Actuarial Risk Management of Investment Guarantees in Life Insurance

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Abstract

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Investment guarantees in life insurance business have generated a lot of research in recent years due to the earlier mispricing of such products. These guarantees generally take the form of exotic options and are therefore difficult to price analytically, even in a simplified setting. A possible solution to the risk management problem of investment guarantees contingent on death and survival is proposed through the use of a conditional lower bound approximation of the corresponding embedded option value. The derivation of the conditional lower bound approximation is outlined in the case of regular premiums with asset-based charges and the implementation is illustrated in a Black-Scholes-Merton setting. The derived conditional lower bound approximation also facilitates verifying economic scenario generator based pricing and valuation, as well as sensitivity measures for hedging solutions.

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Chapter 1

INTRODUCTION

1.1 Investment Guarantees

Life insurance companies have traditionally sold risk products with defined sums at risk and did not compete directly with investment products by way of life insurance savings policies. The taxation benefits on life endowment policies could in some way offset the potential loss in returns but the life insurance industry inevitably revised products to allow policyholders greater investment flexibility and, as a consequence, more investment risk. Variable annuity (VA) or unit-linked (UL) products offer policyholders added investment benefits as well as deferred taxation benefits. The investment benefits include flexibility on asset allocation through a choice of diverse funds to select from and, in some cases, allowance for diversification across funds by permitting policyholders to hold several funds linked to the policyholder's benefits. Policyholders can typically choose one fund or a range of funds in which to invest. Premiums, less charges for expenses and ancillary mortality and morbidity benefits, are allocated to an individual's account. The policyholder sub-account portrays the chosen risk appetite of the policyholder. The cash value of a policyholder's account at any time is determined by the number of units purchased multiplied by the value of the underlying investment fund at that time. Life-contingent risk benefits on these types of policy are usually expressed as a function of the cash value of the policyholder's account or as a function of the contributions made by the policyholder.

Life-contingent risk benefits were initially the only guaranteed liabilities considered on VA and UL contracts, which meant that policyholders had to bear the investment risk in full. The volatility in asset prices, especially in the equity market, soon necessitated the introduction of investment guarantees, i.e. some transfer of investment risk from the policyholder to the insurer. Equity-based life business originated in the Netherlands in the early 1950s, whereas VA business commenced in the USA in the late 1950s and grew substantially to annual sales of $155.7 billion in 2008 as estimated by the Life Insurance Marketing and Research Association (LIMRA,

2009) . Although the focus of this text is on the investment guarantees offered on VA business, similarities to investment guarantees on UL business allow the wider application of the methods introduced in the following. UL business generally offers policyholders a linked endowment with the option of a death and/or maturity guarantee. The linked endowment can also take the form of a retirement annuity, i.e. an investment plan to convert into an immediate living annuity or life annuity at retirement. The retirement annuity type of UL business is almost identical to the core VA business options. VA business offers policyholders a variety of investment options pre-retirement and post-retirement coupled with a range of life-contingent investment guarantees. VA products reached Japan in 1999 and Europe in 2006 (Cudmore and Claffey, 2008).

According to Hardy (2003), UL business with guaranteed death and maturity benefits rose to popularity in the United Kingdom in the late 1960s. These product types soon migrated to Australia and South Africa, both of which still had strong ties with the British actuarial profession. Hardy further describes how maturity guarantees of 100% of premium contributions were a common feature of the unit-linked contracts offered in the United Kingdom during the mid 1970s. To price these guarantees, insurers either resorted to an approach where the distribution of the guarantee was converted into a quantile-based price or where the guarantee was considered negligible out-of-the-money. The depressed markets of 1973 and 1974 in the United Kingdom soon proved these methods to be inappropriate when investment guarantees moved to in-the-money positions. South African firms have predominantly used a deterministic approach to pricing and reserving for maturity guarantees according to a survey done by Foroughi et al. (2003). Only a few respondents used a real-world stochastic approach. Regulation in South Africa has changed in 2005 by requiring life insurance offices to apply stochastic modelling in their reserving requirements. The new regulatory requirement contributed to more suitable pricing and reserving for the risks of investment guarantees.

The investment guarantees offered on VA contracts are optional rider benefits. Policyholders are mainly offered four types of life-contingent investment guarantees during the accumulation period:

1. a Guaranteed Minimum Death Benefit (GMDB),

2. a Guaranteed Minimum Income Benefit (GMIB),

3. a Guaranteed Minimum Accumulation Benefit (GMAB), and

4. a Guaranteed Minimum Withdrawal Benefit (GMWB).

The Guaranteed Minimum Death Benefit is the most common rider on deferred annuities offered in the USA and was introduced in 1980. The basic GMDB guarantees to pay on death of the life assured during the accumulation period the greater of (a) the fund value at death or (b) the return of contributions minus withdrawals made prior to death. Enhancements to the basic GMDB were made in, for example, a ratchet or high-water mark option that guarantees policyholders the greater of the basic GMDB or the highest values attained by the fund on specified dates, e.g. policy anniversary, prior to death. According to the 2009 Annuity Fact Book published by the Insured Retirement Institute (IRI, 2009), ratchet benefits are offered on 87% of all VA products. Other enhancements to the basic GMDB include an interest benefit (47% of all VA products) that guarantees the greater of the GMDB or the return of contributions with specified interest minus withdrawals made prior to death, and an enhanced earnings benefit (64% of all VA products) that provides an additional amount to offset capital gains taxation payable upon death. The cost for guaranteed benefits are charged annually as a percentage of the policyholder's account value. Charges depend on the investment risk (i.e. the asset allocation opted by the policyholder) and mortality risk (i.e. age, sex, smoker status, etc.) attributes of each policy. The IRI report (IRI, 2009) estimates the charges to range from 5 to 115 basis points (bps) p.a. for the ratchet benefit, 5 to 135 bps for the interest benefit, and 10 to 95 bps for the enhanced earnings benefit.

VA Living Benefits or Guaranteed Minimum Living Benefits are terms used to describe all benefits contingent on the survival of the policyholder, and consist of the GMIB, GMAB and GMWB. The Guaranteed Minimum Income Benefit (GMIB) was introduced in the USA in 1996. The benefit is akin to the Guaranteed Annuity Option (GAO) in UL business and guarantees the interest rate at which the policyholder can annuitise the value at maturity or at a specified time (typically after a minimum 10 year holding period). The IRI report (IRI, 2009) estimates the charge for the GMIB, which is offered on 28% of all VA contracts, to range from 15 to 115 bps p.a.

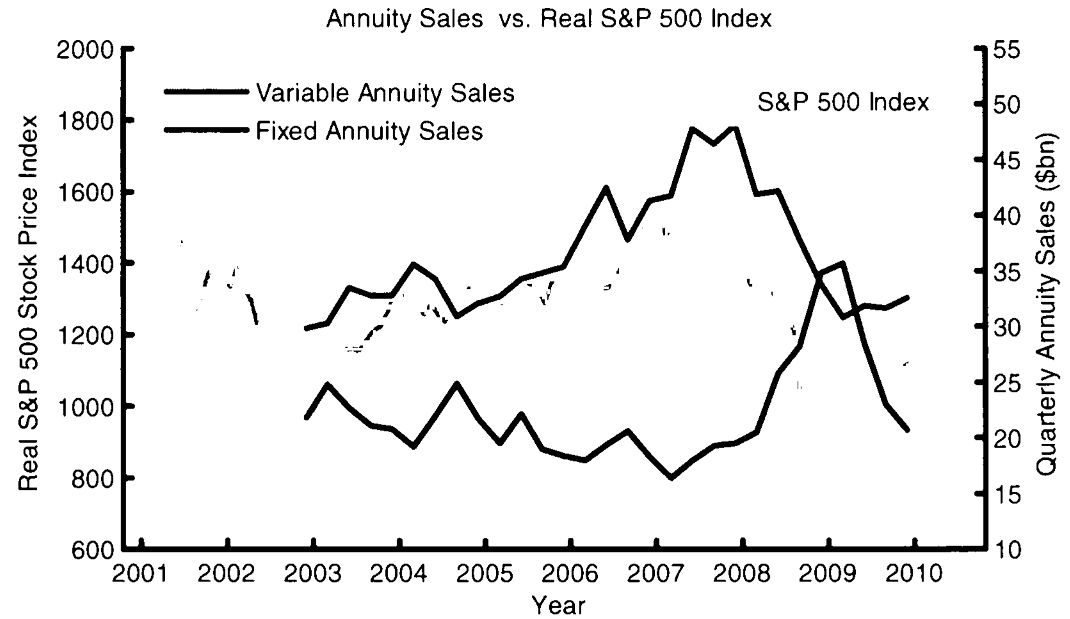
The Guaranteed Minimum Accumulation Benefit (GMAB) was introduced in the USA in 2002. The GMAB effectively replaced the Guaranteed Minimum Maturity Benefit (GMMB) on many product offerings. The GMAB guarantees that the fund value will be at least a specified percentage of the initial premium or total regular premiums after a specified number of years. The GMMB only guarantees the fund value at maturity of the contract. When the fund value greatly exceeded the specified percentage of the fund value guaranteed, policyholders were effectively paying dearly for an out-of-the-money option and hence lapsed their contracts. The cost of the GMAB, which is offered on 32% of all VA contracts, ranges from 15 to 150 bps p.a. and depends heavily on the chosen asset allocation of the policyholder's account (IRI, 2009).

The Guaranteed Minimum Withdrawal Benefit (GMWB) was introduced in mid-2002 in the USA. This benefit guarantees that a certain percentage (usually between 5% to 7%) of the initial contribution can be withdrawn annually until the contribution is fully recovered, regardless of whether the fund value might be insufficient due to market performance. The cost of the GMWB, which is offered on 64% of all VA contracts, ranges from 15 to 150 bps p.a. (IRI, 2009). An enhancement to the GMWB, which is called a Guaranteed Lifetime Withdrawal Benefit (GLWB), was introduced in 2004 and guarantees a certain percentage (usually between 2% to 8%) of the initial contribution to be withdrawn for life. The GLWB is hugely popular with an election rate in the fourth quarter of 2008 of 58% as compared to the 5% election rates for both the GMAB and the basic GMWB. The cost of the GLWB ranges from 25 to 150 bps p.a. and the benefit is offered on 51% of all VA contracts (IRI, 2009). In 2008, the Standalone Lifetime Benefit (SALB) was introduced in the USA. The benefit is effectively a GLWB offered directly on managed portfolios and serves as an enhancement on asset management products, rather than VA products.

VA product sales in the USA are strongly positively correlated to the equity market performance. For example, LIMRA (2009) estimates that total year-to-date VA sales decreased by approximately 26% over the turbulent period of economic crisis from mid 2008 to mid 2009, whereas total fixed deferred annuity sales increased by approximately 46% over the same period. Revised taxation rules also recently reduced the competitive advantage that deferred annuities offer in the United States. Despite a decrease of 15% from 2007 to 2008, total annual VA product sales increased from \$13.5 billion in 1989 to \$156 billion in 2008 (IRI, 2009). The relationship between deferred VA quarterly sales and fixed annuity quarterly sales relative to the real Standard & Poors 500 stock price index is shown in Figure 1.1.

Note that an unintended consequence of the displayed behaviour is that policyholders lock themselves in at low yielding benefit rates, while insurance firms are selling large volumes of

Figure 1.1: Relationship between fixed and variable deferred annuity sales and the equity market.



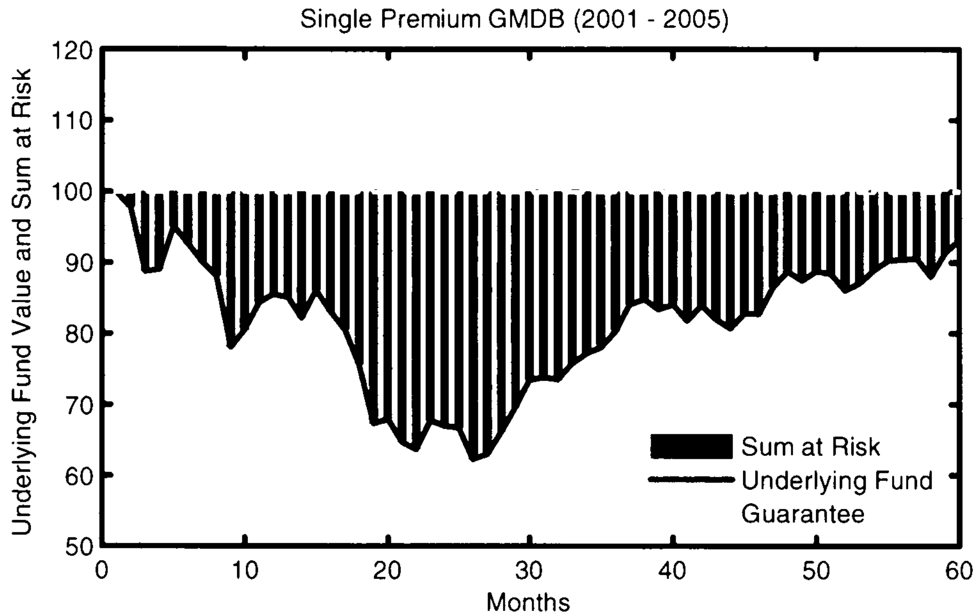
investment guarantees at potentially underpriced rates. The latter consequence arises from insurance firms not adequately accounting for accurate downside risk in high yielding times, thereby underpricing the downside risk being transferred to the firm.

1.2 Illustrative Example

In the following, we illustrate the potential pure financial liability that investment guarantees on death and survival pose to the insurer. We also discuss the effect that dollar-cost averaging (DCA) has on the financial risks posed by the guarantees. The product under consideration provides a guarantee equal to the contributions made during the policy term. The benefit is payable on death in the case of the GMDB and is payable on survival in the cases of the GMMB and the GMWB. The term of the policy is five years and changes in the underlying fund value and sum at risk are monitored on a monthly basis. The underlying fund is assumed to be fully invested in the Standard and Poors 500 equity index, for which the data were obtained from Shiller (2010). No allowance is made for tax or fund charges.

First, we consider a single contribution of 100 made at inception over the 5 year period from 2001 to 2005 for a VA product with a GMDB selected by the policyholder. The S&P 500 equity index commenced its fall with the Latin American crisis and the global uncertainty after the

Figure 1.2: GMDB over prolonged bear market.



11 September 2001 attacks on New York, U.S.A. The subsequent information technology sector equity crisis in 2002 prolonged this fall. It is evident from Figure 1.2 that the insurer is at risk for the entire policy term, albeit that the payoff is contingent on death of the life insured.

The GMDB case over the same 5 year period (2001 to 2005) results in only a sum at risk at maturity, since the contingent event is survival of the policyholder for the duration of the policy term. Although the time that the insurer is at risk is less for offering a GMMB instead of the GMDB in Figure 1.2, the probability of the contingent event is almost always greater for the GMMB than it is for the GMDB. Examples of when the probability of death would be greater than survival are cases of very old age and significant excess mortality due to pandemics such as HIV/AIDS. The sum at risk at maturity is illustrated in Figure 1.3.

The underlying fund value is extremely sensitive to the economic environment at inception of the single premium contract. The price at which the single contribution is unitised will determine the relative growth of the underlying fund throughout the policy term. For example, consider the same 5 year GMDB contract as in Figure 1.2 but commencing one year later, i.e. the policy term is taken from 2002 to 2006. It is evident from Figure 1.4 that the lower equity price at which the single contribution is unitised leads to less risk exposure for the insurer.

Figure 1.3: GMAB over prolonged bear market.

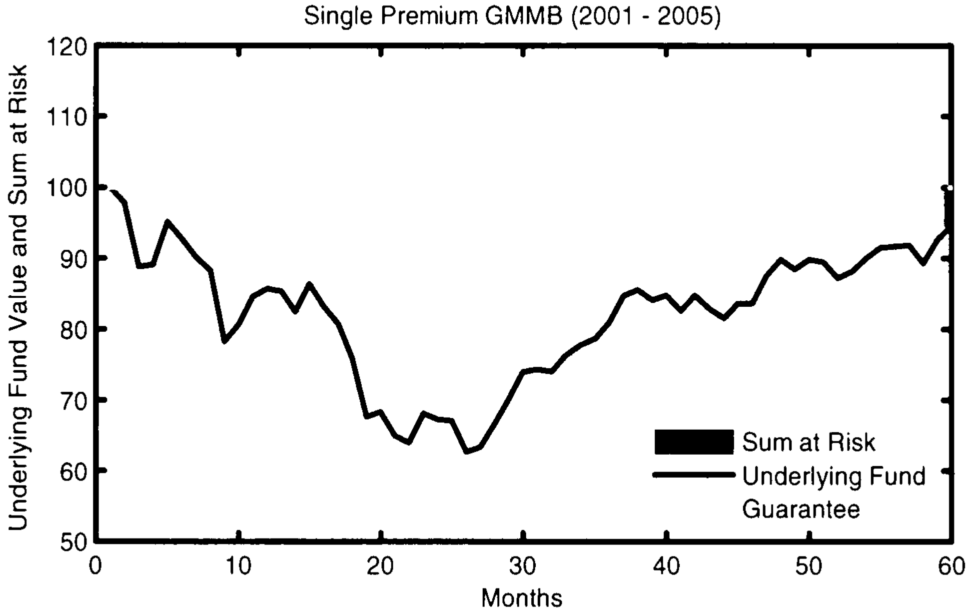


Figure 1.4: GMDB one year later.

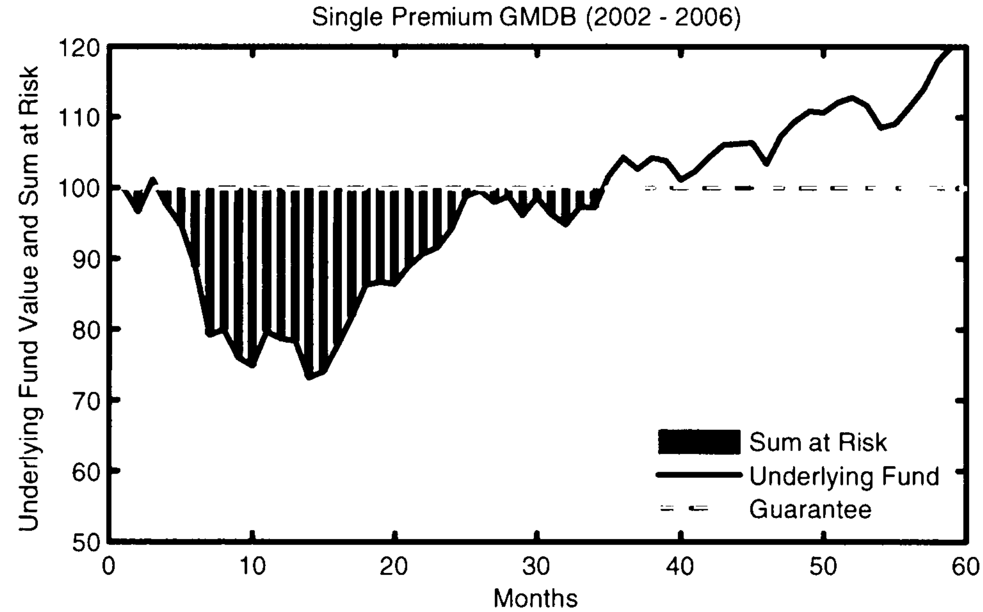
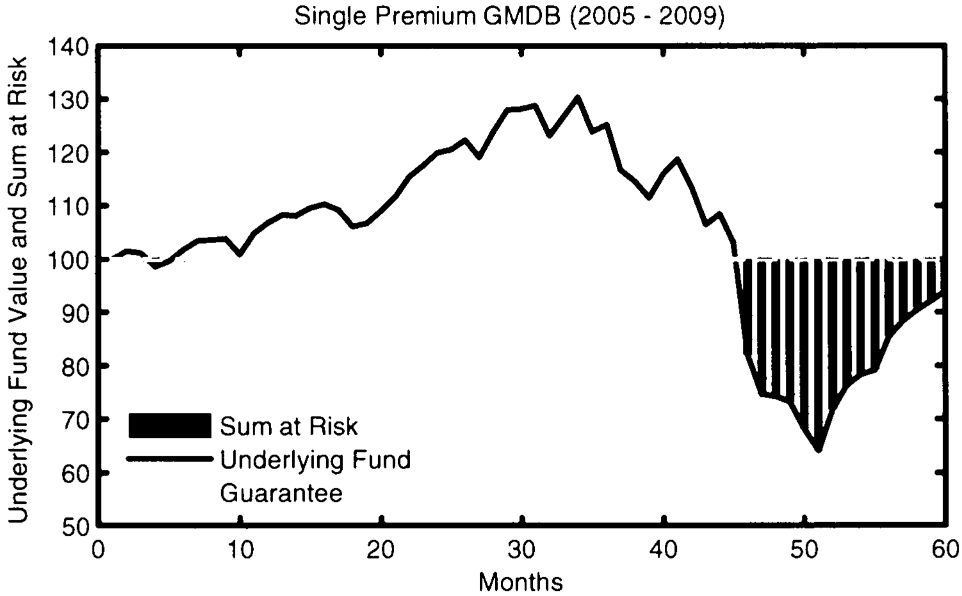


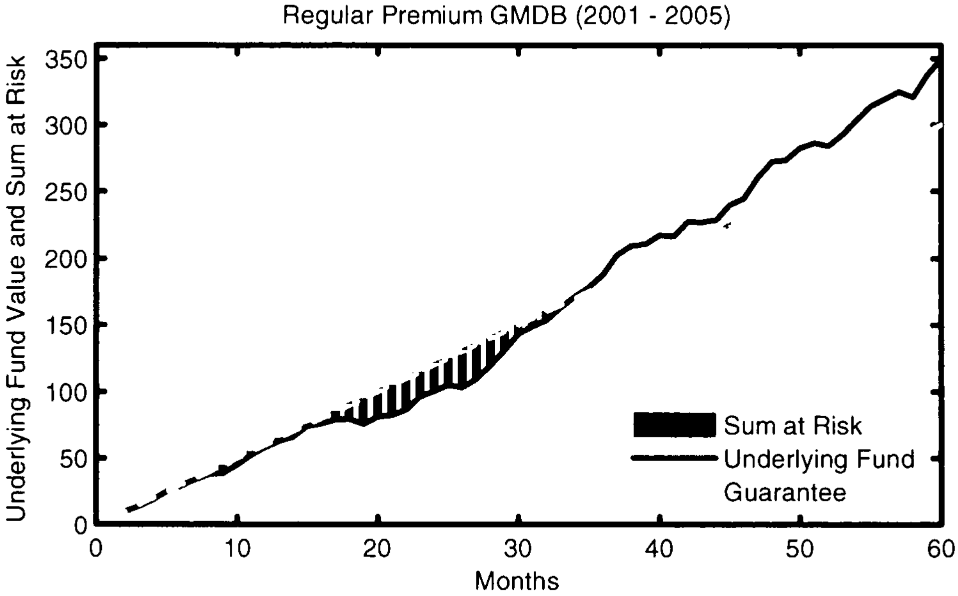
Figure 1.5: GMDB over financial crisis term.



A more recent example, taken from 2005 to 2009, of the sensitivity of the insurer's risk exposure to the price of the underlying fund at inception, i.e. the unitising price, is given in Figure 1.5. The prolonged bull market after 2005 led to significantly reduced risk exposure until the recent severe credit crisis.

In its context to VA and UL products, dollar-cost averaging (DCA) refers to the way in which contributions are made at regular intervals as opposed to a large single contribution at inception. DCA is done by phasing-in a single contribution over multiple intervals by investing a specified portion of the lump sum at each interval and leaving the remainder invested in risk-free assets. Policyholders also allow DCA by taking out a regular premium policy. Although DCA is often cited as being an inefficient strategy, Milevsky and Posner (2003) show that a DCA strategy is mathematically similar to purchasing a zero strike arithmetic Asian option on the underlying security. Policyholders who engage in DCA are implicitly purchasing a path-dependent contingent claim. The authors further show that the expected return from the DCA strategy, conditional on knowing the final value or target value of the security, uniformly exceeds the return from the underlying security for sufficiently large volatilities. In Figure 1.6 the GMDB is set in the same prolonged bear market case of Figure 1.3, but with regular contributions. It is evident that the risk exposure to the insurer is dramatically reduced with the introduction of DCA.

Figure 1.6: DCA over prolonged bear market.



The unitising base, i.e. the weighted average price at which contributions are unitised, decreases over prolonged bear markets but increases over booming periods. A DCA strategy can be implemented by allowing the phasing-in of a single premium or through regular contributions. It is evident from Figure 1.7 and Figure 1.5 that a DCA strategy does not serve as panacea to the financial risk exposure to the insurer, but that the financial risks of investment guarantees to both the insurer and policyholder can be mitigated by offering phasing-in options for single contributions and by marketing regular premium business.

In Figure 1.8, a GMWB benefit that guarantees an annual withdrawal of 20% over a 5 year policy term is illustrated. The 5 year term corresponds to the prolonged bear market considered in Figure 1.3. The underlying fund is depleted by the second last withdrawal and the insurer has to fund the shortfall of the withdrawals in the last two years of the policy. The shortfall at the end of year 4 amounts to only 0.53, while the shortfall at the end of the policy amounts to the full guaranteed withdrawal amount of 20 (20% of the initial contribution of 100). The cost of the guarantee also includes the loss of charge income due to the absorbing barrier at zero and the asset-based charging structure that the industry follows.

Figure 1.7: DCA over financial crisis term.

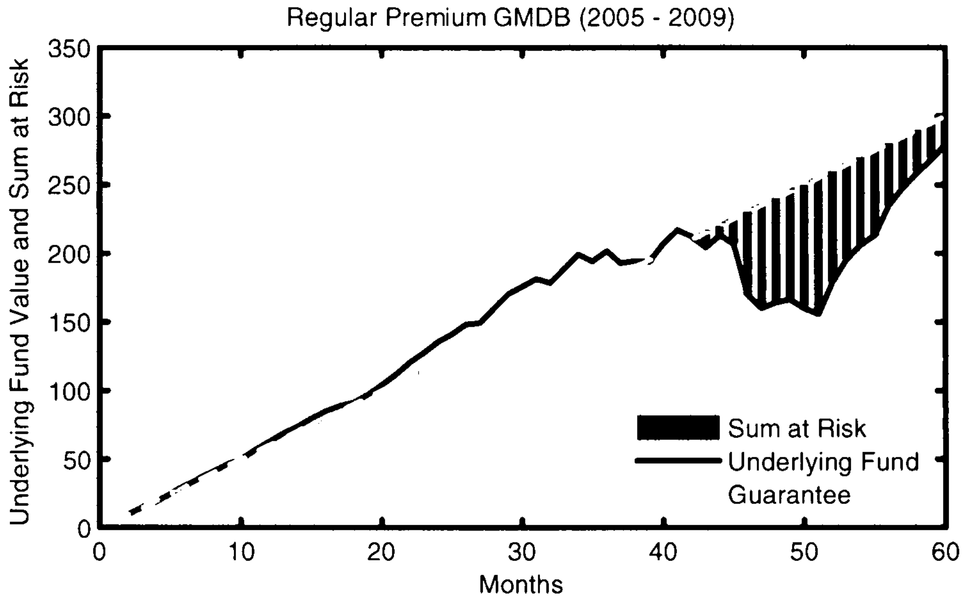
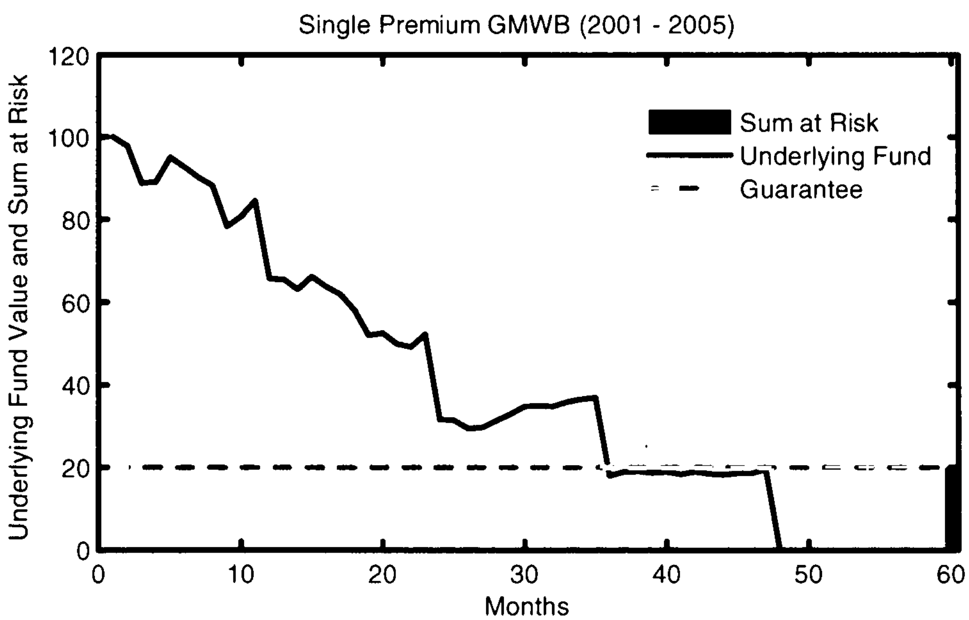


Figure 1.8: GMWB over prolonged bear market.



1.3 Objectives

Life insurers have traditionally concerned themselves with mortality risk, which is diversifiable if the insurer is able to aggregate a large number of independent insured lives. The risks inherent to investment guarantees, however, are largely dependent and require an approach to fair pricing of these guarantees and a hedging programme to effectively transfer risk to third parties or market. According to Hardy (2003), mainly three aspects of risk management specific to investment guarantees need closer attention from insurers, i.e.:

1. What price to charge for the benefit guarantee?
2. How much capital to hold in respect of the benefits through the term of the contract?
3. How to invest this capital?

The pricing of investment guarantees has since its inception received a lot of attention by actuarial practitioners and by academia. The initial attempts at pricing investment guarantees were mainly statistical in nature, e.g. Wilkie (1978) used a lognormal distribution for portfolio returns and adjusted the parameters for monthly premiums and autocorrelation. Significant improvements in the field of finance with the papers by Black and Scholes (1973) and Merton (1973) led to attempts to combine the fields of actuarial science and finance. Of the first substantial attempts were the publications by Boyle and Schwartz (1977), Brennan and Schwartz (1977) and Brennan and Schwartz (1979). The last mentioned authors, Brennan and Schwartz (1979), not only considered fair pricing of both single and recurring premium structures in a Black-Scholes-Merton framework, but also looked at a possible delta hedging strategy and the sensitivities of the hedging strategy to the assumed parameters for the risk-free rate of return and the implied volatility.

The martingale approach to risk-neutral pricing by Harrison and Kreps (1979) and Harrison and Pliska (1981) further facilitated the adoption of modern finance techniques by the actuarial profession. The standard Black-Scholes-Merton framework under the martingale approach was applied to minimum guarantees at death and maturity by Delbaen (1986) and Aase and Persson (1994). The complexity in benefits as well as in the assumptions underlying the approaches considered in literature have grown substantially. Bacinello and Ortu (1993a) consider endogenous minimum guarantees, i.e. minimum guarantees that depend functionally on the premium of the policies. Bacinello and Ortu (1993b), Nielsen and Sandmann (1995) and Nielsen and Sandmann (1996) extend existing results to include stochastic interest rate risk. Ekern and Persson (1996) discuss caps or limits on the guarantees offered as well as allowing the investment guarantee to be based on multiple underlying funds, while Hipp (1996) consider cliquet or ratchet (high-water mark) benefits. Milevsky and Posner (2001) consider a constant force of mortality for pricing the GMDB, which is referred to by the authors as a Titanic option since it has a payoff structure between an European and American option but is triggered at death.

The wider risk management of investment guarantees contingent on death or survival has also received attention. Boyle and Hardy (1997) consider two approaches to effectively manage the guarantee liability of the embedded options: (a) an actuarial approach that is similar to the Value at Risk (VaR) methodology in that a distribution of the guarantee liability is used to determine the necessary retained capital to be held in risk-free assets, and (b) a dynamic approach that aims to set up a replicating portfolio that is frequently rebalanced to match the embedded option's payoff at death or maturity. Møller (1998) considers risk-minimising strategies set in a Black-Scholes-Merton framework, while Møller (2001) proposes a more pragmatic risk-minimising strategy in a discrete Cox-Ross-Rubinstein framework. Risk-minimising strategies aim to minimise the variance of an insurer's future costs, which are defined to be the difference between the guarantee claim and the gains made from trading on the financial market. Møller and Steffensen (2007) as well as Hardy (2003) collated their respective contributions in monographs.

The GMIB or GAO benefit exhibits significant interest rate risk and is therefore considerably different to the minimum guarantees on death and survival that are more sensitive to the performance of the underlying fund. The pricing and risk management of the GMIB or GAO benefit is considered, for example, by Milevsky and Promislow (2001), Boyle and Hardy (2003), Ballotta and Haberman (2003), Ballotta and Haberman (2006), Pelsser (2003), Wilkie et al. (2003), and Biffis and Millossovich (2006).

The GMWB is influenced by changes in policyholder behaviour and adverse selection. Boyle et al. (2005) allow for policyholder behaviour by way of a multiple state model approach, while Milevsky and Salisbury (2006) consider modelling policyholder behaviour with an American option based approach. The more recent GLWB is considered by, for example, Holz et al. (2007).

We assume in the following that the policyholder contributions are exogenously given, which is largely the case in practice. The payment of regular premiums results in the payoff being dependent on the underlying asset price throughout the duration of the contract and leads to an analogy with path-dependent Asian options. A range of numerical methods have been developed to deal with the case of arithmetic Asian options for which the distribution of the sum of dependent random variables is needed, see e.g. Kemna and Vorst (1990) and Turnbull and Wakeman (1991). The distribution of the marginals are usually assumed as lognormal. Upper and lower bounds in terms of double integrals for approximately valuing Asian options have been developed by Rogers and Shi (1995) and Thompson (1999). The application of stochastic bounds to financial products in actuarial science was introduced mainly by Simon et al. (2000), Dhaene et al. (2002a,b), Nielsen and Sandmann (2003) and Schrager and Pelsser (2004). The last mentioned authors use a change of numeraire technique to derive a general pricing formula in the case of stochastic interest rates in the lognormal case for rate of return guarantees in regular premium business. Hürlimann (2008) considers GMDB and GMAB guarantees in regular premium UL business in the lognormal case with a two-factor fund diffusion process, consisting of a one-factor fund price and a one-factor stochastic interest rate model with deterministic bond price volatilities. He considers the call-type option representation of the guaranteed benefit and determined bounds for the premium payable by the policyholder. The contribution in VA business is typically specified by the policyholder, although rider risk benefits are mostly charged for explicitly through risk premiums before investment of the contribution. The investment guarantee charges are then levied from the policyholder's underlying fund in the form of asset-based charges, i.e. the charges are expressed as a percentage of the value of the underlying fund or sub-account. Nielsen et al. (2010) discuss return guarantees on defined contribution equity-linked pension schemes where investors pay periodic contributions, typically over a long period. A proportion of the periodic premium serves to finance the return guarantee. The authors consider three types of pension scheme where each scheme involves pricing an Asian option by means of tight upper and lower bounds. Bekker and Dhaene (2010) consider pricing, through an asset-based charging structure, and hedging of investment guarantees on death and survival in VA products by using a conditional lower bound approximation.

In the following sections, we derive the conditional lower bound approximation for the value of the different types of embedded option implied by investment guarantees. The approximate value of an embedded option is then used to derive the asset-based investment guarantee charge.

We consider regular premium VA business and investment guarantee rider benefits that are contingent on death and survival, i.e. mainly the GMDB and GMAB / GMMB investment guarantee types. We further show how the conditional lower bound approximation can be used to validate sensitivity measures, the so-called Greeks, for hedging. We assume a Black-Scholes-Merton setting throughout, although it is important to note that the conditional lower bound is a versatile approach and can be determined in a model dependent or model independent case. An example of the model independent case is the recent paper by Chen et al. (2008) in which the authors investigate static super-replicating strategies for European-type call options written on a weighted sum of asset prices. For complex models or in some model independent cases, the conditional lower bound might result in an approximation that is not analytically tractable. In such cases and where a point estimate suffices, numeric solutions can be used.

One of the many possible applications of the conditional lower bound approximation is to assist firms in reasonableness checks of their periodic pricing and valuation of investment guarantees. Most firms make use of proprietary Economic Scenario Generators (ESGs) to allow for market consistent pricing and valuation of investment guarantees. According to a PriceWaterhouseCoopers survey performed in 2008 among South African life insurers to determine the approaches to regulatory market consistent valuation of embedded investment derivatives, five out of the six participants used the Barrie and Hibbert proprietary ESG compared to one participant who used the local Maitland model (PriceWaterhouseCoopers, 2008). The survey also mentions examples of ESGs that the remainder of South African firms employ, i.e. the Wilkie, Thompson and Smith models. ESGs are integrated models that use either a risk-neutral approach or a state price deflator approach to model projected asset prices and key economic indicators, e.g. inflation. Some insurers purchase only sets of simulated scenario results from consulting firms due to the extensive hardware requirements imposed by ESGs. ESG models can also be used in an asset liability modelling (ALM) exercise to determine estimates of the embedded option values for each scenario. Firms need to perform multiple nested stochastic simulations and several additional simulations for each nested run to minimise the standard error of the estimate. Although regulatory and / or shareholder requirements typically demand at least one full ALM exercise per risk period, the nested embedded option values can be substituted by their conditional lower bound approximations to save time on interim or sensitivity ALM studies. The freed time and resources can then be applied to rather testing sensitivities and hedging programmes.

Investment guarantee offerings pose several challenges to insurance firms. The guarantees of individual policies are largely dependent, although the diversification benefit arising from the various underlying fund choices offsets the dependency to some extent. The guarantee charges are mostly asset-based in practice. Insurers will carry the initial cost of setting up a hedging portfolio or reserving requirement at inception and then recoup the cost over the policy term. The obvious risk implied by this strategy is that the policyholder's fund decreases and so with the charge income. The income at risk poses an additional cost in the price of the guarantee. The complexity of the embedded options of investment guarantees adds further challenges to the risk management programmes of insurers. The embedded options take the form of exotic path-dependent derivatives and require multiple simulations to be valued. As explained in the previous paragraph, these simulations become nested in larger valuation exercises such as an ALM study. Finally, the combined market of financial risk and biometric risk in which these guarantees are offered do not naturally allow assumptions that hold in the real world to also hold in the risk-neutral world. This problem is further discussed in section 3.2.

The conditional lower bound is derived in Chapter 2 for a simplified financial product where the benefit considered is not contingent on death or survival. In Chapter 3, we show that the conditional lower bound theory easily extends to more sophisticated benefits and costing structures by adding typical costs as well as mortality to the simplified financial product considered in Chapter 2. The popular GMDB and GMAB / GMMB structures are covered in Chapter 3. In Chapter 4, we propose a possible dynamic hedging solution for the aforementioned structures. We conclude in Chapter 5.

Chapter 2

A SIMPLIFIED PRODUCT

We first consider a simplified financial contract for ease of exposition. Consider an investor who invests a regular contribution π_k, $k = 0, 1, ..., n$,with a firm. In exchange for these contributions, the firm guarantees the investor the greater of a guaranteed minimum benefit of b_n and the fund value V_n^- at maturity of the contract, i.e. the benefit payout at time n is $B_n = \max(b_n, V_n^-)$. We assume that the investor, aged x at inception, survives the contract with probability 1, i.e. ${}_np_x = 1$, and we do not allow for other decrements such as surrenders. Let F_k denote the value at time k of one unit of the fund in which the contributions are invested by the firm. The value of the investor's portfolio V_{k+1}^- at the end of the period $(k, k+1)$, before the next period's contribution, is given by the recursive formula:

$$V_{k+1}^- = [V_k + \pi_k] \frac{F_{k+1}}{F_k} \tag{2.1}$$

where V_k is used to denote the fund value at time k, i.e. at the the beginning of the year $(k, k+1)$, after payment of a fund management charge, which will be introduced in section 2.6.

Equation (2.1) states that the value at the end of a period during the contract term is equal to the regular contribution added to the existing value of the fund and then accumulated to the end of the period by the fund growth factor $\frac{F_{k+1}}{F_k}$. We assume $V_0^- = 0$ by construction. Under the assumption that the investor survives each period and that all contributions have been paid, we can express equation (2.1) as:

$$V_{k+1}^- = \sum_{j=0}^{k} \pi_j \frac{F_{k+1}}{F_j} , \quad k = 0, ..., n-1 \tag{2.2}$$

The minimum guaranteed benefit at maturity can be written as follows:

$$B_n = b_n + \max\left(0, V_n^- - b_n\right) \tag{2.3}$$

or:

$$B_n = V_n^- + \max\left(b_n - V_n^-, 0\right) \tag{2.4}$$

Equation (2.3) describes the guaranteed benefit as the value of the guarantee as redemption value of a zero-coupon bond plus a call option on the fund value at maturity with a strike price equal to the value of the guarantee. Equation (2.4), on the other hand, describes the guaranteed benefit as the fund value at maturity plus a put option on the fund value at maturity with a strike price equal to the value of the guarantee. The expressions in equations (2.3) and (2.4) naturally allow for the use of financial economics in the pricing of the guaranteed benefit and option-based risk management strategies. The call-type payoff allows the insurer to hold a zero-coupon bond with maturity equal to that of the contract and the call option, whereas the put-type payoff allows the insurer to supplement the policyholder's sub-account with a put option.

The concept of comonotonicity, which provides the theoretical framework for the conditional lower bound approximation, is briefly reviewed in section 2.1. The assumptions to the underlying asset process and therefore to the sub-account process, and the embedded option expressions are discussed in section 2.2. The complexity problem of the embedded option taking the mathematical form of an arithmetic Asian option is outlined in section 2.3. The conditional lower bound approximation of the simple financial product is derived in section 2.4, and the value of the embedded option is determined in section 2.5. The value of the embedded option is the price that the market requires at time 0 for accepting the risk, whereas the price of the embedded option to the policyholder is typically an annual charge deductible from the sub-account. This charge is determined in section 2.6. In section 2.7 a numerical example illustrates the efficacy of the conditional lower bound approximation in comparison to simulated results under the same Black-Scholes-Merton framework.

2.1 The Concept of Comonotonicity

In the following, we give an overview of some of the main results related to the concept of comonotonicity or "common monotonicity". This section draws on the recent overview of the concept of comonotonicity by Deelstra et al. (2010) and presents the main results needed for the derivation and optimisation of the conditional lower bound of Kaas et al. (2000) that stems from mathematical physics. This bound acts as a convex order lower bound to the true distribution of the policyholder's underlying fund value. The conditional lower bound has the convenient property that it is a comonotonic sum of the conditional random vector $\underline{X}\,|\Lambda$, provided that all conditional terms are increasing or decreasing functions of the conditional random variable Λ. In the case of comonotonicity, the quantiles and stop-loss premiums (option-type payoff functions) of the conditional lower bound follow immediately from the additivity properties of comonotonic sums. We first describe the concept of comonotonicity and its additivity properties. Thereafter, the conditional lower bound, in the convex order sense, is introduced with reference to the relevant additivity properties of comonotonic vectors and inverse distribution functions.

The concept of comonotonicity can be described by the well-known upper bound by Hoeffding (1940) and Fréchet (1951), which is also known as the Fréchet-Hoeffding copula upper limit. For any n-dimensional random vector $\underline{X} = (X_1, X_2, ..., X_n)$ with multivariate cumulative distribution function (cdf) $F_{\underline{X}}(\underline{x})$ and marginal univariate cdf's $F_{X_1}, F_{X_2}, ..., F_{X_n}$ and for any $\underline{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, it holds that:

$$F_{\underline{X}}(\underline{x}) \leq \min \left\{ F_{X_1}(x_1), F_{X_2}(x_2), ..., F_{X_n}(x_n) \right\} \tag{2.5}$$

The upper bound in equation (2.5) is reachable in the case of a very strong dependency structure such as in the case of comonotonicity, i.e. a random vector is said to be comonotonic if and only if:

$$F_{\underline{X}}(\underline{x}) = \min \left\{ F_{X_1}(x_1), F_{X_2}(x_2), ..., F_{X_n}(x_n) \right\} \tag{2.6}$$

for any $\underline{x} \in \mathbb{R}^n$.

Equivalent definitions of a comonotonic vector are given in Dhaene et al. (2002b). In addition to the definition in equation (2.6), a random vector $\underline{X} = (X_1, X_2, ..., X_n)$ is comonotonic if and only if one of the following equivalent conditions holds:

1. $\underline{X}$ has a comonotonic support. This results from the fact that the elements of the support are ordered componentwise.

2. For $U \sim$Uniform$(0,1)$, we have:

$$\underline{X} = \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U) \right) \tag{2.7}$$

3. There exists a random variable Z and non-decreasing functions f_i, $i = 1, 2, ..., n$, such that:

$$\underline{X} \stackrel{d}{=} \left(f_1(Z), f_2(Z), ..., f_n(Z) \right) \tag{2.8}$$

where the notation $\stackrel{d}{=}$ denotes equality in distribution.

Consider a random vector $\underline{X} = (X_1, X_2, ..., X_n)$ and a comonotonic vector $\underline{X}^c$ with the same marginals as $\underline{X}$, which is also known as the comonotonic counterpart of $\underline{X}$. The sum of the components of $\underline{X}^c = (X_1^c, X_2^c, ..., X_n^c)$ is denoted by S^c, where:

$$S^c = X_1^c + X_2^c + ... + X_n^c \tag{2.9}$$

In the case where the marginal distributions follow a well defined distribution such as the lognormal distribution, the usual non-decreasing and left-continuous property applies and the definition of the inverse of a distribution function $F_{S^c}^{-1}(p)$ is given by:

$$F_{S^c}^{-1}(p) = \inf\{s \in \mathbb{R} \,|F_{S^c}(s) \geq p\} \tag{2.10}$$

For any real-valued random variables X and $g(X)$ and for the case when g is a non-decreasing and left-continuous function, we have:

$$F_{g(X)}^{-1}(p) = g\left(F_X^{-1}(p)\right) \tag{2.11}$$

The result in equation (2.11) can be used to show that for a comonotonic sum S^c of the vector $\underline{X}^c = (X_1^c, X_2^c, ..., X_n^c)$, the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions, i.e.:

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p) \tag{2.12}$$

where $p \in [0, 1]$.

The distribution of the comonotonic sum S^c can also be given in terms of the stop-loss premiums of the components of the comonotonic random vector $\underline{X}^c = (X_1^c, X_2^c, ..., X_n^c)$. Dhaene et al. (2000) showed that the following additivity property holds:

$$E\left[(S^c - d)_+\right] = \sum_{i=1}^{n} E\left[(X_i - d_i)_+\right] \tag{2.13}$$

where $F_{S^c}^{-1}(0) \leq d \leq F_{S^c}^{-1}(1)$. The d_i, for $i = 1, 2, ..., n$, are given by:

$$d_i = F_{X_i}^{-1(\alpha_d)}(F_{S^c}(d)) \tag{2.14}$$

where $\alpha_d \in [0, 1]$ is determined by:

$$F_{S^c}^{-1(\alpha_d)}(F_{S^c}(d)) = d \tag{2.15}$$

The inverse distribution function of the comonotonic sum S^c determined by the value of α_d is given by:

$$F_{S^c}^{-1(\alpha_d)}(p) = \alpha_d F_{S^c}^{-1}(p) + (1-p)\,\alpha_d F_{S^c}^{-1+}(p) \tag{2.16}$$

where $F_{S^c}^{-1+}(p)$ is the cádlág (right-continuous and limited on the left) inverse:

$$F_{S^c}^{-1+}(p) = \sup\{s \in \mathbb{R}\,|F_{S^c}(s) \leq p\} \tag{2.17}$$

For the lognormal case, the inverse distribution function is left-continuous and strictly monotonic and the definition in equation (2.10) applies, i.e. the value of α_d in equation (2.15) is equal to one. The financial interpretation of the decomposition formula in equation (2.13) is that the value of an European call option on a stochastic sum S^c of share prices with strike price d is equal to the sum of the values of European call options on the constituent shares with strike prices d_i as determined in equation (2.14). In fact, Jamshidian (1989) proved that in the case of one-factor mean-reverting Gaussian interest rate models, such as the Vasicek (1977) model, a European option on a portfolio of discount coupon-bearing bond decomposes into a portfolio of European options on the individual discount bonds in the portfolio. In the Vasicek model, the prices of all discount bonds at some future time T are decreasing functions of a single random variable, namely the spot rate at that time.

Separate to the Fréchet-Hoeffding bounds on the multivariate cdf of any random vector $\underline{X} = (X_1, X_2, ..., X_n)$, stochastic order bounds have been derived for random sums $S = X_1 + X_2 + ... + X_n$ where the cdf's of the X_i are known but where the joint distribution of the random vector $\underline{X}$ is either unspecified or too cumbersome to work with. Assuming that only the marginal distributions of the random variables are given, the largest sum in convex order occurs when the random variables are comonotonic. This property is used to determine the conditional lower bound in a convex order for the value of the embedded investment guarantee option.

A random variable X is said to precede Y in the convex order sense, written as $X \leq_{cx} Y$, if and only if:

$$E[X] = E[Y] \tag{2.18}$$

and:

$$E\left[(X-d)_+\right] \leq E\left[(Y-d)_+\right] \tag{2.19}$$

or equivalently:

$$E\left[(d-X)_+\right] \leq E\left[(d-Y)_+\right] \tag{2.20}$$

for $d \in \mathbb{R}$. Note that it is sufficient for only the conditions in equation (2.19) and equation (2.20) to hold, since the condition in equation (2.18) is implied should the previous two conditions hold. The relation $X \leq_{cx} Y$ implies that the variance of X is less than the variance of Y and that, compared to the random variable X, the random variable Y has more probability mass in its lower as well as its upper tails.

The conditional lower bound follows from the observation that for any random vector $\underline{X}$ and any random variable Λ, we have:

$$S^l = E\left[X_1 \,|\Lambda\right] + E\left[X_2 \,|\Lambda\right] + \ldots + E\left[X_n \,|\Lambda\right] \leq_{cx} X_1 + X_2 + \ldots + X_n \tag{2.21}$$

or:

$$S^l = E\left[S \,|\Lambda\right] \leq_{cx} S \tag{2.22}$$

In the following, the lower bound in equation (2.22) is applied to the underlying fund value of a typical VA and UL contract as the conditional lower bound. This bound can be optimised by an adequate choice of the conditional random variable Λ. A range of possible optimisation techniques can be applied that consist mainly of globally optimal and locally optimal choices. One way of globally optimising the lower bound is by maximising the first order approximation of the variance of the lower bound with respect to the choice of the conditional random variable Λ. This approach is taken in the following sections. The tail of the distribution of S is often of interest in an actuarial or financial context. For such cases, Vanduffel et al. (2008) propose locally optimal approximations in the sense that the relevant tail of the distribution of $E\left[S \,|\Lambda\right]$ is an accurate approximation for the corresponding tail of the distribution of S.

The lower bound in equation (2.22) is applied in Dhaene et al. (2002b) to derive accurate approximations for European type arithmetic Asian options in the Black-Scholes-Merton setting. In the following, the mathematical similarity between arithmetic Asian options and the investment guarantees found in life insurance is illustrated as well as the consequent application of the conditional lower bound in equation (2.22) to pricing, reserving and hedging of investment guarantees.

2.2 Pricing the Guaranteed Benefit

A first step in pricing the guaranteed benefit is to determine an estimate of the distribution of the fund value. We assume that the parameters of the contract $(b_n,\ \pi_k)$ are fixed at inception of the contract, i.e. these parameters are treated as deterministic. We also assume a Black-Scholes-Merton setting, i.e. the investment fund price process $\{F(t),\ t \geq 0\}$ evolves according to a geometric Brownian motion process with constant drift and risk-free rate δ and constant volatility σ:

$$\frac{dF(t)}{F(t)} = \delta dt + \sigma dW(t), \quad t \geq 0 \tag{2.23}$$

with initial value $F(0) > 0$, and where $\{W(t),\ t > 0\}$ is a standard Brownian motion. Under a unique equivalent martingale measure Q, see e.g Harrison and Pliska (1981), we have:

$$F(t) = F(0) \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right\}, \quad t \geq 0 \tag{2.24}$$

The construction in equation (2.24) above implies that the random variables $\log\left(\frac{F(t)}{F(0)}\right)$, $t \geq 0$, follow a normal distribution with mean $\left(\delta - \frac{\sigma^2}{2}\right) t$ and variance $t\sigma^2$. By combining the results in equation (2.2) and in equation (2.24), we find the following expression for the fund value V_{k+1}^-:

$$V_{k+1}^- = \sum_{j=0}^{k} \pi_j \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right)(k+1-j) + \sigma\left(W(k+1) - W(j)\right)\right\} \tag{2.25}$$

The price of the put-type option, P_t, is given by the expectation of the risk-free discounted payoff of this option, conditional on all information of the process up to and including time t, i.e.:

$$P_t = e^{-\delta(n-t)} E_Q\left[\left(b_n - V_n^-\right)_+ \mid \mathcal{F}_t\right] \tag{2.26}$$

Under the equivalent martingale measure Q, we now have:

$$V_n^- = \sum_{j=0}^{n-1} \pi_j \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right)(n-j) + \sigma\left(W(n) - W(j)\right)\right\} \tag{2.27}$$

By combining equations (2.26) and (2.27), we find that the price of the option is given by:

$$P_t = e^{-\delta(n-t)} E_Q\left[\left(b_n - \sum_{j=0}^{n-1} \pi_j \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right)(n-j) + \sigma\left(W(n) - W(j)\right)\right\}\right)_+ \middle| \mathcal{F}_t\right] \tag{2.28}$$

For the call-type payoff defined in equation (2.3), we have:

$$C_t = e^{-\delta(n-t)} E_Q \left[\left(\sum_{j=0}^{n-1} \pi_j \exp\left\{ \left(\delta - \frac{\sigma^2}{2}\right)(n-j) + \sigma\left(W(n) - W(j)\right) \right\} - b_n \right)_+ \middle| \mathcal{F}_t \right] \quad (2.29)$$

In the following sections, we first derive an accurate approximation of the distribution function of the fund value V_n^-, which is then used to derive an accurate approximation of the option values P_t and C_t. Unless otherwise specified, the expected value operator in the following refers to the expectation taken with respect to the risk-neutral probability measure Q.

2.3 Embedded Asian Option

The price of the embedded guarantee at maturity is in the form of a European put-type option payoff or in the form of a European call-type option, i.e. we have:

$$B_n = V_n^- + P_n = b_n + C_n \quad (2.30)$$

where P_n represents the payoff of the put option at maturity and C_n represents the payoff of the call option at maturity.

The put-type and call-type options are arithmetic sums of dependent random variables and are therefore path dependent. This path dependency takes the same mathematical form of an arithmetic Asian option. Schrager and Pelsser (2004), for example, showed that P_n is equal in distribution to the risk-neutral expectation of an arithmetic Asian option within the Black-Scholes-Merton framework, i.e. under constant volatility and a constant force of interest the embedded option is an Asian option that runs in an opposite direction in time, i.e. starts at maturity and expires at zero. The representation of the guaranteed benefit by an arithmetic Asian option is illustrated in the Black-Scholes-Merton setting below by applying the time reversal property of Brownian motion. This representation follows roughly the same steps as the proof of Asian option duality of dollar-cost averaging by Milevsky and Posner (2003). In a Black-Scholes-Merton framework, the fund value is given by:

$$\begin{aligned} V_n^- &= \sum_{k=0}^{n-1} \pi_k \frac{F_n}{F_k} \\ &= \sum_{k=0}^{n-1} \pi_k \exp\left\{ \left(\delta - \frac{\sigma^2}{2}\right)(n-k) + \sigma\left(W(n) - W(k)\right) \right\} \end{aligned} \quad (2.31)$$

The time reversal property of Brownian motion allows us to state a new process that maintains the exact same probabilistic structure as the old process, i.e.

$$\tilde{W}(\tau) = W(n) - W(k) \tag{2.32}$$

where $\tau = n - k$. By substituting the time reversed Brownian motion of equation (2.32) in equation (2.31), we find:

$$\begin{aligned} V_n^- &= \sum_{\tau=0}^{n-1} \pi_\tau \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right)\tau + \sigma\tilde{W}(\tau)\right\} \\ &= \sum_{\tau=0}^{n-1} \pi_\tau F_\tau \end{aligned} \tag{2.33}$$

Assume now that the contributions are level across the policy term so that we have:

$$\begin{aligned} \pi_k &= \frac{1}{n}\sum_{j=0}^{n-1} \pi_j \\ &= \frac{1}{n}\Pi \end{aligned} \tag{2.34}$$

where $k = 0, 1, ..., n-1$ and Π denotes the total premium paid over the policy term. Combining the results of equation (2.33) and equation (2.34), we have:

$$V_n^- = \Pi \sum_{\tau=0}^{n-1} \frac{1}{n} F_\tau \tag{2.35}$$

The result in equation (2.35) above implies that any contingent claim on the policyholder's fund will take the form of a contingent claim on the arithmetic average of the underlying fund prices across the policy term, i.e. take the mathematical form of an arithmetic Asian option on the underlying fund.

The values of Asian options do not have analytical solutions and various approximations are available in the form of numerical methods or double integral type bounds. Dhaene et al. (2002a) derived comonotonic bounds for an arithmetic Asian option and illustrated the efficacy of these bounds in a Black-Scholes-Merton setting, while Rogers and Shi (1995) derived a conditional lower bound in the continuous averaging case. Dhaene et al. (2002a) demonstrated, in particular, the incredible accuracy of the conditional lower bound as an approximation to the exact value of an arithmetic Asian option. The conditional lower bound aims to approximate the distribution of a sum of random variables of which the dependency structure is unknown with a conditional expectation, i.e. we derive $V_n^{-(l)}$ such that:

$$V_n^{-(l)} \leq_{cx} V_n^- \tag{2.36}$$

where the operator $\leq_{cx}$ indicates that V_n^- stochastically dominates $V_n^{-(l)}$ in the convex stochastic order. The conditional lower bound is defined to be the expection of V_n^- conditioned on some information variable Λ, i.e.:

$$V_n^{-(l)} = E\left[V_n^- \mid \Lambda\right] \tag{2.37}$$

Ideally, the choice of Λ is such that V_n^- and Λ are as alike as possible. This choice reduces to opting for a significant level of dependence between V_n^- and Λ. In our case, we initially choose Λ such that it is a linear combination of the variability of V_n^-, i.e.:

$$\Lambda = \sum_{k=0}^{n-1} \gamma_k \left[W(n) - W(k)\right] \tag{2.38}$$

where γ_k is a deterministic constant.

This initial choice can be optimised by attempting to mimic as much of V_n^- by Λ. This optimisation turns out to be a matter of maximising the variance of $E\left[V_n^- \mid \Lambda\right]$ with respect to Λ. To verify this rule, we first note that the use of the tower property of conditional expectations results in:

$$E\left[V_n^-\right] = E\left[E\left[V_n^- \mid \Lambda\right]\right] = E\left[V_n^{-(l)}\right] \tag{2.39}$$

Equation (2.39) implies that under our normality assumption, which enables the respective distributions to be uniquely specified by the mean and variance alone, we find that the only difference between our approximation and the actual value of the fund is explained by the variance, more specifically:

$$\begin{aligned} Var\left[V_n^-\right] &= E\left[Var\left[V_n^- \mid \Lambda\right]\right] + Var\left[E\left[V_n^- \mid \Lambda\right]\right] \\ &> Var\left[V_n^{-(l)}\right] \end{aligned} \tag{2.40}$$

Equation (2.40) shows that to find the variances of V_n^- and $V_n^{-(l)}$ as alike as possible, we have to maximise $Var\left[E\left[V_n^- \mid \Lambda\right]\right]$ with respect to Λ. In the following section, we derive the conditional lower bound $V_n^{-(l)}$ of V_n^- such that we have an approximate distribution of V_n^- under the equivalent martingale measure Q. This approximate distribution then allows us to price the discounted payoff of the embedded option of the guaranteed benefit.

2.4 Conditional Lower Bound

Recall that the value of the fund at time n is given as:

$$V_n^- = \sum_{k=0}^{n-1} \pi_k \frac{F(n)}{F(k)} = \sum_{k=0}^{n-1} \pi_k e^{(\delta - \frac{1}{2}\sigma^2)(n-k) + \sigma(W(n) - W(k))} \tag{2.41}$$

As discussed in section 2.3, the conditional lower bound is based on the expectation of the fund value conditional on the information variable Λ. The random variables of interest in both V_n^- and Λ are the dependent Brownian motion differences:

$$\{W(n) - W(0), W(n) - W(1), ..., W(n) - W(n-1)\}$$

Each difference variable $Y_k = W(n) - W(k)$, for $k = 0, 1, ..., n-1$, follows a normal distribution. This implies that the distribution of any Brownian difference Y_k given Λ follows a conditional bivariate normal distribution, which means that we have:

$$E[Y_k \mid \Lambda = \lambda] = E[Y_k] + r_k \frac{\sigma_{Y_k}}{\sigma_\Lambda} (\lambda - E[\Lambda]) \tag{2.42}$$

and:

$$Var[Y_k \mid \Lambda = \lambda] = \sigma_{Y_k}^2 \left(1 - r_k^2\right) \tag{2.43}$$

where r_k is Pearson's correlation coefficient for the couple (Y_k, Λ).

Since $Y_k \sim N(0, n-k)$, equations (2.42) and (2.43) reduce to:

$$E[Y_k \mid \Lambda = \lambda] = r_k \frac{\sqrt{n-k}}{\sigma_\Lambda} \lambda \tag{2.44}$$

and:

$$Var[Y_k \mid \Lambda = \lambda] = (n-k)\left(1 - r_k^2\right) \tag{2.45}$$

We have now specified the conditional distribution of Y_k given Λ. Our main goal, however, is to find an approximate distribution for the path dependent sum V_n^- such that:

$$V_n^{-(l)} = \sum_{k=0}^{n-1} \pi_k E\left[\frac{F(n)}{F(k)} \mid \Lambda\right] \leq_{cx} V_n^- = \sum_{k=0}^{n-1} \pi_k \frac{F(n)}{F(k)} \tag{2.46}$$

Consider the conditional marginal distributions of the stochastic fund growth $\frac{F(n)}{F(k)}, k = 0, ..., n-1$. Using the fact that the log of the fund growth is a function of Y_k and therefore has a conditional bivariate normal distribution, we find from equation (2.41), equation (2.44) and equation (2.45) that:

$$
\begin{aligned}
E\left[\frac{F(n)}{F(k)} \mid \Lambda=\lambda\right] &= E\left[e^{\left(\delta-\frac{1}{2}\sigma^2\right)(n-k)+\sigma(W(n)-W(k))} \mid \Lambda=\lambda\right] \\
&= e^{\left(\delta-\frac{1}{2}\sigma^2\right)(n-k)} E\left[e^{\sigma Y_k} \mid \Lambda=\lambda\right] \\
&= e^{\left(\delta-\frac{1}{2}\sigma^2\right)(n-k)} e^{r_k \frac{\sigma\sqrt{n-k}}{\sigma_\Lambda}\lambda+\frac{1}{2}\sigma^2(n-k)\left(1-r_k^2\right)} \\
&= e^{\left(\delta-\frac{1}{2}\sigma^2 r_k^2\right)(n-k)+r_k\frac{\sigma\sqrt{n-k}}{\sigma_\Lambda}\lambda} \qquad (2.47)
\end{aligned}
$$

Since we have that $\Lambda \sim N(0,\sigma_\Lambda^2)$, we can use the standard normal random variable $Z=\frac{\Lambda}{\sigma_\Lambda}$ such that:

$$
E\left[\frac{F(n)}{F(k)} \mid \Lambda\right] = e^{\left(\delta-\frac{1}{2}\sigma^2 r_k^2\right)(n-k)+\sigma r_k\sqrt{n-k}Z} \qquad (2.48)
$$

The result obtained in equation (2.48) now allows an expression for the conditional lower bound:

$$
\begin{aligned}
V_n^{-(l)} &= \sum_{k=0}^{n-1} \pi_k E\left[\frac{F(n)}{F(k)} \mid \Lambda\right] \\
&= \sum_{k=0}^{n-1} \pi_k e^{\left(\delta-\frac{1}{2}\sigma^2 r_k^2\right)(n-k)+\sigma r_k\sqrt{n-k}Z} \qquad (2.49)
\end{aligned}
$$

The expression in equation (2.49) is not an optimal bound as discussed in section 2.3. We still need to maximise the variance of $V_n^{-(l)}$ with respect to Λ. First, we consider the correlation coefficient r_k. In the following, we derive an explicit expression for the (non-optimal) correlation coefficient r_k as used in equation (2.49) . It turns out that the optimal correlation coefficient has a similar form.

The Pearson's correlation coefficient is defined as:

$$
r_k = \frac{Cov\left[Y_k,\Lambda\right]}{\sigma_{Y_k}\sigma_\Lambda}
$$

The covariance of the pair (Y_k, Λ) is given by:

$$
\begin{aligned}
&Cov\left[Y_k, \Lambda\right] \\
&= Cov\left[W(n) - W(k),\ \sum_{l=0}^{n-1} \gamma_l W(n) - W(l)\right] \\
&= \sum_{l=0}^{n-1} \gamma_l Cov\left[W(n) - W(k),\ W(n) - W(l)\right] \\
&= \sum_{l=0}^{n-1} \gamma_l \min(n-k, n-l)
\end{aligned} \tag{2.50}
$$

The variance of Λ can be found in a similar way, i.e.:

$$
\begin{aligned}
Var[\Lambda] &= Cov\left[\Lambda, \Lambda\right] \\
&= Cov\left[\sum_{j=0}^{n-1} \gamma_j (W(n) - W(j)),\ \sum_{l=0}^{n-1} \gamma_l (W(n) - W(l))\right] \\
&= \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \gamma_j \gamma_l Cov[W(n) - W(j),\ W(n) - W(l)] \\
&= \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \gamma_j \gamma_l \min(n-j, n-l)
\end{aligned} \tag{2.51}
$$

From equation (2.50) and equation (2.51) we now have:

$$
r_k = \frac{\sum_{l=0}^{n-1} \gamma_l \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \gamma_j \gamma_l \min(n-j, n-l)}} \tag{2.52}
$$

We optimise our choice of Λ by maximising the variance of the conditional lower bound $V_n^{-(l)}$ with respect to Λ, where the variance of $V_n^{-(l)}$ is given as:

$$
\begin{aligned}
&Var\left[V_n^{-(l)}\right] \\
&= Cov\left[\sum_{k=0}^{n-1} \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z},\ \sum_{l=0}^{n-1} \pi_l e^{\left(\delta - \frac{1}{2}\sigma^2 r_l^2\right)(n-l) + \sigma r_l \sqrt{n-l} Z}\right] \\
&= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \pi_k \pi_l e^{\delta(2n-k-l) - \frac{1}{2}\sigma^2\left(r_k^2(n-k) + r_l^2(n-l)\right)} Cov\left[e^{\sigma r_k \sqrt{n-k} Z}, e^{\sigma r_l \sqrt{n-l} Z}\right]
\end{aligned} \tag{2.53}
$$

Evaluating the covariance term on the left, we find:

$$\begin{aligned}
Cov\left[e^{\sigma r_k\sqrt{n-k}Z}, e^{\sigma r_l\sqrt{n-l}Z}\right] &= E\left[e^{\sigma\left[r_k\sqrt{n-k}+r_l\sqrt{n-l}\right]Z}\right] - E\left[e^{\sigma r_k\sqrt{n-k}Z}\right]E\left[e^{\sigma r_l\sqrt{n-l}Z}\right] \\
&= e^{\left[\frac{1}{2}\sigma^2\left(r_k\sqrt{n-k}+r_l\sqrt{n-l}\right)^2\right]} - e^{\left[\frac{1}{2}\sigma^2 r_k^2(n-k)\right]}e^{\left[\frac{1}{2}\sigma^2 r_l^2(n-l)\right]} \\
&= e^{\left[\frac{1}{2}\sigma^2\left(r_k^2(n-k)+r_l^2(n-l)\right)\right]}\left(e^{\sigma^2 r_k r_l\sqrt{n-k}\sqrt{n-l}} - 1\right) \qquad (2.54)
\end{aligned}$$

Substituting result (2.54) into equation (2.53), we find:

$$Var\left[V_n^{-(l)}\right] = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\pi_k\pi_l e^{\delta(2n-k-l)}\left(e^{\sigma^2 r_k r_l\sqrt{n-k}\sqrt{n-l}} - 1\right)$$

Expanding the exponential term in brackets by a first order Taylor series and by using the definition of the correlation coefficient r_k, we can approximate the variance of $V_n^{-(l)}$ by:

$$\begin{aligned}
Var\left[V_n^{-(l)}\right] &\approx \left[\sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}\sigma r_k\sqrt{n-k}\right]\left[\sum_{l=0}^{n-1}\pi_l e^{\delta(n-l)}\sigma r_l\sqrt{n-l}\right] \\
&= \sigma^2\left[\sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)} r_k\sqrt{n-k}\right]^2 \\
&= \sigma^2\left[\sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}\frac{Cov\left[Y_k,\Lambda\right]}{\sigma_\Lambda\sqrt{n-k}}\sqrt{n-k}\right]^2 \\
&= \sigma^2\left[\frac{Cov\left[\sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}Y_k,\ \Lambda\right]}{\sigma_\Lambda}\right]^2 \\
&= \sigma^2 r_S^2\sigma_S^2 \qquad (2.55)
\end{aligned}$$

where $S = \sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}Y_k$, and the correlation coefficient of the pair (S,Λ) is denoted by r_S.

The expression in equation (2.55) can therefore be maximised by maximising the correlation coefficient r_S. This is only the case if the pair (S,Λ) is perfectly correlated, negatively or positively. Therefore, we obtain the optimum choice of Λ as:

$$\Lambda = S = \sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}Y_k \qquad (2.56)$$

Comparing our optimal choice in equation (2.56) to our initial choice in equation (2.38), it is evident that the optimal choice remains a linear combination of the Brownian differences Y_k but with the constant γ_k defined as:

$$\gamma_k = \pi_k e^{\delta(n-k)} \tag{2.57}$$

We show in Chapter 3 that it is relatively simple to add real world complexities to our initial contract without changing the form of the optimal choice of Λ. In fact, the only change necessary is to find the new optimal constant γ_k. To find the optimal correlation coefficient, we replace the value of the constant γ_k in equation (2.52), i.e.:

$$r_k = \frac{\sum_{l=0}^{n-1} \pi_l e^{\delta(n-l)} \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j \pi_l e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{2.58}$$

2.5 Value of the Embedded Options

Now that we have an approximate distribution for the value of the fund V_n^- under the equivalent martingale measure Q, we use this distribution to approximate the discounted payoff or value of the embedded guarantee of our simple financial contract. We therefore need to evaluate the payoff functions:

$$P_n = E_Q\left[\left(b_n - V_n^{-(l)}\right)_+ \mid \mathcal{F}_t\right] \tag{2.59}$$

or:

$$C_n = E_Q\left[\left(V_n^{-(l)} - b_n\right)_+ \mid \mathcal{F}_t\right] \tag{2.60}$$

depending which of equation (2.3) or equation (2.4) is used for pricing and hedging purposes.

The conditional lower bound $V_n^{-(l)}$ is given as:

$$\begin{aligned} V_n^{-(l)} &= \sum_{k=0}^{n-1} \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \\ &= \sum_{k=0}^{n-1} \alpha_k e^{\sigma r_k \sqrt{n-k} Z}, \text{ with } \alpha_k = \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} \end{aligned} \tag{2.61}$$

From equation (2.61), we see that $V_n^{-(l)}$ is a function of a multiple of the standard normal random variable Z. This observation provides us with a way of solving an otherwise difficult expression, i.e. obtaining a solution for the stop-loss function of a sum of maximally dependent lognormal variables or a multiple of a lognormal random variable. Although we have the exact distribution of the information random variable Λ, it is not simple to split the exponent term in such a way as to separate Z from the summation. Dhaene et al. (2002b) derived an expression for the stop-loss function of a sum of maximally dependent random variables such that the stop-loss function of the sum can be expressed in terms of the sum of the stop-loss functions, namely:

$$\begin{aligned} E\left[\left(\sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}Z}-b_n\right)_+\right] &= \sum_{k=0}^{n-1} E\left[\left(\alpha_k e^{\sigma r_k\sqrt{n-k}Z}-F^{-1}_{\alpha_k e^{\sigma r_k\sqrt{n-k}Z}}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right)_+\right] \\ &= \sum_{k=0}^{n-1} E\left[\left(X_k - F^{-1}_{X_k}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right)_+\right] \end{aligned} \tag{2.62}$$

where $F_X(x)$ and $F_X^{-1}(p)$ denote the cumulative distribution function of the random variable X in the value x and the inverse of the function in the value p, respectively, and where X_k is the lognormal random variable $X_k = \alpha_k e^{\sigma r_k\sqrt{n-k}Z}$.

In order to evaluate the result in equation (2.62), we first need to determine the cumulative distribution function and the inverse distribution function of a function of a random variable. If $g(Z)$ is an increasing and continuous function of Z, the following equality holds:

$$F^{-1}_{g(Z)}(p) = g\left(F_Z^{-1}(p)\right) \tag{2.63}$$

The proof of this result can be found in Dhaene et al. (2002b). By using the result in equation (2.63), we find that:

$$F^{-1}_{X_k}(p) = \alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}(p)} \tag{2.64}$$

and therefore:

$$F^{-1}_{V_n^{-(l)}}(p) = \sum_{k=0}^{n-1} F^{-1}_{X_k}(p) = \sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}(p)} \tag{2.65}$$

We now have the tools needed in order to determine the value of the discounted payoffs. First, consider the call-type payoff in equation (2.60). By using the results from equations (2.62) and (2.64), we obtain:

$$E\left[\left(\sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}Z}-b_n\right)_+\right] = \sum_{k=0}^{n-1}\alpha_k E\left[\left(e^{\sigma r_k\sqrt{n-k}Z}-e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right] \tag{2.66}$$

By using the fact that a lognormal random variable X with distribution $\log X \sim N\left(\mu, \sigma^2\right)$ has the following stop-loss function:

$$E\left[(X-d)_+\right] = e^{\mu+\frac{1}{2}\sigma^2}\Phi\left[\frac{\mu+\sigma^2-\log(d)}{\sigma}\right] - d\Phi\left[\frac{\mu-\log(d)}{\sigma}\right] \tag{2.67}$$

we have:

$$\begin{aligned}
E\left[\left(V_n^{-(l)}-b_n\right)_+\right] &= \sum_{k=0}^{n-1}\alpha_k\left\{e^{\frac{1}{2}\sigma^2 r_k^2(n-k)}\Phi\left[\frac{\sigma^2 r_k^2(n-k)-\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}{\sigma r_k\sqrt{n-k}}\right]\right. \\
&\quad \left. + e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\Phi\left[\frac{-\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}{\sigma r_k\sqrt{n-k}}\right]\right\} \\
&= \sum_{k=0}^{n-1}\alpha_k e^{\frac{1}{2}\sigma^2 r_k^2(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&\quad - \sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\Phi\left[-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&= \sum_{k=0}^{n-1}\alpha_k e^{\frac{1}{2}\sigma^2 r_k^2(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&\quad - \sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\left[1-F_{V_n^{-(l)}}(b_n)\right] \\
&= \sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] - b_n\left[1-F_{V_n^{-(l)}}(b_n)\right]
\end{aligned} \tag{2.68}$$

where the last step follows from equation (2.65) such that $F^{-1}_{V_n^{-(l)}}\left(F_{V_n^{-(l)}}(b_n)\right) = b_n$.

A put-call parity relationship exists that allows us to obtain a result for the put-type payoff from the above derived call-type payoff, i.e.:

$$E\left[\left(b_n - V_n^{-(l)}\right)_+\right] = E\left[\left(V_n^{-(l)}-b_n\right)_+\right] + b_n - E\left[V_n^{-(l)}\right] \tag{2.69}$$

First, observe that the expectation of $V_n^{-(l)}$ is given by:

$$\begin{aligned}
E\left[V_n^{-(l)}\right] &= \sum_{k=0}^{n-1}\pi_k e^{\left(\delta-\frac{1}{2}\sigma^2 r_k^2\right)(n-k)}E\left[e^{\sigma r_k\sqrt{n-k}Z}\right] \\
&= \sum_{k=0}^{n-1}\pi_k e^{\delta(n-k)}
\end{aligned} \tag{2.70}$$

Substituting the results from equations (2.70) and (2.68) into equation (2.69), we have:

$$\begin{aligned}
E\left[\left(b_n - V_n^{-(l)}\right)_+\right] &= \sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} \left\{\Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] - 1\right\} \\
&\quad - b_n\left[1 - F_{V_n^{-(l)}}(b_n)\right] + b_n \\
&= -\sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} \left\{1 - \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right\} + b_n F_{V_n^{-(l)}}(b_n) \\
&= -\sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} \Phi\left[-\sigma r_k \sqrt{n-k} + \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] + b_n F_{V_n^{-(l)}}(b_n)
\end{aligned} \tag{2.71}$$

In both equations (2.68) and (2.71), we have the unknown quantity $F_{V_n^{-(l)}}(b_n)$. By using the result in equation (2.63), we find:

$$\sum_{k=0}^{n-1} \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} e^{\sigma r_k \sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)} - b_n = 0 \tag{2.72}$$

It is evident from equation (2.72) that the value of $F_{V_n^{-(l)}}(b_n)$ can be obtained by finding the root of the expression. Most mathematical libraries of software packages have the functionality to evaluate such a function. The first of the questions posed by Hardy (2003) can be answered by having a market-based price of the embedded option at inception of the policy for both the call-type and put-type payoffs. To fairly price the guarantee, the discounted payoff is to be paid by only those policyholders who survive the term of the policy, i.e.:

$$\begin{aligned}
P_G &= {}_t p_x P_0 \\
&= {}_t p_x e^{-\delta n} P_n \\
&\approx {}_t p_x e^{-\delta n} E\left[\left(b_n - V_n^{-(l)}\right)_+\right]
\end{aligned}$$

where ${}_t p_x$ is the standard actuarial notation for the t-year survival probability of a life aged x. We assumed that ${}_t p_x$ is equal to one for the simple financial contract. The addition of mortality arguments to the simple financial product is considered in section 3.2.

2.6 Price of the Embedded Options

In the previous section, the value of the embedded option at inception was determined. In this section, we determine the periodic price charged in respect of the embedded option. Charging schedules depend on the preferences of each individual insurance firm offering these types of guarantee. This section follows the solution commonly found in practice, i.e. that the charge for the guarantee is an annual management fee charged on the value of the fund at the end of each respective year or contribution period. The charge can be stated annually but deducted monthly or quarterly. At inception, the embedded option has to be purchased by the insurer or a replicating portfolio has to be set up in order to effectively manage the risk of the guarantee. The management charges essentially recover the initial guarantee liability outgo. The annual management fee solution is a discrete case of the continuous pricing solution, which is a deduction in the yield of the underlying fund. Milevsky (2006) gives a structured approach to obtaining the price of the guarantee when recouping the cost of the guarantee continuously from the underlying fund.

2.6.1 The Continuous Case: Single Contribution

In the first part of this section, we consider the pricing problem of a simple single premium product in the Black-Scholes-Merton setting, i.e. the fund follows the stochastic differential equation:

$$dF_t = \delta F_t dt + \sigma F_t dW_t$$

where δ denotes the risk-free rate and σ denotes the volatility as before. These parameters are assumed to be constant in the Black-Scholes-Merton setting. We assume that the price of the embedded option is deducted on a continuous basis from the yield of the underlying fund at time t, i.e.:

$$dF_t = (\delta - c) F_t dt + \sigma F_t dW_t \tag{2.73}$$

where c is the constant continuous expense.

By using the boundary condition that $F_0 = 1$, we can solve the stochastic differential equation (SDE) in equation (2.73) to give:

$$F_n = F_t e^{\left(\delta - c - \frac{1}{2}\sigma^2\right)(n-t) + \sigma(W_n - W_t)} \tag{2.74}$$

The expression in equation (2.74) mirrors the fund growth assumption in section 2.2. The value of the embedded put option for single premium business is simply the European put option with a strike price equal to the guarantee, i.e.:

$$\begin{aligned} P_0 &= e^{-\delta n} E_Q \left[\left(b_n - \pi_0 \frac{F_n}{F_0} \right)_+ \right] \\ &= \pi_0 e^{-\delta n} E_Q \left[\left(\frac{b_n}{\pi_0} - \frac{F_n}{F_0} \right)_+ \right] \end{aligned}$$

where the log of the fund growth $\log\left(\frac{F_n}{F_0}\right)$ follows a normal distribution with mean $\left(\delta - c - \frac{1}{2}\sigma^2\right) n$ and standard deviation $\sigma\sqrt{n}$. Therefore, we have:

$$\begin{aligned} P_0 &= \pi_0 e^{-\delta n} \int_0^{b_n/\pi_0} \left(\frac{b_n}{\pi_0} - s \right) f_Q(s)\, ds \\ &= \pi_0 e^{-\delta n} \left\{ \frac{b_n}{\pi_0} \Phi \left(\frac{\log(b_n/\pi_0) - \left(\delta - c - \frac{1}{2}\sigma^2\right) n}{\sigma\sqrt{n}} \right) \right. \\ &\quad \left. - e^{\left(\delta - c - \frac{1}{2}\sigma^2\right)n + \frac{1}{2}\sigma^2 n} \Phi \left(\frac{\log(b_n/\pi_0) - \left(\delta - c + \frac{1}{2}\sigma^2\right) n}{\sigma\sqrt{n}} \right) \right\} \\ &= e^{-\delta n} b_n \Phi(d_1) - e^{-cn} \pi_0 \Phi(d_2) \end{aligned} \tag{2.75}$$

where:

$$d_1 = \frac{\log(b_n/\pi_0) - \left(\delta - c - \frac{1}{2}\sigma^2\right) n}{\sigma\sqrt{n}} \tag{2.76}$$

and:

$$d_2 = d_1 - \sigma\sqrt{n}$$

The price of the guarantee c can now be found by fair value principles, i.e. equating the expected present value of the premium and the expected present value of the benefits:

$$\begin{aligned} \pi_0 &= e^{-\delta n} E \left[\max\left(\pi_0 \frac{F_n}{F_0}, b_n \right) \right] \\ &= e^{-\delta n} E \left[\pi_0 \frac{F_n}{F_0} \right] + P_0 \\ &= \pi_0 e^{-cn} + P_0. \end{aligned} \tag{2.77}$$

We can solve for the value of the put option in terms of the single premium and the price of the option, i.e.:

$$P_0 = \pi_0 \left(1 - e^{-cn}\right). \tag{2.78}$$

The result in equation (2.78) corresponds to the result derived by Milevsky (2006) in the simplified Black-Scholes-Merton setting. However, the price of the guarantee is still some unknown constant c. The value of c can be solved numerically by substituting the value of the put option P_0 in equation (2.77), i.e.:

$$b_n e^{-\delta n} \Phi(d_1) - \pi_0 \left[1 - e^{-cn} \Phi(-d_2)\right] = 0 \tag{2.79}$$

2.6.2 The Discrete Case: Single Contribution

The pricing approach developed for the continuous case is extended to the discrete case in the following. If the guarantee cost is recouped on a discrete basis, the guarantee cost is deducted as an annual charge of the fund value. We denote the annual charge by e,where $0 \leq e \leq 1$. The value of the fund at time n is then given by:

$$V_n^- = \frac{F_n}{F_0}\pi_0 (1-e)^n \tag{2.80}$$

The log of the fund growth, $\log\left(\frac{F_n}{F_0}\right)$, follows a normal distribution with mean $\left(\delta - \frac{1}{2}\sigma^2\right)n$ and standard deviation $\sigma\sqrt{n}$. The value of the European put option is given by:

$$\begin{aligned} P_0 &= \pi_0 (1-e)^n e^{-\delta n} E_Q\left[\left(\frac{b_n}{\pi_0 (1-e)^n} - \frac{F_n}{F_0}\right)_+\right] \\ &= e^{-\delta n} b_n \Phi(d_1) - \pi_0 (1-e)^n \Phi(d_2) \end{aligned} \tag{2.81}$$

where:

$$d_1 = \frac{\log(b_n/\pi_0) - \left(\delta + \log(1-e) - \frac{1}{2}\sigma^2\right)n}{\sigma\sqrt{n}} \tag{2.82}$$

and:

$$d_2 = d_1 - \sigma\sqrt{n}$$

By comparing the continuous case in equation (2.75) with the discrete case in equation (2.81), it is evident that the following relation holds:

$$e^{-cn} = (1-e)^n$$

or alternatively:

$$c = -\log(1-e) \tag{2.83}$$

The price of the guarantee is found in a similar way for the continuous case by equating the expected present value of the premium and the expected present value of the benefits:

$$\begin{aligned} \pi_0 &= e^{-\delta n} E\left[\pi_0 (1-e)^n \frac{F_n}{F_0}\right] + P_0 \\ \pi_0 &= \pi_0 (1-e)^n + P_0 \end{aligned} \tag{2.84}$$

The price of the guarantee can be solved numerically by substituting the value of the put option P_0 in equation (2.84), i.e. solving for e in the expression:

$$b_n e^{-\delta n}\Phi(d_1) - \pi_0 \left[1 - (1-e)^n \Phi(-d_2)\right] = 0 \tag{2.85}$$

The single premium case, in both the continuous and discrete scenarios, allows us to solve easily for the unknown charges. This approach can be extended to recurring premium business. In order to do so, we first need to extend our simple product to a product that allows for periodic charges of the sub–account value.

2.6.3 Regular Contributions

Consider again the simple investment product of section 2.3. Assume that we now have an annual charge e deducted at the end of each year. At the end of year $(k, k+1)$, the fund value is then given by:

$$V_{k+1}^{-} = [V_k + \pi_k] \frac{F_{k+1}}{F_k} (1 - e) \tag{2.86}$$

The management charge e is deducted from the fund value at the end of each year and if we assume no decrements and no withdrawals the fund value at maturity is given by:

$$\begin{aligned} V_n^{-} &= \sum_{k=0}^{n-1} \pi_k \prod_{j=k}^{n-1} \frac{F_{j+1}}{F_j} (1 - e) \\ &= \sum_{k=0}^{n-1} \pi_k \frac{F_n}{F_k} (1 - e)^{n-k} \end{aligned} \tag{2.87}$$

By following the same steps as in section 2.4, we can use the derivation of the conditional lower bound to allow for guarantee charges. We still assume a Black-Scholes-Merton setting. The fund value therefore becomes:

$$V_n^{-} = \sum_{k=0}^{n-1} \pi_k (1 - e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2\right)(n-k) + \sigma(W(n) - W(k))} \tag{2.88}$$

Comparing equation (2.88) above to equation (2.49) we see that only the premium vector changes, i.e. the guarantee charge effectively reduces the contribution that participates in the fund growth. The information variable Λ remains unchanged from its definition in Chapter 2, i.e.:

$$\Lambda = \sum_{k=0}^{n-1} \gamma_k [W(n) - W(k)] \tag{2.89}$$

where the value of the γ_k can be solved as before by maximising the variance of the conditional lower bound.

Using the result obtained in equation (2.88), we have an expression for the conditional lower bound:

$$V_n^{-(l)} = \sum_{k=0}^{n-1} \pi_k (1-e)^{n-k} E\left[\frac{F(n)}{F(k)} \mid \Lambda\right]$$
$$= \sum_{k=0}^{n-1} \pi_k (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \tag{2.90}$$

The conditional lower bound needs to be optimised by maximising the variance of $V_n^{-(l)}$ with respect to Λ. First, we consider the expression for the non-optimal correlation coefficient r_k in the case of guarantee charges. The optimal correlation coefficient has a similar form, but with the constant γ_k defined according to the optimising strategy of maximising the variance of $V_n^{-(l)}$ with respect to Λ. Recall that the Pearson's correlation coefficient is defined as:

$$r_k = \frac{Cov\left[Y_k, \Lambda\right]}{\sigma_{Y_k} \sigma_\Lambda}$$

where the covariance of the pair (Y_k, Λ) is given by:

$$Cov\left[Y_k, \Lambda\right] = \sum_{l=0}^{n-1} \gamma_l \min(n-k, n-l)$$

Since the information variable does not contain the guarantee charge e, the variance of Λ is as before given by:

$$Var[\Lambda] = \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \gamma_j \gamma_l \min(n-j, n-l) \tag{2.91}$$

Therefore, we have the non-optimal correlation coefficient as:

$$r_k = \frac{\sum_{l=0}^{n-1} \gamma_l \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \gamma_j \gamma_l \min(n-j, n-l)}} \tag{2.92}$$

It is important to note that the addition of an annual guarantee charge did not change the correlation coefficient as derived for the simple investment product in section 2.4. We now optimise the conditional lower bound by maximising the variance of the conditional lower bound $V_n^{-(l)}$ with respect to Λ. The variance of $V_n^{-(l)}$ is given as:

$$Var\left[V_n^{-(l)}\right]$$
$$= Cov\left[\sum_{k=0}^{n-1} \pi_k (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z}, \sum_{l=0}^{n-1} \pi_l (1-e)^{n-l} e^{\left(\delta - \frac{1}{2}\sigma^2 r_l^2\right)(n-l) + \sigma r_l \sqrt{n-l} Z}\right]$$
$$= \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \pi_k \pi_l (1-e)^{2n-k-l} e^{\delta(2n-k-l) - \frac{1}{2}\sigma^2\left(r_k^2(n-k) + r_l^2(n-l)\right)} Cov\left[e^{\sigma r_k \sqrt{n-k} Z}, e^{\sigma r_l \sqrt{n-l} Z}\right] \tag{2.93}$$

where the covariance term on the left is given by equation (2.54) as:

$$Cov\left[e^{\sigma r_k\sqrt{n-k}Z}, e^{\sigma r_l\sqrt{n-l}Z}\right] = e^{\left[\frac{1}{2}\sigma^2\left(r_k^2(n-k)+r_l^2(n-l)\right)\right]}\left(e^{\sigma^2 r_k r_l\sqrt{n-k}\sqrt{n-l}} - 1\right)$$

Therefore, we find:

$$Var\left[V_n^{-(l)}\right] = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\pi_k\pi_l\,(1-e)^{2n-k-l}\,e^{\delta(2n-k-l)}\left(e^{\sigma^2 r_k r_l\sqrt{n-k}\sqrt{n-l}} - 1\right)$$

By expanding the exponential term in brackets and by using the definition of the correlation coefficient r_k, we can approximate the variance of $V_n^{-(l)}$ by:

$$\begin{aligned}
Var\left[V_n^{-(l)}\right] &\approx \left[\sum_{k=0}^{n-1}\pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)}\sigma r_k\sqrt{n-k}\right]\left[\sum_{l=0}^{n-1}\pi_l\,(1-e)^{n-l}\,e^{\delta(n-l)}\sigma r_l\sqrt{n-l}\right] \\
&= \sigma^2\left[\sum_{k=0}^{n-1}\pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)}r_k\sqrt{n-k}\right]^2 \\
&= \sigma^2\left[\sum_{k=0}^{n-1}\pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)}\frac{Cov\left[Y_k,\Lambda\right]}{\sigma_\Lambda\sqrt{n-k}}\sqrt{n-k}\right]^2 \\
&= \sigma^2\left[\frac{Cov\left[\sum_{k=0}^{n-1}\pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)}Y_k,\ \Lambda\right]}{\sigma_\Lambda}\right]^2 \\
&= \sigma^2 r_S^2\sigma_S^2
\end{aligned} \tag{2.94}$$

where $S = \sum_{k=0}^{n-1}\pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)}Y_k$, and the correlation coefficient of the pair (S,Λ) is denoted by r_S.

Therefore, we can maximise the expression in equation (2.94) by maximising the correlation coefficient r_S. This is only the case if the pair (S,Λ) is perfectly correlated, negatively or positively. Therefore, we obtain the optimum choice of Λ as:

$$\Lambda = S = \sum_{k=0}^{n-1}\pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)}Y_k \tag{2.95}$$

Comparing our optimal choice of the information variable Λ in equation (2.95) to our initial choice in equation (2.89), it is evident that the optimal choice remains a linear combination of the Brownian differences Y_k but with the constant γ_k defined as:

$$\gamma_k = \pi_k\,(1-e)^{n-k}\,e^{\delta(n-k)} \tag{2.96}$$

To find the optimal correlation coefficient, we simply replace the value of the constant γ_k in equation (2.92), therefore:

$$r_k = \frac{\sum_{l=0}^{n-1} \pi_l (1-e)^{n-l} e^{\delta(n-l)} \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j \pi_l (1-e)^{2n-j-l} e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{2.97}$$

Now that we have an approximate distribution for the value of the fund V_n^- under the equivalent martingale measure Q and with a guarantee charge, we can use this distribution to approximate the discounted payoff, i.e. the price, of the embedded guarantee of our contract. Therefore, we need to evaluate the payoff functions:

$$P_n = E_Q\left[\left(b_n - V_n^{-(l)}\right)_+ \mid \mathcal{F}_t\right] \tag{2.98}$$

or:

$$C_n = E_Q\left[\left(V_n^{-(l)} - b_n\right)_+ \mid \mathcal{F}_t\right] \tag{2.99}$$

depending on which of equation (2.98) or equation (2.99) one prefers or is feasible to use for pricing and hedging purposes.

By following the steps in section 2.4, we can find the value of the embedded call option as:

$$\begin{aligned} C_0 &= e^{-\delta n} E\left[\left(V_n^{-(l)} - b_n\right)_+\right] \\ &= \sum_{k=0}^{n-1} \pi_k (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad - e^{-\delta n} b_n \left[1 - F_{V_n^{-(l)}}(b_n)\right] \end{aligned} \tag{2.100}$$

and the value of the embedded put option as:

$$\begin{aligned} P_0 &= e^{-\delta n} E\left[\left(b_n - V_n^{-(l)}\right)_+\right] \\ &= -\sum_{k=0}^{n-1} \pi_k (1-e)^{n-k} e^{-\delta k} \Phi\left[-\sigma r_k \sqrt{n-k} + \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad + e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) \end{aligned} \tag{2.101}$$

In both equations (2.100) and (2.101), we have the unknown quantity $F_{V_n^{-(l)}}(b_n)$. By using the result in equation (2.63), we find:

$$\sum_{k=0}^{n-1} \pi_k (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} e^{\sigma r_k \sqrt{n-k}\,\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)} - b_n = 0 \tag{2.102}$$

The annual charge for the guarantee e can now be found by fair value principles, i.e. equating the expected present value of the premium with the expected present value of the benefits:

$$\begin{aligned}\sum_{k=0}^{n-1}\pi_k e^{-\delta k} &= e^{-\delta n}E\left[\max\left(V_n^{-(l)},b_n\right)\right] \\ &= e^{-\delta n}E\left[V_n^{-(l)}\right]+P_0 \\ &= \sum_{k=0}^{n-1}\pi_k\left(1-e\right)^{n-k}e^{-\delta k}+P_0 \\ &= \sum_{k=0}^{n-1}\pi_k\left(1-e\right)^{n-k}e^{-\delta k}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right] \\ &\quad +e^{-\delta n}b_nF_{V_n^{-(l)}}\left(b_n\right) \qquad (2.103)\end{aligned}$$

We can therefore solve for the unknown guarantee charge e by solving the following expression with respect to e:

$$e^{-\delta n}b_nF_{V_n^{-(l)}}\left(b_n\right)-\sum_{k=0}^{n-1}\pi_k e^{-\delta k}\left[1-\left(1-e\right)^{n-k}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right]\right]=0 \quad (2.104)$$

2.7 Illustrative Example

The accuracy of the conditional lower bound derived in section 2.4 can be tested by determining the value of the embedded guarantee using a Monte Carlo method as a proxy of the true value. In Table 2.1 and Table 2.2 below, we calculated the discounted values of the embedded put options for the simplified product. The accuracy of the call options follow by put-call parity. A variance reduction technique, in the form of the antithetic variates method as described by Boyle (1977) and Rubinstein (1981), was used. A good review of simulation techniques and variance reduction can be found in, for example, Dagpunar (2007). Simulated results are based on 50 000 paths for each set of parameter values, which were varied to illustrate the sensitivity of results to both interest rate and volatility risks. First, we assume a continuous risk-free rate of 5% per annum with volatilities of 20%, 30% or 40% per annum. The contributions π_k, $k=0,...,n$, are assumed at 100 per annum and the term of the contract is taken as $n=10$ years. We also vary the guaranteed amount b_n by considering the following five values $(500,\ 750,\ 1000,\ 1250,\ 1500)$. The *CLB* column gives the conditional lower bound approximation, while the *MC* and *s.e.* columns give the Monte Carlo estimate with associated standard error. The results are summarised in Table 2.1.

Table 2.1: Conditional Lower Bound (CLB) estimates of the embedded put option values with $n = 10$, $\delta = 5\%$ and varying volatilities, compared to Monte Carlo (MC) estimates and their standard errors (s.e.).

σ	b_n	CLB	MC	s.e.
20%	500	0.2899	0.3191	0.00061
	750	7.6583	7.7911	0.00368
	1000	39.3632	39.5205	0.00924
	1250	104.2183	104.3376	0.01103
	1500	198.3930	198.5049	0.00816
30%	500	4.6067	4.9362	0.00299
	750	30.2476	30.7541	0.00824
	1000	84.6857	85.1418	0.01132
	1250	164.6151	164.9986	0.01243
	1500	264.0077	264.3668	0.00794
40%	500	15.6902	16.7220	0.00561
	750	60.3649	61.5619	0.01058
	1000	131.4565	132.5241	0.01172
	1250	222.2414	223.1759	0.01005
	1500	327.2443	328.0961	0.00796

Table 2.2: Conditional Lower Bound (CLB) estimates of the embedded put option values with $n = 10$, $\sigma = 20\%$ and varying interest rates, compared to Monte Carlo (MC) estimates and their standard errors (s.e.).

δ	b_n	CLB	MC	s.e.
1%	500	1.9299	2.0269	0.00191
	750	31.1708	31.3591	0.00872
	1000	120.7156	120.8753	0.01294
	1250	266.7567	266.8974	0.00837
	1500	449.5724	449.7517	0.00658
5%	500	0.2899	0.3191	0.00061
	750	7.6583	7.7911	0.00368
	1000	39.3632	39.5205	0.00924
	1250	104.2183	104.3376	0.01103
	1500	198.3930	198.5049	0.00816
10%	500	0.0178	0.0218	0.00012
	750	0.9215	0.9665	0.00105
	1000	7.0577	7.1558	0.00343
	1250	24.3875	24.5078	0.00643
	1500	56.0633	56.1616	0.00947

Table 2.3: Charges in basis points (bps) for the embedded put options with $n = 10$ and varying volatilities and risk-free rates.

	b_n	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
$\delta = 1\%$	500	0.3664	2.8282	7.5429
	750	6.9304	18.7686	32.8233
	1000	51.1506	86.5640	120.8808
$\delta = 5\%$	500	0.06095	0.9881	3.4656
	750	1.6931	7.1870	15.1582
	1000	10.5377	25.1368	41.4655
	1250	48.1448	81.8928	114.5406
$\delta = 10\%$	500	0.00425	0.2254	1.2115
	750	0.2218	2.0356	5.7732
	1000	1.7803	7.3267	15.2321
	1250	6.9371	18.4198	31.8342
	1500	20.2496	40.7130	61.8769

It is evident from Table 2.1 that the conditional bound approximation (CLB) is very accurate for low volatility economies and slightly less accurate for high volatility economies. In Table 2.2, we assume a volatility of 20% with interest rates of 1%, 5% or 10% per annum. It is clear that the value of the embedded option is very sensitive to the volatility and risk-free rate of the economy.

The price of the investment guarantee is assumed to be deducted as a percentage of the sub-account on a periodic basis, which is largely the case in practice. The value of the option as calculated for different scenarios in Table 2.1 and Table 2.2 represents the lump sum amount of the fair value of the total charges deducted. In section 2.6, the periodic asset-based charge was solved for by applying fair value principles, i.e. the charge can be obtained by solving for e in the following:

$$e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) - \sum_{k=0}^{n-1} \pi_k e^{-\delta k} \left[1 - (1-e)^{n-k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right] = 0 \quad (2.105)$$

The annual charge of the simple investment contract example was solved for by using equation (2.105) and is given for different parameter values in Table 2.3. The charges are given in basis points (bps) of the fund value charged annually. The charges for guarantees of b_n equal to 1250 and 1500, for a risk-free rate of 1%, and for guarantees of b_n equal to 1500, for a risk-free rate of 5%, are in excess of or approaching 100% and are therefore omitted. Note that b_n

represents the ratio of the return of contributions guarantee, i.e. if b_n is equal to 1000 then the benefit guarantees 100% return of contributions, whereas if b_n is equal to 1500 then the benefit guarantees 150% return of contributions.

Table 2.3 shows that the charges increase dramatically as the volatility increases or as the risk-free rate decreases. The sensitivity of the option value to these parameters and hence to the price of the option need to be considered in a comprehensive risk management strategy. In Chapter 4, we consider and test a possible hedging strategy that aims to immunise the important risk factors.

Chapter 3

LIFE-CONTINGENT GUARANTEED MINIMUM BENEFITS

In this section, we generalise the simplified pure investment product of Chapter 2 by adding a cost structure that allows for exogenenously given periodic charges as a percentage of the policyholder's sub-account. This section also extends the simplified product by allowing for mortality risk in considering guaranteed death and guaranteed survival benefits. The premiums for ancillary risk benefits such as dread disease, disability and premium waiver benefits are usually deducted from the policyholder's contributions before the contribution is invested or unitised in the sub-account. These benefits do not, therefore, affect the guarantee charges.

We assume that the insurer has a risk-neutral position to mortality risk, i.e. the VA business portfolio consists of a large enough population of independent policyholders such that the aggregate mortality risk is diversified. In the case where mortality risk is fully diversifiable, the random lifetimes of policyholders can be replaced by their expected lifetimes. This assumption does not mean that we model the random lifetimes of policyholders in the risk-neutral world. The Law of Large Numbers implies that the variability in the expected value of the independent random lifetimes of a cohort of identical policyholders tends to zero, i.e. the expected lifetime of a cohort of policyholders becomes deterministic.

We also assume that mortality risk is independent of investment risk, thereby allowing us to express the probability of an integrated risk event as the product of the individual investment risk and mortality risk events. The policyholder sub-account with a typical cost structure and its development over time are described in section 3.1. We assume that all the charges are determined at policy inception, i.e. the charges are fixed contractually. The combined market of financial risk and mortality risk is discussed in a theoretical framework in section 3.2, specifically where care needs to be taken in respect of the independence assumption of financial risk and biometric risk. Benefits payable on maturity, which were briefly introduced at the end of section 2.4, are dealt with in section 3.3. Benefits payable on death are covered in section 3.4. Since benefits on death and survival are contingent on mutually exclusive events, both the benefits can be opted for by policyholders and this combination of benefits is discussed in section 3.5. A numerical example is given in section 3.6 that illustrates the effect of adding additional complexity to the simple product of Chapter 2.

3.1 The Policyholder Sub-Account

The policyholder sub-account serves as an individual policyholder account with inflows and outflows determined by the policy rules and benefits. The sub-account is a separate account linked to the underlying fund or funds opted by the policyholder.

Examples of possible inflows to the sub-account are the contributions payable on a recurring basis or as an initial lump sum, investment returns earned on existing units, rebates received from fund managers or the insurer at a specified policy anniversary as incentive to keep the contract in force, distributions of dividends or interest on an annual, semi-annual or quarterly basis, and benefits payable such as an GMAB at a specified policy anniversary while the policy is still in force.

Examples of outflows from the sub-account are advisory fees or commission payable, allocation charges such as a bid-offer spread, fund management charges specific to the underlying fund(s) selected by the policyholder, activity fees on underlying fund switches, transfers or part surrenders, an annual policy or account fee and annual management charges specific to the insurer's expenses, risk charges to cover the death and survival benefits, and the optional investment guarantee charges. These examples of inflows and outflows are not exhaustive and constant innovation of products allow for new charging structures to accommodate policyholder needs. In this section, we describe a generic product that exhibits much of the sub-account characteristics listed above.

Consider a VA insurance product that is underwritten on a life aged x at time 0 and where the policyholder aims to annuitise the VA in a period of n years. Annual premiums are used to purchase units of an underlying investment fund opted by the policyholder. Although several investment funds can usually be linked to the policy in the form of multiple sub-accounts issued to the policyholder, we assume for simplicity that units are purchased in one underlying investment fund. Note that the fund process F_k, for $k = 0, ..., n$, can also be regarded as the process of an aggregate fund that consists of a range of chosen underlying investment funds. An allocation cost charge c, which is generally around 5%, is charged when purchasing units, which implies that $(1-c)\,\pi_k$ worth of units are available at each premium π_k payment. The allocation cost might be different for single premium and regular premium products. We assume for simplicity that any fractional unit of the underlying fund can be purchased. For most products in practice, the allocation percentage is often lower in the first year in order to cover initial outgo. The allocation cost typically covers the initial expenses, commission or advisory fees and ongoing allocation expenses. The fee for optional risk benefits, e.g. a waiver of premium benefit, can be asset-based, i.e. charged as a percentage of the fund on a periodic basis, or deducted from the received premium before allocation.

The allocated premiums are used to purchase units of the investment fund at the fund's offer price F_k at the start of the year $(k, k+1)$. Contracts are usually structured in such a way that a policyholder is allowed to sell part of his accumulated units back to the insurer at the fund's bid price at the end of each year during which the contract is in force, i.e. the policyholder has a right to withdrawals. Should the contract terminate (at death, upon survival or in the case of surrender), the units are bought back by the insurance company at the bid price. We assume that the bid price of one unit of the investment fund at time t is equal to the value of the fund F_k, i.e. the investment fund prices are given net of the bid-offer spread. The insurer charges the bid-offer spread by increasing the offer price at time k, i.e. setting the offer price at $\frac{F_k}{1-\alpha}$ for some value $0 < \alpha < 1$. This means that the policyholder has to buy the units at a higher price than the value of the fund.

The purchased units are allocated, net of allocation charges, to the sub-account. The sub-account describes the total number of units invested on the policyholder's behalf in the underlying unitised investment fund. Let the total number of units in the sub-account be denoted by N_k at time k. The evolution of the sub-account V_k can be described in terms of its units. The number of units in the sub-account at time k is evaluated at the beginning of year $(k, k+1)$, after payment of a fund management charge e for the year $(k-1, k)$, after withdrawing of units at time k, but before payment of premiums at the beginning of year $(k, k+1)$. The annual management charge e typically includes all asset-based charges, e.g. the investment guarantee charges and the underlying fund management charge. We denote the annual asset-based charge in respect of investment guarantees by e_g and the total annual asset-based charges by e, i.e. the total other annual asset-based charges are equal to $e - e_g$ with all charges being positive The value of the sub-account at time k is therefore given by:

$$V_k = N_k F_k \tag{3.1}$$

We also denote N_{k+1}^- by the number of units of the contract at time $k+1$, evaluated after payment of the fund management charge, e, at the end of year $(k, k+1)$, but before withdrawal of units, before the death benefit payment and before payment of premiums at the beginning of year $(k+1, k+2)$. The corresponding value of the unit account is given by V_{k+1}^-:

$$V_{k+1}^- = N_{k+1}^- F_{k+1} \tag{3.2}$$

Note that the value V_{k+1}^- corresponds to the sub-account value of Chapter 2, except that we now account for both expenses and mortality costs. Should the policyholder die before time n in the case of our endowment contract, the contract ends after the cash value available at the end of the year of death has been paid out to the beneficiary. In case the policyholder is still alive at time n, the contract ends after the cash value V_n^- is paid out to the beneficiary.

The recursive formula of equation (2.1) is generalised in the following to include expenses. Assume that the policyholder is still alive at the beginning of year $(k, k+1)$. The number of units at the end of year $(k, k+1)$, after paying the fund management charge, before payment of death or survival benefits, before withdrawal of units and before payment for premiums at time $k+1$, can be determined by the following recursive relationship:

$$N_{k+1}^- = \left(N_k + \frac{(1-\alpha)(1-c)\,\pi_k - \pi_k^{(r)}}{F_k} \right)(1-e) \tag{3.3}$$

where $\pi_k^{(r)}$ is defined as the risk premium and is paid to cover the mortality expenses.

These mortality expenses relate to the additional non-unit death and survival benefits, i.e. insurance benefits. The risk or insurance premium $\pi_k^{(r)}$ is paid on survival at time k by reducing the number of units available in the unit account at that time.

The savings premium at time k is defined by:

$$\pi_k^{(s)} = (1-\alpha)(1-c)\,\pi_k - \pi_k^{(r)} \tag{3.4}$$

The savings premium is interpreted as the part of the contribution that is effectively used to purchase units of the fund. Note that the savings premium may be negative. This occurs when the risk premium exceeds the allocated bid-offer spread adjusted premium. Equation (3.2) can be expressed as:

$$N_{k+1}^{-}=\left(N_k+\frac{\pi_k^{(s)}}{F_k}\right)(1-e) \tag{3.5}$$

We now return to expressing the sub-account evolution in terms of the underlying fund price. From equations (3.2) and (3.5), we find that the value of the sub-account at the end of year $(k,k+1)$ can be determined by the following recursion:

$$\begin{aligned} V_{k+1}^{-} &= N_{k+1}^{-}F_{k+1} \\ &= \left[V_k+\pi_k^{(s)}\right]\frac{F_{k+1}}{F_k}(1-e) \\ &= \left[V_k+(1-\alpha)(1-c)\pi_k-\pi_k^{(r)}\right]\frac{F_{k+1}}{F_k}(1-e) \end{aligned} \tag{3.6}$$

Let us assume that the policyholder is still alive at time k and that no units have been withdrawn before time k, then we have:

$$V_{k+1}^{-}=\sum_{j=0}^{k}\pi_j^{(s)}\frac{F_{k+1}}{F_j}(1-e)^{k+1-j}\,,\ k=0,...,n-1 \tag{3.7}$$

Therefore, upon survival of the policyholder at time k and assuming that no units have been withdrawn and that the contract contains no non-unit survival benefits except at time n, the cash value V_{k+1}^{-} equals the accumulated value of the savings premiums, corrected for the paid fund management charges. Note the mathematical similarity of the expressions in equation (3.7) above and equation (2.87).

3.2 The Combined Market Framework

The following section draws on the work by Dhaene et al. (2010) in which the authors discuss the potential danger of transferring dependency assumptions from the individual market of biometric or financial risks to the market of combined risks. Biometric risk refers to mortality, morbidity and longevity risks of the policyholder. In the following, the focus is mainly on the GMAB and GMDB types of benefit and, therefore, on mortality risk. Dahl (2004) distinguishes between two types of mortality risk, namely systematic risk and unsystematic risk. Systematic refers to the variability of the underlying mortality intensity, while unsystematic mortality risk refers to the worsening mortality experience of the lives insured by the specific insurer. This section provides a brief but formal overview of the physical probability measures and risk-neutral probability measures in the individual and combined market.

First, consider the financial market of instantaneous security prices. Let the financial market be described by the measurable space $(\Omega^f, \mathcal{F}^f)$ equipped with the filtration $\left\{\mathcal{F}_t^f\right\}_{0 \leq t \leq n}$. The set of all possible outcomes in the financial market during the time interval $0 \leq t \leq n$ is denoted by the non-empty set Ω^f, while the σ-algebra $\mathcal{F}^f$ represents the collection of all possible subsets of Ω such that it contains the empty set $\varnothing$, the complement of any event in $\mathcal{F}^f$, and the union and intersection of any sequence of events in $\mathcal{F}^f$. These requirements allow the probability measure to be defined on a suitable domain. The filtration $\mathcal{F}_t^f$ describes all information about the financial market up to and including time t. Security prices during the time interval $0 \leq t \leq n$ are described by stochastic processes in this probability space. Furthermore, the financial stochastic processes are adapted to the filtration $\left\{\mathcal{F}_t^f\right\}_{0 \leq t \leq n}$, which means that the values of the stochastic processes are observable up to and including time t. A random variable X^f defined on the measurable space $(\Omega^f, \mathcal{F}^f)$ is known as a financial risk.

The biometric risk environment can be described in a similar way. Let the measurable space $(\Omega^m, \mathcal{F}^m)$ be equipped with a filtration $\{\mathcal{F}_t^m\}_{0 \leq t \leq n}$ for describing the observable survival history of all persons under consideration. The stochastic process of interest consists of indicator variables which are 1 for any time t that the person under interest is alive and 0 from the moment of death. Across a cohort of lives, the stochastic process represents the mortality intensity μ_x for some age x. The non-empty set Ω^m contains all possible evolutions for any given life during the time interval $0 \leq t \leq n$, while the filtration $\{\mathcal{F}_t^m\}_{0 \leq t \leq n}$ provides all observable information about death and survival of the lives up to and including time t. A random variable X^m defined on the measurable space $(\Omega^m, \mathcal{F}^m)$ is known as a biometric or actuarial risk.

The combined market of financial and biometric risks under the physical or objective probability measure P is described by the filtered measurable space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq n}, P\right)$, which is defined as the product space of the financial and biometric spaces:

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq n}, P\right) = \left(\Omega^f \times \Omega^m, \mathcal{F}^f \otimes \mathcal{F}^m, \left\{\mathcal{F}_t^f \otimes \mathcal{F}_t^m\right\}_{0 \leq t \leq n}, P\right) \tag{3.8}$$

The tuple in the combined market $(\omega_f, \omega_m) \in \Omega = \Omega^f \times \Omega^m$ represents the feasible outcomes concerning the financial and biometric evolution in the time interval under consideration. The

σ-algebra $\mathcal{F} = \mathcal{F}^f \otimes \mathcal{F}^m$ is the set of all statements that can be made about the combined market in the time interval $[0, T]$.

The definitions of a financial risk X^f and a biometric risk X^m still hold in the filtered measurable space defined in equation (3.8). Any financial risk X^f defined in $\left(\Omega^f, \mathcal{F}^f\right)$ can be considered as a financial risk random variable in the combined measurable space $(\Omega, \mathcal{F})$ by letting:

$$X^f\left(w_1, w_2\right) \equiv X^f\left(w_1\right) \tag{3.9}$$

where the tuple (w_1, w_2) represents an outcome in the product space $\Omega^f \times \Omega^m$ and the equivalency in equation (3.9) is defined for all $(w_1, w_2) \in \Omega^f \times \Omega^m$. Similarly for a biometric risk X^m defined in $(\Omega^m, \mathcal{F}^m)$, we let:

$$X^m\left(w_1, w_2\right) \equiv X^m\left(w_2\right) \tag{3.10}$$

for all $(w_1, w_2) \in \Omega^f \times \Omega^m$.

Independence in the combined market for the physical or objective probability measure P implies that:

$$P\left(A \times B\right) = P^f\left(A\right) \times P^m\left(B\right) \text{ for any } A \in \mathcal{F}^f \text{ and } B \in \mathcal{F}^m \tag{3.11}$$

The independence assumption in equation (3.11) states that any financial risk X^f and any biometric risk X^m defined in the combined market of equation (3.8) are always mutually independent.

Dhaene et al. (2010) gives the following example to illustrate that the assumption of independence for the physical probability measure P in the combined market does not necessarily translate to independence for the risk-neutral probability measure Q. Consider the following probability measure on the combined market:

$$Q'\left(A \times B\right) = Q^f\left(A\right) \times Q^m\left(B\right) \text{ for any } A \in \mathcal{F}^f \text{ and } B \in \mathcal{F}^m \tag{3.12}$$

where Q^f and Q^m are the equivalent martingale measures of the financial sub-market and biometric sub-market, respectively. The independence assumption of equation (3.11) implies that the measure Q' is equivalent to the physical measure P, i.e. the measures Q' and P share the same null sets. However, Q' is not necessarily a martingale measure for the combined market. The measure Q' will be an equivalent martingale measure, i.e. $Q' \equiv Q$, for the combined market by using the simplifying assumption that the combined market contains no combined traded assets, i.e. assets with a payoff depending only on the financial or the biometric process.

Assume now that the equivalent martingale measure Q in the combined market is unique. By the Fundamental Law of Asset Pricing, the equivalent martingale measure used for pricing purposes is unique if and only if the market is complete, thereby ensuring that the combined market is complete. Also, in this case the sub-market measures Q^f and Q^m are unique and, therefore, the sub-markets are complete. The uniqueness of the equivalent martingale measure Q' implies that $Q' \equiv Q$. The simple case of the combined market where no combined assets are traded allows us to conclude from the independence assumption of equation (3.11) that the measure Q' is an equivalent martingale measure for this market. Therefore, the independence assumption is maintained under a change of measure from the physical probability measure to the equivalent martingale measure in the simple case of a complete market where no combined assets are traded.

The independence assumption is not necessarily maintained for the general setting. The biometric market is generally not complete since, for example, no secondary market exist in which to fully trade mortality-dependent securities. Ballotta and Haberman (2006) adopted the approach of letting the parameter of the Girsanov density under the change of measure related to the mortality process be equal to zero. In such a case, the survival probabilities are calculated under the physical probability measure P as if the market were neutral with respect to systematic and unsystematic mortality risk.

In the event that either the financial market or biometric market is incomplete, the combined market is also incomplete. In an incomplete market, there may exist an equivalent martingale measures Q' for the combined measurable space that cannot be represented in the form of equation (3.12). The choice of the martingale measure Q' in such cases is therefore not unique and the choice of which measure to use is then part of the modelling process.

3.3 Guaranteed Minimum Maturity Benefit

This section follows on the brief introduction in section 2.6 of allowing for the random lifetimes of policyholders. We assume that the lifetimes of policyholders are independent of the underlying fund process, i.e. mortality risk and investment risk are independent. In addition, we assume for simplicity that the insurance firm is risk-neutral to mortality risk, which is achieved by pooling a sufficiently large number of independent and identically distributed lives with the same defined age. This assumption enforces the Law of Large Numbers, i.e. the probability that a policyholder aged x survives to time t is replaced by the frequency of survival of the cohort of the population aged x:

$$P\left[T_x > t\right] = {}_tp_x \tag{3.13}$$

where T_x denotes the random lifetime of a policyholder aged x.

We follow the same steps as in section 2.4 for deriving the conditional lower bound of the policyholder's sub-account. The conditional lower bound approximation is then used to determine the discounted payoff of the embedded put option of the GMMB. The asset-based fee e can then be solved for numerically as before. We assume that the maturity benefit is a function of observable and deterministic (exogenously given) values. The maturity benefit is a life contingent claim and provides a guarantee against severe market downturns to policyholders. The risk or insurance premium, $\pi_k^{(r)}$, for mortality benefits is determined as the actuarial value, i.e. the expected present value of the future liabilities. In order to determine the premiums with respect to future liabilities, assumptions have to be made about the discount rate, mortality rate and the performance of the fund over the coming year. As mentioned previously, we assume that the insurer is risk neutral to mortality risk. The estimated value of the fund $\hat{V}_{k+1}^-$ at time $k+1$ by using the premium basis is given by:

$$\hat{V}_{k+1}^- = \left[V_k + (1-\alpha)(1-c)\,\pi_k - \pi_k^{(r)}\right](1+g)(1-e) \tag{3.14}$$

where g is the estimated growth of the fund over year $(k, k+1)$.

The risk premium $\pi_k^{(r)}$ can be charged as a deduction from the units or as an asset-based fee over the term of the contract. We consider both these charging options in section 3.3. For the remainder of this section, we assume that the risk premium is charged as a exogenously given deduction from the unit account.

The survival benefit is payable at maturity of the contract should the policyholder survive. Note the similarity between the simple investment product discussed in the previous section and a pure endowment VA contract. Recall that the contingent benefit considered is a benefit that pays the maximum of the cash value V_n^- and an exogenously determined benefit b_n at time n, the maturity of the contract, on survival of the policyholder. Therefore, we have the following contingency claim at time n:

$$B_n = \ {}_n p_x \max\left(V_n^-, b_n\right) \tag{3.15}$$

Again, the minimum guaranteed benefit at maturity can be written as a call-type option:

$$B_n = \ {}_n p_x b_n + \ {}_n p_x \max\left(V_n^- - b_n, 0\right) \tag{3.16}$$

or a put-type option:

$$B_n = \ {}_n p_x V_n^- + \ {}_n p_x \max\left(b_n - V_n^-, 0\right) \tag{3.17}$$

Assume a Black-Scholes-Merton setting and that the parameters of the contract (b_n, π_k, α, c) are fixed at policy inception, i.e. specified in the policyholder's contract, and the guarantee charge e is solved for. Combining equations (2.27) and (3.7), we find the following expression for the sub-account value V_n^- at time n:

$$\begin{aligned} V_n^- &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} \frac{F_n}{F_k} \\ &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2\right)(n-k) + \sigma(W(n) - W(k))} \end{aligned} \tag{3.18}$$

where $W(n)$ is the Brownian motion process at time n.

We derive the conditional lower bound for the pure endowment VA contract using the same steps as in section 2.4. First, we define the non-optimal conditional variable:

$$\Lambda = \sum_{k=0}^{n-1} \gamma_k \left[W(n) - W(k)\right] \tag{3.19}$$

where γ_k is the constant that optimises the choice of Λ.

The distribution of the Brownian motion differences, $W(n) - W(k)$, given the value of Λ follows a conditional bivariate normal distribution, therefore:

$$E\left[\frac{F(n)}{F(k)} \mid \Lambda = \lambda\right] = e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \tag{3.20}$$

where r_k is Pearson's correlation coefficient for the couple (Y_k, Λ). The non-optimal value of r_k was given in section 2.4 as:

$$r_k = \frac{\sum_{l=0}^{n-1} \gamma_l \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \gamma_j \gamma_l \min(n-j, n-l)}}$$

By using equation (3.20), we can find an expression for the conditional lower bound $V_n^{-(l)}$:

$$
\begin{aligned}
V_n^{-(l)} &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} E\left[\frac{F(n)}{F(k)} \mid \Lambda\right] \\
&= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \qquad (3.21)
\end{aligned}
$$

We can now optimise our choice of Λ by maximising the variance of the conditional lower bound $V_n^{-(l)}$ with respect to Λ. The variance of $V_n^{-(l)}$ is given as:

$$
\begin{aligned}
Var\left[V_n^{-(l)}\right] &= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \pi_k^{(s)}\pi_l^{(s)} (1-e)^{2n-k-l} e^{\delta(2n-k-l) - \frac{1}{2}\sigma^2\left(r_k^2(n-k) + r_l^2(n-l)\right)} \\
&\quad \times Cov\left[e^{\sigma r_k \sqrt{n-k} Z}, e^{\sigma r_l \sqrt{n-l} Z}\right] \qquad (3.22)
\end{aligned}
$$

where the covariance term was derived in the previous section as:

$$
Cov\left[e^{\sigma r_k \sqrt{n-k} Z}, e^{\sigma r_l \sqrt{n-l} Z}\right] = e^{\left[\frac{1}{2}\sigma^2\left(r_k^2(n-k) + r_l^2(n-l)\right)\right]} \left(e^{\sigma^2 r_k r_l \sqrt{n-k}\sqrt{n-l}} - 1\right)
$$

and substituting the result in equation (3.22), we now have:

$$
Var\left[V_n^{-(l)}\right] = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \pi_k^{(s)}\pi_l^{(s)} (1-e)^{2n-k-l} e^{\delta(2n-k-l)} \left(e^{\sigma^2 r_k r_l \sqrt{n-k}\sqrt{n-l}} - 1\right) \qquad (3.23)
$$

By expanding the exponential term in brackets and by using the definition of the correlation coefficient r_k, we can approximate the variance of $V_n^{-(l)}$ by:

$$
\begin{aligned}
Var\left[V_n^{-(l)}\right] &\approx \left[\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \sigma r_k \sqrt{n-k}\right] \left[\sum_{l=0}^{n-1} \pi_l^{(s)} (1-e)^{n-l} e^{\delta(n-l)} \sigma r_l \sqrt{n-l}\right] \\
&= \sigma^2 \left[\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} r_k \sqrt{n-k}\right]^2 \\
&= \sigma^2 \left[\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \frac{Cov\left[Y_k, \Lambda\right]}{\sigma_\Lambda \sqrt{n-k}} \sqrt{n-k}\right]^2 \\
&= \sigma^2 \left[\frac{Cov\left[\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} Y_k,\ \Lambda\right]}{\sigma_\Lambda}\right]^2 \\
&= \sigma^2 r_S^2 \sigma_S^2 \qquad (3.24)
\end{aligned}
$$

where $S = \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} Y_k$, and the correlation coefficient of the pair (S, Λ) is denoted by r_S.

Therefore, we can maximise the expression in equation (3.24) by maximising the correlation coefficient r_S. This is only the case if the pair (S, Λ) is perfectly correlated, negatively or positively. Therefore, we obtain the optimum choice of Λ as:

$$\begin{aligned} \Lambda &= S = \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} Y_k \\ &= \sum_{k=0}^{n-1} \left((1-\alpha)(1-c)\pi_k - \pi_k^{(r)} \right) (1-e)^{n-k} e^{\delta(n-k)} Y_k \end{aligned} \tag{3.25}$$

When comparing the optimal choice in equation (3.25) to the optimal choice of the previous section, equation (2.56), we see that by adding some real world complexities to the simple investment product results in mainly a change in the optimising constant γ_k. The optimising constant γ_k is now defined as:

$$\begin{aligned} \gamma_k &= \left((1-\alpha)(1-c)\pi_k - \pi_k^{(r)} \right) (1-e)^{n-k} e^{\delta(n-k)} \\ &= \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \end{aligned} \tag{3.26}$$

Finally, the value of the optimal correlation coefficient is given by:

$$r_k = \frac{\sum_{l=0}^{n-1} \pi_l^{(s)} (1-e)^{n-l} e^{\delta(n-l)} \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j^{(s)} \pi_l^{(s)} (1-e)^{2n-j-l} e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{3.27}$$

Recall that the payoff functions are the same as for the simple financial contract, i.e. the put-type option is given by:

$$P_n = E_Q \left[\left(b_n - V_n^{-(l)} \right)_+ \mid \mathcal{F}_t \right] \tag{3.28}$$

and the call-type option is given by:

$$C_n = E_Q \left[\left(V_n^{-(l)} - b_n \right)_+ \mid \mathcal{F}_t \right] \tag{3.29}$$

We consider the put-type payoff P_n first and then the call-type payoff C_n. The expression of $V_n^{-(l)}$ in equation (3.23) is a function of a multiple of the standard normal variable, i.e.:

$$\begin{aligned} V_n^{-(l)} &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \\ &= \sum_{k=0}^{n-1} \alpha_k e^{\sigma r_k \sqrt{n-k} Z}, \text{ with } \alpha_k = \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} \end{aligned} \tag{3.30}$$

Note that this expression is mathematically identical to the result in equation (2.62) of the previous section. By again using the result of Dhaene et al. (2002b), it follows from equation (2.66) that:

$$E\left[\left(\sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}Z}-b_n\right)_+\right]=\sum_{k=0}^{n-1}\alpha_k E\left[\left(e^{\sigma r_k\sqrt{n-k}Z}-e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right]$$

Recall that if a lognormal random variable X with distribution $\log X\sim N\left(\mu,\sigma^2\right)$ has the following stop-loss function:

$$E\left[(X-d)_+\right]=e^{\mu+\frac{1}{2}\sigma^2}\Phi\left[\frac{\mu+\sigma^2-\log(d)}{\sigma}\right]-d\Phi\left[\frac{\mu-\log(d)}{\sigma}\right]$$

we have:

$$\begin{aligned}
E\left[\left(V_n^{-(l)}-b_n\right)_+\right]&=\sum_{k=0}^{n-1}\alpha_k\left\{e^{\frac{1}{2}\sigma^2r_k^2(n-k)}\Phi\left[\frac{\sigma^2r_k^2(n-k)-\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}{\sigma r_k\sqrt{n-k}}\right]\right.\\
&\quad\left.-e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\Phi\left[\frac{-\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}{\sigma r_k\sqrt{n-k}}\right]\right\}\\
&=\sum_{k=0}^{n-1}\alpha_k e^{\frac{1}{2}\sigma^2r_k^2(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\\
&\quad-\sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\Phi\left[-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\\
&=\sum_{k=0}^{n-1}\alpha_k e^{\frac{1}{2}\sigma^2r_k^2(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\\
&\quad-\sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\left[1-F_{V_n^{-(l)}}(b_n)\right]\\
&=\sum_{k=0}^{n-1}\pi_k^{(s)}(1-e)^{n-k}e^{\delta(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\\
&\quad-b_n\left[1-F_{V_n^{-(l)}}(b_n)\right]
\end{aligned}\tag{3.31}$$

The last step in the derivation above follows from equation (2.65) such that:

$$F^{-1}_{V_n^{-(l)}}\left(F_{V_n^{-(l)}}(b_n)\right)=b_n$$

Note that $F^{-1}_{V_n^{-(l)}}$ has the same definition as in the previous section, i.e.:

$$F^{-1}_{V_n^{-(l)}}(p)=\sum_{k=0}^{n-1}F^{-1}_{X_k}(p)=\sum_{k=0}^{n-1}\alpha_k e^{\sigma r_k\sqrt{n-k}\Phi^{-1}(p)}$$

A relationship exists between equations (3.28) and (3.29) that allows us to obtain a result for the call-type payoff from the above derived put-type payoff result, i.e.:

$$E\left[\left(b_n - V_n^{-(l)}\right)_+\right] = E\left[\left(V_n^{-(l)} - b_n\right)_+\right] + b_n - E\left[V_n^{-(l)}\right] \tag{3.32}$$

First, observe that the expectation of $V_n^{-(l)}$ is given by:

$$\begin{aligned} E\left[V_n^{-(l)}\right] &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} E\left[e^{\sigma r_k \sqrt{n-k} Z}\right] \\ &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \end{aligned} \tag{3.33}$$

Substituting in the results from equations (3.31) and (3.33) into equation (3.32), we have:

$$\begin{aligned} E\left[\left(b_n - V_n^{-(l)}\right)_+\right] &= \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \left\{\Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] - 1\right\} \\ &\quad - b_n \left[1 - F_{V_n^{-(l)}}(b_n)\right] + b_n \\ &= -\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \left\{1 - \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right\} \\ &\quad + b_n F_{V_n^{-(l)}}(b_n) \\ &= -\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \Phi\left[-\sigma r_k \sqrt{n-k} + \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad + b_n F_{V_n^{-(l)}}(b_n) \end{aligned} \tag{3.34}$$

In both equations (3.31) and (3.34), we have the unknown quantity $F_{V_n^{-(l)}}(b_n)$. As before, the following result is used to find values of $F_{V_n^{-(l)}}(b_n)$:

$$\sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} e^{\sigma r_k \sqrt{n-k} \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)} - b_n = 0 \tag{3.35}$$

The present value of the embedded call option is given by:

$$\begin{aligned} C_0 &= {}_n p_x e^{-\delta n} E\left[\left(V_n^{-(l)} - b_n\right)_+\right] \\ &= \sum_{k=0}^{n-1} {}_n p_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad - {}_n p_x e^{-\delta n} b_n \left[1 - F_{V_n^{-(l)}}(b_n)\right] \end{aligned} \tag{3.36}$$

and the present value of the embedded put option is given by:

$$\begin{aligned}
P_0 &= {}_np_x e^{-\delta n} E\left[\left(b_n - V_n^{-(l)}\right)_+\right] \\
&= -\sum_{k=0}^{n-1} {}_np_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[-\sigma r_k \sqrt{n-k} + \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&\quad + {}_np_x e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) \qquad (3.37)
\end{aligned}$$

The value of the asset-based guarantee charge e can be determined by fair value principles, i.e. solve for e in the fair value equation:

$$E\left[\text{Present value of contributions}\right] = E\left[\text{Present value of benefits}\right] \qquad (3.38)$$

The fair value principle was applied in section 2.5 for both single premium and regular premium business. In the case of the put-type option representation of the GMMB, the fair value principle results in:

$$\begin{aligned}
\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} &= {}_np_x e^{-\delta n} E\left[\max\left(V_n^{-(l)}, b_n\right)\right] \\
&= {}_np_x e^{-\delta n} E\left[V_n^{-(l)}\right] + {}_np_x P_0 \\
&= \sum_{k=0}^{n-1} {}_np_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} + {}_np_x P_0 \\
&= \sum_{k=0}^{n-1} {}_np_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&\quad + {}_np_x e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) \qquad (3.39)
\end{aligned}$$

Note that the present value of contributions, left-hand side of equation (3.39), excludes the contribution attributable to ancillary risk benefits since these benefits are also excluded from the present value of benefits, right-hand side. The unknown guarantee charge e is found by solving the expression in equation (3.39) above.

The call-type option representation of the GMMB, which equals a n-term bond and a call option, can also be used in finding the guarantee charge e. By applying the fair value principle, the call-type option representation results in:

$$\begin{aligned}
\sum_{k=0}^{n-1} {}_k p_x \pi_k^{(s)} e^{-\delta k} &= {}_n p_x e^{-\delta n} E\left[\max\left(V_n^{-(l)}, b_n\right)\right] \\
&= {}_n p_x e^{-\delta n} b_n + {}_n p_x C_0 \\
&= {}_n p_x e^{-\delta n} b_n + \sum_{k=0}^{n-1} {}_n p_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&\quad + e^{-\delta n} b_n \left[1 - F_{V_n^{-(l)}}(b_n)\right] \\
&= \sum_{k=0}^{n-1} {}_n p_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&\quad + {}_n p_x e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) \qquad (3.40)
\end{aligned}$$

The unknown guarantee charge e is again found by solving the expression in equation (3.40) above. It is evident from equations (3.39) and (3.40) that the solution for the guarantee charge e is identical for the call-type option and put-type option representations.

3.4 Guaranteed Minimum Death Benefit

VA contracts typically contain a guaranteed minimum death benefit payable on the death of the policyholder during the term of the contract. This benefit usually guarantees the repayment of some function of the initial premium in single premium contracts, or the repayment of some function of the premiums paid until time of death in periodic premium contracts. The process of determining the risk premiums involves a choice of an appropriate mortality table, technical interest rate, charges for expenses and the assumed growth of the fund. We consider two types of death benefit: a death benefit that equals the sum of the cash value V_{k+1}^- and a deterministic amount b_{k+1}, and a death benefit that equals the maximum of the cash value V_{k+1}^- and a deterministic amount b_{k+1}. These benefits are paid to the beneficiaries at the end of year $(k, k+1)$.

Consider the death benefit that equals the sum of the cash value and a deterministic amount. The death benefit is defined as:

$$B_{k+1} = V_{k+1}^- + b_{k+1} \qquad (3.41)$$

The death benefit contains the fund value, which is linked to the liability, and a non-unit part in the additional specified amount b_{k+1} that implies the only liability to the insurer. The non-unit death benefit can grow by a specified rate of interest over the term of the contract.

The risk premium $\pi_k^{(r)}$, payable at the beginning of year $(k, k+1)$, can be determined without assumptions about the fund growth, i.e.:

$$\pi_k^{(r)} = b_{k+1} A\,(x+k) \tag{3.42}$$

where $A\,(x+k)$ is the cost-loaded actuarial premium for a one-year term insurance with death benefit equal to 1 and sold to the policyholder of age $x+k$. Recall that the fund value at the end of year $(k, k+1)$ is given by:

$$V_{k+1}^{-} = \left[V_k + (1-\alpha)(1-c)\,\pi_k - \pi_k^{(r)}\right] \frac{F_{k+1}}{F_k}(1-e)$$

or by using equation (3.42):

$$V_{k+1}^{-} = \left[V_k + (1-\alpha)(1-c)\,\pi_k - b_{k+1} A\,(x+k)\right] \frac{F_{k+1}}{F_k}(1-e) \tag{3.43}$$

The following upper bound on the choice of b_{k+1} is implied by the assumption that the fund value V_{k+1}^{-} is required to be non-negative throughout the term of the contract:

$$b_{k+1} \leq \frac{1}{A\,(x+k)}\left[V_k + (1-\alpha)(1-c)\,\pi_k\right] \tag{3.44}$$

The risk premium $\pi_k^{(r)}$ is deducted from the annual contribution after allocation costs. The cost-loaded premium includes costs incurred to administer the policy and to settle claims in the event of death. The deterministic death benefit is not seen as an investment guarantee since it contains only mortality risk and not investment risk. Should the sub-account performance realise significant losses, the sum at risk to the insurer remain the additional value b_{k+1}.

We now consider the investment guarantee on death in the death benefit that equals the maximum of the cash value and a deterministic amount. We consider two types of charging structures. The first charging structure deducts the risk premium from the annual contribution after allocation costs. The second charging structure is an asset-based structure, i.e. the risk premium is deducted at the end of the year as a percentage of the fund value. The asset-based fee typically forms part of the annual management charge if the death benefit is not an optional benefit. The value of the deterministic amount b_{k+1} has to be chosen by time k or before. The benefit is defined as:

$$B_{k+1} = \max\left(V_{k+1}^{-}, b_{k+1}\right) \tag{3.45}$$

The value of b_{k+1} in equation (3.45) can be chosen as a set proportion of the premiums paid until time k, e.g. 75%, 100% or 110%, thereby resulting in a money-back-upon-death guarantee.

Another possible choice for b_{k+1} could be the accumulated value of the premiums paid until time k, where the premiums accumulate at some guaranteed specified return r_g. The death benefit can be written as:

$$B_{k+1} = V_{k+1}^{-} + \max\left(0, b_{k+1} - V_{k+1}^{-}\right) \tag{3.46}$$

or

$$B_{k+1} = b_{k+1} + \max\left(V_{k+1}^{-} - b_{k+1}, 0\right) \tag{3.47}$$

The expression of equation (3.46) denotes the benefit as the fund value and a put-option on the fund with strike price b_{k+1}, whereas the expression of equation (3.47) denotes the benefit as the deterministic benefit value and a call-option on the fund with strike price b_{k+1}. In order to price these option-type guaranteed benefits, the methodology of the previous sections can be applied.

We assume again a Black-Scholes-Merton framework for the evolution of the underlying fund of the VA sub-account. Assume also that the insurance firm is risk-neutral with respect to mortality, i.e. that the random lifetime T_x of a policyholder aged x is replaced by the expected lifetime of the policyholder. Let $K+1$ denote the time of the death benefit payment made on the death of a policyholder aged x and assume for simplicity that this payment is made at the end of the year of death $(K, K+1)$. The discounted value at inception of the contract of a death benefit payment of 1 payable at the end of the year of death of a policyholder aged x for the term of the contract is given by:

$$v^{K+1} \tag{3.48}$$

where K is considered during the period $(0, n)$, i.e. for the duration of the contract.

For an insurer that is risk-neutral with respect to mortality, the benefit becomes:

$$\begin{aligned} A_{x:\overline{n}\rceil}^{1} &= E\left[v^{K+1}\right] \\ &= \sum_{k=0}^{n-1} {}_{k}p_x \; q_{x+k} \end{aligned}$$

where K is defined on $(0, n)$ and ${}_{k}p_x \; q_{x+k}$ represents the probability that the random lifetime T_x within the year $(k, k+1)$, i.e. that the policyholder aged x at inception survives to the beginning of year k and die within the year $(k, k+1)$:

$$P\left[k \leq T_x \leq k+1\right] =_k p_x \; q_{x+k} \tag{3.49}$$

We first consider the case where the deduction for the death benefit is made directly from the annual contribution for regular premium business, or where the deduction is made from the VA sub-account annually for single premium business. The evolution of the VA sub-account in the case of regular premium business is given by:

$$V_{k+1}^{-}=\left[V_k+(1-\alpha)(1-c)\pi_k-\pi_k^{(r)}\right]\frac{F_{k+1}}{F_k}(1-e) \tag{3.50}$$

The risk premium $\pi_k^{(r)}$ is determined at the beginning of the year and deducted directly from the contribution after allocation costs. The risk premium covers the mortality risk for the year $(k, k+1)$ only. Therefore, the death benefit can be seen as an annually renewable death benefit under the above-mentioned construction. The risk premium for the annual death benefit is determined by assuming that investment risk is independent of mortality risk, therefore the risk premium is given as the product of the expectation that a life aged x dies in the year $(k, k+1)$ and the sum at risk.

A typical death benefit guarantees that the death benefit is the minimum of the contributions paid and the underlying sub-account value. Note that the guaranteed amount could also be any function of the contributions paid, e.g. 75% or increased at a specified rate per annum. The death benefit for the year $(k, k+1)$ is given by:

$$B_{k+1}=\max\left(V_{k+1}^{-},b_{k+1}\right) \tag{3.51}$$

The guaranteed benefit in equation (3.51) above can be expressed as a put-type option and a call-type option as in the previous section, i.e. the put-type option is given by:

$$B_{k+1}=V_{k+1}^{-}+\max\left(0,b_{k+1}-V_{k+1}^{-}\right) \tag{3.52}$$

and the call-type option is given by:

$$B_{k+1}=b_{k+1}+\max\left(V_{k+1}^{-}-b_{k+1},0\right) \tag{3.53}$$

The more natural representation of the death benefit is the put-type option expression of equation (3.52). At the end of the year of death, the nominated beneficiaries receive the fund value and a top-up value in the case of the fund value being less than the guaranteed amount b_{k+1}.

The mortality risk only applies over the year $(k, k+1)$, therefore the sum at risk is the difference of the guaranteed minimum sum assured at time $k+1$ and the fund value at the end of year $(k, k+1)$, i.e.:

$$S_R = \max\left(b_{k+1} - \left[V_k + (1-\alpha)(1-c)\pi_k - \pi_k^{(r)}\right](1-e)\, e^{\left(\delta - \frac{1}{2}\sigma^2\right) + \sigma(W(k+1) - W(k))},\ 0\right) \quad (3.54)$$

Note that the risk premium $\pi_k^{(r)}$ has to be solved for from the expression:

$$\pi_k^{(r)} = A(x+k)\, e^{-\delta} E[S_R] \quad (3.55)$$

where $A(x+k)$ is the cost-loaded actuarial premium for a one-year term insurance with death benefit equal to 1 and sold to the policyholder of age $x+k$. The payoff function in equation (3.54) is solved by applying the standard Black-Scholes formula. The standard Black-Scholes formula for an European put option with underlying fund value F_k and strike price b_{k+1} over the period $(k, k+1)$ is given by:

$$P_k = b_{k+1} e^{-\delta} \Phi(-d_2) - F_k \Phi(-d_1) \quad (3.56)$$

where d_1 and d_2 are given by:

$$d_1 = \frac{\log(F_k / b_{k+1}) + \left(\delta + \frac{1}{2}\sigma^2\right)}{\sigma} \quad (3.57)$$

and:

$$d_2 = d_1 - \sigma \quad (3.58)$$

The expected value of the expression in equation (3.54) at the start of year $(k, k+1)$ is given by:

$$P_k = \left[V_k + \pi_k^{(s)}\right](1-e)\, e^{-\delta} E\left[\left(\frac{b_{k+1}}{\left[V_k + \pi_k^{(s)}\right](1-e)} - X\right)_+\right] \quad (3.59)$$

where X follows a lognormal distribution with parameters $\left(\delta + \frac{1}{2}\sigma^2\right)$ and σ^2. By applying the Black-Scholes formula given in equation (3.56), we find:

$$P_k = e^{-\delta} b_{k+1} \Phi(-d_2) - \left[V_k + \pi_k^{(s)}\right](1-e)\, \Phi(-d_1) \quad (3.60)$$

where d_1 and d_2 are given by:

$$d_1 = \frac{\log\left(\left[V_k + \pi_k^{(s)}\right](1-e)\right) - \log(b_{k+1}) + \left(\delta + \frac{1}{2}\sigma^2\right)}{\sigma} \quad (3.61)$$

and:

$$d_2 = d_1 - \sigma$$

Table 3.1: Risk premium deduction for the embedded put options with $k = 3$, $b_{k+1} = 4$ and varying starting underlying sub-account values and economic parameters.

	V_k	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
$\delta = 1\%$	1	0.393027	0.393050	0.393491
	2	0.205996	0.212991	0.225308
	3	0.062351	0.088725	0.114789
	4	0.011317	0.031733	0.055999
$\delta = 5\%$	1	0.363505	0.363552	0.364219
	2	0.177504	0.186652	0.200656
	3	0.046929	0.072885	0.098513
	4	0.007393	0.024546	0.046503
$\delta = 10\%$	1	0.328226	0.328332	0.329401
	2	0.144466	0.156535	0.172467
	3	0.031867	0.056211	0.080747
	4	0.004172	0.017517	0.036544

To determine the value of the risk premium $\pi_k^{(r)}$, we substitute the above result in equation (3.55):

$$\pi_k^{(r)} = A(x+k)\,P_k \tag{3.62}$$

where

$$P_k = A(x+k)\left\{e^{-\delta} b_{k+1}\Phi(-d_2) - \left[V_k + (1-\alpha)(1-c)\,\pi_k - \pi_k^{(r)}\right](1-e)\,\Phi(-d_1)\right\} \tag{3.63}$$

Note that the risk premium $\pi_k^{(r)}$ is present in the expressions for d_1 and d_2. As an illustrative example, consider the case of a policyholder aged 30 who took out a VA contract that requires a contribution of 1 per annum and guarantees repayment of paid contributions on death. The contract is in year 3 since inception. The allocation costs are given such that $\alpha = 4\%$ and $c = 5\%$, while the management fee e is given as 1%. We consider different values for the initial fund value V_k and economic parameters to illustrate the sensitivity of the risk premium to the starting sub-account value at time $k = 3$. The cost-loaded actuarial premium for one year's term assurance $A(x+k)$ is given by 0.16. The results of the example are given in Table 3.1 from which it is evident that the deduction for the guaranteed minimum death benefit is very sensitive to the starting value of the sub-account, but less sensitive to the economic parameters.

We now consider the second GMDB fee construction where the GMDB expense over the term of the contract is calculated at inception. The GMDB expense represents the initial outgo that is required as part of the risk management strategy, e.g. entering into hedging arrangements at inception. The initial risk management expense is then recouped by charging an annual asset-based fee e_g. The GMDB fee e_g typically forms part of the regular management fee e. The management fee e in practice consists of investment management charges. insurance charges to cover mortality and expense risks, and investment guarantee charges. In Chapter 2, we assumed that the annual management fee e is used to fully fund for the GMMB expense. We assume in the following that the annual management fee is a composite fee allowing for the investment guarantee charge e_g, and all other asset-based charges e_a. Note that the magnitude of the charges rely on the order in which these asset-based fees are deducted. To simplify the exposition, we assume that a single asset-based fee e is charged at the end of each contribution period, i.e.:

$$e = e_a + e_g \tag{3.64}$$

To ease the exposition on how to solve for the GMDB asset-based fee e_g, we assume in the following that the other asset-based fees e_a are exogenously given. The GMDB in the year $(k, k+1)$ is given by:

$$B_{k+1} = \max\left(V_{k+1}^{-}, b_{k+1}\right)$$

and can be expressed in a put-type option payoff and call-type option payoff as given in equations (3.52) and (3.53), respectively. The key difference between the annual deduction of a risk premium $\pi_k^{(r)}$ at time k and the charging of an annual asset-based charge e_g is the filtration assumed. We assume in the former case that the filtration is at time k, therefore allowing us to observe the value of the starting fund value V_k. In the latter case, the asset-based fee e_g aims to recoup the expense calculated at inception, therefore assuming a filtration at time 0. The annual deduction can be changed to allow for worse mortality experience, e.g. incidence of HIV/AIDS, whereas the asset-based fee e_g is usually contractual for the term of the contract. The asset-based structure also incorporates significant investment risk, e.g. a lower fee income is generated should the sub-account value decrease substantially. The risk premium deduction, on the other hand, is premium based that allows the risk to be fully funded for, i.e. the risk premium deduction increases as the underlying fund value decreases, whereas the asset-based charge decreases as the underlying fund value decreases.

The GMDB for a death in year $(k, k+1)$ in the case of a put-type option payoff is given by:

$$\begin{aligned} B_{k+1} &= V_{k+1}^{-} + \max\left(b_{k+1} - V_{k+1}^{-}, 0\right) \\ &= \sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{n-j} \frac{F_{k+1}}{F_j} + \max\left(b_{k+1} - \sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{n-j} \frac{F_{k+1}}{F_j}, 0\right) \end{aligned} \tag{3.65}$$

The payoff function in equation (3.65) consist of an Asian put-type option with a strike price equal to the minimum guaranteed death benefit. This option is available to the beneficiaries of the policyholder annually. To be able to solve for the asset-based fee e_g, we need to approximate the distribution of the sub-account value V_{k+1}^{-} at each time $k+1$ for $0 \leq k \leq n$. The derivation of the conditional lower bound approximation to the sub-account value $V_{k+1}^{-(l)}$ follows the same steps as in section 2.4. First, we define the non-optimal conditional variable:

$$\Lambda = \sum_{j=0}^{k} \gamma_j \left[W(n) - W(j)\right] \tag{3.66}$$

where γ_j is the constant that optimises the choice of Λ.

The distribution of the Brownian motion differences, $W(n) - W(j)$, given the value of Λ follows a conditional bivariate normal distribution, therefore:

$$E\left[\frac{F(n)}{F(j)} \mid \Lambda = \lambda\right] = e^{\left(\delta - \frac{1}{2}\sigma^2 r_j^2\right)(n-j) + \sigma r_j \sqrt{n-j} Z} \tag{3.67}$$

where r_j is Pearson's correlation coefficient for the couple (Y_k, Λ). The non-optimal value of r_j was given in section 2.4 as:

$$r_j = \frac{\sum_{l=0}^{k} \gamma_l \min(n-j, n-l)}{\sqrt{n-j}\sqrt{\sum_{i=0}^{k}\sum_{l=0}^{k} \gamma_i \gamma_l \min(n-i, n-l)}}$$

By using equation (3.67), we can find an expression for the conditional lower bound $V_{k+1}^{-(l)}$:

$$\begin{aligned} V_{k+1}^{-(l)} &= \sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{n-j} E\left[\frac{F(n)}{F(j)} \mid \Lambda\right] \\ &= \sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{n-j} e^{\left(\delta - \frac{1}{2}\sigma^2 r_j^2\right)(n-j) + \sigma r_j \sqrt{n-j} Z} \end{aligned} \tag{3.68}$$

The conditional variable Λ is optimised by maximising the variance of the conditional lower bound $V_{k+1}^{-(l)}$ with respect to Λ, where the variance of $V_{k+1}^{-(l)}$ is given as:

$$\begin{aligned} Var\left[V_{k+1}^{-(l)}\right] &= \sum_{j=0}^{k}\sum_{l=0}^{k} \pi_j^{(s)} \pi_l^{(s)} (1-e)^{2n-j-l} e^{\delta(2n-j-l) - \frac{1}{2}\sigma^2\left(r_j^2(n-j) + r_l^2(n-l)\right)} \\ &\quad \times Cov\left[e^{\sigma r_j \sqrt{n-j} Z}, e^{\sigma r_l \sqrt{n-l} Z}\right] \end{aligned} \tag{3.69}$$

The covariance term was derived in the section 2.4 as:

$$Cov\left[e^{\sigma r_j\sqrt{n-j}Z}, e^{\sigma r_l\sqrt{n-l}Z}\right] = e^{\left[\frac{1}{2}\sigma^2\left(r_j^2(n-j)+r_l^2(n-l)\right)\right]}\left(e^{\sigma^2 r_j r_l\sqrt{n-j}\sqrt{n-l}}-1\right)$$

By substituting the result in equation (3.69), we now have:

$$Var\left[V_{k+1}^{-(l)}\right] = \sum_{j=0}^{k}\sum_{l=0}^{k}\pi_j^{(s)}\pi_l^{(s)}\left(1-e\right)^{2n-j-l}e^{\delta(2n-j-l)}\left(e^{\sigma^2 r_j r_l\sqrt{n-j}\sqrt{n-l}}-1\right) \tag{3.70}$$

By expanding the exponential term in brackets and by using the definition of the correlation coefficient r_j, we can approximate the variance of $V_{k+1}^{-(l)}$ by:

$$\begin{aligned}
Var\left[V_{k+1}^{-(l)}\right] &\approx \left[\sum_{j=0}^{k}\pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}\sigma r_j\sqrt{n-j}\right]\left[\sum_{l=0}^{k}\pi_l^{(s)}\left(1-e\right)^{n-l}e^{\delta(n-l)}\sigma r_l\sqrt{n-l}\right] \\
&= \sigma^2\left[\sum_{j=0}^{k}\pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}r_j\sqrt{n-j}\right]^2 \\
&= \sigma^2\left[\sum_{j=0}^{k}\pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}\frac{Cov\left[Y_j,\Lambda\right]}{\sigma_\Lambda\sqrt{n-j}}\sqrt{n-j}\right]^2 \\
&= \sigma^2\left[\frac{Cov\left[\sum_{j=0}^{k}\pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}Y_j,\ \Lambda\right]}{\sigma_\Lambda}\right]^2 \\
&= \sigma^2 r_S^2\sigma_S^2
\end{aligned} \tag{3.71}$$

where $S=\sum_{j=0}^{k}\pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}Y_j$, and the correlation coefficient of the pair (S,Λ) is denoted by r_S.

Recall that we can maximise the expression in equation (3.71) by maximising the correlation coefficient r_S. This is only the case if the pair (S,Λ) is perfectly correlated, negatively or positively. The optimum choice of Λ is given by:

$$\begin{aligned}
\Lambda &= S = \sum_{j=0}^{k}\pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}Y_j \\
&= \sum_{j=0}^{k}\left(\left(1-\alpha\right)\left(1-c\right)\pi_j-\pi_j^{(r)}\right)\left(1-e\right)^{n-j}e^{\delta(n-j)}Y_j
\end{aligned} \tag{3.72}$$

Therefore, the solution for the optimising constant γ_j is given by:

$$\begin{aligned}
\gamma_j &= \left(\left(1-\alpha\right)\left(1-c\right)\pi_j-\pi_j^{(r)}\right)\left(1-e\right)^{n-j}e^{\delta(n-j)} \\
&= \pi_j^{(s)}\left(1-e\right)^{n-j}e^{\delta(n-j)}
\end{aligned} \tag{3.73}$$

Finally, the value of the optimal correlation coefficient is given by:

$$r_j = \frac{\sum_{l=0}^{k} \pi_l^{(s)} \left(1-e\right)^{n-l} e^{\delta(n-l)} \min(n-j, n-l)}{\sqrt{n-j}\sqrt{\sum_{i=0}^{k}\sum_{l=0}^{k} \pi_i^{(s)}\pi_l^{(s)} \left(1-e\right)^{2n-i-l} e^{\delta(2n-i-l)} \min(n-i, n-l)}} \tag{3.74}$$

We now consider the put-type payoff P_{k+1} at time $k+1$. The expression of $V_{k+1}^{-(l)}$ in equation (3.68) is a function of a multiple of the standard normal variable, i.e.:

$$\begin{aligned} V_{k+1}^{-(l)} &= \sum_{j=0}^{k} \pi_j^{(s)} \left(1-e\right)^{n-j} e^{\left(\delta - \frac{1}{2}\sigma^2 r_j^2\right)(n-j) + \sigma r_j \sqrt{n-j} Z} \\ &= \sum_{j=0}^{k} \alpha_j e^{\sigma r_j \sqrt{n-j} Z}, \text{ with } \alpha_j = \sum_{j=0}^{k} \pi_j^{(s)} \left(1-e\right)^{n-j} e^{\left(\delta - \frac{1}{2}\sigma^2 r_j^2\right)(n-j)} \end{aligned} \tag{3.75}$$

By using the result of Dhaene et al. (2002b), we have that the expectation of the stop-loss function of a sum of the random variable Z simply becomes the sum of the expectations of the stop-loss function of the random variable Z:

$$E\left[\left(\sum_{j=0}^{k} \alpha_j e^{\sigma r_j \sqrt{n-j} Z} - b_{k+1}\right)_+\right] = \sum_{k=0}^{k} \alpha_j E\left[\left(e^{\sigma r_j \sqrt{n-k} Z} - e^{\sigma r_j \sqrt{n-j}\Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right)}\right)_+\right] \tag{3.76}$$

Recall that a lognormal random variable X with distribution $\log X \sim N\left(\mu, \sigma^2\right)$ has the following stop-loss function:

$$E\left[(X-d)_+\right] = e^{\mu + \frac{1}{2}\sigma^2} \Phi\left[\frac{\mu + \sigma^2 - \log(d)}{\sigma}\right] - d\Phi\left[\frac{\mu - \log(d)}{\sigma}\right] \tag{3.77}$$

The call-type option payoff under the conditional lower bound approximation therefore follows as:

$$
\begin{aligned}
&E\left[\left(V_{k+1}^{-(l)} - b_{k+1}\right)_+\right] \\
&= \sum_{j=0}^{k} \alpha_j \left\{ e^{\frac{1}{2}\sigma^2 r_j^2 (n-j)} \Phi \left[\frac{\sigma^2 r_j^2 (k-j+1) - \sigma r_j \sqrt{k-j+1} \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right)}{\sigma r_j \sqrt{k-j+1}} \right] \right. \\
&\left. -e^{\sigma r_j \sqrt{k-j+1} \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right)} \Phi \left[\frac{-\sigma r_j \sqrt{k-j+1} \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right)}{\sigma r_j \sqrt{k-j+1}} \right] \right\} \\
&= \sum_{j=0}^{k} \alpha_j e^{\frac{1}{2}\sigma^2 r_j^2 (k-j+1)} \Phi \left[\sigma r_j \sqrt{k-j+1} - \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right) \right] \\
&- \sum_{j=0}^{k} \alpha_j e^{\sigma r_j \sqrt{n-j} \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right)} \left[1 - F_{V_{k+1}^{-(l)}}(b_{k+1}) \right] \\
&= \sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{k-j+1} e^{\delta(k-j+1)} \Phi \left[\sigma r_j \sqrt{k-j+1} - \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right) \right] \\
&- b_{k+1} \left[1 - F_{V_{k+1}^{-(l)}}(b_{k+1}) \right] \qquad (3.78)
\end{aligned}
$$

The put-type option payoff can be found from the result in equation (3.78) above by using the equality in equation (2.55), i.e.:

$$
\begin{aligned}
&E\left[\left(b_{k+1} - V_{k+1}^{-(l)}\right)_+\right] \\
&= \sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{k-j+1} e^{\delta(k-j+1)} \left\{ \Phi \left[\sigma r_j \sqrt{k-j+1} - \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right) \right] - 1 \right\} \\
&- b_{k+1} \left[1 - F_{V_{k+1}^{-(l)}}(b_{k+1}) \right] + b_{k+1} \\
&= -\sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{k-j+1} e^{\delta(k-j+1)} \left\{ 1 - \Phi \left[\sigma r_j \sqrt{k-j+1} - \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right) \right] \right\} \\
&+ b_{k+1} F_{V_{k+1}^{-(l)}}(b_{k+1}) \\
&= -\sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{k-j+1} e^{\delta(k-j+1)} \Phi \left[-\sigma r_j \sqrt{k-j+1} + \Phi^{-1}\left(F_{V_{k+1}^{-(l)}}(b_{k+1})\right) \right] \\
&+ b_{k+1} F_{V_{k+1}^{-(l)}}(b_{k+1}) \qquad (3.79)
\end{aligned}
$$

Note that $F^{-1}_{V^{-(l)}_{k+1}}$ has the same definition as in section 2.3, i.e.:

$$F^{-1}_{V^{-(l)}_{k+1}}(p) = \sum_{j=0}^{k} \alpha_j e^{\sigma r_j \sqrt{k-j+1}\Phi^{-1}(p)}$$

The unknown quantity $F_{V^{-(l)}_{k+1}}(b_{k+1})$ is solved for in the equation:

$$\sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{k-j+1} e^{\left(\delta - \frac{1}{2}\sigma^2 r_j^2\right)(k-j+1)} e^{\sigma r_j \sqrt{k-j+1}\Phi^{-1}\left(F_{V^{-(l)}_{k+1}}(b_{k+1})\right)} - b_{k+1} = 0 \tag{3.80}$$

The present value of the embedded put option is given by:

$$\begin{aligned} P_0 &= e^{-\delta(k+1)} E\left[\left(b_{k+1} - V^{-(l)}_{k+1}\right)_+\right] \\ &= -\sum_{j=0}^{k} \pi_j^{(s)} (1-e)^{k-j+1} e^{-\delta j} \Phi\left[-\sigma r_j \sqrt{k-j+1} + \Phi^{-1}\left(F_{V^{-(l)}_{k+1}}(b_{k+1})\right)\right] \\ &\quad + e^{-\delta(k+1)} b_{k+1} F_{V^{-(l)}_{k+1}}(b_{k+1}) \end{aligned} \tag{3.81}$$

The value of the asset-based guarantee charge e_g can be determined by fair value principles, i.e. solve for e_g in the fair value equation:

$$E\left[\text{Present value of contributions}\right] = E\left[\text{Present value of benefits}\right]$$

The fair value principle was applied in the GMMB case for both single premium and regular premium business. In the case of the put-type option representation of the GMDB, the expected present value of the payoffs is:

$$B_0 = \sum_{k=0}^{n-1} e^{-\delta(k+1)} E\left[\left(b_{k+1} - V^{-(l)}_{k+1}\right)_+\right] \tag{3.82}$$

The payoff for each year is contingent on the life aged $x+k$ for $k = 0, ..., n-1$ dying within year $(k, k+1)$. Since we assumed mortality risk to be independent of investment risk, the expected present value of the total benefit is:

$$\sum_{k=0}^{n-1} {}_k p_x \; q_{x+k} e^{-\delta(k+1)} E\left[V^{-(l)}_{k+1} + \left(b_{k+1} - V^{-(l)}_{k+1}\right)_+\right] \tag{3.83}$$

where:

$$e^{-\delta(k+1)} E\left[V^{-(l)}_{k+1}\right] = \sum_{j=0}^{k+1} \left((1-\alpha)(1-c)\pi_j\right)(1-e)^{k-j+1} e^{-\delta j} \tag{3.84}$$

Note that a risk premium deduction $\pi_j^{(r)}$ can be added to equation (3.85) to cover other risk benefits, e.g. a waiver of premium benefit. In a fair value premium scenario, the expected value of the benefit added should equal the risk premium deduction.

The asset-based GMDB charge e_g is solved by equating the expected present value of the contributions and the expected present value of the benefits in equation (3.83). The expected present value of the contributions is given by:

$$\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} \tag{3.85}$$

Note that the premium is the original premium π_k and not the savings premium $\pi_k^{(s)}$ of equation (3.83). By expanding the expected value term of equation (3.83), the GMDB charge e_g can be solved in the following equation:

$$\begin{aligned}
&\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} \\
= &\sum_{k=0}^{n-1}\sum_{j=0}^{k} {}_kp_x \; q_{x+k}\pi_j^{(s)} e^{-\delta j}\left(1-e_a-e_g\right)^{k-j+1}\Phi\left[\sigma r_j\sqrt{k-j+1}+\Phi^{-1}\left(F_{V_{k+1}^{-(l)}}\left(b_{k+1}\right)\right)\right] \\
&+\sum_{k=0}^{n-1} e^{-\delta(k+1)} b_{k+1} F_{V_{k+1}^{-(l)}}\left(b_{k+1}\right)
\end{aligned} \tag{3.86}$$

The asset-based charge for other fees besides the investment guarantee charge is assumed to be exogenously given. Therefore, the only unknown in equation (3.86) above is the GMDB asset-based charge e_g.

3.5 Guaranteed Minimum Death and Survival Benefits

Endowment type VA products typically offer a guaranteed minimum benefit on both death and survival. Since these life-contingent events are mutually exclusive, the investment guarantee benefit simplifies to being the sum of the two life-contingent benefits, i.e.:

$$\begin{aligned}
B_0 &= B_0^q + B_0^p \\
&= \sum_{k=0}^{n-1} {}_kp_x \; q_{x+k} e^{-\delta(k+1)} E\left[\max\left(b_{k+1}^q, V_{k+1}^{-(l)}\right)_+\right] + {}_tp_x e^{-\delta n} E\left[\max\left(V_n^{-(l)}, b_n^p\right)\right]
\end{aligned} \tag{3.87}$$

where b^q_{k+1} denotes the guaranteed minimum death benefit and b^p_n denotes the guaranteed minimum survival benefit. By using the split of the annual management charge of equation (3.64), we again solve for the unknown asset-based guarantee charge e_g from the expression:

$$\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} = B_0^q + B_0^p \tag{3.88}$$

where the death benefit B_0^q is given by the right-hand side of equation (3.86), i.e.:

$$\begin{aligned} B_0^q &= \sum_{k=0}^{n-1}\sum_{j=0}^{k} {}_kp_x\ q_{x+k}\pi_j^{(s)} e^{-\delta j}\left(1-e_a-e_g\right)^{k-j+1}\Phi\left[\sigma r_j\sqrt{k-j+1}+\Phi^{-1}\left(F_{V_{k+1}^{-(l)}}\left(b_{k+1}\right)\right)\right] \\ &\quad + \sum_{k=0}^{n-1} e^{-\delta(k+1)} b_{k+1} F_{V_{k+1}^{-(l)}}\left(b_{k+1}\right) \end{aligned} \tag{3.89}$$

The survival benefit B_0^p can expressed in terms of the asset-based investment guarantee charge e_g:

$$\begin{aligned} B_0^p &= \sum_{k=0}^{n-1} {}_np_x \pi_k^{(s)}\left(1-e_a-e_g\right)^{n-k} e^{-\delta k}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right] \\ &\quad + {}_np_x e^{-\delta n} b_n F_{V_n^{-(l)}}\left(b_n\right) \end{aligned} \tag{3.90}$$

Note that we assume one asset-based guarantee charge for both the death and survival benefits. The portion of the total guarantee charge applicable to the death benefit can be established by:

$$\frac{B_0^q}{B_0} e_g \tag{3.91}$$

and similarly for the portion of the guarantee charge applicable to the survival benefit:

$$\frac{B_0^p}{B_0} e_g \tag{3.92}$$

3.6 Illustrative Example

In the following, we demonstrate the application of the results obtained for solving for the asset-based guarantee charge e_g. We first consider the asset-guarantee charge for the survival benefit and then for the death benefit. The economic assumptions are again varied to illustrate the sensitivity of the asset-based guarantee charge to economic parameters. The contributions π_k, $k = 0, ..., n$, are assumed at 100 per annum and the term of the contract is taken as $n = 10$ years. The GMDB, b_{k+1}, and GMMB, b_n, are assumed to be return of contributions type benefits, therefore comprising the vector $\{b_1 = 100, b_2 = 200, ..., b_{10} = 1000\}$. We also consider the case of a benefit equal to 50%, 75%, 125% and 150% of contributions.

The policyholder is assumed to be a male aged 30 at inception of the policy, while mortality is assumed to follow the PMA92 mortality tables, year of use 2010, as published by the Continuous Mortality Investigation Bureau (CMIB) of the Institute and Faculty of Actuaries, U.K. The aforementioned mortality tables were used for simplicity and a more complex approach can be adopted, e.g. Ballotta and Haberman (2006) considered a stochastic mortality framework in the valuation of guaranteed annuity options. A deterministic mortality assumption can pose serious consequences to an insurer's solvency if its book of policies has few policyholders or if the book of policies consists of largely homogeneous policyholders that are susceptible to the same mortality shocks. In such cases, the Law of Large Numbers does not apply and the added mortality risk implies an additional cost, which must be charged to the policyholder for the insurer to accept the additional risk. This cost stems from either more aggressive reinsurance treaties or more stringent reserving requirements.

An extreme upper bound is considered for the embedded option value in the case of the GMMB. If the underlying fund value were to fall to zero, the payoff becomes certain and the only randomness relates to the survival of the policyholder. The benefit therefore becomes a pure endowment contract with a sum assured equal to the certain payoff. It is evident from Table 3.2 that the value of the embedded option in the combined market case will approach the extreme upper bound for large values of the volatility parameter and small values of the risk-free rate parameter, i.e. implying no time-value of money.

In Table 3.3, the extreme upper bound to the value of the embedded option is also given. If the underlying fund value were to fall to zero, the payoff becomes a certainty and the only randomness relates to the mortality of the policyholder. Therefore, in this case the benefit would be the term life death benefit with a sum assured equal to the certain payoff in the event of death.

Table 3.2: Conditional Lower Bound (CLB) results for the discounted values of the GMDB embedded put options with $n = 10$ for a policyholder aged 30.

			σ		Pure
δ	b_n	20%	30%	40%	Endowment
1%	50%	1.9260	14.2503	36.3826	451.5138
	75%	31.1084	76.2113	125.1575	677.2707
	100%	120.4741	189.4874	255.4479	903.0276
	125%	266.2231	340.6167	413.5743	1128.7845
	150%	448.6732	516.8435	590.2473	1354.5413
5%	50%	0.2893	4.5975	15.6588	302.6587
	75%	7.6430	30.1871	60.2442	453.9881
	100%	39.2845	84.5163	131.1935	605.3175
	125%	104.0098	164.2858	221.7969	756.6468
	150%	197.9962	263.4797	326.5898	907.9762
10%	50%	0.0178	0.9375	4.9634	183.5718
	75%	0.9197	8.1576	22.2728	275.3577
	100%	7.0436	26.9571	53.0722	367.1436
	125%	24.3388	58.5864	95.3486	458.9295
	150%	55.9512	101.8676	146.7768	550.7154

Table 3.3: Conditional Lower Bound (CLB) results for the discounted values of the GMMB embedded put options with $n = 10$ for a policyholder aged 30.

			σ		Term Life
δ	b_n	20%	30%	40%	Benefit
1%	50%	0.0012	0.0106	0.0298	0.5309
	75%	0.0265	0.0696	0.1183	0.7964
	100%	0.1242	0.1943	0.2620	1.0618
	125%	0.2997	0.3720	0.4451	1.3273
	150%	0.5229	0.5840	0.6542	1.5927
5%	50%	0.0002	0.0038	0.0144	0.4014
	75%	0.0079	0.0320	0.0650	0.6021
	100%	0.0511	0.1024	0.1550	0.8028
	125%	0.1487	0.2135	0.2764	1.0035
	150%	0.2907	0.3547	0.4202	1.2043
10%	50%	0.00002	0.0010	0.0055	0.2871
	75%	0.0014	0.0113	0.0297	0.4306
	100%	0.0148	0.0437	0.0785	0.5741
	125%	0.0565	0.1025	0.1497	0.7177
	150%	0.1297	0.1843	0.2384	0.8612

Chapter 4

HEDGING STRATEGIES

4.1 Introduction

In order to fully manage the risks associated with investment guarantees, a pricing exercise in isolation is not sufficient. To effectively manage the guarantee liability, sufficient capital needs to be held and invested according to a suitable risk-mitigating or hedging strategy. The risks throughout the term of the policy need to be addressed. In the following, a dynamic hedging strategy is considered. Although referred to as dynamic, we assume that rebalancing of the hedging portfolio is done on a discrete, periodic basis, e.g. monthly. Life insurance policies are typically long term in nature and the frictional costs erode the profitability of a product if the frequency of rebalancing is performed on a more frequent, e.g. daily, basis. The hedging portfolio is also not a self-financing portfolio due to the path-dependency of the embedded option. As mentioned in Chapter 1, several possible hedging strategies are considered in literature. Due to the variety of investment guarantees offered on VA business, certain hedging strategies might prove more suitable and optimal for respective investment guarantees.

Possible solutions to transfer the market risk of investment guarantees include reinsurance, structured solutions, and hedging methods. Hedging methods include, among others, dynamic hedging, super-hedging, risk-minimising strategies, mean-variance hedging and quantile hedging. Sun (2009) discusses the availability of reinsurance and structured solutions to insurers after the financial crisis of 2008 and generally following severe market conditions. Reinsurance solutions are not always available and are very dependent on market conditions. After the financial crisis of 2002, reinsurers left the VA reinsurance market and caused major difficulties for some VA insurers. Reinsurance was not available for many years thereafter, but gradually reinsurers re-entered the market and reinsurance pricing became competitive at times. Most reinsurers again withdrew from the VA market during the sustained financial crisis that commenced in 2007. Consequently, the pricing of the reinsurers who remained were generally not attractive to VA insurers. It follows, therefore, that severe market downturns generally result in a reinsurance market failure. A similar conclusion applies to over the counter structured solutions of investment banks. Many banks exit the market or quote prohibitively high prices due to a lower risk appetite during market downturns. Investment banks still play a valuable role in assisting insurers who develop their internal hedging programmes to manage the hedging programme, i.e. the operational risk associated with managing the investment risk is effectively transferred to the investment bank. The management of the hedging programme can also be performed by the insurer.

Møller and Steffensen (2007) give an overview of super-hedging, risk minimising and mean-variance hedging strategies and derive results for UL contracts. Super-hedging or super-replicating strategies were suggested by El Karoui and Quenez (1995), where such strategies propose an optimal self-financing strategy such that the hedge portfolio value almost surely exceeds the contingent claim at expiry of the contract. Static strategies, such as super–hedging strategies, are usually not viable for hedging integrated risks in the combined market. The self-financing property typically does not allow the payoff resulting from an integrated risk to be exactly matched to the value of the hedging portfolio at expiry of the contract. Therefore, an alternative approach is necessary to minimise the additional costs resulting from maintaining the hedging strategy, i.e. a risk minimising strategy needs to be followed. The combined market setting also implies that a change in the assumed actuarial basis, i.e. decrement assumptions relating to biometric risk and policyholder behaviour, leads to a necessary change in the assumed cash flow profile of the guaranteed liability and poses an additional cost to the hedging programme since rebalancing of the static strategy is needed.

Møller (1998) first suggested that risk minimising could be applied to UL life insurance contracts. Møller and Steffensen (2007) further analysed and evaluated the aforementioned hedging strategies. The authors conclude that super-hedging might not be the correct approach for the handling of the integrated risk found in UL and VA products, while risk minimising strategies prove to have a more natural form since it involves the survival probability. However, the derivation of a risk minimising strategy requires a subjective choice of the equivalent martingale measure Q due to the market being incomplete. Also, the risk minimising criterion in minimising the variance is a quadratic criterion that penalises additional costs and additional earnings in the same way. A special case of a risk minimising strategy is when the risk criterion is an indicator function on the interval $[0, \infty)$ and is known as quantile hedging. A quantile hedging strategy considers how to maximise the probability in order for the terminal value of the hedging portfolio to exceed the liability in the event that the initial capital is insufficient to set up a super replicating strategy to meet the contingent claim at expiry of the contract. An alternative criterion is to minimize the expected squared loss over all possible self-financing strategies, i.e. follow a mean-variance hedging strategy. Mean-variance hedging uses a self-financing strategy without using a specific martingale measure, although it also has a symmetric criterion that penalises gains and losses equally.

As mentioned, it might be possible for an insurer to transfer its exposure to investment risk to a third party, such as a reinsurer or an investment bank, although these markets might prove infeasible during and after financial turmoil. Reinsurance firms typically have a set risk appetite for investment business and do not readily take on the investment risk of UL and VA business. Investment banks can offer a structured product to the insurer to transfer some or all of the investment risk or aid in the management of an existing hedging programme. For example, a suitable structured product that transfers the full investment risk to the investment bank in the case of an idealised Black-Scholes-Merton economy is a portfolio of ${}_np_x$ forward-starting options with strike prices equal to $e^{\sigma r_k \sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}$, for $k = 0, ..., n-1$. This strategy follows directly from the result of Dhaene et al. (2002b):

$$E\left[\left(\sum_{k=0}^{n-1} \alpha_k e^{\sigma r_k \sqrt{n-k} Z} - b_n\right)_+\right] = \sum_{k=0}^{n-1} \alpha_k E\left[\left(e^{\sigma r_k \sqrt{n-k} Z} - e^{\sigma r_k \sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right]$$

The application of using existing market instruments, including over the counter instruments, to set up a static hedging portfolio is formally considered by, for example, Chen et al. (2008) and Chen et al. (2009).

In the following, we derive the so-called Greeks to implement a dynamic hedging strategy. A good review of dynamic hedging strategies and the various measures used can be found in Taleb (1997) and in Hull (2008). The delta Δ_t of a portfolio at time t measures the change in the value of the embedded option due to a change in the price of the underlying fund. According to a survey done by Matterson et al. (2008), most of the North American, Asian and European participants follow some form of a delta hedging strategy, while almost a third of the participants follow a hedging strategy that includes delta, vega and rho hedging strategies. Life companies might opt for static hedging strategies, as opposed to dynamic hedging strategies, if the investment guarantee structure allow it. For example, a recurring premium product with a GMMB requires a portfolio of forward-starting European put options in order to be hedged with a static hedging strategy where these options might not be available in the market. In this section, we consider approximate measures for the three Greeks typically considered by life insurance firms in their dynamic hedging programmes, i.e. the delta, vega and rho of a portfolio. The vega, which is an invented Greek letter, at time t measures the change in the value of the embedded option due to a change in the volatility parameter.

Hedging with respect to the volatility parameter mainly aims to mitigate model risk. The rho ρ_t at time t measures the change in the value of the embedded option due to a change in the interest rate parameter δ, i.e. it assumes a parallel shift in rates at all times. A time-dependent rate $\delta(t)$ can be used to include the whole term structure of interest rates. We assume the constant interest rate assumption of the Black-Scholes-Merton economy in our derivations. We base the derivation of the Greeks on the results of the embedded option payoffs derived in Chapter 3.

4.2 A Proposed Hedging Solution

In order to derive the delta at time t of the embedded options, we need to express the values of the embedded options in terms of the fund price at time t, where t could be any moment in time on the continuous interval $(0, n)$. In our derivations in the previous sections, we assumed that time elapses in a discrete way, i.e. that the time variable k measures the discrete moments in time $k = 0, 1, ..., n-1$. The Greek measures are derivatives, which implies that instantaneous changes in the embedded option value due to changes in the respective variables or parameters are measured. In the following, we first derive an expression for the value of the embedded option at some instantaneous time t, where $0 \leq t \leq n$ and then find the delta measure by taking the derivative of the derived expression with respect to the underlying fund price at time t, F_t.

Consider the GMMB call-type option value as defined in Chapter 3:

$$\begin{aligned} C_t &= e^{-\delta(n-t)} E_Q \left[\left(\sum_{k=0}^{n-1} \pi_k^{(s)} \frac{F(n)}{F(k)} - b_n \right)_+ \middle| \mathcal{F}_t \right] \\ &= e^{-\delta(n-t)} E_Q \left[\left(\sum_{k=0}^{n-1} \pi_k^{(s)} \exp\left\{ \left(\delta - \frac{\sigma^2}{2} \right)(n-k) + \sigma \left(W(n) - W(k) \right) \right\} - b_n \right)_+ \right] \quad (4.1) \end{aligned}$$

where the savings premium $\pi_k^{(s)}$ is defined as in equation (3.4):

$$\pi_k^{(s)} = (1-\alpha)(1-c)\,\pi_k - \pi_k^{(r)}$$

The expression in equation (4.1) above does not give us an adequate expression to consider dynamic hedging. Note that in our derivation of the call-type option expression, the initial investment fund value at time 0, $F(0)$, cancelled out, i.e.

$$\begin{aligned}\frac{F(n)}{F(k)} &= \frac{F(0)\exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)n+\sigma W(n)\right\}}{F(0)\exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)k+\sigma W(k)\right\}} \\ &= \exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)(n-k)+\sigma\left(W(n)-W(k)\right)\right\} \qquad (4.2)\end{aligned}$$

Therefore, the fund value is insensitive to the value of $F(0)$ for $k>0$. Assume the filtration $\mathcal{F}_t$ and $k<t$. We now have:

$$\begin{aligned}\frac{F(n)}{F(k)} &= \frac{F(t)\exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)(n-t)+\sigma\left(W(n)-W(t)\right)\right\}}{F(k)} \\ &= \frac{F(t)}{F(k)}\exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)(n-t)+\sigma\left(W(n)-W(t)\right)\right\} \qquad (4.3)\end{aligned}$$

The expression in equation (4.3) above shows that, given the filtration $\mathcal{F}_t$, the fund is only sensitive to the value of $F(t)$ for all $k<t$ and takes the history of the investment fund into account, i.e. it is path-dependent. In this section, we derive the conditional lower bound for the above option again in order to obtain the delta of the option. The derivation follows the exact same steps of the previous sections. The call option given the filtration $\mathcal{F}_t$ is given by:

$$C_t = e^{-\delta(n-t)}E_Q\left[\left(FV_t+FV^t-b_n\right)_+\middle|\,\mathcal{F}_t\right] \qquad (4.4)$$

where FV_t represents the case where $k<t$:

$$FV_t = \sum_{k=0}^{\lceil t\rceil-1}\pi_k^{(s)}\frac{F(t)}{F(k)}\exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)(n-t)+\sigma\left(W(n)-W(t)\right)\right\} \qquad (4.5)$$

and FV^t represents the case where $k\geq t$:

$$FV^t = \sum_{k=\lceil t\rceil}^{n-1}\pi_k^{(s)}\exp\left\{\left(\delta-\frac{\sigma^2}{2}\right)(n-k)+\sigma\left(W(n)-W(k)\right)\right\} \qquad (4.6)$$

To derive the bounds, first recall that:

$$W\left(n-t\right) \stackrel{d}{=} W\left(n\right) - W\left(t\right) \tag{4.7}$$

The fund value given the filtration $\mathcal{F}_t$ is given by:

$$V_n^- = FV_t + FV^t \tag{4.8}$$

The conditional expectation of the fund growth, first for $k < t$, is given by:

$$\begin{aligned}
E\left[\frac{F(n)}{F(k)} \mid \Lambda\right] &= E\left[\frac{F\left(t\right)}{F\left(k\right)} e^{\left(\delta - \frac{\sigma^2}{2}\right)(n-t) + \sigma W(n-t)} \mid \Lambda\right] \\
&= \frac{F\left(t\right)}{F\left(k\right)} e^{\left(\delta - \frac{1}{2}\sigma^2\right)(n-t)} E\left[e^{\sigma W(n-t)} \mid \Lambda\right] \\
&= \frac{F\left(t\right)}{F\left(k\right)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_t^2\right)(n-t) + \sigma r_t \sqrt{n-t} Z}
\end{aligned} \tag{4.9}$$

where

$$r_t = \frac{Cov\left[Y_t, \Lambda\right]}{\sqrt{n-t}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j^{(s)} \pi_l^{(s)} e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{4.10}$$

The covariance of the pair (Y_t, Λ) can be found in the usual way:

$$\begin{aligned}
&Cov\left[Y_t, \Lambda\right] \\
&= Cov\left[W(n) - W(t),\ \sum_{l=0}^{n-1} \gamma_l \left[W(n) - W(l)\right]\right] \\
&= \sum_{l=0}^{n-1} \gamma_l Cov\left[W(n) - W(t),\ W(n) - W(l)\right] \\
&= \sum_{l=0}^{n-1} \gamma_l \min(n-t, n-l) \\
&= \sum_{l=0}^{n-1} \pi_l^{(s)} e^{\delta(n-l)} \min(n-t, n-l)
\end{aligned} \tag{4.11}$$

Therefore, the correlation coefficient r_t is defined as:

$$r_t = \frac{\sum_{l=0}^{n-1} \pi_l^{(s)} e^{\delta(n-l)} \min(n-t, n-l)}{\sqrt{n-t}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j^{(s)} \pi_l^{(s)} e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{4.12}$$

For $k \geq t$, the results from the previous sections hold, i.e.:

$$\begin{aligned} FV^{t(l)} &= \sum_{k=\lceil t \rceil}^{n-1} \pi_k^{(s)} E\left[\frac{F(n)}{F(k)} \mid \Lambda\right] \\ &= \sum_{k=\lceil t \rceil}^{n-1} \pi_k^{(s)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \end{aligned} \tag{4.13}$$

Therefore, we find the conditional lower bound for the fund value as:

$$\begin{aligned} & V_n^{-(l)} \\ &= FV_t^{(l)} + FV^{t(l)} \\ &= \sum_{k=0}^{\lceil t \rceil - 1} \pi_k^{(s)} \frac{F(t)}{F(k)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_t^2\right)(n-t) + \sigma r_t \sqrt{n-t} Z} + \sum_{k=\lceil t \rceil}^{n-1} \pi_k^{(s)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \\ &= \sum_{k=0}^{n-1} \pi_k^{(s)} g_k(Z) \end{aligned} \tag{4.14}$$

where $g_k(Z)$ is defined as follows:

$$g_k(Z) = \begin{cases} \frac{F(t)}{F(k)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_t^2\right)(n-t) + \sigma r_t \sqrt{n-t} Z} & \text{for } k < t \\ e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} & \text{for } k \geq t \end{cases} \tag{4.15}$$

Using result (2.62), we find:

$$E\left[\left(\sum_{k=0}^{n-1} \pi_k^{(s)} g_k(Z) - b_n\right)_+\right] = \sum_{k=0}^{n-1} E\left[\left(\pi_k^{(s)} g_k(Z) - F^{-1}_{\pi_k g_k(Z)}\left(F_{V_n^{-(l)}}(b_n)\right)\right)_+\right] \tag{4.16}$$

where $F^{-1}_{\pi_k g_k(Z)}$ is given by:

$$F^{-1}_{\pi_k g_k(Z)}(p) = \pi_k^{(s)} g_k\left(F_Z^{-1}(p)\right) \tag{4.17}$$

and:

$$F^{-1}_{V_n^{-(l)}} = \sum_{k=0}^{n-1} F^{-1}_{\pi_k g_k(Z)}(p) \tag{4.18}$$

Finally, we have:

$$
\begin{aligned}
&E\left[\left(\sum_{k=0}^{n-1}\pi_k^{(s)}g_k(Z)-b_n\right)_+\right]\\
&=\sum_{k=0}^{\lceil t\rceil-1}\pi_k^{(s)}\frac{F(t)}{F(k)}e^{\left(\delta-\frac{1}{2}\sigma^2r_t^2\right)(n-t)}E\left[\left(e^{\sigma r_t\sqrt{n-t}Z}-e^{\sigma r_t\sqrt{n-t}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right]\\
&+\sum_{k=\lceil t\rceil}^{n-1}\pi_k^{(s)}e^{\left(\delta-\frac{1}{2}\sigma^2r_k^2\right)(n-k)}E\left[\left(e^{\sigma r_k\sqrt{n-k}Z}-e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right]\\
&=\sum_{k=0}^{\lceil t\rceil-1}\alpha_kE\left[\left(e^{\sigma r_t\sqrt{n-t}Z}-e^{\sigma r_t\sqrt{n-t}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right]\\
&+\sum_{k=\lceil t\rceil}^{n-1}\alpha_kE\left[\left(e^{\sigma r_k\sqrt{n-k}Z}-e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right]
\end{aligned}
$$

where

$$
\alpha_k=\begin{cases}\pi_k^{(s)}\frac{F(t)}{F(k)}e^{\left(\delta-\frac{1}{2}\sigma^2r_t^2\right)(n-t)} & \text{for } k<t\\ \pi_k^{(s)}e^{\left(\delta-\frac{1}{2}\sigma^2r_k^2\right)(n-k)} & \text{for } k\geq t\end{cases}\tag{4.19}
$$

Therefore,

$$
\begin{aligned}
E\left[\left(V_n^{-(l)}-b_n\right)_+\right]=&\sum_{k=0}^{\lceil t\rceil-1}\alpha_k\left\{e^{\frac{1}{2}\sigma^2r_t^2(n-t)}\Phi\left[\sigma r_t\sqrt{n-t}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right.\\
&\left.-e^{\sigma r_t\sqrt{n-t}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\Phi\left[-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right\}\\
&+\sum_{k=\lceil t\rceil}^{n-1}\alpha_k\left\{e^{\frac{1}{2}\sigma^2r_k^2(n-k)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right.\\
&\left.-e^{\sigma r_k\sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\Phi\left[-\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]\right\}
\end{aligned}\tag{4.20}
$$

By further simplifying the expression in equation (4.20). we obtain:

$$
\begin{aligned}
E\left[\left(V_n^{-(l)} - b_n\right)_+\right] &= F(t) \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} e^{\delta(n-t)} \Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&+ \sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{\delta(n-k)} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&- \left[1 - F_{V_n^{-(l)}}(b_n)\right] \left\{\sum_{k=0}^{\lceil t\rceil -1} F^{-1}_{\pi_k g_k(Z)}\left(F_{V_n^{-(l)}}(b_n)\right) + \sum_{k=\lceil t\rceil}^{n-1} F^{-1}_{\pi_k g_k(Z)}\left(F_{V_n^{-(l)}}(b_n)\right)\right\} \\
&= F(t) \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} e^{\delta(n-t)} \Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&+ \sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{\delta(n-k)} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&- \left[1 - F_{V_n^{-(l)}}(b_n)\right] b_n \qquad (4.21)
\end{aligned}
$$

We now have the conditional lower bound approximation for the option value C_t at time t, i.e.:

$$
\begin{aligned}
C_t &= F(t) \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} \Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&+ \sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{-\delta(k-t)} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\
&- e^{-\delta(n-t)} \left[1 - F_{V_n^{-(l)}}(b_n)\right] b_n \qquad (4.22)
\end{aligned}
$$

The result of equation (4.22) exhibits only the price of transferring investment risk and makes no allowance for the mortality of the policyholder.

Therefore, the delta of a pure financial call option is given by:

$$
\Delta_t = \frac{\partial C_t}{\partial F(t)} = \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} \Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \qquad (4.23)
$$

where the value of $F_{V_n^{-(l)}}(b_n)$ can be found in the usual way, viz.:

$$
\sum_{k=0}^{n-1} F^{-1}_{\pi_k g_k(Z)}\left(F_{V_n^{-(l)}}(b_n)\right) - b_n = 0 \qquad (4.24)
$$

The delta of the embedded call option in the combined market of financial and mortality risk can be found in a similar way as the result in equation (3.36). The value of the GMMB embedded call option is given by:

$$C_t = {}_tp_x F(t) \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} \Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]$$
$$+ \sum_{k=\lceil t\rceil}^{n-1} {}_tp_x \pi_k^{(s)} e^{-\delta(k-t)} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right]$$
$$- {}_tp_x e^{-\delta(n-t)}\left[1 - F_{V_n^{-(l)}}(b_n)\right] b_n \tag{4.25}$$

Similarly, the delta of the embedded call option of the GMMB in the combined market is given by:

$$\Delta_t = \frac{\partial C_t}{\partial F(t)} = {}_tp_x \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} \Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \tag{4.26}$$

The delta of the put-type option can be found by using the following relationship:

$$E\left[\left(b_n - V_n^{-(l)}\right)_+\right] = E\left[\left(V_n^{-(l)} - b_n\right)_+\right] + b_n - E\left[V_n^{-(l)}\right] \tag{4.27}$$

where the value of $E\left[V_n^{-(l)}\right]$ can be found in a similar way to the previous section, i.e.:

$$E\left[V_n^{-(l)}\right] = E\left[FV_t + FV^t\right]$$
$$= E\left[\sum_{k=0}^{\lceil t\rceil -1} \pi_k^{(s)} \frac{F(t)}{F(k)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_t^2\right)(n-t) + \sigma r_t \sqrt{n-t} Z}\right] + E\left[\sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z}\right]$$
$$= \sum_{k=0}^{\lceil t\rceil -1} \pi_k^{(s)} \frac{F(t)}{F(k)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_t^2\right)(n-t)} E\left[e^{\sigma r_t \sqrt{n-t} Z}\right] + \sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} E\left[e^{\sigma r_k \sqrt{n-k} Z}\right]$$
$$= \sum_{k=0}^{\lceil t\rceil -1} \pi_k^{(s)} \frac{F(t)}{F(k)} e^{\delta(n-t)} + \sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{\delta(n-k)} \tag{4.28}$$

Therefore, we have:

$$E\left[\left(b_n - V_n^{-(l)}\right)_+\right] = F(t) \sum_{k=0}^{\lceil t\rceil -1} \frac{\pi_k^{(s)}}{F(k)} e^{\delta(n-t)} \left\{\Phi\left[\sigma r_t \sqrt{n-t} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] - 1\right\}$$
$$+ \sum_{k=\lceil t\rceil}^{n-1} \pi_k^{(s)} e^{\delta(n-k)} \left\{\Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] - 1\right\}$$
$$- \left[1 - F_{V_n^{-(l)}}(b_n)\right] \left\{\sum_{k=0}^{\lceil t\rceil -1} F^{-1}_{\pi_k g_k(Z)}\left(F_{V_n^{-(l)}}(b_n)\right) + \sum_{k=\lceil t\rceil}^{n-1} F^{-1}_{\pi_k g_k(Z)}\left(F_{V_n^{-(l)}}(b_n)\right)\right\}$$
$$+ b_n \tag{4.29}$$

Finally, the value of P_t is given by:

$$P_t = -F(t) \sum_{k=0}^{\lceil t \rceil - 1} \frac{\pi_k^{(s)}}{F(k)} \Phi \left[-\sigma r_t \sqrt{n-t} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right]$$

$$- \sum_{k=\lceil t \rceil}^{n-1} \pi_k^{(s)} e^{-\delta(k-t)} \Phi \left[-\sigma r_k \sqrt{n-k} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] + e^{-\delta(n-t)} b_n F_{V_n^{-(l)}}(b_n) \quad (4.30)$$

by using the fact that:

$$\begin{aligned} b_n &= \sum_{k=0}^{n-1} F^{-1}_{\pi_k g_k(Z)} \left(F_{V_n^{-(l)}}(b_n) \right) \\ &= \sum_{k=0}^{\lceil t \rceil - 1} F^{-1}_{\pi_k g_k(Z)} \left(F_{V_n^{-(l)}}(b_n) \right) + \sum_{k=\lceil t \rceil}^{n-1} F^{-1}_{\pi_k g_k(Z)} \left(F_{V_n^{-(l)}}(b_n) \right) \end{aligned} \quad (4.31)$$

Therefore, the delta of the put-type option is given by:

$$\Delta_t = \frac{\partial P_t}{\partial F(t)} = - \sum_{k=0}^{\lceil t \rceil - 1} \frac{\pi_k^{(s)}}{F(k)} \Phi \left[-\sigma r_t \sqrt{n-t} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] \quad (4.32)$$

The embedded put option value in the combined market can be found in a similar way to the result in equation (3.37). The GMMB embedded put option value is given by:

$$\begin{aligned} P_t = &- {}_t p_x F(t) \sum_{k=0}^{\lceil t \rceil - 1} \frac{\pi_k^{(s)}}{F(k)} \Phi \left[-\sigma r_t \sqrt{n-t} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] \\ &- {}_t p_x \sum_{k=\lceil t \rceil}^{n-1} \pi_k^{(s)} e^{-\delta(k-t)} \Phi \left[-\sigma r_k \sqrt{n-k} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] \\ &+ {}_t p_x e^{-\delta(n-t)} b_n F_{V_n^{-(l)}}(b_n) \end{aligned} \quad (4.33)$$

Similarly, the delta of the embedded put option follows as:

$$\Delta_t = \frac{\partial P_t}{\partial F(t)} = - {}_t p_x \sum_{k=0}^{\lceil t \rceil - 1} \frac{\pi_k^{(s)}}{F(k)} \Phi \left[-\sigma r_t \sqrt{n-t} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] \quad (4.34)$$

The delta measure for the call and put representations of the GMDB embedded option can be derived by the same approach as for the GMMB results derived above. First, we need the valuation result of an expected embedded option payoff with maturity j, for $t \leq j \leq n$, at any time t during the policy term $0 \leq t \leq n$. This result is found in a similar way to the embedded option valuation result of equation (4.21), i.e.:

$$E\left[\left(V_{j+1}^{-(l)}-b_n\right)_+\right] = F(t)\sum_{k=0}^{\lceil t\rceil-1}\frac{\pi_k^{(s)}}{F(k)}e^{\delta(j-t+1)}\Phi\left[\sigma r_t\sqrt{n-t}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]$$
$$+\sum_{k=\lceil t\rceil}^{j}\pi_k^{(s)}e^{\delta(j-k+1)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]$$
$$-\left[1-F_{V_{j+1}^{-(l)}}(b_{j+1})\right]b_{j+1} \qquad (4.35)$$

The call option value at valuation time t is given by:

$$C_t^j = F(t)\sum_{k=0}^{\lceil t\rceil-1}\frac{\pi_k^{(s)}}{F(k)}\Phi\left[\sigma r_t\sqrt{n-t}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]$$
$$+\sum_{k=\lceil t\rceil}^{j}\pi_k^{(s)}e^{-\delta(k-t)}\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]$$
$$-e^{-\delta(j-t+1)}\left[1-F_{V_{j+1}^{-(l)}}(b_{j+1})\right]b_{j+1} \qquad (4.36)$$

The expected value of the total GMDB embedded option value at time t now follows from equation (3.83):

$$\sum_{j=0}^{n-1}{}_jp_x\ q_{x+j}e^{-\delta(j+1)}E\left[b_{j+1}+\left(V_{j+1}^{-(l)}-b_{j+1}\right)_+\right]$$
$$=\sum_{j=\lceil t\rceil-1}^{n-1}{}_jp_x\ q_{x+j}e^{-\delta(j+1)}b_{j+1}+\sum_{j=\lceil t\rceil-1}^{n-1}{}_jp_x\ q_{x+j}C_t^j \qquad (4.37)$$

The left-hand term in equation (4.37) is a constant in the derivation of the GMDB embedded option value with respect to the underlying fund price at time t, $F(t)$, while the right-hand side is a weighted sum of the fund price $F(t)$. The delta of the GMDB embedded call option therefore follows from equation (4.23) and equation (4.36), assuming that the policyholder is alive at time t:

$$\Delta_t = \frac{\partial C_t^j}{\partial F(t)} = \frac{\partial}{\partial F(t)}\sum_{j=0}^{n-1}{}_jp_x\ q_{x+j}C_t^j$$
$$=\sum_{j=\lceil t\rceil-1}^{n-1}{}_jp_x\ q_{x+j}\sum_{k=0}^{\lceil t\rceil-1}\frac{\pi_k^{(s)}}{F(k)}\Phi\left[\sigma r_t\sqrt{n-t}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right] \qquad (4.38)$$

The expected put option payoff follows from the result of equation (3.32) and equation (4.29):

$$E\left[\left(b_{j+1}-V_{j+1}^{-(l)}\right)_+\right] = F(t)\sum_{k=0}^{\lceil t\rceil-1}\frac{\pi_k^{(s)}}{F(k)}e^{\delta(j-t+1)}\left\{\Phi\left[\sigma r_t\sqrt{n-t}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]-1\right\}$$
$$+\sum_{k=\lceil t\rceil}^{n-1}\pi_k^{(s)}e^{\delta(j-k+1)}\left\{\Phi\left[\sigma r_k\sqrt{n-k}-\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]-1\right\}$$
$$+b_{j+1}F_{V_{j+1}^{-(l)}}(b_{j+1}) \tag{4.39}$$

The value of the embedded put option at valuation time t is given by:

$$P_t^j = -F(t)\sum_{k=0}^{\lceil t\rceil-1}\frac{\pi_k^{(s)}}{F(k)}\Phi\left[-\sigma r_t\sqrt{n-t}+\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]$$
$$-\sum_{k=\lceil t\rceil}^{j}\pi_k^{(s)}e^{-\delta(k-t)}\Phi\left[-\sigma r_k\sqrt{n-k}+\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right]+e^{-\delta(j-t+1)}b_{j+1}F_{V_{j+1}^{-(l)}}(b_{j+1}) \tag{4.40}$$

The put option value of the total GMDB benefit is given by:

$$\sum_{j=\lceil t\rceil-1}^{n-1}{}_jp_x\ q_{x+j}e^{-\delta(j+1)}E\left[V_{j+1}^{-(l)}+\left(b_{j+1}-V_{j+1}^{-(l)}\right)_+\right]$$
$$=\sum_{j=\lceil t\rceil-1}^{n-1}{}_jp_x\ q_{x+j}\sum_{k=0}^{j+1}\pi_k^{(s)}(1-e)^{j-k+1}e^{-\delta k}+\sum_{j=0}^{n-1}{}_jp_x\ q_{x+j}P_t^j \tag{4.41}$$

The left-hand side in equation (4.41) is a constant in the differentiation with respect to the price of the underlying fund at time t, $F(t)$. Therefore, the delta of the embedded put option of the total GMDB benefit is given by:

$$\Delta_t = \frac{\partial P_t^j}{\partial F(t)} = \frac{\partial}{\partial F(t)}\sum_{j=0}^{n-1}{}_jp_x\ q_{x+j}P_t^j \tag{4.42}$$
$$= -\sum_{j=\lceil t\rceil-1}^{n-1}{}_jp_x\ q_{x+j}\sum_{k=0}^{\lceil t\rceil-1}\frac{\pi_k^{(s)}}{F(k)}\Phi\left[-\sigma r_t\sqrt{n-t}+\Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right] \tag{4.43}$$

It is evident from the results of equation (4.26), equation (4.34), equation (4.38), and equation (4.43) that the delta of an embedded option in the combined market is a weighted sum of the delta measures, where the weights are determined by the probability of the event that drives the benefit, i.e. survival or death.

The delta of the embedded option measures a change in the value of the embedded option due to a change in a market variable, the price of the underlying fund. The vega and rho of the embedded option aim to immunise the portfolio with respect to the model parameters σ and r, thereby mitigating model risk. Although the delta measure was found analytically, a numeric finite differencing approach could have been used. A point estimate approximation of the embedded option value allows up and down shifts in the embedded option value to be calculated with speed and ease. The vega and rho of the embedded option measure changes in the embedded option value due to changes in the model parameters. The expression for the embedded option values contain the unknown CDF of the approximated fund value, $F_{V_n^{-(l)}}(x)$, and the vega and rho are therefore not analytically tractable. The vega and rho are calculated using the finite differencing approach. A common approach is to consider both an up and a down shift in the option value, see e.g. Jäckel (2002) and Wilmott (2006) for overviews on the central difference approach. The vega for the put-type option is calculated as:

$$\begin{aligned} \mathcal{V} &= \frac{\partial P_t}{\partial \sigma} \\ &\approx \frac{P_t(\sigma + \Delta\sigma) - P_t(\sigma - \Delta\sigma)}{2\Delta\sigma} \end{aligned} \tag{4.44}$$

where $P_t(\sigma)$ denotes the put-type option value as a function of the given parameter value and $\Delta\sigma$ denotes the magnitude of the small shift in the parameter value. Likewise, we find the rho for the put-type option as:

$$\begin{aligned} \rho_t &= \frac{\partial P_t}{\partial \delta} \\ &\approx \frac{P_t(\delta + \Delta\delta) - P_t(\delta - \Delta\delta)}{2\Delta\delta} \end{aligned} \tag{4.45}$$

The magnitude of the shift in the parameter value depends on the firm's risk management framework and is usually a function of expected future volatility. In the following, we consider a shift of 1% in the parameter value. The rho calculated in equation (4.45) above measures the sensitivity of the option to a change in the risk-neutral rate of return δ. In the case of extending the Black-Scholes-Merton model to allow for a term structure of interest rates, the rho needs to be modified. Taleb (1997) discusses a simple method of relative weightings that drastically outperforms an unweighted rho.

Other measures of sensitivity that are also typically used in risk management are the gamma and theta of the embedded option and can be derived in a similar way as above. The gamma Γ_t of an option measures a change in the value of the delta of the embedded option due to a change in the underlying fund. The gamma essentially measures the stability of a delta-hedging programme, i.e. high values of gamma correspond to a need for more frequent rebalancing of the portfolio. Taleb (1997) illustrates that in certain cases the gamma can be positive for an increase in the underlying asset price but negative for a downward movement, which is referred to as risk reversal. By averaging over the gamma measures derived from an upward and downward movement, the aggregate gamma measure would give a deceiving result. It is prudent therefore to calculate both an up-gamma Γ^u and a down-gamma Γ^d, i.e.:

$$\Gamma_t^u \approx \frac{\Delta_t (F_t + \Delta F_t) - \Delta_t (F_t)}{\Delta F_t} \tag{4.46}$$

and:

$$\Gamma_t^d \approx \frac{\Delta_t (F_t - \Delta F_t) - \Delta_t (F_t)}{\Delta F_t} \tag{4.47}$$

where F_t is short-hand for the $F(t)$, the price of the underlying fund at time t, and $\Delta_t (F_t)$ denotes the delta measure around fund price F_t. In case the gamma measure is consistent and no risk reversal is present, gives an overview of the finite differencing approach with the following approximation for the gamma measure:

$$\begin{aligned} \Gamma_t &= \frac{\partial^2 P_t}{\partial F_t^2} \\ &\approx \frac{P_t (F_t + \Delta F_t) - 2 (F_t) - P_t (F_t - \Delta F_t)}{\Delta F_t^2} \end{aligned} \tag{4.48}$$

The theta Θ of the embedded option measures the rate of change in the option price with time, i.e.:

$$\Theta_t \approx \frac{P_t - P_{t+\Delta t}}{\Delta t} \tag{4.49}$$

The derivation for theta Θ does not consider the downward movement in time, i.e. only a positive progression in time is considered. The theta is also known as the time decay of an option and is therefore mostly negative. The theta of an option is closely related to its gamma. Taleb (1997) explains that the gamma per theta ratio is the same regardless of the number of days to expiration, i.e. the loss in time value of an option is offset by its rate of change in delta. Hull (2008) demonstrates this relationship in the Black-Scholes-Merton framework from its differential equation, which is given as:

$$\frac{\partial P_t}{\partial t} + \delta F_t \frac{\partial P_t}{\partial F_t} + \frac{1}{2}\sigma^2 F_t^2 \frac{\partial^2 P_t}{\partial F_t^2} = \delta P_t \tag{4.50}$$

Since $\Delta_t = \frac{\partial P_t}{\partial F_t}$, $\Gamma_t = \frac{\partial^2 P_t}{\partial F_t^2}$ and $\Theta_t = \frac{\partial P_t}{\partial t}$, we have:

$$\Theta_t + \delta F_t \Delta_t + \frac{1}{2}\sigma^2 F_t^2 \Gamma_t = \delta P_t \tag{4.51}$$

For a delta-neutral portfolio with $\Delta_t = 0$, it follows that:

$$\Theta_t + \frac{1}{2}\sigma^2 F_t^2 \Gamma_t = \delta P_t \tag{4.52}$$

The expression in equation (4.52) shows that the theta Θ_t and the gamma Γ_t largely mirror each other in sign and magnitude given a delta-neutral portfolio.

It is important to note that the Greek measures above are for a single option. In practice, insurers have various products contributing to embedded option risk. Aggregating Greek measures over a portfolio of heterogeneous embedded options pose additional challenges to the effective risk management of investment guarantees. A nearly homogeneous group of risks can be constructed by bundling or bucketing neighbouring maturities. Special care should then be taken to determine the Greek measures for each bucket, e.g. Taleb (1997) describes a method of relative weightings that can be applied to individual vega measures in order to aggregate them.

A possible alternative approach to treating each liability separately is to theoretically construct a replicating portfolio for the liability cash flows and then to determine the relevant sensitivity measures of the replicating portfolio. This is achieved by matching the liability cash flows in market value and sensitivity to a portfolio containing common market instruments. The sensitivity attributes of the replicating portfolio can then be used to inform on a possible hedging strategy for the aggregate liability.

4.3 Illustrative Example

Hull (2008), in the chapter on exotic options, describes the relative simplicity of hedging arithmetic Asian options by noting that as time passes we observe more of the underlying investment fund prices. Hence, the uncertainty of the eventual option payoff decreases over time, thereby facilitating the hedging strategy. VA and UL products, as well as other similar insurance products, have long maturities that require less frequent rebalancing of the portfolio in order to moderate the transaction costs typically associated with a dynamic hedging programme. The large portfolio sizes of VA and UL products further absorb the hedging costs. In this section, the efficacy of a dynamic hedge strategy in the Black-Scholes-Merton setting is demonstrated. First, we consider a simple case of delta hedging by using the derived delta measure of equation (4.32). We then add complexity by illustrating a dynamic hedging programme in the combined market of financial and mortality risk.

Consider an investor who contributes a deterministic monthly contribution of 1 at times, $k = 0, 1, ..., n$, for a term of 5 years, i.e. $n = 60$. For simplicity, we assume that all payments are paid with certainty and there is a zero probability of lapses and withdrawals. The insurer offers a pure return of contributions guarantee on the investment, i.e. the insurer offers a European put option on the underlying fund with strike price $b_n = 60$. The cost of the guarantee is therefore P_0. In Figure (4.1), the simulated fund values at maturity ($n = 60$) are illustrated for one estimate run of 10,000 paths with the parameter set ($\delta = 5\%,\ \sigma = 20\%$). The ROC indicator on the x-axis indicates the return of contributions value of $b_n = 60$. The kernel probability density estimate, the smooth line, is also given to illustrate the possible distribution. The kernel density estimate consists of the superposition of standard normal distributions centered over the observations, where the degree of smoothness is determined by the bandwidth parameter h and corresponds to the bin width of the histogram. Rice (1995) provides a brief introduction to kernel probability density estimates.

Figure (4.2) illustrates the sensitivity of the value of the investment guarantee to changes in the implied volatility σ. The risk-free rate is fixed at $\delta = 5\%$ p.a. The moneyness of options, i.e. whether the price of the underlying fund is significantly above or below the strike price of the embedded option, significantly influences the implied volatility of shorter term options. The volatility smile is less dramatic for longer terms to maturity and should not play such a pivotal role in approximating calculations. Recall that the Black-Scholes-Merton setting assumes a deterministic constant volatility regardless of the value of the underlying fund relative to the strike price. Gatheral (2006) shows that even by assuming a simple stochastic structure for volatility in the form of the Heston model, the implied volatility surface resulting from the model still fails to follow the dramatic volatility skew over shorter terms to maturity of the empirical implied volatility surface. However, the Heston model provides a good fit to the empirical volatility surface for longer expirations where the skew is far less pronounced.

Figure (4.3) illustrates the sensitivity of the value of the investment guarantee to the risk-free rate δ. The implied volatility is fixed at $\sigma = 20\%$ p.a. It is evident from the graph that the

Figure 4.1: Distribution of simulated fund values at maturity.

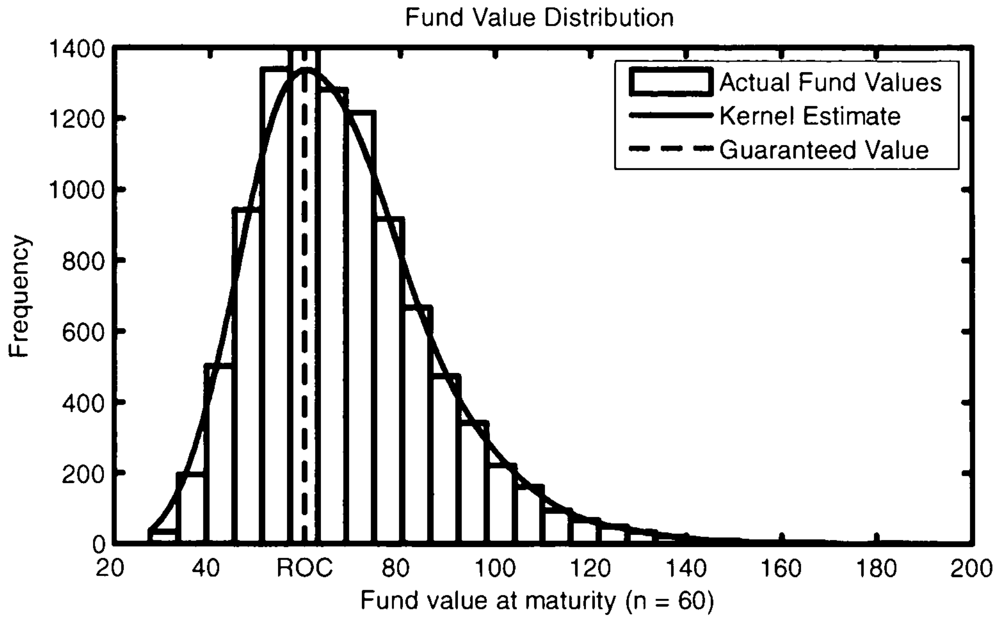


Figure 4.2: Sensitivity of embedded option value to implied volatility.

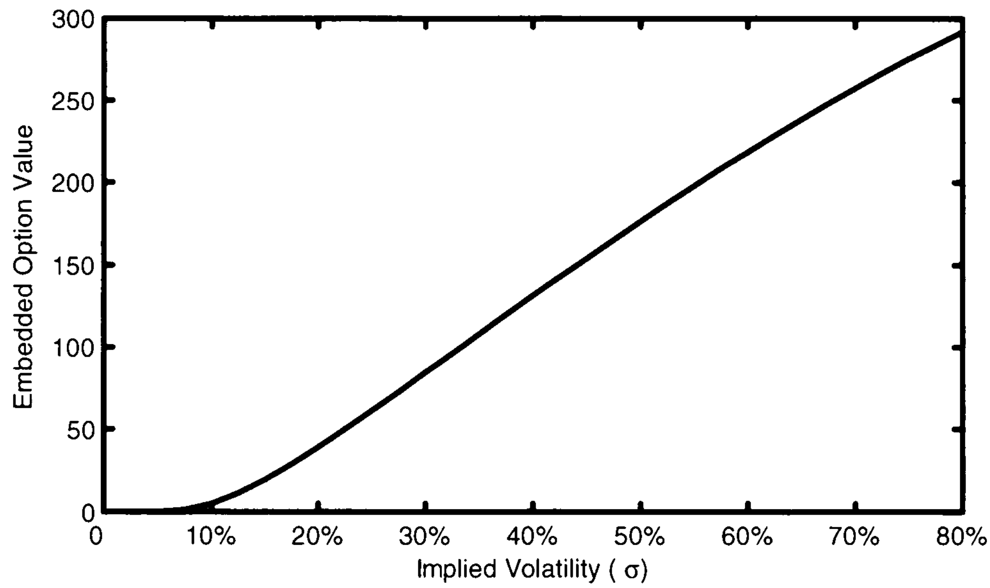
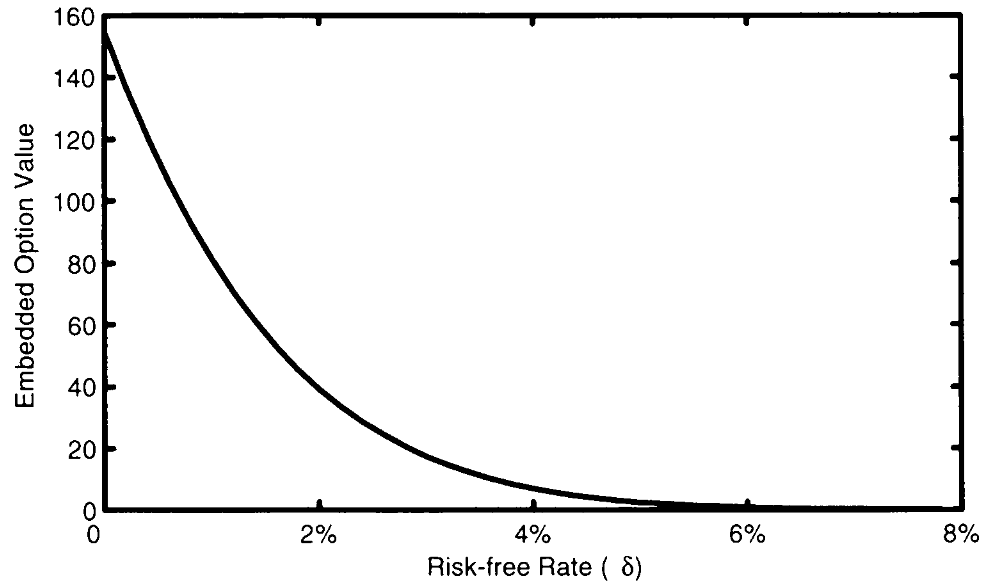


Figure 4.3: Sensitivity of embedded option value to the risk-free rate.



risk-free parameter δ greatly affects the cost of the guarantee. A stochastic term structure model can be added to the analysis to allow for greater accuracy, see e.g. Schrager and Pelsser (2004). The assumption or model risk associated with the risk-free rate can be mitigated by hedging the rho of the embedded option.

Due to the path-dependency of the option, we can only approximate the self-financing property that is otherwise valid in the Black-Scholes-Merton sub-economy. In practice, it is impossible or simply too expensive to adjust the replicating portfolio continuously, i.e. at extremely short intervals. We selected twelve rebalancing events per year, i.e. we rebalance the hedge portfolio each month after a premium contribution. At each rebalancing time t, we adjust the portfolio to have Δ_t of the underlying investment fund and the remainder of the risk-neutral option price at time t in a risk-free bond, i.e. $\Psi_t = P_t - \Delta_t F(t)$. Since the portfolio is not fully replicating, we possibly hold less in a risk-free bond, i.e.:

$$\Psi_t = \min(P_t, H_t^-) - \Delta_t F(t) \tag{4.53}$$

where H_t^- is the value of the hedge portfolio at time t before rebalancing. Hardy (2003) provides a good overview of this approach. The hedge portfolio at time t before rebalancing is given by the recursive formula:

$$\begin{aligned} H_t^- &= \Delta_{t-1} F(t) \frac{F(t+1)}{F(t)} + e^{\delta} \Psi_{t-1} \\ &= \Delta_{t-1} F(t+1) + e^{\delta} \left[\min(P_{t-1}, H_{t-1}^-) - \Delta_t F(t) \right] \end{aligned}$$

with $H_0^- = 0$. For simplicity, we also assume that the hedging portfolio will be positive at all times, i.e.

$$H_t^- \geq 0 \text{ for all } t = 0, ..., n \tag{4.54}$$

In the following, the performance of a dynamic delta hedging portfolio is illustrated for the 100% return of contributions minimum maturity guarantee in the case of the pure investment product. The performance of the hedging portfolio is considered in three possible market scenarios, namely poor, neutral and good. These market scenarios were simulated from the assumed Black-Scholes-Merton economy. An added scenario is considered where the underlying fund does not conform to the assumed underlying distribution. Figure (4.4) illustrates the performance of the hedge portfolio given a weak fund return, i.e. a scenario where the guarantee bites. The hedge portfolio manages to approximately meet the liability at maturity of the contract. In practice, the insurer might decide to rebalance less in the months before maturity to avoid a lot of smaller trades due to the volatility of the Greek measures close to maturity. As mentioned previously, a lookback option such as an arithmetic Asian option becomes less sensitive to changes in the market value of the underlying fund as time emerges. Although it seems that the hedge portfolio adequately covers the shortfall, i.e. cost of the guarantee, we did not account for transaction costs and costs of borrowing securities, i.e. scrip lending fees, for taking short sell positions. These costs form part of determining the total cost of a risk management programme.

Figure (4.5) illustrates a more neutral scenario, i.e. a scenario where the underlying fund value is barely sufficient to cover the guarantee. It is evident that the hedge portfolio tends to zero as the sum at risk tends to zero near the maturity of the contract.

The last scenario, as illustrated in Figure (4.6), considers the case of an exceedingly positive underlying fund, i.e. the fund value comfortably covers the guarantee. The hedge portfolio should cover the cost of the guarantee on average, but it is sensitive enough to allow adjustments in the event of poor fund performance. The additional adjustments necessary when the fund's

Figure 4.4: Performance of the hedge portfolio for weak fund return.

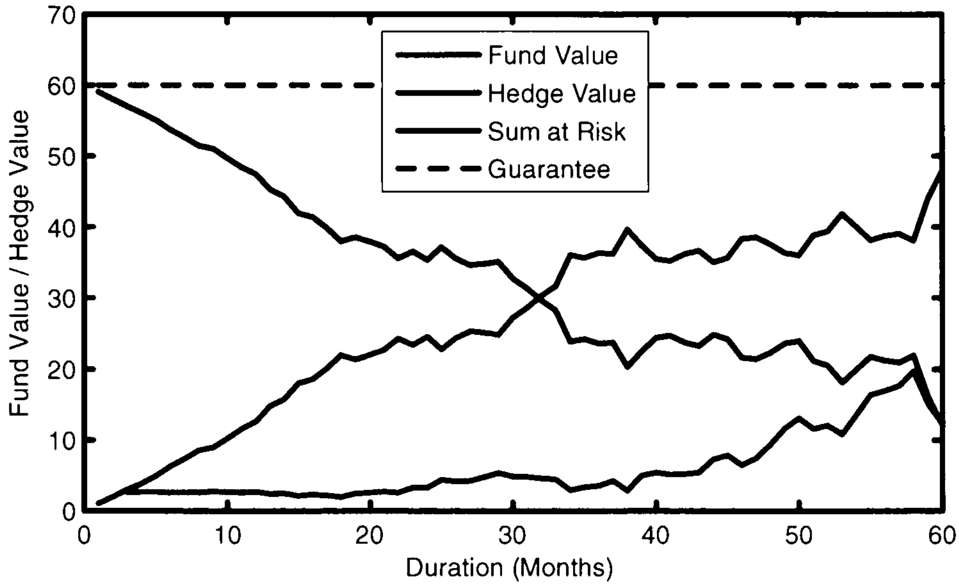


Figure 4.5: Performance of the hedge portfolio for neutral fund return.

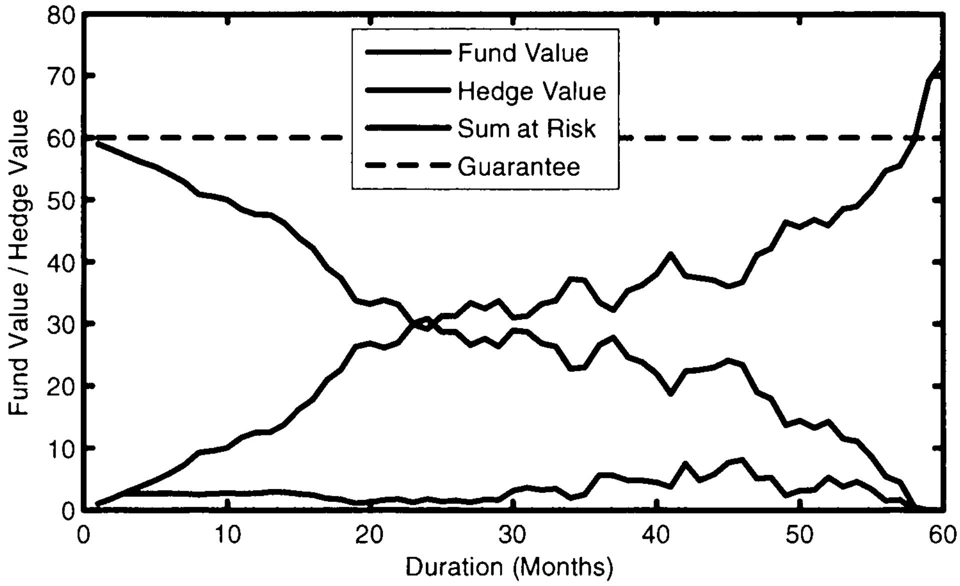
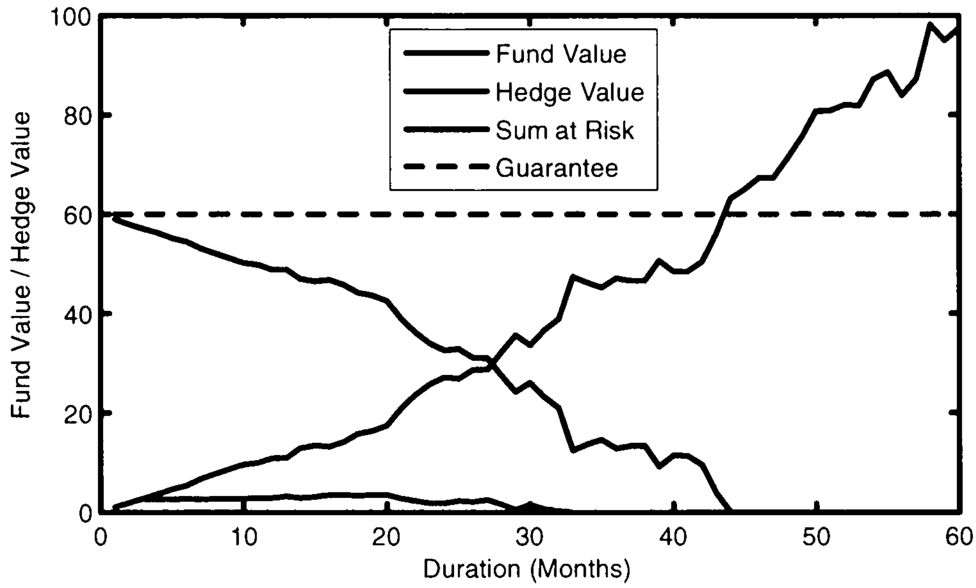


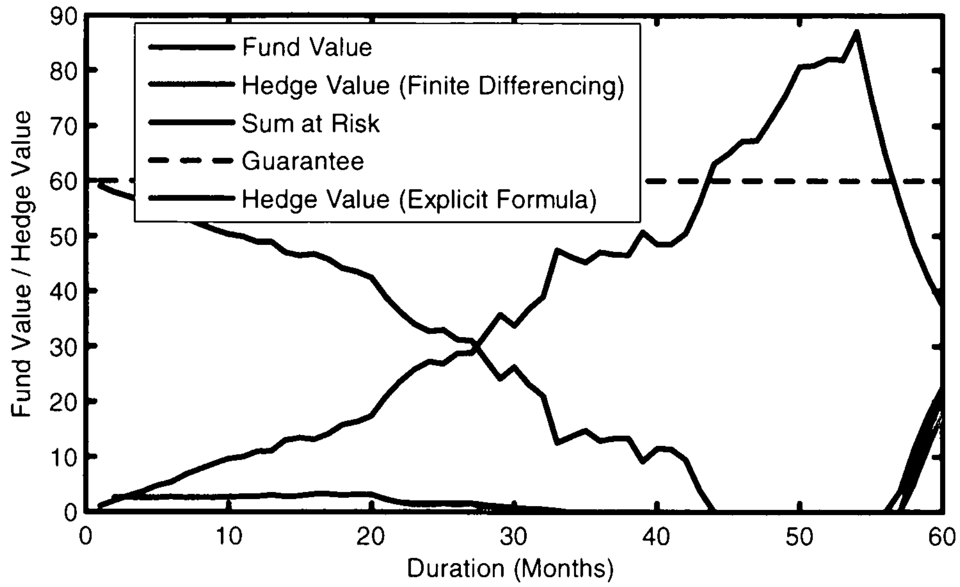
Figure 4.6: Performance of the hedge portfolio for strong fund return.



market value falls, e.g. in month 28 in the given example, come at a cost and a thorough analysis of the unhedged liability, which include all costs, should be conducted to assess the viability of a proposed hedging programme.

The conditions imposed on our hedging strategy, equation (4.53) and equation (4.54) , imply that there will be no additional inflows or outflows during the term of the contract, i.e. only the initial value of the embedded option is used to set up and maintain the hedging portfolio during the term of the contract. In Figure (4.7), we illustrate the potential shortfall of the hedging fund that might arise if markets behave outside of the assumed market parameters. The underlying fund performance of Figure (4.6) was adjusted by assuming a 15% fall month to month from month 55 to maturity of the contract. It is also evident that the hedging portfolio reacts to the falling market with a month lag due to the discrete rebalancing. A closer view of the shortfall is given in Figure (4.8). The derived delta measure for the embedded option value in equation (4.32) can of course be substituted by the finite differencing approach as discussed in section 4.2. Figure (4.7) and Figure (4.8) also show the comparative performance of the centre differencing approach delta. It is evident that the centre differencing approach performs marginally worse than the explicit formula approach. The centre differencing was done for 1% shifts in the underlying fund price and the performance of the approach can possibly

Figure 4.7: Performance of the hedge portfolio for a market crash.

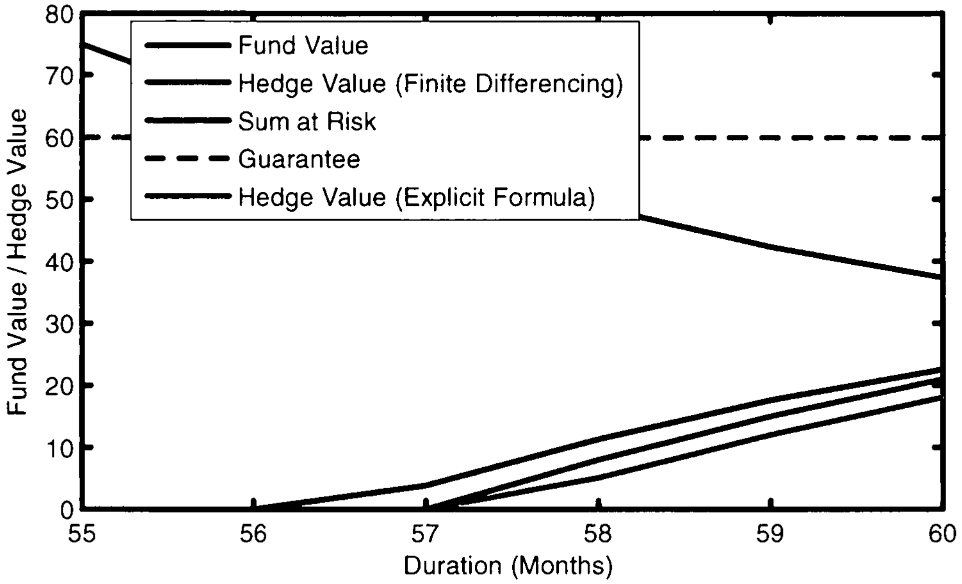


be improved by selecting an alternative magnitude of change in the underlying fund price. The centre differencing approach for the delta of the embedded option is given by:

$$\begin{aligned} \Delta_t &= \frac{\partial P_t}{\partial F(t)} \\ &\approx \frac{P_t\left[F(t)+\Delta F(t)\right]-P_t\left[F(t)-\Delta F(t)\right]}{2\Delta F(t)} \end{aligned} \tag{4.55}$$

Matterson et al. (2008) compares, by way of an illustrative example, the impact of different dynamic hedging strategies on the profit and loss and its variability. The authors considered hedging strategies that consist of delta hedging only, delta and rho hedging, and delta, rho and vega hedging. Simpler strategies, i.e. delta hedging only, might be useful for smaller portfolio sizes or where a cost / benefit analysis supports it. The illustrative example showed that delta hedging only reduces the potential loss by 24.4% and the variability of the expected profit and loss by 13.6%. By adding rho hedging to a delta hedging strategy, the expected loss can be reduced by 72.0% and the variability by 56.0%. Finally, by adding vega hedging to the delta-rho hedging strategy, the expected loss can be reduced by 81.4% and the variability of the expected loss by 75.5%. The aforementioned changes considers the reduction in the expected loss and the profitability from a no hedging approach to the respective implemented hedging strategy. It is clear that adding hedging with respect to the model parameters, i.e. rho and vega hedging,

Figure 4.8: Close-up: Performance of the hedge portfolio.



leads to a substantial reduction in both the expected loss and its variability. The rho measure, Figure (4.14), and the vega measure, Figure (4.15), are given below for the in the money case where the return of contributions guarantee is assumed at 200%. As expected, the embedded put option value becomes less sensitive to the model parameters over time as the filtration includes more observed underlying fund prices and model dependency therefore reduces.

The higher order sensitivity of the embedded option to the underlying fund price is given by the gamma measure, which measures the convexity or curvature of the delta measure. As discussed in section 4.2, the gamma measure can be calculated from either equation (4.46) and equation (4.47), or from directly from the underlying embedded option values from equation (4.48). The gamma measure for the in the money case, i.e. 200% return of contributions guarantee, is given for both cases in Figure (4.16). The difference in the measures mainly relate to the magnitude of change considered.

In Figure (4.9) ,the delta measure as derived in section 4.2, referred to as the formulaic approach, is compared to the delta measures resulting from a centre differencing approach with various magnitudes of change in the underlying fund price. The embedded put option value across the policy term is also given in Figure (4.9) on the left y-axis. It is evident that both approaches to deriving delta measures react to large changes in the underlying fund price equally well, but that the discrepancy in the smaller changes is due to the choice of the magnitude of change considered in the centre differencing approach. The moneyness of the option leads to volatility in the sensitivity estimates, especially for higher order sensitivities such as the gamma measure, due to the first order sensitivity measures being piecewise continuous.

The case of a significantly in the money embedded put option is illustrated in the following where the level of the return of contributions guarantee is assumed at 200%. Also, the neutral fund scenario of Figure (4.5) is assumed throughout. In Figure (4.10), the performance of the delta hedge portfolio for the finite differencing approach of equation (4.55) with a 5% magnitude change is compared to the performance given the explicit formula of equation (4.32). It is evident from the graph that both hedge portfolios perform reasonably well. A closer view is provided at maturity in Figure (4.11) from which it is clear that the explicit formula approach performs marginally better than the finite differencing approach. The associated delta measures are given in Figure (4.12), which shows that the delta measures of the two approaches progress in an almost identical way across the policy term. However, for the case of an at the money embedded put option, i.e. the case where the return of contributions guarantee is assumed at 100%, the difference in the progression of the two delta measures become evident. The delta measures for the 100% return of contributions guarantee are illustrated in Figure (4.13).

The time decay of the embedded option is given by the theta measure. The theta measure for the in the money case is given in Figure (4.17). It is evident from the graph that the change of the embedded option per year is reducing over time. The theta of an embedded option is not hedgeable, since the passage of time is certain. The theta is still a useful measure since it can be regarded as a proxy for gamma, more specifically the gamma of a delta neutral portfolio is large and positive when the theta of the portfolio is large and negative, and vice versa.

Figure 4.9: Comparison of delta measures for neutral fund return.

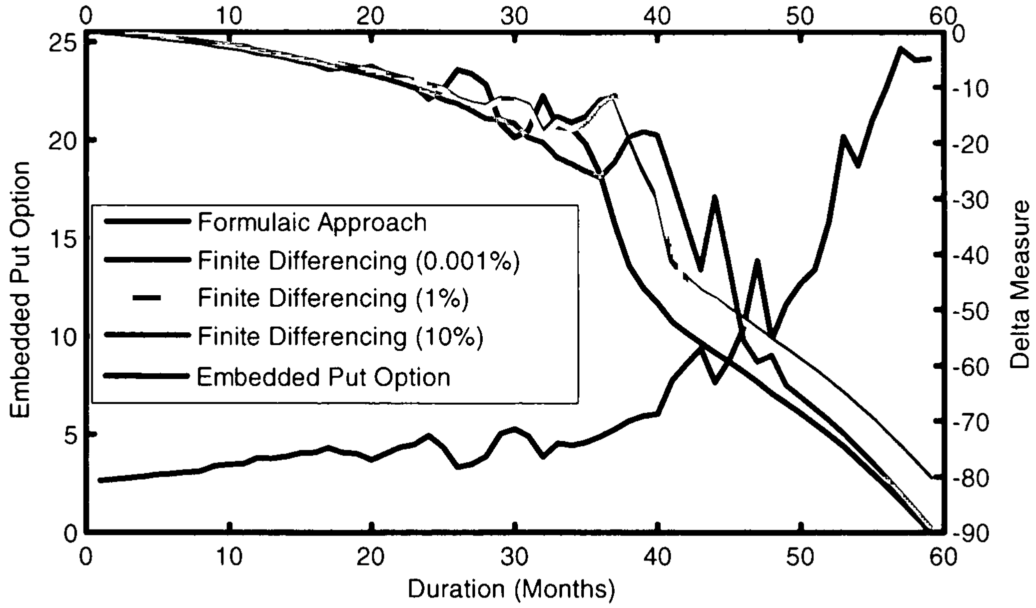


Figure 4.10: Comparison of the performance of delta measures.

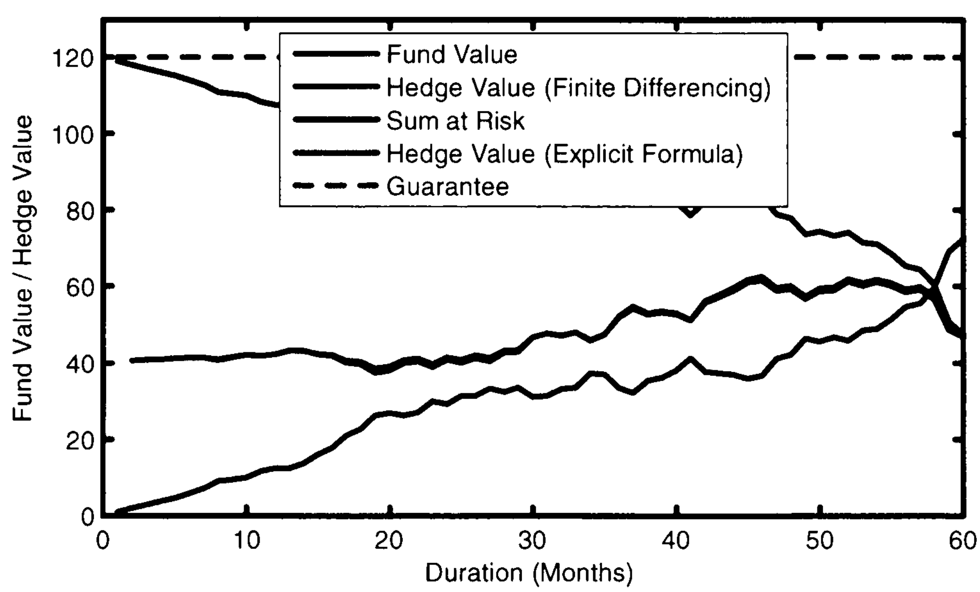


Figure 4.11: Close-up: Comparison of the performance of delta measures.

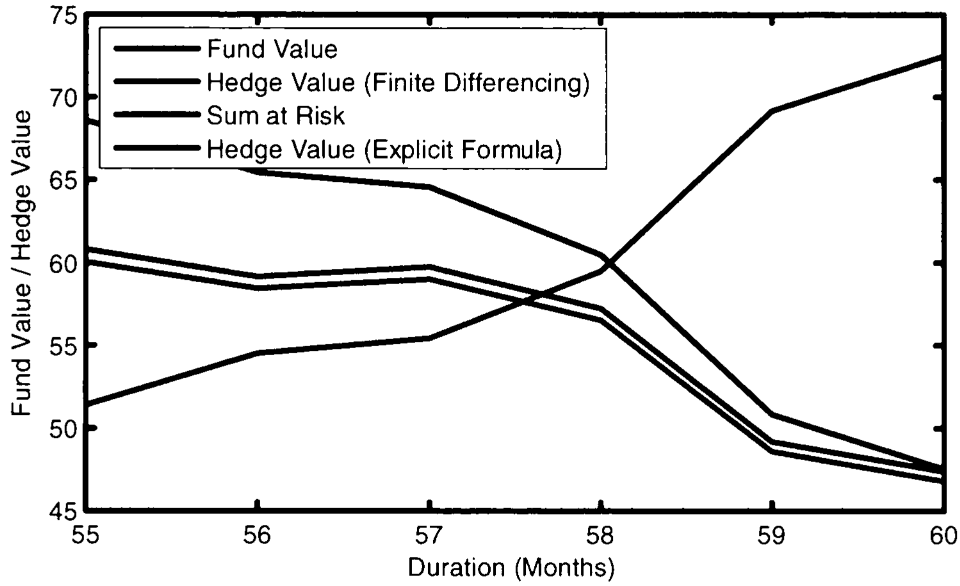


Figure 4.12: Comparison of delta measures for out-of-the-money neutral fund return.

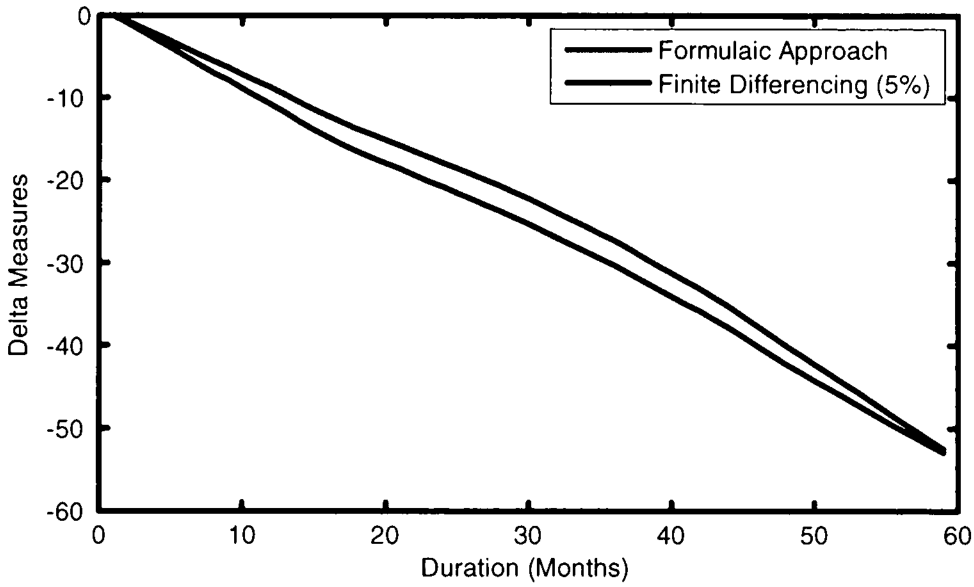


Figure 4.13: Comparison of delta measures for at-the-money neutral fund return.

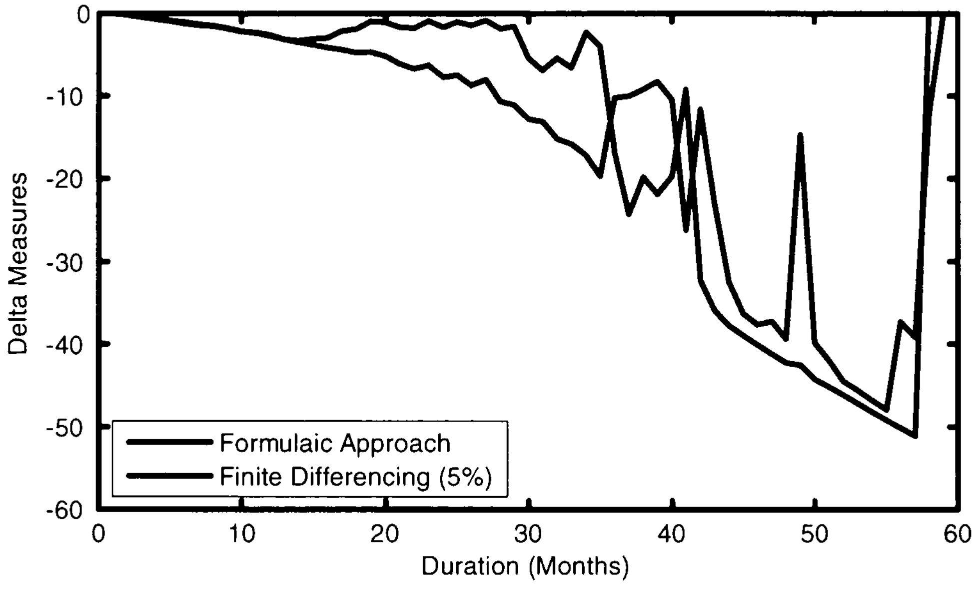


Figure 4.14: Rho measure for neutral fund return.

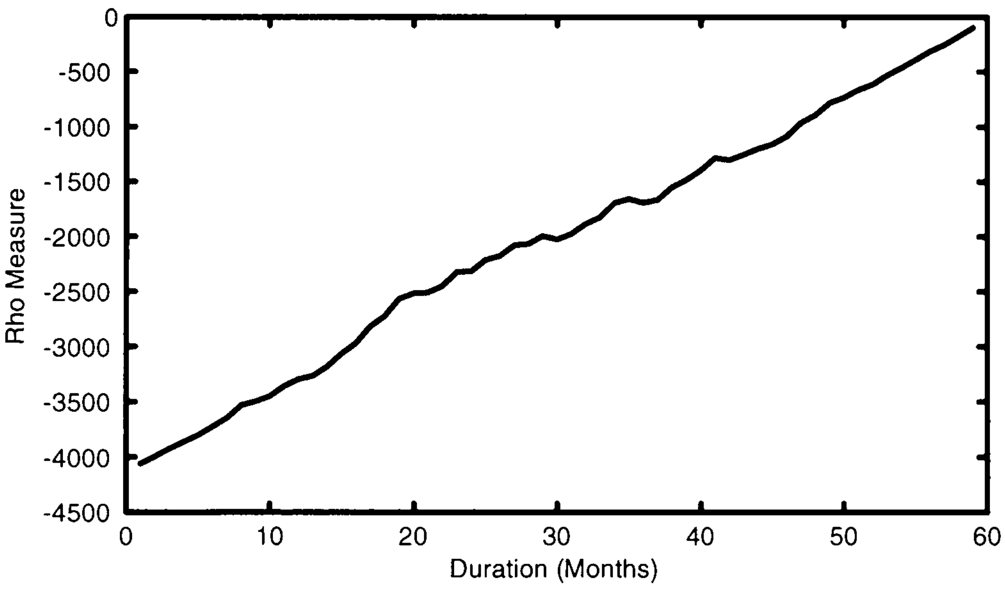


Figure 4.15: Vega measure for neutral fund return.

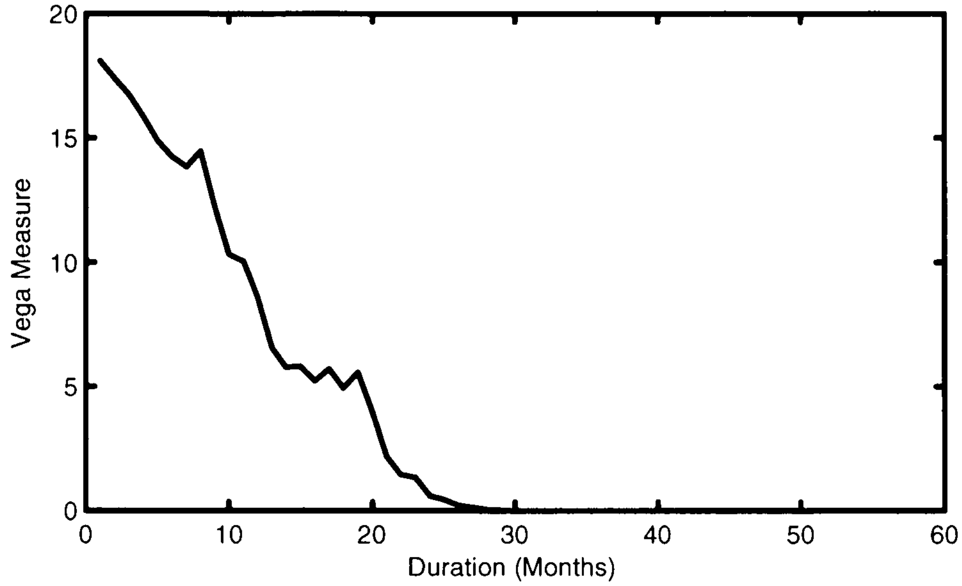


Figure 4.16: Comparison of the gamma measures for neutral fund return.

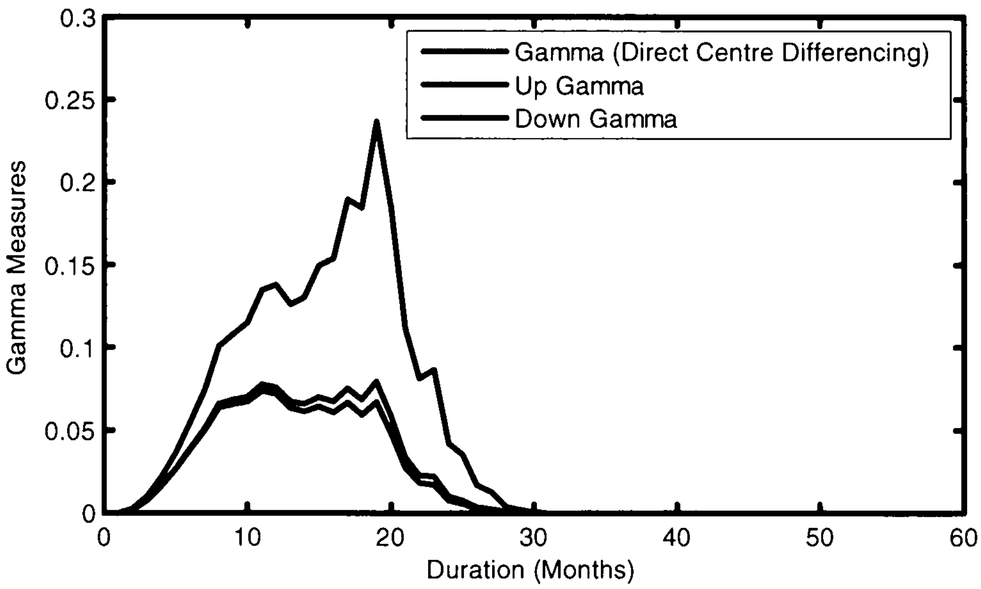


Figure 4.17: Theta measure for neutral fund return.

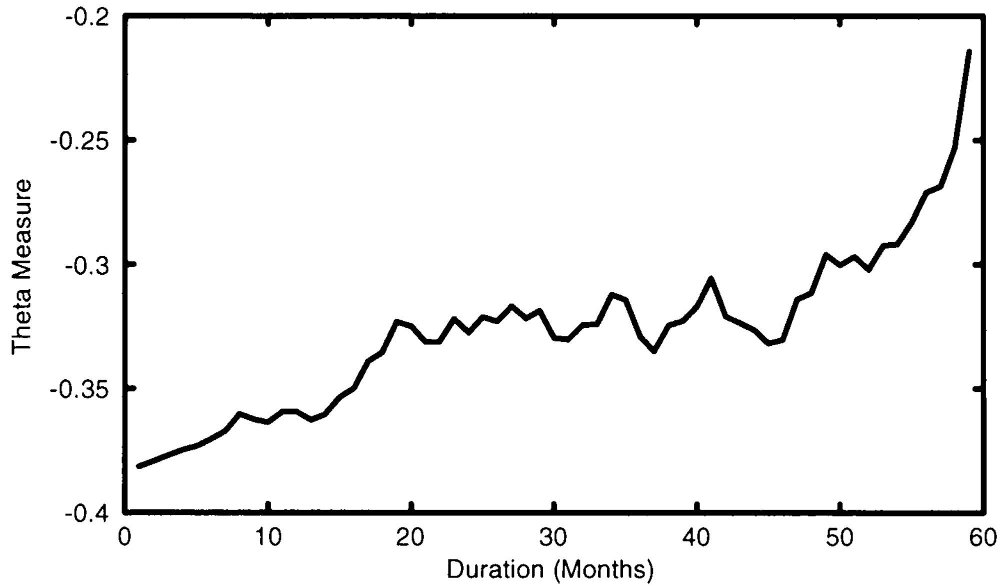


Figure 4.18: Comparison of performance of delta measures for GMMB.

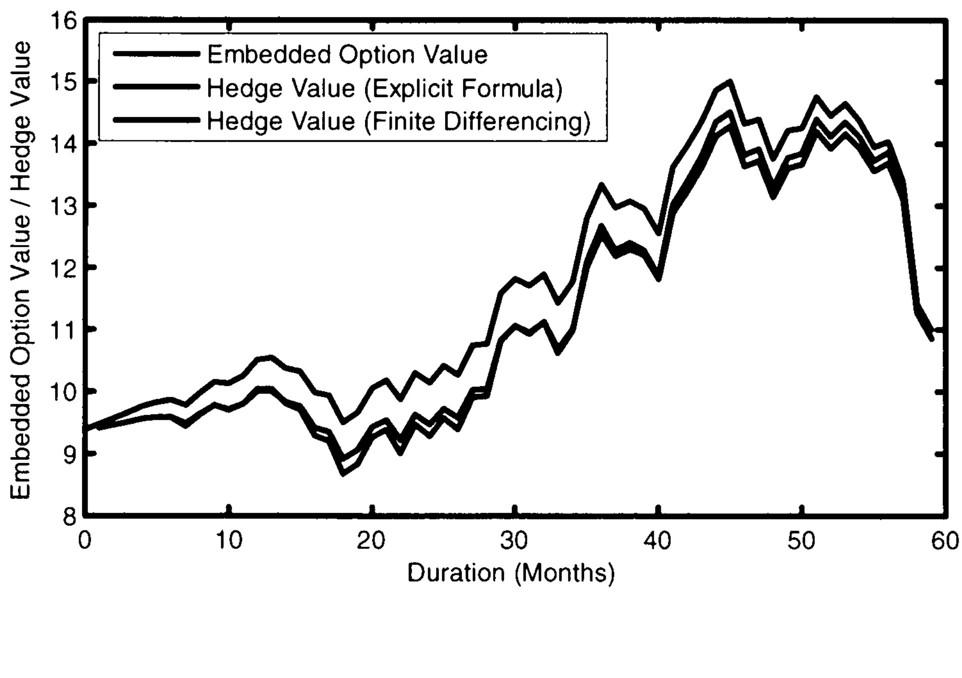
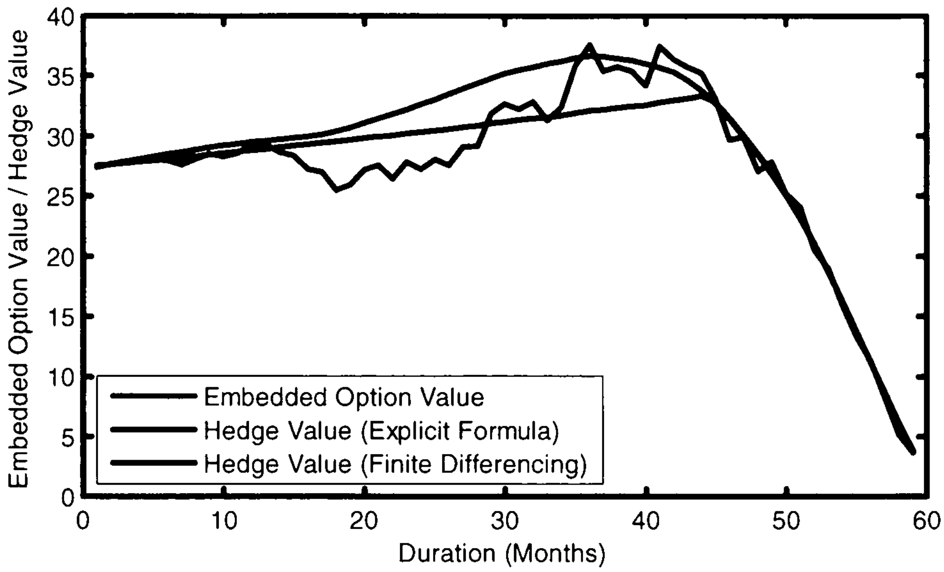


Figure 4.19: Comparison of performance of delta measures for GMDB.



The examples given in the figures above illustrate the efficacy of the derived sensitivity measures for the embedded arithmetic Asian put option of the pure financial product. The hedging strategy therefore aims to eliminate the financial risk of the combined product. For the benefits of the GMMB and the GMDB, the financial risk needs to be hedged in a combined market of financial and mortality risk. The embedded put option value in the case of the GMMB is multiplied by the survival probability of the policyholder surviving the policy term due to the assumed independence of the financial risk and mortality risk. The trends in the above examples therefore still hold true for the GMMB, since the difference would be due to a constant mulitplier. An example of a delta hedging strategy in the case of the GMMB for a return of contributions guarantee of 200% is given in Figure (4.18). Also, the explicit formula delta measure is compared to the finite differencing delta measure.

The embedded put option value of the GMDB, given in equation (4.37), consist of a portfolio of forward starting put options that expire over the duration of the policy term. The illustrative examples given above consist of only one put option that expires at maturity of the contract. The embedded options of the GMDB and GMMB therefore evolve in different ways over time. The embedded put option value of the GMDB and the delta hedge portfolio value in the case of the explicit formula of equation (4.43) and the finite differencing approach for a guarantee of 200% return of contributions are given in Figure (4.19). It is evident from the graph that the delta measures do not perform in the same way for the GMDB as for the GMMB and the pure financial contract. The finite differencing approach, with 5% magnitude change assumed for the example in Figure (4.19), allows for a smoother path across the policy term due to a smaller delta, i.e. a smaller short position in the underlying fund. In contrast, the explict formula approach tracks the increase in the embedded put option value from month 30 to month 45 quite well, but fails to track the embedded put option earlier in the policy term from month 15 to month 30.

The assumption that mortality experience is approximately deterministic, i.e.that mortality experience follows a specified mortality table due to the Law of Large Numbers, means that the difference between the actual mortality basis that emerges over time and the assumed basis leads to a different set of expected future cash flows. This difference makes it difficult to set up a static hedge for guarantees contingent on risks of the combined market. Mortality risk can be rendered approximately deterministic by hedging with mortality swaps. Mortality swaps allow an insurer to exchange its own random mortality experience for the experience of a target population, e.g. population mortality of South Africa. Mortality swaps are largely offered by reinsurers but any firm can enter into a mortality swap arrangement as either the floating party or fixed party.

Chapter 5

CONCLUSION

Risk management strategies for traditional risk products, such as term and whole life insurance, evolve in a natural way from the Law of Large Numbers . By adequately increasing the product's book of policies and contract clauses that manage adverse selection, the risks are reduced substantially. The risks attributable to investment guarantees are set in the combined market of financial risk and mortality risk, which does not allow traditional reserving methods as sufficient. Hardy (2003) sets out three main aspects to be considered in the risk management of investment guarantees:

1. What price to charge for the benefit guarantee?

2. How much capital to hold in respect of the benefits through the term of the contract?

3. How to invest this capital?

The concept of comonotonicity and the conditional lower bound approximation of the value of the underlying fund in the case of regular contributions were introduced in Chapter 2. The price of an arithmetic Asian option at inception of a pure financial contract was dervived in section 2.5. The price at inception of an embedded option is not charged to the policyholder as this would result in the policyholder possibly foregoing some of first contributions. According to Table 2.1, an economy with a risk-free rate of 5% and a volatility of 30% would mean that approximately 85% of the frst contribution is necessary to fund the price of the embedded option. At a volatility of 40%, approximately 132% of the initial contribution is necessary to fund the setting up of a suitable replicating portfolio (assuming a frictionless economy). It is therefore not viable nor always marketable to charge the full initial value of the embedded option to the policyholder at inception of the contract or even as regular amounts over the duration of the contract. In section 2.6, the conditional lower bound approximation is extended to allow for asset-based charges that are solved for by applying the fair-value principle. The asset-based approach is applied to the combined market setting in Chapter 3 for both the Guaranteed Minimum Accumulation Benefit (GMAB) or Guaranteed Minimum Maturity Benefit (GMMB) and Guaranteed Minimum Death Benefit (GMDB) cases. The conditional lower bound approximation therefore aids in addressing the first question posed by Hardy (2003). Pricing embedded guarantees will typically involve pricing with various models over a range of possible economic parameters and scenarios and is extremely time consuming, especially where large sets of simulations are involved. The conditional lower bound approximation aids in quickly dismissing unreasonable benefit structures and underlying fund options from the product development and pricing exercises.

A methodology was outlined in Chapter 3 for determining the price for the embedded arithmetic Asian option of the guaranteed minimum maturity benefit (GMMB) and the guaranteed minimum death benefit (GMDB). For simplicity, the mortality risk of the combined market was assumed to follow a mortality table, i.e. the Law of Large Numbers allow a situation whereby expected mortality rates can be treated as deterministic. This stict mortality assumption can be relaxed to allow a deterministic or stochastic mortality model. In section 3.2, the key issues related to the combined market of mortality risk and financial risk are discussed. One of the main issues to bear in mind when a detailed mortality model is included in the pricing and reserving analyses is that an assumption of independence of mortality risk and financial risk for the physical probability measure is not necessarily maintained when replacing the physical probability measure P with the equivalent martingale measure Q.

In Chapter 4, a potential hedging strategy was considered. The embedded options of investment guarantees are path-dependent and do not allow for self-replicating portfolios to be set up, not even in a simplified setting. Therefore, a real cost is implied by the need for frequent rebalancing of a dynamic hedging portfolio or the cost of a structured solution in the case of a static hedging strategy. This additional cost forms part of the cost of the risk transfer between policyholder and insurer and therefore should be included in the pricing of the product. If an insurer has sufficient risk appetite to accept the combined risks of investment guarantees, a reserving approach can be followed, i.e. a potential shortfall reserve can be set up and maintained during the policy term. A reserving approach can be extremely capital intensive. The cost of capital relates to the cost of the risk transfer and should therefore be included in the pricing of the product.

The last two questions posed by Hardy (2003) consider the ongoing strategy that the insurer needs to follow in order to adequately manage the risk of investment guarantees. These risks can either be transferred to a third party, e.g. reinsurer, investment bank or the open market, or kept by the insurer in the case of reserving. Insurers can follow either of these approaches or a combination, e.g. an insurer can opt to transfer some of the risk by way of a static or dynamic hedging strategy and keep some of the risk on its balance sheet, i.e. set up a reserve. A possible dynamic hedging strategy was outlined in Chapter 4. Although the sensitivity measures, the so-called Greeks, are usually derived from a more comprehensive model than assumed in a Black-Scholes-Merton setting, the Greeks derived from the conditional lower bound approximation can be used to verify the other model outputs. The mortality assumptions in Chapter 4 were from standard mortality tables. Significant fluctuations in mortality experience will alter the profile of the expected aggregate benefit outgo of an insurer, thereby changing the required hedging strategy. Sensitivity to mortality experience fluctuations should be tested for new or unstable portfolios. Static hedging strategies can be considered where the actuarial basis, e.g. mortality assumptions, lapse and surrender assumptions and expense assumptions, is very stable. Static hedging strategies have the advantage that no rebalancing is necessary after inception but typically require structured products that might be prohibitively expensive during and directly after market turmoil.

The amount of capital to be held during the term of the contract therefore relies on the hedging or reserving approach adopted by the insurer. Hedging typically require less initial capital than reserving, since a hedging portfolio aims to replicate the expected maturity payoff of the embedded options. A reserving approach aims to hold sufficient capital to cover a potential loss at a specified sufficiency level, e.g. a 1 in 200 level. This capital is invested according to the maturity profile of the liabilities and the insurer will aim to maximise returns on the capital so as to minimise the cost of capital associated with the reserving requirement. In South Africa, the Professional Guidance Note (PGN) 110 deals with the minimum reserving requirement for the embedded investment derivatives of investment guarantees. The initial requirement amounted to the discounted conditional tail expectation across a recommended 2,000 stochastic economic scenarios. This requirement was very capital intensive and the latest version, ASSA (2008), allows insurers to use the discounted expectation across a recommended 2,000 stochastic economic scenarios. The way in which the capital in a hedging portfolio is invested depends on the number of Greeks that are considered in the dynamic strategy. In section 3.3, the findings by Matterson et al. (2008) were cited and showed that adding additional Greeks to a dynamic hedging strategy reduces the expected loss and the volatility of the expected loss substantially. For example, the put option representation of the embedded option can be delta hedged by shorting the underlying fund or selling futures, rho hedged by entering interest rate swap contracts and vega hedged by purchasing listed options with the required vega measures.

The complexity of insurance contracts set in the combined market of financial risk and mortality risk pose signficant challenges to risk management frameworks of insurers. The conditional lower bound approximation helps to address each of the three questions posed by Hardy (2003) by allowing quick feasiblity studies of product structures during product development, and estimates of the cost of capital in the case of reserving or the cost of a suitable hedging programme in the case of dynamic or static hedging. The use of the conditional lower bound approximation further allows insurers to verify the reasonableness of estimates arising from more complex and resource intensive pricing and reserving models. The conditional lower bound approximation can be extended to allow for more complex asset pricing models and mortality models. In benefit structures where policyholder behaviour plays a significant role, e.g. the GMWB, the conditional lower bound approximation can be used to facilitate the development of an adequate pricing methodology.

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