 3,

$$\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta}) = \nabla_t'(\boldsymbol{\theta})\sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma^2 \end{bmatrix}$$

$$= \sigma^2 exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}\begin{bmatrix} 2 & \frac{2\sigma^2\{2exp(\sigma^2) - 1\}}{exp(\sigma^2) - 1} \end{bmatrix}$$

$$\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t\,(\boldsymbol{\theta})$$

$$= \sigma^2 exp(4\mu + 2\sigma^2)\{exp(\sigma^2) - 1\}^2\left[4 + \frac{2\sigma^2\{2exp(\sigma^2) - 1\}^2}{\{exp(\sigma^2) - 1\}^2}\right]$$

$$\left\{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t\,(\boldsymbol{\theta})\right\}^{1/2}$$

$$= \sigma exp(2\mu + \sigma^2)\{exp(\sigma^2) - 1\}\;\; 2\left[1 + \frac{\sigma^2\{2exp(\sigma^2) - 1\}^2}{2\{exp(\sigma^2) - 1\}^2}\right]^{1/2}$$

$$\frac{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})}{\left\{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t\,(\boldsymbol{\theta})\right\}^{1/2}} = \frac{2\sigma}{2\left[1+\ \dfrac{\sigma^2\{2exp(\sigma^2)-1\}^2}{2\{exp(\sigma^2)-1\}^2}\right]^{1/2}}\left[1 \quad \frac{\sigma^2\{2exp(\sigma^2)-1\}}{exp(\sigma^2)-1}\right]$$

Therefore,

$$\varphi(\boldsymbol{\theta}) = \frac{2\sigma}{2\left[1+\ \dfrac{\sigma^2\{2exp(\sigma^2)-1\}^2}{2\{exp(\sigma^2)-1\}^2}\right]^{1/2}} \times \frac{\sigma^2\{2exp(\sigma^2)-1\}}{exp(\sigma^2)-1}$$

$$= \frac{\sigma^3\{2exp(\sigma^2)-1\}}{\left[\dfrac{2\{exp(\sigma^2)-1\}^2+\ \sigma^2\{2exp(\sigma^2)-1\}^2}{2\{exp(\sigma^2)-1\}^2}\right]^{1/2}}\,\frac{1}{exp(\sigma^2)-1}$$

$$= \frac{\sigma^3\{2exp(\sigma^2)-1\}\sqrt{2}}{\{2\{exp(\sigma^2)-1\}^2+\ \sigma^2\{2exp(\sigma^2)-1\}^2\}^{1/2}}$$

Therefore,

$$p_m(\boldsymbol{\theta}) \ \propto\ \sigma^{-3}\left\{\frac{2(exp(\sigma^2)-1)^2}{\{2exp(\sigma^2)-1\}^2}+\ \sigma^2\right\}^{1/2}$$

Derivation of the Reference Prior Distribution for the Variance of a Lognormal Distribution

Define:

$$t(\boldsymbol{\theta}) =\ exp(2\mu+\ \sigma^2)\{exp(\sigma^2)-1\}$$

$$\therefore\ ln\big(t(\boldsymbol{\theta})\big) =\ (2\mu+\ \sigma^2)+ln\{exp(\sigma^2)-1\}$$

$$\therefore\ \mu = \frac{1}{2}ln\big(t(\boldsymbol{\theta})\big) - \frac{1}{2}\sigma^2 - \frac{1}{2}ln\{exp(\sigma^2)-1\}$$

$$\frac{\partial\mu}{\partial t(\boldsymbol{\theta})} = \frac{1}{2t(\boldsymbol{\theta})}$$

$$\frac{\partial\mu}{\partial\sigma^2} = \ -\frac{1}{2} - \frac{1}{2}\frac{exp(\sigma^2)}{exp(\sigma^2)-1} = -\frac{1}{2}\left(1+\frac{exp(\sigma^2)}{exp(\sigma^2)-1}\right) = \ -\frac{1}{2}\left(\frac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right)$$

$$A = \frac{\partial(\mu, \sigma^2)}{\partial\{t(\boldsymbol{\theta}), \sigma^2\}} = \begin{bmatrix} \dfrac{\partial \mu}{\partial t(\boldsymbol{\theta})} & \dfrac{\partial \mu}{\partial \sigma^2} \\ \dfrac{\partial \sigma^2}{\partial t(\boldsymbol{\theta})} & \dfrac{\partial \sigma^2}{\partial \sigma^2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2t(\boldsymbol{\theta})} & -\dfrac{1}{2}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right) \\ 0 & 1 \end{bmatrix}$$

$$F(t(\boldsymbol{\theta}), \sigma^2) = A'F(\mu, \sigma^2)\,A = \begin{bmatrix} \dfrac{1}{2t(\boldsymbol{\theta})} & 0 \\ -\dfrac{1}{2}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right) & 1 \end{bmatrix} \frac{1}{\sigma^2}\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{2\sigma^2} \end{bmatrix} A$$

$$= \frac{1}{\sigma^2}\begin{bmatrix} \dfrac{1}{2t(\boldsymbol{\theta})} & 0 \\ -\dfrac{1}{2}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right) & \dfrac{1}{2\sigma^2} \end{bmatrix} \begin{bmatrix} \dfrac{1}{2t(\boldsymbol{\theta})} & -\dfrac{1}{2}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right) \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sigma^2}\begin{bmatrix} \dfrac{1}{4t^2(\boldsymbol{\theta})} & -\dfrac{1}{4}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right) \\ -\dfrac{1}{4}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right) & -\dfrac{1}{4}\left(\dfrac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right)^2 + \dfrac{1}{2\sigma^2} \end{bmatrix}$$

$$\therefore p_R(t(\boldsymbol{\theta}), \sigma^2) \propto \frac{1}{t(\boldsymbol{\theta})}\frac{1}{\sigma}\sqrt{\frac{1}{4}\left(\frac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right)^2 + \frac{1}{2\sigma^2}}$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} = 2t(\boldsymbol{\theta})\frac{1}{t(\boldsymbol{\theta})}\frac{1}{\sigma}\sqrt{\frac{1}{4}\left(\frac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right)^2 + \frac{1}{2\sigma^2}}$$

$$\therefore p_R(t(\boldsymbol{\theta}), \sigma^2) \propto \frac{1}{\sigma}\sqrt{\frac{1}{4}\left(\frac{2exp(\sigma^2)-1}{exp(\sigma^2)-1}\right)^2 + \frac{1}{2\sigma^2}}$$

CHAPTER 6

Bivariate Lognormal Distribution

Introduction

As mentioned before, lognormally distributed data presents itself in a number of scientific fields. According to Limpert et al. (2001), the distribution may be used to approximate right skewed data that arises in a wide variety of scientific settings. Particularly in the area of health costs the log distribution has been extensively used by other authors and numerous statistical methods have been developed. In previous chapters the application to health costs has been firmly studied. However, in each case the setting was that of either a single population mean or the ratio of two means from independent samples. This chapter proposes a slightly different setting, that of dependent samples from two lognormal distributions. However, in this chapter the possibility of zero values is not considered.

As mentioned by Bebu and Mathew (2008) the bivariate (and multivariate) lognormal distribution is also particularly suited for the study of the size distribution of aerosol particles and airborne fibres. The distribution of asbestos fibre sizes generated by grinding bulk material or mechanically releasing particles into the air often results in a two-dimensional (bivariate) lognormal distribution. For more details see Schneider and Holst (1983) and Ramachandran, Werner and Vincent (1996).

Hawkins (2002) on the other hand used the bivariate lognormal distribution on a quantitative assay problem dealing with 56 assay pairs for cyclosporin from blood samples of organ transplant recipients obtained by a standard method and an alternative radio-immunoassay method.

In this chapter the Bayesian model that has been developed and discussed in previous chapters will be applied to the bivariate lognormal distribution. The purpose of this study is therefore to develop Bayesian procedures for computing confidence (credibility) intervals for the ratio of the means of the bivariate lognormal distribution. The same procedures can also be used to obtain credibility intervals for the ratio of variances.

The choice of prior distributions is the factor of interest. Specifically the choice of different prior distributions in different parameters setting and the appropriateness of each is of primary importance.

The credibility intervals will be compared with the generalized confidence intervals approach (GCI) used by Bebu and Mathew (2008) and the "method of variance estimates recovery" (MOVER) proposed by Zou et al (2009a). Berger and Sun (2006, 2008) also proved that the GCI approach is a Bayesian procedure if the right-Haar prior is used. The MOVER was designed in order to apply to a general scenario and also to provide adequate coverage rates in estimation procedures relating to lognormally distributed data. The advantage of the MOVER method is that it can easily be applied to many different settings with little more than a basic knowledge of introductory statistical texts.

In the next section we begin with a formulation of the model and a specification of all parameters and distributions of interest. In further sections we compare the performance of the method for different prior distributions by conducting a simulation study to assess some quantities of the proposed credibility intervals in pre-defined finite sample sizes (the same as those used by Bebu and Mathew (2008)).

6.1 Notation and Description of the Setting

Let $[X_1 \quad X_2]'$ follow a bivariate lognormal distribution so that $[Y_1 \quad Y_2]' = [lnX_1 \quad lnX_2]'$ follows a bivariate normal distribution with mean parameters $\boldsymbol{\mu} = [\mu_1, \mu_2]'$ and covariate matrix $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, where ρ is the correlation between X_1 and X_2.

Thus

$$E(Y_i) = exp\left(\mu_i + \frac{1}{2}\sigma_i^2\right), \qquad i = 1,2$$

Bayesian confidence intervals for the parameters

$$\tilde{\theta} = (\mu_1 - \mu_2) + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)$$

will be constructed in order to compare the ratio of the lognormal means. Note that θ is the log ratio of the population means.

For the ratio of the variances Bayesian confidence intervals for

$$\tilde{\delta} = \frac{Var(Y_1)}{Var(Y_2)} = exp(2\tilde{\theta})\left\{\frac{exp(\sigma_1^2) - 1}{exp(\sigma_2^2) - 1}\right\}$$

will be obtained.

Let $[X_{1k}, X_{2k}]', k = 1,2,\dots,n$ denote a random sample from the bivariate lognormal distribution and let $\boldsymbol{Y}_k = [Y_{1k}, Y_{2k}]' = [lnX_{1k}, lnX_{2k}]'$. The sufficient statistics (for $n \geq 3$) are $\bar{\boldsymbol{Y}} = [\bar{Y}_1, \bar{Y}_2]'$ and $S = \sum_{k=1}^{n}(\boldsymbol{Y}_k - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_k - \bar{\boldsymbol{Y}})' = \begin{bmatrix} s_{11} & r\sqrt{s_{11}s_{22}} \\ r\sqrt{s_{11}s_{22}} & s_{22} \end{bmatrix}$

where

$$\bar{Y}_i = \frac{1}{n}\sum_{k=1}^{n} Y_{ik},$$

$$s_{ij} = \sum_{k=1}^{n}(Y_{ik} - \bar{Y}_i)(Y_{jk} - \bar{Y}_j)' \text{ and}$$

$$r = \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}}$$

Also

$$[\bar{Y}_1, \bar{Y}_2]' \sim N\left(\boldsymbol{\mu}, \frac{1}{n}\Sigma\right)$$

$$S \sim W_2(\Sigma, \mathrm{n} - 1)$$

the bivariate Wishart distribution with scale matrix Σ and degrees of freedom $n - 1$.

Since $S \sim W_2(\Sigma, \mathrm{n} - 1)$ we shall use the following properties of the Wishart distribution (see Berger and Sun (2006 & 2008) and Bebu and Mathew (2008)) to construct the generalized pivot statistics:

$$T_1 = \frac{s_{11}}{\sigma_1^2} \sim \chi_{n-1}^2$$

(6.1)

$$T_2 = \frac{s_{22}(1-r^2)}{\sigma_2^2(1-\rho^2)} \sim \chi_{n-2}^2$$

(6.2)

$$T_3 = \left[\frac{s_{11}}{\sigma_2^2(1-\rho^2)}\right]^{1/2} \left[\frac{r\sqrt{s_{22}}}{\sqrt{s_{11}}} - \frac{\rho\sigma_2}{\sigma_1}\right] = Z \sim N(0,1)$$

(6.3)

Here Z is a standard normal random variable and χ_{n-1}^2 and χ_{n-2}^2 are chi-squared random variables with the indicated degrees of freedom.

From (6.1) to (6.3) it follows that

$$\frac{s_{11}}{\chi_{n-1}^{2*}} = \sigma_1^{2*}$$

(6.4)

$$\sigma_2^{2*} = s_{22}(1-r^2)\left\{\frac{1}{\chi_{n-2}^{2*}} + \frac{1}{\chi_{n-1}^{2*}}\left(\frac{Z^*}{\sqrt{\chi_{n-2}^{2*}}} - \frac{r}{\sqrt{1-r^2}}\right)^2\right\}$$

(6.5)

$$\rho = \varphi(\tilde{Y}^*)$$

(6.6)

where

$$\varphi(\tilde{Y}^*) = \frac{\tilde{Y}^*}{\sqrt{1+\tilde{Y}^{*2}}}$$

and

$$\tilde{Y}^* = \frac{-Z^*}{\sqrt{\chi^{2*}_{n-1}}} + \frac{\sqrt{\chi^{2*}_{n-2}}}{\sqrt{\chi^{2*}_{n-1}}} \frac{r}{\sqrt{1 - r^2}} \tag{6.7}$$

The asterisk is used to represent a random realized observation from the implied distribution. Equations (6.4) to (6.6) are similar to R_{22}, R_{12} and R_{11} defined on page 2687 of Bebu and Mathew (2008) except that ρ is simulated in (6.4) to (6.6), while Bebu and Mathew simulated the covariance σ_{12}.

According to Berger and Sun (2006, 2008) there is no reference in the literature to the fact that the fiducial distributions of ρ (equations (6.6) and (6.7)) and derived by Fisher (1930) are exact frequentist matching (proved in Theorem 5 of their paper). They also mentioned that standard statistical software utilises various approximations to arrive at frequentist confidence sets for ρ, missing the fact that a simple exact confidence set exists.

The question is whether the fiducial distributions ((6.6) and (6.7)) can be derived in a Bayesian fashion. Berger and Sun (2006, 2008) proved that the answer is yes.

6.2 Prior and Posterior Distributions

Berger and Sun (2006 & 2008) considered the following important class of prior densities (a subclass of the generalized Wishart distribution of Brown et al (1994)):

$$\pi_{ab}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \propto \frac{1}{\sigma_1^{3-a}\sigma_2^{2-b}(1-\rho^2)^{2-b/2}} \tag{6.8}$$

They also mentioned the use of certain prior densities depending on what the parameter of interest is.

Let us examine a few special cases of this general prior distribution.

If we take $a = 1$ and $b = 0$ then we have the following prior distribution

$$\pi_{10} = \pi_J(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \propto \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)^2} \tag{6.9}$$

which is Jeffreys' dependence prior.

Then if $a = 1$ and $b = 1$ then prior distribution is

$$\pi_{11} = \pi_{RO}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \propto \frac{1}{\sigma_1^2\sigma_2\,(1-\rho^2)^{3/2}} \tag{6.10}$$

The independence Jeffreys' prior

$$\pi_{21} = \pi_{IJ}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \propto \frac{1}{\sigma_1 \ \sigma_2 \ (1 - \rho^2)^{3/2}} \tag{6.11}$$

follows from using a constant prior for the means and then the Jeffreys' prior for the covariance matrix with means given.

For the general prior distribution (6.8) Berger and Sun (2006, 2008) proved that the constructive posterior distribution of $(\sigma_1^2, \sigma_2^2, \rho)$, given the data, can be expressed as

$$\frac{s_{11}}{\chi_{n-a}^{2*}} = \sigma_1^{2*} \tag{6.12}$$

$$\sigma_2^{2*} = s_{22}(1 - r^2)\left\{\frac{1}{\chi_{n-b}^{2*}} + \frac{1}{\chi_{n-a}^{2*}}\left(\frac{Z^*}{\sqrt{\chi_{n-b}^{2*}}} - \frac{r}{\sqrt{1 - r^2}}\right)^2\right\} \tag{6.13}$$

$$\rho^* = \varphi(\tilde{Y}^*) \tag{6.14}$$

where, as before,

$$\varphi(\tilde{Y}^*) = \frac{\tilde{Y}^*}{\sqrt{1 + \tilde{Y}^{*2}}}$$

and

$$\tilde{Y}^* = \frac{-Z^*}{\sqrt{\chi_{n-a}^{2*}}} + \frac{\sqrt{\chi_{n-b}^{2*}}}{\sqrt{\chi_{n-a}^{2*}}} \frac{r}{\sqrt{1 - r^2}} \tag{6.15}$$

Furthermore

$$\Sigma^* = \begin{bmatrix} \sigma_1^{2*} & \rho\sigma_1{}^*\sigma_2{}^* \\ \rho\sigma_1{}^*\sigma_2{}^* & \sigma_2^{2*} \end{bmatrix}$$

(6.16)

Once Σ^* is simulated we know that given the data

$$\boldsymbol{\mu}^* \sim N\left(\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}, \frac{1}{n}\Sigma^*\right)$$

(6.17)

As mentioned, the asterisk represents a random draw from the implied distribution. Thus $\boldsymbol{\mu}^*$ will represent a random draw from its posterior distribution, Z^* represents a draw from the standard normal distribution and χ^{2*}_{n-a} and χ^{2*}_{n-b} will be independently drawn random variables from the chi-squared distribution with the indicated degrees of freedom.

It is clear that if $a = 1$ and $b = 2$ the posterior distributions defined in equations (6.12) to (6.13) will be the same as the fiducial distributions given in equations (6.4) to (6.7) and on page 2688 of Bebu and Mathew (2008).

The resulting prior if $a = 1$ and $b = 2$ is

$$\pi_H = \pi_{12}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \propto \frac{1}{\sigma_1^2(1 - \rho^2)}$$

(6.18)

the right-Haar prior.

From equation (6.17) it follows that

$$\tilde{\theta}|\Sigma^*, \text{data} \sim \mathrm{N}\left\{(\bar{x}_1 - \bar{x}_2) + \frac{1}{2}(\sigma_1^{2*} - \sigma_2^{2*}), \frac{1}{\mathrm{n}}(\sigma_1^{2*} + \sigma_2^{2*} - 2\rho\sigma_1{}^*\sigma_2{}^*)\right\}$$

(6.19)

where, as before

$$\tilde{\theta} = (\mu_1 - \mu_2) + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)$$

In our simulation experiment of $\tilde{\theta}$, using (6.19), the following nine priors will be compared with respect to coverage probabilities and interval lengths:

$$\pi_{21} = \pi_{IJ} \propto \sigma_1^{-1}\sigma_2^{-1}(1 - \rho^2)^{-3/2}$$

$$\pi_{10} \propto \sigma_1^{-2}\sigma_2^{-2}(1 - \rho^2)^{-2}$$

$$\pi_{12} \propto \sigma_1^{-2}(1 - \rho^2)^{-1}$$

$$\pi_{11} = \pi_{RO} \propto \sigma_1^{-2}\sigma_2^{-1}(1 - \rho^2)^{-2}$$

$$\pi_{MS} \propto \sigma_1^{-1}\sigma_2^{-1}(1 - \rho^2)^{-1/2}$$

$$\pi_{RP} \propto \sigma_1^{-1}\sigma_2^{-1}(1 - \rho^2)^{-1}$$

$$\pi_S \propto \sigma_1^{-1}\sigma_2^{-1}$$

$$\pi_{R\sigma} \propto \sqrt{1 + \rho^2}\sigma_1^{-1}\sigma_2^{-1}(1 - \rho^2)^{-1}$$

$$\tilde{\pi}_{R\sigma} \propto \sigma_1^{-1}\sigma_2^{-1}(1 - \rho^2)^{-1}(2 - \rho^2)^{-1/2}$$

According to Berger and Sun the independence Jeffreys prior, $\pi_{21} = \pi_{IJ}$, is virtually never optimal, while the dependence Jeffreys prior, π_{10}, is very often optimal in contradiction to the common perception that independence Jeffreys prior is better than the Jeffreys prior. π_{11} is useful for certain parameters. The right-Haar prior, $\pi_{12} = \pi_H$, is exact Probability-Matching for at least seven parameters including ρ and $|\Sigma|$. Berger and Sun (2006 & 2008) recommend π_{RP} as a "general purpose" objective prior. The prior $\pi_{R\sigma}$ will be suggested for use with inferences concerning σ_{12}. The motivation that is often given for π_S is that it is "standard" to use σ_i^{-1} as the prior for the standard deviation while $-1 \leq \rho \leq 1$ is on a bounded set and so one can use a constant prior in ρ. A related prior is π_{MS} whose power of $(1-\rho^2)$ is between π_S and π_{RP}. For further details see Berger and Sun (2006, 2008).

6.3 Marginal Posterior Distributions

For the class of prior densities $\pi_{ab}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ independent samples can easily be obtained from the marginal posterior $\pi \quad (\sigma_1, \sigma_2, \rho | data)$ (see equations (6.12) – (6.15)).

In the case of priors π_{RP}, $\pi_{R\sigma}$, $\tilde{\pi}_{R\sigma}$, π_S and π_{MS} simulations can also easily be obtained by the following acceptance-rejection algorithm (Berger and Sun (2006, 2008)):

1. Simulation step: Generate $(\sigma_1, \sigma_2, \rho)$ from the independence Jeffreys posterior $\pi_{IJ}(\sigma_1, \sigma_2, \rho | data)$ and independently sample $U \sim Uniform(0,1)$.
2. Rejection step: Suppose $M = sup_{(\sigma_1, \sigma_2, \rho)} \frac{\pi \quad (\sigma_1, \sigma_2, \rho)}{\pi_{IJ}(\sigma_1, \sigma_2, \rho)} < \infty$. If $U \leq \pi \quad (\sigma_1, \sigma_2, \rho) / [M \pi_{IJ}(\sigma_1, \sigma_2, \rho)]$, report $(\sigma_1, \sigma_2, \rho)$ otherwise go back to the simulation step.

In Table 76 (Table 5 in Berger and Sun) it is shown that the rejection algorithm is quite efficient for sampling these posteriors.

Table 75: Ratio $\dfrac{\pi}{\pi_{IJ}}$ upper bound M, rejection step and acceptance probability for ρ=0.8,0.95 and 0.99 for π_{RP}, $\pi_{R\sigma}$, $\tilde{\pi}_{R\sigma}$, π_S and π_{MS}

Prior	Ratio	Bound M	Rejection Step	Acceptance Probability		
				$\rho = 0.8$	$\rho = 0.95$	$\rho = 0.99$
π_{RP}	$\sqrt{1-\rho^2}$	1	$u \leq \sqrt{1-\rho^2}$	0.6000	0.3122	0.1410
$\pi_{R\sigma}$	$\sqrt{1-\rho^4}$	1	$u \leq \sqrt{1-\rho^4}$	0.7684	0.4307	0.1985
$\tilde{\pi}_{R\sigma}$	$\sqrt{\dfrac{1-\rho^2}{2-\rho^2}}$	$\dfrac{1}{\sqrt{2}}$	$u \leq \sqrt{\dfrac{1-\rho^2}{2-\rho^2}}$	0.7276	0.4215	0.1975
π_S	$(1-\rho^2)^{\frac{3}{2}}$	1	$u \leq (1-\rho^2)^{\frac{3}{2}}$	0.2160	0.0304	0.0028
π_{MS}	$1-\rho^2$	1	$u \leq 1-\rho^2$	0.3600	0.0975	0.0199

6.4 Method of Variance Estimates Recovery (MOVER)

Instead of adapting a simulation approach for making inferences on $\tilde{\theta} = (\mu_1 - \mu_2) + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)$, Zou, Taliban and Huo (2009a) proposed procedures involving the so-called "method of variance estimates recovery" (MOVER). The MOVER method was designed in order to apply to a general scenario and also to provide adequate coverage rates in estimation procedures relating to lognormally distributed data. As mentioned, the advantage of the MOVER is therefore that it is easily applicable to many different settings with little more than a basic knowledge of introductory statistical texts.

The $(1-\alpha)100\%$ confidence limits for $\mu_i + \frac{1}{2}\sigma_i^2$, $i = 1,2$, using the MOVER as given by Zou et al (2009a) on page 3758 are:

$$\tilde{L}_i = \hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2} - \sqrt{Z_{\alpha/2}^2 \frac{\hat{\sigma}_i^2}{n_i} \left\{ \frac{\hat{\sigma}_i^2}{2} \left(1 - \frac{\nu_i}{\chi_{1-\alpha/2,\nu_i}^2} \right) \right\}^2}$$

$$\tilde{U}_i = \hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2} + \sqrt{Z_{\alpha/2}^2 \frac{\hat{\sigma}_i^2}{n_i} \left\{ \frac{\hat{\sigma}_i^2}{2} \left(\frac{\nu_i}{\chi_{\alpha/2,\nu_i}^2} - 1 \right) \right\}^2}$$

Underlying these limits is the well known result that the $(1-\alpha)100\%$ confidence interval for σ_i^2 is given by $\left[{\nu_i s_i^2}/{\chi_{1-\alpha/2,\nu_i}^2} ; {\nu_i s_i^2}/{\chi_{\alpha/2,\nu_i}^2} \right]$ where $\chi_{\alpha/2,\nu_i}^2$ is the ${\alpha}/{2}$th percentile from the chi-squared distribution with ν degrees of freedom where $\nu_i = n - 1$. $Z_{\alpha/2}$ is the upper ${\alpha}/{2}\,th$ quantile of the standard normal distribution and $\hat{\mu}_i = \bar{x}_i$ and $\hat{\sigma}_i^2 = \frac{1}{\nu} s_{ii}$.

The $(1-\alpha)100\%$ confidence interval for $\tilde{\theta}$ is therefore given by:

$$L = \hat{\theta}_1 - \hat{\theta}_2 - \sqrt{(\hat{\theta}_1 - \tilde{L}_1)^2 + (\tilde{U}_2 - \hat{\theta}_2)^2 - 2r\,(\hat{\theta}_1 - \tilde{L}_1)(\tilde{U}_2 - \hat{\theta}_2)}$$

$$U = \hat{\theta}_1 - \hat{\theta}_2 + \sqrt{(\hat{\theta}_2 - \tilde{L}_2)^2 + (\tilde{U}_1 - \hat{\theta}_1)^2 - 2r\,(\hat{\theta}_2 - \tilde{L}_2)(\tilde{U}_1 - \hat{\theta}_1)}$$

where

$$\hat{\theta}_i = \hat{\mu}_i + \frac{1}{2}\hat{\sigma}_i^2, \qquad (i = 1,2)$$

and

$$r = \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}}$$

6.5 Comparison of Variances

The Bayesian simulation procedure described above can easily be used to obtain credibility intervals for any scalar function of $\boldsymbol{\mu}$ and Σ. For example, to compare the variances of the bivariate lognormal distribution we can construct credibility intervals for the ratio of the variances, say $\tilde{\delta}$, given by

$$\tilde{\delta} = \frac{Var(Y_1)}{Var(Y_2)} = exp\left(2\tilde{\theta}\right)\left\{\frac{exp(\sigma_1^2) - 1}{exp(\sigma_2^2) - 1}\right\}$$

(see Bebu and Mathew (2008)) for any of the nine prior distributions defined in Section 6.2.

As far as we know, it is not possible to use the MOVER to construct a confidence interval for $\tilde{\delta}$.

6.6 Results from Simulation Studies

Based on the methodology previously described a simulation study was performed using the same designs as those proposed by Bebu and Mathew (2008). The results from individual parameter setting are given in Tables 80 to 89. For the sake of brevity these have been presented in the appendix to this chapter. Aggregated results are given below (aggregation was performed excluding value of "Inf" for average interval length for the variance). The simulation study was performed to evaluate credibility intervals of both θ and δ. As mentioned previously, the MOVER could not be presented for the latter case, to the best of our knowledge. Thus a comparison of the MOVER will be made only for the ratio of means and not for the ratio of variances.

Table 76: Results by Sample Size

		$\tilde{\theta}$				$\tilde{\delta}$			
Method	n	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
MOVER	5	94.0678	26.9783						
π_{21}		94.2578	55.4800	98.1156	41.1144	93.5489	1.25E+159	91.7333	3.33E+276
π_{10}		93.0000	25.2844	95.5978	20.2322	92.0822	1.11E+214	92.8200	1.11E+150
π_{12}		94.7411	32.5488	97.3300	27.2381	93.9511	1.50E+135	96.3733	1.29E+84
π_{RO}		93.7956	26.9144	96.1956	22.0011	93.0289	1.11E+277	94.6044	3.33E+178
π_{MS}		95.7544	26.5189	97.3089	22.2567	95.6589	1.25E+71	95.7978	3.33E+209
π_{RP}		95.3578	33.1967	97.8133	26.7178	95.0000	1.11E+252	94.4178	3.33E+158
π_{S}		95.9333	24.6289	96.7889	21.0100	96.1856	2.22E+231	96.9478	6.67E+155
$\pi_{R\sigma}$		95.3678	34.5844	98.1344	27.7144	94.8211	2.22E+291	94.0944	1.11E+192
$\tilde{\pi}_{R\sigma}$		95.3511	34.5833	98.0511	27.6844	94.8844	1.11E+298	94.0978	3.33E+182
MOVER	10	94.6078	10.6871						
π_{21}		94.8511	12.1234	97.1533	10.8180	94.5156	3.33E+70	93.1311	6.67E+54
π_{10}		94.4533	9.8200	95.7022	8.8960	94.1178	2.22E+38	94.4156	1.11E+31
π_{12}		94.7467	10.5210	96.0222	9.5938	94.4511	2.22E+64	95.6844	1.11E+49
π_{RO}		94.5911	10.0373	95.8822	9.1216	94.3400	4.44E+44	94.8844	1.11E+34
π_{MS}		95.2367	9.9548	96.3678	9.0937	95.2633	2.22E+45	95.3856	5.56E+34
π_{RP}		94.9733	10.7874	96.7533	9.7636	94.9533	4.44E+56	94.5178	1.11E+44
π_{S}		95.2800	9.3731	95.6433	8.6132	95.5100	2.22E+51	96.1800	8.89E+39
$\pi_{R\sigma}$		95.1467	10.9351	96.9400	9.8853	95.0200	6.67E+47	94.2600	4.44E+36
$\tilde{\pi}_{R\sigma}$		95.0800	10.9809	96.9956	9.9244	94.9689	1.11E+60	94.2022	2.22E+47
MOVER	20	95.4089	6.0731						
π_{21}		95.0000	6.0153	96.2000	5.7101	94.8444	1.11E+16	93.9444	1.11E+13
π_{10}		94.6311	5.5602	95.5222	5.2979	94.5956	8.89E+15	94.5911	3.33E+13
π_{12}		95.0422	5.7659	95.6556	5.4981	94.8800	1.11E+21	95.5933	3.33E+17
π_{RO}		94.8267	5.6294	95.5511	5.3669	94.8244	1.11E+17	95.0222	3.33E+14
π_{MS}		94.9844	5.5978	95.5600	5.3469	95.0044	2.22E+19	95.1378	7.78E+15
π_{RP}		95.2911	5.7830	96.1667	5.5156	95.1333	5.56E+14	94.7689	1.11E+12
π_{S}		94.9122	5.4377	95.2022	5.1983	95.0322	1.00E+15	95.5322	1.11E+12
$\pi_{R\sigma}$		95.0689	5.8067	96.1689	5.5379	95.0333	1.11E+18	94.4267	5.56E+14
$\tilde{\pi}_{R\sigma}$		95.1511	5.8232	96.0289	5.5526	95.0444	3.33E+17	94.4600	2.22E+14

The results indicate that in the small sample size situation the MOVER performs well when comparing the ratio of the means. Efficiency in terms of average interval length does improve as the sample size increases. The same trend is evident for the coverage probability as well. In the small sample situation it is surprising that the Jeffreys prior achieves the desired coverage, but with an increased average interval length. The independence Jeffreys prior, on the other hand, results in a greater efficiency, but at a somewhat worse level of coverage in the confidence interval. As seen in previous chapters, the advantage of the Bayesian framework is the construction of HPD intervals.

For all choices of prior distributions both the coverage and the average interval length are improved with the HPD intervals. In most cases the tendency is towards overcoverage. It is with regards to the HPD intervals that the Bayesian framework is proven to have acceptable and increased performance. Adequate coverage is achieved with the largest reduction in efficiency.

With regards to coverage alone some choices of prior distributions performed admirably, given that the intention was not specifically for application to the ratio of means from two populations, but more towards the variance. It is clear that π_{RP}, $\pi_{R\sigma}$, $\tilde{\pi}_{R\sigma}$, π_S and π_{MS} all achieve better coverage than the Independence Jeffreys prior and in particular, π_S achieves this with an average interval length that is better than both the MOVER and all Jeffreys priors.

As the sample size increases the distinctions between the different prior distributions begins to decrease. We see better coverage probabilities and an improvement in the average interval length. The Jeffreys rule prior achieves the most accurate coverage, yet the interval length is only a slight improvement on the MOVER. The MOVER itself in the larger samples seems to be the most inefficient of methods. The best performing prior distributions are the independence Jeffreys prior and again the π_S prior. As with the small sample scenario HPD intervals improve both the coverage and the interval length, which further illustrates the usefulness of the Bayesian approach to the situation.

When considering the situation of the ratio of variances it is no longer possible to compare the results to the MOVER. So the usefulness of the Bayesian framework is even further enhanced not only in performance, but in the applicability to a wider variety of settings.

With regards to the small sample setting we see that both Jeffreys priors result in substantial undercoverage and average interval width. The performance does improve as the sample size increases, though the tendency to undercover still remains, even when considering the HPD intervals, which are an improvement on equal-tailed intervals. The priors π_S and π_{MS} seem to be the most practical choices, with the latter giving better coverage and the former resulting in a better average interval width. The interval lengths for the ratio of variances are large, but nevertheless the choice of these two as the natural choice of prior distribution for the ratio of variances is evident.

Table 77: Results by Correlation

Method	ρ	$\bar{\theta}$				$\bar{\delta}$			
		Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
MOVER	-0.9	97.754	16.748						
π_{21}		94.836	23.267	96.958	18.073	94.609	2.22E+148	92.820	1.11E+93
π_{10}		94.536	13.309	95.540	11.352	94.264	3.33E+74	94.171	5.56E+50
π_{12}		94.910	14.265	96.008	12.379	94.880	1.03E+17	95.516	1.25E+77
π_{RO}		94.736	13.589	95.713	11.649	94.633	1.11E+84	94.929	1.11E+53
π_{MS}		95.129	11.935	95.680	10.567	95.628	1.00E+67	96.000	2.22E+45
π_{RP}		95.169	14.650	96.444	12.446	95.371	1.00E+93	94.804	2.22E+61
π_S		94.871	10.931	94.789	9.816	95.772	1.11E+76	96.801	7.78E+38
$\pi_{R\sigma}$		95.273	14.947	96.689	12.670	95.282	2.22E+83	94.629	3.33E+52
$\tilde{\pi}_{R\sigma}$		95.364	15.022	96.733	12.720	95.307	4.44E+85	94.469	1.00E+55
MOVER	0.1	94.539	14.744						
π_{21}		94.784	28.826	97.333	23.352	94.164	3.75E+70	93.151	3.33E+276
π_{10}		93.607	15.810	95.767	13.622	92.907	1.11E+214	93.578	1.11E+150
π_{12}		94.424	21.120	96.618	18.529	93.616	7.50E+130	96.236	1.25E+72
π_{RO}		94.000	17.045	96.002	14.965	93.342	1.11E+277	94.896	3.33E+178
π_{MS}		94.914	19.590	97.152	17.051	94.370	2.50E+45	94.081	3.33E+209
π_{RP}		94.940	21.984	97.396	18.770	94.420	1.11E+252	93.907	3.33E+158
π_S		94.912	18.780	97.037	16.463	94.608	2.22E+231	94.326	6.67E+155
$\pi_{R\sigma}$		94.824	23.002	97.398	19.524	94.216	2.22E+291	93.584	1.11E+192
$\tilde{\pi}_{R\sigma}$		94.907	22.915	97.324	19.445	94.349	1.11E+298	93.720	3.33E+182
MOVER	0.9	91.791	12.246						
π_{21}		94.489	21.525	97.178	16.218	94.136	1.11E+159	92.838	7.78E+99
π_{10}		93.942	11.545	95.516	9.452	93.624	1.11E+98	94.078	6.67E+63
π_{12}		95.196	13.451	96.382	11.422	94.787	1.00E+135	95.900	1.00E+84
π_{RO}		94.478	11.947	95.913	9.875	94.218	7.78E+91	94.687	1.11E+63
π_{MS}		95.932	10.546	96.404	9.079	95.929	1.11E+71	96.240	4.44E+44
π_{RP}		95.513	13.133	96.893	10.781	95.296	4.44E+88	94.993	4.44E+57
π_S		96.342	9.729	95.809	8.543	96.348	1.11E+61	97.533	7.78E+39
$\pi_{R\sigma}$		95.486	13.378	97.157	10.944	95.377	1.11E+90	94.568	3.33E+59
$\tilde{\pi}_{R\sigma}$		95.311	13.451	97.018	10.997	95.242	1.11E+105	94.571	6.67E+67

From the above results it is apparent that in situations of both high negative and positive correlations the MOVER results in substantial overcoverage for the ratio of two means, even though the interval width is markedly less than the Jeffreys rule prior. The independence Jeffreys prior and the right-Haar prior are both improvements on both the MOVER and the Jeffreys rule prior with regards to the ratio of the two means. The trend is again seen when comparing the HPD intervals. Of particular interest again is the performance of the π_S prior, which results in the best coverage as well as the most efficient intervals in terms of average length.

Furthermore, in situations of low correlation between the two means the MOVER appears to perform rather well. The only improvement on the MOVER is the HPD interval for the independence Jeffreys prior. Thus, both the MOVER and the independence Jeffreys prior are particularly well suited to situations of small correlation.

When comparing the two variances we once again see that HPD intervals are an improvement on the standard equal-tailed intervals with regards to both coverage and interval length. Both of the Jeffreys priors result in undercoverage. As expected again, the π_S prior results in the required coverage as well as the best interval length, further re-inforcing it as a natural choice of prior when there is a high degree of correlation between the variables when analysing variances in the bivariate lognormal distribution model we have presented in previous sections of this chapter. However, when there is not a high degree of correlation it seems that the right-Haar prior is the most likely choice.

The following table presents the overall aggregated results of the different prior distributions. Overall, it would appear as though the right-Haar prior is the best choice, rather than π_S. This statement needs some qualification. The right-Haar prior is only adequate in situations of low correlation. In settings with high correlation the interval widths were exceptionally wide, resulting in values tending towards infinitely large, which were excluded for practical reasons and may thus give a skewed impression in the below table. Refer to the appendices to this chapter for the complete analyses.

Table 78: Overall Results

Method	$\tilde{\theta}$ Cover	Length	Cover (HPD)	Length (HPD)	$\bar{\delta}$ Cover	Length	Cover (HPD)	Length (HPD)
MOVER	94.695	14.580						
π_{21}	94.703	24.540	97.156	19.214	94.303	3.70E+158	92.936	1.11E+276
π_{10}	94.028	13.555	95.607	11.475	93.599	3.70E+213	93.942	3.70E+149
π_{12}	94.843	16.279	96.336	14.110	94.427	3.33E+134	95.884	3.33E+83
π_{RO}	94.404	14.194	95.876	12.163	94.064	3.70E+276	94.837	1.11E+178
π_{MS}	95.325	14.024	96.412	12.232	95.309	3.70E+70	95.440	1.11E+209
π_{RP}	95.207	16.589	96.911	13.999	95.029	3.70E+251	94.568	1.11E+158
π_{S}	95.375	13.147	95.878	11.607	95.576	7.41E+230	96.220	2.22E+155
$\pi_{R\sigma}$	95.194	17.109	97.081	14.379	94.958	7.41E+290	94.260	3.70E+191
$\bar{\pi}_{R\sigma}$	95.194	17.129	97.025	14.387	94.966	3.70E+297	94.253	1.11E+182

Appendix to Chapter 6

Results of Simulation Studies

Table 79: Results for MOVER

				$\tilde{\theta}$				$\tilde{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	96.69	20.47						
		5	5	98.26	33.22						
		1	10	95.99	36.32						
	0.1	1	5	94.55	18.29						
		5	5	95.06	29.44						
		1	10	94.50	34.12						
	0.9	1	5	90.07	15.27						
		5	5	98.87	25.56						
		1	10	82.62	30.14						
10	-0.9	1	5	97.41	8.88						
		5	5	99.09	14.04						
		1	10	97.27	15.23						
	0.1	1	5	94.30	7.43						
		5	5	94.33	11.19						
		1	10	94.63	13.81						
	0.9	1	5	85.84	5.69						
		5	5	97.92	7.82						
		1	10	90.68	12.09						
20	-0.9	1	5	97.92	5.29						
		5	5	99.63	8.29						
		1	10	97.53	9.01						
	0.1	1	5	94.54	4.33						
		5	5	93.82	6.13						
		1	10	95.12	7.97						
	0.9	1	5	91.09	3.26						
		5	5	96.01	3.44						
		1	10	93.02	6.95						

Table 80: Results for π_{21}

				$\bar{\theta}$				$\bar{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	94.70	39.29	98.10	27.71	94.56	2.00E+10	90.78	5.00E+04
		5	5	96.18	30.46	98.64	27.39	94.56	2.00E+149	94.62	1.00E+94
		1	10	93.22	85.36	97.40	57.71	93.70	7.00E+05	89.72	4.29E+03
	0.1	1	5	95.72	44.78	98.42	34.05	94.18	4.00E+62	92.08	9.00E+37
		5	5	93.22	62.10	98.50	54.77	91.68	Inf	92.82	3.00E+277
		1	10	94.32	89.14	98.00	62.98	94.24	1.00E+57	91.54	1.00E+33
	0.9	1	5	94.00	38.01	97.74	25.98	93.66	2.50E+09	90.96	1.50E+04
		5	5	92.82	27.16	98.74	23.96	91.22	1.00E+160	92.32	7.00E+100
		1	10	94.14	83.02	97.50	55.48	94.14	4.00E+04	90.76	4.38E+01
10	-0.9	1	5	94.62	9.12	96.78	8.16	94.52	1.70E+00	91.98	8.88E-01
		5	5	95.38	8.74	96.84	8.59	95.04	7.00E+20	94.78	3.00E+16
		1	10	94.58	17.93	97.02	15.41	94.60	1.98E+00	92.00	1.21E+00
	0.1	1	5	95.28	9.94	97.66	8.90	94.70	3.00E+06	93.12	3.60E+04
		5	5	94.84	13.88	98.04	13.46	93.84	3.00E+71	94.34	6.00E+55
		1	10	94.68	18.81	96.98	16.19	94.58	1.40E+05	92.62	2.56E+03
	0.9	1	5	95.28	7.97	96.76	6.79	95.18	1.54E-01	92.76	9.27E-02
		5	5	94.26	5.94	97.90	5.75	93.38	1.00E+28	93.88	2.00E+21
		1	10	94.74	16.78	96.40	14.12	94.80	2.78E-02	92.70	1.60E-02
20	-0.9	1	5	94.76	4.76	95.98	4.55	94.58	9.88E-02	92.84	6.16E-02
		5	5	95.28	4.98	95.80	4.95	95.02	7.00E+06	94.80	4.00E+05
		1	10	94.80	8.76	96.06	8.19	94.90	3.70E-03	93.86	2.10E-03
	0.1	1	5	95.38	4.88	96.34	4.63	95.10	1.32E+00	94.36	5.54E-01
		5	5	94.62	6.79	96.46	6.71	94.28	1.00E+17	94.22	1.00E+14
		1	10	95.00	9.13	95.60	8.49	94.88	4.61E-02	93.26	1.98E-02
	0.9	1	5	95.54	3.82	96.42	3.54	95.42	2.34E-02	94.30	1.65E-02
		5	5	94.64	2.85	96.96	2.82	94.38	1.00E+07	94.58	3.00E+05
		1	10	94.98	8.18	96.18	7.52	95.04	9.00E-04	93.28	5.00E-04

Table 81: Results for π_{10}

				$\bar{\theta}$				$\bar{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	93.32	18.48	95.28	14.60	93.78	8.41E+02	92.94	8.39E+01
		5	5	93.60	16.17	95.96	15.18	91.56	3.00E+75	92.90	5.00E+51
		1	10	94.58	38.24	95.36	28.70	94.62	1.01E+03	93.74	2.68E+02
	0.1	1	5	93.70	20.52	96.12	16.64	92.38	1.00E+27	93.52	1.00E+17
		5	5	90.04	28.40	96.16	25.76	87.36	1.00E+215	89.20	1.00E+151
		1	10	93.66	40.04	95.34	30.62	93.28	1.00E+23	93.96	6.00E+13
	0.9	1	5	94.14	16.90	95.10	12.66	94.34	1.55E+03	94.60	2.52E+01
		5	5	89.64	12.26	96.42	11.11	87.10	1.00E+99	89.70	6.00E+64
		1	10	94.32	36.55	94.64	26.82	94.32	8.53E+00	94.82	1.86E+00
10	-0.9	1	5	95.10	7.54	95.82	6.89	94.82	1.57E+00	94.72	8.71E-01
		5	5	94.98	7.63	96.20	7.53	93.98	1.00E+15	94.66	9.00E+11
		1	10	94.80	14.45	94.92	12.69	95.16	5.05E-01	94.92	2.78E-01
	0.1	1	5	94.58	8.07	95.70	7.33	93.86	6.50E+05	94.54	8.50E+03
		5	5	93.18	11.08	96.76	10.76	92.40	2.00E+39	92.78	1.00E+32
		1	10	94.52	15.03	95.14	13.17	94.42	3.04E+02	95.08	2.72E+01
	0.9	1	5	94.68	6.32	94.70	5.50	94.74	1.58E-01	95.24	1.01E-01
		5	5	93.32	4.74	97.32	4.60	92.72	6.00E+14	92.92	1.00E+11
		1	10	94.92	13.52	94.76	11.60	94.96	2.96E-02	94.88	1.82E-02
20	-0.9	1	5	94.78	4.43	95.62	4.25	95.02	9.95E-02	94.50	6.38E-02
		5	5	94.98	4.73	95.42	4.70	94.52	4.00E+07	94.72	2.00E+06
		1	10	94.68	8.12	95.28	7.62	94.92	5.70E-03	94.44	3.20E-03
	0.1	1	5	94.20	4.54	95.46	4.32	94.24	7.91E-01	94.22	3.66E-01
		5	5	94.04	6.25	96.00	6.18	93.70	8.00E+16	94.20	3.00E+14
		1	10	94.54	8.36	95.22	7.81	94.52	1.47E-01	94.70	6.41E-02
	0.9	1	5	94.76	3.54	94.96	3.30	94.70	2.66E-02	95.16	1.92E-02
		5	5	94.56	2.61	96.60	2.58	94.44	5.00E+05	94.08	3.00E+04
		1	10	95.14	7.47	95.14	6.90	95.30	1.10E-03	95.30	7.00E-04

Table 82: Results for π_{12}

				$\tilde{\theta}$				$\tilde{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	95.01	18.92	96.53	15.82	95.06	1.00E+16	95.62	1.00E+09
		5	5	95.46	23.05	98.12	21.08	94.32	1.00E+137	97.40	1.00E+78
		1	10	94.54	38.71	96.34	30.02	95.04	2.00E+16	94.94	5.00E+07
	0.1	1	5	93.90	25.48	97.24	23.19	92.70	6.00E+116	96.62	5.00E+68
		5	5	92.12	63.65	98.40	52.00	89.60	Inf	96.58	Inf
		1	10	95.18	43.81	96.96	38.07	94.72	6.00E+131	96.86	1.00E+73
	0.9	1	5	95.38	22.36	96.68	18.67	94.96	1.00E+30	96.44	1.00E+16
		5	5	95.02	20.20	99.08	18.16	93.34	9.00E+135	96.96	9.00E+84
		1	10	96.06	36.75	96.62	28.15	95.82	2.00E+10	95.94	7.00E+04
10	-0.9	1	5	94.48	7.64	95.38	7.00	94.76	3.06E+00	94.82	1.51E+00
		5	5	95.40	8.12	96.52	8.00	94.92	7.00E+17	96.60	1.00E+14
		1	10	94.54	14.52	95.12	12.77	94.90	3.84E-01	94.42	1.96E-01
	0.1	1	5	95.14	8.29	96.40	7.71	94.08	6.00E+06	96.24	3.00E+04
		5	5	93.80	13.83	97.56	13.30	92.88	2.00E+65	96.60	1.00E+50
		1	10	94.54	15.29	95.30	13.59	94.54	7.00E+06	95.58	2.00E+04
	0.9	1	5	94.78	8.03	94.90	7.05	94.76	1.55E+00	95.78	4.64E-01
		5	5	95.14	5.48	98.00	5.33	94.50	1.00E+21	96.16	2.00E+16
		1	10	94.90	13.49	95.02	11.59	94.72	4.42E-02	94.96	2.33E-02
20	-0.9	1	5	94.80	4.44	95.20	4.27	94.90	1.35E-01	94.98	8.63E-02
		5	5	95.28	4.82	95.76	4.80	95.02	2.00E+07	95.96	2.00E+06
		1	10	94.68	8.16	95.10	7.66	95.00	7.20E-03	94.90	4.10E-03
	0.1	1	5	95.20	4.55	95.60	4.36	94.62	4.14E+00	95.72	1.31E+00
		5	5	94.96	6.78	96.98	6.69	94.46	1.00E+22	96.84	3.00E+18
		1	10	94.98	8.39	95.12	7.86	94.94	8.73E-02	95.08	3.38E-02
	0.9	1	5	95.30	4.48	95.12	4.18	95.18	2.44E-02	95.56	1.57E-02
		5	5	95.24	2.80	97.04	2.77	95.02	2.00E+06	96.12	1.00E+05
		1	10	94.94	7.48	94.98	6.91	94.78	1.20E-03	95.18	7.00E-04

Table 83: Results for π_{RO}

				$\bar{\theta}$				$\bar{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	93.94	18.56	95.52	14.85	93.92	5.00E+05	94.30	2.30E+03
		5	5	94.50	17.81	96.94	16.73	92.84	1.00E+85	94.60	1.00E+54
		1	10	94.72	38.60	95.74	29.15	95.06	3.00E+04	94.94	2.15E+02
	0.1	1	5	93.60	21.53	96.26	18.37	92.26	1.00E+54	94.86	7.00E+34
		5	5	92.26	37.12	97.86	33.25	90.06	1.00E+278	93.86	3.00E+179
		1	10	93.50	39.90	94.94	31.87	93.10	1.00E+44	95.16	1.00E+26
	0.9	1	5	93.94	16.92	95.08	12.94	94.18	7.00E+04	95.28	2.23E+02
		5	5	92.74	14.30	97.92	13.11	90.72	7.00E+92	93.70	1.00E+64
		1	10	94.96	37.49	95.50	27.74	95.12	5.40E+04	94.74	2.85E+02
10	-0.9	1	5	94.72	7.59	95.78	6.94	95.02	1.98E+00	94.84	1.06E+00
		5	5	95.32	7.82	96.02	7.71	94.78	4.00E+21	95.56	2.00E+16
		1	10	94.70	14.53	95.18	12.76	94.80	1.21E+00	94.82	6.46E-01
	0.1	1	5	94.64	8.14	95.76	7.47	94.02	4.00E+06	94.64	4.00E+04
		5	5	93.42	12.20	97.28	11.82	92.50	4.00E+45	95.02	1.00E+35
		1	10	94.42	15.01	94.88	13.23	94.56	5.04E+03	95.32	1.23E+02
	0.9	1	5	94.80	6.44	95.68	5.61	94.56	1.82E-01	94.44	1.12E-01
		5	5	94.24	5.05	97.50	4.91	93.46	4.00E+19	94.20	6.00E+14
		1	10	95.06	13.56	94.86	11.64	95.36	3.96E-02	95.12	2.37E-02
20	-0.9	1	5	95.26	4.43	95.74	4.26	95.42	1.06E-01	95.10	6.71E-02
		5	5	94.94	4.78	95.36	4.75	95.20	9.00E+06	95.74	5.00E+05
		1	10	94.52	8.18	95.14	7.69	94.66	5.90E-03	94.46	3.30E-03
	0.1	1	5	95.10	4.55	95.68	4.34	95.12	1.33E+00	95.52	5.57E-01
		5	5	94.36	6.52	96.32	6.45	93.92	1.00E+18	94.94	3.00E+15
		1	10	94.70	8.44	95.04	7.90	94.54	3.92E-02	94.74	1.67E-02
	0.9	1	5	95.06	3.54	95.08	3.30	95.24	2.66E-02	95.00	1.90E-02
		5	5	94.20	2.70	96.30	2.67	94.00	1.00E+06	94.54	6.00E+04
		1	10	95.30	7.52	95.30	6.95	95.32	1.30E-03	95.16	8.00E-04

Table 84: Results for π_{MS}

n	ρ	σ_{11}	σ_{22}	$\bar{\theta}$ Cover	Length	Cover (HPD)	Length (HPD)	$\bar{\delta}$ Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	94.96	15.41	96.35	12.93	96.00	3.20E+04	96.33	3.94E+02
		5	5	96.27	17.79	97.47	17.23	96.71	9.00E+67	96.67	2.00E+46
		1	10	95.42	29.91	95.85	23.31	96.11	1.90E+04	97.09	1.30E+02
	0.1	1	5	95.11	26.47	97.77	21.98	94.07	2.00E+38	94.05	4.00E+23
		5	5	94.19	40.47	98.69	37.68	92.79	Inf	93.09	3.00E+210
		1	10	95.15	49.94	97.47	38.56	94.71	4.00E+41	94.21	5.00E+25
	0.9	1	5	96.63	14.29	96.85	11.61	96.83	5.60E+04	97.47	6.02E+01
		5	5	98.25	15.71	99.71	15.15	97.74	1.00E+72	95.97	4.00E+45
		1	10	95.81	28.68	95.62	21.86	95.97	3.90E+04	97.30	1.78E+02
10	-0.9	1	5	94.82	6.97	95.38	6.45	95.42	1.97E+00	95.80	1.01E+00
		5	5	95.59	7.60	96.27	7.51	96.17	1.00E+17	95.73	1.00E+13
		1	10	94.98	13.00	95.42	11.56	95.38	6.07E-01	95.84	2.99E-01
	0.1	1	5	94.90	9.06	97.08	8.24	94.62	1.10E+07	94.06	1.10E+05
		5	5	94.70	13.11	97.80	12.78	94.00	2.00E+46	94.74	5.00E+35
		1	10	95.04	16.84	96.66	14.68	94.64	1.73E+03	94.24	4.65E+01
	0.9	1	5	95.42	5.79	95.44	5.12	95.54	2.14E-01	96.08	1.23E-01
		5	5	96.53	5.19	98.65	5.09	96.17	7.00E+14	95.37	3.00E+11
		1	10	95.15	12.03	94.61	10.42	95.43	3.61E-02	96.61	1.97E-02
20	-0.9	1	5	94.78	4.26	94.38	4.10	95.04	1.20E-01	95.86	7.58E-02
		5	5	94.76	4.71	95.36	4.68	95.12	1.00E+07	95.20	7.00E+05
		1	10	94.58	7.77	94.64	7.33	94.70	4.70E-03	95.48	2.60E-03
	0.1	1	5	95.42	4.81	96.54	4.58	95.16	1.13E+00	94.26	4.32E-01
		5	5	94.78	6.72	96.42	6.66	94.54	2.00E+20	94.46	7.00E+16
		1	10	94.94	8.90	95.94	8.31	94.80	4.51E-02	93.62	1.79E-02
	0.9	1	5	94.88	3.38	94.76	3.16	95.20	2.85E-02	95.98	2.01E-02
		5	5	95.48	2.73	97.20	2.70	95.26	1.20E+05	95.62	1.20E+04
		1	10	95.24	7.11	94.80	6.60	95.22	1.30E-03	95.76	8.00E-04

Table 85: Results for π_{RP}

				$\tilde{\theta}$				$\tilde{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	94.76	20.91	96.88	16.49	95.54	5.00E+05	94.18	1.37E+03
		5	5	96.00	20.70	98.12	19.72	95.68	9.00E+93	95.80	2.00E+62
		1	10	95.40	42.02	97.00	30.88	95.72	1.10E+04	94.78	3.15E+02
	0.1	1	5	95.68	31.32	98.52	25.39	94.82	5.00E+45	93.66	5.00E+27
		5	5	93.86	45.92	98.66	42.20	92.10	1.00E+253	93.20	3.00E+159
		1	10	94.92	59.58	97.72	44.77	94.52	3.00E+37	93.22	1.00E+22
	0.9	1	5	95.94	19.35	97.56	14.67	95.98	5.00E+05	95.18	2.07E+02
		5	5	95.60	17.55	98.74	16.56	94.46	4.00E+89	94.92	4.00E+58
		1	10	96.06	41.42	97.12	29.78	96.18	3.97E+02	94.82	5.79E+00
10	-0.9	1	5	94.82	7.84	96.22	7.16	95.02	3.92E+00	94.26	1.63E+00
		5	5	95.02	8.07	96.46	7.97	95.16	2.00E+22	95.22	7.00E+17
		1	10	95.08	14.82	96.12	12.98	95.42	4.14E-01	94.48	2.09E-01
	0.1	1	5	94.96	9.41	97.04	8.51	94.92	7.10E+04	94.20	1.07E+03
		5	5	94.80	13.49	97.86	13.15	94.22	4.00E+57	94.28	1.00E+45
		1	10	94.00	17.55	96.86	15.22	94.10	3.28E+03	93.20	1.49E+02
	0.9	1	5	95.18	6.61	96.14	5.75	95.06	2.18E-01	94.58	1.26E-01
		5	5	95.46	5.40	98.16	5.28	94.98	6.00E+20	95.02	1.00E+16
		1	10	95.44	13.89	95.92	11.85	95.70	3.00E-02	95.42	1.76E-02
20	-0.9	1	5	94.76	4.48	95.64	4.31	95.20	1.06E-01	94.20	6.62E-02
		5	5	95.54	4.80	95.94	4.78	95.32	1.50E+06	95.44	1.30E+05
		1	10	95.14	8.21	95.62	7.72	95.28	8.20E-03	94.88	4.70E-03
	0.1	1	5	95.28	4.84	96.28	4.62	94.96	1.43E+00	93.98	5.50E-01
		5	5	95.60	6.73	97.50	6.68	95.00	5.00E+15	95.30	1.00E+13
		1	10	95.36	9.00	96.12	8.40	95.14	9.41E-02	94.12	4.03E-02
	0.9	1	5	95.34	3.60	95.72	3.36	95.62	2.60E-02	95.40	1.82E-02
		5	5	95.16	2.78	96.86	2.76	94.50	4.20E+05	94.82	3.00E+04
		1	10	95.44	7.60	95.82	7.03	95.18	1.10E-03	94.78	6.00E-04

Table 86: Results for π_S

n	ρ	σ_{11}	σ_{22}	$\tilde{\theta}$ Cover	$\tilde{\theta}$ Length	$\tilde{\theta}$ Cover (HPD)	$\tilde{\theta}$ Length (HPD)	$\tilde{\delta}$ Cover	$\tilde{\delta}$ Length	$\tilde{\delta}$ Cover (HPD)	$\tilde{\delta}$ Length (HPD)
5	-0.9	1	5	94.51	13.66	94.77	11.68	96.24	5.10E+04	97.90	5.19E+02
		5	5	95.85	17.87	97.01	17.36	97.40	1.00E+77	97.03	7.00E+39
		1	10	95.64	25.77	94.80	20.50	96.47	1.30E+06	98.15	2.51E+03
	0.1	1	5	95.46	25.13	97.87	21.04	94.82	1.00E+38	94.54	2.00E+24
		5	5	94.62	38.18	98.81	35.76	93.94	2.00E+232	94.60	6.00E+156
		1	10	94.67	47.00	97.07	36.70	94.49	3.00E+31	94.61	3.40E+20
	0.9	1	5	97.10	12.72	96.22	10.67	97.34	1.50E+05	98.96	1.18E+02
		5	5	99.43	16.32	99.84	15.82	98.70	1.00E+62	98.48	7.00E+40
		1	10	96.12	25.01	94.71	19.56	96.27	3.80E+04	98.26	4.32E+01
10	-0.9	1	5	94.20	6.36	94.27	5.93	95.19	2.55E+00	96.52	1.32E+00
		5	5	95.00	7.21	95.55	7.13	96.16	5.00E+20	96.19	6.00E+16
		1	10	95.06	11.53	93.95	10.36	95.65	3.29E-01	97.84	1.66E-01
	0.1	1	5	95.02	8.88	96.66	8.09	94.42	4.90E+05	94.24	7.15E+03
		5	5	95.38	12.84	98.08	12.54	94.98	2.00E+52	94.48	8.00E+40
		1	10	94.62	16.66	96.40	14.56	94.74	1.09E+03	93.84	2.28E+01
	0.9	1	5	95.49	5.22	93.79	4.66	95.61	4.05E-01	97.99	2.06E-01
		5	5	97.78	5.05	99.03	4.96	97.58	1.00E+13	96.76	1.00E+10
		1	10	94.97	10.62	93.06	9.30	95.26	4.58E-02	97.76	2.49E-02
20	-0.9	1	5	94.83	4.08	94.41	3.94	94.93	1.29E-01	96.24	8.19E-02
		5	5	93.99	4.58	94.30	4.55	94.97	4.80E+05	94.73	5.10E+04
		1	10	94.76	7.31	94.04	6.91	94.94	5.80E-03	96.61	3.20E-03
	0.1	1	5	94.52	4.78	95.76	4.56	94.54	1.11E+00	94.02	4.72E-01
		5	5	95.00	6.71	96.96	6.65	94.54	9.00E+15	94.54	1.00E+13
		1	10	94.92	8.85	95.72	8.27	95.00	5.53E-02	94.06	2.56E-02
	0.9	1	5	95.04	3.24	93.98	3.03	95.00	3.07E-02	96.97	2.16E-02
		5	5	96.15	2.70	97.51	2.66	96.23	9.50E+04	95.45	8.64E+03
		1	10	95.00	6.70	94.14	6.22	95.14	1.20E-03	97.17	8.00E-04

Table 87: Results for $\pi_{R\sigma}$

n	ρ	σ_{11}	σ_{22}	$\bar{\theta}$ Cover	$\bar{\theta}$ Length	$\bar{\theta}$ Cover (HPD)	$\bar{\theta}$ Length (HPD)	$\bar{\delta}$ Cover	$\bar{\delta}$ Length	$\bar{\delta}$ Cover (HPD)	$\bar{\delta}$ Length (HPD)
5	-0.9	1	5	95.00	21.27	97.36	16.74	95.14	2.10E+05	94.12	5.63E+02
		5	5	96.34	20.94	98.26	19.96	95.68	2.00E+84	95.58	3.00E+53
		1	10	95.50	43.74	97.96	32.08	95.56	3.20E+06	94.14	4.74E+03
	0.1	1	5	95.16	32.86	98.32	26.47	94.00	8.00E+51	92.98	2.00E+32
		5	5	93.72	48.78	98.48	44.72	91.80	2.00E+292	93.06	1.00E+193
		1	10	94.80	63.73	98.00	47.49	94.42	8.00E+42	92.60	2.00E+25
	0.9	1	5	96.17	19.64	98.01	14.80	96.41	4.10E+04	95.35	6.79E+01
		5	5	95.40	17.61	99.12	16.60	94.46	1.00E+91	94.64	3.00E+60
		1	10	96.22	42.69	97.70	30.57	95.92	4.80E+07	94.38	5.07E+03
10	-0.9	1	5	95.28	7.90	96.50	7.21	95.38	1.64E+00	94.10	8.51E-01
		5	5	94.92	8.12	96.30	8.02	94.72	1.00E+19	95.02	6.00E+14
		1	10	94.96	14.97	96.24	13.12	95.52	4.76E-01	94.28	2.50E-01
	0.1	1	5	95.48	9.51	97.50	8.58	94.72	1.00E+06	93.74	1.10E+04
		5	5	94.88	13.56	97.88	13.19	93.92	6.00E+48	94.40	4.00E+37
		1	10	94.82	17.93	97.16	15.52	94.88	8.28E+03	93.30	1.01E+02
	0.9	1	5	95.06	6.75	96.30	5.86	95.40	1.94E-01	94.26	1.12E-01
		5	5	95.22	5.46	98.30	5.33	94.88	2.00E+20	94.62	2.00E+15
		1	10	95.70	14.23	96.28	12.14	95.76	3.71E-02	94.62	2.19E-02
20	-0.9	1	5	94.82	4.53	95.94	4.35	94.92	1.27E-01	94.40	7.97E-02
		5	5	95.64	4.85	96.08	4.83	95.40	1.30E+06	95.06	1.00E+05
		1	10	95.00	8.22	95.56	7.73	95.22	4.30E-03	94.96	2.40E-03
	0.1	1	5	94.44	4.88	96.02	4.65	94.36	1.35E+00	93.88	5.48E-01
		5	5	95.22	6.76	97.06	6.70	94.86	1.00E+19	95.00	5.00E+15
		1	10	94.90	9.01	96.16	8.40	94.98	4.55E-02	93.30	1.96E-02
	0.9	1	5	95.20	3.63	96.34	3.39	95.54	2.43E-02	94.54	1.70E-02
		5	5	95.62	2.78	97.14	2.76	95.18	1.20E+05	94.42	1.10E+04
		1	10	94.78	7.62	95.22	7.05	94.84	9.00E-04	94.28	5.00E-04

Table 88: Results for $\tilde{\pi}_{R\sigma}$

				$\tilde{\theta}$				$\tilde{\delta}$			
n	ρ	σ_{11}	σ_{22}	Cover	Length	Cover (HPD)	Length (HPD)	Cover	Length	Cover (HPD)	Length (HPD)
5	-0.9	1	5	95.44	21.55	97.74	16.93	95.66	3.10E+06	94.10	7.29E+03
		5	5	96.18	20.83	98.20	19.83	95.64	4.00E+86	95.34	9.00E+55
		1	10	95.00	43.92	97.50	32.17	95.28	2.30E+03	94.02	1.65E+02
	0.1	1	5	95.02	32.72	98.14	26.37	94.04	1.00E+46	93.60	6.00E+26
		5	5	94.14	48.48	98.66	44.37	92.48	1.00E+299	93.44	3.00E+183
		1	10	95.32	63.35	97.90	47.18	94.74	4.00E+38	93.26	4.00E+23
	0.9	1	5	96.06	20.08	97.58	15.11	96.08	1.70E+07	94.98	1.71E+03
		5	5	95.22	17.90	99.02	16.85	93.86	1.00E+106	93.74	6.00E+68
		1	10	95.78	42.42	97.72	30.35	96.18	4.19E+03	94.40	2.98E+01
10	-0.9	1	5	95.10	7.88	96.52	7.19	94.90	1.82E+00	94.20	8.90E-01
		5	5	95.44	8.17	96.54	8.08	95.56	7.00E+21	95.34	2.00E+16
		1	10	94.98	15.14	96.50	13.26	95.10	7.98E-01	93.82	3.97E-01
	0.1	1	5	95.10	9.59	97.30	8.66	94.76	3.30E+07	93.56	3.60E+04
		5	5	94.54	13.56	97.72	13.20	93.66	1.00E+61	93.98	2.00E+48
		1	10	95.20	17.83	97.28	15.43	95.16	4.80E+02	93.38	1.87E+01
	0.9	1	5	95.56	6.73	96.92	5.85	95.74	1.75E-01	94.50	1.03E-01
		5	5	95.00	5.44	97.98	5.32	94.66	1.00E+25	95.20	4.00E+19
		1	10	94.80	14.48	96.20	12.34	95.18	2.67E-02	93.84	1.42E-02
20	-0.9	1	5	95.46	4.54	95.88	4.36	94.70	9.66E-02	93.66	5.99E-02
		5	5	95.46	4.84	95.96	4.82	95.50	2.80E+06	95.46	1.70E+05
		1	10	95.22	8.33	95.76	7.83	95.42	4.80E-03	94.28	2.70E-03
	0.1	1	5	94.96	4.87	96.02	4.64	94.72	1.29E+00	94.24	5.18E-01
		5	5	95.18	6.76	96.92	6.71	94.66	3.00E+18	94.72	2.00E+15
		1	10	94.70	9.07	95.98	8.45	94.92	3.70E-02	93.30	1.52E-02
	0.9	1	5	95.10	3.61	95.12	3.37	95.34	2.65E-02	94.80	1.86E-02
		5	5	95.72	2.79	97.62	2.76	95.40	2.80E+06	95.14	1.80E+05
		1	10	94.56	7.61	95.00	7.04	94.74	1.10E-03	94.54	7.00E-04

CHAPTER 7

Random Effect Model: Balanced Case

Introduction

Lognormally distributed data are found in many settings including occupational health. It is not just confidence intervals that one is interested in though. Many standard statistical models can be adapted to facilitate the use of lognormal data.

One such setting has been proposed by Krishnamoorthy and Mathew (2002), whereby they applied a one-way random effects model to balanced lognormally distributed data. This was done as a means to assess occupational exposure. The parameter of interest in these cases was the occupational exposure limit (OEL). The reader is referred to the original articles by Krishnamoorthy and Mathew (2002) for a more complete description of the medical applications of this method as well as the texts by Rappaport, Kromhout and Symanski (1993), Heederik and Hurley (1994) and Lyles, Kupper and Rappaport (1997).

Krishnamoorthy and Mathew (2002), in almost an extension of their other work on lognormally distributed data, attempted to analyse the data using generalized confidence intervals and generalized p-values in order to test hypotheses by means of confidence intervals on the overall mean exposure limits. The intention here is not to further describe the application to medical exposure limits (although a brief description of the setting will be given in sections that follow, as well as offering other potential settings that relate to the field of finance), but merely to develop a similar method using Bayesian methodology in order to improve the method and to determine the performance of different prior distributions for the data. All prior distributions will be non-informative, since we have no subjective prior opinion as to the distribution of the data.

The application of other non-informative priors, particularly the Reference prior (as developed by Berger) and Probability-Matching priors can be applied to the geometric mean and not just the arithmetic mean of the exposure data. In many instances the geometric mean can minimize the influence of outliers in the data and in this specific case also has implications for the computational simplicity of the procedures. A derivation of these prior distributions is also given in the appendix to the chapter.

7.1 Description of the Setting

As mentioned in the Introduction to this chapter, there have been several authors that have proposed the use of random effects models to model the exposure levels of workers to potential workplace contaminants. One of these methods proposed has been this model to the log-transformed shift-long exposures and it attempts to incorporate the between- and within-worker sources of variability. In this case we are interested in workers or groups of workers for whom the long-term exposure levels exceed the specific OEL.

We are in this section also looking at specifically the balanced case where the exposure levels for each worker are lognormally distributed. The analysis of problems in this area is rather different from the traditional problems associated with the analysis of mixed models. Deriving exact tests is perhaps a slightly more complicated procedure since the analysis involves all the regular parameters of the traditional mixed model analysis, namely the mean, as well as the two variance components, that is the within- and between-subject variability.

As mentioned in the Introduction, Krishnamoorthy and Mathew proposed a generalized confidence interval and generalized p-value method to the analysis of these problems. Lyles et al. (1997) proposed large sample tests, such as Wald, likelihood ratio and score-type tests.

To represent this data situation refer to the following diagrammatic representation:

Table 89: Representation of Shift Exposure Data

	Shift-Long Exposure Measurements				
Workers	1	2	...	n	Worker Means
1	x_{11}	x_{12}	...	x_{1n}	$\bar{X}_{1\cdot}$
2	x_{21}	x_{22}	...	x_{2n}	$\bar{X}_{2\cdot}$
...	...	...	...	...	...
k	x_{k1}	x_{k2}	...	x_{kn}	$\bar{X}_{k\cdot}$

Therefore, there are essentially n measurements per worker. The overall mean of all the data can be represented as $\bar{X}_{\cdot\cdot}$ The X_{ij} are lognormally distributed and therefore $Y_{ij} = ln(X_{ij})$ is distributed normally. The assumed one-way random effects model is:

$$Y_{ij} = \mu + \tau_i + e_{ij}, \qquad i = 1, \dots, k; j = 1, \dots, n. \tag{7.1}$$

where μ is the general mean, $\tau_i \sim N(0, \sigma_\tau^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$. All the random variables are independent of each other and here τ_i represents the random effect due to the i-th worker.

According to Krishnamoorthy and Mathew (2002) let

$$\mu_{x_i} = E\left(X_{ij}|\tau_i\right) = E\left(exp\left[Y_{ij}\right]|\tau_i\right) = exp\left(\mu + \tau_i + {\sigma_e^2}/{2}\right) \tag{7.2}$$

and μ_{x_i} is the mean exposure for the i-th worker. Let θ denote the probability that μ_{x_i} exceeds the OEL. Thus,

$$\theta = P\left(\mu_{x_i} > OEL\right) = P\left(ln\left(\mu_{x_i}\right) > ln(OEL)\right) = 1 - \Phi\left(\frac{ln(OEL) - \mu - {\sigma_e^2}/{2}}{\sigma_\tau}\right) \tag{7.3}$$

where $\Phi(.)$ denotes the c.d.f. of the standard normal distribution. The kinds of hypothesis that are going to be considered here are:

$$H_0: \theta \geq A \; vs \; H_1: \theta < A$$

where A is a specific quantity that is usually small, according to Krishnamoorthy and Mathew (2002).

As mentioned previously, Krishnamoorthy and Mathew (2002) apply a technique based on generalized confidence limits and generalized p-values in order to test the above hypothesis. In this chapter we will apply a Bayesian methodology to the problem as well as evaluating the performance of other prior distributions on a geometric mean.

The following is an example data set of simulated "styrene exposures" that will serve as a basis for discussion in this chapter and will help us define and illustrate the objectives of the chapter:

Table 90: Simulated "Styrene Exposures"

Worker	Observations / Measurements 1	2	3	4	5	6	7	8	9	10
1	95.58	64.72	50.91	87.36	82.27	149.90	33.45	77.48	70.81	60.95
2	57.40	82.27	174.16	107.77	98.49	129.02	121.51	95.58	92.76	132.95
3	84.77	214.86	132.95	79.84	169.02	149.90	164.02	84.77	84.77	114.43
4	68.72	77.48	66.69	54.05	41.26	64.72	46.53	59.15	45.15	54.05
5	114.43	101.49	49.40	101.49	90.02	52.46	114.43	79.84	68.72	87.36
6	87.36	242.26	145.47	132.95	174.16	214.86	137.00	129.02	169.02	179.47
7	54.05	75.19	84.77	55.70	90.02	70.81	60.95	101.49	64.72	95.58
8	64.72	95.58	57.40	95.58	82.27	101.49	92.76	60.95	101.49	98.49
9	137.00	208.51	92.76	159.17	92.76	82.27	121.51	90.02	159.17	174.16
10	125.21	87.36	121.51	90.02	154.47	107.77	117.92	179.47	129.02	129.02
11	84.77	42.52	87.36	72.97	66.69	75.19	50.91	59.15	49.40	66.69
12	57.40	68.72	59.15	64.72	55.70	60.95	92.76	52.46	42.52	52.46
13	101.49	149.90	111.05	77.48	111.05	84.77	64.72	62.80	149.90	70.81
14	68.72	101.49	111.05	179.47	82.27	174.16	174.16	87.36	145.47	114.43
15	121.51	77.48	145.47	174.16	77.48	92.76	159.17	129.02	104.58	77.48

The above table represents the X_{ij} data points and the following table represents the corresponding $Y_{ij} = ln(X_{ij})$:

Table 91: Log of Simulated "Styrene Exposures"

Worker	Observations / Measurements									
	1	2	3	4	5	6	7	8	9	10
1	4.56	4.17	3.93	4.47	4.41	5.01	3.51	4.35	4.26	4.11
2	4.05	4.41	5.16	4.68	4.59	4.86	4.8	4.56	4.53	4.89
3	4.44	5.37	4.89	4.38	5.13	5.01	5.1	4.44	4.44	4.74
4	4.23	4.35	4.2	3.99	3.72	4.17	3.84	4.08	3.81	3.99
5	4.74	4.62	3.9	4.62	4.5	3.96	4.74	4.38	4.23	4.47
6	4.47	5.49	4.98	4.89	5.16	5.37	4.92	4.86	5.13	5.19
7	3.99	4.32	4.44	4.02	4.5	4.26	4.11	4.62	4.17	4.56
8	4.17	4.56	4.05	4.56	4.41	4.62	4.53	4.11	4.62	4.59
9	4.92	5.34	4.53	5.07	4.53	4.41	4.8	4.5	5.07	5.16
10	4.83	4.47	4.8	4.5	5.04	4.68	4.77	5.19	4.86	4.86
11	4.44	3.75	4.47	4.29	4.2	4.32	3.93	4.08	3.9	4.2
12	4.05	4.23	4.08	4.17	4.02	4.11	4.53	3.96	3.75	3.96
13	4.62	5.01	4.71	4.35	4.71	4.44	4.17	4.14	5.01	4.26
14	4.23	4.62	4.71	5.19	4.41	5.16	5.16	4.47	4.98	4.74
15	4.8	4.35	4.98	5.16	4.35	4.53	5.07	4.86	4.65	4.35

From these data we have the following definitions and associated results:

$$k = 15\,; n = 10$$

$$v_1 = k(n-1)\,;\ v_2 = k-1$$

$$\bar{Y}_{i\cdot} = \frac{1}{n} Y_{i\cdot} = \frac{1}{n}\sum_{j=1}^{n} Y_{ij} =$$

$$= [4.28\ 4.65\ 4.79\ 4.04\ 4.42\ 5.05\ 4.29\ 4.42\ 4.83\ 4.80\ \ 4.16\ 4.09\ 4.54\ 4.77\ 4.71]'$$

$$\bar{Y}_{\cdot\cdot} = \frac{1}{kn}\sum_{i=1}^{k}\sum_{j=1}^{n} Y_{ij} = 4.5228$$

$$SS_\tau = v_2 m_2 = n\sum_{i=1}^{k}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 = 13.2283 = \text{"}between\ workers\ sum\ of\ squares\text{"}$$

$$SS_e = v_1 m_1 = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(Y_{ij} - \bar{Y}_{i\cdot}\right)^2 \quad = 11.3897$$

$$= \text{"}within\ workers\ sum\ of\ squares\text{"}$$

7.2. Bayesian Methodology

As mentioned previously, the basis for analyzing any situation from a Bayesian perspective is the following result:

$$Posterior \propto Likelihood \times Prior$$

Due to the setting we already know what the likelihood function looks like (in matrix form):

$$L(\mu, \sigma_\tau^2, \sigma_e^2, \boldsymbol{u}|\boldsymbol{Y}) \propto \left(\frac{1}{\sigma_e^2}\right)^{\frac{1}{2}kn} exp\left\{-\frac{1}{2\sigma_e^2}(\boldsymbol{Y} - \theta\boldsymbol{1} - Z\boldsymbol{u})'(\boldsymbol{Y} - \theta\boldsymbol{1} - Z\boldsymbol{u})\right\}$$
$$\times \left(\frac{1}{\sigma_\tau^2}\right)^{\frac{1}{2}k} exp\left\{-\frac{1}{2\sigma_\tau^2}\,\boldsymbol{u}'\boldsymbol{u}\right\} \tag{7.4}$$

where $\boldsymbol{Y}$ is a vector of the data, $\boldsymbol{u} \sim N(\boldsymbol{0}, I\sigma_\tau^2)$ and the other vectors and matrices are as follows:

Z is the $kn \times k$ matrix: $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & & 0 \\ \vdots & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & 0 \\ \vdots & \vdots & & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$, $\boldsymbol{1}$ is a $kn \times 1$ vector of ones, $\boldsymbol{u} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_k \end{bmatrix}$, and $\boldsymbol{e} \sim N(\boldsymbol{0}, I\sigma_e^2)$.

Before looking at the Reference and Probability-Matching priors we will begin our discussion by choosing the following non-informative prior distribution:

$$p(\mu, \sigma_\tau^2, \sigma_e^2) = p(\mu)p(\sigma_\tau^2, \sigma_e^2) \propto \frac{1}{\sigma_e^2} \frac{1}{(\sigma_e^2 + n\sigma_\tau^2)}$$

(7.5)

The above prior distribution is the non-informative distribution that has been discussed at length by Box and Tiao (1973) a full criticism can be found in this text.

By combining (7.4) and (7.5) we arrive at the posterior distribution of $\mu, \sigma_\tau^2, \sigma_e^2$ and $\boldsymbol{u}$:

$$L(\mu, \sigma_\tau^2, \sigma_e^2, \boldsymbol{u}|\boldsymbol{Y}) \propto \left(\frac{1}{\sigma_e^2}\right)^{2kn} exp\left\{-\frac{1}{2\sigma_e^2}(\boldsymbol{Y} - \theta\boldsymbol{1} - Z\boldsymbol{u})'(\boldsymbol{Y} - \theta\boldsymbol{1} - Z\boldsymbol{u})\right\}$$

$$\times \left(\frac{1}{\sigma_\tau^2}\right)^{2k} exp\left\{-\frac{1}{2\sigma_\tau^2}\, \boldsymbol{u}'\boldsymbol{u}\right\} \times \frac{1}{\sigma_e^2} \frac{1}{(\sigma_e^2 + n\sigma_\tau^2)}$$

(7.6)

This can then be integrated over $\boldsymbol{u}$ by completing the square with respect to $\boldsymbol{u}$. Therefore, the joint posterior density function of $(\mu, \sigma_\tau^2, \sigma_e^2)$ can be written as:

$$p(\mu, \sigma_\tau^2, \sigma_e^2 |\, \boldsymbol{Y})$$

$$\propto (\sigma_e^2)^{-\frac{1}{2}(\nu_1 + 2)}(\sigma_e^2 + n\sigma_\tau^2)^{-\frac{1}{2}(\nu_2 + 3)}$$

$$\times\, exp\left\{-\frac{1}{2}\left[\frac{kn(\bar{Y}_{..} - \mu)^2}{\sigma_e^2 + n\sigma_\tau^2} + \frac{\nu_2 m_2}{\sigma_e^2 + n\sigma_\tau^2} + \frac{\nu_1 m_1}{\sigma_e^2}\right]\right\}$$

(7.7)

Now, to obtain the joint posterior distribution of $(\sigma_\tau^2, \sigma_e^2)$ we integrate (7.7) with respect to μ and arrive at the desired result:

$$p(\sigma_\tau^2, \sigma_e^2 |\, \boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}(\nu_1 + 2)}(\sigma_e^2 + n\sigma_\tau^2)^{-\frac{1}{2}(\nu_2 + 2)} \times exp\left\{-\frac{1}{2}\left[\frac{\nu_2 m_2}{\sigma_e^2 + n\sigma_\tau^2} + \frac{\nu_1 m_1}{\sigma_e^2}\right]\right\}$$

(7.8)

It is important to notice as well that $\sigma_\tau^2 > 0$.

7.3. Simulation Study

A simulation study was performed using the results obtained in the previous section. However, there was a departure from the analysis that Krishnamoorthy and Mathew (2002) performed. In particular Krishnamoorthy and Mathew applied to the generalized confidence limits and p-values to the following quantity:

$$\mu_{x_i} = e^{\left(\mu + \tau_i + \frac{\sigma_e^2}{2}\right)}$$

which is the mean exposure of the i-th worker. As it turns out, their "mean exposure for the i-th worker" did not include the actual row mean (mean for that specific worker) anywhere in the simulation, but rather the simulation was based only on the overall mean and distribution of τ_i. So in actual fact, Krishnamoorthy and Matthew had more a "general" or "overall" mean exposure for each worker. In this section though we suggest an enhancement of this technique, whereby the mean for the i-th worker is estimated (from a Bayesian ideology) whereby the actual row means also influence the mean exposure level for that specific worker. So this section in fact presents various simulation studies in the following order:

1. Mean exposure of the i-th worker accounting for the row mean (enhancement on Krishnamoorthy and Mathew technique).
2. Mean exposure of the i-th worker not accounting for the row mean (comparable technique).
3. Overall mean exposure.

7.3.1 Simulation 1

The following is a description of this simulation study:

The intention is to simulate the mean exposure per worker from the posterior distribution using the prior mentioned by Box and Tiao (1973).

Let

$$\mu_{x_i} = e^{\left(\mu + \tau_i + \frac{\sigma_e^2}{2}\right)}$$

represent the mean exposure level of the i-th worker. For each worker the probability that the mean exposure exceeds a certain pre-defined limit can be simulated from the posterior distribution as follows:

1. Simulate λ from a $\chi^2_{\nu_1}$ distribution where $\nu_1 = k(n-1)$ and using this calculate: $\sigma_e^2 = \frac{SSE}{\lambda}$
2. Simulate δ from a $\chi^2_{\nu_2}$ distribution where $\nu_2 = k - 1$ and using this calculate: $\sigma_e^2 + n\sigma_\tau^2 = \frac{SS\tau}{\delta} = \sigma_{12}^2$
3. Calculate $\sigma_\tau^2 = \frac{\sigma_{12}^2 - \sigma_e^2}{n}$. This implies that we have simulated σ_e^2 and σ_τ^2 from their joint posterior distribution as given in (7.8).
4. If a negative value is obtained in step 3 we disregard both σ_e^2 and σ_τ^2 and repeat steps 1 - 3 until we find a pair where both are positive. This is somewhat different to Krishnamoorthy and Mathew's technique, whereby in their method negative estimates were simply set equal to zero and not totally disregarded.

5. For each pair of $(\sigma_e^2, \sigma_\tau^2)$ simulate $\zeta^* \sim N(I\ ,II)$ (the posterior distribution of $\mu + \tau_i + \frac{\sigma_e^2}{2}$ given the variance components has this normal distribution) where
   a. $I = \frac{n\sigma_\tau^2}{\sigma_e^2 + n\sigma_\tau^2}\bar{Y}_{i\cdot} + \frac{\sigma_e^2}{\sigma_e^2 + n\sigma_\tau^2}\bar{Y}_{\cdot\cdot} + \frac{\sigma_e^2}{2}$ and
   b. $II = \frac{\sigma_e^2}{\sigma_e^2 + n\sigma_\tau^2}\left\{\frac{\sigma_e^2}{nk} + \sigma_\tau^2\right\}$.

   The posterior distribution of $\mu + \tau_i + \frac{\sigma_e^2}{2}$ given the variance components follows from the fact that $\mu + \tau_i | \sigma_\tau^2, \sigma_e^2, \boldsymbol{Y} \sim N\left(\frac{n\sigma_\tau^2}{\sigma_e^2 + n\sigma_\tau^2}\ ; \frac{\sigma_e^2}{\sigma_e^2 + n\sigma_\tau^2}\left\{\frac{\sigma_e^2}{nk} + \sigma_\tau^2\right\}\right)$. For further details see van der Merwe and Bekker (2007).
6. Calculate $\mu_{x_i} = e^{(\zeta^*)}$
7. Repeat steps 1 - 6 $l\ (= 1000\ or\ 10000)$ times for each of the 15 workers. The result is a 15 $\times\ l$ matrix of simulated observations.

Using the data from steps 1 - 7 above the following can be calculated:

1. For each worker (i.e. each row of the 15 $\times\ l$ matrix of simulated observations mentioned in step 7 above):
   a. Draw up a histogram
   b. Calculate $P(\mu_{x_i} > OEL) = \frac{\#\ Simulated\ Values > OEL}{l}$ where OEL is a pre-defined value of "clinical" interest.

Assume, for the purposes of illustration that $OEL = 130$ and furthermore take $l = 10000$. Histograms of individual worker simulations are presented in the appendix to this chapter as Figures 48 to 62.

From the above simulations the following descriptive statistics and Bayesian credibility intervals (CI) were calculated:

Table 92: Simulation Summary Results – Simulation 1

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
Worker 1	0.0000	66.42293	88.76973	64.51248	91.44171	77.12465	76.74997	76.25
Worker 2	0.0185	93.85262	125.052	91.19376	128.7124	108.6394	108.2715	108.25
Worker 3	0.2553	106.1538	142.1947	103.1265	146.5288	123.3331	122.7541	121.25
Worker 4	0.0000	53.34025	71.93241	51.90571	74.2143	62.13377	61.80065	60.25
Worker 5	0.0000	75.36018	100.801	73.30483	103.7646	87.52919	87.11573	86.25
Worker 6	0.9717	133.1064	179.1255	129.3266	184.1964	155.433	154.9561	150.25
Worker 7	0.0000	67.60246	91.04797	65.74694	93.72037	78.78893	78.39056	77.25
Worker 8	0.0000	75.96121	101.4618	73.82028	104.4091	88.01604	87.59079	87.25
Worker 9	0.4020	110.4973	147.6873	107.3347	152.1216	127.9384	127.2589	124.25
Worker 10	0.2824	106.8612	143.1709	103.6013	147.4745	124.1411	123.6203	125.25
Worker 11	0.0000	59.44985	79.83156	57.78559	81.94631	69.18085	68.89727	68.75
Worker 12	0.0000	55.73752	74.84502	54.06075	76.96428	64.90277	64.66764	63.75
Worker 13	0.0010	84.68986	113.1525	82.36906	116.6028	98.1874	97.81163	98.25
Worker 14	0.1787	103.6231	138.6366	100.8463	142.5568	120.4003	119.8748	118.75
Worker 15	0.0698	98.45825	131.9492	95.61997	136.1275	114.4943	113.9588	110.75

As mentioned previously, the results here are rather different from the results obtained Krishnamoorthy and Mathew (2002). Using their method it seems not to be possible to investigate the mean exposure levels of individual workers. For this reason, they found the probability (or tested the hypothesis) that $(100 \times A)\%$ of these workers had exposure levels in excess of a certain pre-defined threshold. However, using the Bayesian methodology as presented above, it is clear that one can examine the probability that a *specific* worker's exposure levels exceed a pre-defined threshold. The above table also provides credibility intervals for each worker's mean exposure levels. For example, it is evident that the mean exposure level for Worker 6, specifically, is considerably higher than that of his/her fellow workers and the probability that his/her exposure levels are in excess of 130 is in excess of 90%. So using this Bayesian framework we are able to examine the exposure of each worker.

7.3.2 Simulation 2

As mentioned previously, the method given by Krishnamoorthy and Mathew is an overall representation of the mean exposure level for each worker and since it does not incorporate the actual row means from the data it is somewhat different to the simulation mentioned previously, which offers the opportunity to look specifically at an individual worker. In this simulation the Bayesian equivalent of this "overall" method is described: The quantity we are interested in simulating is again

$$\mu_{x_i} = e^{\left(\mu + \tau_i + \frac{\sigma_e^2}{2}\right)}$$

1. Let $\tilde{\lambda} = \mu + \tau_i + \frac{\sigma_e^2}{2}$
2. The conditional distribution of $\tilde{\lambda}$ is as follows: $\tilde{\lambda}|\boldsymbol{Y}, \tau_i, \sigma_e^2, \sigma_\tau^2 \sim N\left(\bar{Y}_{..} + \tau_i + \frac{\sigma_e^2}{2}, \frac{\sigma_e^2 + n\sigma_\tau^2}{nk}\right)$.
3. To simulate a value from the unconditional distribution, $\tilde{\lambda}|\boldsymbol{Y}$, we do the following:
    a. Simulate a pair of $(\sigma_e^2, \sigma_\tau^2)$ values from their joint posterior distribution as follows:
        i. Simulate $\tilde{\lambda}$ from a $\chi^2_{v_1}$ distribution where $v_1 = k(n-1)$ and using this calculate: $\sigma_e^2 = \frac{SSE}{\tilde{\lambda}}$
        ii. Simulate δ from a $\chi^2_{v_2}$ distribution where $v_2 = k - 1$ and using this calculate: $\sigma_e^2 + n\sigma_\tau^2 = \frac{SS\tau}{\delta} = \sigma_{12}^2$
        iii. Calculate $\sigma_\tau^2 = \frac{\sigma_{12}^2 - \sigma_e^2}{n}$

iv. If a negative value is obtained in step (iii) we disregard both σ_e^2 and σ_τ^2 and repeat steps (i) - (iii) until we find a pair where both are positive.

b. Simulate a τ_i observation. Since $\tau_i \sim N(0, \sigma_\tau^2)$ this distribution will be the same for all the workers and therefore we have a "general" mean exposure over all workers, as in Krishnamoorthy and Mathew (2002).

c. Using the values simulated in a) and b), i.e. $\tau_i, \sigma_e^2, \sigma_\tau^2$, simulate an observation (λ) from the normal distribution described in Step 2..

d. Calculate $\mu_{x_i} = e^{(\tilde{\lambda})}$.

e. Repeat #a to #d $l\ (= 1000\ or\ 10000)$ times. So now we will have a $1 \times l$ matrix representing an overall picture for all workers.

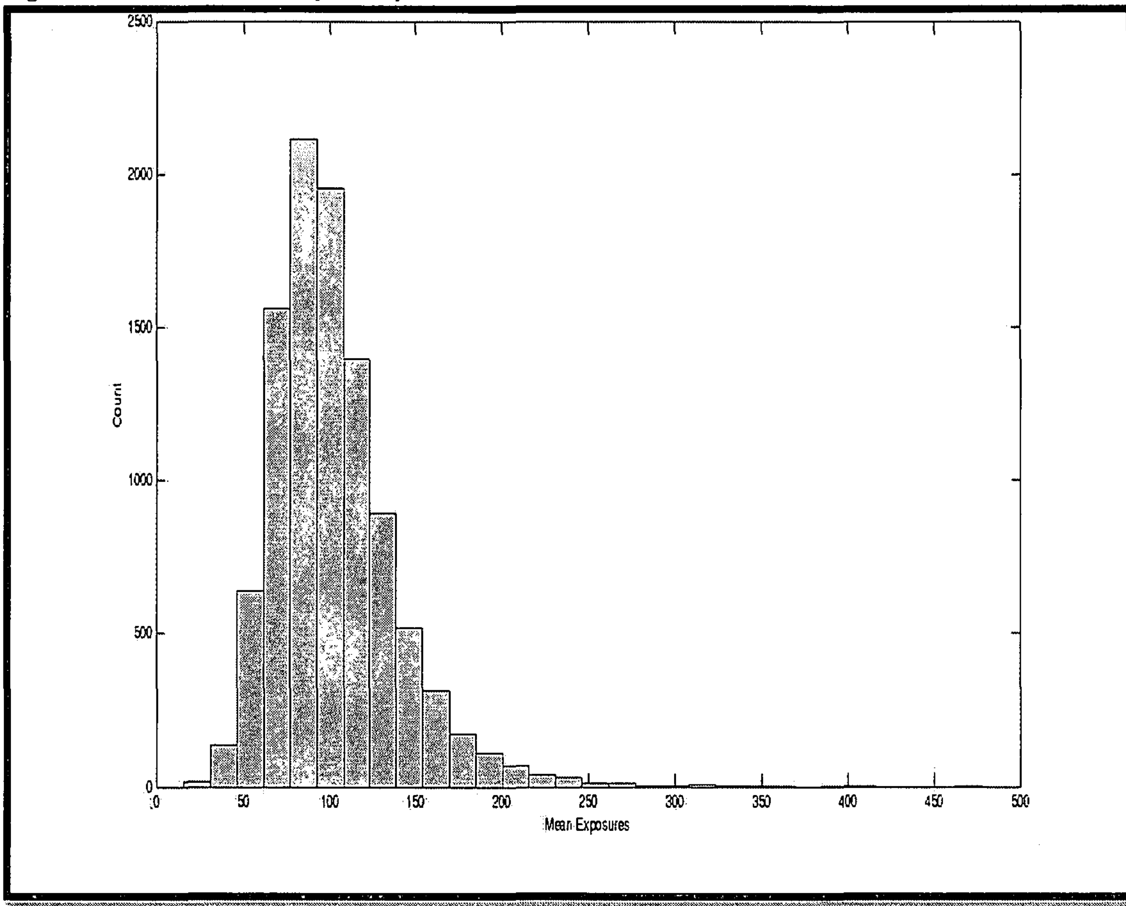
4. For the data (i.e. the $1 \times l$ matrix of simulated observations mentioned in sub-step e) above):

    a. Draw up a histogram - so there will be only a single histogram representing an overall picture for all workers.

    b. Calculate $P\left(\mu_{x_i} > OEL\right) = \frac{\#\,Simulated\ Values > OEL}{l}$

Assume, for the purposes of illustration that $OEL = 130$ and furthermore take $l = 10000$. In this case, the following results were obtained:

Figure 41: Overall Mean Exposure per Worker



*Probability that mean exposure > OEL (130) = 0.1762
*90% Bayesian Confidence Interval: [56.5422; 166.8839]
*95% Bayesian Confidence Interval: [49.8624; 189.4470]
*Mean = 102.2833; Median = 96.4365; Mode = 83.25

The following table represents the results from the above graph:

Table 93: Simulation Summary Results – Simulation 2

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
All Workers	0.1762	56.5422	166.8839	49.8624	189.4470	102.2833	96.4365	83.25

We can see that the credibility intervals for the overall mean exposure per worker are rather wide, even though the mean values are well below the OEL.

Krishnamoorthy and Mathew (2002) also simulate the following statistic (and for the purposes of comparison will be analysed using the Bayesian methodology developed previously):

$$T = \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2$$

where A is a suitably chosen parameter between 0 and 1 and Z is the cumulative distribution function of the standard normal distribution. Using a specific value of OEL the following hypothesis can be tested:

$$H_0: \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2 \geq ln(OEL)$$

against the alternative hypothesis

$$H_1: \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2 < ln(OEL)$$

For example, if our choice of A is 0.05 then essentially we are testing (one-sided) whether at least 5% of the workers have mean exposure levels in excess of the chosen OEL. The OEL is chosen to be a clinically relevant value. The specific choice of OEL is not the primary concern of this research, but primarily a demonstration of the Bayesian methodology.

In order to replicate the methodology of Krishnamoorthy and Mathew from a Bayesian perspective the following simulation study was undertaken for a range of both OEL and A values:

1. Let $T = \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2$
2. To simulate a value of T we do the following:
    a. Simulate a pair of $(\sigma_e^2, \sigma_\tau^2)$ values from their joint posterior distribution as follows:
        i. Simulate $\tilde{\lambda}$ from a $\chi^2_{\nu_1}$ distribution where $\nu_1 = k(n-1)$ and using this calculate: $\sigma_e^2 = \frac{SSE}{\lambda}$

ii. Simulate δ from a $\chi^2_{\nu_2}$ distribution where $\nu_2 = k - 1$ and using this calculate: $\sigma_e^2 + n\sigma_\tau^2 = \frac{SS\tau}{\delta} = \sigma_{12}^2$

iii. Calculate $\sigma_\tau^2 = \frac{\sigma_{12}^2 - \sigma_e^2}{n}$

iv. If a negative value is obtained in step (iii) we disregard both σ_e^2 and σ_τ^2 and repeat steps (i) - (iii) until we find a pair where both are positive.

b. Using the values simulated in a) i.e. $\sigma_e^2, \sigma_\tau^2$, simulate μ from the following normal distribution $\mu|\boldsymbol{Y}, \sigma_e^2, \sigma_\tau^2 \sim N\left(\bar{Y}_{..}, \frac{\sigma_e^2 + n\sigma_\tau^2}{nk}\right)$

c. Calculate $T = \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2$, where Z is a standard normal variable and Z_{1-A} is the inverse of the cumulative distribution function.

d. Repeat #a to #c $l\ (= 1000\ or\ 10000)$ times. So now we will have a $1 \times l$ matrix.

3. For the data (i.e. the $1 \times l$ matrix of simulated observations mentioned in sub-step d) above):

   a. Order the observations from smallest to largest.

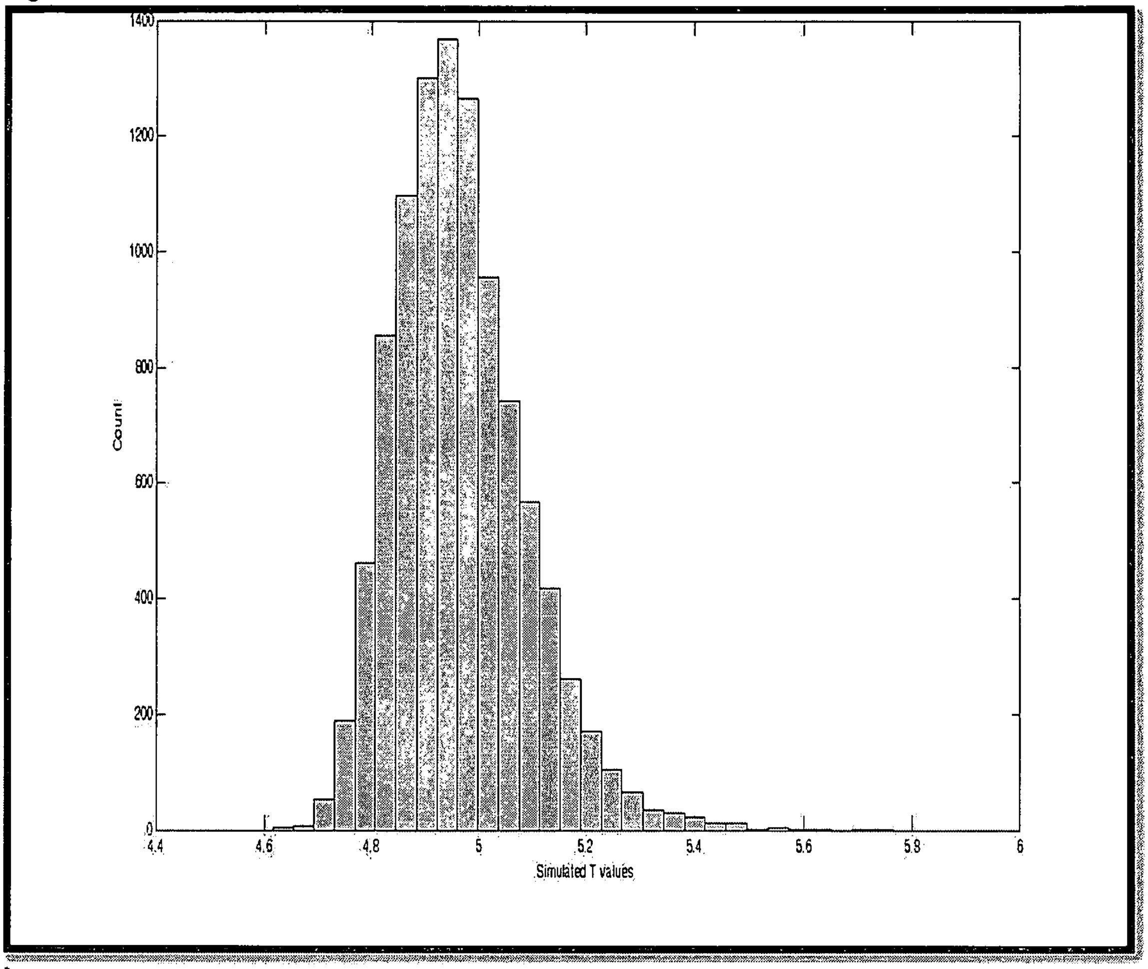
   b. Taking $\alpha = 0.05$ find the $100(1 - \alpha)$th percentile.

   c. For a specific choice of OEL determine whether $ln(OEL)$ falls in the critical region (i.e. test the hypothesis described earlier).

This procedure was performed for several choices of OEL (= [130; 140; 150; 160; 170; 180]) and for several choices of A (= [0.1; 0.05; 0.025; 0.001]).

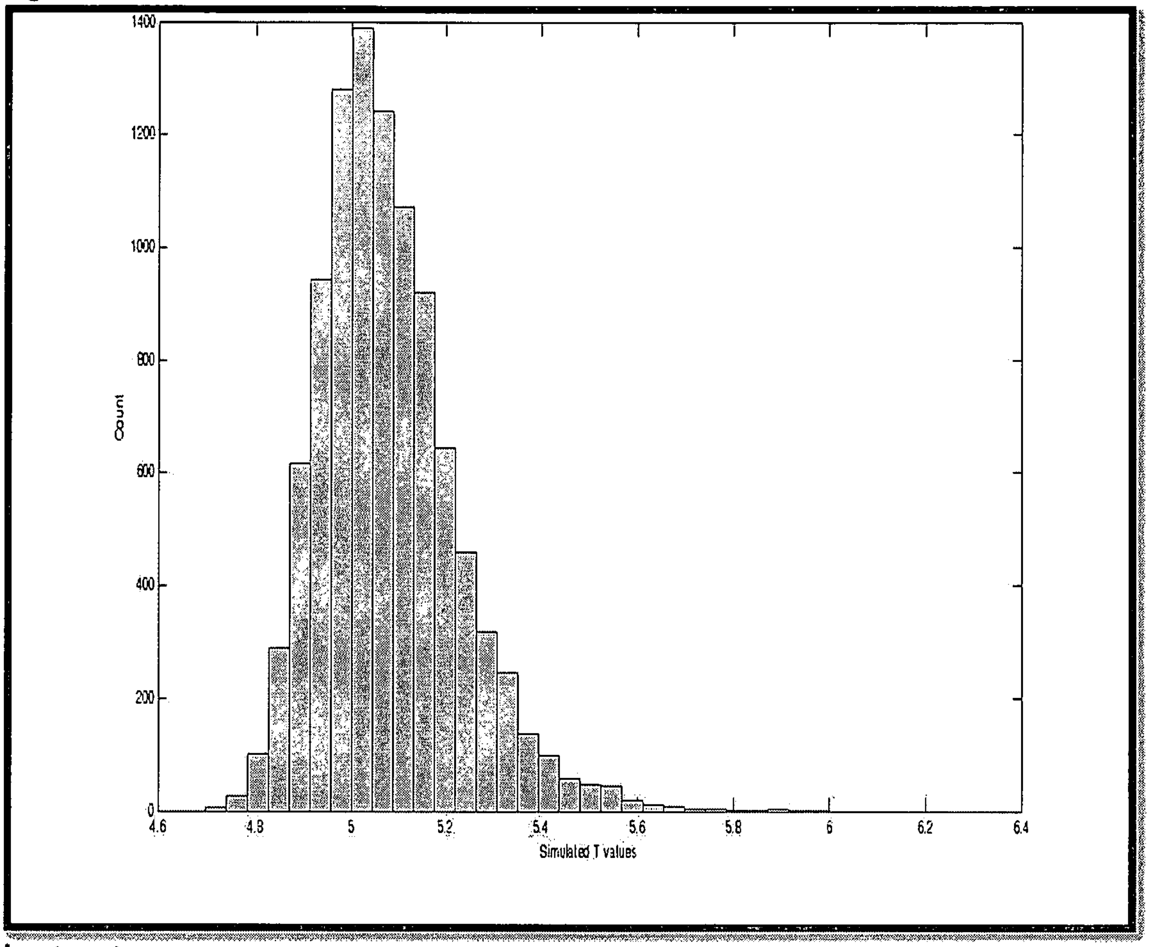
The following results were obtained:

Figure 42: Simulated T-values for A = 0.1 – Simulation 2



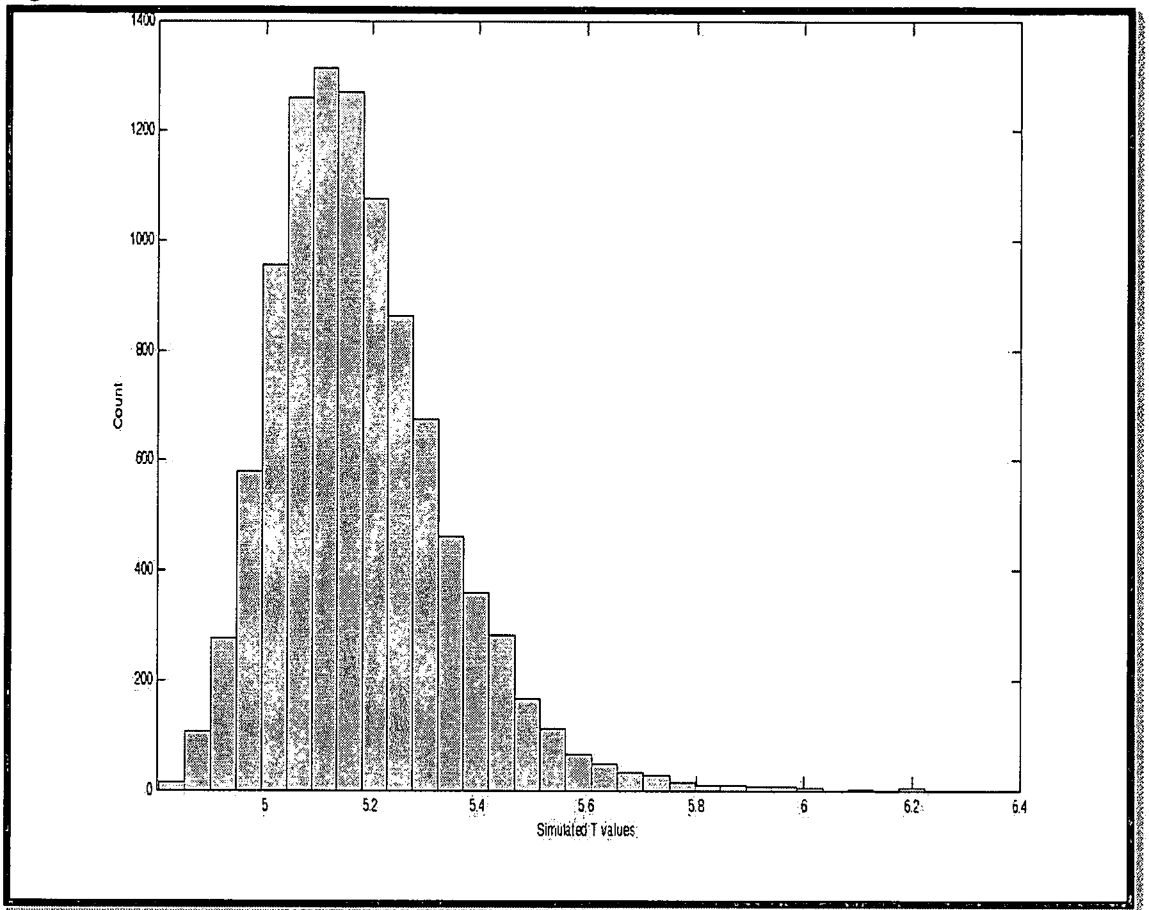
*$100(1-\alpha)th$ percentile = 5.1807
*$\ln(130) = 4.8675$
*$\ln(140) = 4.9416$
*$\ln(150) = 5.0106$
*$\ln(160) = 5.0752$
*$\ln(170) = 5.1356$
*$\ln(180) = 5.1930$

Figure 43: Simulated T-values for A = 0.05 – Simulation 2



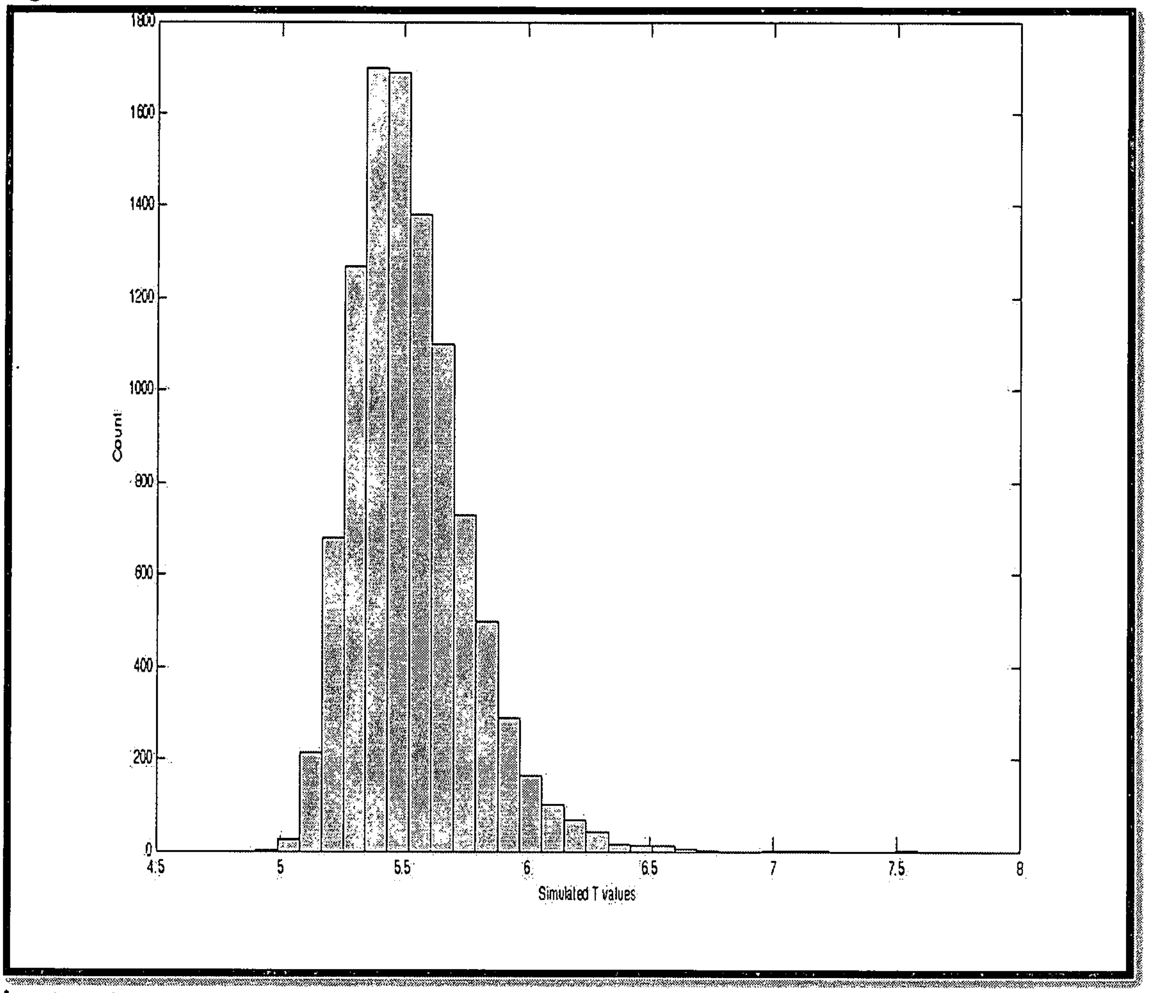
*$100(1-\alpha)th$ percentile = 5.3389

Figure 44: Simulated T-values for A = 0.025 – Simulation 2



*$100(1-\alpha)th$ percentile = 5.4681

Figure 45: Simulated T-values for A = 0.001 – Simulation 2



$^{\bullet}100(1-\alpha)th$ percentile = 5.9388

So for the simulated "Styrene Exposures" we can, by way of example arrive at the following: for the first figure we can say that 10% (or more) of the workers had exposure values in excess of 170 (95th percentile = 5.1907 and $\ln(170)$ = 5.1358) whereas we cannot say that 10% (or more) of the workers had exposure values in excess of 180 ($\ln(180)$ = 5.193) at a 5% significance level.

7.3.3 Simulation 3

We now shift our attention to the overall mean exposure, i.e. the mean exposure for all the workers.

So we now define the overall mean exposure as:

$$\mu_x = e^{\left(\mu + \frac{\sigma_e^2 + \sigma_\tau^2}{2}\right)}$$

To simulate mean overall exposure levels we do the following:

1. Define $\beta = \mu + \frac{\sigma_e^2 + \sigma_\tau^2}{2}$.
2. Now, $\beta | \boldsymbol{Y}, \sigma_e^2, \sigma_\tau^2 \sim N\left(\bar{Y}_{..} + \frac{\sigma_e^2 + \sigma_\tau^2}{2}, \frac{\sigma_e^2 + n\sigma_\tau^2}{nk}\right)$. This is the conditional posterior distribution.
3. To simulate an observation from the unconditional posterior distribution, $\beta | \boldsymbol{Y}$, we can do the following:
   a. Simulate a pair of $(\sigma_e^2, \sigma_\tau^2)$ values from their joint posterior distribution as follows:
      i. Simulate λ from a $\chi_{\nu_1}^2$ distribution where $\nu_1 = k(n-1)$ and using this calculate: $\sigma_e^2 = \frac{SSE}{\lambda}$
      ii. Simulate δ from a $\chi_{\nu_2}^2$ distribution where $\nu_2 = k - 1$ and using this calculate: $\sigma_e^2 + n\sigma_\tau^2 = \frac{SS\tau}{\delta} = \sigma_{12}^2$
      iii. Calculate $\sigma_\tau^2 = \frac{\sigma_{12}^2 - \sigma_e^2}{n}$
      iv. If a negative value is obtained in step (iii) we disregard both σ_e^2 and σ_τ^2 and repeat steps (i) - (iii) until we find a pair where both are positive.

b. Using this pair of $(\sigma_e^2, \sigma_\tau^2)$ values and the data simulate a $\beta|\, \boldsymbol{Y}$. Since we have the variance components we can plug them in and simulate the values.

c. Calculate $\mu_x \;= e^{(\beta)}$.

d. Repeat sub-steps a) to c) $l\,(= 1000\; or\; 10000)$ times. So now we will have a $1\; \times l$ matrix representing simulated overall mean exposure levels.

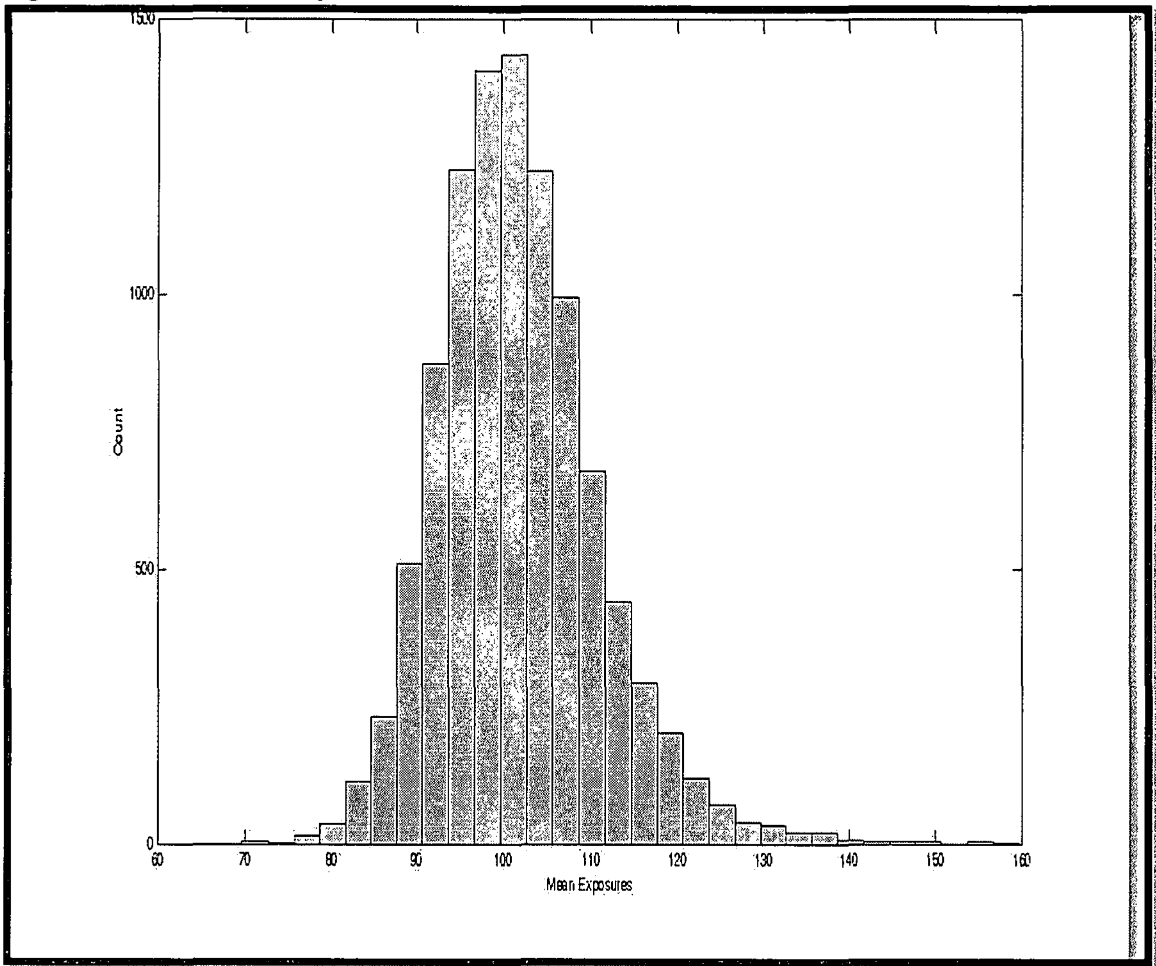
4. For the data (i.e. the $1\; \times l$ matrix of simulated observations mentioned in sub-step d) above):

   a. Draw up a histogram - so there will be only a single histogram representing all simulated mean exposure levels.

   b. Calculate $P\left(e^{(\beta)} > OEL\right) = \frac{\#\, Simulated\; Values > \; OEL}{l}$

Assume, for the purposes of illustration that $OEL = 130$ and furthermore take $l = 10000$. In this case, the following results were obtained:

Figure 46: Overall Mean Exposure – Simulation 3



*Probability that mean exposure > OEL (130) = 0.0091
*90% Bayesian Confidence Interval: [88.3341; 117.9463]
*95% Bayesian Confidence Interval: [85.8763; 122.4390]
*Mean = 101.6887; Median = 100.9088; Mode = 97.75

Table 94: Simulation Summary Results – Simulation 3

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
All Workers	0.0091	88.3341	117.9463	85.8763	122.439	101.6887	100.9088	97.75

Thus, the overall mean exposure can be easily simulated. The above distribution is substantially narrower than the overall mean per each worker as discussed in the relevant simulation study.

7.3.4 The Geometric Mean

In finance particularly there is an abundant number of settings where one could have a random effects model and where the data are lognormally distributed. Consider the following as two elementary examples:

1. You have a sample of k insurance companies operating in the insurance industry and for each of these companies you have n years worth total claims data. In this case the "exposure" is not so much a biological exposure or threshold that is of risk to the company, but rather a financial risk. Larger claims for an individual company result in a larger chance of bankruptcy or liquidity shortages (analogous to death or disability in the previous sections).
2. A sample of k investment portfolios is drawn and for each of these investment portfolios there are n periods (annual, quarterly or even monthly) worth of returns data for each investment portfolio. In this case a larger return is of value to a portfolio manager, even though it is not a negative characteristic.

The geometric mean is often of interest in these situations, since it minimizes the effect of outliers. Whereas the arithmetic mean is very sensitive to outliers, the geometric mean is less so. In addition to these advantages financial institutions calculate the geometric mean for portfolios since it is particularly suited to percentages and the actual amount invested does not have to be known. In addition to this, many actuarial texts (not mentioned here) reveal that returns and interest rates generally follow a lognormal distribution. Thus, the possibility of applying the method mentioned before to the geometric mean is plausible. The data for these situations may diagrammatically be arranged as:

Table 95: Data Representation – Insurance Claims

	Annual Insurance Claims / Portfolio Returns				
Companies / Investment Portfolios	**1**	**2**	**...**	**n**	**Company / Portfolio Means**
1	x_{11}	x_{12}	...	x_{1n}	$\bar{X}_{1\cdot}$
2	x_{21}	x_{22}	...	x_{2n}	$\bar{X}_{2\cdot}$
...	...	...	...	...	...
k	x_{k1}	x_{k2}	...	x_{kn}	$\bar{X}_{k\cdot}$

Given the previous discussion the extension of the Bayesian method to the geometric mean exposures will be applied to the simulated "styrene exposures", but with the knowledge that the application may not be limited to just this kind of biological setting.

The following is an introductory definition of the geometric mean (hereafter referred to as "GM") and a few derivations:

$$GM = \left(x_{f_1} x_{f_2} \dots x_{f_n}\right)^{1/n} = \left(\prod_{j=1}^{n} x_{f_j}\right)^{1/n} \tag{7.9}$$

and as before the $x_{f_j}, j = 1, \dots, n$ have a lognormal distribution.

The expected value can be derived as:

$$E(GM) = E\left(\prod_{j=1}^{n} x_{f_j}\right)^{1/n} = E\left(e^{Y_{f1}}\, e^{Y_{f2}} \dots e^{Y_{fn}}\right)^{1/n} = E\left(e^{\frac{1}{n}\sum_{j=1}^{n} Y_{f_j}}\right) = E\left(e^{\bar{Y}_{f\cdot}}\right) \tag{7.10}$$

Now,

$$Y_{f_j} | \mu, \sigma_\tau^2, \sigma_e^2 \sim N(\mu, \sigma_\tau^2 + \sigma_e^2)$$

and

$$\bar{Y}_{f.} | \mu, \sigma_\tau^2, \sigma_e^2 \sim N\left(\mu, \frac{n\sigma_\tau^2 + \sigma_e^2}{n}\right)$$

Therefore,

$$E\left(e^{\bar{Y}_{f.}}\right) = e^{\left(\mu + \frac{n\sigma_\tau^2 + \sigma_e^2}{2n}\right)} = \mu_x(GM) \tag{7.11}$$

So we are interested in the above mean exposure for the f-th (future) worker/company/portfolio.

Using the results obtained in Section 7.2 it is a simple matter to simulate mean exposures from the posterior distribution, given that the prior distribution applied is the non-informative prior distribution given by Box and Tiao (1973). The procedure for simulating is as follows:

The prior distribution is:

$$p(\mu, \sigma_e^2, \sigma_\tau^2) \propto \sigma_e^{-2}(\sigma_e^2 + n\sigma_\tau^2)^{-1}$$

This was the same prior as used in previous simulations. So to simulate $\mu_x(GM)$ values do the following:

1. Define $\beta = \mu + \frac{\sigma_e^2 + n\sigma_\tau^2}{2n}$.
2. Now, $\beta | \boldsymbol{Y}, \sigma_e^2, \sigma_\tau^2 \sim N\left(\bar{Y}_{..} + \frac{\sigma_e^2 + n\sigma_\tau^2}{2n}, \frac{\sigma_e^2 + n\sigma_\tau^2}{nk}\right)$. This is the conditional posterior distribution.

3. To simulate an observation from the unconditional posterior distribution, $\beta | \boldsymbol{Y}$, we can do the following:

   a. Simulate a pair of $(\sigma_e^2, \sigma_\tau^2)$ values from their joint posterior distribution as follows:

      i. Simulate λ from a $\chi^2_{\nu_1}$ distribution where $\nu_1 = k(n-1)$ and using this calculate: $\sigma_e^2 = \frac{SSE}{\lambda}$

      ii. Simulate δ from a $\chi^2_{\nu_2}$ distribution where $\nu_2 = k - 1$ and using this calculate: $\sigma_e^2 + n\sigma_\tau^2 = \frac{SS\tau}{\delta} = \sigma_{12}^2$

      iii. Calculate $\sigma_\tau^2 = \frac{\sigma_{12}^2 - \sigma_e^2}{n}$

      iv. If a negative value is obtained in step (iii) we disregard both σ_e^2 and σ_τ^2 and repeat steps (i) - (iii) until we find a pair where both are positive.

   b. Using this pair of $(\sigma_e^2, \sigma_\tau^2)$ values and the data simulate a $\beta | \boldsymbol{Y}$. Since we have the variance components we can substitute them in 2) above and simulate the values.

   c. Calculate $\mu_x = e^{(\beta)}$.

   d. Repeat sub-steps a) to c) $l\ (= 1000\ or\ 10000)$ times. So now we will have a $1 \times l$ matrix representing simulated overall geometric mean exposure levels.

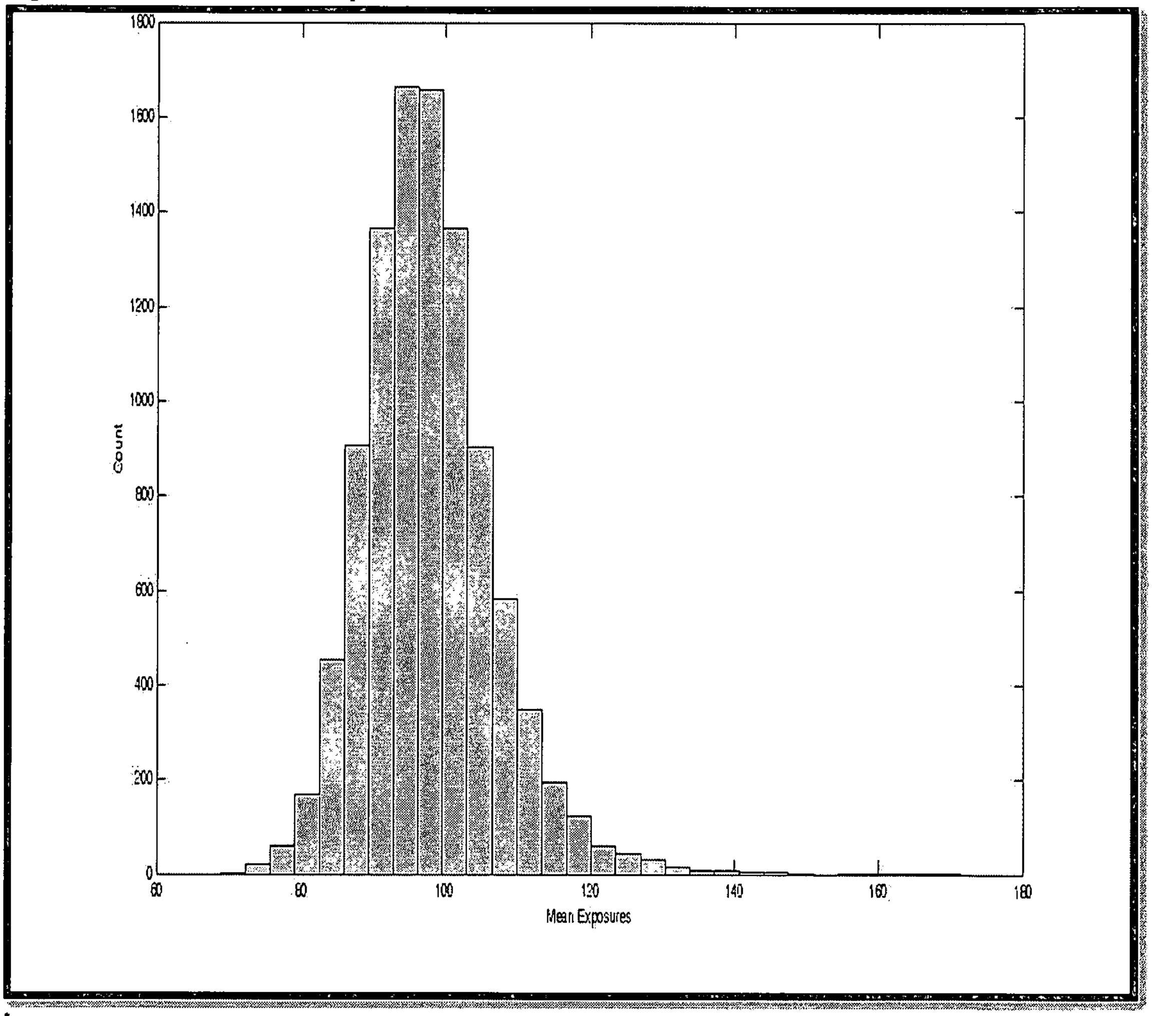
4. For the data (i.e. the $1 \times l$ matrix of simulated observations mentioned in d) above):

   a. Draw up a histogram.

b. Calculate $P\left(e^{(\beta)} > OEL\right) = \frac{\#\,Simulated\,Values > OEL}{l}$

The following results were obtained:

Figure 47: Geometric Mean Exposure - Box and Tiao Prior Distribution



•Probability that mean exposure > OEL (130) = 0.0047
•90% Bayesian Confidence Interval: [84.6968; 113.2535]
•95% Bayesian Confidence Interval: [82.5039; 118.1787]
•Mean = 97.6988; Median = 96.9063; Mode = 96.25

Table 96: Simulation Summary Results – Geometric Mean

Worker	$P(\mu_{exposure} > 130)$	90% CI Low	90% CI High	95%CI Low	95%CI High	Mean	Median	Mode
All Workers	0.0047	84.6968	113.2535	82.5039	118.1787	97.6988	96.9063	96.25

Other Prior Distributions

As before, in this section we will use the method described by Datta and Ghosh (1995) to obtain Probability-Matching priors for the parameters of the lognormal distribution.

Probability-Matching Prior Distribution

In this section the Probability-Matching prior (PMP) will be derived for $e^{\left(\mu+\frac{n\sigma_\tau^2+\sigma_e^2}{2n}\right)}$ and this will be compared to the Box and Tiao (1973) prior as well as the Reference prior.

The Probability-Matching prior is

$$\pi_1(\boldsymbol{\theta}) = \pi_1(\mu,\sigma_\tau^2,\sigma_e^2) \propto (n\sigma_\tau^2+\sigma_e^2)^{-1}\left[1+\frac{2n}{(n\sigma_\tau^2+\sigma_e^2)}\right]^{\frac{1}{2}} \tag{7.12}$$

because

$$\frac{\partial}{\partial\mu}\{\pi(\boldsymbol{\theta})\zeta_1(\boldsymbol{\theta})\} + \frac{\partial}{\partial\sigma_\tau^2}\{\pi(\boldsymbol{\theta})\zeta_{21}(\boldsymbol{\theta})\} + \frac{\partial}{\partial\sigma_e^2}\{\pi(\boldsymbol{\theta})\zeta_3(\boldsymbol{\theta})\} = 0+0+0=0.$$

The proof is given in the appendix to this chapter.

In addition,

$$\pi_2(\boldsymbol{\theta}) = \pi_2(\mu,\sigma_\tau^2,\sigma_e^2) \propto \sigma_e^{-2}(n\sigma_\tau^2+\sigma_e^2)^{-1}\left[1+\frac{2n}{(n\sigma_\tau^2+\sigma_e^2)}\right]^{\frac{1}{2}}$$

(7.13)

will also be a Probability-Matching prior.

It is interesting to note that if we substitute $n = 1, \sigma_\tau^2 = 0\ and\ \sigma_e^2 = \sigma^2$ into (7.12) then it follows that

$$\pi_1(\boldsymbol{\theta}) = \frac{1}{\sigma^2}\sqrt{1 + \frac{2}{\sigma^2}}$$

which is the same as the Probability-Matching prior derived in Chapter 3, Theorem 3.2.

<u>Reference Prior Distribution</u>

In this section the Reference prior will be derived for $e^{\left(\mu + \frac{n\sigma_\tau^2 + \sigma_e^2}{2n}\right)}$.

$$\therefore p_R(\mu, \sigma_\tau^2, \sigma_e^2) \propto t(\boldsymbol{\theta})[t(\boldsymbol{\theta})]^{-1}\left(\frac{1}{2(n\sigma_\tau^2 + \sigma_e^2)} + \frac{n}{(n\sigma_\tau^2 + \sigma_e^2)^2}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2}$$

$$\propto (n\sigma_\tau^2 + \sigma_e^2)^{-\frac{1}{2}}\left(\frac{1}{2} + \frac{n}{(n\sigma_\tau^2 + \sigma_e^2)}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2}$$

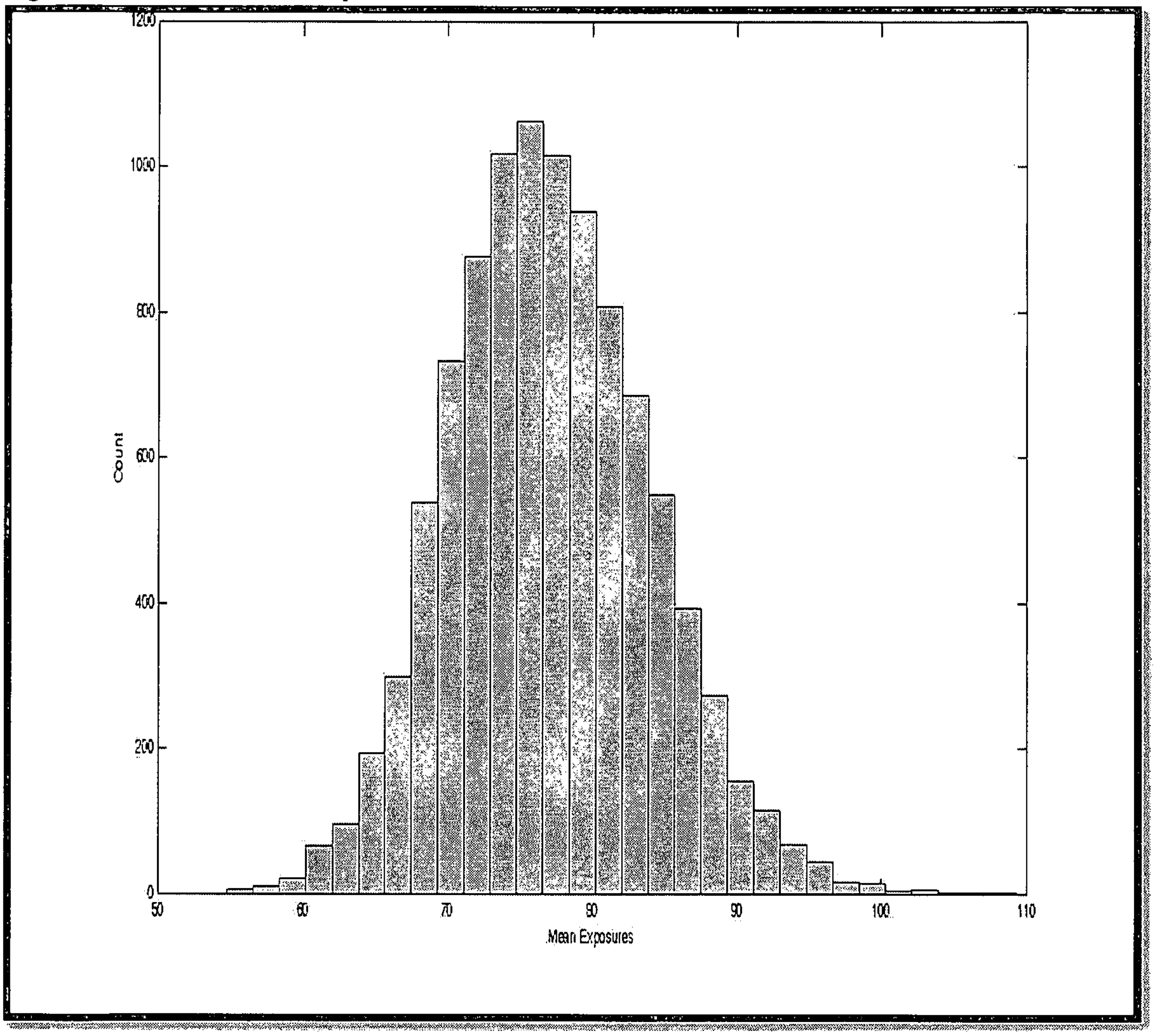
$$\propto (n\sigma_\tau^2 + \sigma_e^2)^{-\frac{1}{2}}\left(1 + \frac{2n}{(n\sigma_\tau^2 + \sigma_e^2)}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2}$$

This result is once again similar to the Reference prior distribution derived in Chapter 3, Theorem 3.1. The proof is given in the appendix to this chapter.

Appendix to Chapter 7

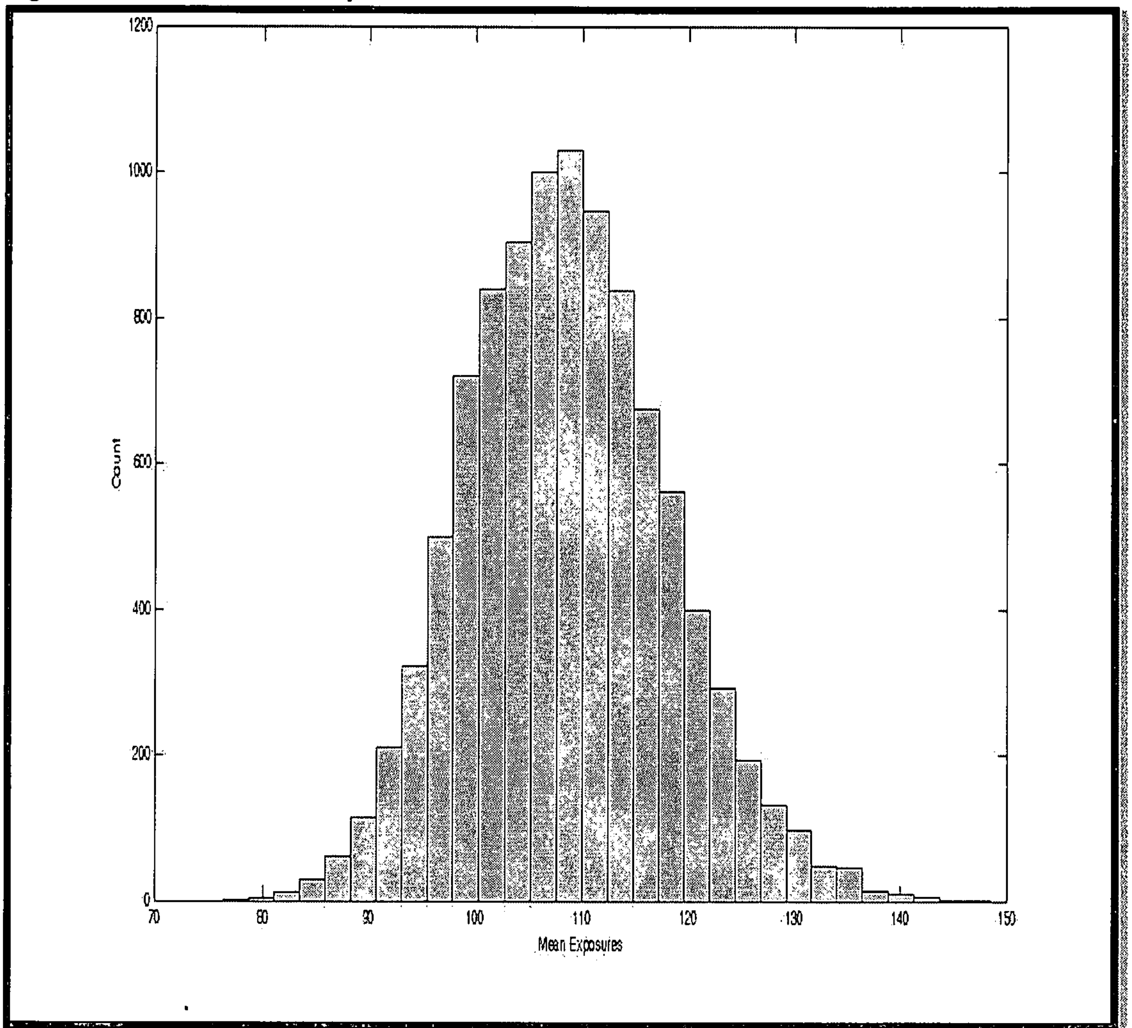
Results of Simulation Studies

Figure 48: Simulated Mean Exposures for the Worker 1



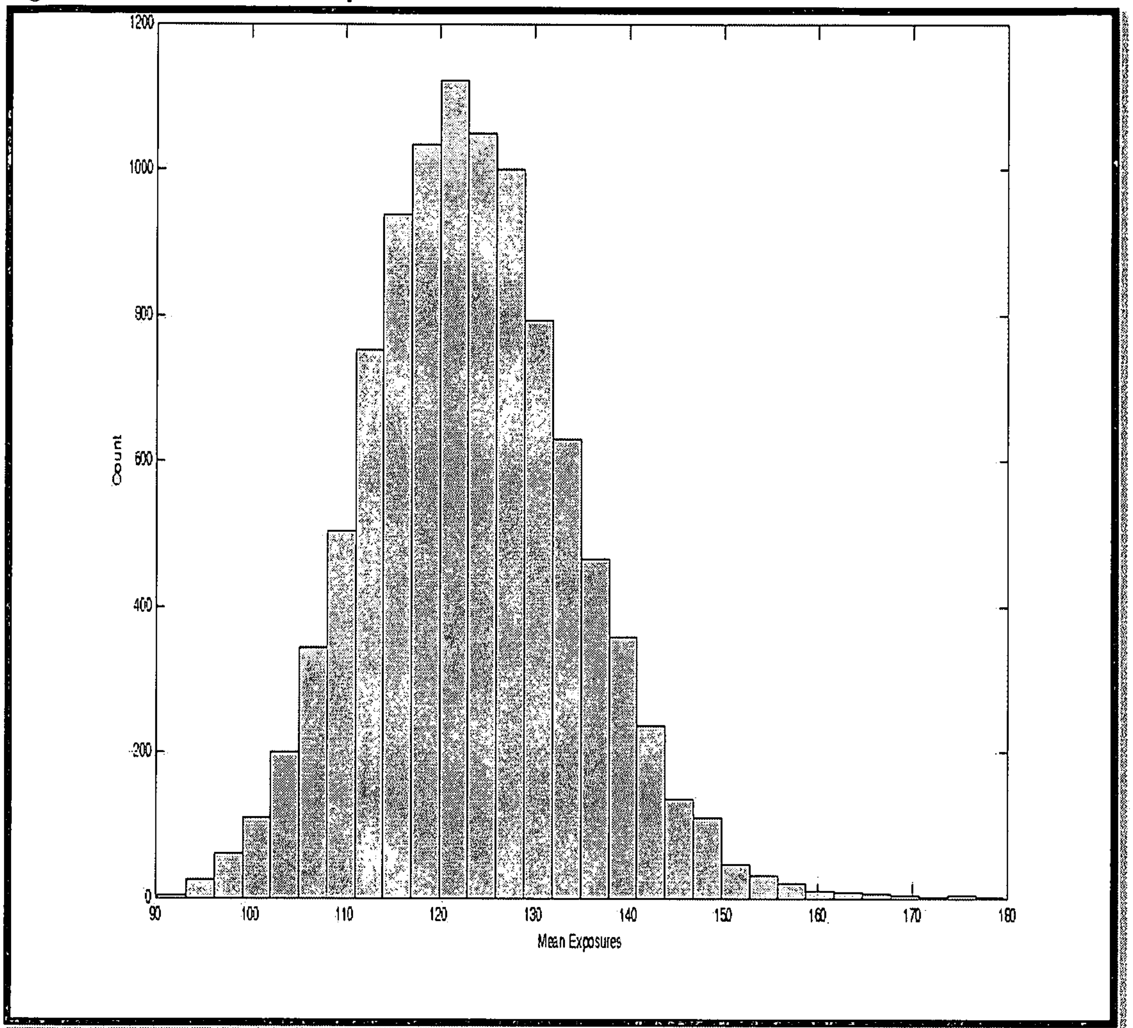
- Worker 1
- Probability that mean exposure > OEL (130) = 0.0000
- 90% Bayesian Confidence Interval: [66.4229; 88.7697]
- 95% Bayesian Confidence Interval: [64.5125; 91.4417]
- Mean = 77.1247; Median = 76.74997; Mode = 76.25

Figure 49: Simulated Mean Exposures for the Worker 2



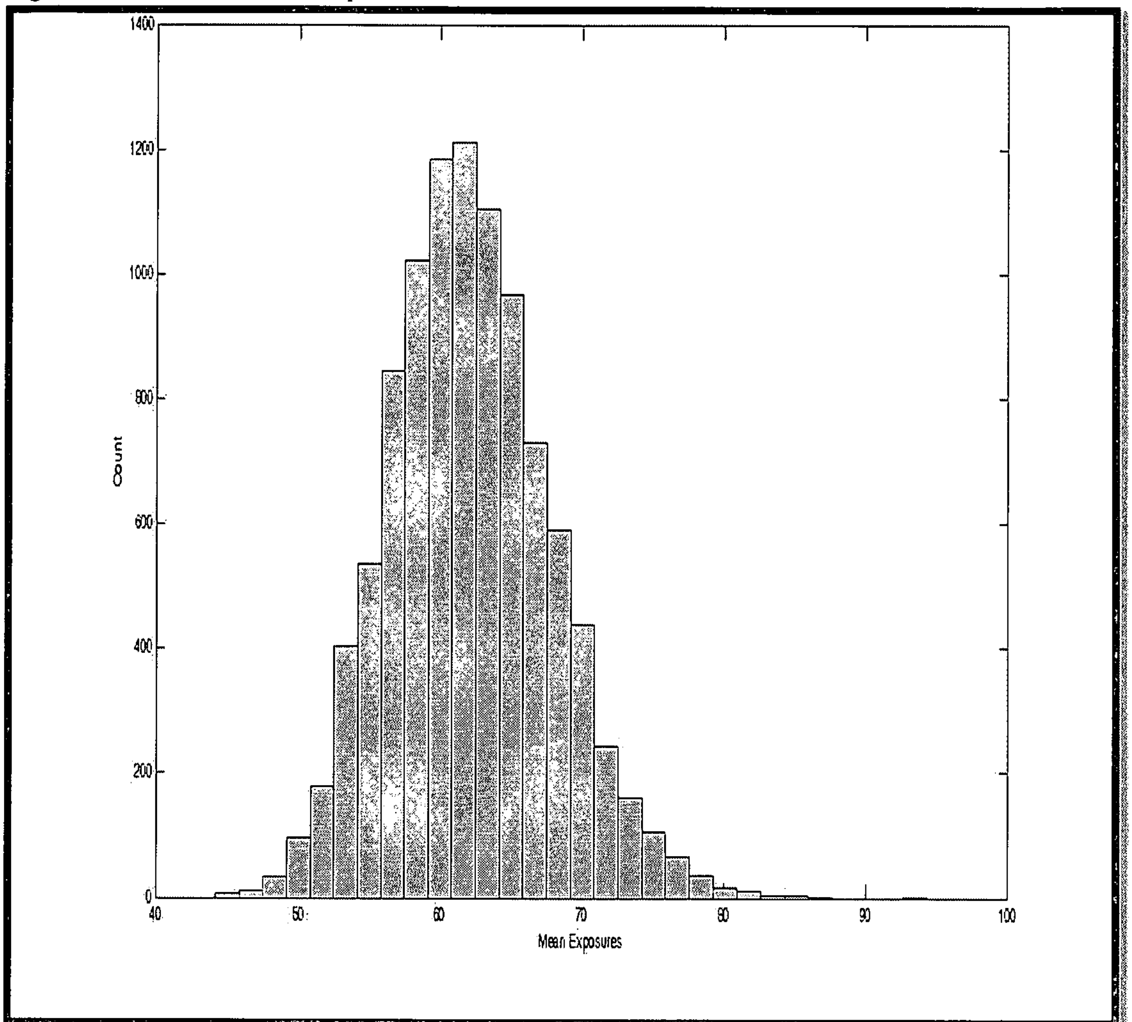
- Worker 2
- Probability that mean exposure > OEL (130) = 0.0185
- 90% Bayesian Confidence Interval: [93.8526; 125.052]
- 95% Bayesian Confidence Interval: [91.1934; 128.712]
- Mean = 108.6394; Median = 108.2715; Mode = 108.25

Figure 50: Simulated Mean Exposures for the Worker 3



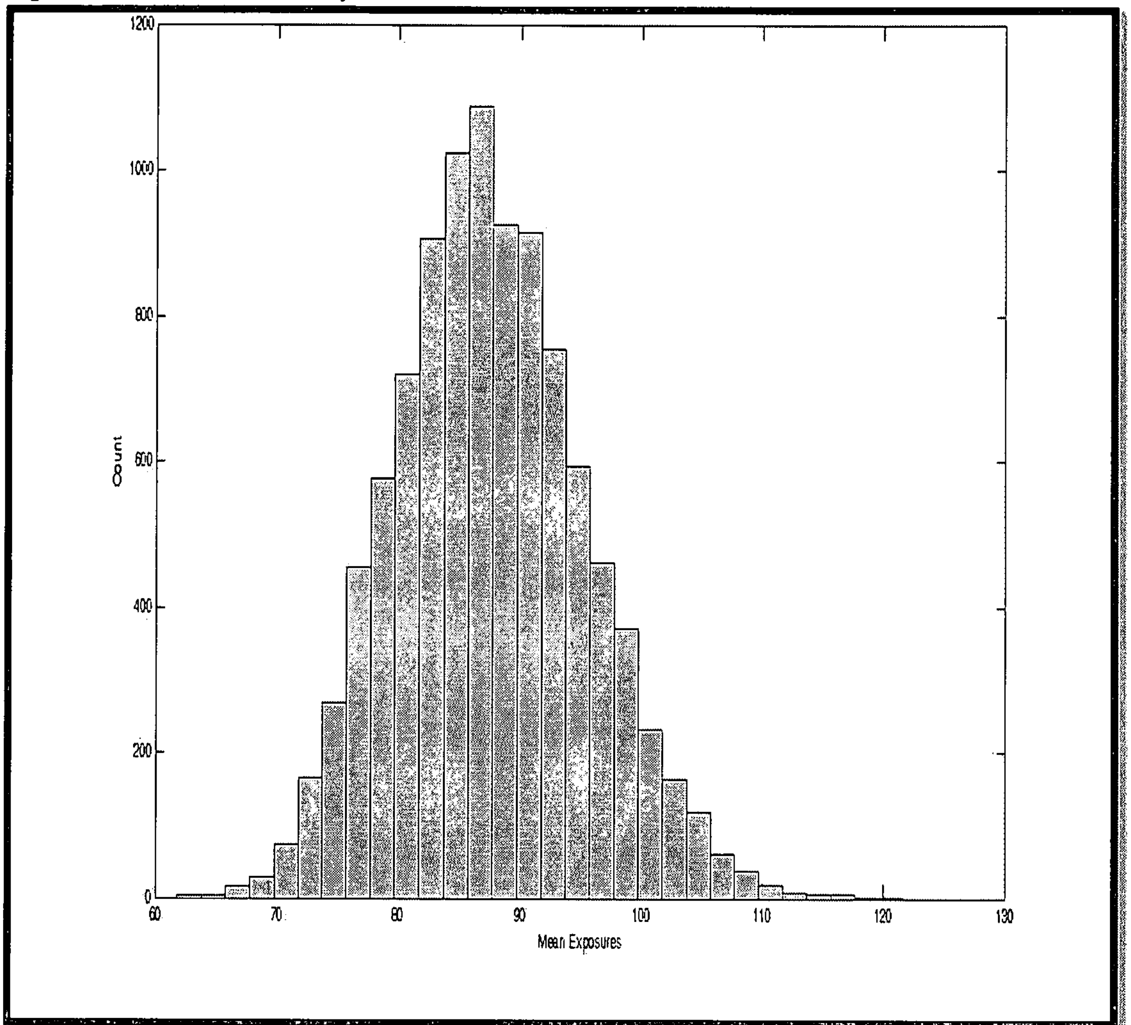
- Worker 3
- Probability that mean exposure > OEL (130) = 0.2553
- 90% Bayesian Confidence Interval: [106.1538; 142.1947]
- 95% Bayesian Confidence Interval: [103.1265; 146.5288]
- Mean = 123.3331; Median = 122.7541; Mode = 121.25

Figure 51: Simulated Mean Exposures for the Worker 4



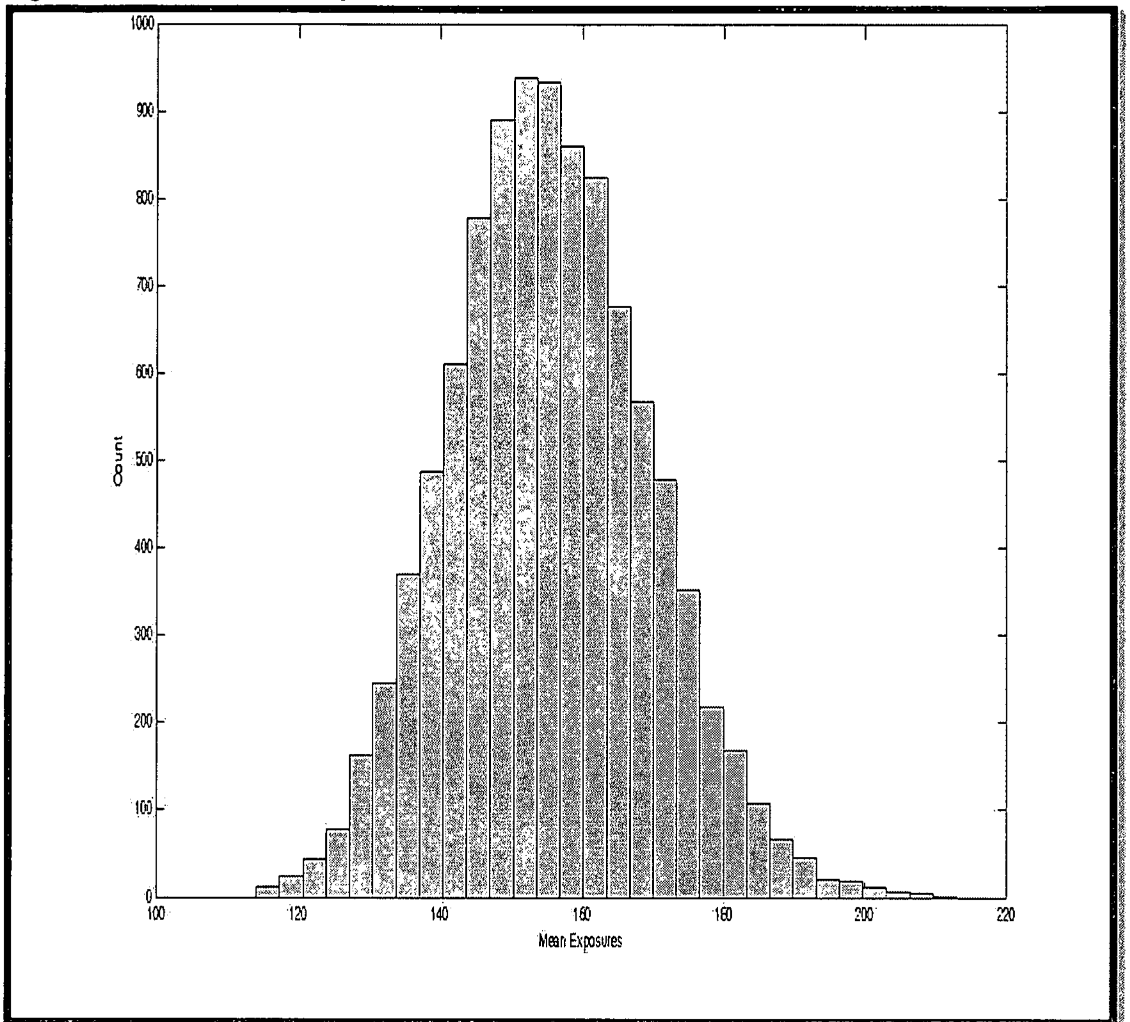
- Worker 4
- Probability that mean exposure > OEL (130) = 0.0000
- 90% Bayesian Confidence Interval: [53.3403; 71.9324]
- 95% Bayesian Confidence Interval: [51.9057; 74.2143]
- Mean = 62.1338; Median = 61.8007; Mode = 60.25

Figure 52:Simulated Mean Exposures for the Worker 5



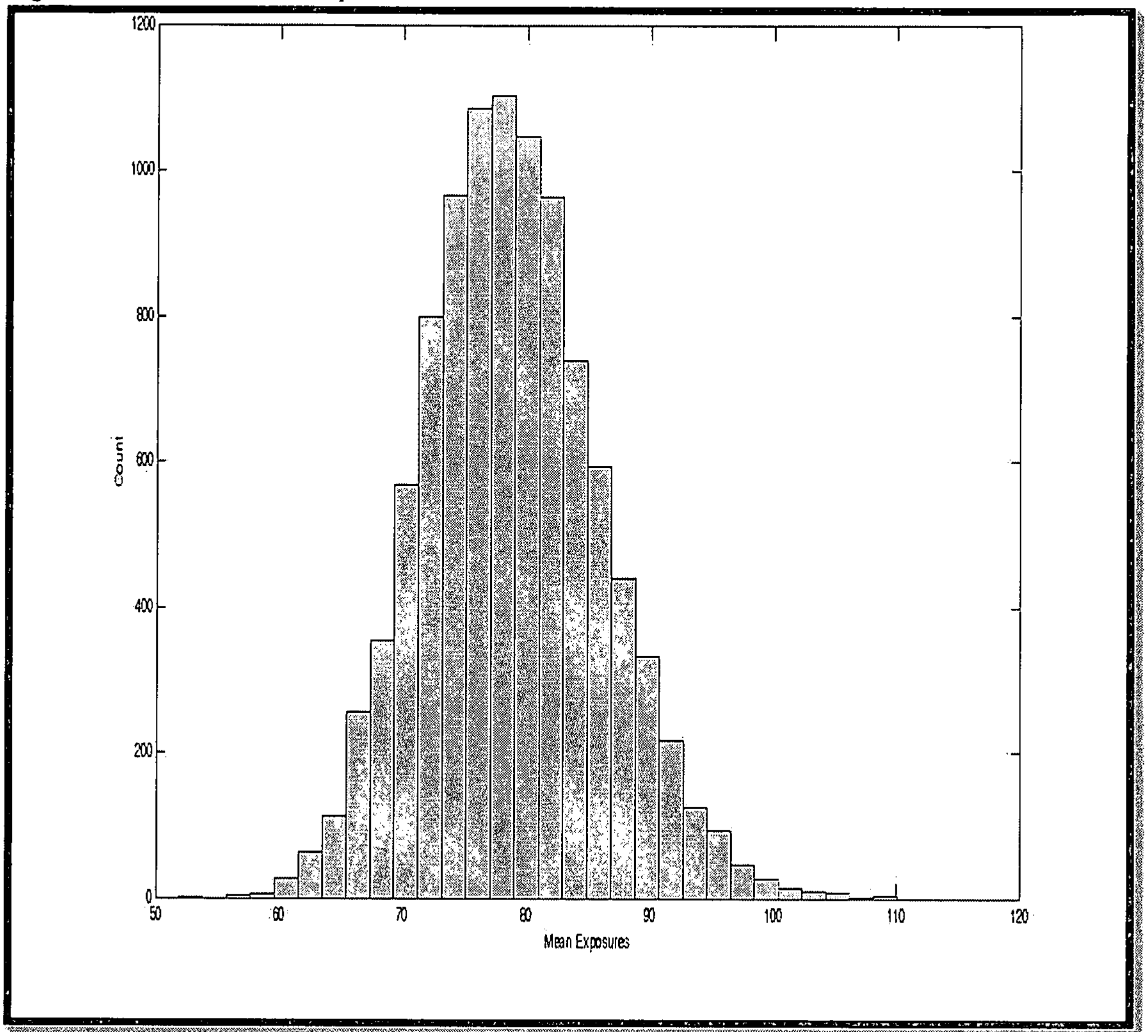
•Worker 5
•Probability that mean exposure > OEL (130) = 0.0000
•90% Bayesian Confidence Interval: [75.3602; 100.801]
•95% Bayesian Confidence Interval: [73.3048; 103.765]
•Mean = 87.5292; Median = 87.1157; Mode = 86.25

Figure 53:Simulated Mean Exposures for the Worker 6



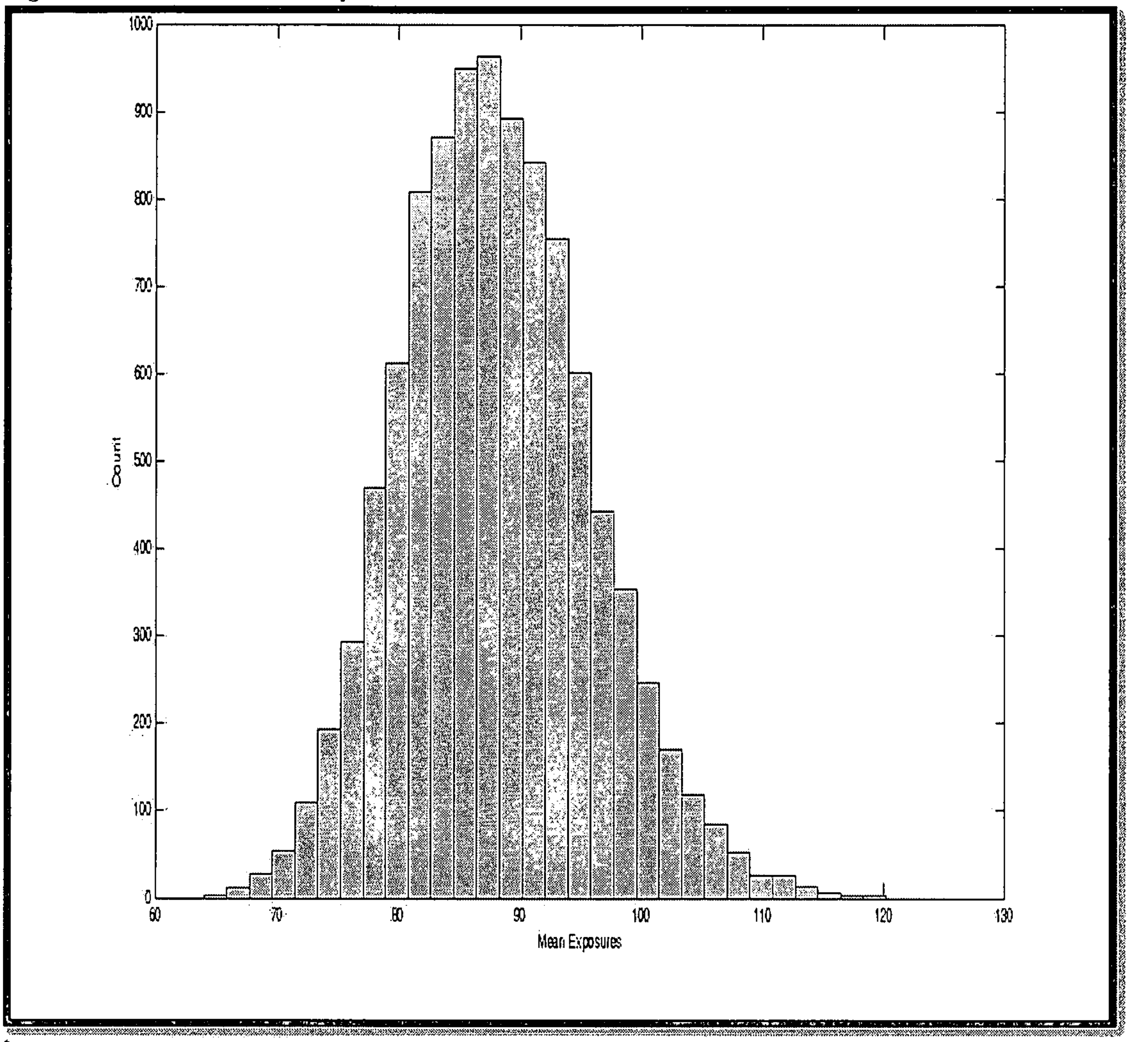
*Worker 6
*Probability that mean exposure > OEL (130) = 0.9717
*90% Bayesian Confidence Interval: [133.1064; 179.1255]
*95% Bayesian Confidence Interval: [129.3266; 184.1964]
*Mean = 155.433; Median = 154.9561; Mode = 150.25

Figure 54:Simulated Mean Exposures for the Worker 7



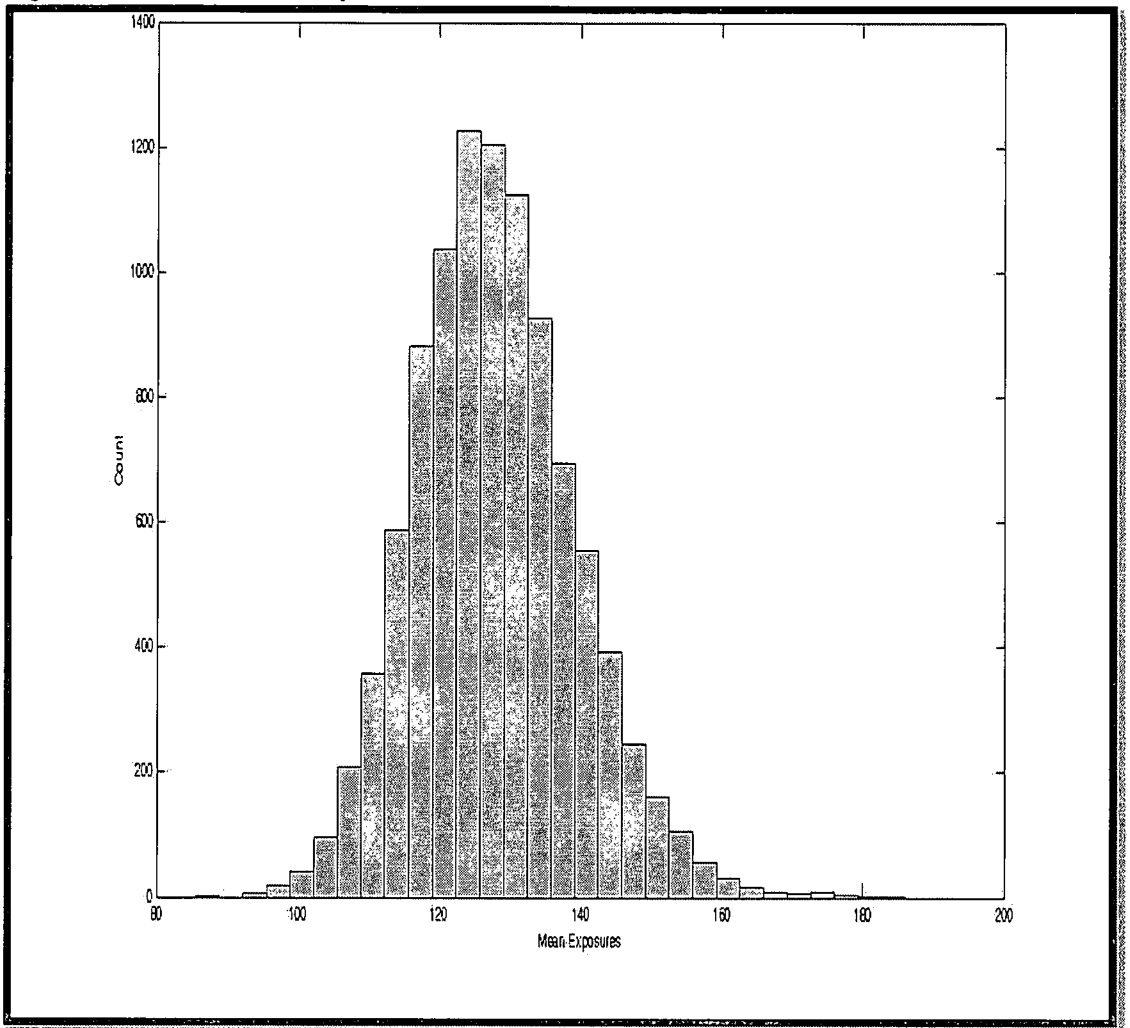
- Worker 7
- Probability that mean exposure > OEL (130) = 0.0000
- 90% Bayesian Confidence Interval: [67.6025; 91.0479]
- 95% Bayesian Confidence Interval: [65.7469; 93.7203]
- Mean = 78.7889; Median = 78.3905; Mode = 77.25

Figure 55:Simulated Mean Exposures for the Worker 8



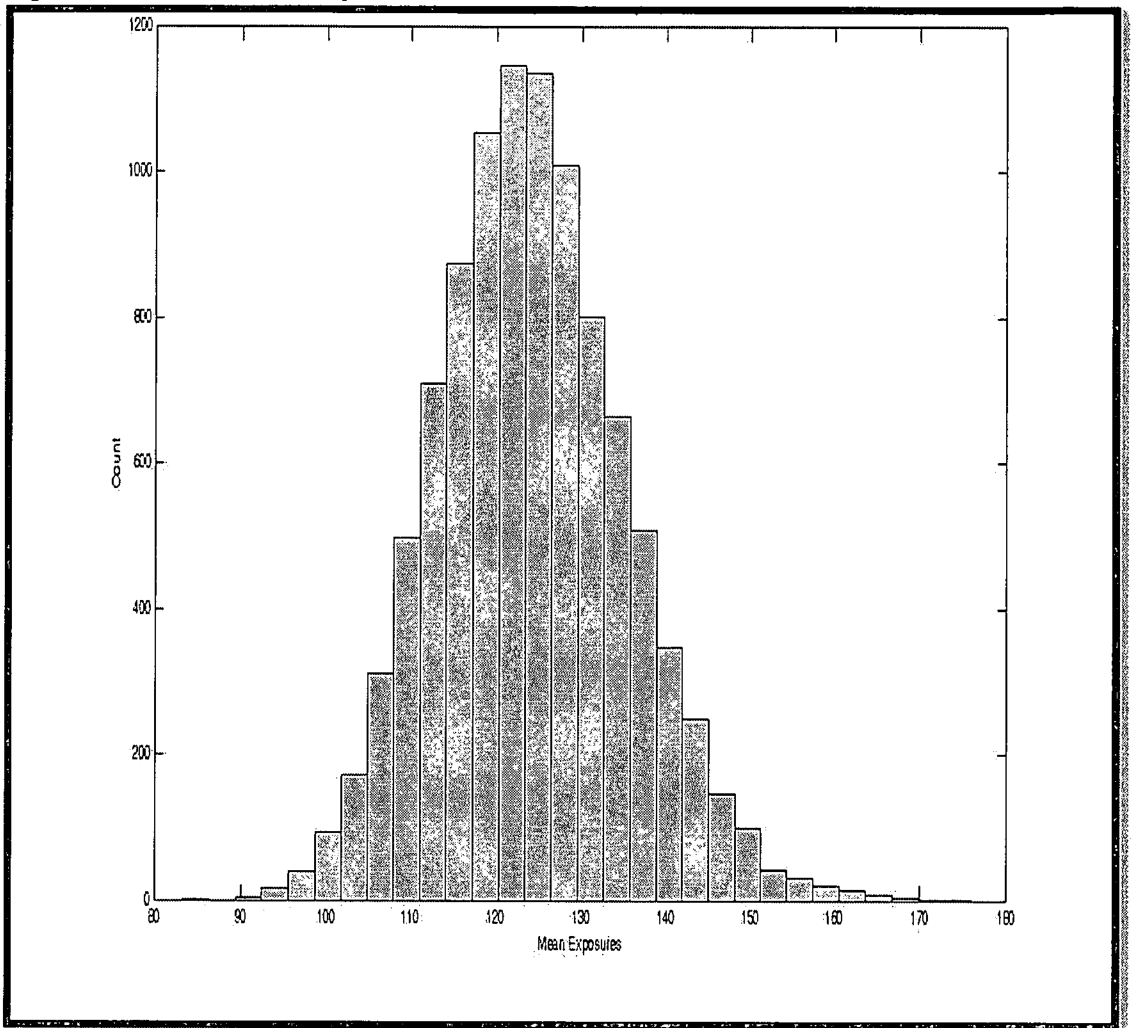
- Worker 8
- Probability that mean exposure > OEL (130) = 0.0000
- 90% Bayesian Confidence Interval: [75.9612; 101.4618]
- 95% Bayesian Confidence Interval: [73.8203; 104.4091]
- Mean = 88.0160; Median = 87.5908; Mode = 87.25

Figure 56:Simulated Mean Exposures for the Worker 9



- Worker 9
- Probability that mean exposure > OEL (130) = 0.4020
- 90% Bayesian Confidence Interval: [110.4973; 147.6873]
- 95% Bayesian Confidence Interval: [107.3347; 152.1216]
- Mean = 127.9384; Median = 127.2589; Mode = 124.25

Figure 57:Simulated Mean Exposures for the Worker 10



*Worker 10

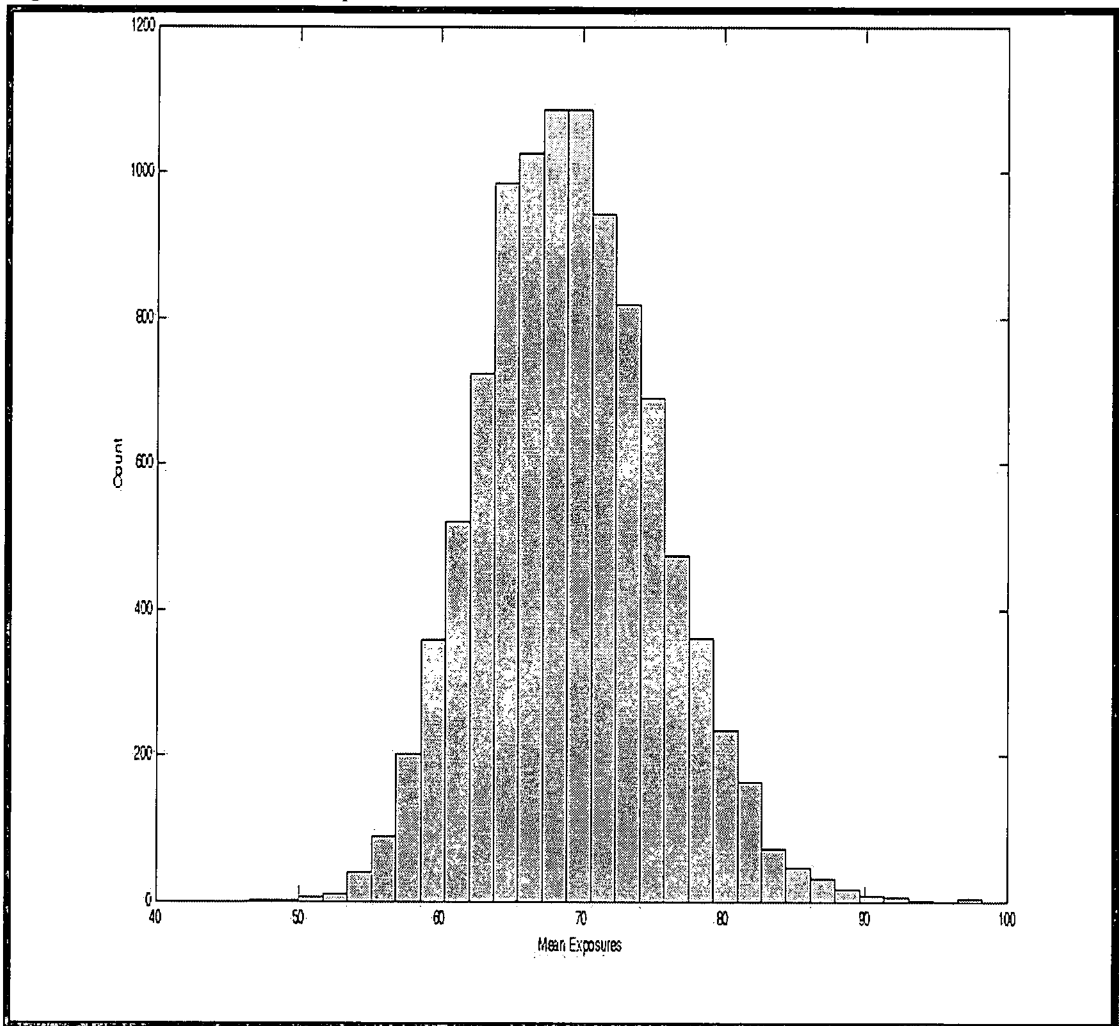
*Probability that mean exposure > OEL (130) = 0.2824

*90% Bayesian Confidence Interval: [106.8612; 143.1709]

*95% Bayesian Confidence Interval: [103.6013; 147.4745]

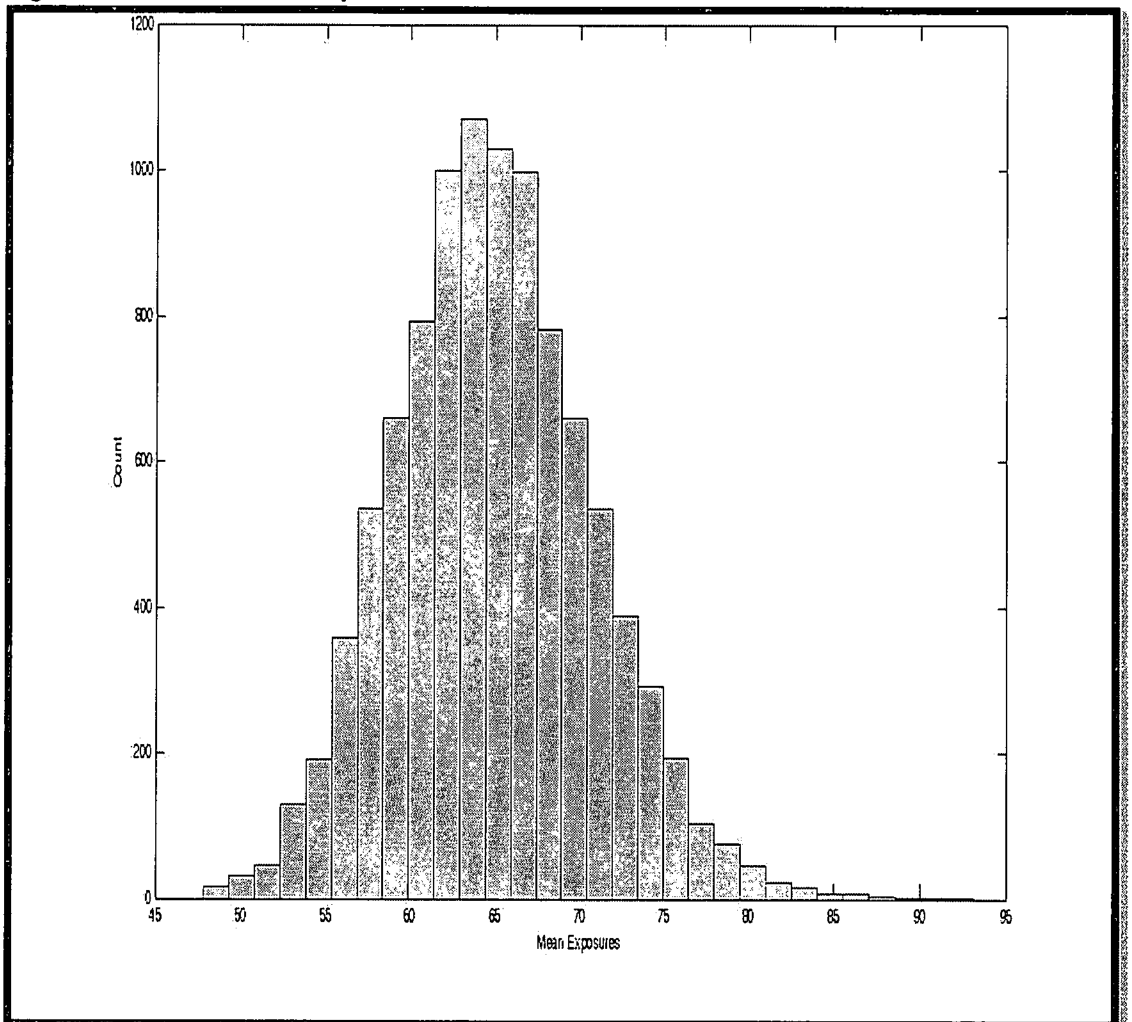
*Mean = 124.1411; Median = 123.6203; Mode = 125.25

Figure 58:Simulated Mean Exposures for the Worker 11



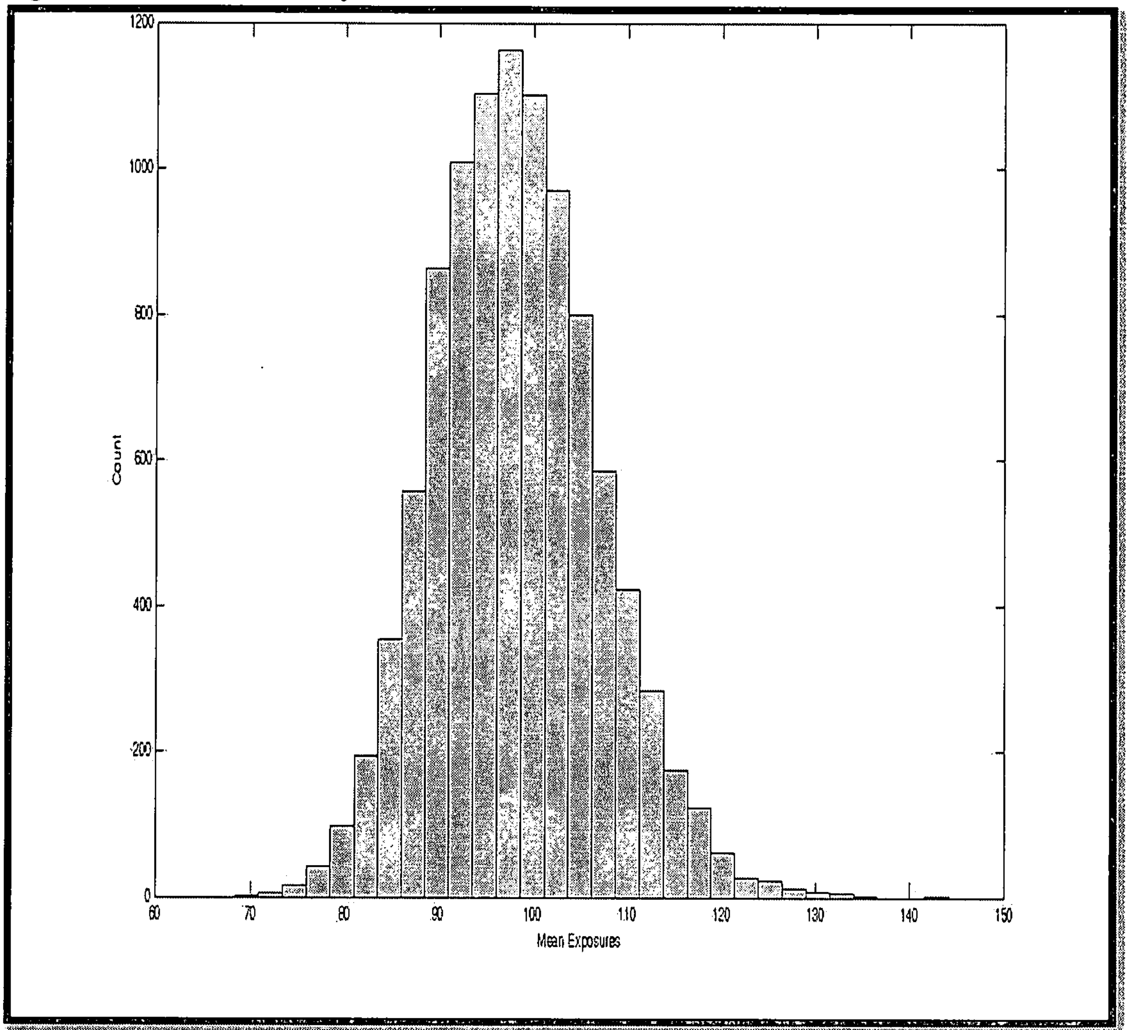
- Worker 11
- Probability that mean exposure > OEL (130) = 0.0000
- 90% Bayesian Confidence Interval: [59.4499; 79.8315]
- 95% Bayesian Confidence Interval: [57.7856; 81.9463]
- Mean = 69.1806; Median = 68.8973; Mode = 68.75

Figure 59:Simulated Mean Exposures for the Worker 12



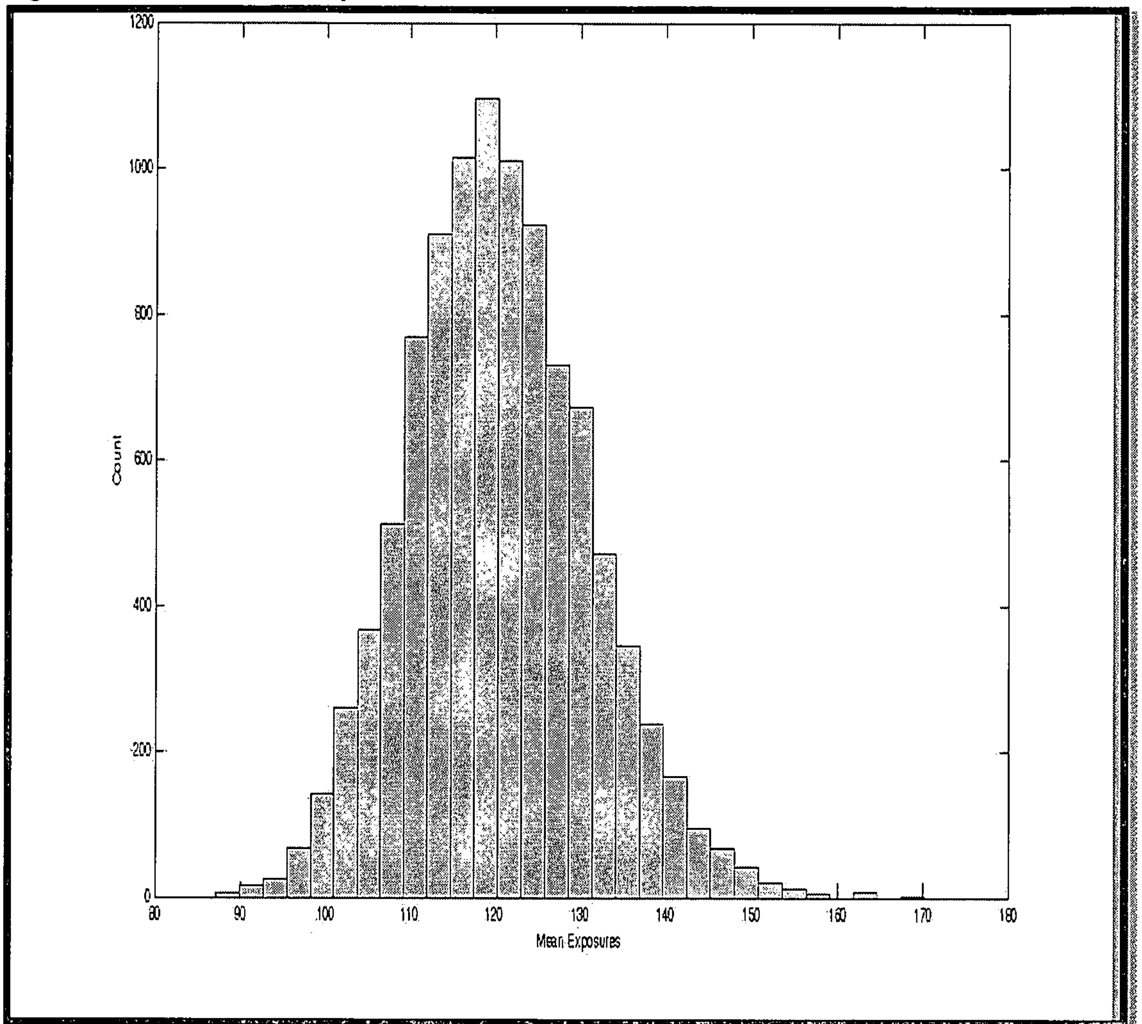
- Worker 12
- Probability that mean exposure > OEL (130) = 0.0000
- 90% Bayesian Confidence Interval: [55.7375; 74.8450]
- 95% Bayesian Confidence Interval: [54.0608; 76.9643]
- Mean = 64.9028; Median = 64.6676; Mode = 63.75

Figure 60:Simulated Mean Exposures for the Worker 13



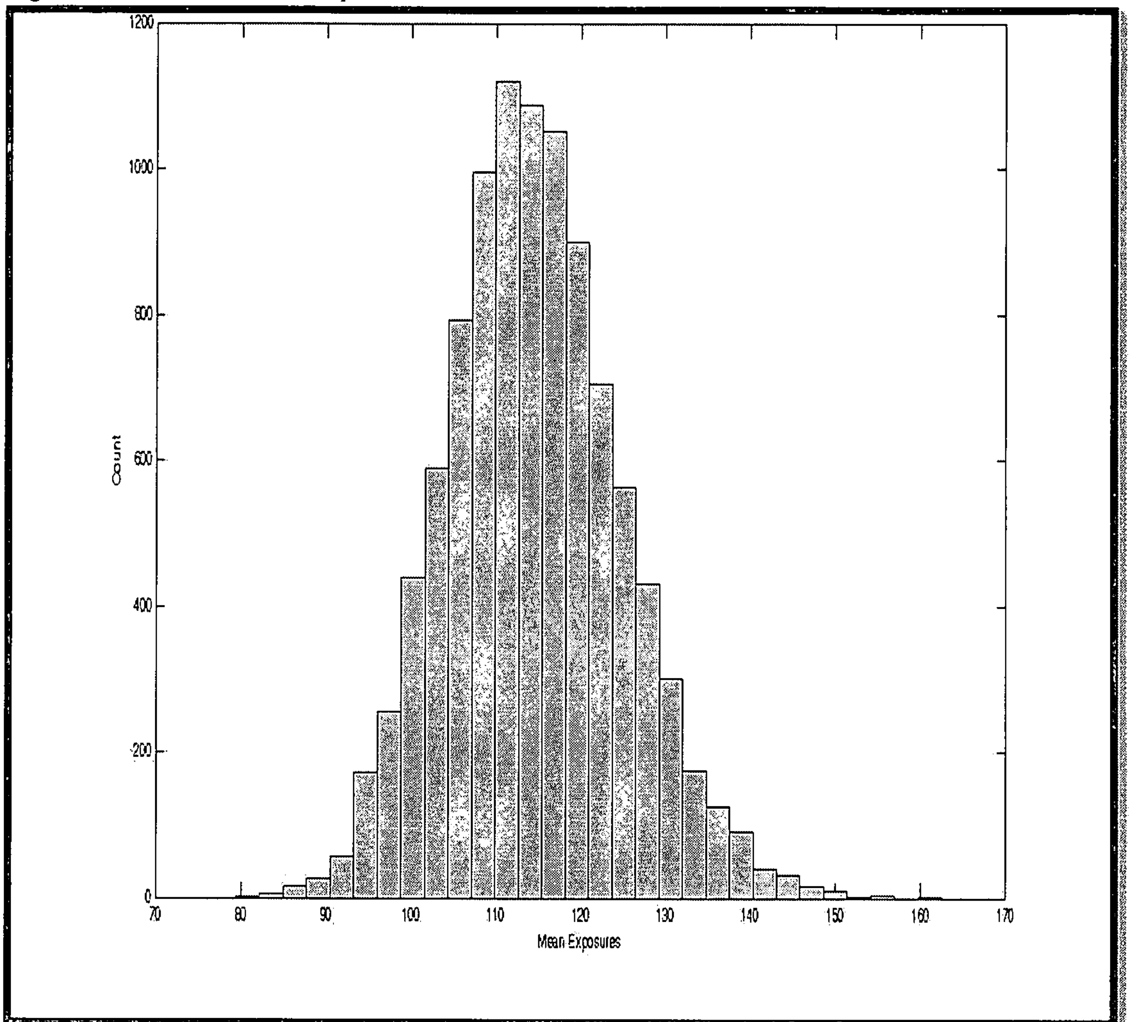
- Worker 13
- Probability that mean exposure > OEL (130) = 0.0010
- 90% Bayesian Confidence Interval: [84.6900; 113.1525]
- 95% Bayesian Confidence Interval: [83.3691; 116.6028]
- Mean = 98.1874; Median = 97.8116; Mode = 98.25

Figure 61:Simulated Mean Exposures for the Worker 14



- Worker 14
- Probability that mean exposure > OEL (130) = 0.1787
- 90% Bayesian Confidence Interval: [103.6231; 138.6366]
- 95% Bayesian Confidence Interval: [100.8463; 142.5568]
- Mean = 120.4003; Median = 119.8748; Mode = 118.75

Figure 62:Simulated Mean Exposures for the Worker 15



*Worker 15
*Probability that mean exposure > OEL (130) = 0.0698
*90% Bayesian Confidence Interval: [98.4583; 131.9492]
*95% Bayesian Confidence Interval: [95.6199; 136.1275]
*Mean = 114.4943; Median = 113.9588; Mode = 110.75

Derivation of the Probability-matching Prior

The Fisher Information Matrix for the above parameters is given by

$$F(\mu, \sigma_\tau^2, \sigma_e^2) = F(\boldsymbol{\theta}) = \begin{bmatrix} \dfrac{kn}{n\sigma_\tau^2 + \sigma_e^2} & 0 & 0 \\ 0 & \dfrac{kn^2}{2(n\sigma_\tau^2 + \sigma_e^2)^2} & \dfrac{kn}{2(n\sigma_\tau^2 + \sigma_e^2)^2} \\ 0 & \dfrac{kn}{2(n\sigma_\tau^2 + \sigma_e^2)^2} & \dfrac{k(n-1)}{2(\sigma_e^2)^2} + \dfrac{k}{2(n\sigma_\tau^2 + \sigma_e^2)^2} \end{bmatrix}$$

and the inverse is found to be

$$F^{-1}(\mu, \sigma_\tau^2, \sigma_e^2) = F^{-1}(\boldsymbol{\theta})$$

$$= \begin{bmatrix} \dfrac{n\sigma_\tau^2 + \sigma_e^2}{kn} & 0 & 0 \\ 0 & \dfrac{2\{(n-1)(n\sigma_\tau^2 + \sigma_e^2)^2 + (\sigma_e^2)^2\}}{kn^2(n-1)} & \dfrac{-2(\sigma_e^2)^2}{kn(n-1)} \\ 0 & \dfrac{-2(\sigma_e^2)^2}{kn(n-1)} & \dfrac{2(\sigma_e^2)^2}{k(n-1)} \end{bmatrix}$$

We are interested in the Probability-Matching prior for $t(\boldsymbol{\theta}) = e^{\mu + \frac{1}{2n}(n\sigma_\tau^2 + \sigma_e^2)} = e^B$ where $B = + \frac{1}{2n}(n\sigma_\tau^2 + \sigma_e^2)$.

Therefore,

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} = e^B$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\tau^2} = \frac{1}{2} e^B$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_e^2} = \frac{1}{2n} e^B$$

Now it is possible to find

$$\nabla_{\mathrm{t}}'(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} & \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\tau^2} & \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_e^2} \end{bmatrix} = e^B \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2n} \end{bmatrix}$$

$$\nabla_{\mathrm{t}}'(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) = e^B \begin{bmatrix} \frac{(n\sigma_\tau^2 + \sigma_e^2)}{kn} & \frac{(n-1)(n\sigma_\tau^2 + \sigma_e^2)^2}{kn^2(n-1)} & 0 \end{bmatrix}$$

$$\nabla_{\mathrm{t}}'(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) \nabla_{\mathrm{t}}(\boldsymbol{\theta}) = e^{2B} \left[\frac{(n\sigma_\tau^2 + \sigma_e^2)}{kn} + \frac{(n\sigma_\tau^2 + \sigma_e^2)^2}{2kn^2} \right]$$

$$= e^{2B} \left(\frac{n\sigma_\tau^2 + \sigma_e^2}{kn} \right) \left[1 + \frac{(n\sigma_\tau^2 + \sigma_e^2)}{2n} \right]$$

$$\{\nabla_{\mathrm{t}}'(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) \nabla_{\mathrm{t}}(\boldsymbol{\theta})\}^{\frac{1}{2}} = e^B \left(\frac{n\sigma_\tau^2 + \sigma_e^2}{kn} \right)^{\frac{1}{2}} \left[1 + \frac{(n\sigma_\tau^2 + \sigma_e^2)}{2n} \right]^{\frac{1}{2}}$$

$$\zeta'(\boldsymbol{\theta}) = \frac{\begin{bmatrix} \frac{(n\sigma_\tau^2 + \sigma_e^2)}{kn} & \frac{(n\sigma_\tau^2 + \sigma_e^2)^2}{kn^2} & 0 \end{bmatrix}}{\left(\frac{n\sigma_\tau^2 + \sigma_e^2}{kn} \right)^{\frac{1}{2}} \left[1 + \frac{(n\sigma_\tau^2 + \sigma_e^2)}{2n} \right]^{\frac{1}{2}}} = [\zeta_1(\boldsymbol{\theta}) \quad \zeta_2(\boldsymbol{\theta}) \quad \zeta_3(\boldsymbol{\theta})]$$

Now since $\zeta_1(\boldsymbol{\theta})$ does not contain μ and $\zeta_3(\boldsymbol{\theta})$ is zero, we are only interested in $\zeta_2(\boldsymbol{\theta})$:

$$\zeta_2(\boldsymbol{\theta}) = \frac{\frac{(n\sigma_\tau^2 + \sigma_e^2)^2}{kn^2}}{\left(\frac{n\sigma_\tau^2 + \sigma_e^2}{kn} \right)^{\frac{1}{2}} \left[1 + \frac{(n\sigma_\tau^2 + \sigma_e^2)}{2n} \right]^{\frac{1}{2}}}$$

$$= \frac{\frac{1}{kn^2} (n\sigma_\tau^2 + \sigma_e^2)^2}{\sqrt{2n} \left(\frac{1}{kn} \right)^{\frac{1}{2}} (n\sigma_\tau^2 + \sigma_e^2) \left[1 + \frac{2n}{(n\sigma_\tau^2 + \sigma_e^2)} \right]^{\frac{1}{2}}}$$

$$= \frac{(kn)^{\frac{1}{2}}}{kn^2 \sqrt{2n}} (n\sigma_\tau^2 + \sigma_e^2) \left[1 + \frac{2n}{(n\sigma_\tau^2 + \sigma_e^2)} \right]^{-\frac{1}{2}}$$

Therefore, the Probability-Matching prior is

$$\pi_1(\boldsymbol{\theta}) = \pi_1(\mu, \sigma_\tau^2, \sigma_e^2) \propto (n\sigma_\tau^2 + \sigma_e^2)^{-1}\left[1 + \frac{2n}{(n\sigma_\tau^2 + \sigma_e^2)}\right]^{\frac{1}{2}}$$

Derivation of the Reference Prior

We are interested in the Fisher Information Matrix for $t(\boldsymbol{\theta})$, σ_τ^2 and σ_e^2. Since $t(\boldsymbol{\theta}) = e^{\mu+\frac{1}{2n}(n\sigma_\tau^2+\sigma_e^2)}$ it follows that $\mu + \frac{1}{2n}(n\sigma_\tau^2 + \sigma_e^2) = ln(t(\boldsymbol{\theta}))$.

$$\therefore \ \mu = ln(t(\boldsymbol{\theta})) - \frac{1}{2n}(n\sigma_\tau^2 + \sigma_e^2)$$

Therefore,

$$\frac{\partial\mu}{\partial t(\boldsymbol{\theta})} = \frac{1}{t(\boldsymbol{\theta})}\ ; \ \frac{\partial\mu}{\partial\sigma_\tau^2} = -\frac{1}{2}\ ; \ \frac{\partial\mu}{\partial\sigma_e^2} = -\frac{1}{2n}$$

Let

$$A = \frac{\partial(\mu, \sigma_\tau^2, \sigma_e^2)}{\partial(t(\boldsymbol{\theta}), \sigma_\tau^2, \sigma_e^2)} = \begin{bmatrix} \frac{\partial\mu}{\partial t(\boldsymbol{\theta})} & \frac{\partial\mu}{\partial\sigma_\tau^2} & \frac{\partial\mu}{\partial\sigma_e^2} \\ \frac{\partial\sigma_\tau^2}{\partial t(\boldsymbol{\theta})} & \frac{\partial\sigma_\tau^2}{\partial\sigma_\tau^2} & \frac{\partial\sigma_\tau^2}{\partial\sigma_e^2} \\ \frac{\partial\sigma_e^2}{\partial t(\boldsymbol{\theta})} & \frac{\partial\sigma_e^2}{\partial\sigma_\tau^2} & \frac{\partial\sigma_e^2}{\partial\sigma_e^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{t(\boldsymbol{\theta})} & -\frac{1}{2} & -\frac{1}{2n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Fisher Information Matrix for $t(\boldsymbol{\theta})$, σ_τ^2 and σ_e^2 is

$$F(t(\boldsymbol{\theta}),\sigma_\tau^2,\sigma_e^2) = A'F(\mu,\sigma_\tau^2,\sigma_e^2)A$$

$$= \begin{bmatrix} \frac{1}{t(\boldsymbol{\theta})} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2n} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{kn}{n\sigma_\tau^2+\sigma_e^2} & 0 & 0 \\ 0 & \frac{kn^2}{2(n\sigma_\tau^2+\sigma_e^2)^2} & \frac{kn}{2(n\sigma_\tau^2+\sigma_e^2)^2} \\ 0 & \frac{kn}{2(n\sigma_\tau^2+\sigma_e^2)^2} & \frac{k(n-1)}{2(\sigma_e^2)^2}+\frac{k}{2(n\sigma_\tau^2+\sigma_e^2)^2} \end{bmatrix} A$$

$$= \begin{bmatrix} \frac{kn}{t(\boldsymbol{\theta})(n\sigma_\tau^2+\sigma_e^2)} & 0 & 0 \\ \frac{-kn}{2(n\sigma_\tau^2+\sigma_e^2)} & \frac{kn^2}{2(n\sigma_\tau^2+\sigma_e^2)^2} & \frac{kn}{2(n\sigma_\tau^2+\sigma_e^2)^2} \\ \frac{-k}{2(n\sigma_\tau^2+\sigma_e^2)} & \frac{kn}{2(n\sigma_\tau^2+\sigma_e^2)^2} & \frac{k(n-1)}{2(\sigma_e^2)^2}+\frac{k}{2(n\sigma_\tau^2+\sigma_e^2)^2} \end{bmatrix} A$$

$$= \begin{bmatrix} \frac{kn}{t^2(\boldsymbol{\theta})(n\sigma_\tau^2+\sigma_e^2)} & \frac{-kn}{2t(\boldsymbol{\theta})(n\sigma_\tau^2+\sigma_e^2)} & \frac{-k}{2t(\boldsymbol{\theta})(n\sigma_\tau^2+\sigma_e^2)} \\ \frac{-kn}{2t(\boldsymbol{\theta})(n\sigma_\tau^2+\sigma_e^2)} & \frac{kn^2}{2(n\sigma_\tau^2+\sigma_e^2)^2}+\frac{kn}{4(n\sigma_\tau^2+\sigma_e^2)} & \frac{kn}{2(n\sigma_\tau^2+\sigma_e^2)^2}+\frac{k}{4(n\sigma_\tau^2+\sigma_e^2)} \\ \frac{-k}{2t(\boldsymbol{\theta})(n\sigma_\tau^2+\sigma_e^2)} & \frac{kn}{2(n\sigma_\tau^2+\sigma_e^2)^2}+\frac{k}{4(n\sigma_\tau^2+\sigma_e^2)} & \frac{k(n-1)}{2(\sigma_e^2)^2}+\frac{k}{2(n\sigma_\tau^2+\sigma_e^2)^2}+\frac{k}{4n(n\sigma_\tau^2+\sigma_e^2)} \end{bmatrix}$$

We shall derive the Reference prior distribution for the ordering $[t(\boldsymbol{\theta}),(\sigma_\tau^2,\sigma_e^2)]$, which implies that $t(\boldsymbol{\theta})$ is the most important parameter and $(\sigma_\tau^2,\sigma_e^2)$ is of the same importance.

By re-arranging we can write the Fisher Information matrix as

$$F(t(\boldsymbol{\theta}),\sigma_\tau^2,\sigma_e^2) = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

where

$$F_{11} = \frac{kn}{t^2(\boldsymbol{\theta})C}$$

$$F_{12} = \begin{bmatrix} \frac{-kn}{2t(\boldsymbol{\theta})C} & \frac{-k}{2t(\boldsymbol{\theta})C} \end{bmatrix}$$

$$F_{21} = \begin{bmatrix} \frac{-kn}{2t(\boldsymbol{\theta})C} \\ \frac{-k}{2t(\boldsymbol{\theta})C} \end{bmatrix}$$

$$F_{22} = \begin{bmatrix} \frac{kn^2}{2C^2} + \frac{kn}{4C} & \frac{kn}{2C^2} + \frac{k}{4C} \\ \frac{kn}{2C^2} + \frac{k}{4C} & \frac{k(n-1)}{2(\sigma_e^2)^2} + \frac{k}{2C^2} + \frac{k}{4nC} \end{bmatrix}$$

where $C = (n\sigma_\tau^2 + \sigma_e^2)$.

$$h_1 = F_{11} - F_{12}F_{22}^{-1}F_{21}$$

$$\therefore p_R\big(t(\boldsymbol{\theta})\big) \propto h_1^{\frac{1}{2}} = [t(\boldsymbol{\theta})]^{-1}$$

and

$$\begin{aligned} h_2 = |F_{22}| &= \frac{kn}{2C}\left(\frac{1}{2} + \frac{n}{C}\right)\frac{k}{2C}\left(\frac{1}{2n} + \frac{1}{C}\right) + \frac{kn}{2C}\left(\frac{1}{2} + \frac{n}{C}\right)\frac{k(n-1)}{2(\sigma_e^2)^2} \\ &- \frac{k}{2C}\left(\frac{1}{2} + \frac{n}{C}\right)\frac{k}{2C}\left(\frac{1}{2} + \frac{n}{C}\right) \\ &= \left(\frac{k}{2C}\right)^2\left(\frac{1}{2} + \frac{n}{C}\right)^2 - \left(\frac{k}{2C}\right)^2\left(\frac{1}{2} + \frac{n}{C}\right)^2 + \frac{k^2n(n-1)}{4C}\left(\frac{1}{2} + \frac{n}{C}\right)\frac{1}{(\sigma_e^2)^2} \\ &= \frac{k^2n(n-1)}{4C}\left(\frac{1}{2} + \frac{n}{C}\right)\frac{1}{(\sigma_e^2)^2} \end{aligned}$$

Therefore,

$$h_2^{\frac{1}{2}} = \left(\frac{k^2n(n-1)}{4}\right)^{\frac{1}{2}}\left(\frac{1}{2C} + \frac{n}{C^2}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2}$$

$$\therefore p_R(\sigma_\tau^2, \sigma_e^2) \propto \left(\frac{1}{2C} + \frac{n}{C^2}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2} = \left(\frac{1}{2(n\sigma_\tau^2 + \sigma_e^2)} + \frac{n}{(n\sigma_\tau^2 + \sigma_e^2)^2}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2}$$

and

$$\therefore p_R(t(\boldsymbol{\theta}), \sigma_\tau^2, \sigma_e^2) \propto [t(\boldsymbol{\theta})]^{-1}\left(\frac{1}{2(n\sigma_\tau^2 + \sigma_e^2)} + \frac{n}{(n\sigma_\tau^2 + \sigma_e^2)^2}\right)^{\frac{1}{2}}\frac{1}{\sigma_e^2}$$

Now,

$$t(\boldsymbol{\theta}) = e^{\mu + \frac{1}{2n}\left(n\sigma_\tau^2 + \sigma_e^2\right)}$$

and

$$\frac{\partial}{\partial t(\boldsymbol{\theta})} = e^{\mu + \frac{1}{2n}\left(n\sigma_\tau^2 + \sigma_e^2\right)} = t(\boldsymbol{\theta})$$

$$\therefore p_R(\mu, \sigma_\tau^2, \sigma_e^2) \propto t(\boldsymbol{\theta})[t(\boldsymbol{\theta})]^{-1}\left(\frac{1}{2(n\sigma_\tau^2 + \sigma_e^2)} + \frac{n}{(n\sigma_\tau^2 + \sigma_e^2)^2}\right)^{\frac{1}{2}} \frac{1}{\sigma_e^2}$$

$$\propto (n\sigma_\tau^2 + \sigma_e^2)^{-\frac{1}{2}}\left(\frac{1}{2} + \frac{n}{(n\sigma_\tau^2 + \sigma_e^2)}\right)^{\frac{1}{2}} \frac{1}{\sigma_e^2}$$

$$\propto (n\sigma_\tau^2 + \sigma_e^2)^{-\frac{1}{2}}\left(1 + \frac{2n}{(n\sigma_\tau^2 + \sigma_e^2)}\right)^{\frac{1}{2}} \frac{1}{\sigma_e^2}$$

CHAPTER 8

Random Effect Model: Unbalanced Case

Introduction

In this chapter the extension to the balanced case, presented in Chapter 7, is given. In many instances the assumption of balanced data (for example, equal observations for a set of companies or the same number of styrene exposures for a group of different workers) is overly simplistic. Unbalanced data can arise due to a number of different factors and in this case the Bayesian methodology is similar to the previous chapter, however additional derivations are required.

Again, this situation has been approached by some authors (e.g. Krishnamoorthy and Guo, 2005), but the methods proposed involve generalized p-value approaches. The problem statement is nevertheless the same: we would like to estimate the proportion of exposure measurements exceeding a pre-specified limit or perhaps the probability of an insurance claim exceeding a pre-specified boundary. According to Krishnamoorthy and Guo (2005) the one-way random effects model incorporates both within and between sources of variation in measurements. Since we are dealing with data that have a lognormal distribution (i.e. the logged exposure levels are normally distributed) we are interested in the overall mean effect and the two variance components in the model. Once again, for a complete description of the methods and medical applications of this method please refer to Krishnamoorthy and Guo (2005) as well as the texts by Rappaport, Kromhout and Symanski (1993), Heederik and Hurley (1994) and Lyles, Kupper and Rappaport (1997).

In this chapter we extend the discussion regarding the one-way balanced random effects model to the unbalance case. Particularly, this will be approached from the Bayesian perspective. In order to complete the Bayesian specification of the model prior distributions have to be derived and this forms a large part of this chapter. The selection and determination of non-informative priors in multi-parameter settings is not an easy task and it has been observed that the selection of a specific prior could have unexpectedly dramatic effects on the posterior distribution. In this chapter, the derivation of suitable priors will be considered, where the Reference prior (Berger and Bernardo, 1992) is one such option and the second is the Probability-Matching prior. A simulation study will also be presented to show the effectiveness of the proposed prior distributions.

8.1 Description of the Setting

The setting for this chapter is nearly identical to the setting described in Chapter 7 and will not be repeated at length here. Essentially, we have the following:

- Several authors have proposed the use of the one-way random effects model to model exposure levels, even insurance claims.
- The natural logarithm of the data is normally distributed.
- Between- and within source of variability are accounted for.
- A pre-specific exposure level (OEL) has to be selected.

The only significant difference, conceptually, is that we are dealing here with the unbalance case, e.g. not all workers have the same number of observations. Even though this is a minor conceptual change all prior distributions and the derivation thereof will be considerably different. So in this case we have the following situation (diagrammatic representation):

Table 97: Representation of Shift Exposure Data

	Shift-Long Exposure Measurements				
Workers	**1**	**2**	**...**	n_i	**Worker Means**
1	x_{11}	x_{12}	...	x_{1n_1}	$\bar{X}_{1\cdot}$
2	x_{21}	x_{22}	...	x_{2n_2}	$\bar{X}_{2\cdot}$
...	...	...	...	...	...
k	x_{k1}	x_{k2}	...	x_{kn_k}	$\bar{X}_{k\cdot}$

Therefore, there are n_i measurements for the i-th worker. The overall mean of all the data can be represented as $\bar{X}_{\cdot\cdot}$ The X_{ij} are lognormally distributed and therefore $Y_{ij} = ln(X_{ij})$ are distributed normally. The assumed one-way random effects model is:

$$Y_{ij} = \mu + \tau_i + e_{ij}, \qquad i = 1, \dots, k; j = 1, \dots, n_i.$$

where μ is the general mean, $\tau_i \sim N(0, \sigma_\tau^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$. All the random variables are independent of each other and here τ_i represents the random effect due to the i-th worker.

According to Krishnamoorthy and Guo (2005) let

$$\mu_{x_i} = E(X_{ij}|\tau_i) = E(exp[Y_{ij}]|\tau_i) = exp\left(\mu + \tau_i + {\sigma_e^2}/{2}\right)$$

and μ_{x_i} is the mean exposure for the i-th worker. Let θ denote the probability that μ_{x_i} exceeds the OEL. Thus,

$$\theta = P(\mu_{x_i} > OEL) = P\left(ln(\mu_{x_i}) > ln(OEL)\right) = 1 - \Phi\left(\frac{ln(OEL) - \mu - {\sigma_e^2}/{2}}{\sigma_\tau}\right)$$

where $\Phi(.)$ denotes the c.d.f. of the standard normal distribution. The kind of hypotheses that are going to be considered here are:

$$H_0: \theta \geq A \; vs \; H_1: \theta < A$$

where A is a specific quantity that is usually small, according to Krishnamoorthy and Guo (2005).

As mentioned previously, Krishnamoorthy and Guo (2005) apply a technique based on generalized confidence limits and generalized p-values in order to test the above hypothesis. In this chapter we will be applying a Bayesian methodology to the problem.

The following is an example data set of simulated "styrene exposures" that will serve as a basis for discussion in this chapter and will help us define and illustrate the objectives of the chapter (it is the same data set as in Chapter 7, however, random observations have been deleted, resulting in an "unbalanced" design):

Table 98: Simulated "Styrene Exposures"

	Observations / Measurements									
Worker	1	2	3	4	5	6	7	8	9	10
1	95.6	64.7	50.9	87.4	82.3	149.9	33.4	77.5	70.8	60.9
2	57.4	82.3	174.2	107.8	98.5	129.0	121.5	95.6	92.8	133.0
3	84.8	214.9	79.8	169.0	149.9	164.0	84.8	84.8	114.4	
4	68.7	77.5	54.1	41.3	64.7	46.5	59.1	45.2	54.1	
5	114.4	101.5	49.4	101.5	90.0	52.5	114.4	79.8	68.7	87.4
6	87.4	242.3	145.5	133.0	174.2	214.9	137.0	129.0	169.0	179.5
7	54.1	75.2	84.8	55.7	90.0	70.8	60.9	101.5	64.7	95.6
8	64.7	95.6	57.4	95.6	82.3	101.5	92.8	60.9	101.5	98.5
9	137.0	208.5	92.8	159.2	92.8	82.3	90.0			
10	125.2	87.4	121.5	90.0	154.5	107.8	117.9	179.5	129.0	129.0
11	42.5	73.0	50.9	59.1	49.4	66.7				
12	57.4	68.7	59.1	64.7	55.7	92.8	42.5			
13	101.5	149.9	111.1	77.5	111.1	84.8	64.7	62.8		
14	68.7	101.5	111.1	179.5	82.3	174.2	174.2	87.4	145.5	114.4
15	121.5	77.5	145.5	174.2	77.5	92.8	159.2	129.0	104.6	77.5

The above table represents the X_{ij} data points and the following table represents the corresponding $Y_{ij} = ln(X_{ij})$:

Table 99: Log of Simulated "Styrene Exposures"

Worker	Observations / Measurements									
	1	2	3	4	5	6	7	8	9	10
1	4.56	4.17	3.93	4.47	4.41	5.01	3.51	4.35	4.26	4.11
2	4.05	4.41	5.16	4.68	4.59	4.86	4.8	4.56	4.53	4.89
3	4.44	5.37	4.38	5.13	5.01	5.1	4.44	4.44	4.74	
4	4.23	4.35	3.99	3.72	4.17	3.84	4.08	3.81	3.99	
5	4.74	4.62	3.9	4.62	4.5	3.96	4.74	4.38	4.23	4.47
6	4.47	5.49	4.98	4.89	5.16	5.37	4.92	4.86	5.13	5.19
7	3.99	4.32	4.44	4.02	4.5	4.26	4.11	4.62	4.17	4.56
8	4.17	4.56	4.05	4.56	4.41	4.62	4.53	4.11	4.62	4.59
9	4.92	5.34	4.53	5.07	4.53	4.41	4.5			
10	4.83	4.47	4.8	4.5	5.04	4.68	4.77	5.19	4.86	4.86
11	3.75	4.29	3.93	4.08	3.9	4.2				
12	4.05	4.23	4.08	4.17	4.02	4.53	3.75			
13	4.62	5.01	4.71	4.35	4.71	4.44	4.17	4.14		
14	4.23	4.62	4.71	5.19	4.41	5.16	5.16	4.47	4.98	4.74
15	4.8	4.35	4.98	5.16	4.35	4.53	5.07	4.86	4.65	4.35

From these data we have the following definitions and associated results:

$$k = 15$$

$$v_1 = \sum_{i=1}^{k}(n_i - 1) \ ; \ v_2 = k - 1$$

$$\bar{Y}_{i\cdot} = \frac{1}{n_i}Y_{i\cdot} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij} = [4.3\ 4.7\ 4.8\ 4.0\ 4.4\ 5.0\ 4.3\ 4.4\ 4.8\ 4.9\ 4.0\ 4.1\ 4.5\ 4.8\ 4.7]'$$

$$n_i = \begin{bmatrix} 10 \\ 10 \\ 9 \\ 9 \\ 10 \\ 10 \\ 10 \\ 10 \\ 7 \\ 10 \\ 6 \\ 7 \\ 8 \\ 10 \\ 10 \end{bmatrix}$$

$$\bar{Y}_{\cdot\cdot} = \frac{1}{k}\sum_{i=1}^{k}\bar{Y}_{i\cdot} = 4.508 = \hat{\mu}$$

$$SS_e = v_1 m_1 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{ij} - \bar{Y}_{i\cdot}\right)^2 \quad = 10.492 = \text{"}within\ workers\ sum\ of\ squares\text{"}$$

$$SS_\tau = v_2 m_2 = \sum_{i=1}^{k} n_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 = 11.885 = \text{"}between\ workers\ sum\ of\ squares\text{"}$$

8.2. Bayesian Methodology

As mentioned previously, the basis for analyzing any situation from a Bayesian perspective is the following result:

$$Posterior \propto Likelihood \times Prior$$

Given the setting described earlier we are able to represent the combined distribution (or likelihood function) as follows:

$$L(\mu, \boldsymbol{\tau}, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y}) = (2\pi\sigma_e^2)^{-\frac{1}{2}\tilde{n}} exp\left\{-\frac{1}{2\sigma_e^2}(\boldsymbol{Y} - \mu\mathbf{1} - Z\boldsymbol{\tau})'(\boldsymbol{Y} - \mu\mathbf{1} - Z\boldsymbol{\tau})\right\} (2\pi\sigma_\tau^2)^{-\frac{1}{2}k} exp\left\{-\frac{1}{2\sigma_\tau^2}\boldsymbol{\tau}'\boldsymbol{\tau}\right\}$$

where

$$\tilde{n} = \textstyle\sum_{i=1}^{k} n_i$$

$$\boldsymbol{\tau}' = [\tau_1 \quad \tau_2 \quad \dots \quad \tau_k]$$

$$\mu\mathbf{1} = \begin{bmatrix}\mu\\ \mu\\ \vdots\\ \mu\end{bmatrix}$$

$$Z_{\tilde{n}\times k} = \begin{bmatrix} 1_1 & 0 & \cdots & 0 \\ 1_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1_{n_1} & 0 & \cdots & 0 \\ 0 & 1_1 & \cdots & 0 \\ 0 & 1_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_1 \\ 0 & 0 & \cdots & 1_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{n_k} \end{bmatrix}$$

and

$$\boldsymbol{Y} = \begin{bmatrix} y_{11}\, y_{12} \cdots\, y_{1n_1} \cdots\, y_{k1}\, y_{k2} \cdots\, y_{kn_k} \end{bmatrix}'$$

Now, we already know, from the specification of the random effects model, that $\tau_i \sim N(0, \sigma_\tau^2)$ with $i = 1, 2, \dots, k$. Since this is the case we would therefore like to define prior distributions for μ, σ_e^2 and σ_τ^2. For the sake of convenience though we will define

$$\tilde{r} = \frac{\sigma_\tau^2}{\sigma_e^2}$$

and then define prior distributions for μ, σ_e^2 and $\tilde{r}$ instead.

In order to derive prior distributions for this though we first need to derive the integrated likelihood function, $L(\mu, \sigma_e^2, \sigma_\tau^2 | \boldsymbol{Y})$.

Theorem 8.1

The integrated likelihood function, $L(\mu, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y})$ is given by the following:

$$L(\mu, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}(\tilde{n}-k)} \prod_{i=1}^{k} \left(\frac{1}{n_i\sigma_\tau^2 + \sigma_e^2}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\sigma_\tau^2 + \sigma_e^2}\right]\right\}$$

(8.1)

<u>Proof:</u> The proof is given in the appendix to this chapter.

Now, if $\tilde{r} = \frac{\sigma_\tau^2}{\sigma_e^2}$ then it follows that

$$L(\mu, \sigma_e^2, \tilde{r}|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}\tilde{n}} \prod_{i=1}^{k} \left(\frac{1}{n_i\tilde{r} + 1}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2\sigma_e^2}\left[v_1 m_1 + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\tilde{r} + 1}\right]\right\}$$

(8.2)

We can now prove the following theorems:

Theorem 8.2

$$\bar{y}_{i\cdot}|\mu, \sigma_e^2, \sigma_\tau^2 \sim N\left(\mu, \frac{n_i\sigma_\tau^2 + \sigma_e^2}{n_i}\right)$$

<u>Proof:</u> The proof is given in the appendix to this chapter.

Theorem 8.3

For the model:

$$\boldsymbol{Y} = \mu\boldsymbol{1} + Z\boldsymbol{\tau} + \boldsymbol{e}$$

where $\boldsymbol{e} \sim N(\boldsymbol{0}, \sigma_e^2 I_{\tilde{n}})$ and $\boldsymbol{\tau} \sim N(\boldsymbol{0}, \sigma_\tau^2 I_k)$, then the Fisher Information Matrix for the parameters $(\mu, \tilde{r}, \sigma_e^2)$ is given by

$$F(\mu,\tilde{r},\sigma_e^2) = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & 0 & 0 \\ 0 & \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \\ 0 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2 \end{bmatrix}$$

Proof: The proof is given in the appendix to this chapter.

Theorem 8.4

For the model:

$$\boldsymbol{Y} = \mu\mathbf{1} + Z\boldsymbol{\tau} + \boldsymbol{e}$$

where $\boldsymbol{e} \sim N(\mathbf{0},\sigma_e^2 I_{\tilde{n}})$ and $\boldsymbol{\tau} \sim N(\mathbf{0},\sigma_\tau^2 I_k)$, then the Probability-Matching Prior for the parameters $(\mu,\tilde{r},\sigma_e^2)$ is given by

$$P(\mu,\tilde{r},\sigma_e^2) \propto \frac{1}{\sigma_e^2}\left\{\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} - \frac{1}{n}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2\right\}^{\frac{1}{2}} \tag{8.3}$$

Proof: The proof is given in the appendix to this chapter.

Theorem 8.5.1

For the model:

$$\boldsymbol{Y} = \mu\mathbf{1} + Z\boldsymbol{\tau} + \boldsymbol{e}$$

where $\boldsymbol{e} \sim N(\mathbf{0},\sigma_e^2 I_{\tilde{n}})$ and $\boldsymbol{\tau} \sim N(\mathbf{0},\sigma_\tau^2 I_k)$, then the Reference Prior for the parameter groupings $(\mu,\tilde{r},\sigma_e^2)$, $(\tilde{r},\mu,\sigma_e^2)$ and $(\tilde{r},\sigma_e^2,\mu)$ is given by

$$P_{R_1}(\mu, \tilde{r}, \sigma_e^2) \propto \frac{1}{\sigma_e^2}\left\{\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} - \frac{1}{\tilde{n}}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2\right\}^{\frac{1}{2}}$$

(8.4)

This is coincidentally the same as the Probability-Matching Prior and therefore the Probability-Matching Prior is also the Reference Prior.

Proof: The proof is given in the appendix to this chapter.

Theorem 8.5.2

For the model:

$$\boldsymbol{Y} = \mu\boldsymbol{1} + Z\boldsymbol{\tau} + \boldsymbol{e}$$

where $\boldsymbol{e} \sim N(\boldsymbol{0}, \sigma_e^2 I_{\tilde{n}})$ and $\boldsymbol{\tau} \sim N(\boldsymbol{0}, \sigma_\tau^2 I_k)$, then the Reference Prior for the parameter groupings $(\mu, \sigma_e^2, \tilde{r})$, $(\sigma_e^2, \mu, \tilde{r})$ and $(\sigma_e^2, \tilde{r}, \mu)$ is given by

$$P_{R_2}(\mu, \sigma_e^2, \tilde{r}) \propto \frac{1}{\sigma_e^2}\left\{\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right\}^{\frac{1}{2}}$$

(8.5)

Proof: The proof is given in the appendix to this chapter.

At this stage it is appropriate to illustrate the following derivation:

In equation (8.4) substitute $n_1 = n_2 = \ldots = n_k = n$ and $\tilde{n} = kn$. Then it follows immediately from this that

$$P(\mu, \tilde{r}, \sigma_e^2) \propto \frac{1}{\sigma_e^2}\left\{\frac{kn^2}{(1+\tilde{r}n)^2} - \frac{1}{kn}\left(\frac{kn}{1+\tilde{r}n}\right)^2\right\}^{\frac{1}{2}}$$

$$\propto \frac{1}{\sigma_e^2}\left\{\frac{kn^2}{(1+\tilde{r}n)^2} - \frac{kn}{(1+\tilde{r}n)^2}\right\}^{\frac{1}{2}}$$

$$\propto \frac{1}{\sigma_e^2}\left\{\frac{kn\,(n-1)}{(1+\tilde{r}n)^2}\right\}^{\frac{1}{2}}$$

$$\propto \frac{1}{\sigma_e^2}\ (1+\tilde{r}n)^{-1}$$

If we now substitute $\tilde{r} = \frac{\sigma_\tau^2}{\sigma_e^2}$ and the Jacobian $\frac{\partial \tilde{r}}{\partial \sigma_\tau^2} = \frac{1}{\sigma_e^2}$ then we get the following:

$$P_{R_1}(\mu, \sigma_\tau^2, \sigma_e^2) \propto \frac{1}{\sigma_e^2}\ \left(1+\frac{n\,\sigma_\tau^2}{\sigma_e^2}\right)^{-1} \frac{1}{\sigma_e^2}$$

$$\propto \frac{1}{\sigma_e^2}\ (\sigma_e^2 + n\,\sigma_\tau^2)^{-1}$$

(8.6)

Equation (8.6) is the same as the prior that was used in the Balanced Case in Chapter 7.

In a similar way we examine equation (8.5) and again we substitute in $n_1 = n_2 = \ldots = n_k = n$ and $\tilde{n} = kn$. Then it follows immediately from this that

$$P_{R_2}(\mu, \sigma_e^2\,, \tilde{r}) \propto \frac{1}{\sigma_e^2}\left\{\frac{kn^2}{(1+\tilde{r}n)^2}\right\}^{\frac{1}{2}}$$

$$\propto \frac{1}{\sigma_e^2}\ (1+\tilde{r}n)^{-1}$$

and once again we can see that

$$P_{R_2}(\mu, \sigma_e^2\,, \sigma_\tau^2) \propto \frac{1}{\sigma_e^2}\ (\sigma_e^2 + n\,\sigma_\tau^2)^{-1}$$

(8.7)

8.2.1 Joint Posterior Distribution for μ, σ_e^2 and $\tilde{r}$

We are now able to examine the distribution of the posterior distribution of μ, σ_e^2 and $\tilde{r}$. This is based on the previous derivations and theorems that have been stated and proved in the appendices to this chapter. From the formulation of the Bayesian model we know the following:

$$p(\mu, \sigma_e^2, \tilde{r} \mid \boldsymbol{Y}) \propto L(\mu, \sigma_e^2, \tilde{r} \mid \boldsymbol{Y}) p(\mu, \sigma_e^2, \tilde{r})$$

where

$$L(\mu, \sigma_e^2, \tilde{r} | \boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}\tilde{n}} \prod_{i=1}^{k} \left(\frac{1}{n_i \tilde{r} + 1}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2\sigma_e^2}\left[v_1 m_1 + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i \tilde{r} + 1}\right]\right\}$$

If we use the Probability-Matching prior as defined by equation (8.3), which is the same as the Reference prior for the first ordering or parameters as described in equation (8.4), then the joint posterior distribution is given by:

$$P_{R_1}(\mu, \tilde{r}, \sigma_e^2 | \boldsymbol{Y})$$

$$\propto \left(\frac{1}{\sigma_e^2}\right)^{\frac{1}{2}(\tilde{n}+2)} \prod_{i=1}^{k} \left(\frac{1}{n_i \tilde{r} + 1}\right)^{\frac{1}{2}}$$
$$\times\ exp\left\{-\frac{1}{2\sigma_e^2}\left[v_1 m_1 + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i \tilde{r} + 1}\right]\right\}$$
$$\times \left\{\sum_{i=1}^{k} \frac{n_i^2}{(1 + \tilde{r} n_i)^2} - \frac{1}{n}\left(\sum_{i=1}^{k} \frac{n_i}{1 + \tilde{r} n_i}\right)^2\right\}^{\frac{1}{2}} \qquad (8.8)$$

where

$$v_1 m_1 = SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij} - \bar{y}_{i\cdot}\right)^2$$

From equation (8.8) it follows that the joint posterior can be expressed hierarchically as

$$P_{R_1}(\mu, \tilde{r}, \sigma_e^2 | \boldsymbol{Y}) = p(\mu | \boldsymbol{Y}, \tilde{r}, \sigma_e^2) \times p(\sigma_e^2 | \tilde{r}, \boldsymbol{Y}) \times p(\tilde{r} | \boldsymbol{Y})$$

where

$$\mu | \boldsymbol{Y}, \tilde{r}, \sigma_e^2 \sim N\left(\hat{\mu}\,,\ \sigma_e^2\left(\sum\nolimits_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^{-1}\right) \tag{8.9}$$

and

$$\hat{\mu} = \frac{\sum\nolimits_{i=1}^{k}\bar{y}_{i\cdot}\dfrac{n_i}{1+\tilde{r}n_i}}{\sum\nolimits_{i=1}^{k}\dfrac{n_i}{1+\tilde{r}n_i}}$$

In addition,

$$P_{R_1}(\sigma_e^2 | \tilde{r}, \boldsymbol{Y}) = K_1\left(\frac{1}{\sigma_e^2}\right)^{\frac{1}{2}(\tilde{n}+1)} exp\left\{-\frac{1}{2\sigma_e^2}\left[v_1 m_1 + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\} \tag{8.10}$$

which is an inverse Gamma distribution. Furthermore, we know that

$$K_1 = \left\{\frac{1}{2}\left[v_1 m_1 + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\}^{\frac{1}{2}(\tilde{n}-1)}$$

and

$$P_{R_1}(\tilde{r}|\,\boldsymbol{Y}) \propto \prod_{i=1}^{k}\left(\frac{1}{n_i\tilde{r}+1}\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^{-\frac{1}{2}} \times \left\{\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}-\frac{1}{n}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2\right\}^{\frac{1}{2}}$$

$$\times\left[v_1 m_1+\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot}-\hat{\mu})^2}{n_i\tilde{r}+1}\right]^{-\frac{1}{2}(n-1)}$$

(8.11)

If we use the alternate ordering for parameters as described in equation (8.5) we find that the joint posterior distribution has the same hierarchical structure, except that

$$P_{R_2}(\tilde{r}|\,\boldsymbol{Y}) \propto \prod_{i=1}^{k}\left(\frac{1}{n_i\tilde{r}+1}\right)^{\frac{1}{2}} \times \left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^{-\frac{1}{2}} \times \left(\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right)^{\frac{1}{2}}$$

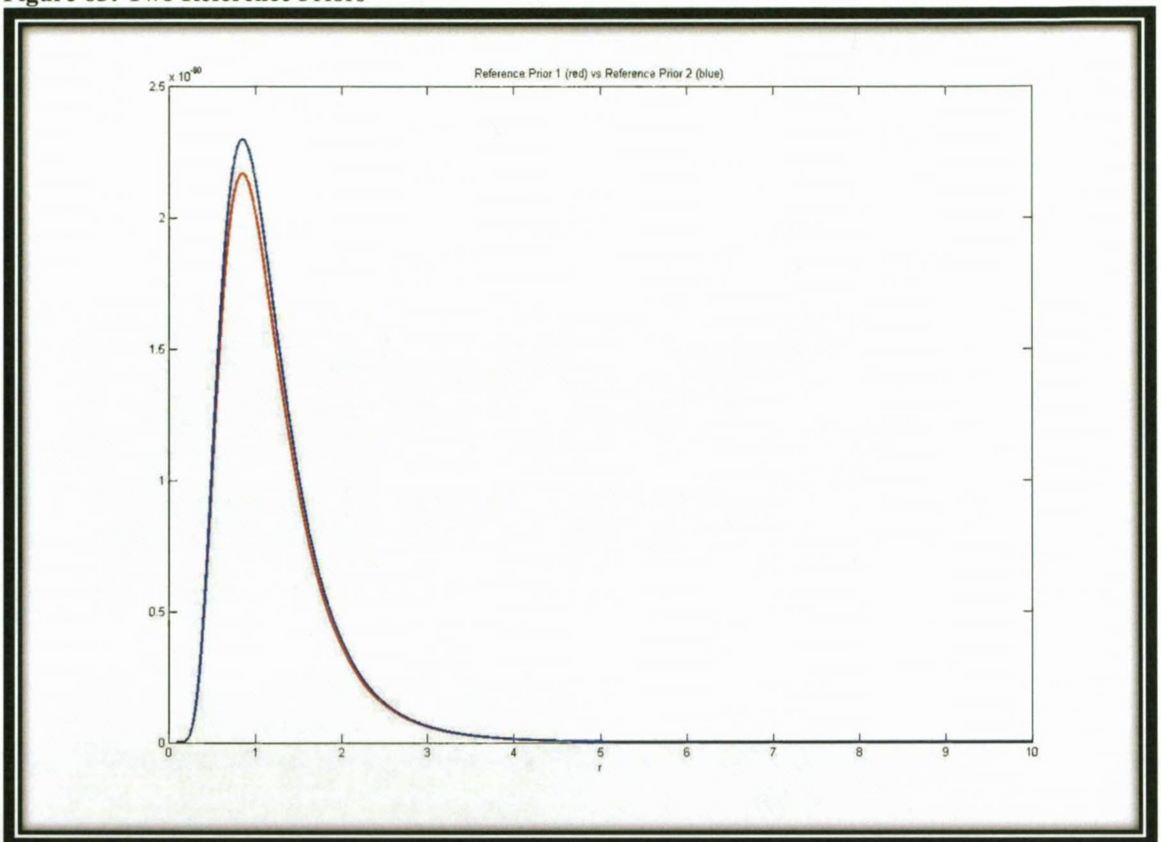
$$\times\left[v_1 m_1+\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot}-\hat{\mu})^2}{n_i\tilde{r}+1}\right]^{-\frac{1}{2}(n-1)}$$

(8.12)

where $0 < \tilde{r} < \infty$.

The following figure depicts these two prior distributions:

Figure 63: Two Reference Priors



*Reference Prior 1 = Red; Reference Prior 2 = Blue.

For further details see van der Merwe, Pretorius and Meyer (2006).

Theorem 8.6

For the model:

$$\boldsymbol{Y} = \mu\mathbf{1} + Z\boldsymbol{\tau} + \boldsymbol{e}$$

where $\boldsymbol{e} \sim N(\mathbf{0}, \sigma_e^2 I_{\tilde{n}})$ and $\boldsymbol{\tau} \sim N(\mathbf{0}, \sigma_\tau^2 I_k)$, the posterior distribution of $\mu + \tau_i$ given σ_e^2 and $\tilde{r}$ is normal with the following mean and variance:

$$E\{(\mu + \tau_i) \mid \boldsymbol{Y}, \sigma_e^2, \tilde{r}\} = \frac{\tilde{r}\, n_i}{1 + \tilde{r} n_i} \bar{y}_{i\cdot} + \frac{1}{1 + \tilde{r} n_i} \hat{\mu}$$

and

$$Var\{(\mu + \tau_i) \mid \boldsymbol{Y}, \sigma_e^2, \tilde{r}\} = \sigma_e^2 \left\{ \tilde{r} + \frac{1}{1 + \tilde{r} n_i} \left(\sum_{i=1}^{k} \frac{n_i}{1 + \tilde{r} n_i} \right)^{-1} \right\}$$

Proof: The proof is given in the appendix to this chapter.

From Theorem 8.6 it follows that

$$\mu + \tau_i + \frac{1}{2}\sigma_e^2 \mid \boldsymbol{Y}, \sigma_e^2, \tilde{r}$$

is distributed normally with

$$E\left\{\left(\mu + \tau_i + \frac{1}{2}\sigma_e^2\right) \mid \boldsymbol{Y}, \sigma_e^2, \tilde{r}\right\} = \frac{\tilde{r}\, n_i}{1 + \tilde{r} n_i}\bar{y}_{i\cdot} + \frac{1}{1 + \tilde{r} n_i}\hat{\mu} + \frac{1}{2}\sigma_e^2 \tag{8.13}$$

and

$$Var\left\{\left(\mu + \tau_i + \frac{1}{2}\sigma_e^2\right) \mid \boldsymbol{Y}, \sigma_e^2, \tilde{r}\right\} = \sigma_e^2 \left\{ \tilde{r} + \frac{1}{1 + \tilde{r} n_i} \left(\sum_{i=1}^{k} \frac{n_i}{1 + \tilde{r} n_i} \right)^{-1} \right\} \tag{8.14}$$

Now, we are interested in the posterior distribution of

$$exp\left(\mu + \tau_i + {\sigma_e^2}/{2}\right) \tag{8.15}$$

for $i = 1, 2, \dots, k$, in other words, for each worker.

Given σ_e^2 and $\tilde{r}$ we can now simulate from (8.15) by simulating from a Normal Distribution with mean and variance specified by equations (8.13) and (8.14) respectively. Using these results we are able to simulate and test hypotheses for individuals (e.g. individual workers). The results will be presented in later sections.

8.2.2 Procedure for Simulation Study

The purpose of this chapter is to describe the behavior of the various prior distributions to the setting described earlier. Although detailed descriptions will be given in relevant sections, here we offer a broad description of the simulation of σ_e^2 and $\tilde{r}$ values from the distributions obtained in previous sections, including the final simulation of μ, which will ultimately enable the simulation of quantities such as defined by equation (8.15). The simulation procedure can broadly be described as follows:

1. Simulate a value for $\tilde{r}$ using either equation (8.11) or (8.12), based on the choice of prior distribution. Since neither (8.11) nor (8.12) is a known distribution and cannot be solved in closed form the use of the Rejection method as described in Rice (1995) will be used.
2. Each value of $\tilde{r}$ simulated in the previous step will then be substituted into equation (8.10) to simulate a value of σ_e^2. In this case the distribution is of a known form, i.e. an Inverse Gamma distribution, and therefore we can simulate σ_e^2 by making use of the fact that:

$$\left\{\frac{1}{\sigma_e^2}\left[v_1 m_1 + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\} \sim \chi_{n-1}^2$$

It follows that a simulated value of σ_e^2 can be obtained from the equation

$$\frac{1}{\chi_{n-1}^2}\left\{\left[v_1 m_1 + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\} = \sigma_e^2$$

Using the values of σ_e^2 and $\tilde{r}$ simulated in the previous steps we can simulate values of μ (if desired) from equation (8.9). All the desired quantities are based on these variables in some manner.

8.3. An Upper Confidence Bound and Test for the Overall Mean Exposure

This chapter is, as mentioned previously, a Bayesian interpretation and application of the problem proposed by Krishnamoorthy and Guo (2005) entitled "Assessing occupational exposure via the one-way random effects model with unbalanced data." Of primary interest is testing the hypothesis of whether the occupational exposure in an individual (discussed previously in (8.13) and (8.14)) or group of workers exceeds a pre-specified or acceptable threshold. If we consider making inferences about the total group, we are interested in the distribution the Overall Mean Exposure, which can be represented as:

$$\mu_x = exp\left\{\mu + \frac{1}{2}(\sigma_e^2 + \sigma_\tau^2)\right\}$$

For the unbalanced case this will for convenience be written as:

$$\mu_x = exp\left\{\mu + \frac{\sigma_e^2}{2}(\tilde{r} + 1)\right\} = e^{\theta} \tag{8.16}$$

Now we know that from (8.9)

$$\mu|\boldsymbol{Y},\tilde{r},\sigma_e^2 \sim N\left(\hat{\mu}\,,\ \sigma_e^2\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^{-1}\right)$$

and therefore,

$$\theta = \mu + \frac{\sigma_e^2}{2}(\tilde{r} + 1)|\boldsymbol{Y},\tilde{r},\sigma_e^2 \sim N\big(E(\theta)\,,\ Var(\theta)\big)$$

is distributed normally with the following mean and variance:

$$E(\theta) = E\left\{\mu + \frac{\sigma_e^2}{2}(\tilde{r} + 1)|\boldsymbol{Y},\tilde{r},\sigma_e^2\right\} = \hat{\mu} + \frac{\sigma_e^2}{2}(\tilde{r} + 1) \tag{8.17}$$

$$Var(\theta) = Var\left\{\mu + \frac{\sigma_e^2}{2}(\tilde{r} + 1)|\boldsymbol{Y}, \tilde{r}, \sigma_e^2\right\} = \sigma_e^2\left(\sum_{i=1}^{k}\frac{n_i}{1 + \tilde{r}n_i}\right)^{-1} \tag{8.18}$$

Thus, given $\tilde{r}$ and σ_e^2 we simulate θ from a normal distribution with mean and variance defined by (8.17) and (8.18) and substitute this into (8.16). We then repeat this process $l\,(= 10000)$ times.

Additionally one of the objectives of the work by Krishnamoorthy and Guo (2005) was to test hypotheses as to whether the overall exposure exceeds a certain limit. The authors also simulate the following statistic (and for the purposes of comparison will be analysed using the Bayesian methodology developed previously):

$$T = \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2$$

where A is a suitably chosen parameter between 0 and 1 and $Z \sim N(0,1)$ is the cumulative distribution function of the standard normal distribution. Using a specific value of OEL the following hypothesis can be tested:

$$H_0: \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2 \geq ln(OEL)$$

against the alternative hypothesis

$$H_1: \mu + Z_{1-A}\sigma_\tau + \frac{1}{2}\sigma_e^2 < ln(OEL)$$

For example, if our choice of A is 0.05 then essentially we are testing (one-sided) whether at least 5% of the workers have mean exposure levels in excess of the chosen OEL. In practice the OEL is chosen to be a clinically relevant value. The specific choice of OEL is not the primary concern of this research, but primarily a demonstration of the Bayesian methodology.

In order to replicate the methodology of Krishnamoorthy and Guo (2005) from a Bayesian perspective the following simulation study was undertaken for a range of both OEL and A values:

Let $T = \mu + \sigma_e^2\left(\frac{1}{2} + Z_{1-A}\tilde{r}\right)$

We know that $T \mid \boldsymbol{Y}, \tilde{r}, \sigma_e^2$ is distributed normally with:

$$E\{T \mid \boldsymbol{Y}, \tilde{r}, \sigma_e^2\} \sim \hat{\mu} + \sigma_e^2\left(\frac{1}{2} + Z_{1-A}\tilde{r}\right)$$

and

$$Var\{T \mid \boldsymbol{Y}, \tilde{r}, \sigma_e^2\} \sim \sigma_e^2\left(\sum_{i=1}^{k} \frac{n_i}{1 + \tilde{r}n_i}\right)^{-1}$$

This procedure was performed for several choices of OEL (= [130; 140; 150; 160; 170; 180]) and for several choices of A (= [0.1; 0.05; 0.025; 0.001]), as was done in Chapter 7.

8.4. Results from the Simulation Study

Using the methodology derived previously a simulation study was conducted to simulate 10000 observations for each particular type of analysis. Using the unbalanced data provided in Table 99 we were able to simulate observations relating to occupational exposure in the workplace. Since 2 Reference priors were derived the simulations were repeated for each of these Reference priors. The results are presented in the following sections.

8.4.1 Results: Individual Worker Means

As mentioned it was possible to simulate observations from the posterior distribution for each of the 15 workers, using both Reference priors. These results are presented in the appendix to this chapter as Figures 79 to 108.

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We can see that the results from the first and second Reference priors are comparable, with no large differences between the choice of Reference prior.

The effect of "unbalancing" has largely been minimized. For example, workers 4 and 11 both had comparable mean exposure levels (55.98 and 55.7 respectively), but were at the two extremes (in this hypothetical data set) with regards to unbalancing (worker 4 had 10 exposure observations, which worker 11 only had 6). It is interesting to note though that in both cases the probability of exceeding the OEL of 130 was 0.0001 (based on 10000 simulated observations). It thus appears that the Bayesian methodology is rather stable with regards to unbalanced data, particularly at a worker-specific level.

Table 100: Simulation Summary Results – Reference Prior 1

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
Worker 1	0.0013	63.64	91.84	59.94	97.16	77.49	77.25	77.75
Worker 2	0.0550	90.61	130.96	85.60	139.11	109.27	108.29	109.75
Worker 3	0.2312	102.05	147.41	97.01	155.86	122.45	121.07	118.75
Worker 4	0.0001	50.34	72.64	47.29	76.177	61.62	61.71	63.25
Worker 5	0.003	72.32	104.14	68.51	109.68	87.64	87.31	88.75
Worker 6	0.9522	130.31	188.23	123.89	199.80	155.65	153.50	152.25
Worker 7	0.0013	64.81	93.68	61.05	96.81	78.69	78.69	78.75
Worker 8	0.0034	72.90	105.40	69.10	111.36	88.29	87.70	87.25
Worker 9	0.1551	99.11	142.28	94.00	150.67	118.80	117.50	117.75
Worker 10	0.2955	103.50	150.46	98.33	160.07	124.74	123.11	151.25
Worker 11	0.0001	51.48	74.27	48.33	77.76	63.17	63.19	61.75
Worker 12	0.0001	55.34	80.22	52.04	84.27	67.89	67.88	67.25
Worker 13	0.0124	80.14	115.55	76.21	122.36	96.68	95.84	94.25
Worker 14	0.2013	100.18	145.38	95.13	154.12	120.86	119.37	117.75
Worker 15	0.1055	95.44	137.77	90.61	145.03	114.83	113.70	111.75

Table 101: Simulation Summary Results – Reference Prior 2

Worker	$P(\mu_{exposure} > 130)$	90% CI Low	90% CI High	95%CI Low	95%CI High	Mean	Median	Mode
Worker 1	0.0010	63.86	92.07	60.19	97.26	77.50	77.24	77.75
Worker 2	0.0487	90.61	129.66	85.67	136.44	108.82	107.88	107.75
Worker 3	0.2271	101.78	146.15	97.07	155.45	122.12	120.76	121.75
Worker 4	0.0002	50.05	72.90	47.35	76.42	61.59	61.61	62.75
Worker 5	0.0040	72.77	104.50	68.81	110.52	87.84	87.30	87.25
Worker 6	0.9521	130.48	188.44	123.69	200.44	155.62	153.33	152.75
Worker 7	0.0014	65.18	93.44	61.36	98.29	78.94	78.71	78.75
Worker 8	0.0036	73.1934	104.88	69.11	110.89	88.24	87.70	87.25
Worker 9	0.1565	98.68	142.71	93.37	149.86	118.52	117.30	113.75
Worker 10	0.2832	103.55	149.47	97.43	157.73	124.44	123.13	123.75
Worker 11	0.0001	51.24	74.07	48.18	77.50	63.15	63.28	64.25
Worker 12	0.0003	55.97	79.97	52.46	83.93	68.02	67.98	66.25
Worker 13	0.0107	79.79	115.25	75.63	121.94	96.45	95.66	95.25
Worker 14	0.1976	100.37	145.30	94.72	153.68	120.85	119.55	117.75
Worker 15	0.0997	95.84	136.99	91.11	144.72	114.76	113.58	111.75

8.4.2 Results: Overall Mean Exposure

The next result relates to the overall mean exposure, i.e. the exposure of the group of 15 workers as a whole. The following results were obtained for the two Reference prior distributions (the relevant information for each histogram is displayed in the subsequent table):

Figure 64: Overall Mean Exposure – Reference Prior 1

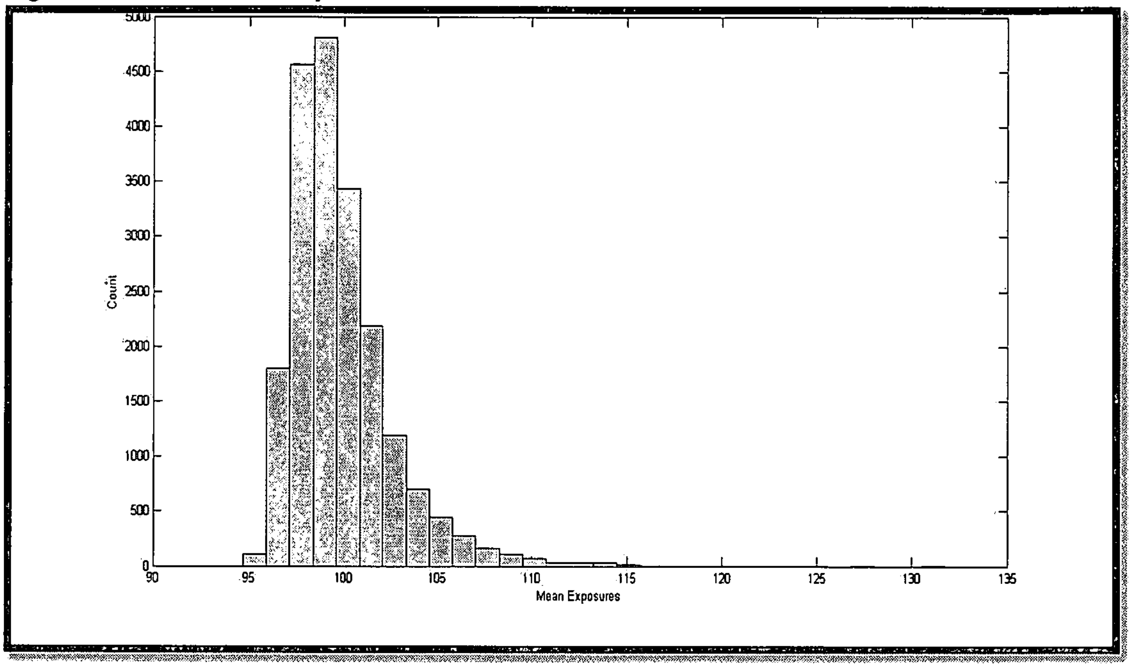


Figure 65:Overall Mean Exposure – Reference Prior 2

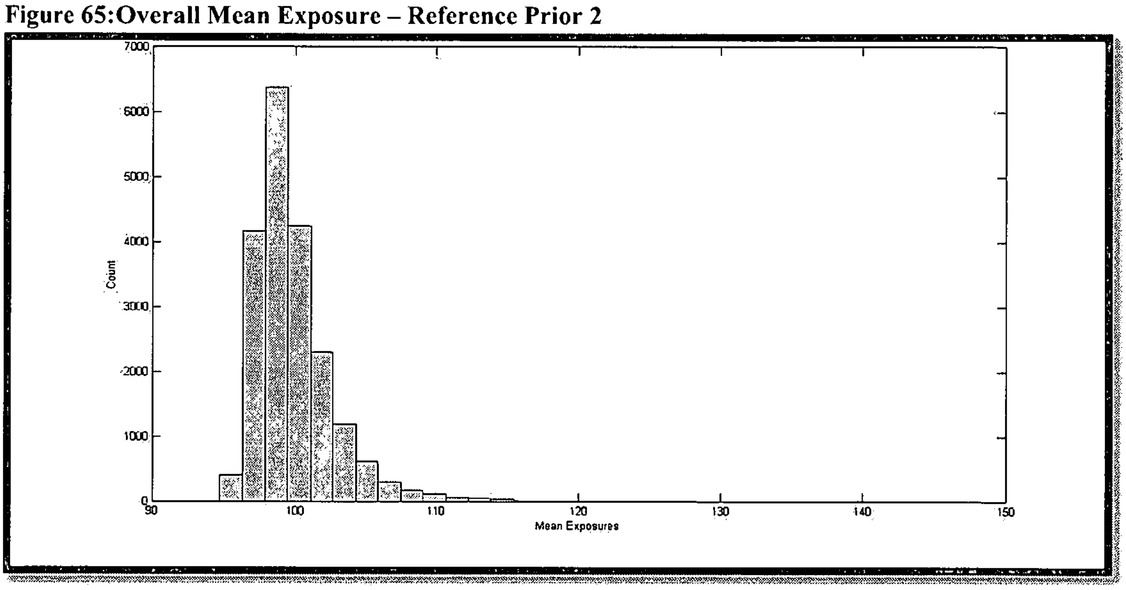


Table 102: Simulation Summary Results of Overall Mean Exposure

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
Reference Prior 1	0.0001	96.75	105.04	96.43	107.04	99.89	99.27	98.75
Reference Prior 2	0.0001	96.73	105.07	96.39	107.11	99.86	99.24	98.25

The above results are based on 20000 simulations. We can see very little difference between the two Reference prior distributions.

8.4.3 Results: Hypothesis Testing

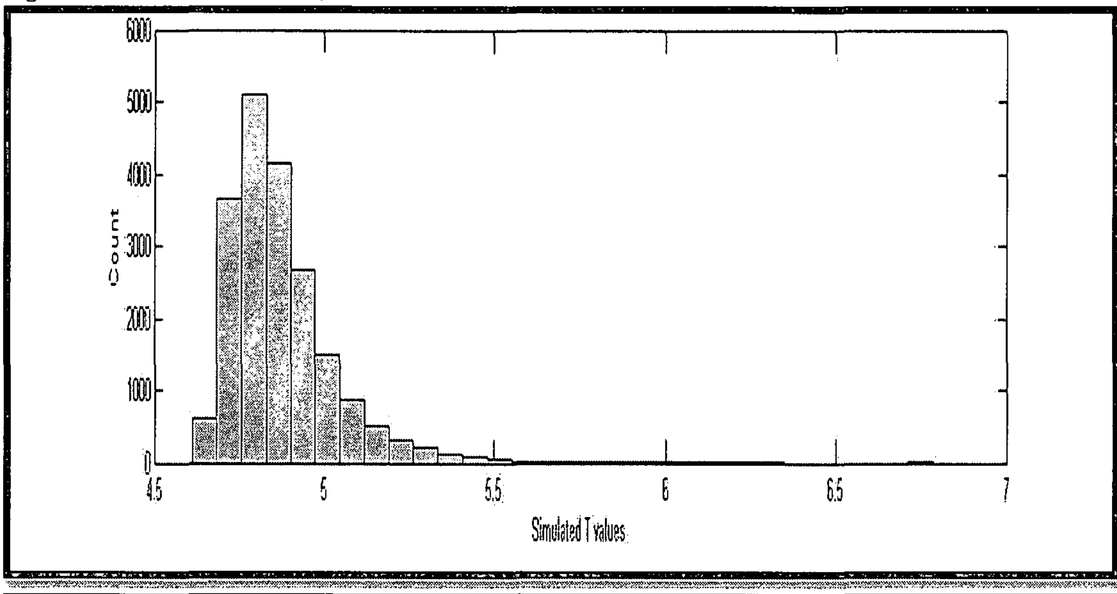
Lastly, and most importantly, Krishnamoorthy and Guo (2005) tested hypotheses regarding the group of workers using the following measure:

$$T = \mu + Z_{1-A}\sigma_{\tau} + \frac{1}{2}\sigma_e^2$$

where A is a suitably chosen parameter between 0 and 1 and Z denotes the standard normal distribution. So, for example, if our choice of A is 0.05 then essentially we are testing (one-sided) whether at least 5% of the workers have mean exposure levels in excess of the chosen OEL.

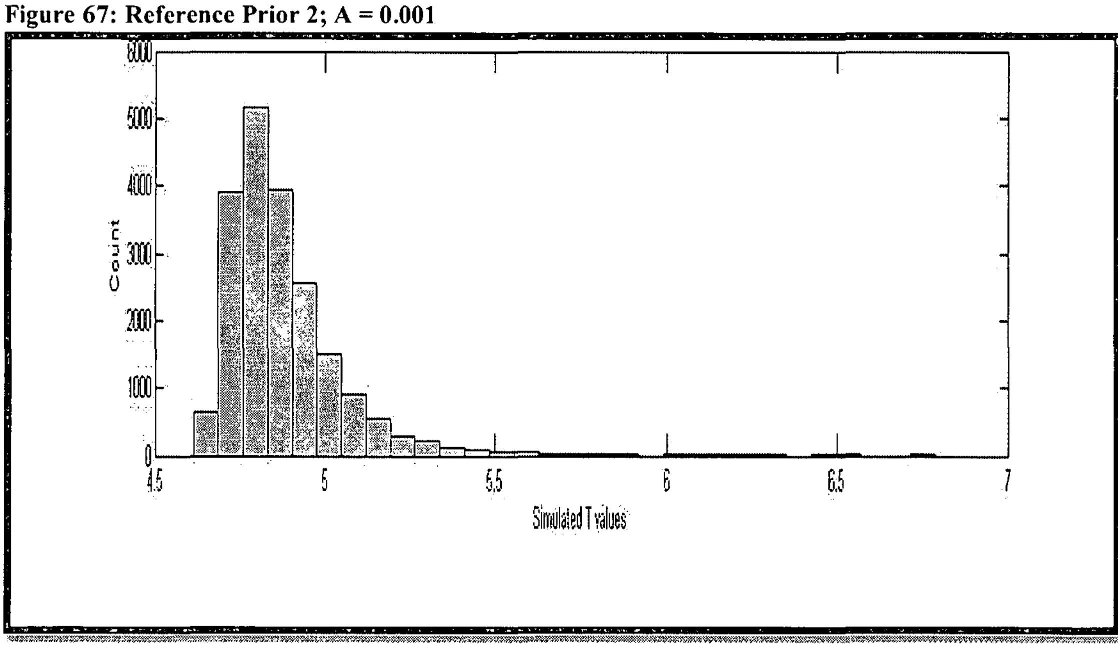
Several different values of A were chosen in addition to several different OEL limits. The results are once again produced for both Reference prior distributions:

Figure 66: Reference Prior 1; A = 0.001



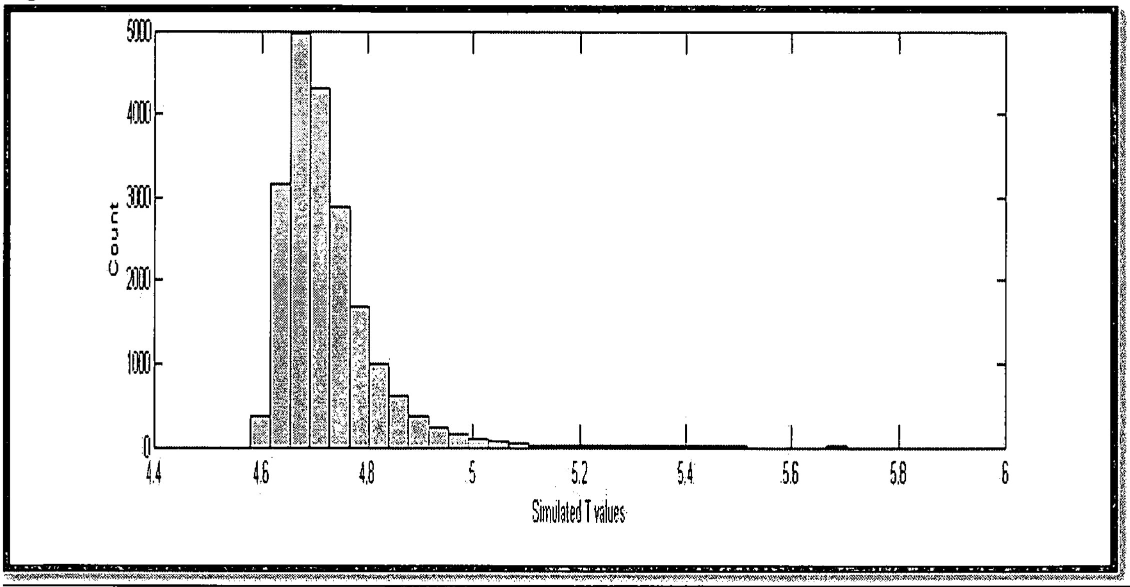
$^{*}100(1-\alpha)th$ percentile = 5.1713

Figure 67: Reference Prior 2; A = 0.001



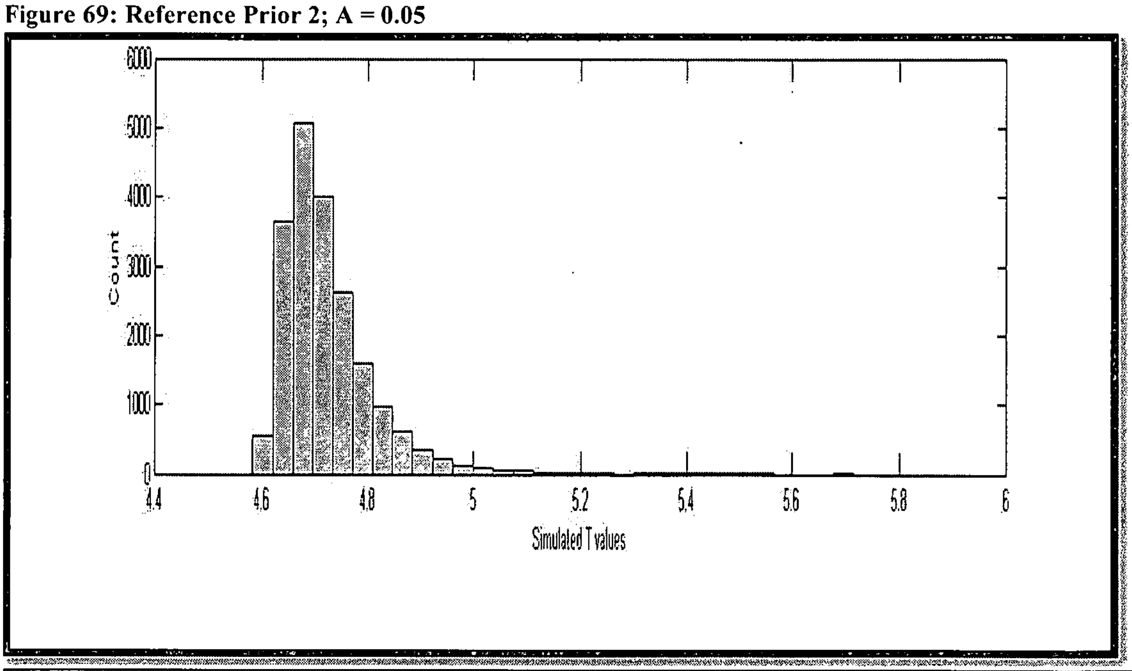
$^{*}100(1-\alpha)th$ percentile = 5.169

Figure 68: Reference Prior 1; A = 0.05



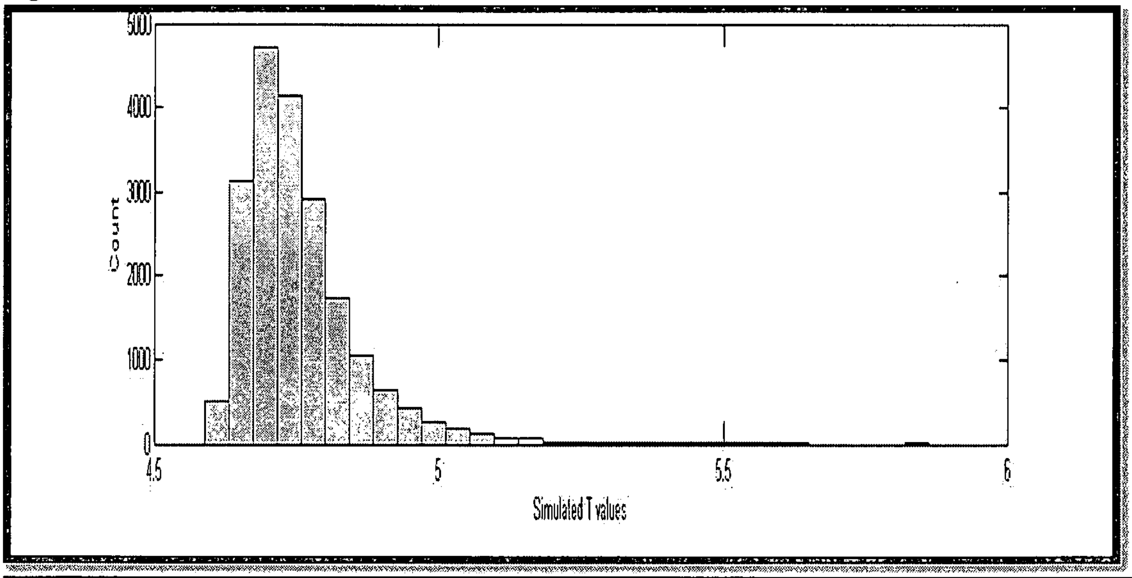
*100$(1-\alpha)th$ percentile = 4.882

Figure 69: Reference Prior 2; A = 0.05



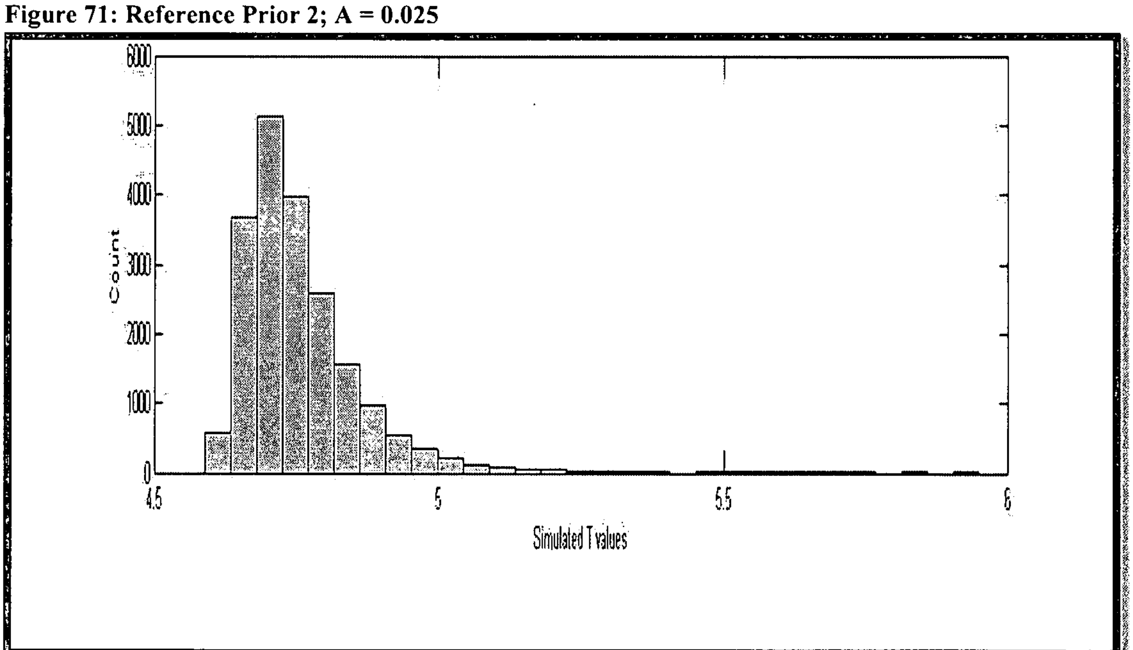
*100$(1-\alpha)th$ percentile = 4.881

Figure 70: Reference Prior 1; A = 0.025



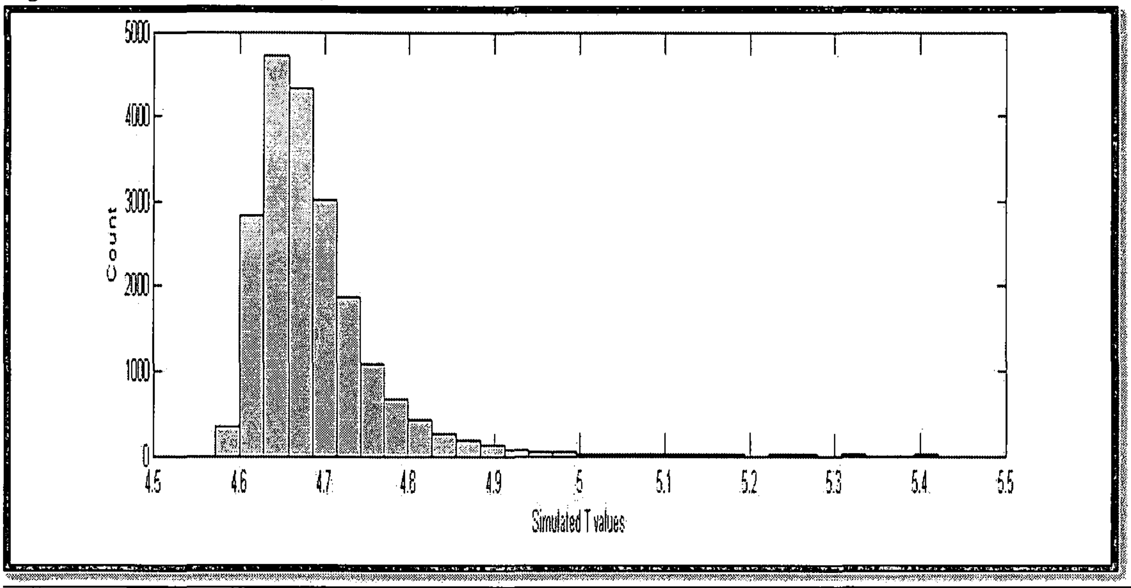
$^{*}100(1-\alpha)th$ percentile = 4.9454

Figure 71: Reference Prior 2; A = 0.025



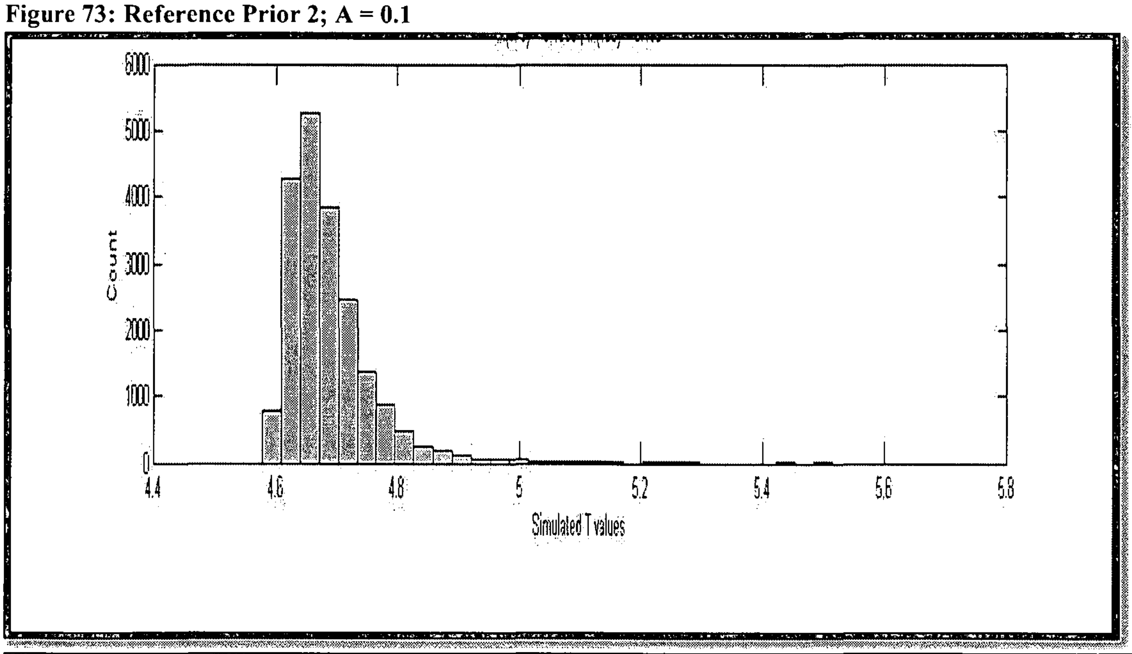
$^{*}100(1-\alpha)th$ percentile 4.9449

Figure 72: Reference Prior 1; A = 0.1



$^{*}100(1-\alpha)th$ percentile = 4.8089

Figure 73: Reference Prior 2; A = 0.1



$^{*}100(1-\alpha)th$ percentile = 4.8086

What is interesting to note is that compared to results obtained in Chapter 7, of which this chapter is merely an extension to the unbalance case, the distributions in the unbalanced case are more skewed, with longer tails. From the above figure we can see that 10% or more of workers had occupational exposure levels in excess of 4.8086 (95th percentile), which corresponds to an OEL of roughly 130.

8.4.1 Prior Distribution Based on Gelman

In addition to the prior distributions applied in the previous results section, Gelman (2006) suggested the following prior for the unbalanced random effects model:

$$p_{Gelman}(\mu, \sigma_e^2, \sigma_\tau^2) \propto \sigma_e^{-2}\, \sigma_\tau^{-1}$$

We know that the use of σ_τ^{-2} would result in an improper posterior distribution.

From (8.1) we know that

$$L(\mu, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}(\tilde{n}-k)} \prod_{i=1}^{k} \left(\frac{1}{n_i\sigma_\tau^2 + \sigma_e^2}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\sigma_\tau^2 + \sigma_e^2}\right]\right\}$$

Now transforming Gelman's prior into the notation of the ratio of σ_e^2 and σ_τ^2 we have the following:

$$p_{Gelman}(\mu, \sigma_e^2, \sigma_\tau^2) \propto \sigma_e^{-2}\, \sigma_\tau^{-1}$$

$$\tilde{r} = \frac{\sigma_\tau^2}{\sigma_e^2}$$

$$\therefore\ p_{Gelman}(\mu, \sigma_e^2, \tilde{r}) \propto \sigma_e^{-3}\, \tilde{r}^{-\frac{1}{2}} \left|\frac{\partial \sigma_\tau^2}{\partial \tilde{r}}\right| = \sigma_e^{-3}\, \tilde{r}^{-\frac{1}{2}}\, \sigma_e^2$$

$$\propto \sigma_e^{-1}\, \tilde{r}^{-\frac{1}{2}}$$

Converting the likelihood to specification in terms of $\tilde{r}$:

$$L(\mu, \sigma_e^2, \tilde{r}|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}\tilde{n}} \prod_{i=1}^{k} \left(\frac{1}{n_i\tilde{r} + 1}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\tilde{r} + 1}\right]\right\}$$

(8.19)

In a similar way to the previous sections we can derive the combined posterior distribution, given Gelman's prior distribution, as follows:

$$p(\mu, \sigma_e^2, \tilde{r}|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}(\tilde{n}+1)} \tilde{r}^{-\frac{1}{2}} \prod_{i=1}^{k} \left(\frac{1}{n_i\tilde{r} + 1}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\tilde{r} + 1}\right]\right\}$$

Therefore,

$$p(\sigma_e^2, \tilde{r}|\boldsymbol{Y}) = \int_{-\infty}^{\infty} p(\mu, \sigma_e^2, \tilde{r}|\boldsymbol{Y})\ d\mu$$

In order to find this integral we have to complete the square with respect to μ (similarly to previously done) and therefore,

$$p(\sigma_e^2, \tilde{r}|\boldsymbol{Y}) = (\sigma_e^2)^{-\frac{1}{2}(\tilde{n}+1)} \tilde{r}^{-\frac{1}{2}} \prod_{i=1}^{k} \left(\frac{1}{n_i\tilde{r} + 1}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\} \int_{-\infty}^{\infty} exp\left\{-\frac{1}{2\sigma_e^2}\left[(\mu - \hat{\mu})^2 \sum_{i=1}^{k} \frac{n_i}{n_i\tilde{r} + 1}\right]\right\} d\mu$$

Therefore,

$$p(\sigma_e^2, \tilde{r}|\boldsymbol{Y}) = (\sigma_e^2)^{-\frac{1}{2}(\tilde{n})} \tilde{r}^{-\frac{1}{2}} \prod_{i=1}^{k} \left(\frac{1}{n_i\tilde{r} + 1}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} \frac{n_i}{n_i\tilde{r} + 1}\right)^{-\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k} \frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\}$$

Therefore,

$$p(\sigma_e^2|\tilde{r},\boldsymbol{Y}) = \widetilde{\boldsymbol{K}}(\sigma_e^2)^{-\frac{1}{2}(\tilde{n})} \, exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\}$$

which is an Inverse Gamma distribution with

$$\widetilde{\boldsymbol{K}} = \left\{\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\}^{\frac{1}{2}(\tilde{n}-2)}$$

and

$$p(\tilde{r}|\boldsymbol{Y}) = \tilde{r}^{-\frac{1}{2}}\prod_{i=1}^{k}\left(\frac{1}{n_i\tilde{r} + 1}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k}\frac{n_i}{n_i\tilde{r} + 1}\right)^{-\frac{1}{2}}\left\{\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right]\right\}^{-\frac{1}{2}(\tilde{n}-2)}$$

Once again, to simulate observations from the posterior distribution, the Rejection Method will be used to simulate observations from $p(\tilde{r}|\boldsymbol{Y})$. Then using this simulated value of $\tilde{r}$ the following relationship will be used to simulate σ_e^2:

$$\frac{1}{\sigma_e^2}\left\{SSE + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \hat{\mu})^2}{n_i\tilde{r} + 1}\right\} \sim \chi^2_{\tilde{n}-2}$$

<u>Simulation Results:</u>

The same simulations that were performed for the two Reference priors previously will also be presented for the prior proposed by Gelman to examine whether any differences between the prior distributions exist. The following situations were examined in the previous simulation study (and will be repeated for Gelman's prior here):

1. Simulation of individual worker means
2. Simulation of overall mean exposure
3. Hypothesis testing

The methodologies are similar to those mentioned previously and will thus not be repeated here. The only deviations from the previous descriptions would be the exact specifications of $p(\tilde{r}|\boldsymbol{Y})$ and $p(\sigma_e^2|\tilde{r}, \boldsymbol{Y})$.

The simulation results for individual workers using Gelman's prior are presented in the appendix as Figures 109to 123. The following table summarises the information therein:

Table 103: Simulation Summary Results – Gelman Prior

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
Worker 1	0.0021	62.49	93.19	58.72	98.36	77.46	77.21	76.75
Worker 2	0.0635	89.12	133.00	83.73	142.34	109.27	108.07	106.75
Worker 3	0.2605	100.74	150.49	94.78	159.32	123.05	121.46	119.75
Worker 4	0.0003	49.37	73.74	46.46	78.29	61.58	61.43	62.25
Worker 5	0.0062	71.26	106.35	67.38	113.18	88.10	87.51	87.75
Worker 6	0.9413	128.16	191.69	121.41	205.56	156.25	153.68	146.25
Worker 7	0.0018	63.48	95.50	59.52	101.63	78.96	78.63	79.25
Worker 8	0.0046	71.39	106.53	66.96	112.30	88.27	87.75	87.25
Worker 9	0.1779	97.55	145.68	92.42	155.18	119.05	117.35	111.25
Worker 10	0.2994	102.37	151.74	96.05	161.17	124.87	123.49	122.25
Worker 11	0.0006	50.19	75.26	46.92	79.38	62.99	63.02	62.75
Worker 12	0.0006	54.55	81.05	51.06	86.31	67.83	67.74	68.75
Worker 13	0.0156	7821	116.80	74.21	124.49	96.49	95.69	95.25
Worker 14	0.2194	99.41	147.88	93.89	156.15	121.46	120.02	118.75
Worker 15	0.1210	93.87	140.48	88.54	148.32	115.00	113.63	109.75

The results from Gelman's Prior distribution are as expected. Given the form of the prior distribution, when compared to the two Reference prior distributions, it appears as though the Reference priors decrease the variance in the posterior distribution more efficiently than the Reference priors. The only possible exception to this would be when $\sigma_\tau^2 < 1$. So it would be expected that Gelman's prior would be better suited when modeling the situation when within-worker variance is less than one. However, this was not the case in this example and consequently the two Reference priors result in narrower 95% confidence intervals, implying a decrease in variance. This is easily seen in Worker 2, for example, whereby the confidence intervals for the first and second Reference priors are [85.61 : 139.11] and [85.67 : 136.44] respectively. In particular, for these two prior distributions it would appear as though the second Reference prior results in upper bounds that are less than for the first Reference prior. Thus, Reference prior 2 perhaps does not place as much emphasis on potential extreme upper values from the distribution. Gelman's prior on the other hand accounts for more variance than both of these and results in a 95% Bayesian confidence interval of [83.73 : 142,33] for Worker 2.

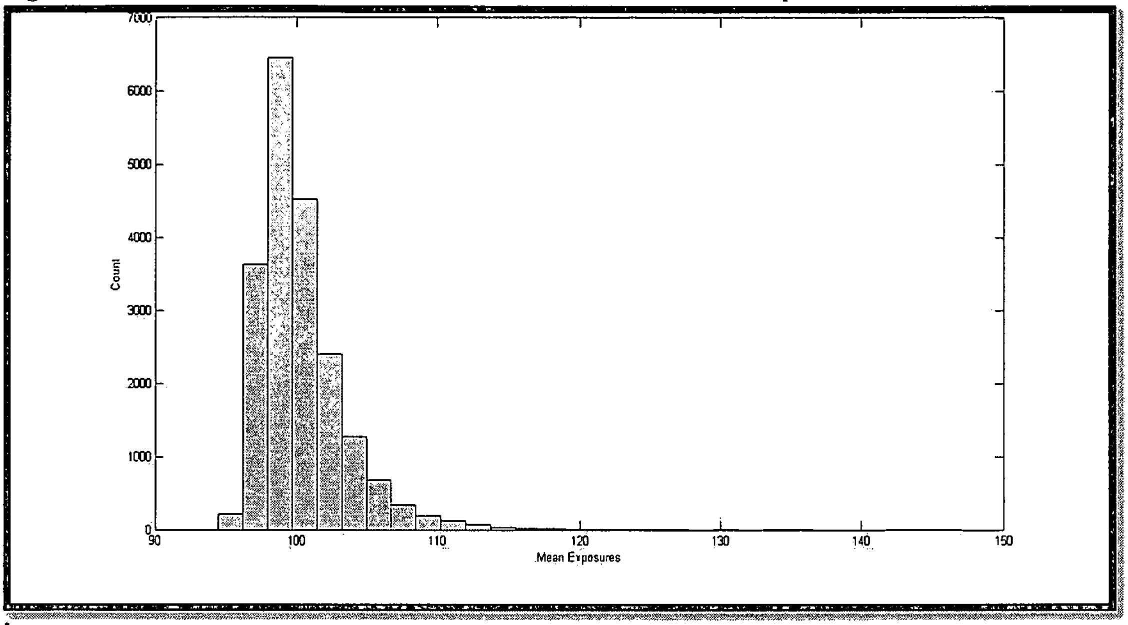
The following table summarizes the results and differences between the 3 priors considered here for the Overall Mean Exposure:

Table 104: Simulation Summary Results

Worker	$P(\mu_{exposure} > 130)$	90% CI		95%CI		Mean	Median	Mode
		Low	High	Low	High			
Gelman	0.0003	96.90	106.18	96.53	108.45	100.35	99.62	98.75
Reference Prior 1	0.0001	96.75	105.04	96.43	107.04	99.89	99.27	98.75
Reference Prior 2	0.0001	96.73	105.07	96.39	107.11	99.86	99.24	98.25

We see that there is very little to distinguish between the two Reference priors. However, it seems that Gelman's prior results in interval lengths that are slightly longer than the two Reference priors.

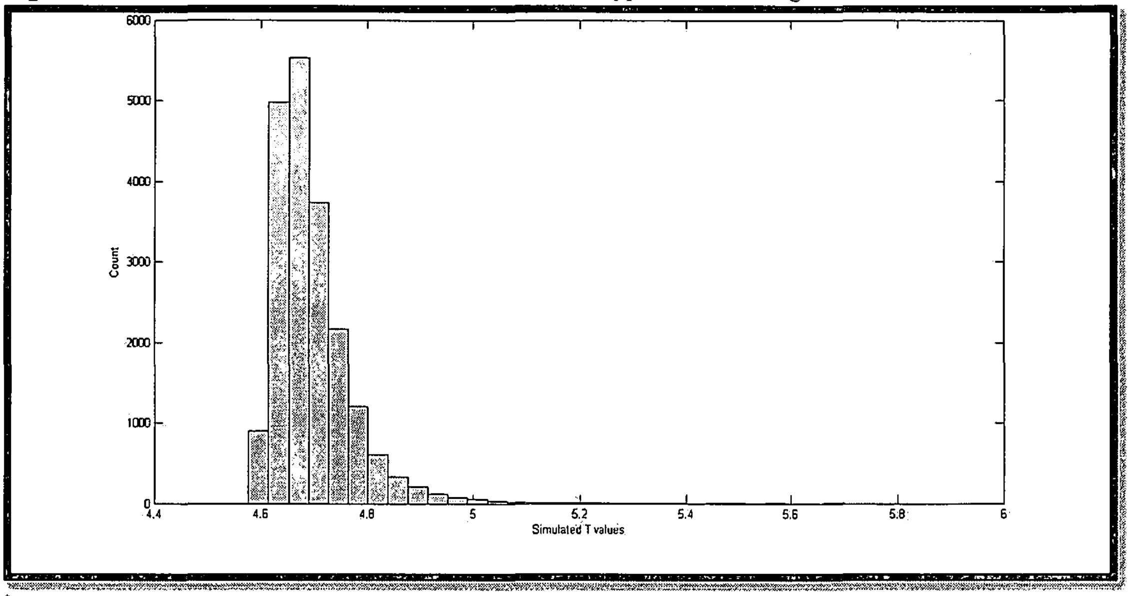
Figure 74: Simulation Results for Gelman's Prior for Overall Mean Exposure



*Refer to Table 105 for summary statistics accompanying this figure

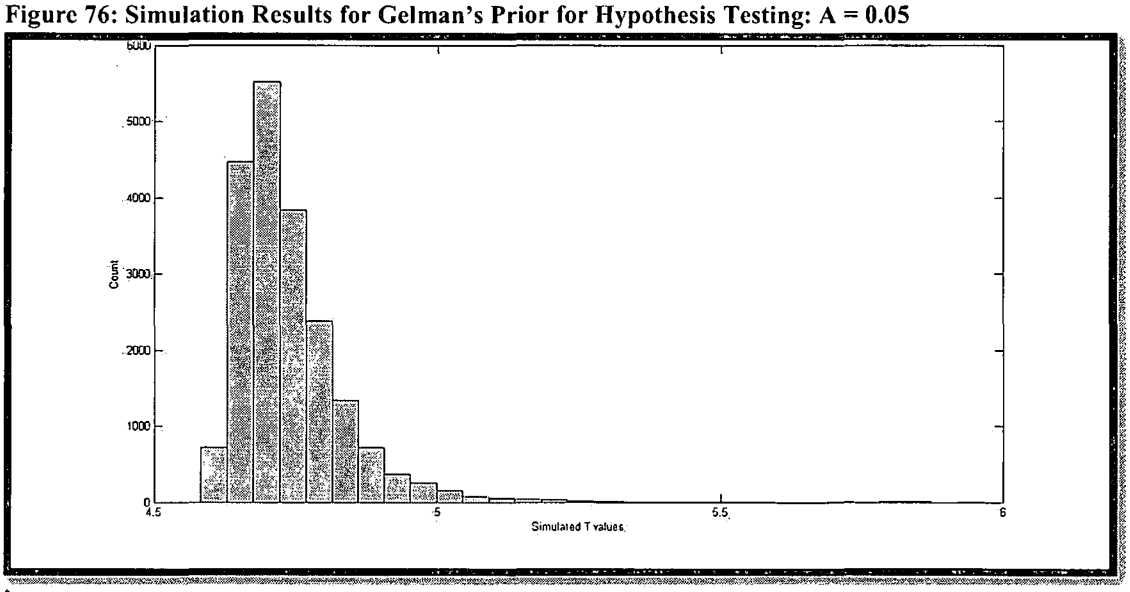
From the above and a comparison of the Reference priors previously discussed it appears as though the posterior distribution derived using Gelman's prior has longer tails. Thus, there is more potential to simulate extreme values for the overall mean exposure. This indicates that perhaps Gelman's prior results in a larger variance component than the Reference prior and therefore accounts for "more" uncertainty.

Figure 75: Simulation Results for Gelman's Prior for Hypothesis Testing: A = 0.1



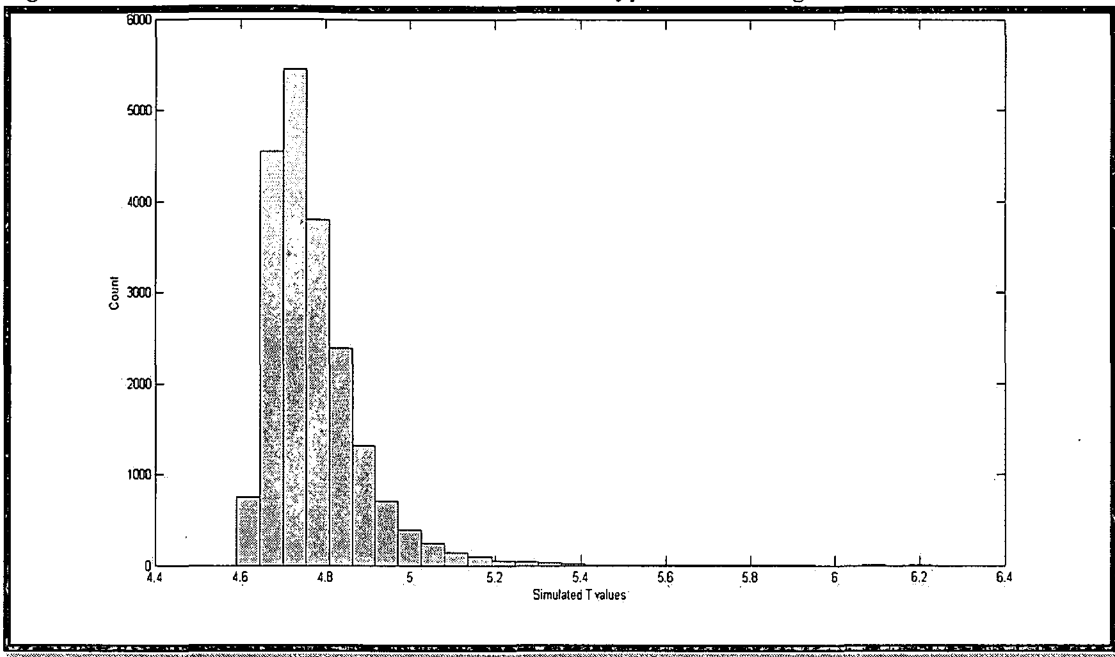
*$100(1-\alpha)th$ percentile = 4.8284

Figure 76: Simulation Results for Gelman's Prior for Hypothesis Testing: A = 0.05



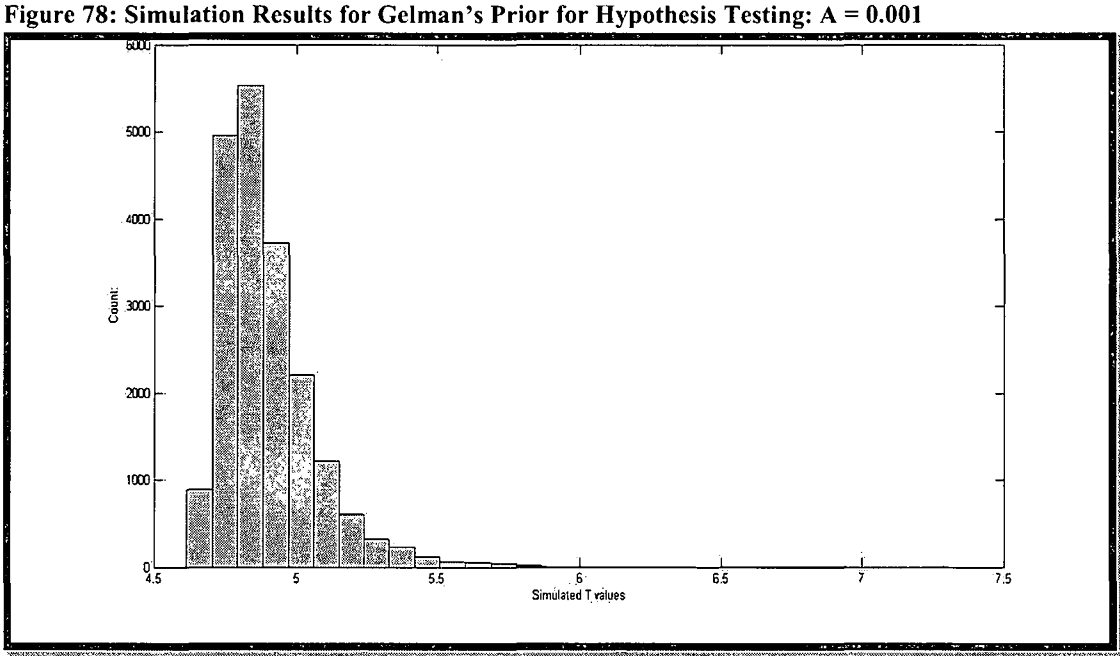
*$100(1-\alpha)th$ percentile = 4.9064

Figure 77: Simulation Results for Gelman's Prior for Hypothesis Testing: A = 0.025



*$100(1-\alpha)th$ percentile = 4.9769

Figure 78: Simulation Results for Gelman's Prior for Hypothesis Testing: A = 0.001



*$100(1-\alpha)th$ percentile = 5.2209

Appendix to Chapter 8

Proof of Theorem 8.1

$$L(\mu, \sigma_e^2, \sigma_\tau^2 | \boldsymbol{Y}) = \int_{\infty}^{\infty} L(\mu, \boldsymbol{\tau}, \sigma_e^2, \sigma_\tau^2 | \boldsymbol{Y}) \partial \boldsymbol{\tau}$$

We begin by completing the square with respect to $\boldsymbol{\tau}$. Let

$S = \frac{1}{\sigma_e^2}\left(\widetilde{\boldsymbol{Y}} - Z\boldsymbol{\tau}\right)'\left(\widetilde{\boldsymbol{Y}} - Z\boldsymbol{\tau}\right) + \frac{1}{\sigma_\tau^2}\boldsymbol{\tau}'\boldsymbol{\tau}$ where $\widetilde{\boldsymbol{Y}} = \boldsymbol{Y} - \mu\boldsymbol{1}$.

Therefore,

$$S = \frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} - \frac{2}{\sigma_e^2}\boldsymbol{\tau}'Z'\widetilde{\boldsymbol{Y}} + \frac{1}{\sigma_e^2}\boldsymbol{\tau}'Z'Z\boldsymbol{\tau} + \frac{1}{\sigma_\tau^2}\boldsymbol{\tau}'\boldsymbol{\tau}$$

$$= \frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} + \boldsymbol{\tau}'\left\{\frac{1}{\sigma_e^2}Z'Z + \frac{1}{\sigma_\tau^2}I_k\right\}\boldsymbol{\tau} - 2\boldsymbol{\tau}'\frac{1}{\sigma_e^2}Z'\widetilde{\boldsymbol{Y}}$$

$$= \frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} + \boldsymbol{\tau}'D\boldsymbol{\tau} - 2\boldsymbol{\tau}'\boldsymbol{C}$$

where

$D = \left\{\frac{1}{\sigma_e^2}Z'Z + \frac{1}{\sigma_\tau^2}I_k\right\}$ and $\boldsymbol{C} = \frac{1}{\sigma_e^2}Z'\widetilde{\boldsymbol{Y}}$

After completing the square it follows that

$$S = \frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} + (\boldsymbol{\tau} - D^{-1}\boldsymbol{C})'D(\boldsymbol{\tau} - D^{-1}\boldsymbol{C}) - \boldsymbol{C}'D^{-1}\boldsymbol{C}$$

Therefore, the integrated likelihood function is

$$L(\mu, \sigma_e^2, \sigma_\tau^2 | \boldsymbol{Y}) = \int_{\infty}^{\infty} L(\mu, \boldsymbol{\tau}, \sigma_e^2, \sigma_\tau^2 | \boldsymbol{Y}) \partial \boldsymbol{\tau}$$

where

$$L(\mu, \boldsymbol{\tau}, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y}) \propto \left(\frac{1}{\sigma_e^2}\right)^{\frac{1}{2}\tilde{n}} \left(\frac{1}{\sigma_\tau^2}\right)^{\frac{1}{2}k} exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} + (\boldsymbol{\tau} - D^{-1}\boldsymbol{C})'D(\boldsymbol{\tau} - D^{-1}\boldsymbol{C}) - \boldsymbol{C}'D^{-1}\boldsymbol{C}\right]\right\}$$

Therefore, the integrated likelihood is:

$$L(\mu, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y}) \propto \left(\frac{1}{\sigma_e^2}\right)^{\frac{1}{2}\tilde{n}} \left(\frac{1}{\sigma_\tau^2}\right)^{\frac{1}{2}k} |D|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} - \boldsymbol{C}'D^{-1}\boldsymbol{C}\right]\right\}$$

(A)

Equation (A) can be simplified as follows:

Consider

$$|D|^{-\frac{1}{2}} = \left|\frac{1}{\sigma_e^2}Z'Z + \frac{1}{\sigma_\tau^2}I_k\right|^{-\frac{1}{2}}$$

$$= \begin{vmatrix} \frac{n_1}{\sigma_e^2} + \frac{1}{\sigma_\tau^2} & 0 & \cdots & 0 \\ 0 & \frac{n_2}{\sigma_e^2} + \frac{1}{\sigma_\tau^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n_k}{\sigma_e^2} + \frac{1}{\sigma_\tau^2} \end{vmatrix}^{-\frac{1}{2}} = \left\{\prod_{i=1}^{k}\left(\frac{n_i}{\sigma_e^2} + \frac{1}{\sigma_\tau^2}\right)\right\}^{-\frac{1}{2}}$$

$$= \prod_{i=1}^{k}\left(\frac{\sigma_e^2\sigma_\tau^2}{n_i\sigma_\tau^2 + \sigma_e^2}\right)^{\frac{1}{2}}$$

(B)

Next, consider $\frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}}$ in the exponent of (A):

$$\frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} = \frac{1}{\sigma_e^2}(\boldsymbol{Y} - \mu\boldsymbol{1})'(\boldsymbol{Y} - \mu\boldsymbol{1}) = \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij} - \mu\right)^2$$

Now,

$$\bar{y}_{i\cdot} = \frac{1}{n_i} y_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$$

and therefore,

$$\frac{1}{\sigma_e^2} \widetilde{\boldsymbol{Y}}' \widetilde{\boldsymbol{Y}} = \frac{1}{\sigma_e^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(y_{ij} - \mu\right)^2 = \frac{1}{\sigma_e^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left\{\left(y_{ij} - \bar{y}_{i\cdot}\right) + \left(\bar{y}_{i\cdot} - \mu\right)\right\}^2$$

$$= \frac{1}{\sigma_e^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(y_{ij} - \bar{y}_{i\cdot}\right)^2 + \frac{1}{\sigma_e^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(\bar{y}_{i\cdot} - \mu\right)^2 \qquad \text{(C)}$$

since the middle term becomes zero.

Furthermore,

$$\boldsymbol{C}' D^{-1} \boldsymbol{C} = \frac{1}{\sigma_e^2} \widetilde{\boldsymbol{Y}}' Z D^{-1} Z' \widetilde{\boldsymbol{Y}} \frac{1}{\sigma_e^2}$$

$$= \frac{1}{\sigma_e^2} (\boldsymbol{Y} - \mu \mathbf{1})' Z D^{-1} Z' (\boldsymbol{Y} - \mu \mathbf{1}) \frac{1}{\sigma_e^2}$$

$$= \left(\frac{1}{\sigma_e^2}\right)^2 \left[\sum_{j=1}^{n_1} \left(y_{1j} - \mu\right) \quad \sum_{j=1}^{n_2} \left(y_{2j} - \mu\right) \quad \cdots \quad \sum_{j=1}^{n_k} \left(y_{kj} - \mu\right) \right]$$

$$\times \begin{bmatrix} \frac{\sigma_e^2\sigma_\tau^2}{n_1\sigma_\tau^2 + \sigma_e^2} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_e^2\sigma_\tau^2}{n_2\sigma_\tau^2 + \sigma_e^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\sigma_e^2\sigma_\tau^2}{n_k\sigma_\tau^2 + \sigma_e^2} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n_1}(y_{1j} - \mu) \\ \sum_{j=1}^{n_2}(y_{2j} - \mu) \\ \vdots \\ \sum_{j=1}^{n_k}(y_{kj} - \mu) \end{bmatrix}$$

$$= \left(\frac{\sigma_\tau^2}{\sigma_e^2}\right)[n_1(\bar{y}_{1\cdot} - \mu) \quad n_2(\bar{y}_{2\cdot} - \mu) \quad \cdots \quad n_k(\bar{y}_{k\cdot} - \mu)]$$

$$\times \begin{bmatrix} \frac{1}{n_1\sigma_\tau^2 + \sigma_e^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{n_2\sigma_\tau^2 + \sigma_e^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n_k\sigma_\tau^2 + \sigma_e^2} \end{bmatrix} \begin{bmatrix} n_1(\bar{y}_{1\cdot} - \mu) \\ n_2(\bar{y}_{2\cdot} - \mu) \\ \vdots \\ n_k(\bar{y}_{k\cdot} - \mu) \end{bmatrix}$$

$$= \left(\frac{\sigma_\tau^2}{\sigma_e^2}\right)\sum_{i=1}^{k}\frac{n_i^2(\bar{y}_{i\cdot} - \mu)^2}{n_i\sigma_\tau^2 + \sigma_e^2}$$

(D)

From (C) and (D) it follows that

$$\frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} - \boldsymbol{C}'D^{-1}\boldsymbol{C}$$

$$= \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i\cdot})^2 + \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_{i\cdot} - \mu)^2 - \left(\frac{\sigma_\tau^2}{\sigma_e^2}\right)\sum_{i=1}^{k}\frac{n_i^2(\bar{y}_{i\cdot} - \mu)^2}{n_i\sigma_\tau^2 + \sigma_e^2}$$

$$= \frac{\nu_1 m_1}{\sigma_e^2} + \frac{1}{\sigma_e^2}\sum_{i=1}^{k}(\bar{y}_{i\cdot} - \mu)^2\left\{n_i - \sigma_\tau^2\frac{n_i^2}{n_i\sigma_\tau^2 + \sigma_e^2}\right\}$$

$$= \frac{v_1 m_1}{\sigma_e^2} + \frac{1}{\sigma_e^2}\sum_{i=1}^{k}(\bar{y}_{i\cdot} - \mu)^2 \left\{\frac{n_i(n_i\sigma_\tau^2 + \sigma_e^2) - n_i^2}{n_i\sigma_\tau^2 + \sigma_e^2}\sigma_\tau^2\right\}$$

$$= \frac{v_1 m_1}{\sigma_e^2} + \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\sigma_\tau^2 + \sigma_e^2}$$

Therefore,

$$L(\mu, \sigma_e^2, \sigma_\tau^2|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}(n-k)} \prod_{i=1}^{k}\left(\frac{1}{n_i\sigma_\tau^2 + \sigma_e^2}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_e^2} + \sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot} - \mu)^2}{n_i\sigma_\tau^2 + \sigma_e^2}\right]\right\}$$

which proves the theorem.

Proof of Theorem 8.2

$$y_{ij}|\,\mu, \tau_i, \sigma_e^2 \sim N(\mu + \tau_i, \sigma_e^2)$$

$$\bar{y}_{i\cdot}|\,\mu, \tau_i, \sigma_e^2 \sim N\left(\mu + \tau_i, \frac{\sigma_e^2}{n_i}\right)$$

Therefore,

$$\bar{y}_{i\cdot}|\,\mu, \sigma_\tau^2, \sigma_e^2 \sim N\left(\mu, \frac{\sigma_e^2}{n_i} + \sigma_\tau^2\right)$$

$$\bar{y}_{i\cdot}|\,\mu, \sigma_\tau^2, \sigma_e^2 \sim N\left(\mu, \frac{n_i\sigma_\tau^2 + \sigma_e^2}{n_i}\right)$$

Therefore,

$$\bar{y}_{i\cdot}|\,\mu, \tilde{r}, \sigma_e^2 \sim N\left(\mu, \frac{\sigma_e^2(n_i\tilde{r} + 1)}{n_i}\right)$$

Proof of Theorem 8.3

From the integrated likelihood function,

$$L(\mu,\sigma_e^2,\tilde{r}|\boldsymbol{Y}) \propto (\sigma_e^2)^{-\frac{1}{2}\tilde{n}}\prod_{i=1}^{k}\left(\frac{1}{n_i\tilde{r}+1}\right)^{\frac{1}{2}} exp\left\{-\frac{1}{2\sigma_e^2}\left[v_1m_1+\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot}-\mu)^2}{n_i\tilde{r}+1}\right]\right\}$$

it follows that

$$l = ln\big(L(\mu,\sigma_e^2,\tilde{r}|\boldsymbol{Y})\big) = -\frac{1}{2}\tilde{n}ln(\sigma_e^2)-\frac{1}{2}\sum_{i=1}^{k}ln(n_i\tilde{r}+1)-\frac{1}{2\sigma_e^2}\left[v_1m_1+\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot}-\mu)^2}{n_i\tilde{r}+1}\right]$$

Now

$$\frac{\partial l}{\partial\sigma_e^2} = -\frac{\tilde{n}}{2\sigma_e^2}+\frac{1}{2}\left(\frac{1}{\sigma_e^2}\right)^2\left\{v_1m_1+\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot}-\mu)^2}{n_i\tilde{r}+1}\right\}$$

$$\frac{\partial^2 l}{(\partial\sigma_e^2)^2} = \frac{\tilde{n}}{2(\sigma_e^2)^2}-\left(\frac{1}{\sigma_e^2}\right)^3\left\{v_1m_1+\sum_{i=1}^{k}\frac{n_i(\bar{y}_{i\cdot}-\mu)^2}{n_i\tilde{r}+1}\right\}$$

Therefore,

$$-E\left(\frac{\partial^2 l}{(\partial\sigma_e^2)^2}\right) = -\frac{\tilde{n}}{2(\sigma_e^2)^2}+\left(\frac{1}{\sigma_e^2}\right)^3\left\{v_1\sigma_e^2+\sum_{i=1}^{k}\frac{n_i}{n_i\tilde{r}+1}\times\frac{\sigma_e^2(n_i\tilde{r}+1)}{n_i}\right\}$$

$$= -\frac{\tilde{n}}{2(\sigma_e^2)^2}+\tilde{n}\left(\frac{1}{\sigma_e^2}\right)^2$$

$$= \frac{\tilde{n}}{2(\sigma_e^2)^2}$$

Furthermore,

$$\frac{\partial l}{\partial\tilde{r}} = -\frac{1}{2}\sum_{i=1}^{k}\frac{n_i}{n_i\tilde{r}+1}+\frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{(n_i)^2(\bar{y}_{i\cdot}-\mu)^2}{(n_i\tilde{r}+1)^2}$$

and

$$\frac{\partial^2 l}{(\partial \tilde{r})^2} = \frac{1}{2}\sum_{i=1}^{k}\frac{(n_i)^2}{(n_i\tilde{r} + 1)^2} - \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{(n_i)^3(\bar{y}_{i\cdot} - \mu)^2}{(n_i\tilde{r} + 1)^3}$$

Therefore,

$$-E\left(\frac{\partial^2 l}{(\partial \tilde{r})^2}\right) = -\frac{1}{2}\sum_{i=1}^{k}\frac{(n_i)^2}{(n_i\tilde{r} + 1)^2} + \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{(n_i)^3}{(n_i\tilde{r} + 1)^3} \times \frac{\sigma_e^2(n_i\tilde{r} + 1)}{n_i}$$

$$= \frac{1}{2}\sum_{i=1}^{k}\frac{(n_i)^2}{(n_i\tilde{r} + 1)^2}$$

Also,

$$\frac{\partial^2 l}{\partial \tilde{r}\partial \sigma_e^2} = -\frac{1}{2}\left(\frac{1}{\sigma_e^2}\right)^2\sum_{i=1}^{k}\frac{(n_i)^2(\bar{y}_{i\cdot} - \mu)^2}{(n_i\tilde{r} + 1)^2}$$

and

$$-E\left(\frac{\partial^2 l}{\partial \tilde{r}\partial \sigma_e^2}\right) = \frac{1}{2}\left(\frac{1}{\sigma_e^2}\right)^2\sum_{i=1}^{k}\frac{(n_i)^2}{(n_i\tilde{r} + 1)^2} \times \frac{\sigma_e^2(n_i\tilde{r} + 1)}{n_i}$$

$$= \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{n_i\tilde{r} + 1}$$

Differentiation with respect to μ gives

$$\frac{\partial l}{\partial \mu} = \frac{2}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i\,(\bar{y}_{i\cdot} - \mu)}{n_i\tilde{r} + 1}$$

and

$$\frac{\partial^2 l}{(\partial \mu)^2} = -\frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{n_i\tilde{r} + 1}$$

Therefore,

$$-E\left(\frac{\partial^2 l}{(\partial \mu)^2}\right) = \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{n_i\tilde{r} + 1}$$

Furthermore,

$$\frac{\partial^2 l}{\partial \mu \partial \sigma_e^2} = -\frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i\,(\bar{y}_{i\cdot} - \mu)}{n_i\tilde{r} + 1}$$

and

$$-E\left(\frac{\partial^2 l}{\partial \mu \partial \sigma_e^2}\right) = 0$$

Also,

$$\frac{\partial^2 l}{\partial \mu \partial \tilde{r}} = -\frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{(n_i)^2\,(\bar{y}_{i\cdot} - \mu)}{(n_i\tilde{r} + 1)^2}$$

and

$$-E\left(\frac{\partial^2 l}{\partial \mu \partial \tilde{r}}\right) = 0$$

Proof of Theorem 8.4

The inverse of the Fisher Information Matrix is given by,

$$F^{-1}(\mu,\tilde{r},\sigma_e^2) = \begin{bmatrix} \sigma_e^2\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^{-1} & 0 & 0 \\ 0 & \frac{n}{2|H|}\left(\frac{1}{\sigma_e^2}\right)^2 & -\frac{1}{2\sigma_e^2|H|}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \\ 0 & -\frac{1}{2\sigma_e^2|H|}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & \frac{1}{2|H|}\sum_{i=1}^{k}\frac{(n_i)^2}{(1+\tilde{r}n_i)^2} \end{bmatrix}$$

The determinant $|H|$ is equal to

$$|H| = \frac{n}{4(\sigma_e^2)^2}\sum_{i=1}^{k}\frac{(n_i)^2}{(1+\tilde{r}n_i)^2} - \frac{1}{4(\sigma_e^2)^2}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2$$

We are interested in the Probability-Matching Prior for $\tilde{r} = \frac{\sigma_\tau^2}{\sigma_e^2}$ (the variance ratio). Let $\boldsymbol{\theta}' = (\mu,\tilde{r},\sigma_e^2)$ and $t(\boldsymbol{\theta}) = \tilde{r}$, then

$$\frac{\partial t(\boldsymbol{\theta})}{\partial\mu} = 0$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial\tilde{r}} = 1$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial\sigma_e^2} = 0$$

$$\boldsymbol{\nabla}_{\mathrm{t}}'(\boldsymbol{\theta}) = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$$

and

$$\nabla_{\mathrm{t}}'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta}) = \begin{bmatrix}0 & \frac{n}{2|H|}\left(\frac{1}{\sigma_e^2}\right)^2 & -\frac{1}{2\sigma_e^2|H|}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\end{bmatrix}$$

Further,

$$\sqrt{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})} = \sqrt{\frac{n}{2|H|}\left(\frac{1}{\sigma_e^2}\right)^2}$$

and

$$\frac{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})}{\sqrt{\nabla_t'(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}} = \left[0 \quad \frac{\tilde{n}^{\frac{1}{2}}}{\sqrt{2}|H|^{\frac{1}{2}}\sigma_e^2} \quad -\frac{\tilde{n}^{-\frac{1}{2}}}{\sqrt{2}|H|^{\frac{1}{2}}}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right]$$

$$= [\eta_1(\boldsymbol{\theta}) \quad \eta_2(\boldsymbol{\theta}) \quad \eta_3(\boldsymbol{\theta})]$$

The prior $p(\boldsymbol{\theta}) = p(\mu, \tilde{r}, \sigma_e^2)$ is a Probability-Matching Prior if the following differential equation is satisfied:

$$\frac{\partial\{\eta_1(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\mu} + \frac{\partial\{\eta_2(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\tilde{r}} + \frac{\partial\{\eta_3(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_e^2} = 0$$

If we take $p(\boldsymbol{\theta}) = |H|^{\frac{1}{2}}$ then

$$\frac{\partial\{\eta_1(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\mu} = 0$$

$$\frac{\partial\{\eta_2(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\tilde{r}} = \frac{\partial}{\partial\tilde{r}}\left\{\frac{\tilde{n}^{\frac{1}{2}}}{\sqrt{2}\sigma_e^2}\right\} = 0$$

$$\frac{\partial\{\eta_3(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_e^2} = \frac{\partial}{\partial\sigma_e^2}\left\{-\frac{\tilde{n}^{-\frac{1}{2}}}{\sqrt{2}}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right\} = 0$$

The differential equation is therefore satisfied and the theorem has been proved. It is further clear that

$$p(\boldsymbol{\theta}) = p(\mu, \tilde{r}, \sigma_e^2) = |H|^{\frac{1}{2}}$$

$$\propto \frac{1}{\sigma_e^2}\left\{\sum_{i=1}^{k}\frac{(n_i)^2}{(1+\tilde{r}n_i)^2}-\frac{1}{n}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2\right\}^{\frac{1}{2}}$$

is a Probability-Matching Prior.

Proof of Theorem 8.5.1

Only the Reference prior for the group ordering $(\mu, \tilde{r}, \sigma_e^2)$ will be derived, since the Reference priors for the group orderings $(\tilde{r}, \mu, \sigma_e^2)$ and $(\tilde{r}, \sigma_e^2, \mu)$ can be derived in a similar way.

The Fisher Information Matrix for the ordering $(\mu, \tilde{r}, \sigma_e^2)$ is given by

$$F(\mu, \tilde{r}, \sigma_e^2) = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & 0 & 0 \\ 0 & \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \\ 0 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & \frac{n}{2}\left(\frac{1}{\sigma_e^2}\right)^2 \end{bmatrix}$$

The functions h_j $(j = 1,2,3)$ are calculated as follows:

$$h_1 = F_{11.2} = \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} - \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \\ \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore\ h_1 = \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}$$

Further

$$\begin{bmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{bmatrix}^{-1} = H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

and

$$H = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & 0 \\ 0 & \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} \end{bmatrix} - \left(\frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2\right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \end{bmatrix} \times$$

$$\begin{bmatrix} 0 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \end{bmatrix}$$

which means that

$$H_{22} = F_{22} - \frac{1}{F_{33}} F_{23} F_{32} = h_2$$

where

$$h_2 = \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} - \frac{1}{2n}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2$$

Further

$$h_3 = F_{33} = \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2$$

Now $p(\mu) \propto h_1^{\frac{1}{2}} = 1$ (because it does not contain μ).

$$p(\tilde{r}|\mu) \propto h_2^{\frac{1}{2}} = \left\{ \sum_{i=1}^{k} \frac{n_i^2}{(1+\tilde{r}n_i)^2} - \frac{1}{n}\left(\sum_{i=1}^{k} \frac{n_i}{1+\tilde{r}n_i} \right)^2 \right\}^{\frac{1}{2}}$$

and

$$p(\sigma_e^2|\tilde{r},\mu) \propto h_3^{\frac{1}{2}} = \frac{1}{\sigma_e^2}$$

Thus, the Reference prior for the group ordering $(\mu, \tilde{r}, \sigma_e^2)$ is given by

$$p_{R_1}(\mu,\tilde{r},\sigma_e^2) = p(\mu)p(\tilde{r}|\mu)p(\sigma_e^2|\tilde{r},\mu) = \frac{1}{\sigma_e^2}\left\{ \sum_{i=1}^{k} \frac{n_i^2}{(1+\tilde{r}n_i)^2} - \frac{1}{n}\left(\sum_{i=1}^{k} \frac{n_i}{1+\tilde{r}n_i} \right)^2 \right\}^{\frac{1}{2}}$$

and also satisfies the Probability-Matching criterion.

Proof of Theorem 8.5.2

Only the Reference prior for the group ordering $(\mu, \sigma_e^2, \tilde{r})$ will be derived, since the Reference priors for the group orderings $(\sigma_e^2, \mu, \tilde{r})$ and $(\sigma_e^2, \tilde{r}, \mu)$ can be derived in a similar way.

The Fisher Information Matrix for the ordering $(\mu, \sigma_e^2, \tilde{r})$ is given by

$$F(\mu,\sigma_e^2,\tilde{r}) = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & 0 & 0 \\ 0 & \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \\ 0 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2} \end{bmatrix}$$

The functions h_j $(j = 1,2,3)$ are calculated as before:

$$h_1 = F_{11.2} = \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} - [0 \quad 0]\begin{bmatrix} \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} \\ \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\end{bmatrix}^{-1}\begin{bmatrix}0\\0\end{bmatrix}$$

$$\therefore\ h_1 = \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}$$

Further

$$\begin{bmatrix} F^{11} & F^{12} \\ F^{21} & F^{22}\end{bmatrix}^{-1} = H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix}$$

and

$$H = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i} & 0 \\ 0 & \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2\end{bmatrix} - \left(\frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right)^{-1}\begin{bmatrix} 0 \\ \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\end{bmatrix} \times$$

$$\begin{bmatrix} 0 & \frac{1}{2\sigma_e^2}\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\end{bmatrix}$$

Therefore

$$h_2 = H_{22} = \frac{\tilde{n}}{2}\left(\frac{1}{\sigma_e^2}\right)^2 - \frac{1}{2}\left(\frac{1}{\sigma_e^2}\right)^2\left\{\left(\frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right)^{-1}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2\right\}$$

$$= \left(\frac{1}{\sigma_e^2}\right)^2\left[\frac{\tilde{n}}{2} - \frac{1}{2}\left\{\left(\frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right)^{-1}\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^2\right\}\right]$$

and

$$h_3 = F_{33} = \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}$$

Now $p(\mu) \propto h_1^{\frac{1}{2}} = 1$ (because it does not contain μ).

$$p(\sigma_e^2|\mu) \propto h_2^{\frac{1}{2}} = \frac{1}{\sigma_e^2}$$

and

$$p(\tilde{r}|\sigma_e^2,\mu) \propto h_3^{\frac{1}{2}} = \left(\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right)^{\frac{1}{2}}$$

Thus, the Reference prior for the group ordering $(\mu, \sigma_e^2, \tilde{r})$ is given by

$$p_{R_2}(\mu,\sigma_e^2,\tilde{r}) = p(\mu)p(\sigma_e^2|\mu)p(\tilde{r}|\sigma_e^2,\mu) = \frac{1}{\sigma_e^2}\left(\sum_{i=1}^{k}\frac{n_i^2}{(1+\tilde{r}n_i)^2}\right)^{\frac{1}{2}}$$

Proof of Theorem 8.6

From Theorem 1 it follows that:

$$p(\mu,\boldsymbol{\tau},\sigma_e^2,\tilde{r}|\boldsymbol{Y})$$

$$\propto \left(\frac{1}{\sigma_e^2}\right)^{\frac{1}{2}\tilde{n}} (\tilde{r}\,\sigma_e^2)^{\frac{1}{2}k}$$

$$\times\, exp\left\{-\frac{1}{2\sigma_e^2}(\boldsymbol{Y}-\mu\boldsymbol{1}-Z\boldsymbol{\tau})'(\boldsymbol{Y}-\mu\boldsymbol{1}-Z\boldsymbol{\tau})\right\} exp\left\{-\frac{1}{2\sigma_\tau^2}\boldsymbol{\tau}'\boldsymbol{\tau}\right\}$$

$$\times\, p(\mu,\sigma_e^2,\tilde{r})$$

To obtain the posterior distribution of $\boldsymbol{\tau}$ the square with respect to $\boldsymbol{\tau}$ in the exponent must be completed. From Theorem 8.1 we know that

$$S = \frac{1}{\sigma_e^2}\widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}} + (\boldsymbol{\tau}-D^{-1}\boldsymbol{C})'D(\boldsymbol{\tau}-D^{-1}\boldsymbol{C}) - \boldsymbol{C}'D^{-1}\boldsymbol{C}$$

Therefore,

$$\boldsymbol{\tau} \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y} \sim N\{D^{-1}\boldsymbol{C},\ D^{-1}\}$$

where D^{-1} and $\boldsymbol{C}$ are defined as in Theorem 8.1. From equation (8.9) we know that

$$\mu|\boldsymbol{Y}, \tilde{r}, \sigma_e^2 \sim N\left(\hat{\mu}\,,\ \sigma_e^2\left(\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}\right)^{-1}\right)$$

and

$$\hat{\mu} = \frac{\sum_{i=1}^{k}\bar{y}_{i\cdot}\frac{n_i}{1+\tilde{r}n_i}}{\sum_{i=1}^{k}\frac{n_i}{1+\tilde{r}n_i}} = \bar{\boldsymbol{Y}}_{..}$$

Since $\boldsymbol{\tau} = \ (\tau_1, \tau_2, \dots, \tau_k)$ it follows from the definitions of D^{-1} and $D^{-1}\boldsymbol{C}$ given in Theorem 1 that

$$E(\tau_i \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y}) = \tilde{r}\ (\bar{y}_{i\cdot} - \ \mu)\frac{n_i}{1+\tilde{r}n_i}$$

and

$$Var(\tau_i \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y}) = \ \frac{\tilde{r}\sigma_e^2}{1+\tilde{r}n_i}$$

It follows from this that

$$E(\mu + \tau_i \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y}) = \ \tilde{r}\ (\bar{y}_{i\cdot} - \ \mu)\frac{n_i}{1+\tilde{r}n_i} + \ \mu$$

$$= \frac{\tilde{r}n_i}{1+\tilde{r}n_i}\bar{y}_{i\cdot} + \frac{1}{1+\tilde{r}n_i}\mu$$

and

$$Var(\mu + \tau_i \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y}) = \ \frac{\tilde{r}\sigma_e^2}{1+\tilde{r}n_i}$$

Therefore, unconditionally on μ it follows that

$$E(\mu + \tau_i \mid \sigma_e^2, \tilde{r}, \boldsymbol{Y}) = \frac{\tilde{r}n_i}{1 + \tilde{r}n_i}\bar{y}_{i\cdot} + \frac{1}{1 + \tilde{r}n_i}\hat{\mu}$$

and

$$Var(\tau_i \mid \sigma_e^2, \tilde{r}, \boldsymbol{Y}) = E_\mu(\mu + \tau_i \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y}) + Var_\mu(\mu + \tau_i \mid \mu, \sigma_e^2, \tilde{r}, \boldsymbol{Y})$$

$$= \frac{1}{1 + \tilde{r}n_i}\left\{\tilde{r}\sigma_e^2 + \frac{1}{1 + \tilde{r}n_i}\sigma_e^2\left(\sum_{i=1}^{k}\frac{n_i}{1 + \tilde{r}n_i}\right)^{-1}\right\}$$

Please refer to Table 104 for the accompanying results to these figures.

Figure 79: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 1

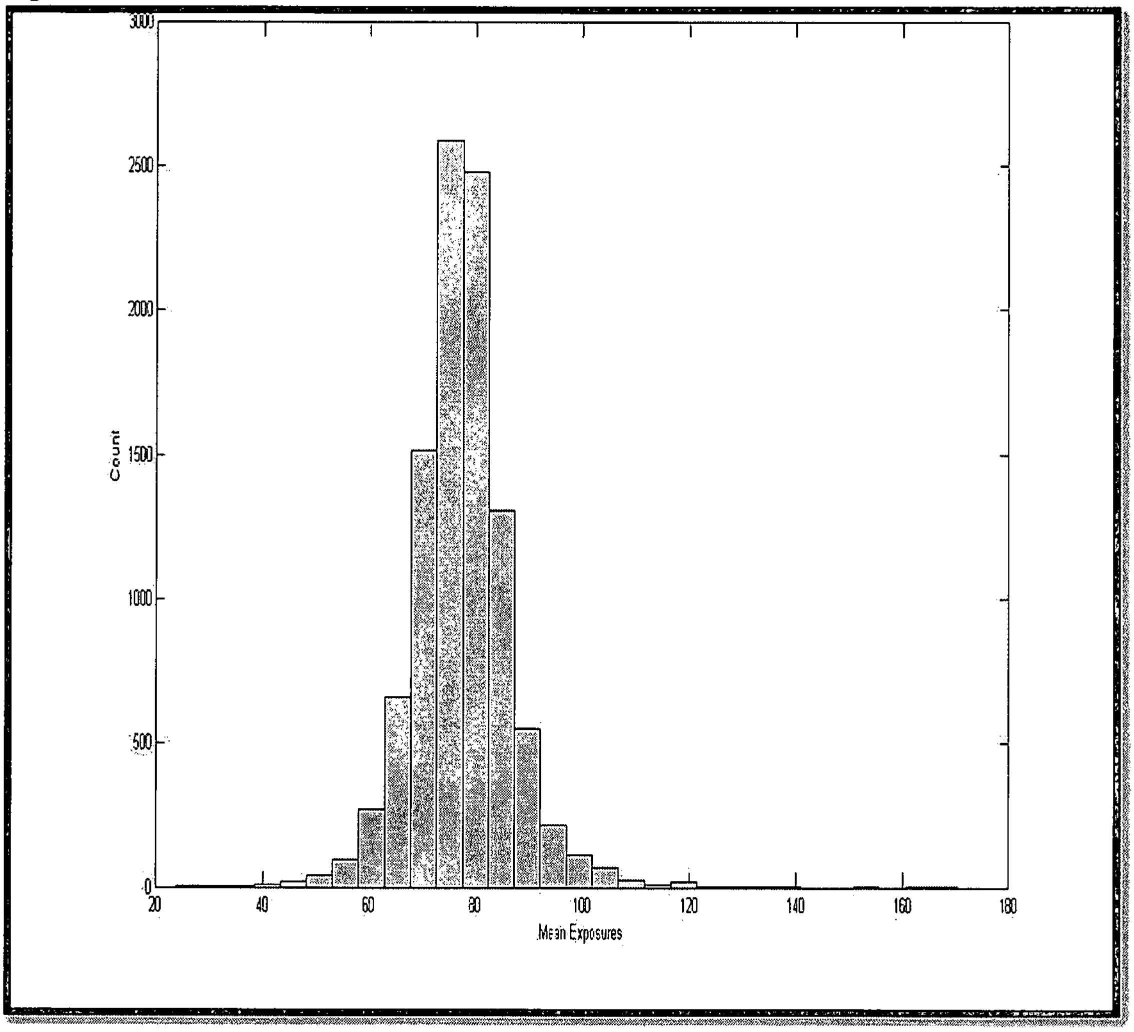


Figure 80: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 2

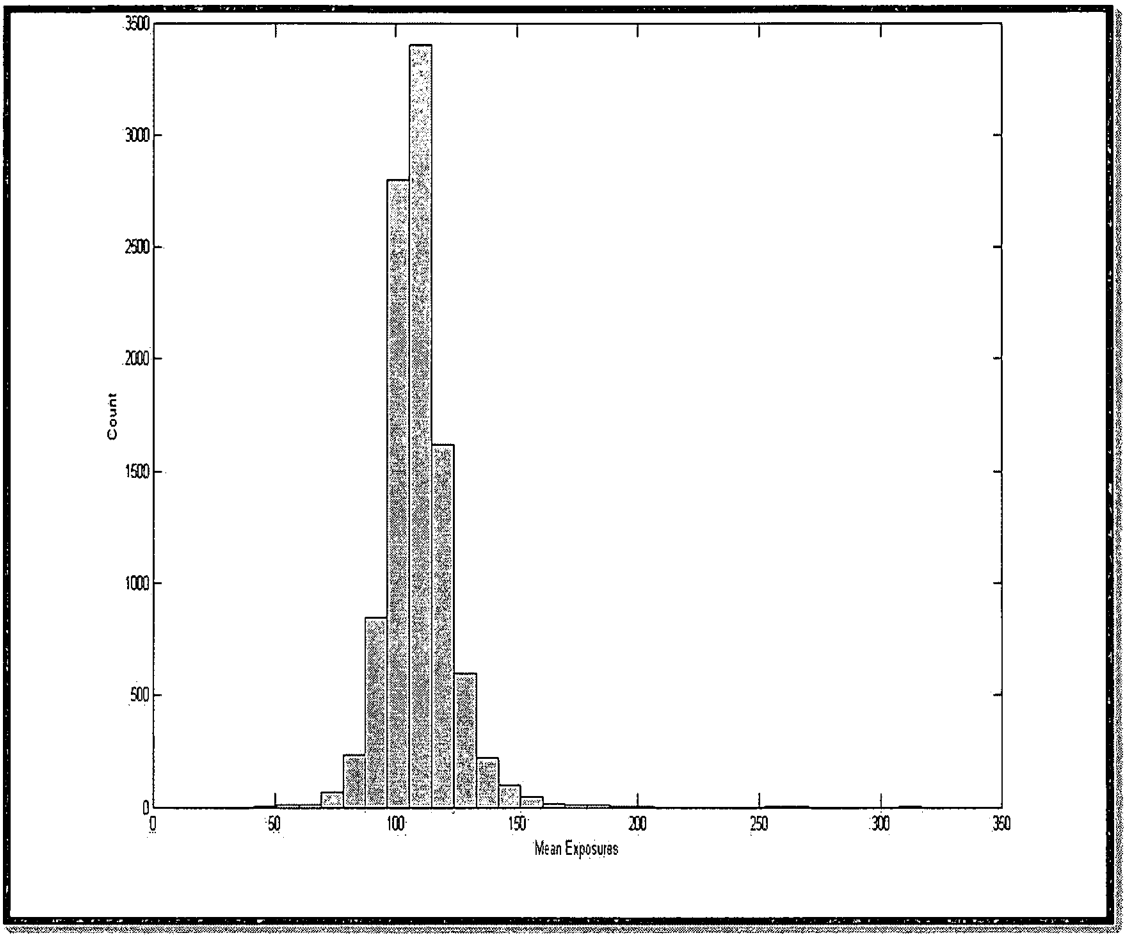


Figure 81: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 3

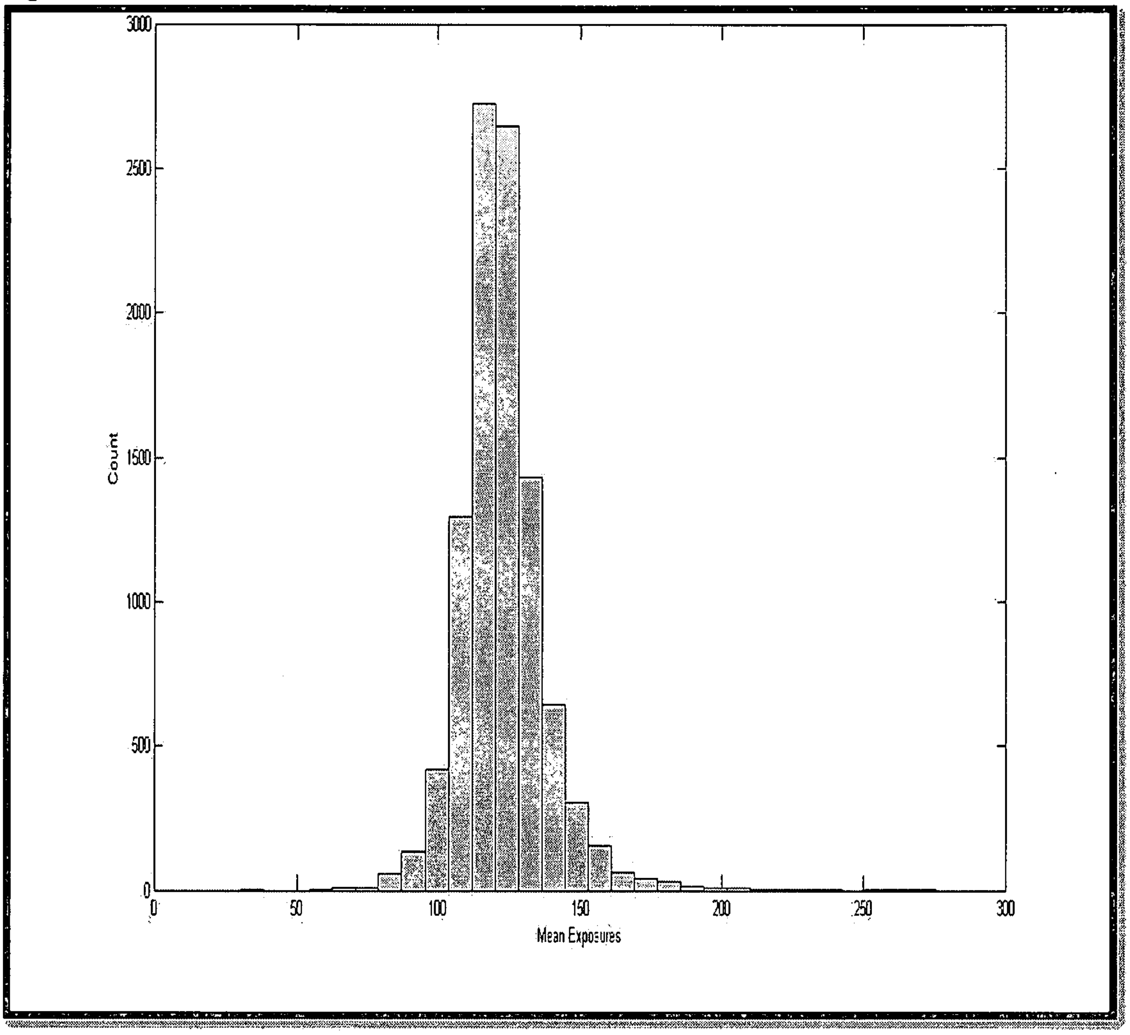


Figure 82: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 4

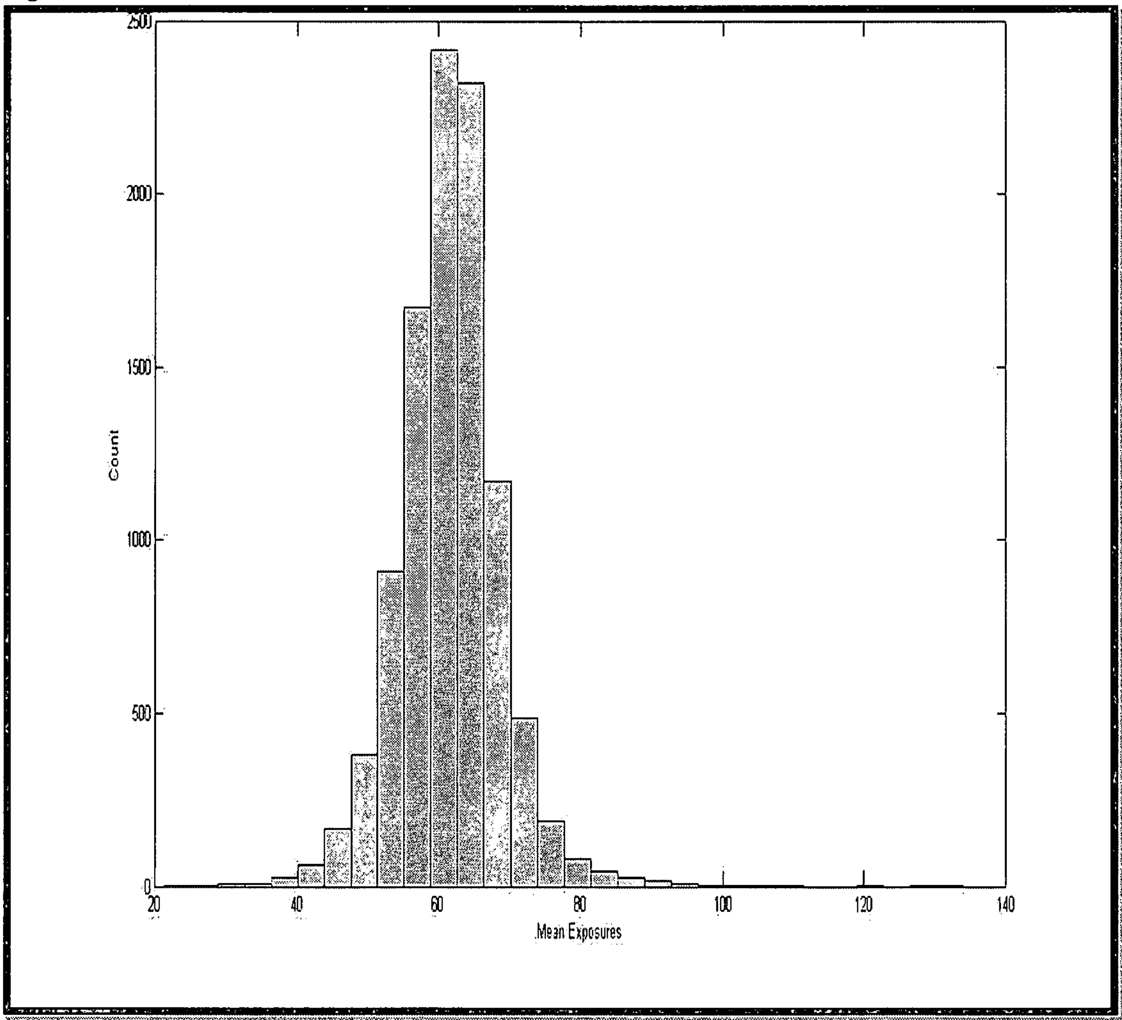


Figure 83: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 5

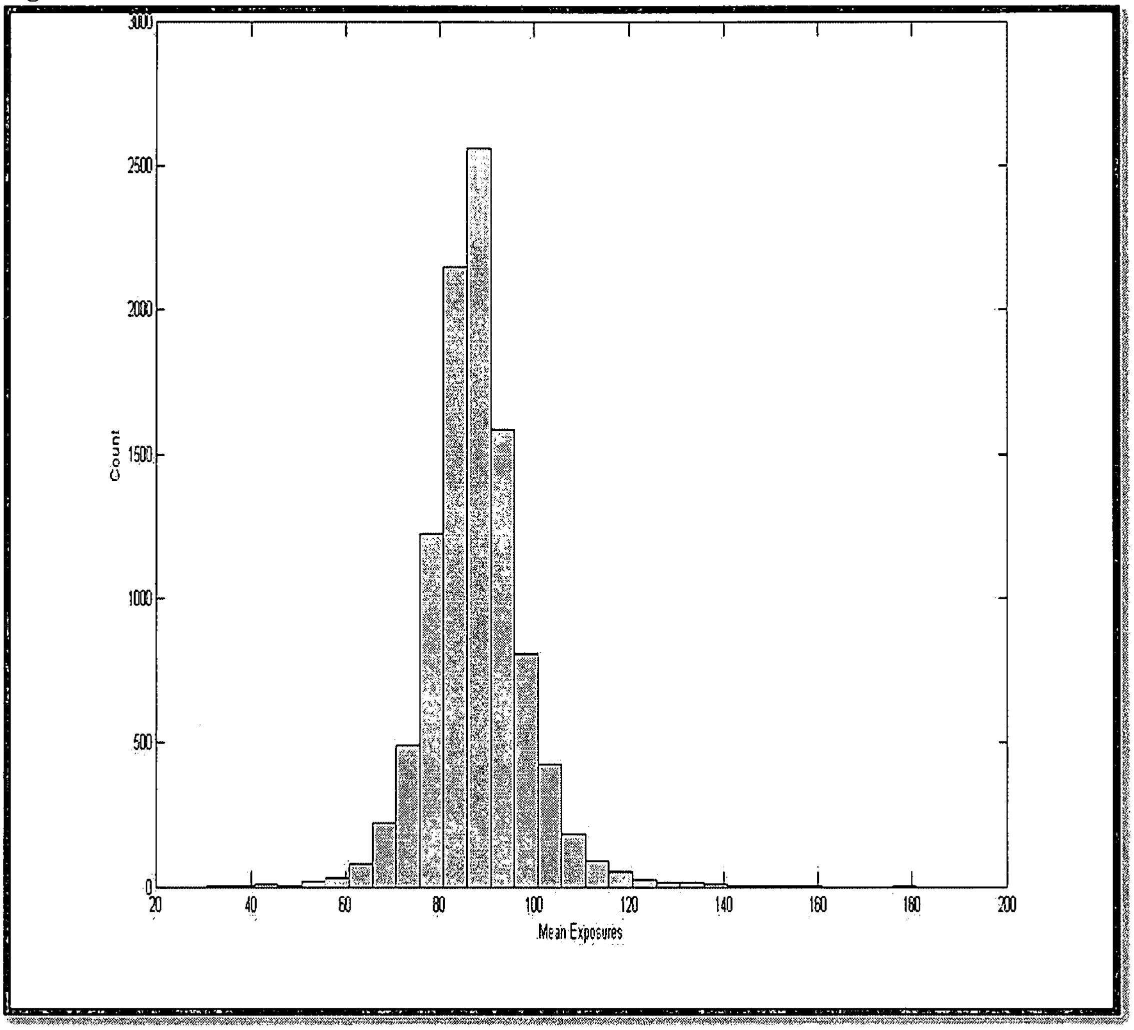


Figure 84: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 6

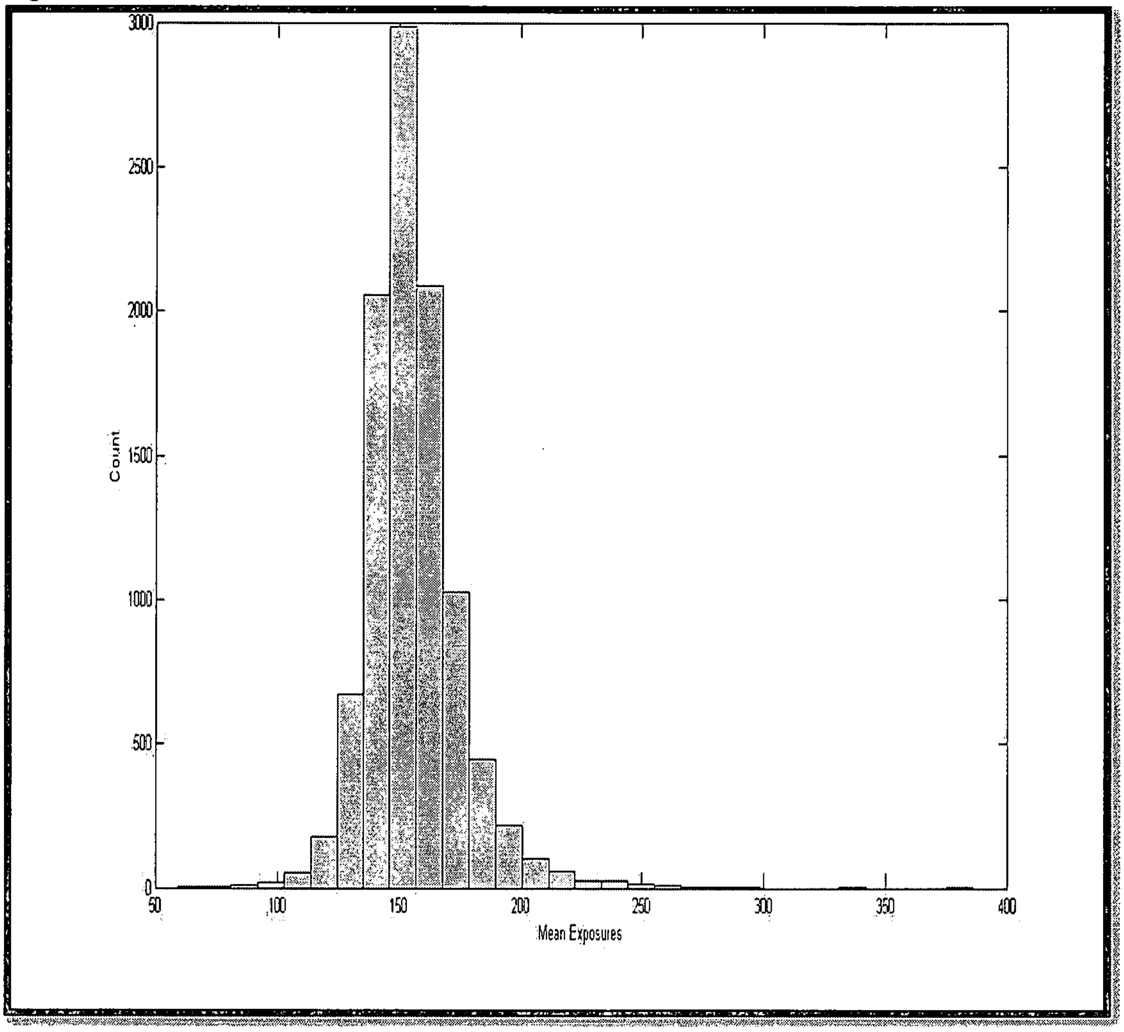


Figure 85: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 7

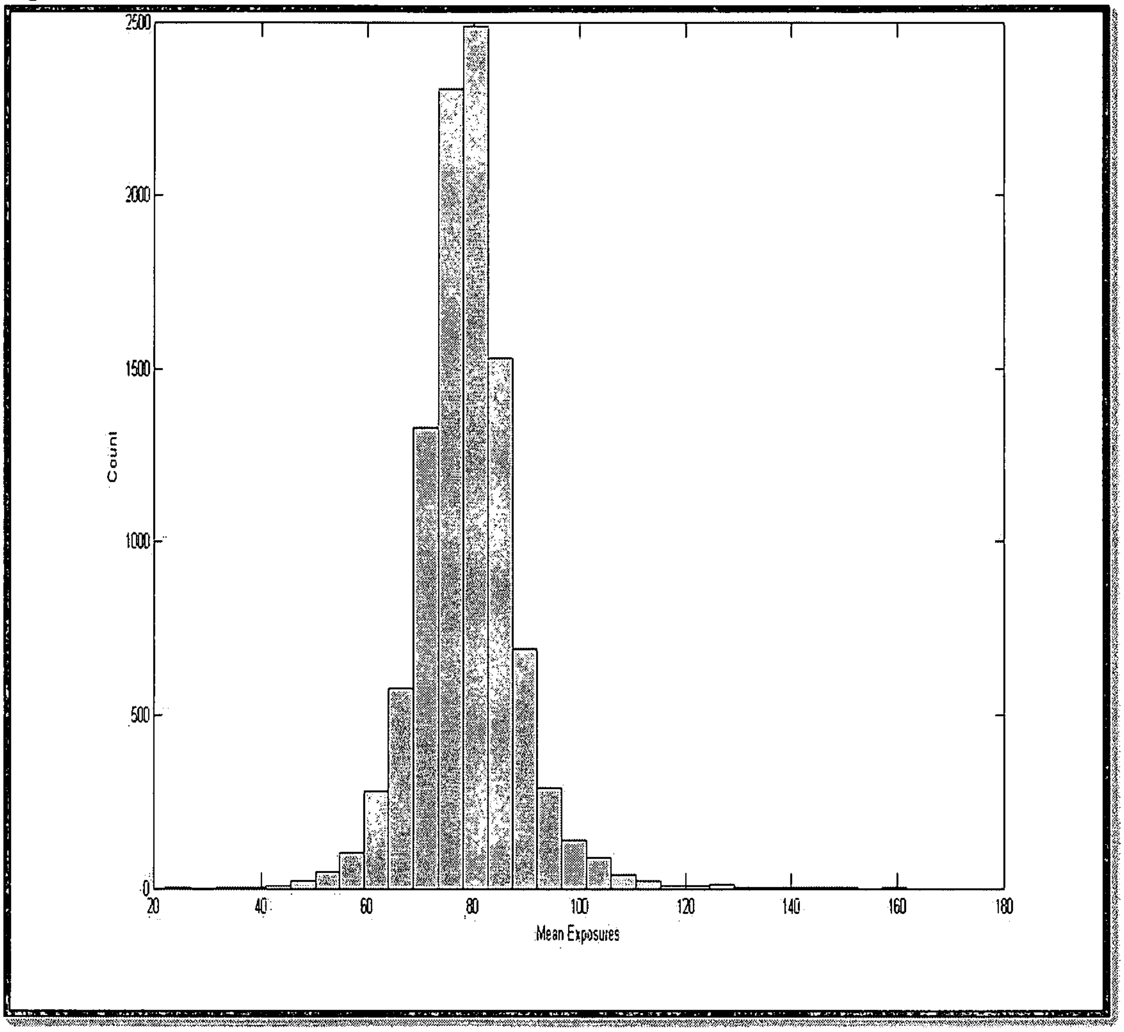


Figure 86:Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 8

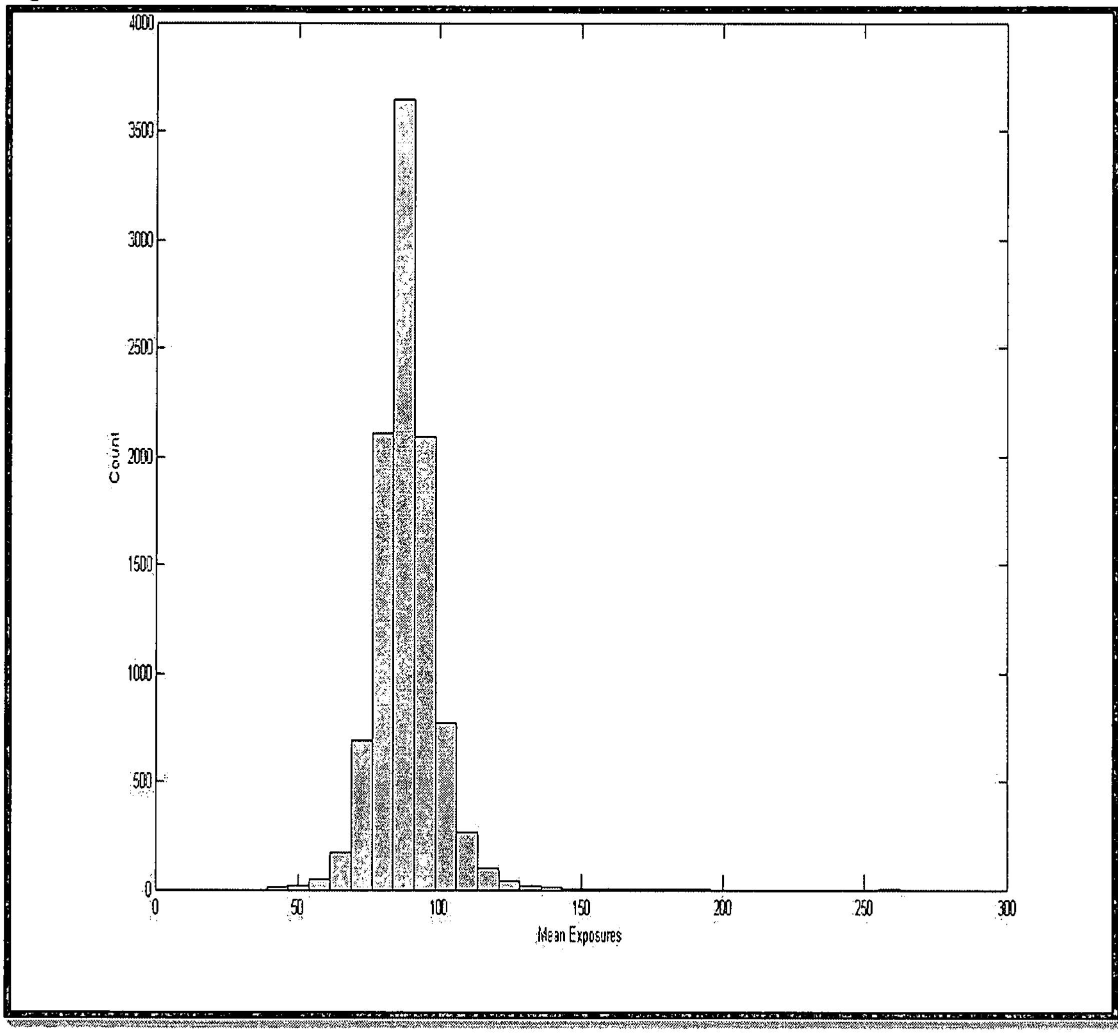


Figure 87:Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 9

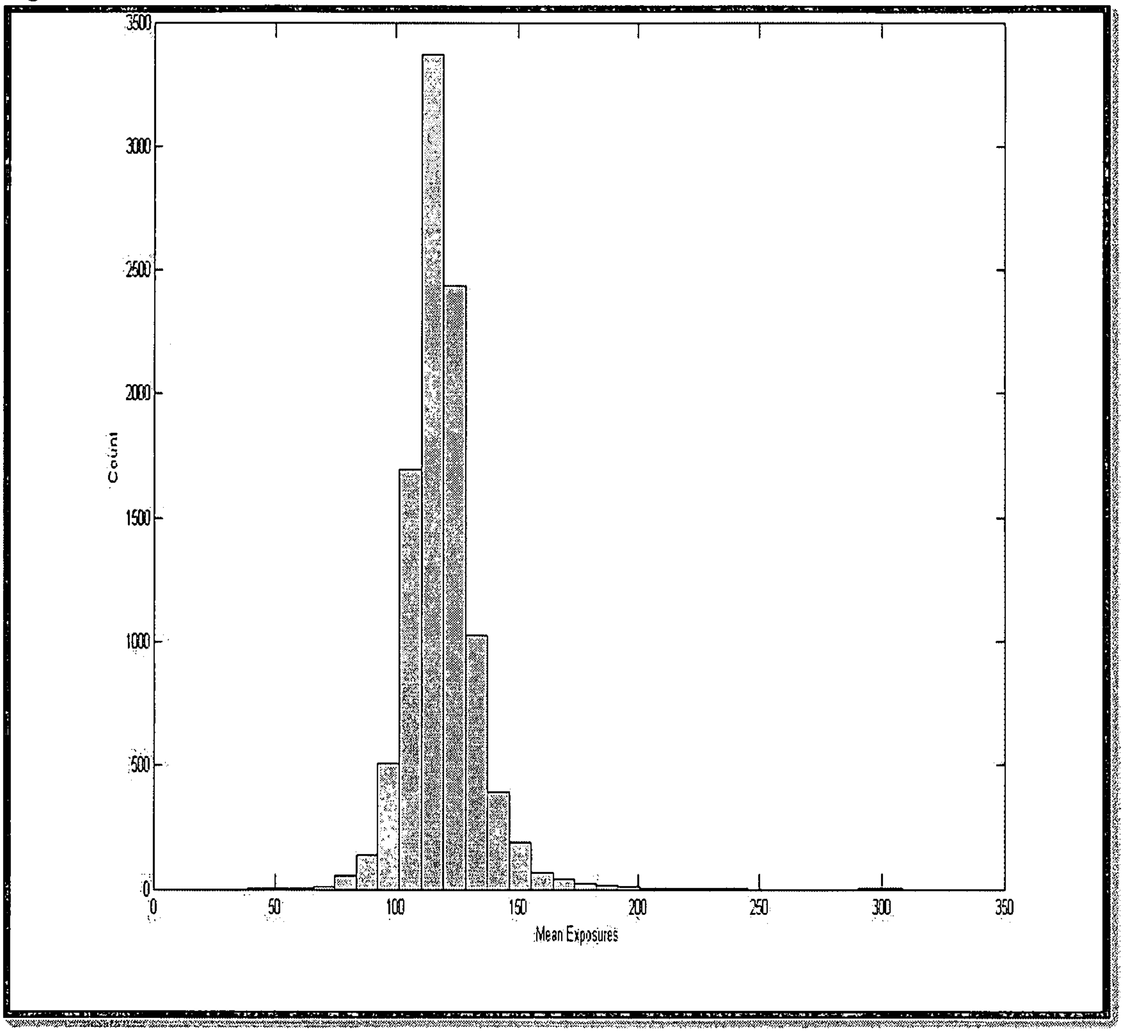


Figure 88: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 10

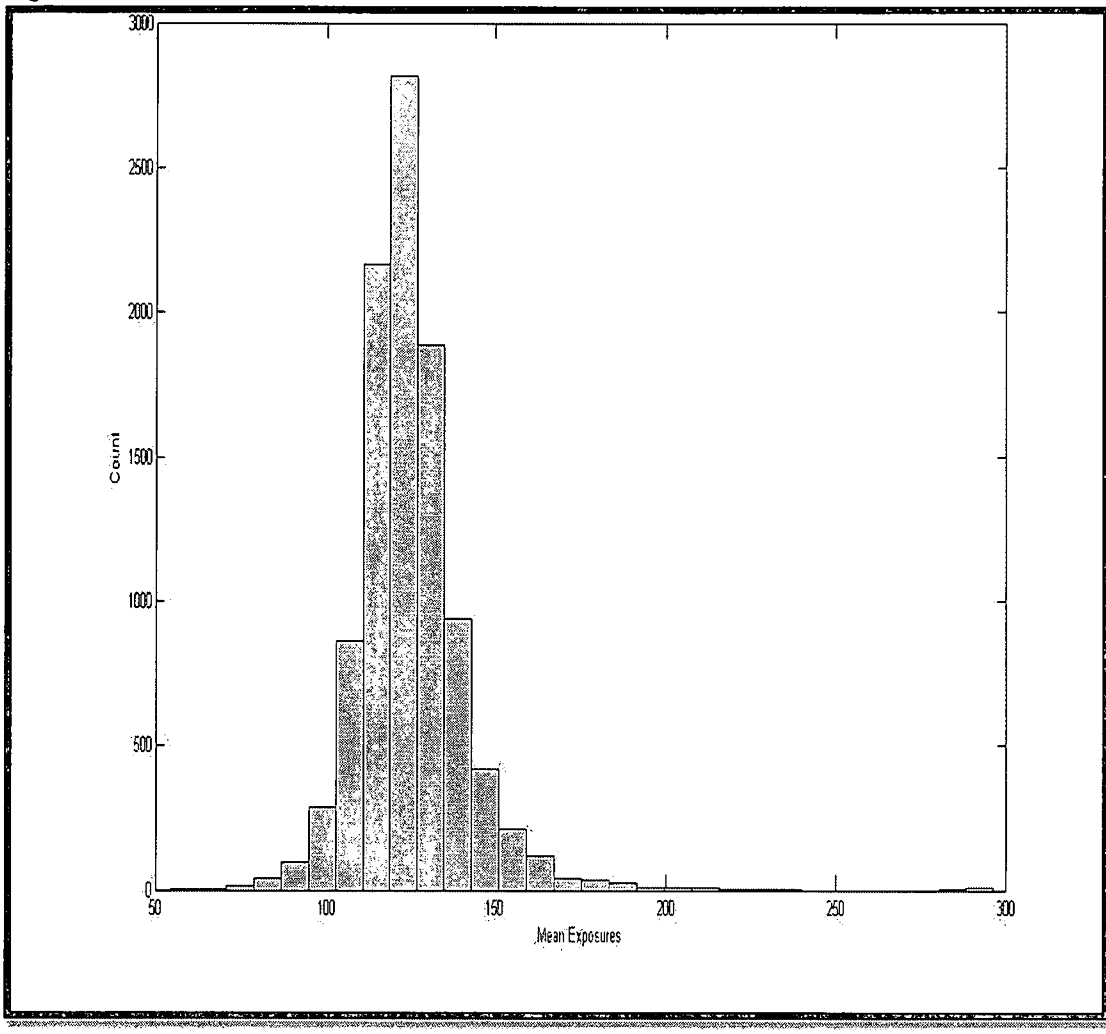


Figure 89: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 11

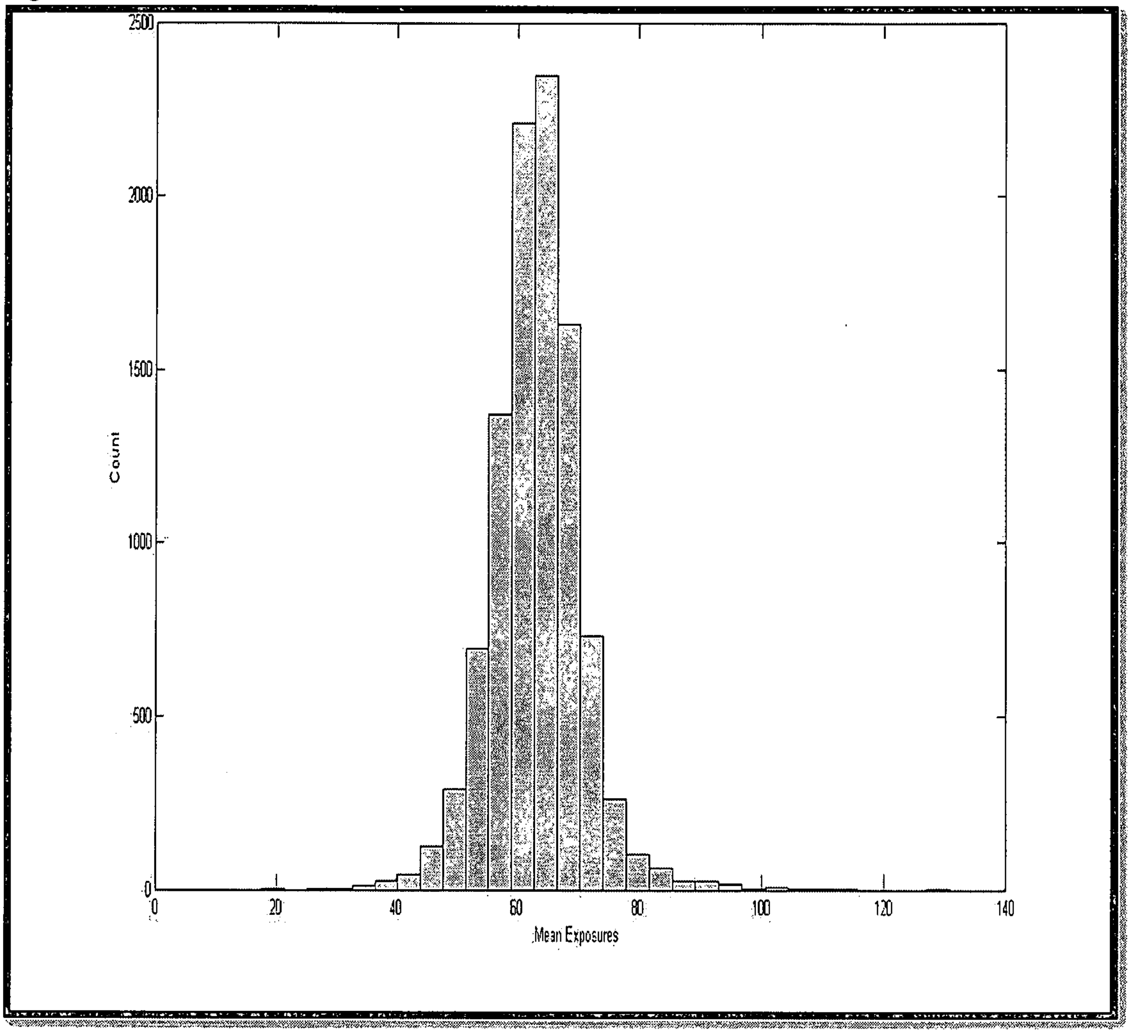


Figure 90: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 12

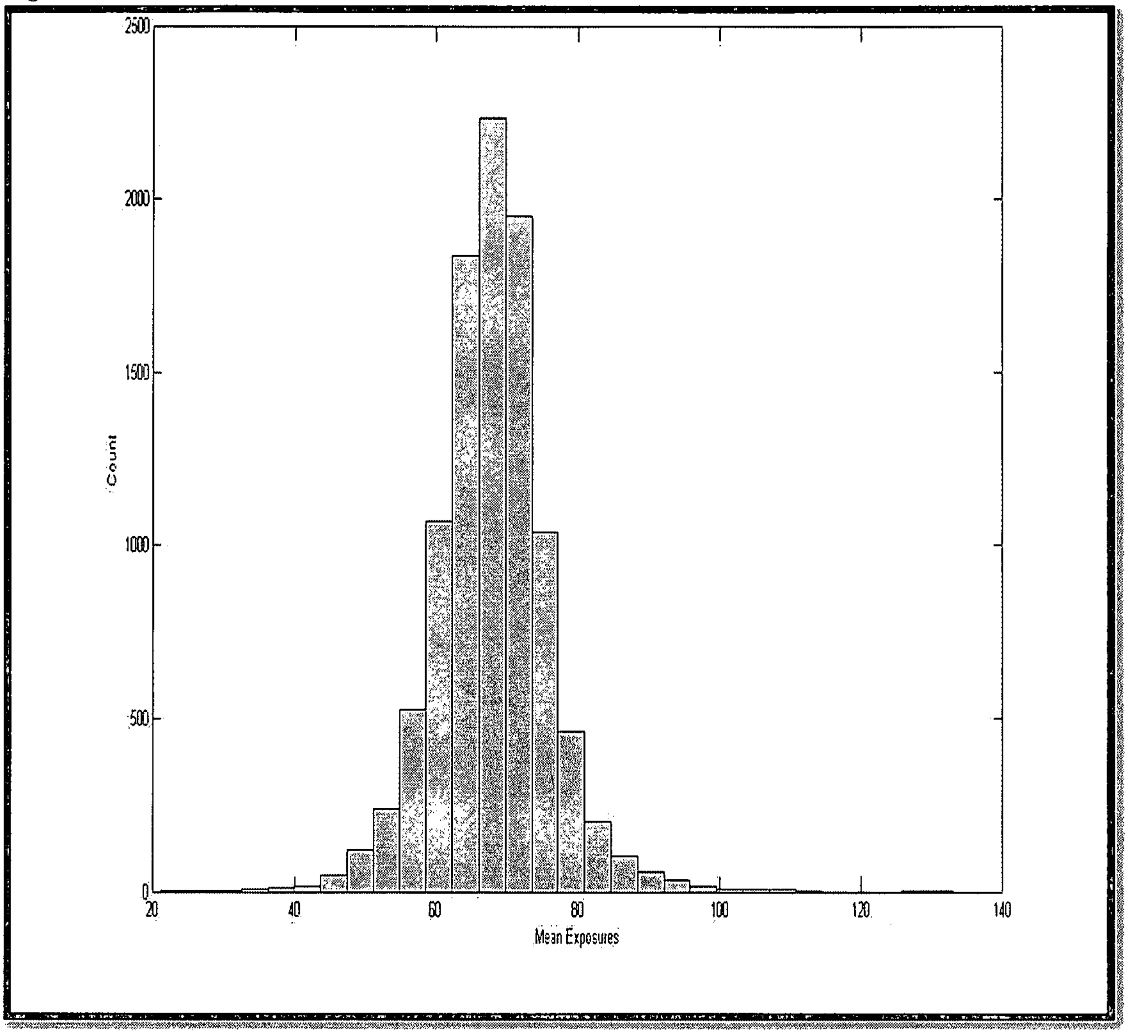


Figure 91: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 13

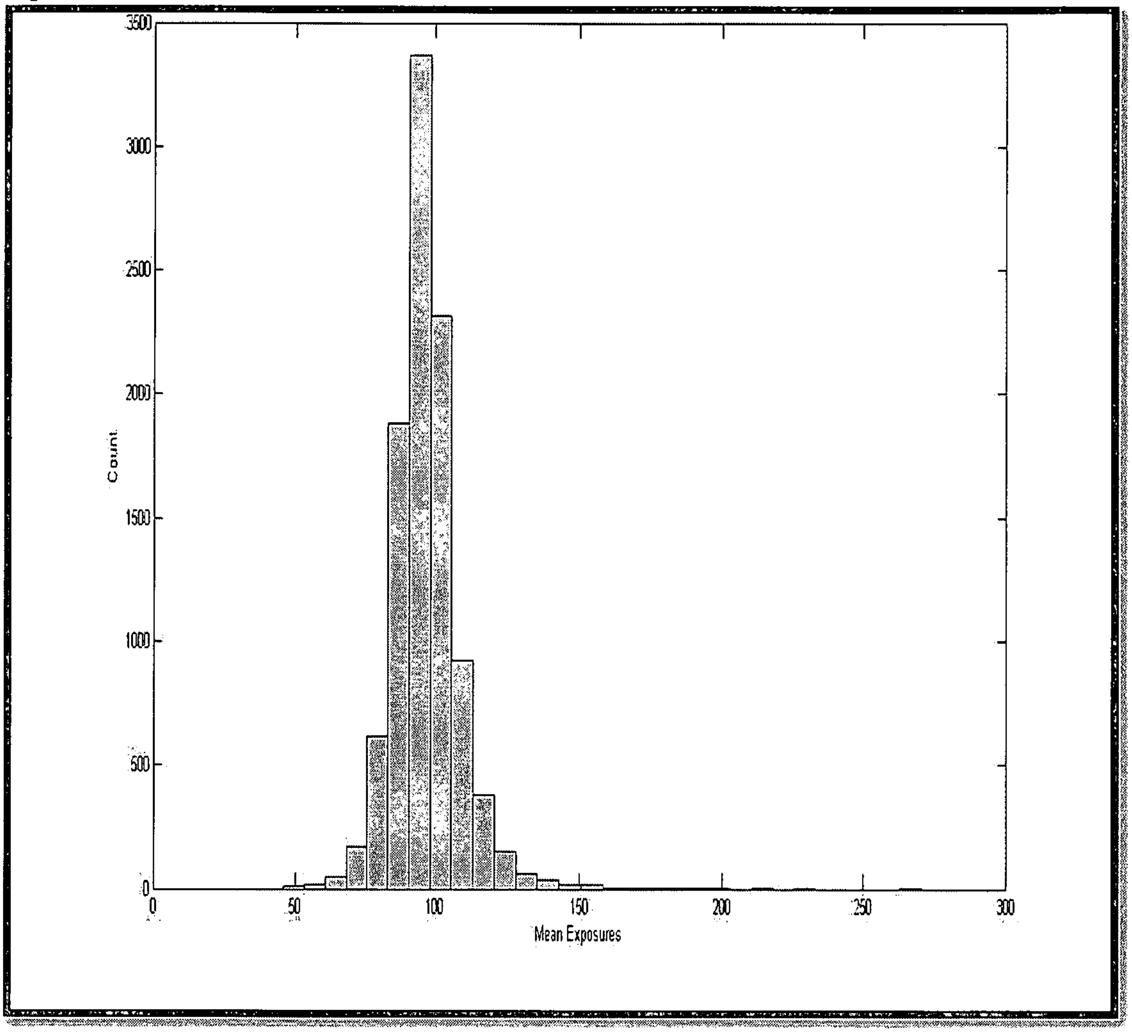


Figure 92: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 14

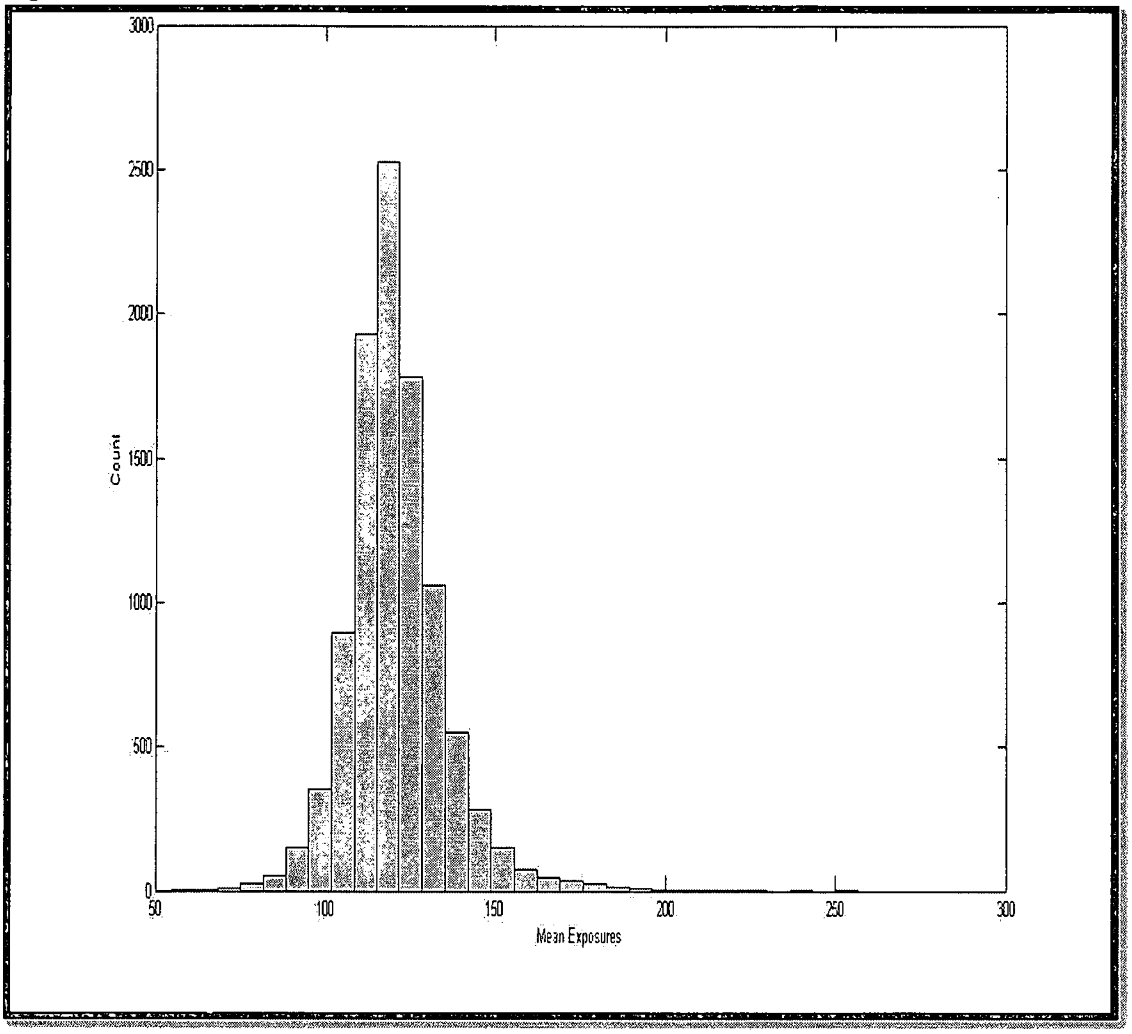
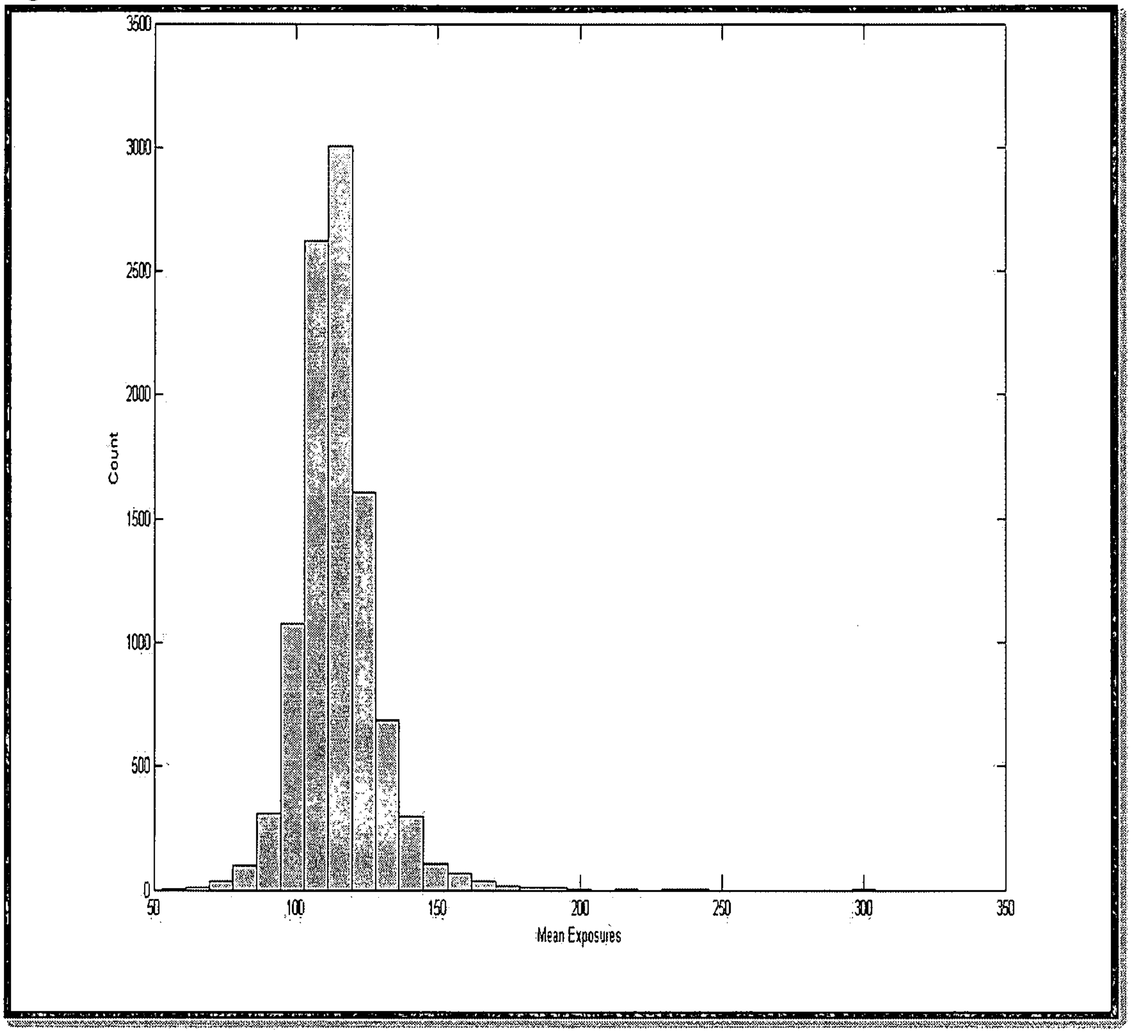


Figure 93: Simulation Results for Reference Prior 1 for Individual Worker Means - Worker 15



Please refer to Table 102 for the accompanying results to these figures.

Figure 94: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 1

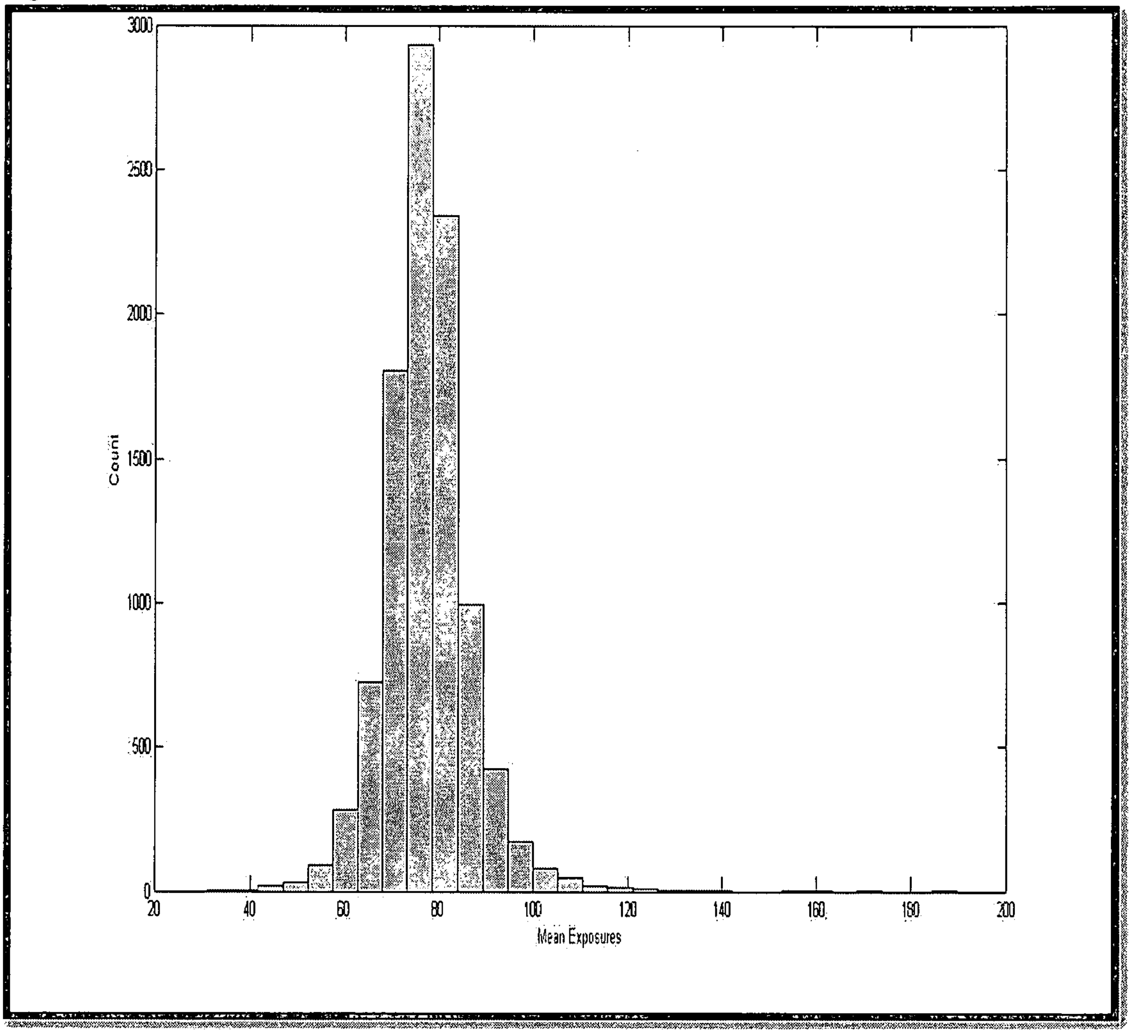


Figure 95: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 2

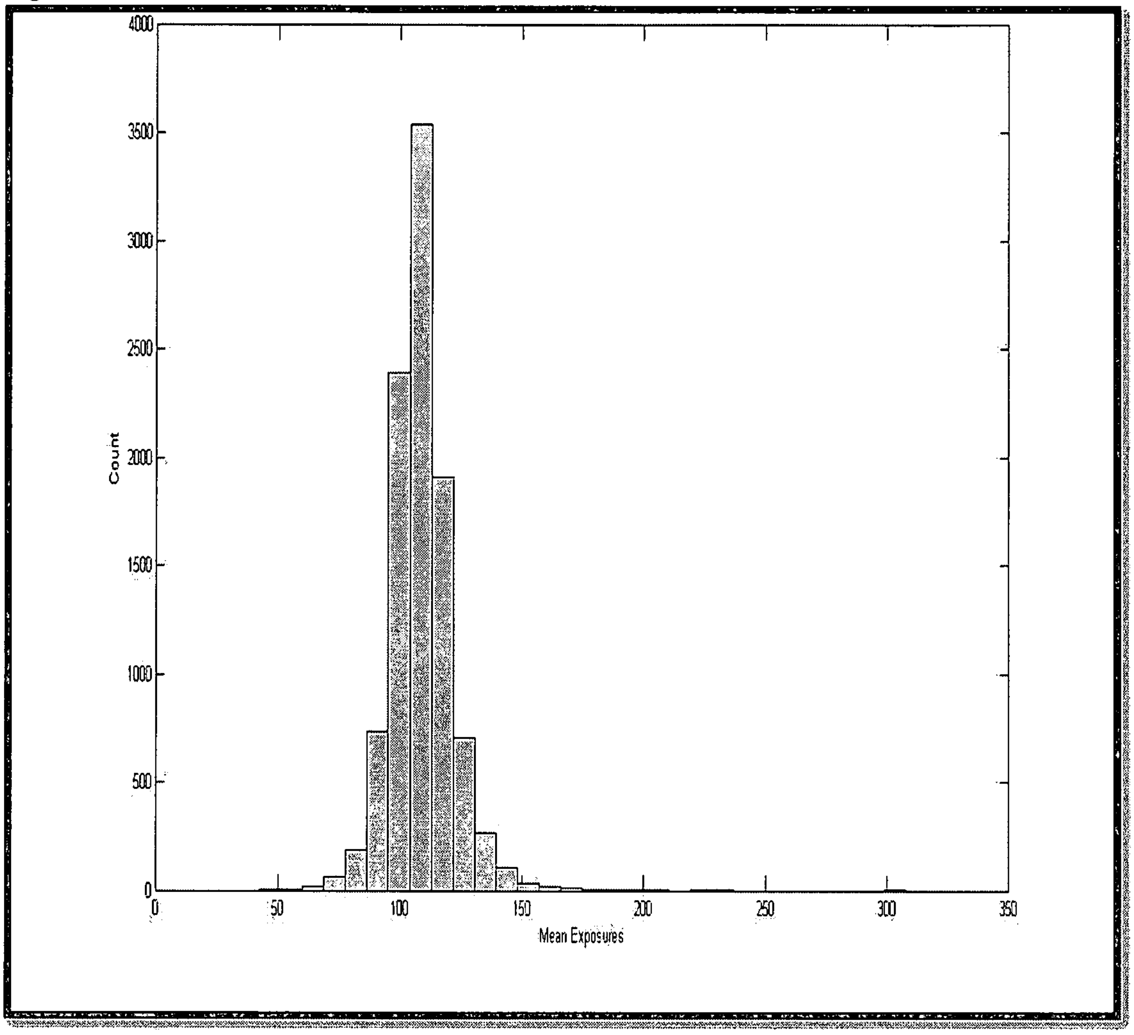


Figure 96:Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 3

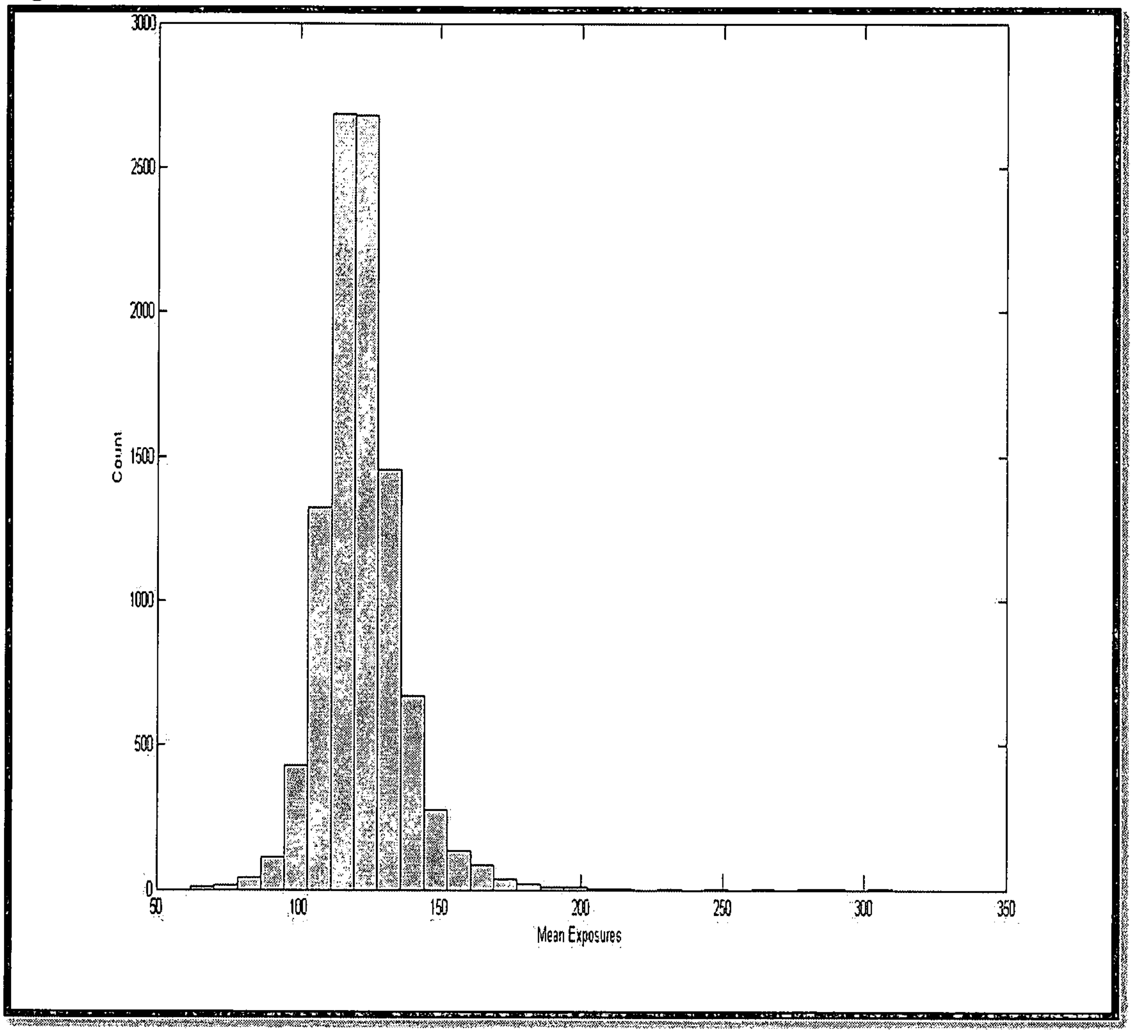


Figure 97:Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 4

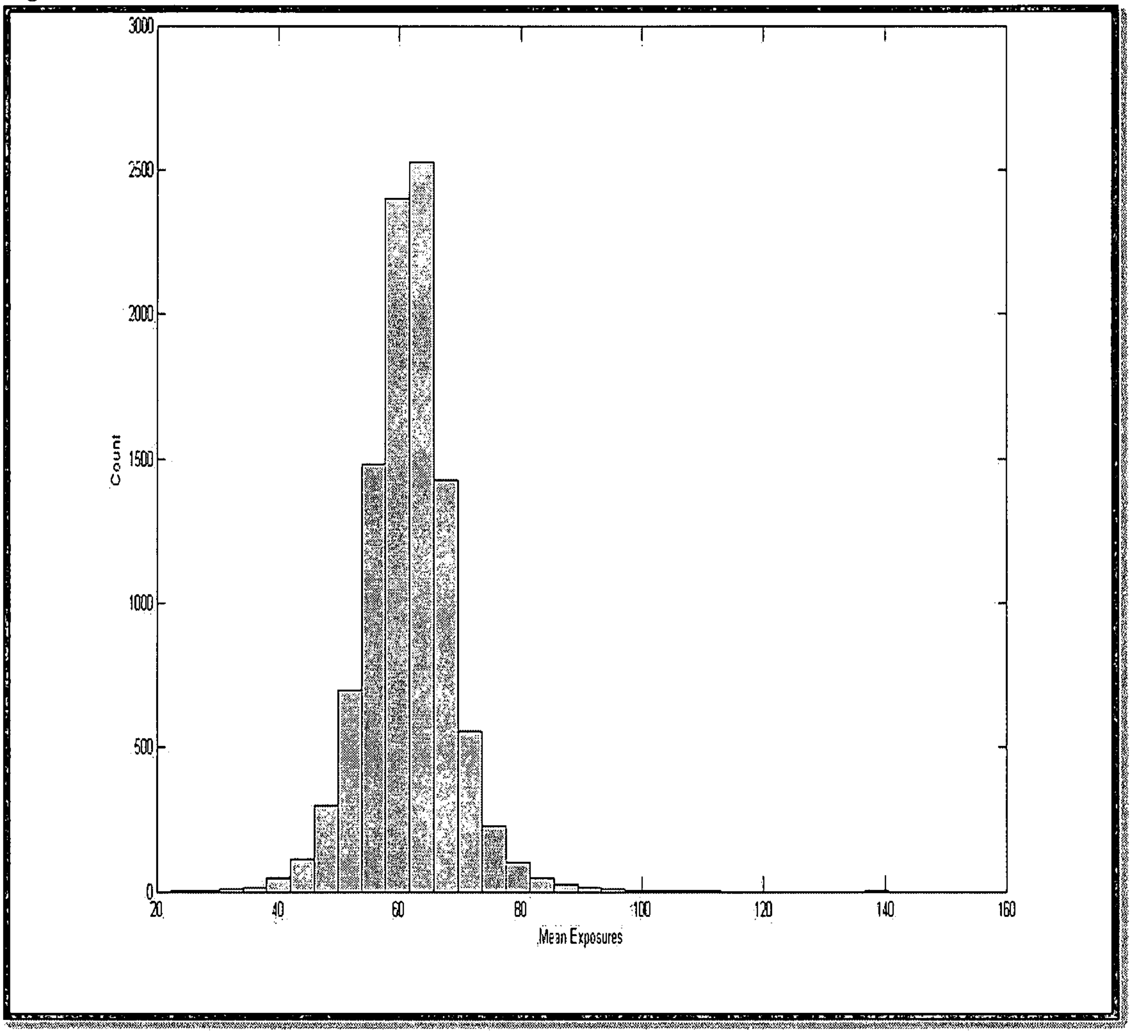


Figure 98: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 5

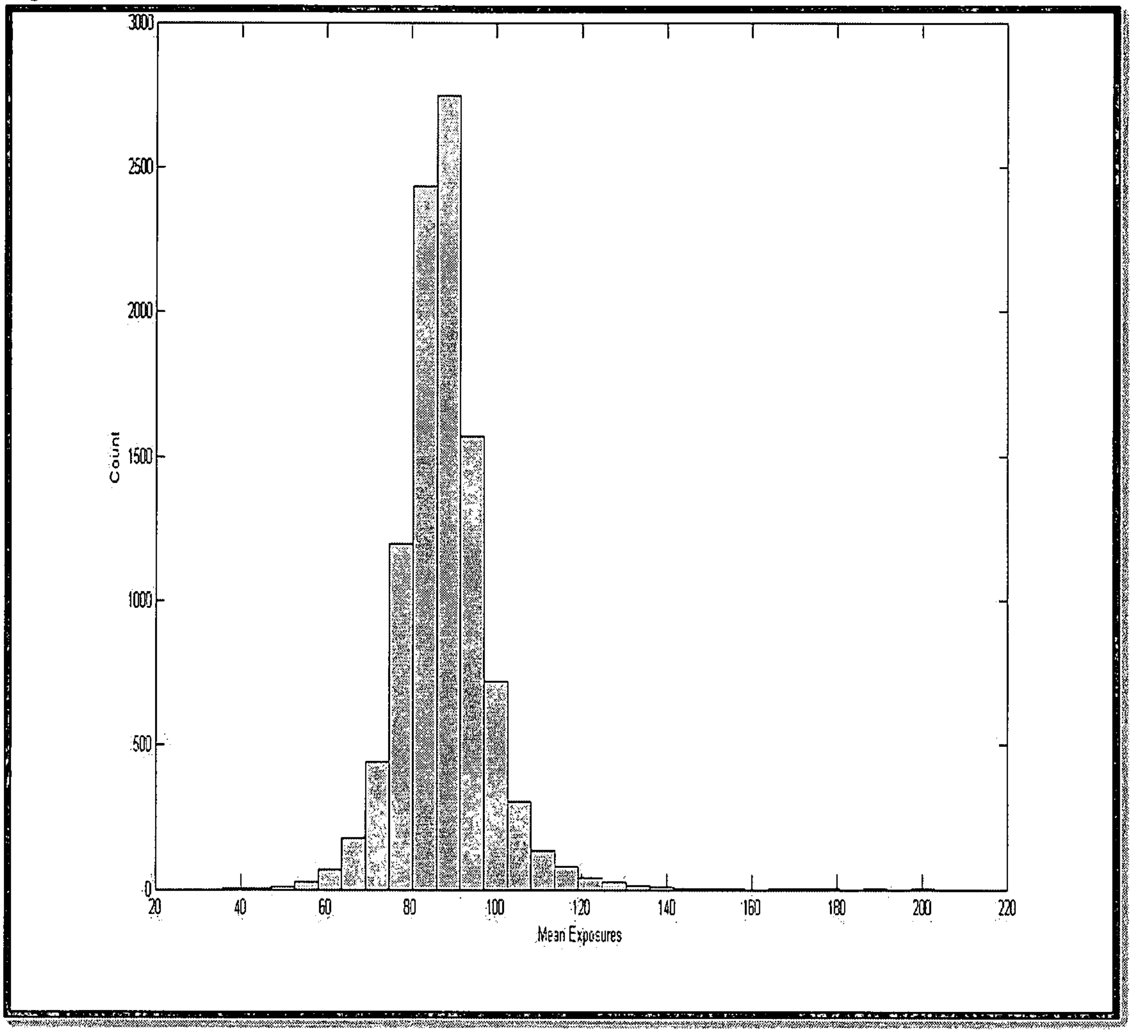


Figure 99: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 6

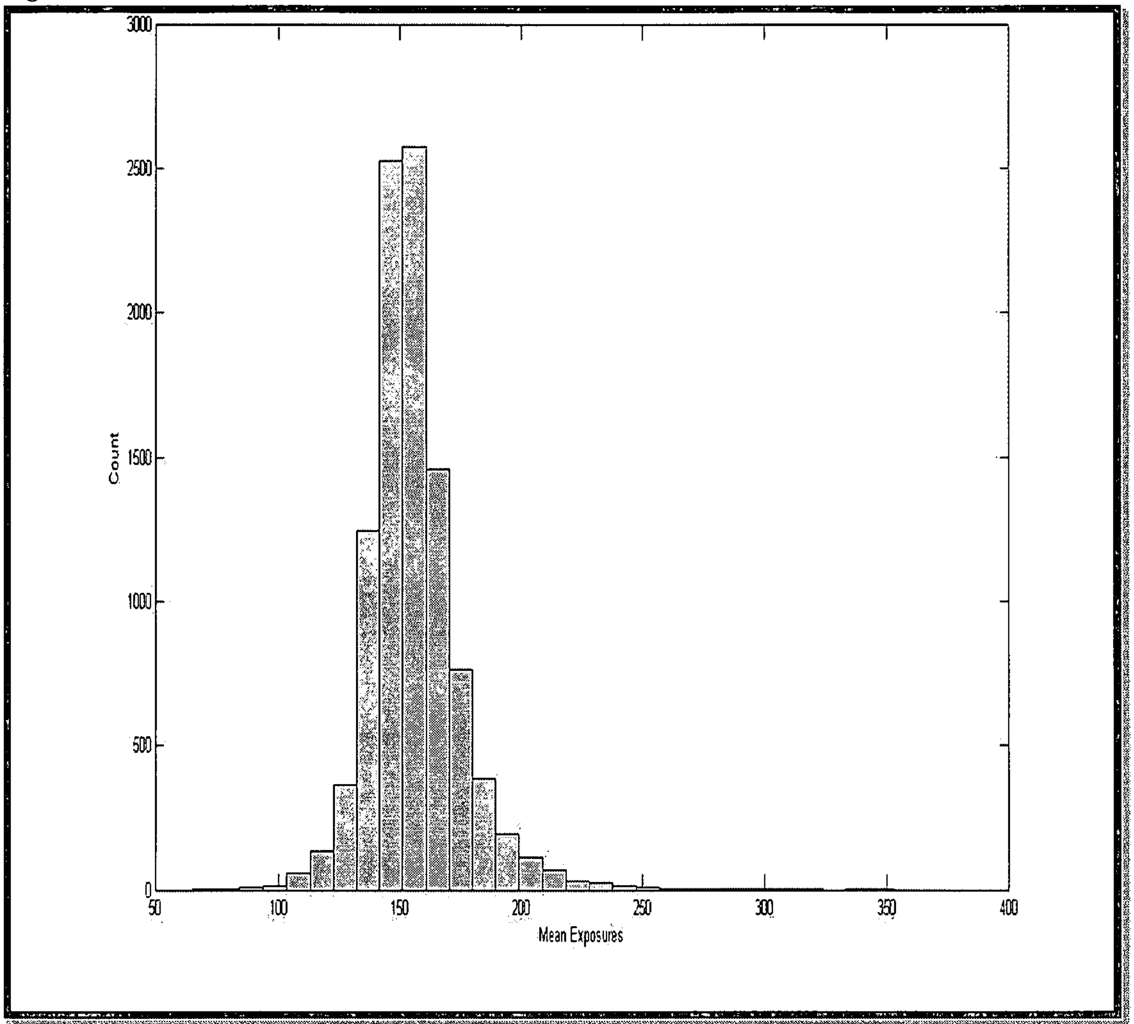


Figure 100: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 7

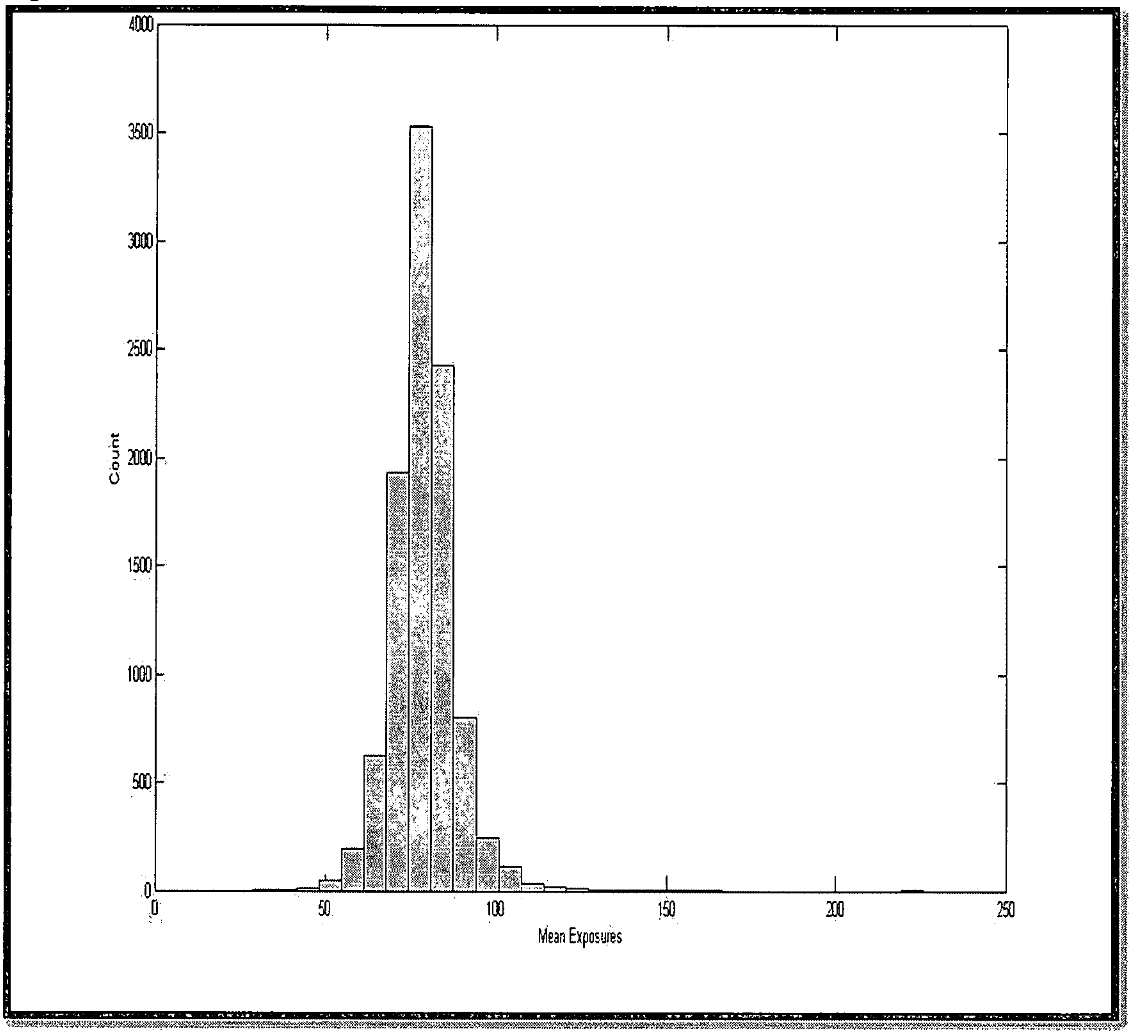


Figure 101: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 8

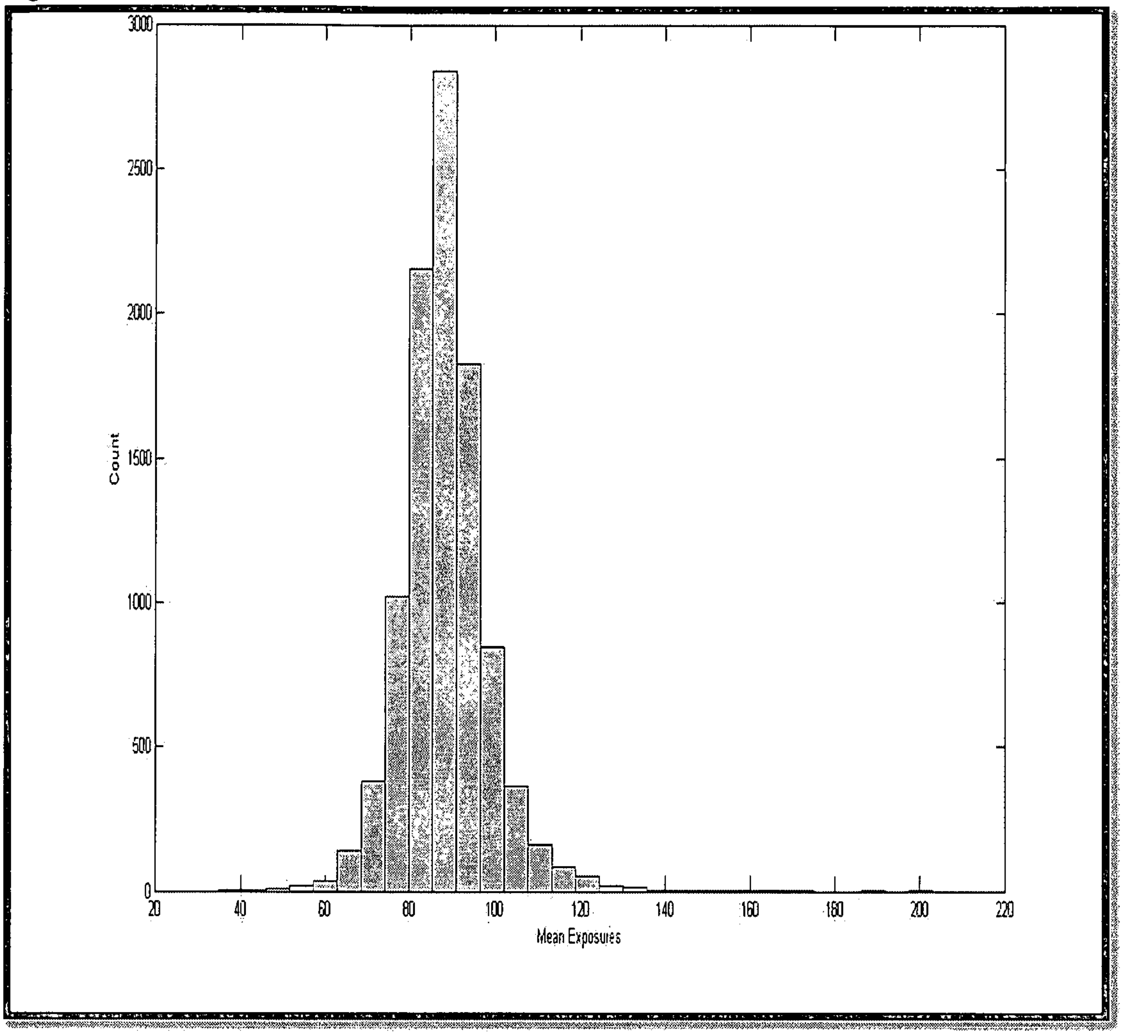


Figure 102: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 9

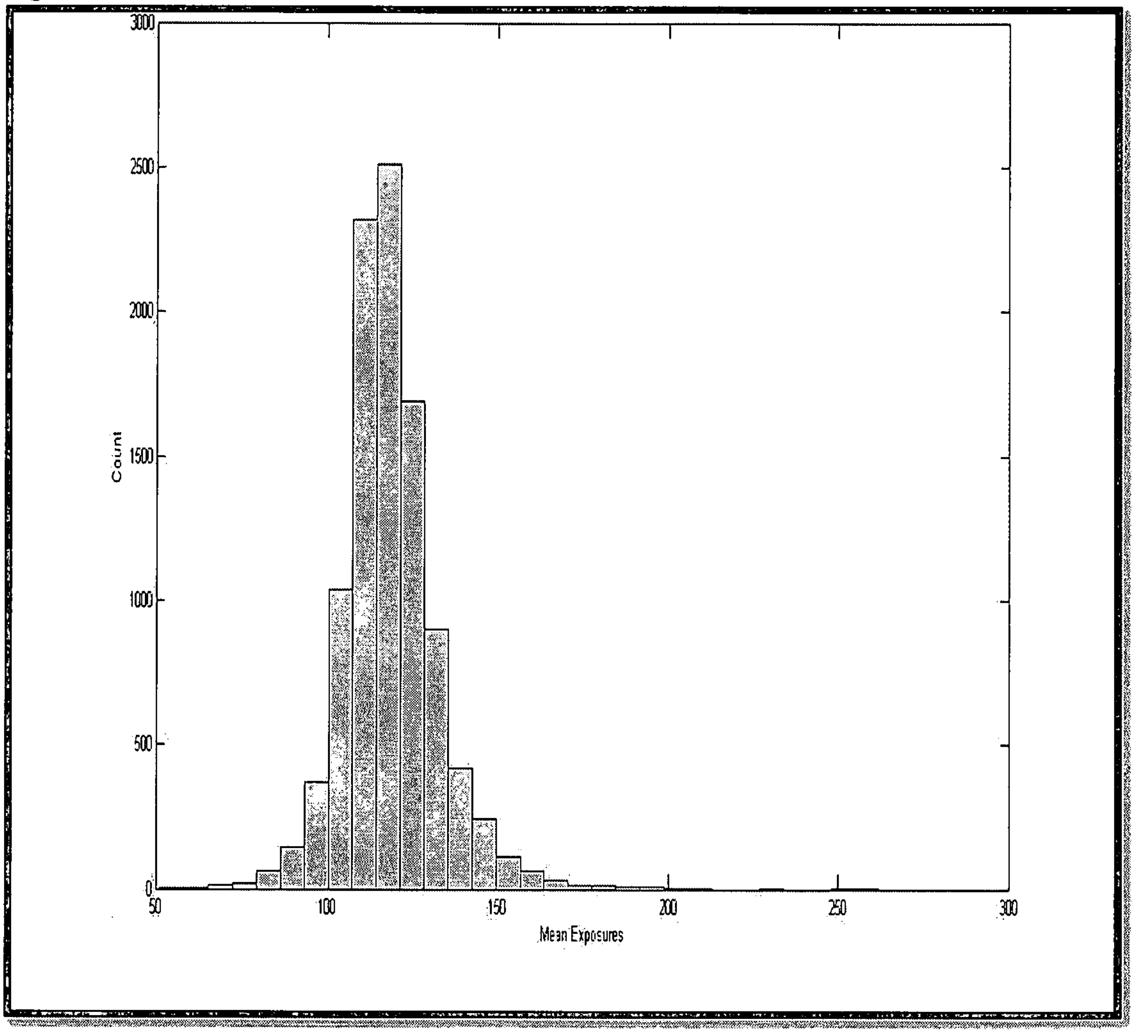


Figure 103: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 10

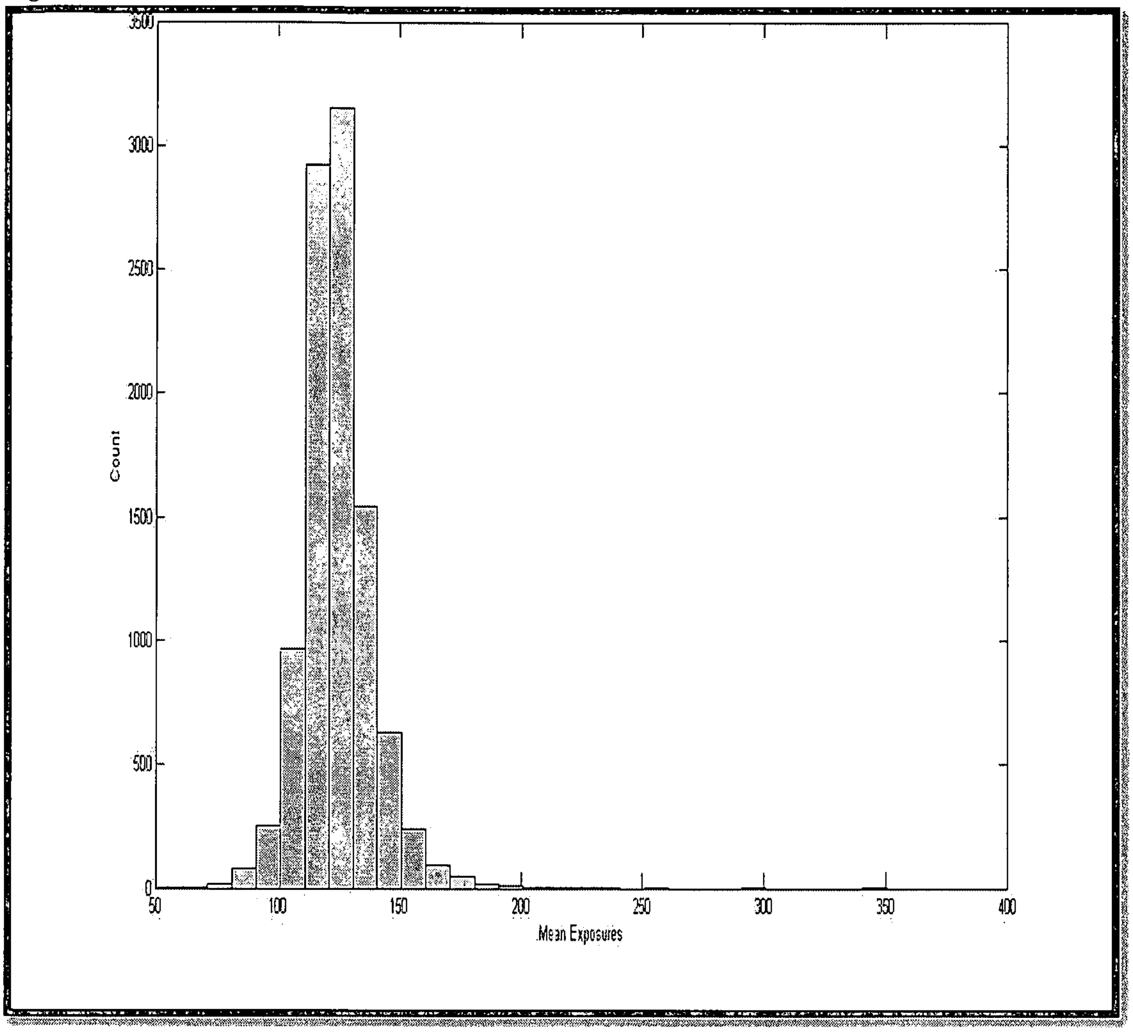


Figure 104: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 11

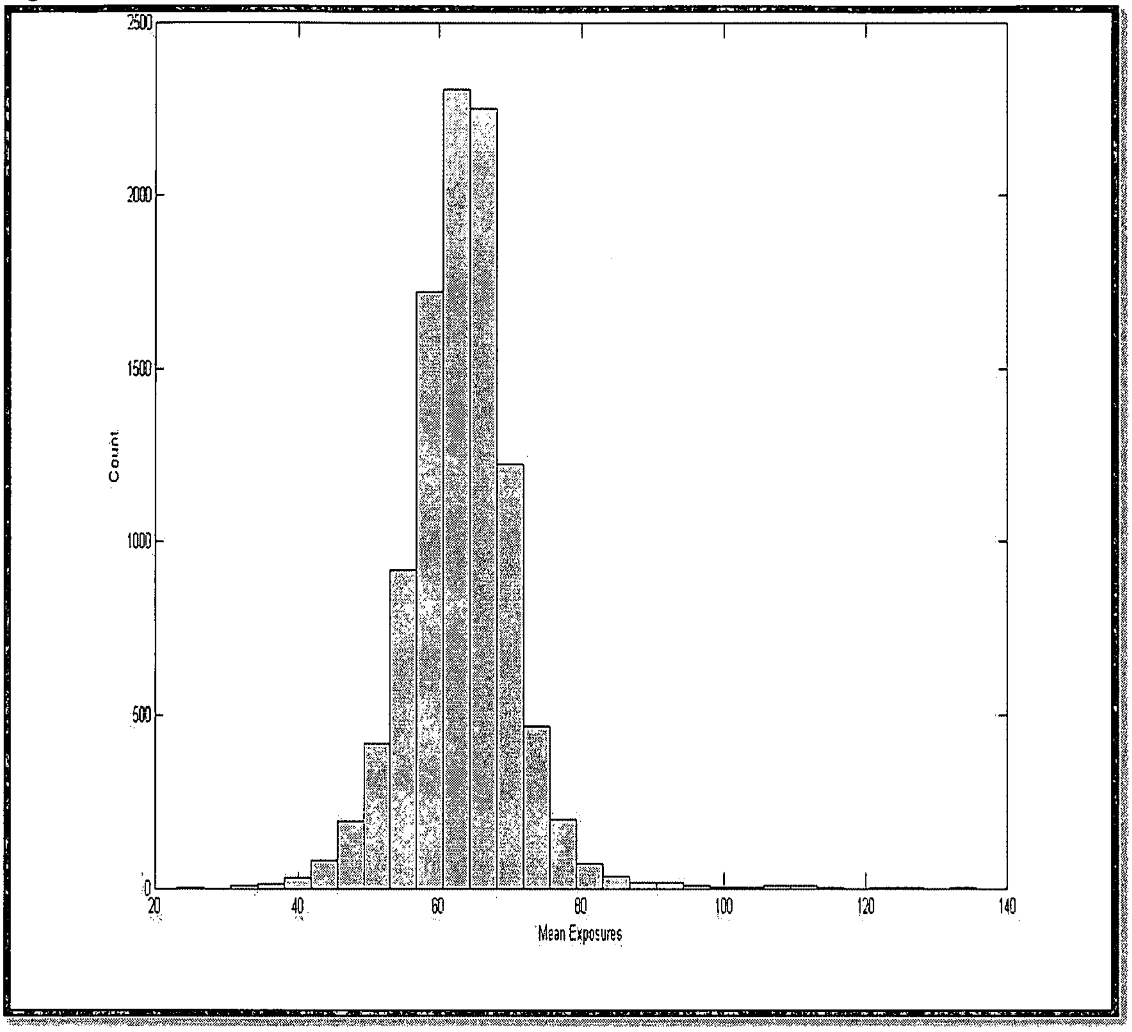


Figure 105: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 12

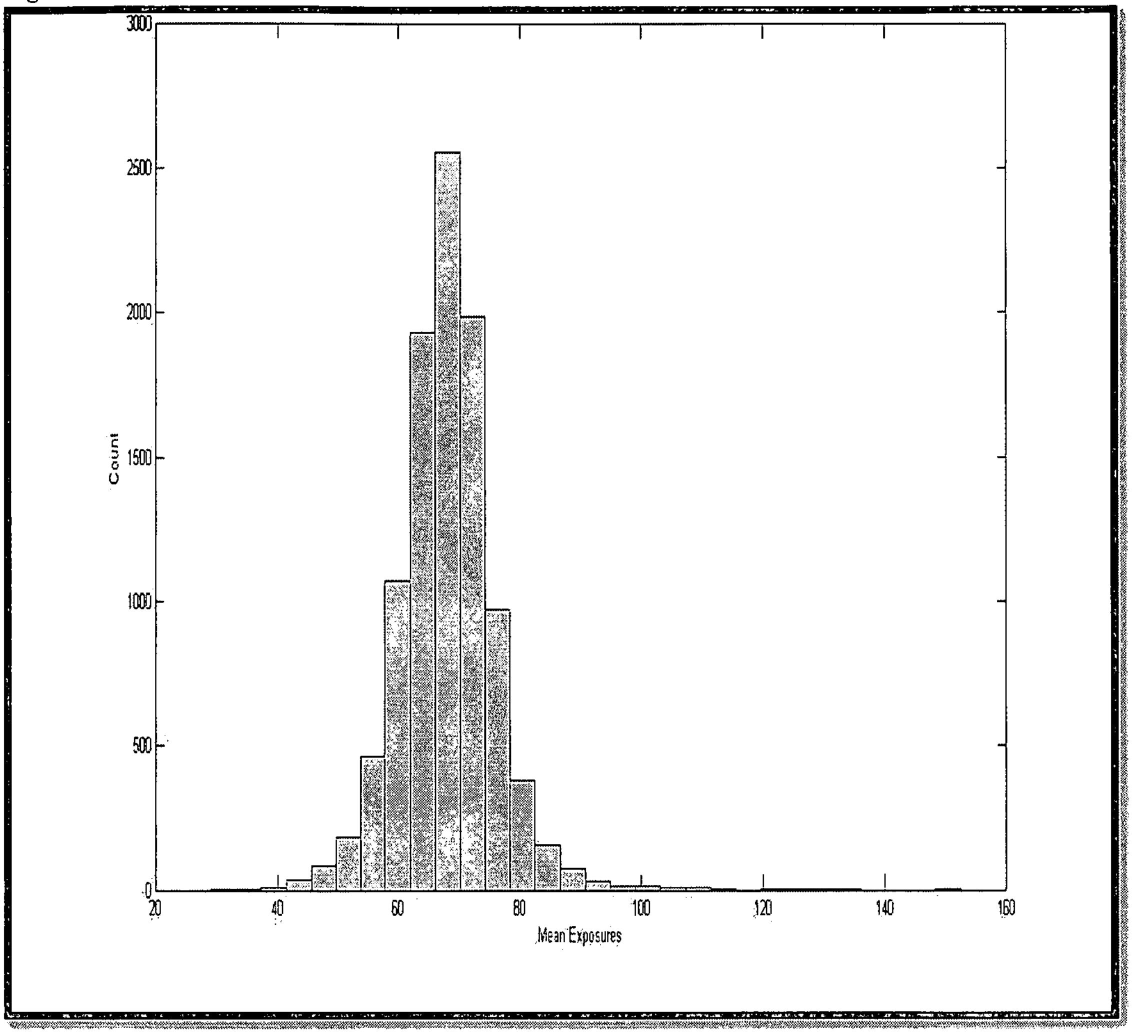


Figure 106: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 13

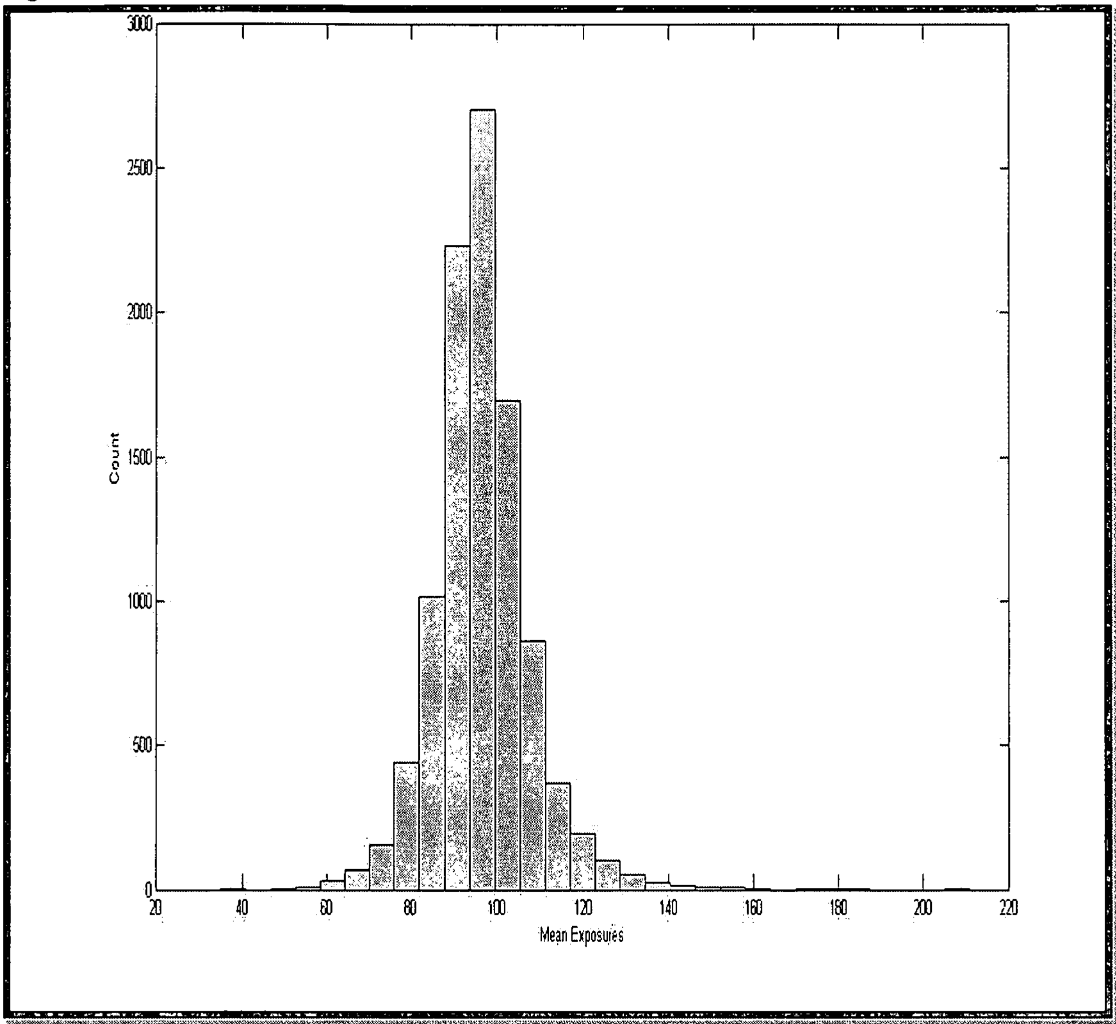


Figure 107: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 14

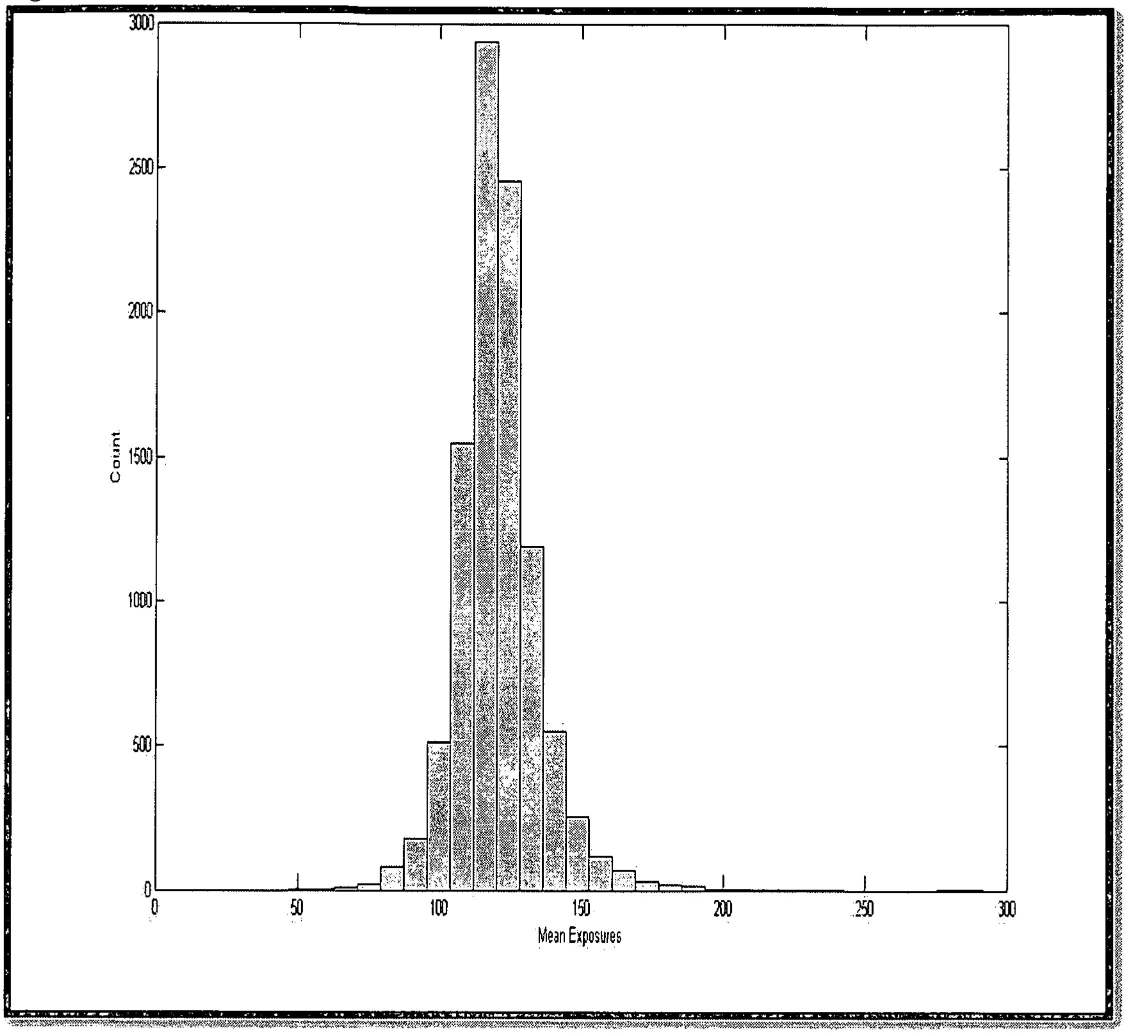
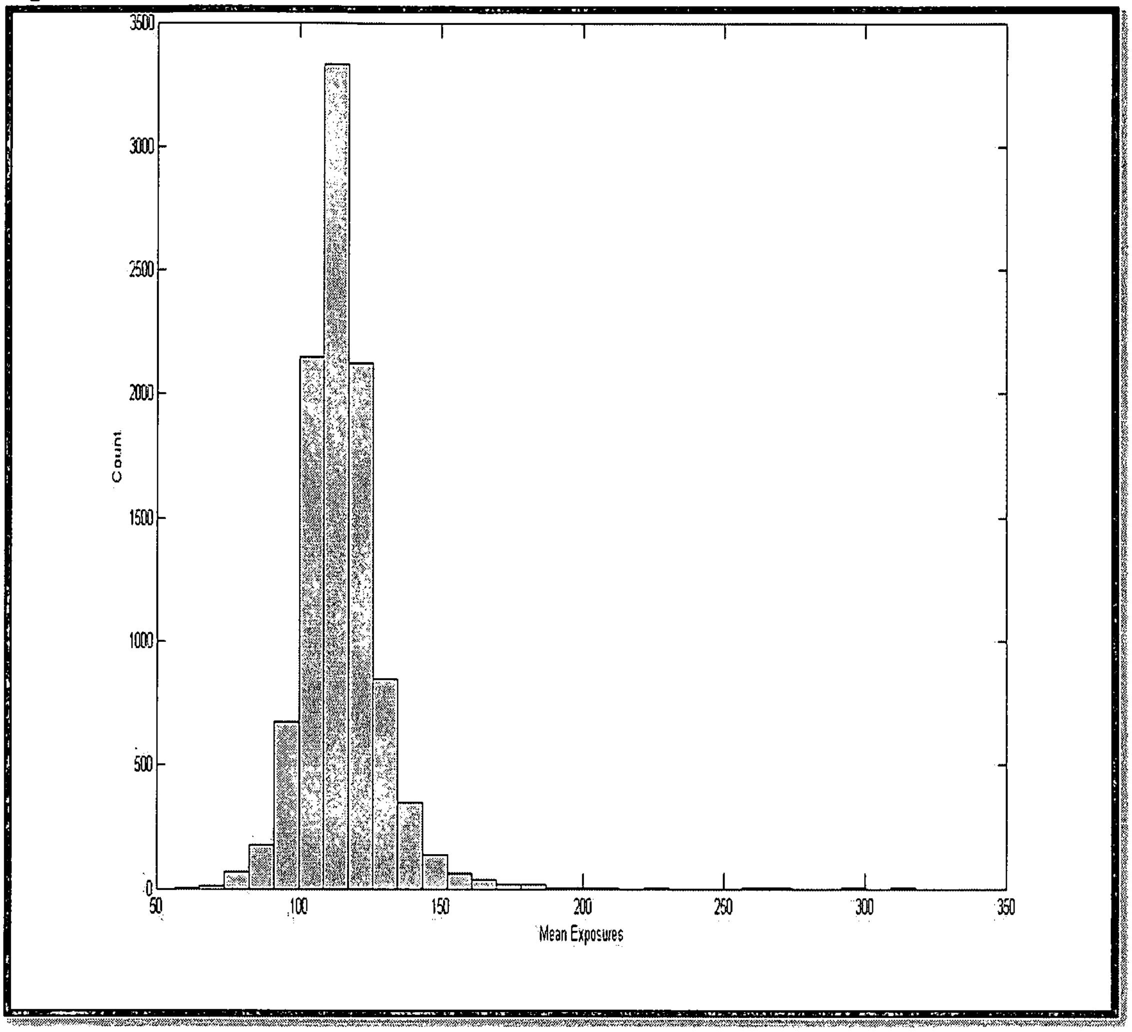


Figure 108: Simulation Results for Reference Prior 2 for Individual Worker Means - Worker 15



Please refer to Table 104 for the accompanying results to these figures.

Figure 109: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 1

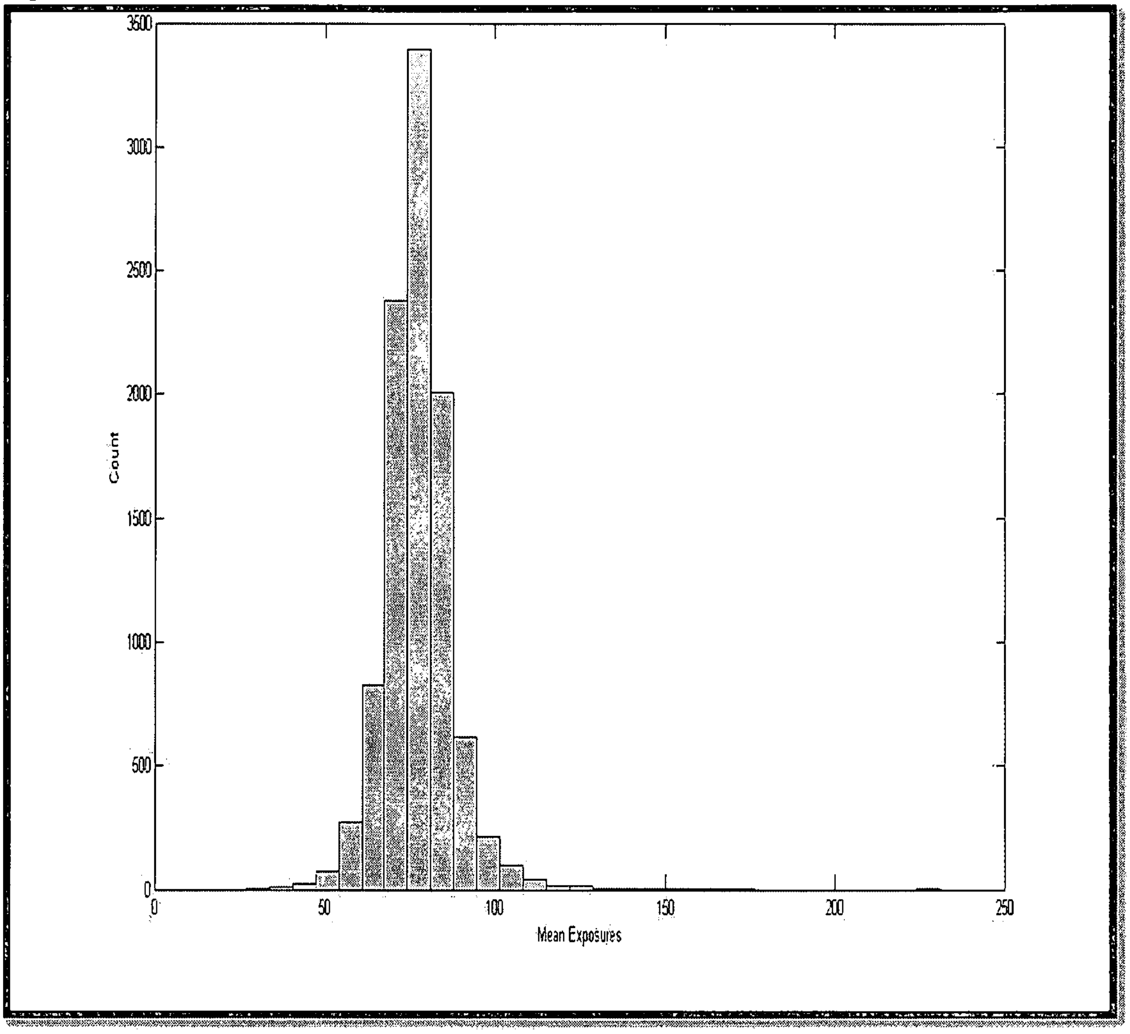


Figure 110: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 2

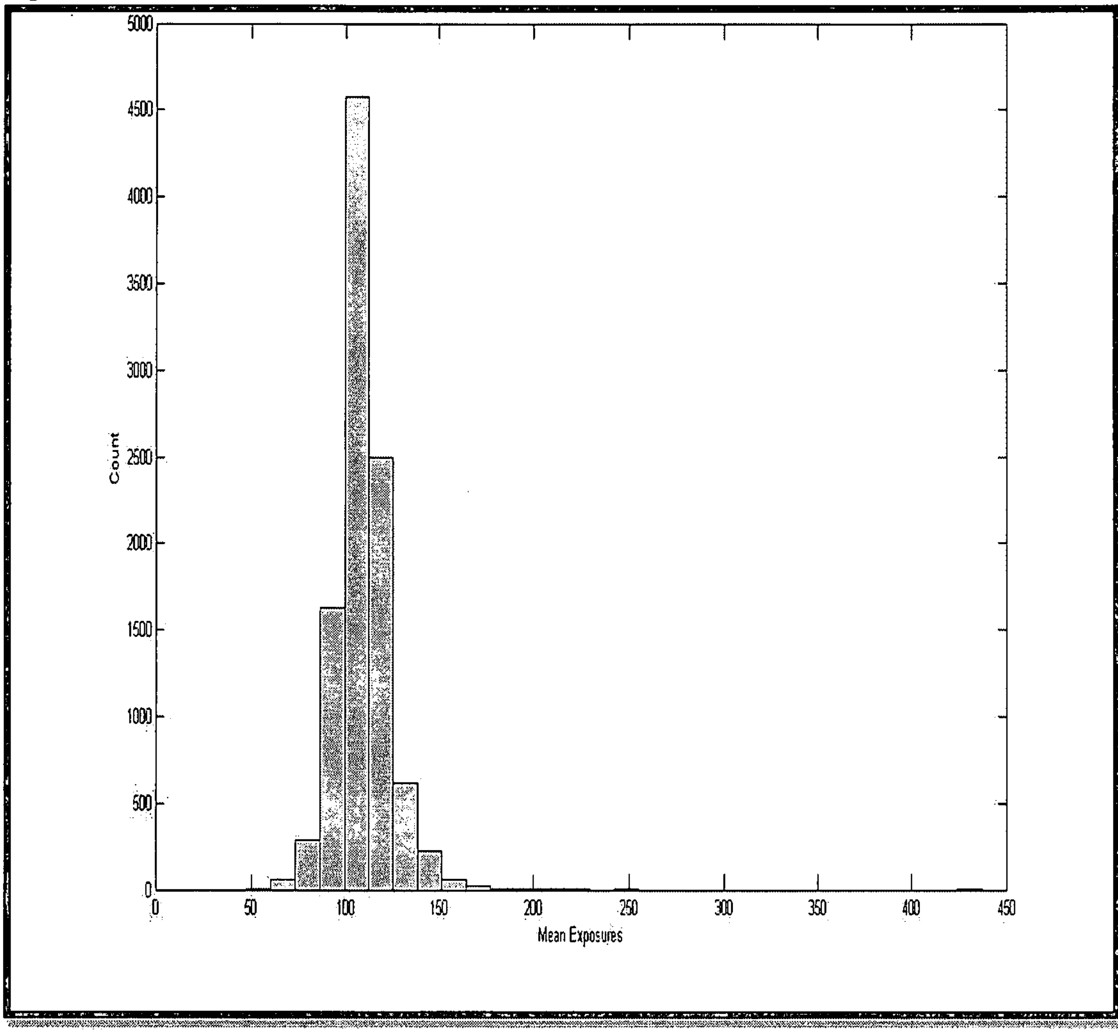


Figure 111: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 3

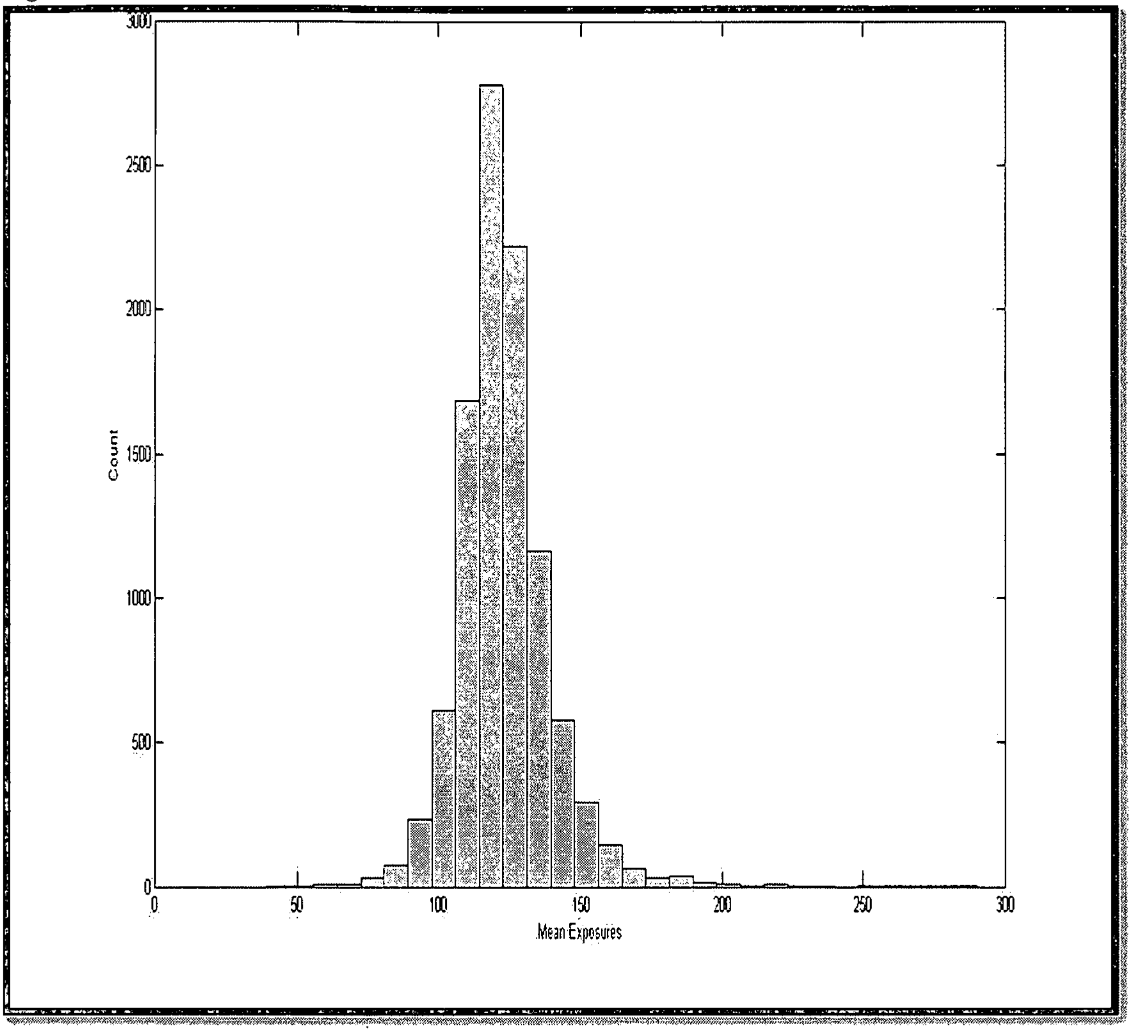


Figure 112: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 4

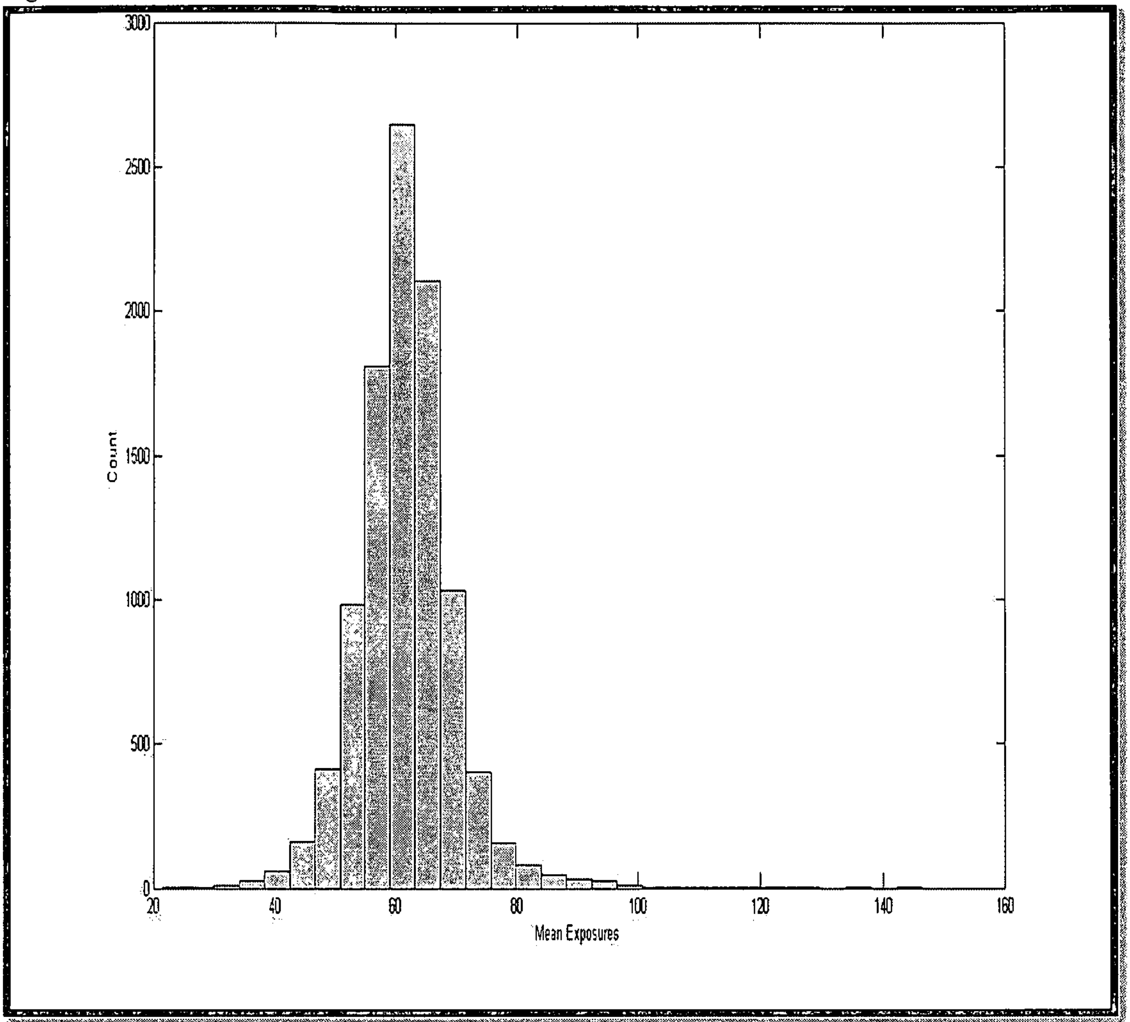


Figure 113: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 5

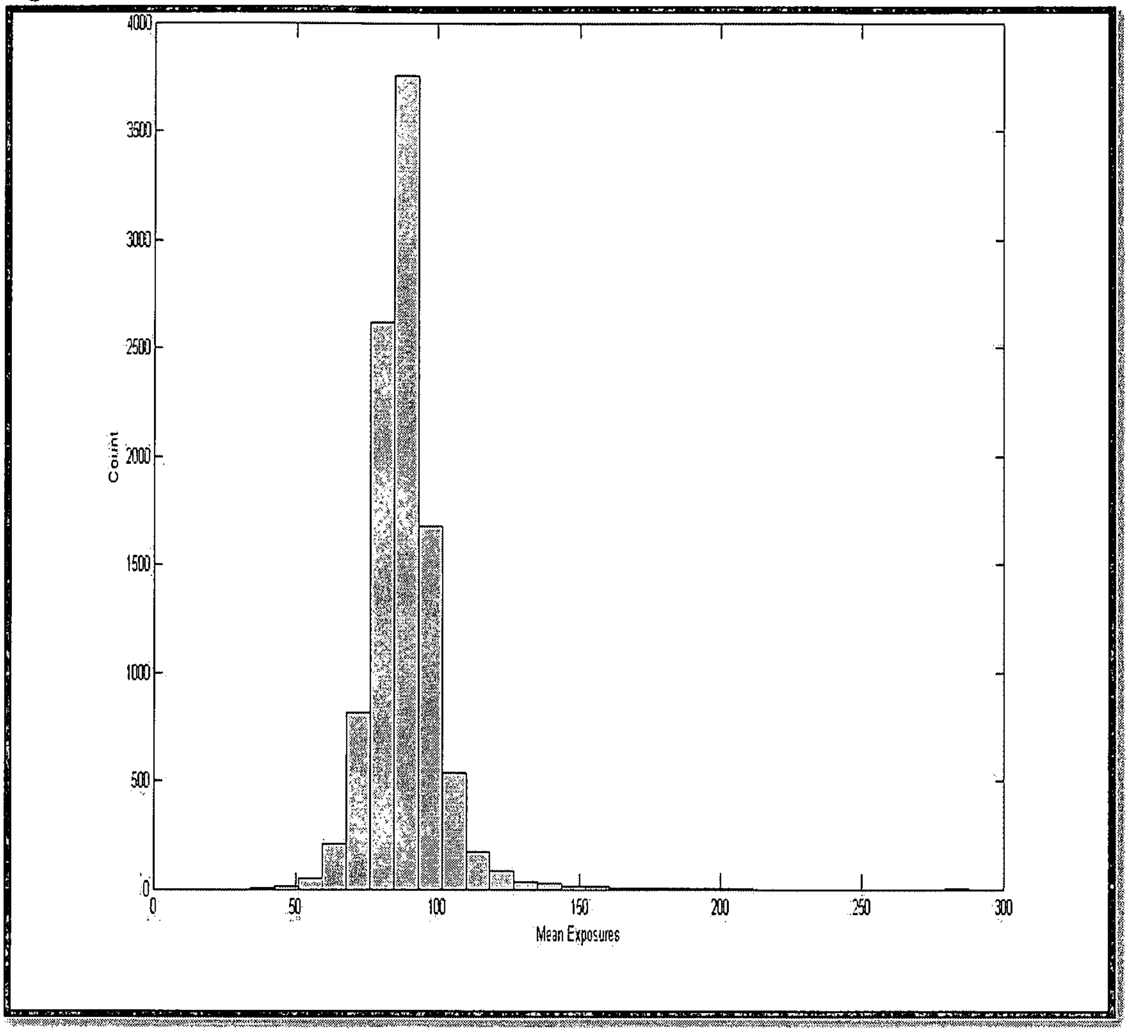


Figure 114: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 6

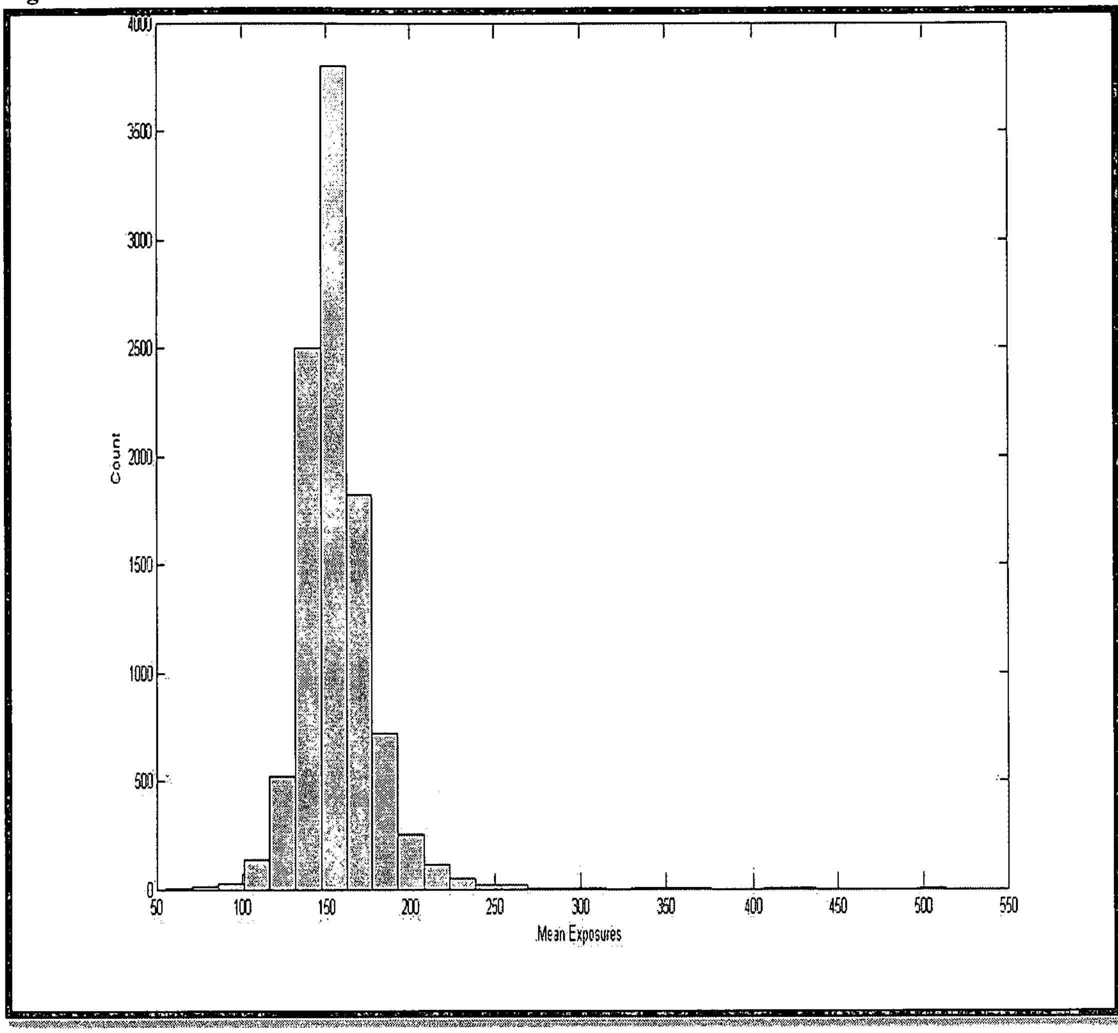


Figure 115: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 7

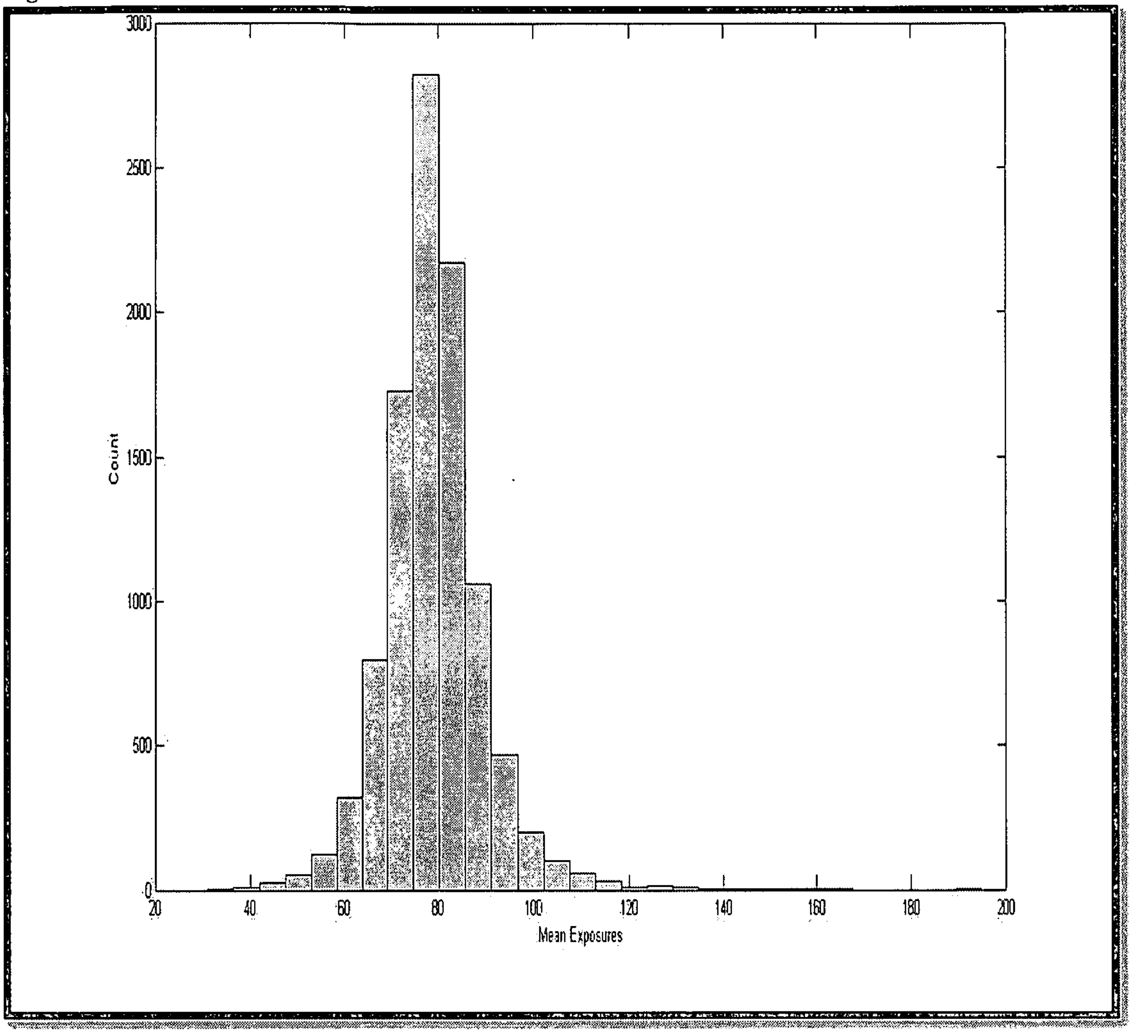


Figure 116: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 8

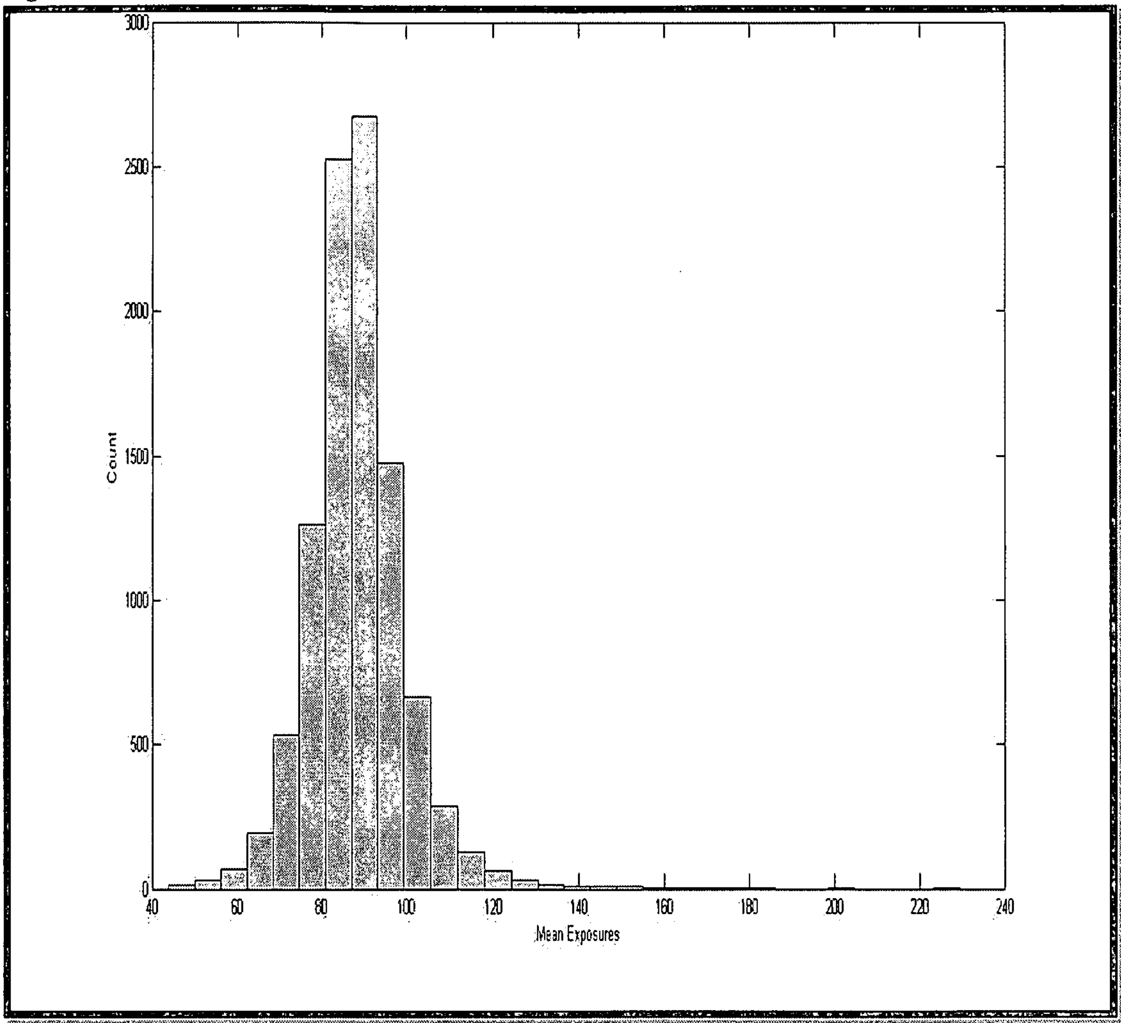


Figure 117: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 9

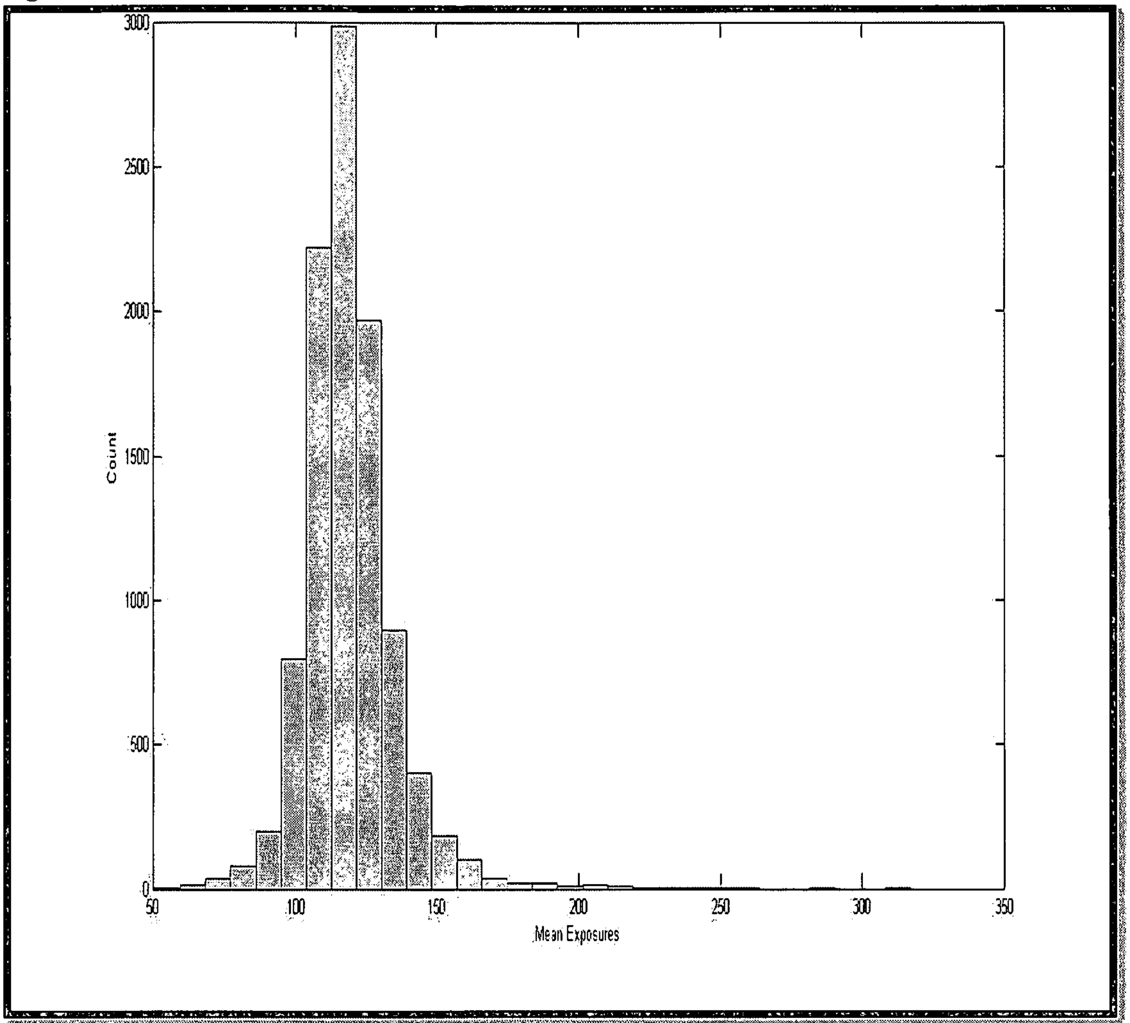


Figure 118: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 10

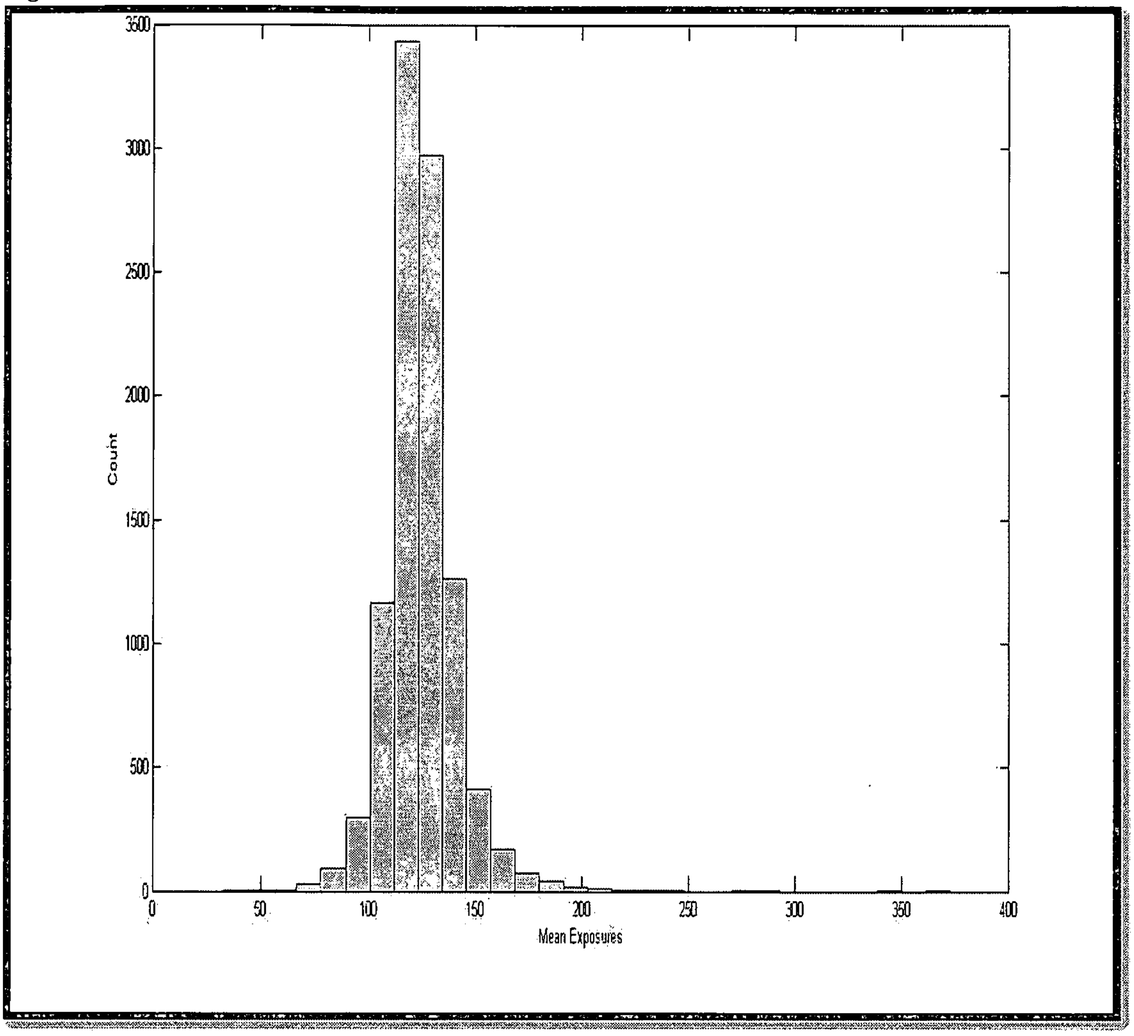


Figure 119: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 11

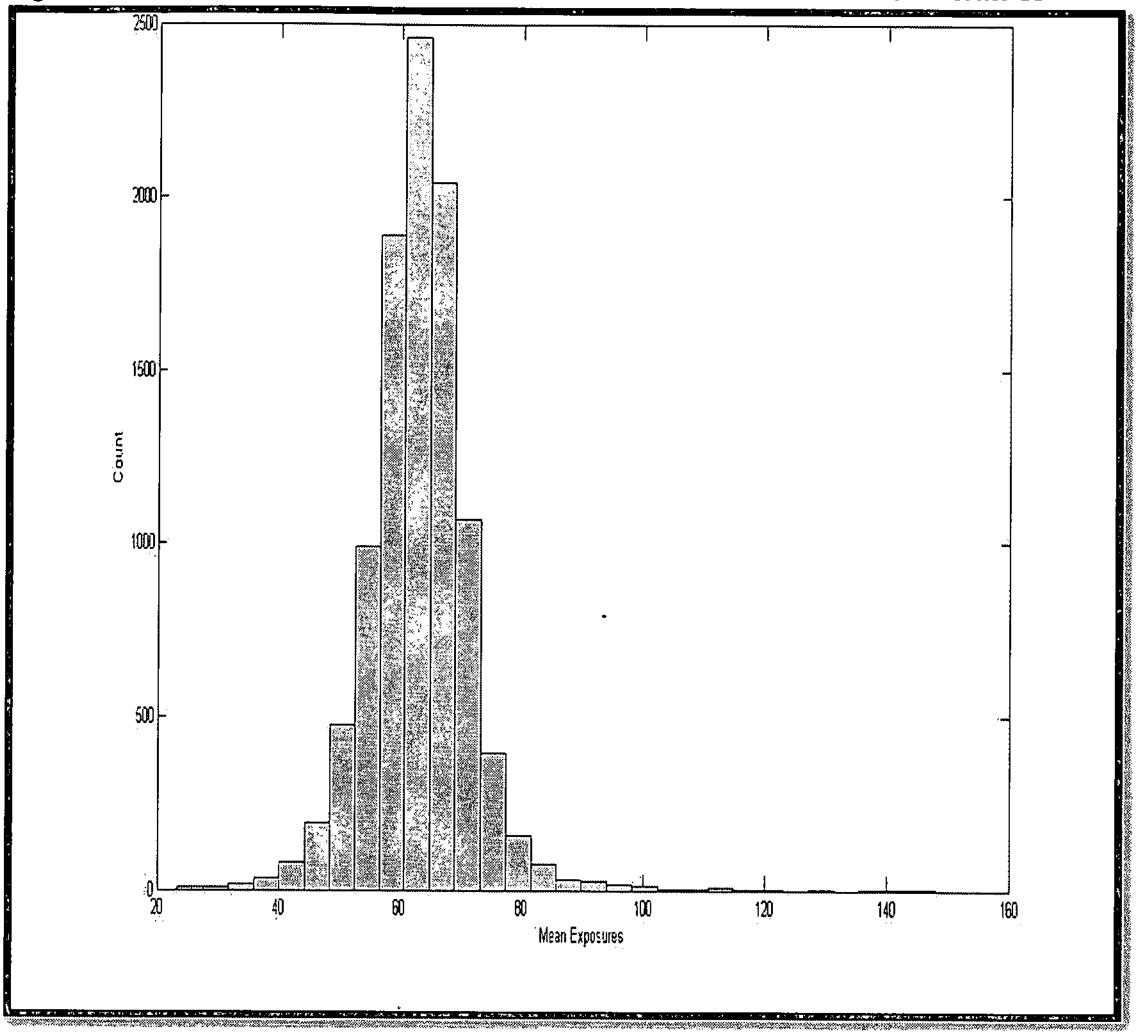


Figure 120: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 12

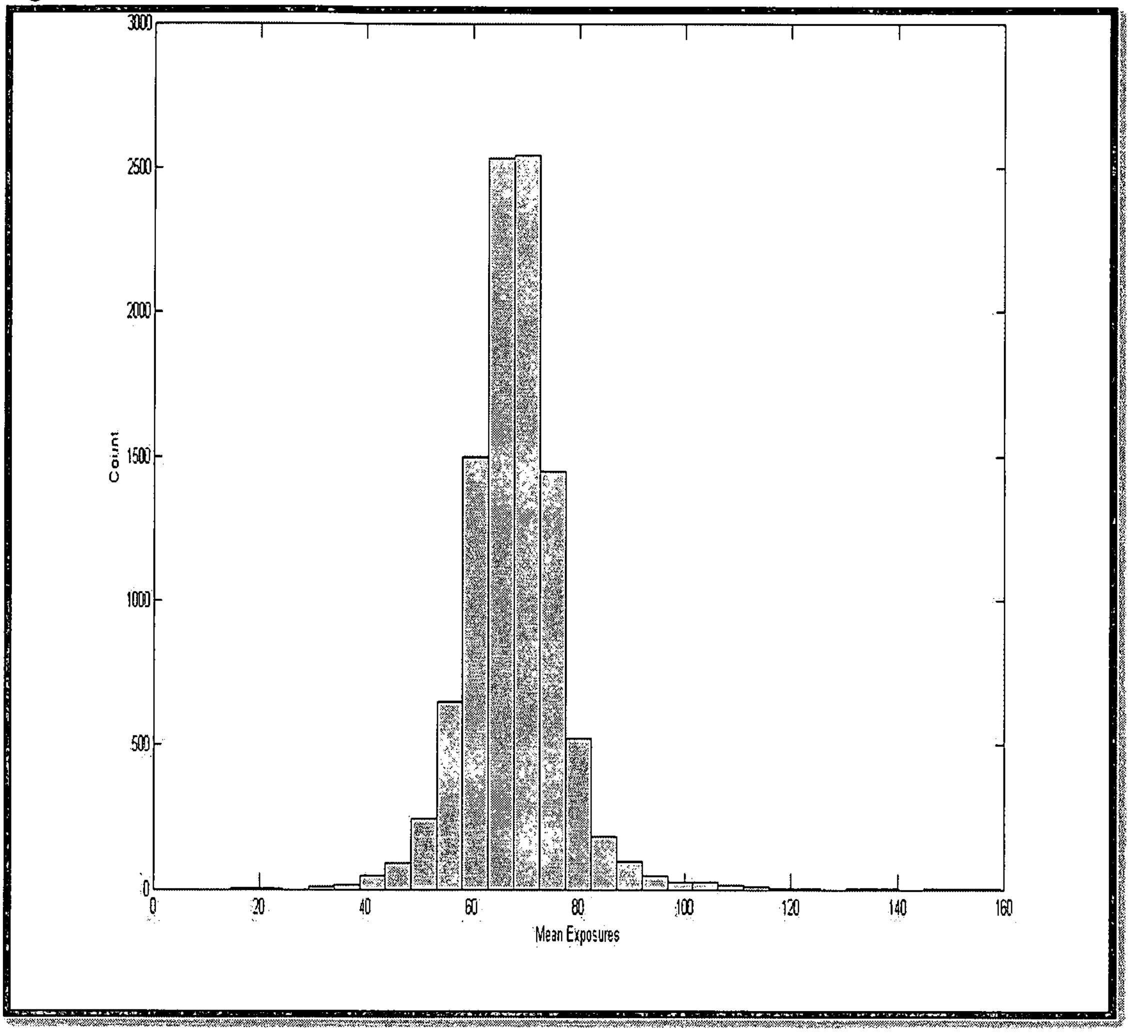


Figure 121: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 13

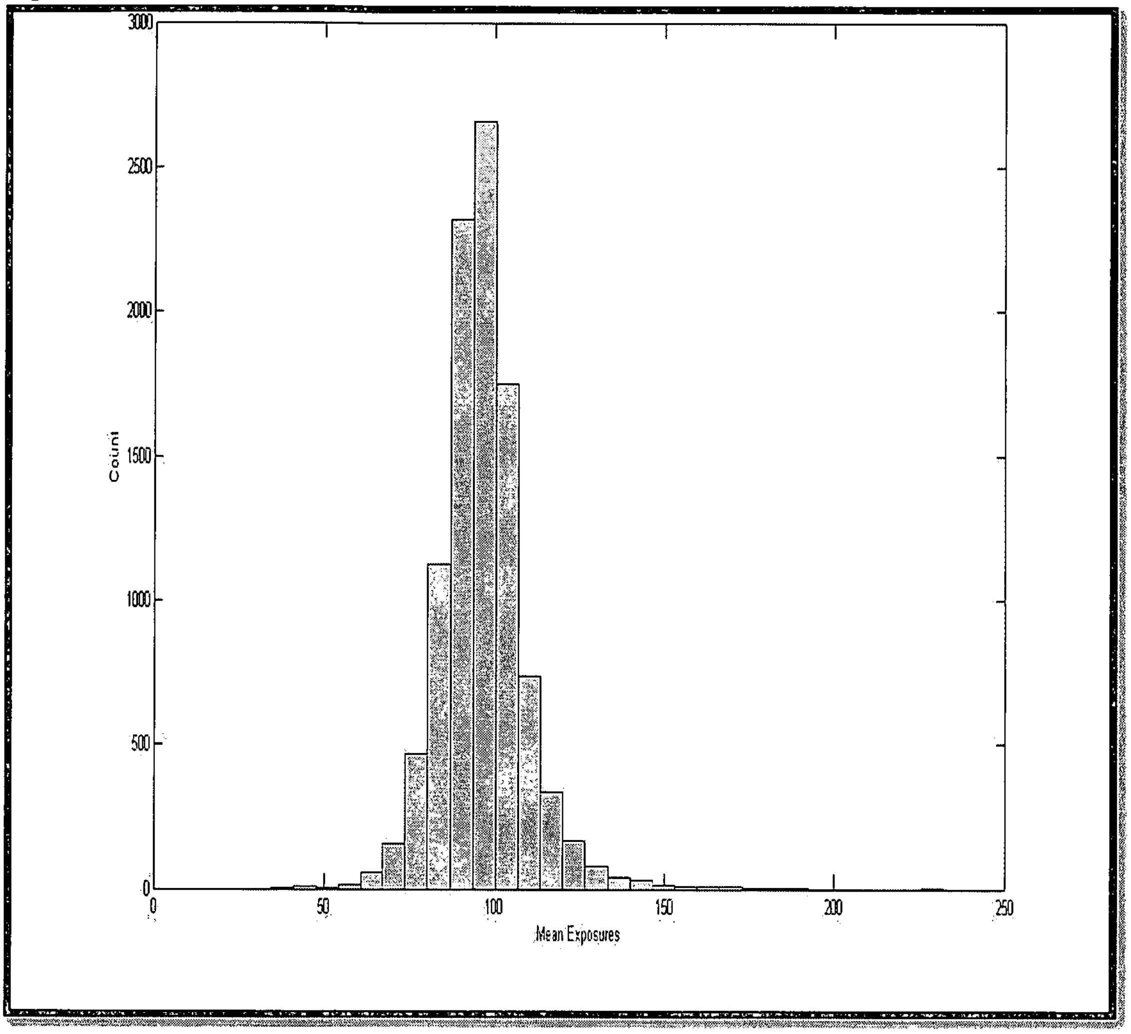


Figure 122: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 14

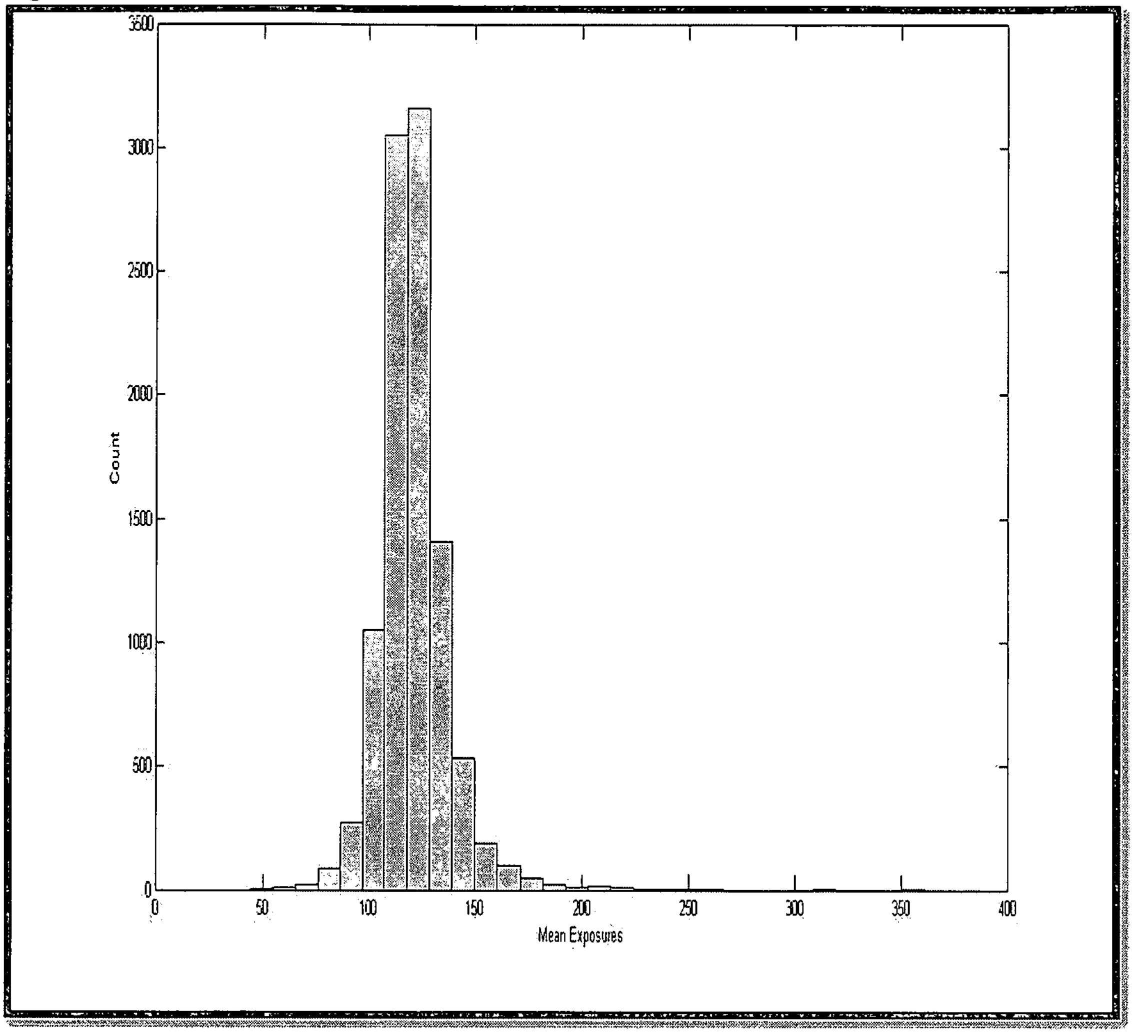
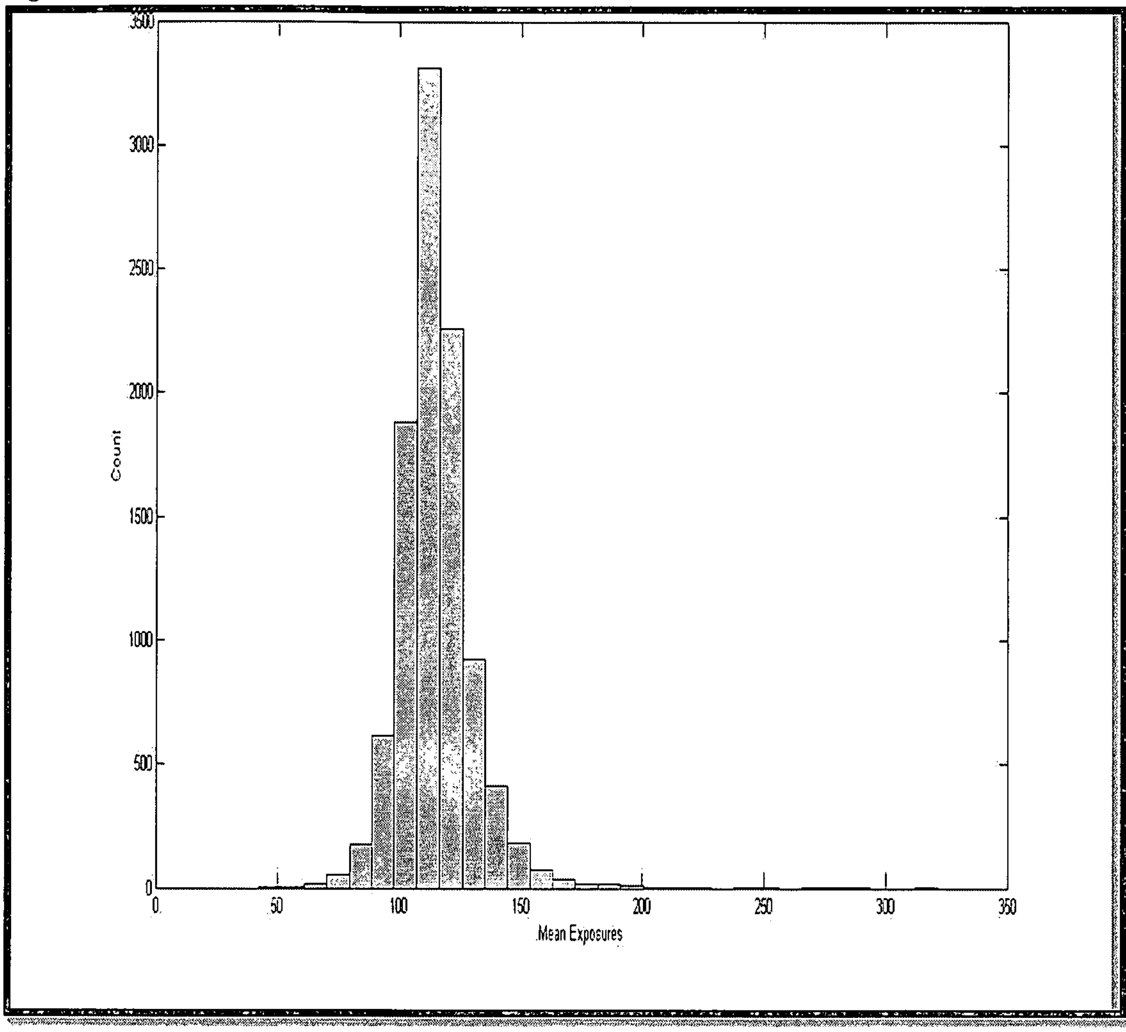


Figure 123: Simulation Results for Gelman's Prior for Individual Worker Means - Worker 15



Summary and Conclusions

The work contained in this thesis is focused on objective Bayesian techniques for various estimation applications, as discussed in the Introduction to this thesis. However, regardless of the particular setting, in each case the objective was to develop the Bayesian framework for analysis and test this framework using a variety of objective prior distributions against known frequentist techniques. In general, the performance was measured by comparing the coverage probabilities as well as the average interval lengths and in some cases the associated bias of a particular technique.

The prior distributions primarily applied were:

1. Jeffreys Rule prior
2. Independence Jeffreys prior
3. Probability-Matching prior
4. Reference prior

In specific chapters, additional priors were considered, such as the uniform and constant priors in Chapters 2 to 4 and a variety of other priors proposed in the literature in Chapter 6 for the bivariate lognormal distribution.

In general, the Bayesian techniques developed compared well to the existing frequentist techniques and in certain cases clearly outperformed certain of the existing frequentist techniques.

In Chapters 2 to 4 for the analysis of the mean of the lognormal distribution the Bayesian techniques proved superior to Maximum Likelihood and Bootstrap techniques. The only exception is perhaps in comparison to the MOVER. Neither method proved to be superior in this setting and motivation for the MOVER was evidenced by means of its simplicity. However, the usefulness of Bayesian techniques was further enhanced in subsequent chapters where the MOVER can currently not be derived.

Furthermore, all priors did not perform equally well. In particular, the Jeffreys priors (Independence Jeffreys and Jeffreys Rule) proved to be most efficient in the majority of settings when compared to the Reference and Probability-Matching priors. Only in a few selected settings did the Reference and Probability-Matching priors show efficiency. For the Probability-Matching prior though, this was not entirely unexpected, due to the nature of the Probability-Matching prior.

Furthermore, it was found that the Bayesian techniques were particular useful in small sample size settings.

When analyzing the means of the bivariate lognormal distribution the Bayesian techniques showed flexibility and usefulness above other techniques. As far as we could tell, in specific settings the MOVER was not derivable and as such the Bayesian techniques proved to be advantageous.

For the one-way random effects models, in both the balanced and unbalanced cases the Bayesian framework added to the analysis of the setting. From the "Styrene exposures" example setting it was clear that in addition to overall mean exposure the Bayesian framework offers the ability to estimate exposure levels for individual workers, which was not previously possible.

From this work it is clear that there are possible suggestions for future research. In particular, future research into hypothesis testing and confidence testing for three or more lognormal means could be undertaken. Of particular interest is the multiple testing problem where other authors have managed to control the simultaneous coverage rate, in what is referred to as the multiple comparison problem.

Opsomming

Die werk waarop in hierdie tesis gefokus is, is die toepassing van objektiewe Bayes tegnieke vir verskeie beramings, soos bespreek in die inleiding van hierdie tesis. Ongeag die spesifieke opset, was die doelwit in elke geval om 'n Bayes raamwerk te ontwerp vir die analisering en toetsing van daarvan deur gebruik te maak van verskeie Bayes prior verdelings teenoor die meer bekende frekwentistiese tegnieke. In die algemeen is die prestasie gemeet deur die dekkings waarskynlikhede te vergelyk sowel as die gemiddelde interval-lengtes en in sekere gevalle die geassosieerde sydigheid van 'n spesifieke tegniek.

Die prior verdelings wat primêr toegepas is, was:

1. Jeffreys Reël prior
2. Onafhanklike Jeffreys prior
3. Waarskynlikheidsafparings prior
4. Referensie prior

In spesifieke hoofstukke is addisionele priors oorweeg, soos die uniforme en konstante priors in hoofstukke 2 tot 4. 'n Verskeidenheid van ander priors in die literatuur is in hoofstuk 6 voorgestel vir die bivariate lognormaal verdeling.

Die Bayes tegnieke wat ontwikkel is, het in die algemeen goed ooreengestem met die huidige frekwentisitiese tegnieke. In sekere gevalle het die Bayes tegnieke beter presteer as die huidige frekwenstistiese tegnieke.

In hoofstukke 2 tot 4 het die Bayes tegnieke, die Maksimum aanneemlike- sowel as Skoenlus tegnieke oortref vir die analise van die gemiddeld van die lognormaal verdeling. Die enigste moontlike uitsondering is in vergelyking met die MOVER. Geeneen van die twee tegnieke was daar beter nie en motivering ten gunste van die MOVER berus slegs op die eenvoudigheid van hierdie metode. Die bruikbaarheid van die Bayes tegnieke is verder uitgelig in latere hoofstukke waar die MOVER tans nie afgelei kan word nie.

Verder is gevind dat alle priors nie ewe goed presteer het nie. Onder andere het dit geblyk dat die Jeffreys priors (onafhanklike Jeffreys en Jeffreys reël) die mees doeltreffende metode is in die meerderheid van ontwerpe as dit vergelyk word met die Referensie and Waarskynlikheidsafparings priors. Slegs in enkele geselekteerde ontwerpe was die Referensie en Waarskynlikheidsafparings priors meer doeltreffendheid. Dit was nie heeltemal onverwags nie as gevolg van die aard van die Waarskynlikheidsafparings prior.

Verder was gevind dat die Bayes tegnieke besonders nuttig is in kleiner steekproef ontwerpe.

By die analisering van die bivariate lognormaal verdeling was die Bayes tegnieke meer buigsaam en nuttig as ander tegnieke. Sover as wat ons kennis strek, kon die MOVER vir sekere ontwerpe nie afgelei word nie en dus is die gebruik van Bayes tegnieke daar meer voordelig.

In die eenrigting ewekansige effekte modelle vir beide gebalanseerde en ongebalanseerde ontwerpe, het die Bayes raamwerk die analise van die ontwerp verbeter. Uit die "Styrene exposures" voorbeeld was dit duidelik dat die Bayes raamwerk die vermoë verskaf om blootstellingsvlakke van individuele werkers te beraam in plaas daarvan om die algehele gemiddelde van alle werkers te beraam, wat voorheen nie moontlik was nie.

Uit die werk is dit duidelik dat daar moontlikhede vir toekomstige navorsing bestaan. Navorsing in hipotesetoetse en vertrouensintervalle vir drie of meer lognormaal gemiddeldes kan ondersoek word. Van spesifieke belang is die meervoudige toetsings probleem, waar verskeie ander outeurs die gelyktydige oordekkings koers kon beheer, ook bekend as die meervoudige vergelykings probleem.

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