DEVELOPMENT OF A GROUNDWATER RECHARGE MODEL

Jescica M. Spannenberg

Submitted in the fulfilment of the requirements for the degree

Magister Scientiae in Geohydrology

in the

Faculty of Natural and Agricultural Sciences

(Institute for Groundwater Studies)

at the

University of the Free State

Supervisor: Prof. Abdon Atangana

January 2017

DECLARATION

I, Jescica M. SPANNENBERG, hereby declare that the dissertation hereby submitted by me to the Institute for Groundwater Studies in the Faculty of Natural and Agricultural Sciences at the University of the Free State, in fulfilment of the degree of Magister Scientiae, is my own independent work. It has not previously been submitted by me to any other institution of higher education. In addition, I declare that all sources cited have been acknowledged by means of a list of references.

In addition, the following two papers have been submitted and are under review at the two corresponding journals:

1. **Spannenberg, J; & Atangana, A., 2016. New Approach to Groundwater Recharge on a Regional Scale: Uncertainties Analysis and Application of Fractional Differentiation. *Arabian Journal of Geosciences*.**
2. **Spannenberg, J; & Atangana, A., 2016. New non-linear model of groundwater recharge: Inclusion of memory, heterogeneity and visco-elasticity. *Journal of Hydrology and Earth System Sciences*.**

Jescica M. SPANNENBERG

January 2017

ACKNOWLEDGMENTS

First and foremost, I would like to thank God for blessing me with the opportunity to do my degree at the Institute for Groundwater Studies. Everything within this thesis would not have been possible without this blessing.

"Most people say that it is the intellect which makes a great scientist. They are wrong: it is character." - Albert Einstein. With that being said, I owe special gratitude and appreciation to my supervisor, Prof. Abdon Atangana because it is not only his intellect, guidance, support and inspiration, but also his character which expanded my intellectual knowledge as well as appreciation of scientific research.

Appreciation and a big thank you is also given to the Institute for Groundwater Studies for accepting me as a student as well as awarding me a bursary.

This dissertation also owes its existences to my friends both within Bloemfontein (Albie Steyn, Pieta Hoffman, Marnus du Toit and Grant Lombard) and in Cape Town (Jason van der Merwe, Tashreeq Samodien and Jehrod Visagie) who all brought me inspiration, laughter and an unforgettable 2016.

Finally, without the love, support, and motivation from my family this thesis would not have been. Therefore, I would like to thank my mother (Susan Spannenberg) and father (Vincent Spannenberg) for their continuous love and support of my goals and passion for scientific research. Additionally, I would like to express gratitude to Virgil, Herschelle, Ashleigh and Spencer for their existence and joy brought from their side.

ABSTRACT

Existing groundwater recharge estimation methods appear to mainly generate site specific groundwater recharge estimates. These methods fail to yield reliable recharge estimates on a regional scale. This is due to failure in accounting for the concepts of heterogeneity, viscoelasticity, and the memory effect. Accordingly, this study was aimed at developing a new approach to groundwater recharge estimation by means of taking these concepts into account. Literature proves that these concepts have been well accounted for in the field of fractional differentiation. This study's methodological approach entailed obtaining an exact solution to a selected groundwater recharge equation by applying the Laplace and inverse Laplace transform. Upon doing an uncertainty analysis and statistical analysis of the parameters within the solution, it was found that storativity and drainage resistance both require accurate estimation when estimating recharge from the selected equation. Following this, the Caputo derivative, Caputo-Fabrizio derivative, and the Atangana-Baleanu derivative were applied and an exact solution was obtained for each derivative; and upon doing a numerical simulation for each of these solutions, the results depict the behaviour of a particular real world problem. It was concluded that groundwater recharge within a heterogeneous and viscoelastic geological formation is well described with the concept of fractional differentiation with the generalised Mittag-Leffler law or the Atangana-Baleanu fractional derivative. To add, recharge via elastic geological formations can be model via the Caputo and Caputo-Fabrizio derivatives. Furthermore, hydraulic head is assumed to be influenced by uncertain factors which are not accounted for in the general recharge equations. The Eton approach was thus applied, and reveals the uncertain function has a significant effect on hydraulic head distribution. Ultimately, this study concludes that a groundwater recharge model incorporating heterogeneity, viscoelasticity, the memory effect, and uncertainties, will generate a new and improved understanding to groundwater recharge investigations.

Keywords: Groundwater recharge model; heterogeneity; viscoelasticity; memory effect; uncertain function; fractional differentiation; numerical simulation.

OPSOMMING

Bestaande grondwater herlaai beraming metodes blyk om hoofsaaklik, op ‘n klein skaal, gebied spesifieke beramings te lewer. As gevolg van dit is hierdie metodes nie betroubaar op 'n plaaslike skaal nie. Dit kan toegeskryf word daartoe dat konsepte soos heterogeniteit, visco-elastiese eienskappe, en die geheue effek nie in ag geneem word nie. Hierdie studie is daarop gemik om 'n nuwe grondwater herlaai beraming metode aan te skaf waar die bogenoemde konsepte in ag geneem word. Literatuur bewys dat hierdie konsepte goed erken word in die veld van fraksionele differensiasie. Hierdie studie se metodologiese benadering het ingesluit om 'n presiese oplossing te vind vir 'n geselekteerde grondwater herlaai vergelyking, deur die Laplace-transform en inverse Laplace-transform toe te pas. Met die uitvoer van ‘n onsekerheid ontleding en statistiese ontleding van die parameters in die oplossing dui daarop dat die bergingsvermoë en dreinering weerstand beide 'n akkurate beraming vereis wanneer die herlaai beraam word met die geselekteerde vergelyking. Daarna was die Caputo afgeleide, Caputo-Fabrizio afgeleide, en die Atangana-Baleanu afgeleide toegepas om ‘n presiese oplossing te kry vir elke afgeleide. Die numeriese simulasie resultate vir elke oplossing wys dat die gedrag van 'n bepaalde werklike probleem uitgebeeld word. Daar was tot die gevolgtrekking gekom dat grondwater herlaai vir 'n heterogene en visco-elastiese geologiese formasie goed beskryf word met die konsep van fraksionele differensiasie deur die Mittag-Leffler wet of die Atangana-Baleanu fraksionele afgeleide. Dit was ook aanvaar dat herlaai in 'n elastiese geologiese formasie deur die Caputo en Caputo-Fabrizio afgeleide beraam kan word. Verder is dit aanvaar dat hidrouliese kop beïnvloed word deur onseker faktore wat nie in ag geneem word in algemene herlaai vergelykings nie. Die Eton benadering was dus toegepas, en wys dat die onsekere funksie 'n beduidende uitwerking op hidrouliese kop verspreiding het. Laastens, hierdie studie dui daarop dat 'n grondwater herlaai model wat heterogeniteit, visco-elastiese eienskappe, die geheue effek, en onsekerhede inkorporeer en in ag neem vir 'n nuwe en verbeterde begrip tot grondwater herlaai ondersoeke.

LIST OF GREEK NOTATIONS

α	Alpha
β	Beta
τ	Tau
∂	Partial differential
δ	Zeta
φ	Phi
Γ	Gamma
Δ	Lambda
θ	Pi
ϕ	Phi
ψ	Psi
Φ	Phi
σ	Sigma
Σ	Sigma
μ	Mu
$\mathcal{L}$	Laplace transform

ABBREVIATIONS AND NOTATIONS

^{14}C	Carbon-14
^{2}H	Deuterium
^{3}H	Tritium
^{18}O	Oxygen-18
AET	Actual Evapotranspiration
Cl^{-}	Chloride
CMB	Chloride Mass Balance
CRD	Cumulative Rainfall Departure
DR	Drainage Resistance
DWAF	Department of Water Affairs and Forestry
DWS	Department of Water and Sanitation
EARTH	Extended Model for Aquifer Recharge and Soil
ET	Evapotranspiration
GIS	Geographic Information System
K	Hydraulic Conductivity
LHS	Latin Hypercube Sampling
MCS	Monte Carlo Simulation
PET	Potential Evapotranspiration
R	Recharge
RIB	Rainfall Infiltration Breakthrough
S	Storativity
Ss	Specific Storage
SVF	Saturated Volume Flux
Sy	Specific Yield
T	Transmissivity
TMG	Table Mountain Group
TU	Tritium Units
WLF	Water Level Fluctuation
WTF	Water Table Fluctuation
WRC	Water Research Commission
ZFP	Zero-Flux Plane

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

1.1. BACKGROUND AND RATIONALE

With water being the most essential element on Earth due to its need in society, the economy and the environment, there is a continuous need for managing the resource adequately. Managing surface water sustainably for the aforementioned needs, requires quantification of the input, as this essentially controls the output (distribution) of the resource. However, with surface water resources declining in both South Africa and the rest of the world (Scanlon, *et al*., 2006; WRC, 2015), groundwater becomes an essential resource. In light of this, adequate distribution of groundwater requires accurate quantification of the input into groundwater systems. This quantification is known as groundwater recharge estimation.

Numerous groundwater recharge estimation methods exist. However, the literatures within which these methods are presented indicate they are all associated with a degree of uncertainty. Unsaturated zone methods are generally associated with limitations of being site specific and dependent on hydraulic conductivity (K) which is difficult to determine accurately (Bredenkamp, *et al*., 1995; Scanlon & Dutton, 2003; Simmers, 2013). This site specific limitation is similar for saturated water balance methods which are more reliant on storativity (S) which is another parameter difficult to determine accurately (Bredenkamp, *et al*., 1995; Xu & Beekman, 2003; Gomo & Van Tonder, 2012; Sun, *et al*., 2013). Furthermore, the latter methods rely on water level data and so become even more inadequate when the cause of water level fluctuation (wlf) is not precisely known as it can be caused by changes in atmospheric pressure; ocean tides; earthquakes; and entrapped air (Miyazaki, *et al*., 2012). Essentially, this means that methods applicable to both the unsaturated and saturated zone become uncertain for studies wanting more than just a point estimate.

Furthermore, tracer methods include deuterium (^{2}H) and oxygen-18 (^{18}O) as stable isotopes, and tritium (^{3}H) as a radioactive isotope. ^{2}H and ^{18}O is however more useful for determining groundwater recharge origin, rather than estimation (Bredenkamp, *et al*., 1995), and therefore it becomes more of a qualitative approach (Xu & Beekman, 2003) rather than fit for present day recharge estimation (Simmers, 1987). On the other hand, since ^{3}H has a half-life of 50 years, it is no longer reliable because it is now more than 50 years past the nuclear bomb testing which introduced H^{3} to the atmosphere (Bredenkamp, *et al*., 1995). Consequently, many researchers rely on the chloride mass balance (CMB) due to its ease in application and efficiency in cost (Xu & Beekman, 2003; Scanlon, *et al*., 2011). Unfortunately, the CMB can only be applied provided there are assumptions of no evapotranspiration; no upward or downward leakage; and no irrigation or recycled waste water (Somaratne & Smettem, 2013).

As a result of the above limitations, researchers tend to rely on groundwater modelling methods. However, these methods have limitations due to their requirement of a large amount of data, making these methods time consuming, expensive and difficult to calibrate (Xu & Beekman, 2003). An example is the EARTH model which requires 11 parameters for recharge estimation. Due to all these parameters, the model becomes inadequate for a study area where very little hydrogeological data is available (Bredenkamp, *et al*., 1995; Van Tonder & Xu, 2000; Wu, 2005). Ultimately, all these limitations signify that all existing methods have some form of uncertainty in their recharge outputs.

Based on the aforementioned, a considerable limitation lies with spatial and temporal variation in recharge (Simmers, 1987; Cherkauer, 2004). This difficulty is significant for semi-arid areas (Conrad, *et al*., 2007), causing recharge estimates in these areas to become questionable. Furthermore, these variations can be associated with the concept of heterogeneity which can refer to variation in aquifer properties as a result of variation in an aquifer system's structure and composition (DWAF, 2006). This can occur on a basis of variation from one aquifer to another or even within the same aquifer system; and since variation occurs everywhere (Harter & Rollins, 2008), this relates back to why several recharge techniques are site specific, rather than suitable for regional scale.

Nevertheless, the Department of Water Affairs and Forestry (DWAF) now known as the Department of Water and Sanitation (DWS) deems it necessary to estimate recharge on a regional scale (DWAF, 2006). As a result, they estimated the 1.5% to 35.7% on a national scale for South Africa. However, since DWAF used Geographic Information System (GIS) techniques in their study, their recharge estimates may be questionable, as it is argued that GIS techniques using digital spatial data may be associated with a high degree of uncertainty (Bogena, *et al*., 2005). This calls for an improved approach to estimating recharge on such a large scale. This could be done by taking account of heterogeneity.

Heterogeneity can be further complicated by the concept of viscoelasticity, a property of both viscosity and elasticity, which has been of interest to various research fields related to flow of water. Engineers have given interest due to the ability of viscoelastic material to reduce both drag and heat transfer in channel flows (Kreith, *et al*., 1999). It also becomes imperative to understand in interfacial water, as it gives an understanding of how water-solid molecular interactions affect molecular transport in biological and environmental processes (Sochi, 2009). This can be inferred as viscoelasticity being a fundamental parameter to aquifers systems – possibly giving insight to rock-water interactions affecting transport processes.

It is reasonable to say that by incorporating viscoelasticity in groundwater recharge models, there will be reduced uncertainty and increased reliability in recharge estimates. First of all, this statement is made because viscoelasticity includes the concept of viscosity, meaning it is associated with an effect on hydraulic properties because increased viscosity decreases K (Freeze & Cherry, 1979; Gribovszki, *et al*., 2010). Secondly, viscoelastic effects causes drawdown curves to vary slightly in the middle stages, giving an analogy to double porosity and an unconfined aquifer having a delayed yield (Ci-qun, 1985). The final reason is that viscoelastic properties in an elastic aquifer with "delay and feed" enhances heterogeneity (Ci-qun, 1987). These reasons indicate that since heterogeneity and viscoelasticity affect data used in recharge estimation, these two concepts become limiting factors as recharge equations do not account for them. Ultimately, these limitations emphasise that recharge is a function of both space and time but since recharge models can be inferred as non-linear and associated with non-constant parameters, there becomes a need for accounting for the memory effect (accounts for history of a function of time) in recharge models.

Although these concepts are not often addressed in groundwater recharge, they have been addressed in other fields of scientific research using fractional differentiation. This is because fractional order differential equations involve real/complex order derivatives which have an extent of memory. To add, fractional operators are non-local, in comparison to ordinary derivatives which are local derivatives (Baleanu, 2013). Firstly, Riemann-Liouville has the ability to account for elasticity of a geological formation and the memory effect. Secondly, the Caputo derivative is also a fractional operator very often used for issues within the real world because it keeps memory of past values as the curve changes over time with the fractional order integration. In addition, it also accounts for the effect of elasticity (Ali, *et al*., 2016; Atangana & Alqahtani, 2016). Next, the Caputo-Fabrizio derivative has the ability to take into account heterogeneity on different scales (Atangana & Alkahtani, 2015; Atangana & Alqahtani, 2016). Finally, the Atangana-Baleanu derivative which is based on the Mittag-

Leffler function is another derivative which appears to address real world problems with even greater complexity, as it accounts for heterogeneity, viscoelasticity and the memory effect (Atangana & Alqahtani, 2016). In light of the phenomena the aforementioned derivatives respectively account for, one could apply these derivatives to a real world problem such as groundwater recharge which requires more accurate estimation.

1.2. PROBLEM STATEMENT

Groundwater recharge estimation plays a fundamental role in groundwater resource management. This is due to its importance in quantifying groundwater resources, issuing abstraction licences, understanding groundwater surface water interaction, and assessing groundwater vulnerability (Misstear, 2000). For these reasons, groundwater recharge becomes essential in the context of the National Water Act of South Africa (DWAF, 2006). Although several methods exist, the literatures within which these methods are presented indicate they are all associated with a degree of uncertainty due to their respective limitations. The main limitations are that these methods rely on parameters which have high uncertainty, and that they fail to estimate recharge on a regional scale. This is related to the concept of heterogeneity, which as mentioned can be further complicated by the property viscoelasticity. The current study addresses this problem by conducting an uncertainty analysis to identify which parameter(s) within a certain recharge equation has considerable uncertainty. This is followed by modifying the particular recharge equation and developing a new groundwater recharge model, by accounting for heterogeneity, viscoelasticity, and the memory effect through fractional differentiation.

1.3. AIMS AND OBJECTIVES

The aim of this study is to develop a new approach to groundwater recharge estimation using a new trend of differentiation

Objectives

1.3.1. To review existing groundwater recharge estimation methods.

1.3.2. To conduct an uncertainty analysis on the parameters controlling groundwater recharge using a combination of the Monte Carlo Sampling (MCS) and Latin Hypercube Sampling (LHS) methods.

1.3.3. To modify a selected existing groundwater recharge equation (EARTH model), using the concept of fractional differentiation.

1.3.4. To obtain exact solutions to the selected groundwater recharge equation using fractional order derivatives.

1.3.5. To produce numerical simulations for each exact solution

1.3.6. To obtain a new equation to the EARTH model, using the concept of an uncertain function.

1.4. RESEARCH FRAMEWORK

The framework used to maintain the aim of this study is given below:

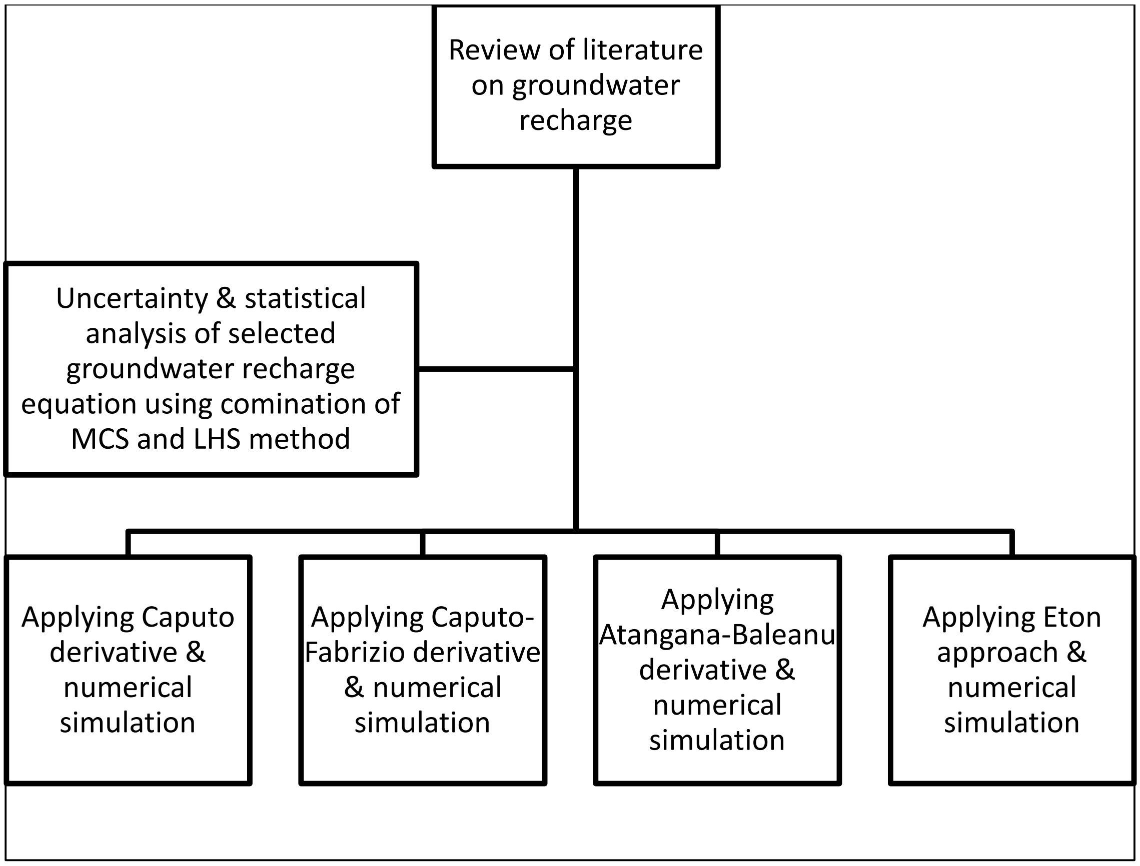

Figure 1: Research framework followed to achieve the aim of this study

1.5. DISSERTATION OUTLINE

This dissertation is divided into seven chapters: Chapter 1 provides a background to the groundwater recharge estimation and the associated problem which will be addressed within the dissertation itself. Chapter 2 and Chapter 3 presents a review on the literature of groundwater recharge occurrence and groundwater recharge estimation, respectively. Moreover, Chapter 3 also addresses the first objective of this study. This is followed by Chapter 4 providing the theory guiding this study. Next, Chapter 5 addresses the second objective of this study by providing the importance of uncertainty analysis and the methods which will be used within this study. To add, Chapter 5 entails the results to the uncertainty analysis, with the addition of the statistical analysis. This is followed by Chapter 6 presenting the mathematical formulations and numerical analysis which addresses the rest of the objectives for this study. Lastly, Chapter 7 closes this study with the conclusion to the study.

CHAPTER 2: LITERATURE REVIEW OF RECHARGE

2.1 INTRODUCTION

Groundwater recharge as a topic is very broad. With that being said, this section gives an overview of what groundwater recharge is, the different types of groundwater recharge occurrences, and the factors controlling groundwater recharge occurrence.

2.2 OVERVIEW OF GROUNDWATER RECHARGE

Groundwater recharge is merely the addition of water to the water table or the replenishment of an aquifer system (Xu & Beekman, 2003; Healy, 2010). The manner in which this occurs differs for different types of groundwater recharge. Therefore the following section discusses the different types of recharge occurrences.

2.3 TYPES OF GROUNDWATER RECHARGE

Groundwater recharge can be divided into namely, natural recharge; artificial recharge; induced recharge and inter-aquifer recharge (Xu & Beekman, 2003; Sun, *et al*., 2013).

Natural Recharge through the Unsaturated Zone

Natural recharge is a process within the water cycle which is merely defined as the movement of water from above Earth's surface through the unsaturated zone to the water table (Sun, *et al*., 2013). The same definition is given for *diffuse, local, and direct recharge* (Simmers, 1987; Healy, 2010) Natural recharge has several influential factors that will determine if natural recharge will make a significant or insignificant contribution to the water table (USDA, 1967). Finally, it is a slow process, and every so often to a point that groundwater discharge exceeds recharge (Yadav, *et al*., 2012). As a result, artificial recharge becomes of interest.

Artificial Recharge

This simply involves directing surface water from an area of great surface water quantity into a groundwater system in an area of water scarcity (Freeze & Cherry, 1979). Therefore, motivation for artificial recharge is storing excess surface water for in cases of drought or where sustainable yield is low, as well as in cases of improving groundwater quality in saline sub-surfaces (Yadav, *et al*., 2012). Artificial recharge can be both direct and indirect, whereby direct is by means of injection wells, and indirect is by means of placing a quantity of surface water on land and ensuring it infiltrates into the subsurface (Bhattacharya, 2010). To add, just as in the case of natural recharge, artificial recharge also has influential factors.

Induced Recharge

Induced recharge is the addition of water to the water table from a surface water body. It is regarded as an indirect type of artificial recharge, as it occurs upon significant abstraction from an aquifer hydraulically connected to a surface water body (Yadav, *et al*., 2012). To expand, as abstraction occurs, the surface water body will act as a *constant-head line*; and so abstracted water will initially be sourced from the groundwater system (Freeze & Cherry, 1979). However, as pumping continues, a cone of depression occurs toward the surface water body; and eventually surface water will be induced into the groundwater system. In the long run, this causes abstracted water to be of surface water content as well.

A similar definition of induced recharge is given for *focused* recharge, as it is the movement of water from a surface water body to an underlying aquifer (Healy, 2010). Focused recharge is emphasised more in arid environments where low precipitation limits the process of diffuse recharge. As a result, when estimating groundwater recharge, methods can be divided into those adequate for diffuse and focused recharge.

Inter-Aquifer Recharge

This type of recharge occurs as water moves laterally or vertically between aquifer systems. A study aimed at investigating groundwater recharge in two distinct aquifers concluded there is a possibility of an alluvial aquifer being recharge by both infiltration of rainwater, and inter-aquifer recharge from an adjacent terrestrial aquifer (Gomo & Van Tonder, 2012). The latter could however not be proved. As previously indicated, groundwater recharge is defined as the addition of water to the water table, and so in this sense, inter-aquifer recharge is not considered groundwater recharge (Healy, 2010).

2.4 FACTORS CONTROLLING GROUNDWATER RECHARGE

Regardless of the type of recharge occurrence, the occurrence of recharge itself depends on the factors controlling recharge. This is why prior development of a model, a conceptual model should first be maintained by considering the factors influencing recharge (Healy 2010). This section expands on the climatic, surface and subsurface factors controlling recharge, as these factors determine the quantity and time taken for recharge to occur (Nolan, *et al*., 2006).

2.4.1. Climatic Factors

Rainfall

Without groundwater recharge, there is no groundwater resource (Gomo & Van Tonder, 2012). This means that without rainfall, there is no resrource. Subsequently, the amount of rainfall becomes a significant control in groundwater recharge. This was seen with a reported range of 1mm/y to 5mm/y recharge in the Kalahari, which was suggested to be due to an increase in rainfall from 350mm to 450mm (De Vries, *et al*., 2000). Additionally, recharge is also controlled by rainfall intensity, duration and distribution (Lerner, *et al*., 1990).

Another way to look at the significance of rainfall in recharge is by means of a rainfall threshold for recharge to occur. This is also known as a recharge threshold. A high threshold is associated with no recharge over a long period which is due to low rainfall conditions. Conversely, a low threshold is associated with quick recharge, and therefore it is generally found in temperate climates (DWAF, 2006). DWAF determined the recharge threshold for South Africa, but it was however on a quaternary catchment scale, so it is somewhat generalised and associated with uncertainty. This uncertainty was simply due to the failure in quantifying the factors affecting the recharge threshold. As a result, they suggested that for a more detailed view on the recharge threshold, there should be determination of spatial variability in recharge per quaternary catchment. In spite on their uncertainty, South Africa is thought to have recharge threshold of 400mm (Bredenkamp, *et al*., 1995).

Evapotranspiration (ET)

The rate or amount of groundwater recharge occurring is influenced by ET which is the removal of moisture from a subsurface (Simmers, 2013). It is considered a water loss, as it does not contribute to groundwater available for supply (Singhal & Gupta, 2010).

Furthermore, ET is controlled by solar radiation, temperature, relative humidity, wind speed, environmental conditions, and crop characteristics and practices (Bredenkamp, *et al*., 1995; DWAF, 2006). To expand, the likelihood of ET increasing does so with increasing temperature – this is eventually associated with less recharge (Nolan, *et al*., 2006). This is also seen on a daily scale, where groundwater tables show diurnal variation in water level fluctuation due to day and night variation in ET (Singhai & Gupta, 1999). ET being high during the day causes a drop in water level, and since no ET occurs at night, no water loss is found, causing insignificant to no decrease in groundwater levels. Consequently, ET is a negligible parameter during night time hours of wlf measurements (Fan, *et al*., 2014).

The occurrence of ET depends largely on depth to water table and soil type. Subsequently, shallow water table aquifers show water table fluctuation being controlled by ET (Singhal & Gupta, 2010). To add, it is reported that high evaporation rates can remove up to about 50% moisture in the first 0.25m of a soil column (Van Wyk, 2011). This is why large ET is associated with water scarce environments, namely arid; semi-arid and sub-humid environments (Singhal & Gupta, 2010). Furthermore, a study conducted in an area of Denmark for estimation and a comparison of recharge and ET between two ecosystems; found that higher recharge estimates were associated with higher precipitation and lower transpiration (Laderkarl, *et al*., 2005). In light of the importance of ET as control of recharge, there is a need for accurate ET measurements.

When considering ET in groundwater recharge studies, it is imperative to understand ET is a component made up of different sources (Healy, 2010), as seen in (1)

$$\mathrm{ET} = ET_{sw} + ET_{gw} + ET_{uz} \tag{1}$$

Where ET is evapotranspiration; ET_{sw} is the evapotranspiration from water stored on the surface of Earth; ET_{gw} is evapotranspiration from the saturated zone; and ET_{uz} is evapotranspiration from the unsaturated zone. This once again justifies reason for accurately measuring the parameter.

ET can be measured in two different ways – potential evapotranspiration (PET) and actual evapotranspiration (AET) (Singhal & Gupta, 2010). PET is the amount of moisture removed when soil has enough water to meet its demand, and AET is the amount essentially removed under existing field conditions. To add, AET is generally 50-90% less than PET. This is seen in literature where PET is greater than AET for hard rock groundwater systems (Van Wyk, 2011). Another study aimed at showing the effect of ET on groundwater recharge using a scintilometer which measures the total ET, used their ET measurements for a HYDRUS-2D model to maintain the aim of their study (Bugan, *et al*., 2011), The study concluded that there is a need for accurately measuring ET in groundwater. Essentially, both these studies suggest more research be conducted on both AET and PET estiamtion, as it is a critical parameter to recharge estimation.

2.4.2. Surface Factors

Groundwater recharge associated with land cover or land use practices

Surface features or activities above ground are also seen as ultimate controls of how groundwater recharge occurs and what is eventually estimated. To expand, urbanisation relates to both decreased recharge due to impervious surfaces and increased recharge due to pipeline leakage from water supply units (Singhal & Gupta, 2010). These urbanisation effects on wlf are found in both unconfined and confined aquifers. Furthermore, land use practice effects on recharge can be seen with irrigation and cultivation. To expand, a study revealed that flood-irrigated land produces more groundwater recharge than non-irrigated or non-cultivated land, and croplands generate less recharge due to more transpiration (Singhal & Gupta, 2010).

Another surface factor being vegetation, can either enhance or reduce infiltration. Enhancement is generally associated with vegetation type. To expand, certain types of vegetation restrict surface runoff, and in doing so, more time is allowed for water to infiltrate the soil. Moreover, infiltration increases even more when roots systems are well developed. This would eventually result in more recharge. This theory is supported elsewhere, whereby boreholes in sparse grasslands, banskia woodland vegetation and pine plantation show average depth to water tables of 1.02m, 0.55m and 0.68m, respectively (Fan, *et al*., 2014). Additionally, recharge was about 20% in the pine and banskia vegetation, which was lower than the grassland. This was due to increased interception from their canopies and their lower depth to water table. This can be related back to what was previously said on ET having a greater influence on shallower depth to water tables. In contrast to this, rain drops falling on bare soil causes compaction of the surface, resulting in less recharge. Furthermore, when referring to vegetation density, thick vegetation cover increases interception, and in doing so it reduces infiltration and percolation (Simmers, 1987). Similarly, denser vegetated land is associated with more ET and thus less recharge (Healy, 2010). Conversely, if permeable soils or fractured rock material has poor vegetation cover, it may be associated with significant recharge. Finally, bare soil may soak up rainfall and enhance recharge, however if the soil swells, no more water will enter the soil (Simmers, 1987).

Topography and hydrology

Steeper slopes are generally associated with greater surface runoff and less infiltration, and therefore less recharge. In contrast, flatter surfaces are usually associated with less runoff and most likely diffuse recharge (Healy, 2010). Furthermore, surface water bodies become an influential factor in groundwater recharge due to groundwater surface water interaction. This interaction is given by losing streams, where a stream generates recharge to an underlying aquifer system. In that occurring, recharge is dependent on temperature of surface water, as it controls viscosity and K which further controls the migration of water (Freeze & Cherry, 1979; Gribovszki, *et al*., 2010). Lastly, losing streams are generally associated with indirect recharge (DWAF, 2006); and so the more tributaries there are, the greater the recharge (Shaban, *et al*., 2005).

2.4.3. Subsurface Factors

Depth to water table/unsaturated zone thickness

Thickness of the unsaturated zone can be determined by the depth to water table. This was found for a study conducted in West Coast of South Africa, where it was suggested that recharge will generally not occur in areas where soil thickness is greater than 4m, as water will store but before percolation occurs, ET occurs (Conrad, *et al*., 2007). This is supported when it is reported that thickness of soil reduces infiltration and thus recharge (Shaban, *et al*., 2005). As a result, a lag time in recharge of up to 4 months was inferred for the West Coast study. Consequently, it was concluded that previous recharge estimates of 8-15% for the area may have inaccuracies.

Soil moisture

Soil moisture content depends on the quantity of water loss or retained which is ultimately controlled by ET and intermolecular forces against the pull of gravity (Singhal & Gupta, 2010). When the soil is dry, infiltration is generally higher; and when soil moisture reaches field capacity, percolation will occur – causing a rise in the water table (Simmers, 1987).

Lithology

First of all, infiltration is a process of water entering soil, be it surface water or precipitation. The rate of infiltration depends on the characteristics of the subsurface material such as its lithology (Singhal & Gupta, 2010). As a result, lithology influences storage and permeability properties of soil that are eventually linked to soil moisture and thus recharge which becomes necessary to understand (Van Wyk, 2011).

As mentioned, fine grained subsurface material is generally associated with higher ET than coarse grained material, and therefore the latter usually produces more recharge. Furthermore, granular material is generally associated with minor water level fluctuation, as recharge is distributed aerially (Lerner, *et al.*, 1990). Also, recharge is more significant in fractured rock and karst formations due to significant permeability (Van Wyk, 2011). This is supported when it is indicated that karst aquifers are related to dissolution of carbonate material which results in sinkholes and solution channels, and this eventually causes significant recharge from the surface (Shaban, *et al.*, 2005). In essence, since lithology varies on a spatial scale, this raises the concept of heterogeneity which will be accounted for in this study.

Storativity (S)

S is the volume of water released from or stored in an aquifer due to hydraulic head changes (Kresic, 2007). It is storativity (S) in a confined aquifer and specific yield (Sy) in an unconfined aquifer. The amount of water eventually reaching an aquifer system or more specifically the water table relies on this property (DWAF, 2006). This is the reason a study conducted in the Aynalem well field of Mekele, Ethiopia shows that a slight change in S has a significant influence on the change in water table (Tegeri, 2009). Furthermore, the results of a study conducted in the Table Mountain Group (TMG) aquifer shows the dependence of recharge on Sy (Sun, *et al.*, 2013). Since it is known that TMG aquifers are heterogeneous in nature, they used Sy values ranging from 0.0021 - 0.21 for recharge estimation. Their findings were daily recharge values ranging from 0.5% - 51.5% and monthly recharge values ranging from 0.15% - 15.7%. This clearly shows that a small change in Sy or S, changes the amount of groundwater recharge estimated.

There are various techniques for determining S, and these range from pumping tests, empirical approaches, and tracer tests (Bredenkamp, *et al.*, 1995). The first technique merely uses analytical models (Theis, Cooper Jacob, and Boulton methods) along with their assumptions to determine S from pumping test data. Most of these assumptions are however defied in fractured rock aquifers where the aquifer does not show a homogenous, isotropic and uniform thickness; and neither do they show an infinite extent, as there always appears to be some form of boundary condition. Another limitation to the pumping test technique is that is it time consuming and thus cost consuming too. In addition, it does not always produce true characteristics of the aquifer's hydraulic behaviour. An example reveals that a 12 day pumping test in the literature did not yield confident results. Consequently, the second technique of using an empirical approach became famous. However, it required confirmation on its accuracy. Tracer test then also became a major interest, in particular for an experiment done on the Karoo aquifer which does not go by the analytical models' assumptions. The findings to these tracer tests presented S ranging from 0.0001 – 0.0077, whereby greater S values were assumed for fractures and lower S for the matrix; and this significant range was suggested to be a result of the heterogeneous nature of the Karoo aquifer. It was concluded that the most adequate way to determine the 'correct' S was using an empirical technique but as mentioned confirmation was required for technique's accuracy, thus they recommended research be done to reduce uncertainty in the technique.

Due to difficulty in obtaining S, researchers often use a constant S/Sy. However, it is argued that constant Sy can lead to overestimation of both groundwater recharge and ET (Fan, *et al*., 2014). Consequently, since Sy can also change in depth, for groundwater systems with a water table greater than 1m, the dependence of Sy on depth to water table needs to be considered. This clearly indicates the need for a groundwater recharge model accounting for heterogeneity.

Groundwater abstraction

Groundwater abstraction accounts for a parameter of recharge. However, quantifying the impact of groundwater abstraction on groundwater recharge is not often seen. As a result, a study was conducted in the Bengal Basin in Bangladesh during a period of significant abstraction (Shamsudduha, *et al*., 2010). The findings to this study indicate that irrigation systems using a significant amount of groundwater caused a decline in water level. In doing so, groundwater recharge increased by means induced groundwater recharge. Ultimately, this indicates groundwater abstraction becomes a fundamental control in the amount and source of recharge, as well as when quantifying it.

CHAPTER 3: REVIEW OF RECHARGE ESTIMATION

3.1. INTRODUCTION

As mentioned, there are several influential factors (refer to Chapter 2), and therefore it becomes reasonable to say that groundwater recharge as a topic is particularly broad. This is the reason the key to recharge studies is the project objective (Simmers, 1987; Healy, 2010). This dissertation is aimed at developing a new approach to groundwater recharge modelling, and therefore the review to follow is on already existing recharge estimation models/methods.

3.2. REVIEW OF GROUNDWATER RECHARGE ESTIMATION METHODS

Numerous ways to classify groundwater recharge estimation methods exist, but this section provides a classification based on a combination of the previously mentioned unsaturated zone methods, saturated water balance methods, tracer methods, and groundwater modelling recharge methods.

3.2.1. Methods applicable to the unsaturated zone

3.2.1.1. Zero-Flux Plane (ZFP) method

The ZFP method is primarily based on the position of the ZFP where there is no hydraulic gradient in the soil profile. It is proposed that groundwater recharge using this method is calculated by summing the changes in water located below the ZFP; and the position of the plan may be given using *tensiometers* – a device measuring soil water retention. Upon locating the ZFP, changes in water content occurring both above and below the plane are associated evaporation/evapotranspiration and drainage, respectively. Moreover, groundwater recharge is calculated by summing the changes occurring below the ZFP. The flux can be given using hydraulic conductivity (K) the potential gradient using (2) which is based on Darcy's Law (Simmers, 1987; Bredenkamp, *et al*., 1995).

$$q = -K(\theta).\frac{d\emptyset}{dz} \tag{2}$$

Where K (θ) is the unsaturated K; $\emptyset$ is the total water potential; z is a reference value.

K is often difficult to determine precisely, as K varies significantly amongst different soil types; and so the ZFP method often become unreliable in recharge estimation (Bredenkamp, *et al*., 1995). Moreover, the method becomes inadequate when infiltration is high causing a downward hydraulic gradient (Simmers, 1987; Bredenkamp, *et al*., 1995). As a result, it is recommended using water balance approaches instead (Simmers, 1987). This is supported elsewhere, where it is suggested the method is therefore best applied in summer months when infiltration is low and the ZFP is more easily located (Cooper, 1980). Finally, it can be said that the method appears to be inadequate for reliable recharge estimation.

3.2.1.2. Darcy's Law in the unsaturated zone

As we know, Darcy's Law describes flow through a porous medium, and using this principle as a means of flow from a source of recharge, groundwater recharge may be estimated (Healy, 2010). As seen in the ZFP method, the unsaturated zone has both an upper and lower zone, which are determined by soil and evapotranspiration properties; and the extent of these properties determine whether or not drainage of upward movement of water within the soil occurs (Bredenkamp, *et al*., 1995).

Just as in the ZFP method, a flux can be related to groundwater recharge. As a result, the downward flux can be obtained when combining Darcy's Law and a concept of mass balance. This combined equation is given in (3)

$$\frac{\delta\theta}{\delta t} = \left[K(\theta)\left(\frac{\delta\psi}{\delta z} - 1\right)\right] \tag{3}$$

Where θ is the volumetric water content; t is the time; z is the depth; K is hydraulic conductivity for the soil matrix; and ψ is the matrix potential.

For application of the Darcy method in recharge estimation, there should be an understanding of a few fundamental factors. These factors are *porosity* which simply determines the amount of water soil can still maintain; *matrix potential* which is gives insight to the amount of water in a a soil profile; and the *residual water* which is merely the fraction of water maintained if suction were to occur (Bredenkamp, *et al*., 1995). It is imperative to understand these factors because it gives an understainding of water content which is has a relationship with matrix potential and K. To expand, when water content slightly changes, a noteworthy change in K can be observed. K therefore becomes the primary limitation to the method due to significant spatial variation in soil moisture for different soils (Scanlon & Dutton, 2003). Since K is difficult to determine directly (Healy, 2010; De Vries & Simmers, 2002), even more inadequacy is associated with the method.

Besides K being the main limitation, there are several other limitations in association. This includes difficulty in determining boundary conditions; lack of precipitation and evapotranspiration data in certain areas; and because K may vary significantly in a soil profile, one cannot make extrapolations (Bredenkamp, *et al*., 1995). Finally, the Darcy approach only provides point estimates (Scanlon & Dutton, 2003); and as mentioned, with point estimates extrapolation cannot be made.

3.2.1.3. Lysimeter method

Lysimeters are often used in ET estimation under varying vegetation cover and soil moisture conditions (Bredenkamp, *et al*., 1995; Singhal & Gupta, 2010). Lysimeters allow the most direct estimation of recharge (Bredenkamp, *et al*., 1995), and the instrument may exist both naturally or through manual construction (Seiler & Gat, 2007). Natural lysimeters can be springs of known catchment size; or karst caves, fractures and tunnels. Conversely, constructed lysimeters are achieved by drilling open end cylinders into the soil at different depths, and at the open ends, there may be placement of *suction cups* for investigating water flow and water quality in the unsaturated zone. Furthermore, an investigation using lysimeters in an area of Pretoria showed that percolation appears to have a linear relationship with precipitation, but only above a precipitation threshold of 400m (Bredenkamp, *et al*., 1995). Consequently, it was established that recharge would only adequately be seen on an annual scale, rather than monthly scale. Another study done to analyse groundwater recharge from precipitation, and groundwater loss due to ET using lysimeters, found that lysimeter measurements were influenced by vegetation, depth to water table, and soil moisture content (Seiler & Gat, 2007).

Although being a direct method for recharge estimation, the method incorporates several limitations. Firstly, for recharge estimation with lysimeters, the soil needs to be in its natural state but this is however rather difficult and time-consuming to achieve (Bredenkamp, *et al*., 1995). It is indicated that when recharge is eventually estimated, it is only a point estimate. Therefore extrapolation is not possible. Moreover, meaningful variation in recharge can only be observed upon doing several years of measurements; and lysimeters are somewhat difficult to maintain, causing studies to be incomplete. Furthermore, it is also suggested that soil thickness exceeding 1.8m show insignificant to no percolation, and this is probably due to water loss through ET. This gives an indication that depth to water table/thickness of the unsaturated zone controls seepage. Other limitations include that lysimeters often disturb the natural water content and that all types of lysimeters tend to either under/overestimate ET and/or groundwater recharge (Seiler & Gat, 2007).

3.2.2. Saturated water balance methods

Saturated water balance methods make use of inputs, outputs, and the change in storage which is a resultant of the inputs and outputs (Bredenkamp, *et al*., 1995). These methods calculate a water balance over a selected time period. For that which is calculated to be valid, the response which could be runoff; baseflow or groundwater level change should occur within that specific time period. However, difficulty is associated with relating a delayed response to a specific precipitation event, as a precipitation event does not always give an instantaneous effect. To add, delays may even extend up to several months. However, even in cases of immediate effect, the rise in water level may continue for longer than the recharge event, and again there is difficulty in relating the response to a particular event. As a result, they suggest the time component should be selected with consideration, but this depends on the method used.

Both groundwater hydrographs and groundwater balance equations may be used to determine a saturated groundwater balance. Ultimately, both show a groundwater system's response to input and output. The following are saturated water balance methods:

3.2.2.1. Spring flow analysis

The spring flow analysis recharge method forms part of a hydrograph saturated water balance method. The basis of this method is that spring discharge forms due to accumulation of recharge sources over a catchment area; and since spring flow tends to increase when rainfall increases, and vice versa, it is possible to relate the spring flow discharge observed on a hydrograph to a precipitation recharge event (Bredenkamp, *et al*., 1995). To expand, springs from karst aquifers depend on groundwater recharge, groundwater storage and groundwater flow (Singhal & Gupta, 2010). It is indicated that a hydrograph for this type of aquifer generally resembles that of a surface water body, whereby there is a rising limb, peak, recession limb, and a lag time between the precipitation and peak of the hydrograph. Furthermore, it is also suggested that when visually observing the hydrograph, interpretations can be made regarding the groundwater system. For example, when the lag time is large and the peak is broad and flat, it may be indicative of a permeable aquifer, such as a karst formation. Moreover, karst formations also tend to show *bumps* in there curve, suggesting a *complex recharge* and *flow mechanism*. On the other hand, in the case of focused recharge with minimum storage, the hydrograph will show a spike in the curve; and in contrast, when diffused recharge and significant storage is present, the hydrograph will show extended spring flow and only a slight peak in the hydrograph.

The method makes use of the discharge of the spring to obtain recharge for a defined catchment area (Simmers, 1987; Bredenkamp, *et al*., 1995). This is done using (4) (Singhal & Gupta, 2010).

$$\mathrm{AR} = \frac{Q_2}{\alpha} - \frac{Q_1}{\alpha} + \int_{t_1}^{t_2} Q\,dt \qquad (4)$$

Where A is area; R is recharge; Q_1 and Q_2 are spring discharge at t_1 and t_2, respectively; t_1 and t_2 are time at the end of one dry season and the beginning of the next one; and α is the recession coefficient. Moreover, α is a function of the groundwater system's transmissivity; storage coefficient or specific yield and the catchment geometry. Large α is indicative of high hydraulic conductivity and more so fracturing.

When a reliable estimate of the average discharge of spring flow is obtained, and the catchment area is accurately defined, the recharge estimate may be attained (Bredenkamp, *et al*., 1995). Subsequently, the method was applied and the findings suggest that spring flow seen on the hydrograph does not always give a true reflection of the total recharge, as water may be lost to subsurface flow which is a component difficult to determine. For that reason, it is stipulated that recharge estimated from this method only yield 50% the recharge actually occurring.

Additional limitations make the method largely inadequate for recharge estimation. This includes that recharge is represented as a product of both recharge and area (Van Tonder & Xu, 2000); and that catchment area is often difficult to define, especially when the spring may be due to a geological structure indicating water is sourced from a larger catchment (Bredenkamp, *et al*., 1995).

3.2.2.2. Water Table Fluctuation (WTF) Method

Another hydrograph method is the WTF method; and these hydrographs are constructed by plotting changes in water level at a specific location over a period of time (Prathapar & Sides, 1993). Moreover, water levels fluctuate due recharge and discharge components, and the extent of the influence of two these components is controlled by Sy. Consequently, the WTF method is based on the premise that the groundwater table in an unconfined aquifer increases as a result of water being added to the water table (Healy, 2010). The equation used for recharge estimation is given in (5).

$$R = \frac{Sy\Delta h}{\Delta t} \tag{5}$$

Where, R is recharge; Sy is specific yield; Δh is change in water table height; and Δt is the time interval (Healy, 2010).

Using this premise, an estimation of 53mm/y and 55mm/y recharge was made for the aforementioned alluvial and terrestrial aquifers, respectively (Gomo & Van Tonder, 2012). Another study which applied the method, reports 210mm/y and 164mm/y for an aquifer located in an area of Argentina called Pampa Plain which receives 908mm average annual precipitation (Varni, *et al.*, 2013). The latter study's difference in recharge estimates were attributed to Sy values of 0.09 and 0.07, respectively. The difference clearly shows that despite the method providing an easy and simple application because of only few parameters required, the method is still sensitive to Sy. The sensitivity of the method to Sy is supported by others (Healy, 2010; Gomo & Van Tonder, 2012). This is due to the difficulty in estimating an accurate Sy as discussed in section 2.4.3.

The WTF method is however only adequate over short time periods (Rushton, 2004), as these data are not significantly impacted by other parameters in a more complex water balance equation such as those given for water balance methods which will be reviewed later in this section. Finally, as previously mentioned, groundwater level does not only necessarily fluctuate due to addition of water to the water table, but also due to changes in atmospheric pressure, ocean tides, earthquakes, and entrapped air – making it even more difficult to have confidence in the estimate.

As mentioned, groundwater balance equations on the other hand are also useful in saturated groundwater balance recharge methods; and they too are defined by inputs, change in storage and output (Healy, 2010). A simple equation for a soil column extending down from the surface of the Earth to some depth below ground is seen in (6.1).

$$P = ET + \Delta S + R_{off} + D \tag{6.1}$$

Where P is precipitation; ET is evapotranspiration; ΔS is the change in storage; R_{off} is precipitation which becomes direct runoff; and D is drainage which ultimately defines the recharge component when the drainage is added to the water table. Furthermore, when a watershed is found above an unsaturated or saturated zone, (6.2) is given.

$$P + Q_{on} = ET + \Delta S + Q_{off} \tag{6.2}$$

Where P, ET and ΔS are the same parameters in (6.1); O_{on} is surface and subsurface water flowing into the watershed; and Q_{off} is surface and subsurface water discharging from of the watershed. However, ET; ΔS and Q are parameters made up of different components and sources, and therefore (6.3) arises. Although it is highlighted that most of these parameters are regarded as negligible, they generally make up water balance approaches for estimating groundwater recharge, provided a parameter of recharge is within the equation. Furthermore, (6.3) can be refined to equation (6.4) which is more suited to an aquifer system. This equation is also adequately used in numeric modelling of groundwater.

$$P + (Q^{sw}_{on} + Q^{gw}_{on}) = (ET^{sw} + ET^{uz} + ET^{gw}) + (\Delta S^{snow} + \Delta S^{uz} + \Delta S^{gw}) + (Q^{gw}_{off} + R_{off} + Q_{bf}) \tag{6.3}$$

$$R + Q^{gw}_{on} = \Delta S^{gw} + Q_{bf} + ET^{gw} + Q^{gw}_{off}) \tag{6.4}$$

Where P is precipitation; and Q^{sw}_{on} and Q^{gw}_{on} are surface and groundwater flow into the watershed, respectively. ET^{sw}, ET^{uz} and ET^{gw} are evapotranspiration from surface water, the unsaturated zone and the saturated zone, respectively. $\Delta S^{snow} + \Delta S^{uz} + \Delta S^{gw}$ are changes in storage within snow, unsaturated zone and saturated zone, respectively. Lastly, $Q^{gw}_{off} + R_{off} + Q_{bf}$ are groundwater abstraction, surface water runoff and groundwater baseflow, respectively.

Essentially, this shows that several equations exist for different groundwater balance methods. They range from simple equations such as (6.1), to equations using more detailed parameters. The equation used solely depends on the parameters considered. The parameters considered are based on the significance of the groundwater recharge influential factors, and therefore it is suggested that whenever using a water balance approach, a conceptual model and uncertainty analysis be used alongside (Healy, 2010). This is however not always the case.

Furthermore, water balance approaches initiated several years ago. Therefore, it is suggested these approaches have significantly developed over time (Bredenkamp, *et al*., 1995). The following are saturated water balance groundwater recharge estimation methods defined by a water balance equation.

3.2.2.3. Saturate Volume Flux (SVF) Method

The SVF method involves obtaining the status of the saturated volume of an aquifer, from the average of groundwater levels of observation boreholes in an area of interest (Bredenkamp, *et al*., 1995; Xu & Beekman, 2003). This can be done by means of constructing a grid network of elements using the water levels from the observation boreholes. Thereafter, an arbitrary value is given for the aquifer's base thickness. This value is selected to ensure the saturated volume remains positive. Upon obtaining the status of the saturated volume of the aquifer, its variation over time is analysed (Bredenkamp, *et al*., 1995). Furthermore, the method can estimate recharge from current data, and is best applicable to unconfined to semi-confined aquifers (Xu & Beekman, 2003).

Based on Bredenkamp, *et al*., (1995), the method's mathematic expression used for recharge estimation from the SVF method is given by (7)

$$RE + I - O - \frac{\Delta \mathrm{v}}{\Delta \mathrm{t}} \cdot S = Q \tag{7}$$

Where RE is recharge into the aquifer system; I is the average lateral inflow; O is the average lateral outflow; Δv is the change in saturated volume of the aquifer; Δt is the period over which the water balance process takes place; S is the storativity of the specific yield; and Q is the discharge in terms of abstraction. Therefore the data required merely incorporates groundwater levels, storativity or specific yield, rainfall data and area size (Adams, *et al*., 2004). To add, when evapotranspiration is known it may be added to the abstraction (Q) for a more accurate estimation of groundwater recharge (Bredenkamp, *et al*., 1995). However, when it is not known, the estimated recharge is only regarded as the effective recharge which is smaller than actual recharge.

A study aimed at quantifying and characterising groundwater recharge in a basement and alluvial aquifer situated in the Central Namaqualand region of South Africa applied the SVF for recharge estimation (Adams, *et al*., 2004). They obtained information about recharge areas from a topographic map, and used Microsoft Excel for simulation of groundwater levels. They obtained S from both pumping test data and the SVF method, whereby the latter is assumed more reliable because pumping test data do not always produce reliable information from fractured rock aquifers. The final conclusions were that the method was successfully applied and generated adequate recharge estimates. However, they make mention that although it is regarded as successful, the estimates may still have some uncertainty due to the dependence on storativity.

Another application of the method is given in a dissertation for a study conducted in Potchefstroom, South Africa (Van Rooyen, 2014). Application was done for both a fractured rock and dolomite aquifer. Simulations of water levels for the two aquifers with the SVF were done using Microsoft Excel. The researcher’s findings suggest recharge estimates of 3% and 12% from the average annual rainfall. No conclusion was made regarding the recharge estimates, as the study was aimed at developing a method for quantifying the risks of urbanisation on groundwater systems. Lastly, another study conducted in the West Coast of South Africa presents 0.3% – 1.7% recharge from the SVF method which is lower than an 8% recharge given in literature for the area (Conrad, *et al*., 2007). This study concludes that the SVF estimates may be more reliable because other methods used within the same study yielded similar estimates. However their final remark was that their study’s results are only preliminary and thus require further assessment.

Although application of this method is widely seen, several limitations are accompanied. This includes the requirement of an accurate estimation of S which we now know is difficult to determine accurately; quantities of groundwater inflow and outflow is not always available; a large dataset of many years is required for adequate recharge estimation; and the method is unreliable in aquifers where there is significant variation in water level response (Adams, *et al*., 2004). Other limitations include inadequacy in application to multi-layered aquifer systems; groundwater levels used to obtain the saturated volume flux need to be representative of the whole aquifer system; there should be knowledge of inflow and outflow boundaries; and that a uniform groundwater recharge estimate is obtained (Xu & Beekman, 2003). Lastly, for adequate results, there should not be uncertainty in the aquifer system's inflow and outflow; and the boreholes should be well spread throughout the study area (Van Tonder & Xu, 2000).

3.2.2.4. Cumulative Rainfall Departure (CRD) Method

Another method showing significant development is the CRD method. Dating several years back, it is indicated that the CRD method is based on the principle of equilibrium conditions eventually occurring in an aquifer over a period of time (Bredenkamp, *et al*., 1995). This is when the average input equals the average output. From these conditions, the CRD method is able to indicate the influence of input and output on the groundwater level fluctuation. As a result, it is suggested that the method responds adequately to groundwater level fluctuation, thus one can compare groundwater level fluctuation and CRD. Furthermore, the equation used to estimate groundwater recharge from the CRD method is given in (8.1)

$$\Delta h_i = \left(\frac{r}{S}\right) \cdot ({}_{av}^{1}CRDi) \qquad \text{(i = 0, 1, 2, 3, ...N)} \qquad (8.1)$$

Where Δh_i is the change in water level over a period of time; r is the fraction of CRD resulting in recharge from rainfall; S is the storativity; and CRD is the cumulative rainfall departures which can be given in (8.2)

$${}_{av}^{1}CRDi = \sum_{n=1}^{i} R_n - k \sum_{n=1}^{i} R_{av} \qquad \text{(i = 0, 1, 2, 3, ...N)} \qquad (8.2)$$

Where R is the amount of rainfall; i is the i-th month; av is the average; and k is a parameter indicating abstraction conditions, whereby k = 1 it shows natural system; and k>1 shows abstraction conditions system. Furthermore, when the CRD is positive the groundwater level will increase and vice versa.

Application of the method is given by Bredenkamp, *et al*., (1987) who regarded it as relatively adequate. However, it was later suggested that despite a negative CRD, groundwater level will still increase, for as long as groundwater recharge exceeds groundwater discharge (Van Tonder & Xu, 2000). As a result the formulae was revised to (8.3) for improved CRD calculation.

$$ {}_{t}^{1}CRDi = \sum_{n=1}^{i} R_n - \left(2 - \frac{1}{R_{av}i} \sum_{n=1}^{i} R_n\right) \sum_{n=1}^{i} R_i \qquad (8.3) $$

Where R_t characterises a threshold value further representing boundary conditions within an aquifer system. This value ranges from 0 to R_{av}, whereby 0 represents a closed system and R_{av} represents an open system; and these values are given during simulation. As we now know CRD is related to groundwater level fluctuation, therefore the new formulae for recharge estimation is given in (8.4)

$$ \Delta h_i = \left(\frac{r}{S}\right) \cdot \left({}_{t}^{1}CRDi\right) - \frac{Q_{pi} + Q_{outi}}{AS} \qquad (i = 0, 1, 2, 3, ...N) \qquad (8.4) $$

The parameters still represent that in (8.1), and $\frac{Qpi + Qouti}{AS}$ is only relevant when an abstraction borehole significantly influences the entire study area. Furthermore, upon simulation, an optimisation process was used to reduce the root means square error (RMSE) between the CRD water levels and the observed water levels. This was done using a Recharge Estimation Model within Microsoft Excel. In doing so, the findings suggest that the previous equation generates greater recharge estimates than the revised equation, and therefore the revised equation is regarded as more adequate. The final conclusions were that the method is adequate in shallow aquifers; however, uncertainty is still associated with deeper aquifers as well as fractured rock aquifers.

Another study aimed at estimating the net groundwater recharge of an aquifer in an area of Palestine where rainfall is assumed as the only factor contributing to groundwater recharge, applied the CRD (Baalousha, 2005). The rationale behind selecting the method was that recharge has several influential factors that have great uncertainty, and since the CRD equation excludes several of these uncertain parameters, it generated a simple yet adequate approach for recharge estimation. Upon reducing the RMSE, 43 million m^3 recharge was estimated for the study area, and this was in line with previous findings on recharge within the area. This essentially meant that the method is satisfactory, despite the exclusion of several parameters. Based on the findings to this study, the CRD approach is only adequate provided there is no uncertainty in other causes of water level fluctuation, such as evapotranspiration and other factors given in 2.2. In essence, both studies indicate the sensitivity of the CRD recharge estimates with Sy. Therefore if no accurate Sy is known, recharge estimates will be unreliable. Finally, the method is limited in aquifers where there is uncertainty in inflow and outflow (Van Tonder & Xu, 2000).

3.2.2.5. Rainfall Infiltration Breakthrough (RIB) Method

The RIB method is a method developed in 2003, and it appears to be similar to the CRD method, as it too simulates groundwater levels, and estimates both recharge and storativity (Xu & Beekman, 2003). As mentioned earlier, it is difficult to associate a recharge event to a particular rainfall event due to the delay associated with recharge. This is a reason for creating the RIB method, as it to some extent accounts for the delay. First of all, the infiltrated rainwater eventually reaching the water table which we know as recharge, in this case is defined as the rainfall infiltration breakthrough (RIB). When the RIB is positive it means there is an increase in the water table, and when negative, a drop in water table is given. This merely defines a relationship between RIB and groundwater level fluctuation, thus they state that the RIB is what causes water level fluctuation. For this reason the method is a model which has an input of rainfall, output as the RIB, and a transfer function being a dynamic weighting factor. This relationship is given in (9.1), as indicated by Sun, *et al*., (2013).

$$\Delta h_i = \left(\frac{1}{Sy}\right) \cdot (RIB(i)_m^n) \tag{9.1}$$

Where Δh_i is the water level fluctuation; Sy is specific yield; RIB(i) is the cumulative recharge from a rainfall event of m to n. However, for the RIB to be the cause of water level fluctuation, other factors influencing water level fluctuation should be constant or insignificant. Thus the method can also account for other influencing factors as seen in (9.2)

$$\Delta h_i = \left(\frac{1}{Sy}\right) \cdot (RIB(i)_m^n) - \frac{Q_p + Q_{out} + Q_{oth}}{A.Sy} \tag{9.2}$$

Where Q_p, Q_{out} and Q_{oth} are abstraction, outflow and volumes changes in groundwater, respectively. Furthermore, recharge is calculated using (9.3) and (9.4)

$$Re(1) = \Delta h(1) \cdot Sy - \frac{Q_{p1} + Q_{out1} + Q_{oth1}}{A} \tag{9.3}$$

$$T_{Re} = Re(1) + \sum_{i=2}^{n} Re(i) \qquad (i = 2, 3, ...I) \tag{9.4}$$

Where Re(1) is recharge for the first time step; re(i) is recharge at the i^{th} time and this could be daily, monthly or annually; and T_{Re} is the sum of recharge for the entire series.

The developed RIB method was applied in three different case studies and compared it to the CRD method. The first being a closed aquifer system yielded RIB simulations being closer to the observed water levels than the CRD. For a dolomite aquifer, the RIB and CRD estimated 5.71% and 11% recharge, respectively. In addition, the RIB was once again closer to the observed water level than the CRD. Lastly, for a Karoo aquifer, an average of 1.67% was produced for the two methods and the RIB water levels was yet again closer than the CRD water levels to the observed water levels.

Later the RIB method was modified to account even more for the delayed response (*lag time*) in recharge so that it may be linked to specific rainfall event(s) (Sun, *et al*., 2013). The method was enhanced in a way that it accounts for lag time on a daily, monthly and annual scale; and upon application, the findings indicate a shallow unconfined coastal sandy aquifer receiving diffuse recharge has a lag time of less than a day. On the other hand, it was found that a TMG/fractured rock aquifer receiving recharge via preferential flow paths, and whose recharge estimates have been presented earlier, has a lag time of about less than 82 days. This clearly indicates that RIB method is adequate in reflecting the different recharge mechanisms associated with a particular formation. In addition, using Spearman's correlation they showed that monthly estimates of recharge with the RIB method are more adequate than daily estimates, as factors such as evapotranspiration, atmospheric pressure and entrapped air (factors highlighted in the WLF method) show a significant impact on water level fluctuation on a daily scale.

Finally, it is indicated that although the RIB method yields more adequate results than the CRD method; and that (9.2) can however only be used provided the abstraction, outflow and change in storage data is available (Xu & Beekman, 2003). The assumption for RIB having a linear relationship with water level fluctuation only becomes valid provided other wlf causing factors are constant or insignificant; and the model can only be used provided it is known that the water level rise is due rainfall. Furthermore, the RIB method cannot account for spatial variation in certain influential factors, thus considerate thought should be given prior application. Another limitation is the RIB method's reliance on an accurate Sy. This is a limitation because as we know now know S is a difficult parameter to obtain. To add, the method has significant uncertainty in areas of rainfall less than 100mm/a (Sun, *et al*., 2013), indicating a threshold is required for the method to operate adequately. Moreover, it is also not recommended to apply the method to aquifers with high transmissivity and hydraulic conductivity, as it will most likely cause outflow to exceed inflow and result in an inadequate view on the groundwater level rise. Lastly, the method is sensitive with depth to water table, and becomes uncertain in deep formations (Xu & Beekman, 2003; Sun, *et al*., 2013).

3.2.3. Tracer methods

3.2.3.1. Chloride Mass Balance (CMB) Method

The CMB method relies on chloride (Cl^-) which is a natural environmental tracer, for groundwater recharge estimation (Healy, 2010). It is reported to be a substantial tracer due its conservative property in steady state mediums. In other words, upon input of wet and dry deposition of Cl^- into a subsurface; when evaporation occurs, only water is lost, as no mass loss in Cl^- occurs when water is transported through an unsaturated zone or saturated zone (Simmers, 2005). Therefore, the method is based on an assumption of a mass flux of Cl^- occurring between input of Cl^- of wet and dry deposition and Cl^- of groundwater; and the outcome of the flux is a result of the Cl^- concentrations within environment through which the water is transported (Xu & Beekman, 2003; Simmers, 2005). Furthermore, (10.1) and (10.2) are used for recharge estimation using the CMB, whereby the latter takes into account dry deposition (Healy, 2010).

$$R = \frac{P*Clp}{Clgw} \tag{10.1}$$

$$R = \frac{P(Clp+D)}{Clgw} \tag{10.2}$$

Where, R is recharge; P is precipitation amount; Cl_p is Cl^- concentration in rainfall; Cl_{gw} is Cl^- concentration in groundwater; and D is the Cl^- of dry deposition.

Advantages to the CMB are its simple and easy application, and particularly its adequacy for use in regions of minimal groundwater level data and semi-arid conditions (Xu & Beekman, 2003). As a result, several studies have applied the method. This includes a study aimed at estimating recharge for the Canary Islands of Spain which has a total average rainfall of 387mm/y, and therefore recharge amounting 24% – where 1% to 2% was obtained for regions with 200mm/y to 300mm/y rainfall, and 12% for areas with less than 600mm/y rainfall (Nanjaro, *et al*., 2015). On a national scale in South Africa, less than 1.5% to 35.7% was estimated using the CMB along with GIS techniques (DWAF, 2006). On a more local scale, in the Sandveld region of the West Coast, 28% for high lying areas with 500mm rainfall, 5% at the foot of the mountains with 250mm to 300mm rainfall, and 1% in low lying areas with 200mm rainfall were presented (Conrad, *et al*., 2007). Comparing these results, it appears the CMB yields comparable results in areas of similar climatic setting.

It is however argued that (10.2) occasionally leads to underestimated groundwater recharge estimates in certain environments. To expand, the CMB becomes inadequate in cases of unquantified sources of such as karst features extending to limestone aquifers, as it is no longer only diffuse recharge occurring but also point recharge (Somaratne & Smettem, 2013). Thus underestimation of groundwater recharge is suggested for this method. This may be supported when it is agreed that the CMB often yields underestimated recharge estimates, where there are sources of Cl^- that is not accounted for (Krásný & Sharp, 2007). As a result, the the CMB was applied with considering point recharge by accounting for the catchment equivalent depth at which point recharge occurs from drainage (Somaratne & Smettem, 2013). This was done using (10.3)

$$R = \frac{P(Clp+D)}{Clgw} + Q_p \tag{10.3}$$

Where the parameters represent that in (10.2); and Q_p is a depth of the catchment. The major difference between the (10.2) and (10.3) is that the former give recharge in volume and the latter give recharge in depth (Somaratne & Smettem, 2013). The findings confirmed that (10.2) yields lower recharge estimates than (10.3). Moreover, (10.3) appeared to agree more with other recharge estimation methods.

As mentioned, a limitation is that Cl- can originate from different unknown sources or sources difficult to quantify. In this case it can come from infiltration of atmospheric deposition of Cl- and seawater intrusion of Cl- along coastal aquifers. As a result, several studies make use of an indicator showing additional input of Cl- concentration. This could range from using chloride/bromide ratios and chloride/sulphate ratios to reduce uncertainty (Scanlon, *et al*., 2011), or using elevated Total Dissolve Solids as a possible indication of unquantified groundwater sources (DWAF, 2006). Although, these indicate the uncertain sites, it still does not give accurate recharge estimation. So how else can one then account for this uncertainty?

Besides its inadequacy in not accounting for unquantified sources of Cl^-, the list goes on for limitations in the CMB method. This includes that the recharge estimate cannot account for the time taken for recharge to occur, as it is based on residence time (Sun, *et al*., 2013).

Furthermore, it can only be applied to the saturated zone provided there is an assumption of no evapotranspiration losses; upward or downward leakage; and nor irrigation or recycled waste water (Somaratne & Smettem, 2013). Lastly, since Cl^- varies significantly in space, the CMB is deemed as yielding only *site specific* recharge estimates (Xu & Beekman, 2003).

3.2.3.2. Stable Isotopes

Stable isotopes, namely deuterium (^2H) and oxygen-18 (^{18}O) are used in groundwater recharge studies. These two naturally occuring isotopes are used for delineating groundwater recharge sources, as well as the processes which groundwater was exposed to (Simmers, 1987; Bredenkamp, *et al.*, 1995; Singhal & Gupta, 2010). To expand, it allows identification of water that underwent *re-evaporation*; recharge sourced from higher alititudes or areas of high rainfall intensity; recharge coming from surface water bodies; recharge from snowmelt and from areas where temperature is high; recharge from cold climates; and whether or not present day or historical recharge is the the cause of groundwater in an aquifer. These are identified by means of how the conncentrattion ratios of these isotopes vary from a standard reference such as the standard mean ocean water (SMOW). Moreover, the concentrations of isotopes may be given using (11) (Singhal & Gupta, 2010).

$$\delta^{18}‰ = \frac{18O/16O_{standard} - 18O/16O_{standard}}{18O/16O_{standard}} * 1000 \qquad (11)$$

Application of this method is given for an alluvial aquifer, where it was identified that groundwater from an alluvial aquifer underwent evavporation before or during groundwater recharge occurrence (Gomo & Van Tonder, 2012).

However, the use of stable isotopes is merely useful for determining groundwater recharge origin, rather than groundwater recharge estimation. Difficulty is associated with quantifying groundwater recharge relaibly due to different mixing ratios eventually yielding the same isotopic ratio. As a result, the use of ^2H and ^{18}O becomes convenient in qualitative groundwater recharge studies (Xu and Beekman, 2003) and inconvenient in estimating present day recharge (Simmers, 1987).

3.2.3.3. Radioactive Isotopes

In the late 90s researchers made use of tritium (^3H) as a radioactive isotope, whereby the high tritium concentration in rainwater due to thermonuclear bomb testing was used as a technique for determining groundwater recharge. To expand, ^3H generally exists in precipitation at a concentration of 20 tritium units (TU). However, during the nuclear bomb testing in 1963, concentration increased to about 1000TU. As we know, precipitation infiltrations the subsurface, and therefore the ^3H concentration in the subsurface in relation to the ^3H concentration in precipitation can be used for groundwater recharge estimation (Bredenkamp, *et al*., 1995; Singhal & Gupta, 2010).

Different methods exist for recharge estimation with this particular tracer (Singhal & Gupta, 2010). This includes an environmental tritium method which makes use of (12.1)

$$R_p = \frac{100S_m}{P} \quad (12.1)$$

Where R_p is the amount of recharge from precipitation; S_m is the total amount of soil moisture within a soil column from the surface to the depth at which the 1963 increase is seen; and P is the total amount of rainfall since 1963.

Another method is called the Integral method which uses (12.2) for recharge estimation.

$$R_p = \frac{\sum a_x m_x}{\sum A_i P_i} \quad (12.2)$$

Where a_x is the ^3H concentration of the soil at depth x; m_x is the amount of soil moisture at depth x; A_i is the ^3H concentration in precipitation; and P_i is the precipitation for different years since 1952.

Lastly, an Injected Tritium method for estimating groundwater recharge may too be used. The assumption behind this method is that water movement through the unsaturated zone occurs by means of *piston flow* – upon adding freshwater to the subsurface, older water will be pushed downward toward the water table. Therefore when injecting ^3H into the subsurface, and after a rainfall event, the vertical movement of ^3H can be monitored by measuring the concentration of ^3H at different depths. From this, an interpretation can be made regarding groundwater recharge rate using an estimation of the average infiltration flux obtained from (12.3).

$$q_i = \frac{\Delta z}{\Delta t}\theta v \quad (12.3)$$

Where Δz is the depth at which the ^3H peak occurs; Δz is the time between sampling and the highest tritium activity in history; and θv is the volumetric soil water content.

An example is given where groundwater recharge was estimated in the Gangetic Plains (Singhal & Gupta, 2010). This was done by injecting ^3H at a depth of 70m beneath the surface for both a region where a paleochannel exists, as well as for an alluvial plain. Injection was made pre-monsoon time. After the monsoon, the peak in ^3H concentration was observed to be 1.6m and 0.4m deeper for the region of the paleochannel and the alluvial plain, respectively. As a result, the paleochannel had a greater recharge estimate of 17-28.7%, in comparison to the alluvial plain which was estimated to have 6.3-11% recharge.

Apart from adequate recharge estimation, another advantage to the method includes its independence on storativity. Despite this, the isotope has a half-life of 50 years. It is now more than 50 years past the nuclear bomb testing, therefore these two methods are no longer seen as reliable for recharge estimation using those ^3H inputs. As a result, researchers have dwelled away from these two techniques (Bredenkamp, *et al*., 1995).

On the other hand, Singhal & Gupta (2010) present another radioactive isotope for groundwater recharge estimation includes ^{14}C which uses (12.4)

$$Q = \frac{n_e H}{t} \ln\frac{H}{h} \quad (12.4)$$

Where Q is the recharge rate; n_e is the effective porosity of the saturated zone; H is the aquifer's total thickness; h is the aquifer's saturated thickness; and t is the ^{14}C age of water. However, unlike ^3H, ^{14}C is influenced by geochemical processes within the subsurface, making it a method associated with unreliability. Finally, as previously mentioned, groundwater dating methods are expensive and sensitive to areas of unknown porosity.

3.2.4. Modelling groundwater recharge

3.2.4.1. Groundwater Modelling

The purpose of groundwater modelling is to forecast piezometric levels for a groundwater system under stressed conditions; and when proceeding with groundwater modelling, it is generally done under using (13.1) (Xu & Beekman, 2003).

$$\frac{\partial}{\partial_{Xi}}\left(K_{ij}\frac{\partial h}{\partial_{Xj}}\right) + q_s = S_s\frac{\partial h}{\partial t} \tag{13.1}$$

Where K is hydraulic conductivity; i,j are coordinate directions; h is the hydraulic head; S_s is specific storage; x is a space coordinate; t is time; and q_s is fluid source or sink. Groundwater recharge can be estimated when the piezometry, K, S, and inflow and outflow components of the groundwater system are known.

Essentially, the above equation indicates groundwater modelling is associated with many parameters. This means that although rated as a method delivering moderately accurate recharge estimates, it requires a large amount of data and can thus become time consuming, expensive and difficult to calibrate. Additionally, it may also fail to yield reliable results because accuracy depends on the level of discretisation and the correctness in the parameters used. Limitations also include its dependence on accurate transmissivity and its sensitivity to boundary conditions. Lastly, most groundwater models assume recharge occurs uniformly over the study area, but this is not always the case.

3.2.4.2. EARTH Model

One of the most common types of groundwater models used in groundwater recharge is called the EARTH model which is a lumped parameter technique for recharge estimation. It simulates groundwater level fluctuation using climatic, soil moisture and groundwater level data (Bredenkamp, *et al*., 1995; Van Tonder & Xu, 2000; Wu, 2005). It is a model made up of different modules, namely MAXIL, SOMOS, and LINRES which make up the direct recharge method component; and SATFLOW which makes up the indirect component. The MAXIL module determines the effective precipitation, or the amount of water Earth's surface intercepts (Toure, *et al*., 2016) or the excess/infiltrated water (Wang, *et al*., 2008). The SOMOS module divides the effective precipitation into AET, percolation, ponding and runoff. Using a transfer function, the LINRES module reallocates the percolation component of the SOMOS module so that it spreads out over time through the unsaturated zone. Lastly, the SATFLOW module makes use of the aforementioned modules outputs and calculates groundwater level fluctuation. Essentially, the direct methods simulate groundwater recharge in the unsaturated zone, and the SATFLOW module calculates groundwater level from the recharge simulated by the direct methods.

The general equation used for recharge estimation is given in (13.2) (Van Tonder & Xu, 2000).

$$S\frac{dh}{dt} = R - \frac{h}{DR} \tag{13.2}$$

Where S is specific yield; dh/dt is change in water level over a period of time; R is recharge; h is groundwater level; and DR is drainage resistance.

Application of the model is given by in a study whose aim was to evaluate the actual groundwater recharge of a semi arid area in Hebei Plain, China (Wang, *et al*., 2008). A range of 7.2% – 26.5% recharge was estimated and assumed due to variation in rainfal/irrigation infiltration, soil type, depth to water table and differences in the unsaturated zone's lithology. Another application was done in The Netherlands, for a dissertation aimed at assessing the effect of meterological forcing on groundwater recharge and water level fluctuation through unsaturated zone water flow modelling (Gebreyohannes, 2008). The findings to the study revealed a slow repsonse in groundwater level fluctuation and it was thought to be a result of a thick unsaturated zone buffering short term rainfall events, as well as due to the groundwater system's high DR and high Sy. Moroever, the estimated recharge was 345mm/y (39%). In essence, the two aforementioned studies along with another (Toure, *et al*., 2016) agree that the model yields reasonably adequate recharge estimates.

However, because the model requires so many parameters (11 in total), it makes it inadequate for a study area where very little hydrogeological data is available (Bredenkamp, *et al*., 1995; Van Tonder & Xu, 2000; Wu, 2005). In addition, just like previous methods relying on S/Sy, this method becomes unreliable in cases of poor knowledge of the parameter (Xu & Beekman, 2003).

3.2.4.3. Mathematical and Regression modelling

Mathematical modelling and regression modelling also form part of groundwater modelling. Both are explained in the *Manual on Quantitative Estimation of Groundwater Recharge and Aquifer Storativity* (Bredenkamp, *et al*., 1995). Mathematical modelling was used in a study aimed at developing a mathematical model for hydraulic properties of a groundwater system (Singh, *et al*., 2005). In essence, the model was used to simulate groundwater recharge occurring through surface drains, and additionally estimate groundwater recharge. This was done for both detained and free flow schemes. Moreover, as part of the approach used, a rainfall threshold for recharge to occur was identified. To add, it was also found that recharge is sensitive to Sy. Other the other hand, another study makes use of a Regional Regression Recharge (RRR) estimation method for a study aimed at regionalising groundwater recharge estimates of a local scale to estimates of a larger scale for an area in the State of Minnesota (Delin, *et al*., 2007). The results to the study compared to results of other methods but it was acknowledged that this only adds confidence to the findings, because similar findings do not indicate accuracy. Essentially, it is agreed that both mathematical modelling approaches are adequate and significantly useful, but eventually the model output still depends on the accuracy of the model input (Singh, *et al*., 2005; Delin, *et al*., 2007).

CHAPTER 4: THEORETICAL FRAMEWORK

When developing a new model for recharge, it is critical to understand the challenges associated with recharge estimation. A considerable challenge identified in Chapter 2 and Chapter 3 is the difficulty in estimating recharge on a regional scale due to spatial and temporal variation, as well as uncertainty. This can be linked to the concepts of heterogeneity, viscoelasticity and the memory effect. Since these three concepts have been well addressed in other scientific research fields using fractional differentiation, the theory guiding this study is that fractional differentiation will improve the accuracy in investigations based on groundwater recharge estimation. Accordingly, this chapter discusses the importance of considering these concepts and how they can be accounted for in groundwater recharge investigations using fractional differentiation and numerical approximations.

4.1. Heterogeneity and groundwater recharge estimation

Despite conditions above ground favouring groundwater recharge, (i.e. high rainfall, low ET, and land use practices and topography promoting infiltration and high permeable soils) recharge occurrence still ultimately depends on what happens below ground; and for water to enter a groundwater system, there should be a deficit in soil moisture capacity and aquifer properties should be sufficiently favourable for migration of water to the water table (refer to Chapter 2). Furthermore, since one cannot always physically see or measure these aforementioned properties below ground, recharge studies become challenging. To add, recharge studies are further complicated if what we cannot see has significant spatial variation.

Heterogeneity refers to the variation in a physical property, in 2D and/or 3D space (Harter, 1994). As previously indicated, heterogeneity in a groundwater system arises as a result of an aquifer system's structure and composition varying in space – be it between two aquifer systems or within the same aquifer system. Characteristics and layering of lithology control the associated heterogeneity. To expand, structural heterogeneity such as cataclastic faults and compaction bands normally affect high porosity aquifers by decreasing K by 3 – 6 orders of magnitude, whereas joints and open fractures generally cause low porosity aquifers to increase in K by a significant order of magnitude relative to the host formation (Antonellini & Mollema, 2016). As a result, K values often used in recharge models need careful consideration prior selecting a value.

Several groundwater recharge studies have concentrated on signifying the effect of spatial variation on groundwater recharge. It is reported that spatial variation in recharge may be controlled by differences in soil and land cover, and may have an eventual effect on the basin's water budget as well as nutrient and contaminant transport processes (Dripps, *et al.*, 2001). Another study reports recharge values ranging between 8% and 30% average annual rainfall and concluded this range is due to variation in soil and its associated K values (Cherry, 2000). A similar conclusion was given for a study in the Yanqing Basin of Beijing, China (Gong, *et al.*, 2012). Essentially, these studies indicate groundwater recharge estimates are notably affected by heterogeneity, and because of that, how then does one accurately give a recharge estimate for a study larger than site/local scale, when it is necessary for management purposes?

Literature gives a response to this question, in which it is often found that recharge studies based on spatial variation as well as on a scale exceeding a local scale are often associated with remote sensing and GIS based techniques. Recharge was estimated in southern California on a regional scale using a water budget which takes into account boundary conditions, aquifer properties, groundwater level data and groundwater abstraction information (Manghi, *et al.*, 2009). Within the study, GIS techniques were used to interpolate and extrapolate grids for Sy, bedrock elevation and raw groundwater data. Another study made use of both remote sensing and GIS, in which they integrated lithological, land cover/use, and topography and hydrology factors from which the weight of these factors were obtained from aerial photos, geological maps, land use databases, field observation (Yeh, *et al.*, 2008). Similar studies are found elsewhere in literature (Simmers, 1987; DWAF, 2006) but as mentioned, GIS and in addition remote sensing techniques are vulnerable to uncertainty. Consequently, further research is required for improving the consideration of heterogeneity in groundwater recharge.

4.2. Viscoelasticity and groundwater recharge estimation

Fluids are classified into non-Newtonian and Newtonian fluids (Kreith, *et al.*, 1999). Ideal Newtonian fluids have a viscosity that is temperature and pressure dependent, and is associated with a linear shear vs stress graph. These fluids are described by Newton's Law of viscosity which accurately describes gases and liquids comprised of small molecules. On the other hand, there also exist complex molecules which have time and rate dependent properties, and these are the non-Newtonian fluids which are associated with non-linear flow curves (Bird, *et al.*, 2002; Callahan, 2011). These latter fluids are regarded as having time-dependent, viscoelastic, and time-dependent behaviour which define their response to deformation and strain rate, elongation and shearing. For this study, viscoelastic behaviour is of interest to groundwater recharge.

As the name suggests, viscoelasticity is a property of flow both viscosity and elasticity; and is associated with the occurrence of deformation (Sochi, 2009). Since it is elastic, it will return to its original form once the force causing deformation is removed. However, because it is associated with viscosity, it will take some time to achieve the original form (Vincent, 2012). Viscoelastic materials are therefore material having time and rate affecting their response to stress (Corapcioglu, 1976). Moreover, these materials' existence is often at a time scale which does not get reached (Sochi, 2009). Subsequently, it is necessary to understand if a particular material within a subsurface has this property.

As to how this is related to geohydrology, reference is made to a study which dealt with Cretaceous shale which has been affected by the property of viscoelasticity (Bredehoeft, *et al.*, 1983). To expand, the results reported for the study indicate that laboratory and in situ estimated specific storage (Ss) are much smaller than Ss estimated from a flow simulation done for a time period of several decades. The larger estimates were concluded to be a result of the effect of viscoelasticity on the Cretaceous shale, where it was indicated that part of this particular geology's response to change in effective stress is time dependent. This in other words meant that a more significant decrease in hydraulic head is accompanied by compression of the shale along with the release of water. With that being said, it is wondered what an effect of viscoelasticity on a geological formation undergoing recharge would yield as a recharge estimate since recharge is signified by change in hydraulic head.

Furthermore, as indicated in the beginning of this write up, viscoelasticity is a property which can further complicate heterogeneity; and because of this, it has been used in fields related to flow of water such as engineering and geohydrology. Chapter 1 highlights the effects viscoelasticity has on data used to understand the factors influencing recharge, as well as the effect it has on the data used to estimate parameters used within groundwater recharge equations. In addition, specific aquifer properties such as hydraulic conductivity (K) and storativity (S) generally vary as a result of the nature of geology, raising the concern of heterogeneity. Since these two aquifer properties are required in almost all of the aforementioned existing recharge models, viscoelasticity alongside heterogeneity becomes essential to incorporate in groundwater recharge models. Ultimately, it is reasonable to say that there is a lack of research done on the effect of viscoelasticity on groundwater recharge, as there is limited literature available regarding this property being incorporated into groundwater recharge models.

4.3. Fractional differentiation

Fractional differentiation may be used for both the concept of heterogeneity and viscoelasticity. This is because it is appropriate for modelling real world scenarios (Atangana, 2015). Before showing why these derivatives may be used to improve groundwater recharge investigations, it is imperative to understand the essence of each derivative. Literature shows that the common fractional order derivatives of fluids associated with these concepts are namely, the Riemann-Liouville derivative, the Caputo derivative, the Caputo-Fabrizio derivative and the Atangana-Baleanu derivative. The definitions for each of these fractional derivatives are given below:

Definition for Riemann-Liouville fractional derivative:

$$ {}_{\alpha}\mathcal{D}_t^{\alpha} f(t) = \frac{d^n}{d^{tn}} {}_{\alpha}D_t^{-(n-\alpha)} f(t) = \frac{d^n}{d^{tn}} {}_{\alpha}I_t^{n-\alpha} f(t) \qquad (14.1)$$

Definition for Caputo fractional derivative:

Let $b > 0, f \in H^1(0,b)$ and $0 < \alpha < 1$, Caputo fractional derivative of f of α is the following:

$$ {}^{C}\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds \qquad t > 0 \qquad (14.2)$$

When changing the kernel $(t-s)^{-\alpha}$ by the function $exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)$ and $\frac{1}{\Gamma(1-\alpha)}$ by $\frac{1}{\sqrt{2\pi(1-\alpha^2)}}$, the Caputo-Fabrizio derivative of order $0 < \alpha < 1$ is obtained:

$$ \mathcal{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_{\alpha}^{t} f'(x) exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \qquad (14.3)$$

Where $M(\alpha)$ is a normalisation function such that $M(0) = M(1) = 1$. However, if the function does not belong to $H^1(a,b)$, then the derivative is given as:

$$ {}^{CF}\mathcal{D}^{\alpha} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s) ds \qquad t \geq 0 \qquad (14.4)$$

Where $M(\alpha)$ is a normalisation constant depending on α. As indicated, the main difference between Caputo and Caputo-Fabrizio derivatives is that the latter is not associated with singularity for $t = s$.

Atangana-Baleanu fractional derivative in the sense of the Liouville-Caputo is defined as:

$$ {}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{x} f(x) = \frac{B(\alpha)}{1-\alpha} \int_{\alpha}^{x} \dot{f}(X)\, E_{\alpha}\left(-\alpha \frac{(x-X)^{\alpha}}{1-\alpha}\right) d_X, \qquad 0 < \alpha \leq 1 \tag{14.5} $$

and

$$ {}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha} \int_{\alpha}^{t} \dot{f}(\theta)\, E_{\alpha}\left(-\alpha \frac{(t-\theta)^{\alpha}}{1-\alpha}\right) d\theta, \qquad 0 < \alpha \leq 1 \tag{14.6} $$

Atangana-Baleanu fractional derivative in the sense of the Riemann-Liouville derivative:

$$ {}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{x} f(x) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dx} \int_{\alpha}^{x} f(X)\, E_{\alpha}\left(-\alpha \frac{(x-X)^{\alpha}}{1-\alpha}\right) d_X, \qquad 0 < \alpha \leq 1 \tag{14.7} $$

$$ {}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dx} \int_{\alpha}^{t} f(\theta)\, E_{\alpha}\left(-\alpha \frac{(t-\theta)^{\alpha}}{1-\alpha}\right) d\theta, \qquad 0 < \alpha \leq 1 \tag{14.8} $$

The Atangana-Beleanu fractional integral of order α of a function $f(t)$ is given as:

$$ {}^{AB}_{\ a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{0}^{t} f(s)\, (t-s)^{\alpha-1} ds \tag{14.9} $$

The Mittag-Leffler function $E_{\alpha,\beta}$ is defined as:

$$ E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m+\beta)'} \qquad (\alpha > 0), \qquad (\beta > 0), \tag{14.10} $$

Due to the differences in definitions, there is a corresponding variation in their respective properties and application. The Riemann-Liouville derivative of a constant term does not equal zero, and when it is associated with the Laplace transform (refer to Chapter 5 for Laplace transform properties), it is argued that it yields insignificant physical terms or initial conditions (Ali, *et al.*, 2016; Gómez-Aguilar, *et al.*, 2016). This argument is however not mathematically supported because the derivative is obtained through the classical differentiation and Cauchy formulation of the multiple integrals. This means that when applying the Laplace transform to the Riemann-Liouville derivative, a formula with initial condition raised to the power becomes acceptable and appropriate when working in the framework of fractional differentiation.

Due to the first argument being that the Riemann-Liouville derivative yields insignificant initial conditions, the Caputo derivative becomes more of use because it is not associated with the aforementioned limitation (Ali, *et al*., 2016; Gómez-Aguilar, *et al*., 2016). However, it is imperative to understand that the Caputo derivative is an adaptive derivative. This means it has no fractional integral and was derived artificially, not following clear mathematical derivation. Nevertheless, the Caputo derivative accounts for the entire history of a function by means of the memory effect (Dzieliński, *et al*., 2011). Accordingly, the derivative was used for numerically simulating anomalous diffusion (e.g. diffusion through porous materials) which has an issue of complexity and heterogeneity that could not be addressed using linear nor non-linear differential equations as they fail to completely simulate real behaviour of anomalous diffusion (Ciesielski & Leszczynski, 2003). In applying this derivative, the diffusion equation's spatio-temporal derivatives were replaced by fractional derivatives. Hence, more complexity could be accounted for. Although used adequately in this real world scenario, the Caputo derivative is of fractional order with singular function, meaning it does not accurately describe the full effect of memory (Atangana & Alkahtani, 2015; Ali, *et al*., 2016). This is similar for the Riemann–Liouville derivative (Gómez-Aguilar, *et al*., 2016).

As a result of the aforementioned downfalls to the two respective derivatives, the Caputo-Fabrizio derivative was developed. This derivative addresses heterogeneity at different scales and avoids singularity, meaning it enhances the memory effect (Atangana & Alkahtani, 2015; Atangana & Alqahtani, 2016). In addition, it is appropriate for use with the Laplace transform (Ali, *et al*., 2016; Atangana & Alqahtani, 2016). It is a derivative based on the convolution of a first order derivative and the exponential function, and is well associated with conventional viscoelastic materials, thermal media, and electromagnetic systems (Atangana & Alkahtani, 2015). Subsequently, the derivative was used within an epidemiology study, in which the ordinary time derivative was replaced by the fractional derivative (Atangana & Alkahtani, 2015). The numerical simulations for the study revealed how different values of alpha depict differences in the solution. Ultimately, this concluded that the solutions to their equation have a great dependence on fractional order.

Another study gives application to the typical advection dispersion equation (Atangana & Alqahtani, 2016). This involved replacing the ordinary time derivative, which further allowed replication of particles flowing through a geological formation. The study entailed numerical simulations using the Crank-Nicolson approximation (definitions and properties given later within this chapter). The findings show that different alpha values signify fractional differentiation has the ability to control variation in the plume's migration within the geological formation. To add, the plume's migration not only occurred within the homogeneous but also heterogeneous formations, and this ultimately means that more than just the typical advection dispersion equation is needed to model a real world scenario like the aforementioned. Essentially, these studies indicate how ordinary equations describing groundwater flow and transport can be used reliably when associated with a derivative of fractional order without a singular function. Furthermore, the Atangana-Baleanu fractional derivative which as mentioned is based on the Mittag-Leffler function given in (14.10), is associated with a non-singular and non-local function, allowing the behaviour of groundwater flow in lithological material showing viscoelastic effects to be addressed (Alkahtani & Atangana, 2016; Ali, *et al*., 2016). To add, it is a derivative appropriate for use with the Laplace transform (Ali, *et al*., 2016).

The Atangana-Baleanu fractional derivative was used for a study on a simple non-linear system, in which the findings to the derivative were not that obtained from a local derivative (Atangana & Koca, 2016). This means the particular derivative yields results of greater understanding. Another study gave application to a model of groundwater migration through an unconfined aquifer (Alqahtani, 2016). Herein the model was solved analytically using the Laplace transform and numerically by means of the Crank-Nicolson approximation. The study concluded that this newly established model will generate a new understanding of groundwater flow in an unconfined aquifer. Furthermore, another study shows how the Atangana derivative was combined with a concept of variable order derivative to formulate a new approach to modelling real world scenarios – this is known as the Eton approach (Alkahtani & Atangana, 2016). For that study, the Eton approach was applied to the Poisson equation which is used to describe the potential energy field caused by a given charge or mass density distribution. The findings to the numerical analysis for the study revealed that this approach is helpful in describing real world scenarios with memory.

Since fractional differentiation has successfully been used in several fields of science, including geohydrology; the theory guiding this study is that fractional differentiation can be used to improve the estimation of groundwater recharge on the basis of accounting for heterogeneity, viscoelasticity and the memory effect which has significant effect on data used within recharge estimation techniques. Subsequently, this study is governed by objectives applying the Riemann-Liouville, Caputo, Caputo-Fabrizio, and Atangana-Baleanu derivatives.

4.4. Numerical simulation and groundwater recharge

Based on the above section, it is seen that after applying fractional differentiation derivatives to formulate an exact solution, researchers usually adopt an approach of numerical simulation. In order to do numerical simulations adequately for a real world scenario, an appropriate numerical approximation should be selected to obtain the approximation which will be simulated. There are however different numerical approximations which may be used to proceed with numerical analysis. Since the theory guiding this study is that fractional order derivative will generate new knowledge and understanding to groundwater recharge, it becomes imperative to provide an overview of the different numerical approximation schemes because selecting an appropriate scheme will give adequate simulations to an exact solution. These are schemes are namely, explicit, implicit and Crank-Nicolson schemes.

The explicit scheme is an unstable scheme, having error that would grow exponentially. This means that after a few time steps the numerical solution will not have significance to the true solution, and so the time step becomes a restriction. Alternatively, the implicit scheme is unconditionally stable, and so any grid dimension can be selected for simulation. This means there is no restriction to the time steps. The implicit scheme however requires simultaneous solutions of a set of linear equations; and as a result, computational time increases. On the other hand, the Crank-Nicolson solution is based on a scheme that equates to a central difference approximation. This means that the spatial derivatives are solved in the centre of two time periods. The Crank-Nicolson scheme is very much similar to the implicit scheme, and so it is associated with the same disadvantages. In addition, the Crank-Nicolson scheme gives a better approximation to the exact solution for small Δt, as well as converges faster than both explicit and implicit schemes (Patankar, 1980; LeVeque, 2005). Ultimately, it is seen that the Crank-Nicolson scheme appears to be a highly appropriate scheme for numerical simulation.

As mentioned, the Crank Nicolson scheme is regarded as most adequate. This is seen in previous studies which adequately and successfully applied the Crank Nicolson scheme. It has been used where partial differentiation equations describing flow and transport of dissolved organic compounds in a 2D domain (Khebchareon, 2012). Moreover, it was used along with the fully implicit and Runge-Kutta schemes for modelling pollutant migration associated with infiltration and groundwater recharge (Shahraiyni & Ataie-Ashtiani, 2012). Another study also concentrating on infiltration used the Crank-Nicolson scheme to obtain numerical simulation results of infiltrated water for various time periods (Borana, *et al*., 2013). Since these studies are all related to groundwater recharge, it can be said that the Crank-Nicolson scheme appears to be an appropriate scheme for numerical analysis of exact solutions to groundwater recharge equations. Accordingly, definitions for the Crank-Nicolson scheme is given below:

Definition for the Crank-Nicolson scheme:

$$\frac{\partial h}{\partial t} \rightarrow \frac{h_i^{j+1} - h_i^j}{\Delta t} \tag{15.1}$$

Definition for the Crank-Nicolson scheme, for a 2nd order derivative:

$$\frac{\partial^2 h}{\partial x^2} \rightarrow \frac{1}{2(\Delta x^2)}\left(\left(h_{i+1}^{j+1} - 2h_i^{j+1} + h_{i-1}^{j+1}\right) + \left(h_{i+1}^{j} - 2h_{i+1}^{j} + h_{i-1}^{j}\right)\right) \tag{15.2}$$

Definition for the Crank-Nicolson scheme, for a 1st order derivative:

$$\frac{\partial h}{\partial x} \rightarrow \frac{1}{2}\left(\frac{\left(h_{i+1}^{j+1} - h_{i-1}^{j+1}\right)}{2(\Delta x)} + \frac{\left(h_{i+1}^{j} - h_{i-1}^{j}\right)}{2(\Delta x)}\right) \tag{15.3}$$

Definition for the Crank-Nicolson scheme, at a particular time:

$$h \rightarrow \frac{1}{2}\left(h_i^{j+1} + h_i^j\right) \tag{15.4}$$

Definition for the Crank-Nicolson scheme, at the previous channel:

$$h_N \rightarrow \frac{1}{2}\left(h_{Ni}^{j+1} + h_{Ni}^j\right) \tag{15.5}$$

Definition for the Crank-Nicolson scheme, at the next channel:

$$h_M \rightarrow \frac{1}{2}\left(h_{Mi}^{j+1} + h_{Mi}^j\right) \tag{15.6}$$

CHAPTER 5: UNCERTAINTY ANALYSIS

As seen in Chapter 3, several methods exist for groundwater recharge estimation. Moreover, the degree of accuracy in the recharge estimate from these methods depends largely on the reliability of the parameter values used within a particular recharge estimate. To provide more confidence in recharge estimates, researchers often use multiple/complimentary methods for recharge investigations (Scanlon, *et al*., 2006; Gomo & Van Tonder, 2012). However, this is time consuming and more costly. As a result, several researchers rely on uncertainty and sensitivity analysis to gain more confidence in recharge estimation investigations. The following section provides an understanding of the importance of uncertainty analysis, as well as insight to the Monte Carlo Sampling and Latin Hypercube Sampling methods for uncertainty analysis.

Uncertainty analysis is a technique used to gain more confidence in a model's output, and this is achieved by gaining more confidence in the input parameters. Uncertainty in a model's output may be due to natural uncertainty caused by intrinsic variation in the real system; model uncertainty due to fault in the mathematic equations governing the model; and/or parameter uncertainty caused by inaccuracy in the input data required for the particular model (Baalousha, 2007). Moreover, uncertainty influences the model's calibration and so the model's output eventually becomes doubtful. To expand, an uncertainty analysis was conducted on recharge from a river and an artificial recharge scheme as part of their broader study (Hashemi, *et al*., 2013). The findings indicate the former had more uncertainty than the latter; and that their model's output was more sensitive to the former. As a result, it was concluded that for reducing uncertainty in the estimated recharge, groundwater migration through the unsaturated zone should be further studied for both the river and artificial recharge schemes. In light of this, it becomes necessary to understand the factors or parameters associated with significant uncertainty, as this will reduce uncertainty in a model output (in this case, recharge estimate).

5.1. MONTE CARLO SAMPLING (MCS)

The MCS method is one of the famous uncertainty analysis methods. It is adequate for solving problems for a variety of methods which include *analytical, numerical, experimental and empirical frequency methods* (Zhen-min, 2011). An advantage to the method is that it is efficient in both time and cost, as it accounts for unavailable or insufficient field data (Zhen-min, 2011; Baalousha, 2016). Furthermore, MCS is used to solve two problems: the first is when a probability distribution is given from which samples are generated, and the second is to estimate probable outcomes of a function under a given distribution (Mackay, 1998).

The MCS method is a random sampling statistical method based on probability distribution (Baalousha, 2016). During model simulation, *fake* stochastic variables are generated and distributed into different portions according to the requirement. After evaluating the density distribution, the final results are created. Furthermore, the MCS method involves running a model a numerous amount of times for a parameter of interest g(x), where $X = (x_1, x_2, ...x_m)$ becomes a vector of *m* random variables (Baalousha, 2009). Moreover, n samples are randomly drawn from corresponding density probability functions f(x). Finally, the mean (μMC) and variance (σ^2MC) are determined based on the equation 16.1 and 16.2, respectively.

$$E[f(X)]MC = \mu MC = \frac{1}{n}\sum_{i=1}^{n} g(xi) \tag{16.1}$$

$$var[g(x)]MC = \sigma^2 MC = \frac{1}{n}\sum_{i=1}^{n}(g(x) - \mu h)^2 \tag{16.2}$$

The MCS method is applicable for different types of groundwater recharge investigations. However, too few researchers try to estimate the contribution of both diffuse and *concentrated* (focused) recharge. As a result, a study was conducted and aimed at estimating both diffuse and focused recharge using a water balance approach, and since the study focused on both diffuse and focused recharge, the water balance was divided into different compartments (*the ditch channel, the ditch banks and bed, the unsaturated inter-ditch zone, and the groundwater zone*) (Dages, *et al*., 2009). However, because water balance methods are vulnerable to uncertainty in input parameters, the MCS method was used to analyse the extent of uncertainty in the recharge estimates obtained. Two simulations were performed, in which one was to evaluate the uncertainty in the diffuse and focused recharge estimates to the uncertainties in each parameter of the water balance; and the second was to determine which parameters of the water balances led to the greatest uncertainty in final output. Thereafter, the distributions obtained were plotted on histograms and a conclusion was made that diffuse recharge was most sensitive to overland flow and change in storage in the unsaturated inter-ditch zone. To add, it was found that focused recharge was more sensitive to change in volume in the saturated zone, overland flow, and change in volume in the unsaturated-inter-ditch zone.

Furthermore, the MCS also appears to be adequate for application with Darcy's Law. To expand, another study applied a water balance application for recharge estimation using (16.3) (Baalousha, 2016). Here both groundwater recharge and groundwater discharge to sea were unknown parameters. Due to inadequate data availability and characterisation of the geohydrology, the MCS method alongside Darcy's Law was applied to determine the discharge to the sea (Fs). MCS was used to create random samples for transmissivity (T) and hydraulic gradient (i) which are parameters of Darcy's Law. Normal distribution was assumed and 15 million realisations were created for T and i. Upon simulation, a mean of 139.75m^2/d and 0.00333 was obtained for T and i, respectively. To add, these were obtained using (16.4) and (16.5), respectively. These values were then used with Darcy's Law (16.6) to obtain Fs. Finally, using 14.3, a recharge estimate of 58.73Mm^3/y was obtained.

$$R = P + F_s - F_L - I_R \tag{16.3}$$

Where, R is recharge; P is abstraction; Fs is discharge to sea; F_L is lateral flow; and I_R is irrigation return flow.

$$E(F_s) = \mu MC = \frac{1}{n}\sum_{i=1}^{n} L\left(T\frac{dh}{dn}\right)i \tag{16.4}$$

$$\sigma^2 MC = \frac{1}{n}\sum_{i=1}^{n} L\ \left(T\frac{dh}{dn} - \mu MC\right)^2 \tag{16.5}$$

Where, μMC and σ^2MC are the mean and variance of the MCS, respectively.

$$F_s = LT\frac{dh}{dn} \tag{16.6}$$

Where, L is length; T is transmissivity; and dh/dn is the hydraulic gradient.

As mentioned in Chapter 1 and Chapter 2, remote sensing and GIS techniques are often relied on for data sources in groundwater recharge investigations when looking at spatial variation in groundwater recharge. However, it is argued that digital spatial data used as a data source is generally associated with a high degree of uncertainty, and this can be due to inadequate compilation of the data. As a result, the MCS method for reducing uncertainty, in essence shows how the MCS method may be used for determining distributed input errors for slope and aspect parameters when using digital data (Bogena, *et al*., 2005).

With the aforesaid, it is seen that the MCS method can be applied for various groundwater investigations, different types of environments, and for different data sources. However, the MCS method only produces reasonable estimates for the distribution of Y, provided the value of n is large (Wyss & Jorgensen, 1998; Baalousha, 2009). This is required to avoid clustering which occurs when a small n is used. Unfortunately, for a large n, there would have to be a significant amount of computations, and this is generally associated with a significant computation expense. Therfore the MCS becomes inadequate for models using a significant number of variables (Baalousha, 2009). To add, it is also suggested that the convergence rate of MCS is slow. Consequently, another sampling method should be made reference to.

5.2. LATIN HYPERCUBE SAMPLING (LHS)

The LHS is a homogeneous stratified method which is developed from the MCS method. Essentially, the MCS is entirely random sampling and the LHS is stratfied. Nonetheless, it is further indicated that the LHS method works by selecting different values from each of k variables x_1, x_2, ... x_k. Thereafter, the range of variables is subdivided into n nonoverlapping intervals, and this is based on equal probability. A value of each interval is then randomly selected but in accordance to its density probability within that particular interval. The n values are drawn from the x_1 variable and randomly paired with n values from the x_2 variable. This pairing produces n pairs which are randomonly paired with n values of the x_3 variable. This produces n-triplets. This pairing process is done until k-tuplets are formed; and these k-tuplets are the LHS. To add, when pairing in this manner, no repetition occurs; and so the sampling output becomes representative of the entire sample space (Wyss & Jorgensen, 1998). Ultimately, the LHS method appears more efficient and requires less time than the MCS method (Wyss & Jorgensen, 1998; Atangana & Van Tonder, 2014).

Although the LHS appear more precise, a proposal was made to use the MCS alongside the LHS, as this yields an even greater adequacy in the output (Atangana & Van Tonder, 2014). With this approach it was indicated that with the assumption of the uncertain parameter being β and it ranges within *a* and *b*, then it would first be required to do sampling using the MCS within a and b. Thereafter, the number of sampling would be condensed by determining the mean, variance and standard deviation of the sample created. Using these determined statistics, the distribution function can be obtained; and with this, the LHS can be applied to create the final samples. Furthermore, with the insufficient knowledge on the impact of hydraulic fracturing on groundwater contamination in the Karoo, Atangana & Van Tonder (2014) looked at the relation between fracture aperture and the rate of groundwater discharge. For this relation they proposed the Parallel Plate Model for approximation of upward flow along inadequate cement structure. However, because aperture is difficult to measure they conducted an uncertainty analysis, using the aforementioned approach. Ultimately, their findings indicate fracking can only be done provided the entire fractured area is plugged with cement, or pollution may occur. In light of the above, this uncertainty analysis approach proves to be an essential tool not only for research but also water resource management. In terms of recharge being an imperative parameter to estimate accurately, and also because all recharge methods are associated with uncertainty, there is a need for investigating the uncertainty of the parameters influencing groundwater recharge before conducting a groundwater recharge investigation. Accordingly, the following sections entail both uncertainty and statistical analysis for a selected groundwater recharge equation.

Additionally, the following sections forms part of: **Spannenberg, J; & Atangana, A., 2016. New Approach to Groundwater Recharge on a Regional Scale: Uncertainties Analysis and Application of Fractional Differentiation. *Arabian Journal of Geosciences*.**

5.3. ANALYTICAL SOLUTION OF THE LAPLACE TRANSFORM

In this work, the following groundwater recharge equation taken from the EARTH Model is considered:

$$S\frac{dh}{dt} = R - \frac{h}{DR} \tag{17.1}$$

With initial condition $h(0)$ which can be a constant or a fraction of space. To solve the above equation, the Laplace transform is applied on both sides of the equation to obtain:

$$S\left[P\bar{h}(P) - h(0)\right] = \frac{R}{P} - \frac{\bar{h}(P)}{DR} \tag{17.2}$$

Rearranging, the following is obtained:

$$\bar{h}(P) = \frac{R}{P(PS+\frac{1}{DR})} + \frac{Sh(0)}{PS+\frac{1}{DR}} = \frac{R}{P(PS+\frac{1}{DR})} + \frac{h(0)}{P+\frac{1}{DR}} \tag{17.3}$$

The exact solution of the recharge equation using the Laplace transform operation.

To accommodate for the reader who is not familiar with this operation, the definition of the Laplace transform is given.

$$\mathcal{L}(f(t)(P) = \int_0^\infty e^{-P,t} f(t)dt \tag{17.4}$$

The following are some properties of the Laplace transformation

$$\mathcal{L}\big(af(t) + bh(t)\big) = a\mathcal{L}\big(f(t)\big)(P) + b\mathcal{L}\big(h(t)\big)(P) \tag{17.5}$$

$$\mathcal{L}\left(\frac{dh}{dt}\right)(P) = \mathcal{L}\big(f(t)\big)(P)P - f(0) \tag{17.6}$$

The definition of the inverse Laplace transform is given as:

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t) \tag{17.7}$$

Now applying the inverse Laplace transformation, the following is obtained:

$$\left[DR - \exp\left[-\frac{1}{DRS}t\right]DR\right]R + h(0)\exp\left[-\frac{1}{DR}t\right] \tag{17.8}$$

Therefore, the exact solution of the recharge equation using the Laplace transform is:

$$h(t) = R.DR + \exp\left[-\frac{1}{SDR}t\right](h(0) - R.DR) \tag{17.9}$$

For an understanding of the extent of the effect of uncertainty of both S and DR on R in (17.9), it is necessary to understand the parameters S and DR. Therefore, the following section provides a brief overview of S and DR.

5.4. PARAMETER UNCERTAINTY ANALYSIS

Storativity (S)

As mentioned in Chapter 2, S is a dimensionless property known as the volume of water an aquifer system will store or release from storage per unit surface area per unit change in hydraulic head. As indicated, the mathematical equations used to obtain this parameter differs for unconfined and confined aquifer systems, whereby for unconfined aquifer systems it is specific yield (Sy) and for confined aquifer systems it is storativity (S). This is due to the physical mechanism controlling releases and storage of water (Kresic, 2007). Nonetheless, in relation to groundwater recharge, S/Sy controls the amount of water eventually reaching the water table (DWAF, 2006).

Drainage resistance (DR)

DR is a lumped, site specific parameter (Nyende, *et al*., 2013; Toure, *et al*., 2016), and is given by (18.1).

$$DR = L^2/\beta T \qquad (18.1)$$

Where DR is drainage resistance (days) L is length of the flow path (m), β = 2 for radial flow or β = 4 for parallel flow, and T is transmissivity (m^2/d)

T is given as the rate at which water is transmitted through a unit width of an aquifer system. Therefore it is given by (18.2).

$$T = KB \qquad (18.2)$$

Where, B is aquifer thickness (m).

In light of the above mentioned, it is fundamental to make correct assumptions regarding the aquifer type and the geological nature of the system of interest, when estimating groundwater recharge. This is due to T, K, and S differing for different geology and aquifer type.

Using the exact solution given in (17.8) and applying a selected range of values for both S and DR, the correspondent hydraulic head change over time is given. This is depicted in **Figure 2** and **Figure 3**. The former depicts hydraulic head change over time for S ranging from 0.0021 to 0.35, with a constant DR of 100 days and recharge of 40%. On the other hand, the latter depicts hydraulic head change over time for DR ranging from 10 days to 500 days, with a constant S of 0.02 and recharge of 40%.

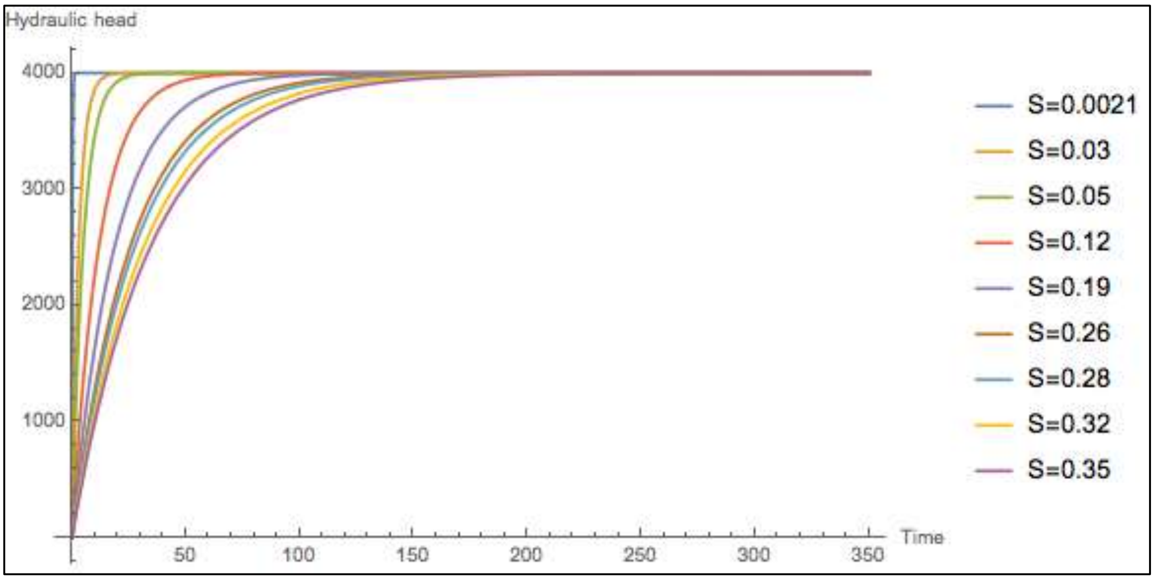

Figure 2: Selected range of S values and their associated hydraulic head over time

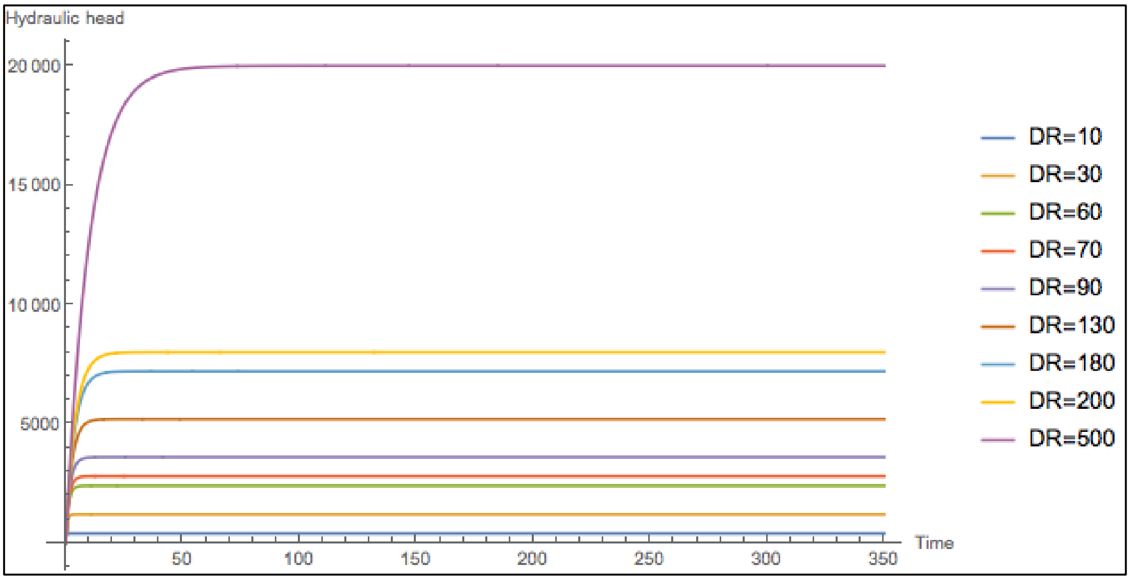

Figure 3: Selected range of DR values and their associated hydraulic head over time

Water level fluctuation signifies groundwater recharge or groundwater discharge. In the scenarios presented above, water level fluctuations are depicted as a basis of variation in S and DR with a constant recharge of 40%. The results depict that minor changes in both S and DR have a significant effect on the change in hydraulic head. To expand, with an increasing S and constant DR, the hydraulic head undergoes a more gradual increase over time. In contrast, with increasing DR and constant S, the hydraulic head has a more rapid and peaked rise. This is noteworthy when DR changes from a value of 200 days to 500 days.

Furthermore, when recharge occurs, slow water migration into the subsurface causes a rapid rise into the capillary fringe, allowing the soil pores to become saturated. Once fully saturated, the water table reaches ground surface and there is an occurrence of surface runoff. This entire process occurs provided there is no exploitation of the aquifer system (Liu & Zhang, 1993). This concept can be related to the change in hydraulic head over time reaching a constant, regardless of the change in S or change in DR. To expand, at about 120 days to 130 days, regardless of the value of S, the hydraulic head reaches a point where no change occurs. This is assumed due to the aquifer being recharged by the hypothetical 40% while no water loss (abstraction and/or evapotranspiration) occurs. In other words, it could mean that the aquifer has reached its capacity to store water. To add, the value in days where all DR values yield a constant hydraulic head is significantly lower (about 30 days) for the selected DR ranges.

In light of the aforementioned, it become imperative to understand the error associated with these parameters because it is these errors which indicate the effect of a value of either S or DR, or both, on the hydraulic head which is further related to groundwater recharge. The following section presents a statistical analysis of the results presented above.

5.5. STATISTICAL ANALYSIS

The following equations are taken out of *Applied Statistics* (Hillenmeyer, 2005)

5.5.1. Harmonic mean

The mean is a typical value found within a data set over time series, and can often be seen as the operating point of a physical system generating the series of data. Furthermore, the harmonic mean is a kind of mean used when the numbers are defined in relation to some unit; when a sample contains extreme values; and when more stability is needed regarding outliers. It is given using (19.1).

$$H_X = \frac{n}{\sum_{i=1}^{n} \frac{1}{xi}} \qquad (19.1)$$

Where H_X is harmonic mean; n is number of data points; x_i is the i^{th} sample point

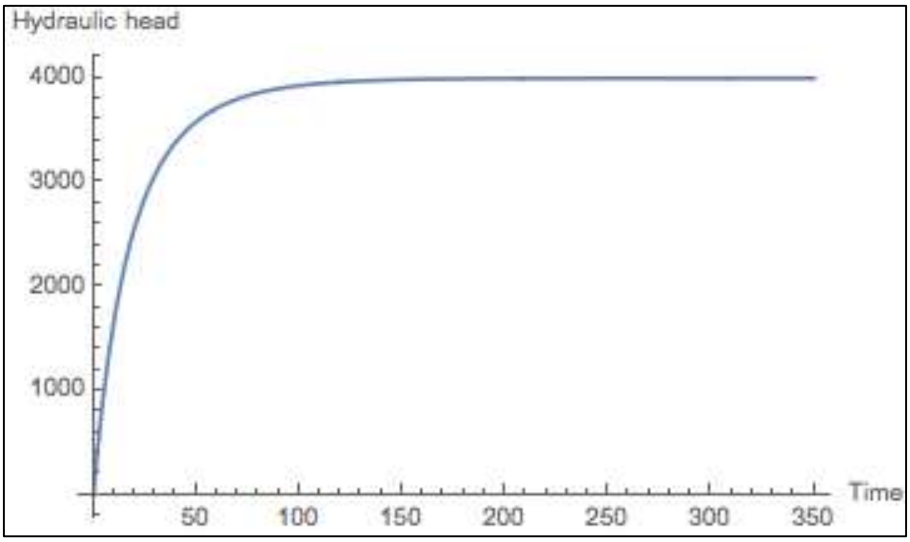

Figure 4: Harmonic Mean for S distribution

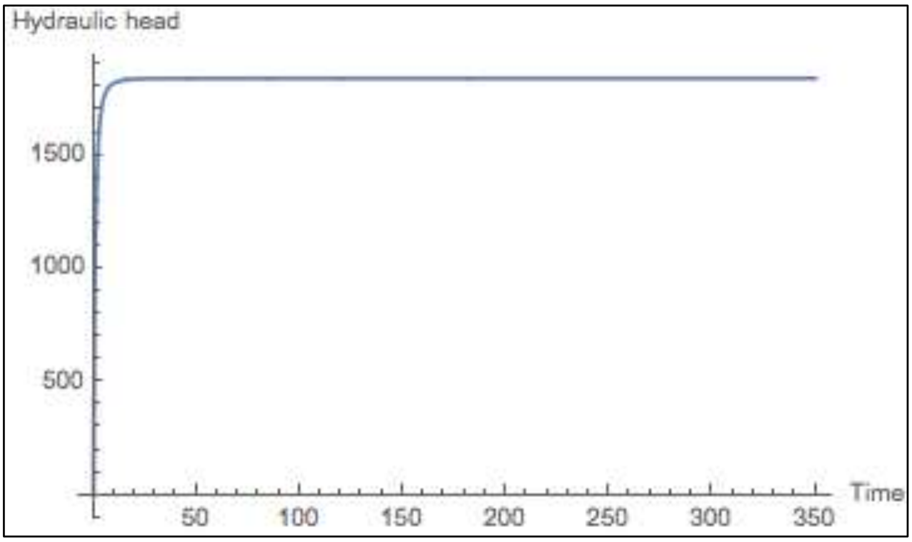

Figure 5: Harmonic Mean for DR distribution

5.5.2. Standard deviation (SD)

SD is a measure of dispersion round the mean value, for a generated time series of data. In other words, it gives insight to uncertainty and error based on how concentrated or scattered the samples are from the mean value. SD is given by the square root of the variance using (19.2).

$$s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}} \qquad (19.2)$$

Where s is the standard deviation; X_i is a data point; and $\bar{X}$ is the mean

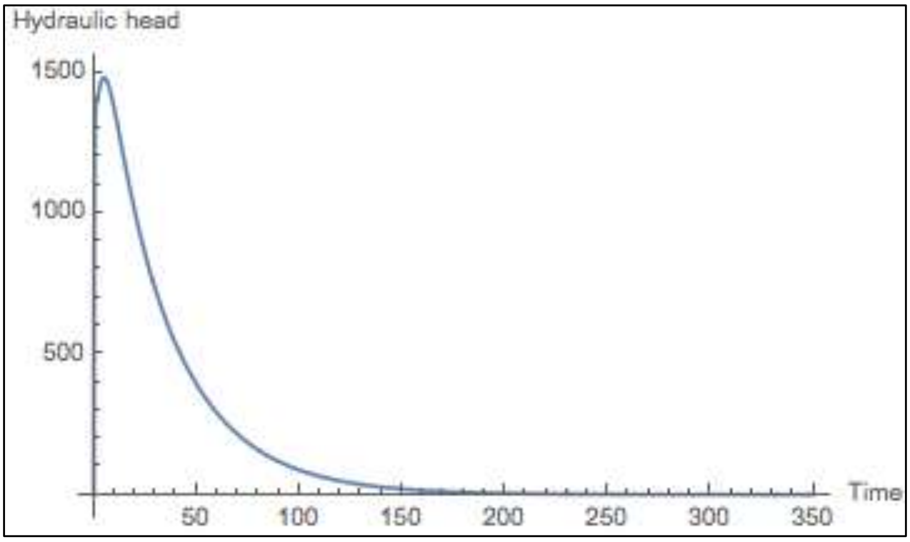

Figure 6: Standard Deviation for S distribution

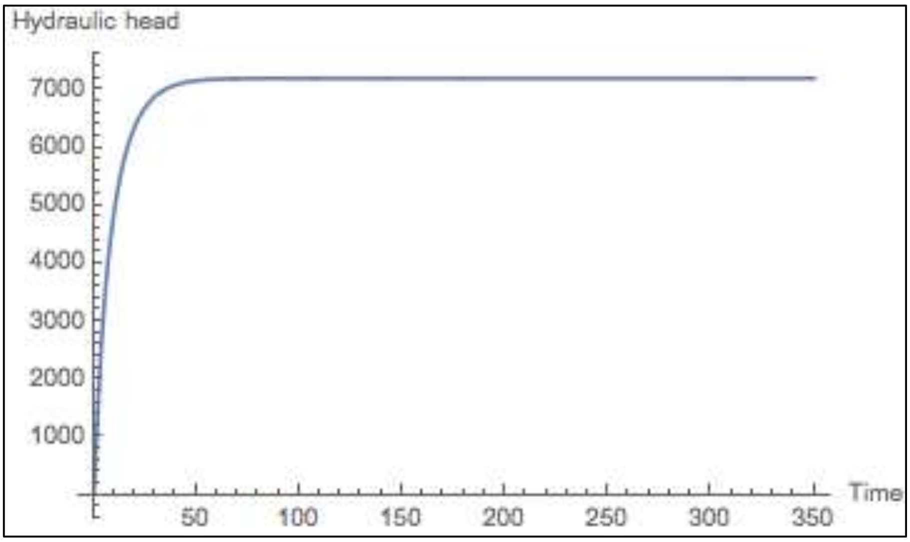

Figure 7: Standard Deviation for DR distribution

5.5.3. Skewness

Skewness is higher-order statistical attribute of a time series. It is essentially used for measure of symmetry of the probability density function of the amplitude of a time series and the assessment of the departure from normality of the data. To expand, when the number of large and small amplitude values is equal within a time series of data, there is a value of zero skewness. Subsequently, a skewed distribution (positive or negative) has a mean and median that is not identical. Furthermore, skewness can be quantified using (19.3) to define the extent to which the times series distribution differs from a normal distribution.

$$g_1 = \frac{1}{ns^3} \sum_{i=1}^{n} (Y_i - \bar{Y})^3 \tag{19.3}$$

Where g_1 is the skewness; Y_i a data point; and $\bar{Y}$ is the mean.

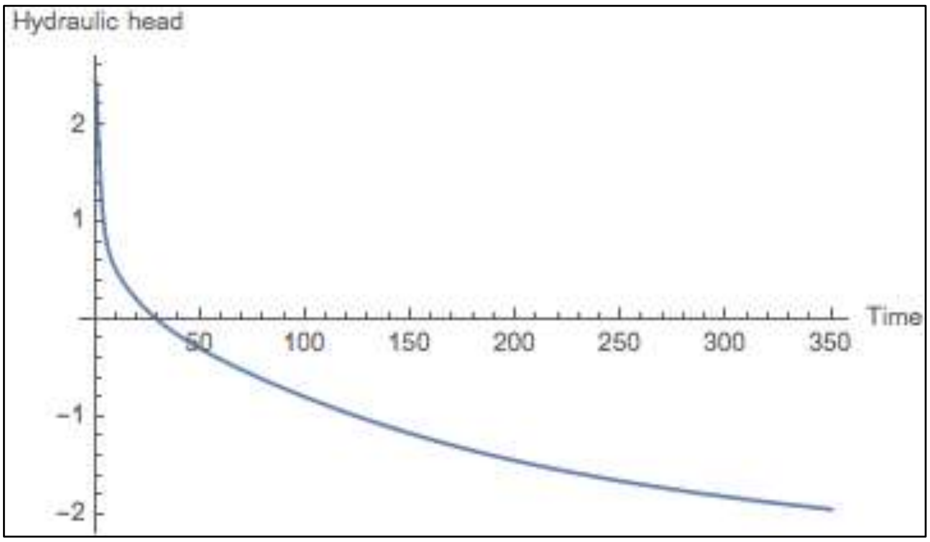

Figure 8: Skewness for S distribution

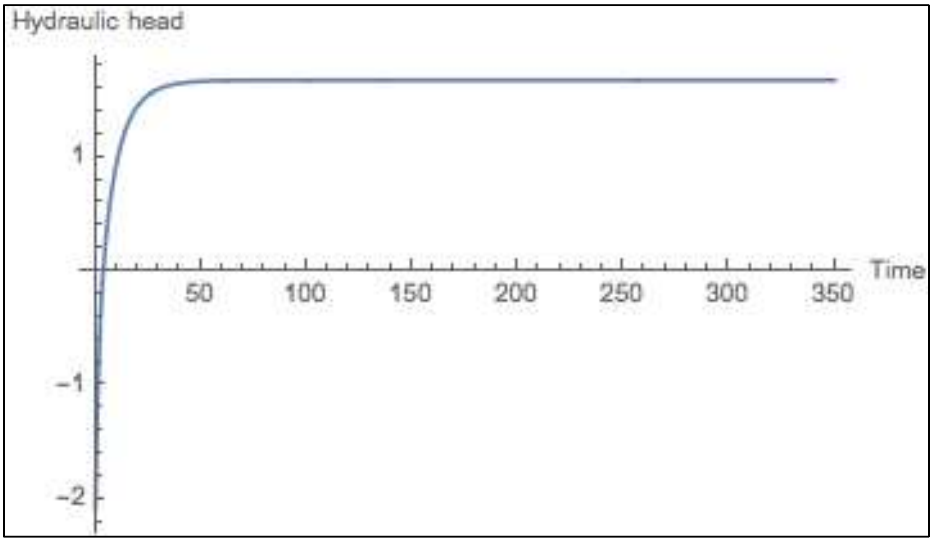

Figure 9: Skewness for DR distribution

5.5.4. Kurtosis

Kurtosis is measures the peakedness of a distribution, or in other words how 'heavy-tailed' or 'light-tailed' the data is relative to a normal distribution. To expand, when a data set has a high kurtosis, it is associated with heavy tails, or outliers. Alternatively, when the measure of kurtosis is low, it is associated with a lack of outliers.

$$g_1 = \frac{1}{ns^4}\sum_{i=1}^{n}(Y_i - \bar{Y})^4 \tag{19.4}$$

Where g_2 is the kurtosis

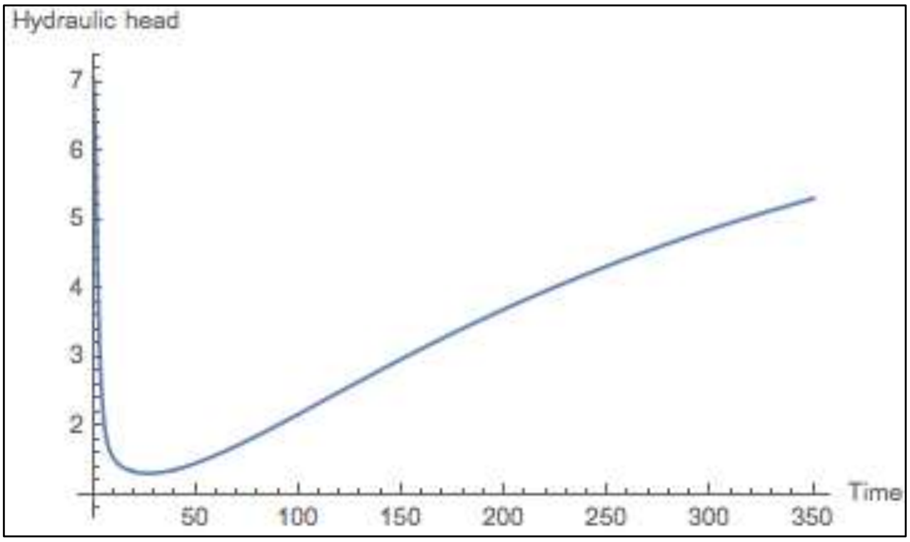

Figure 10: Kurtosis for S distribution

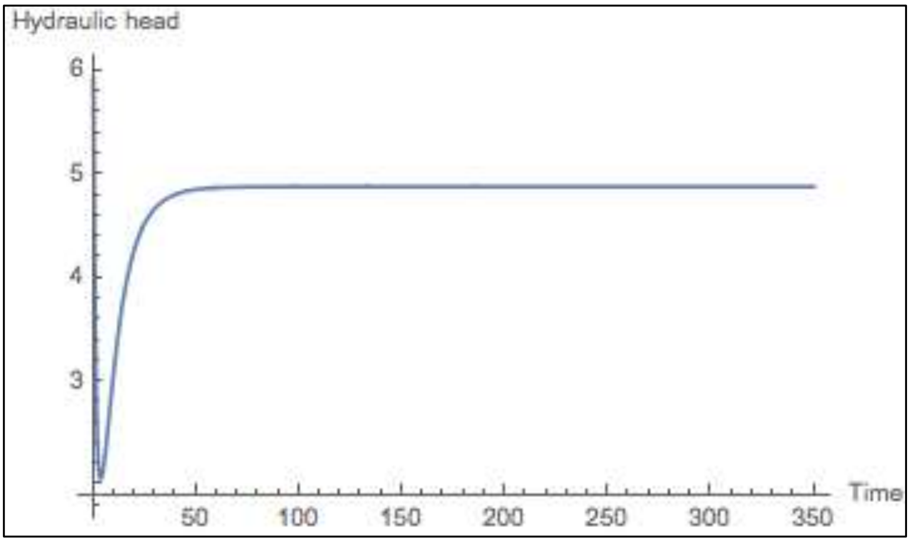

Figure 11: Kurtosis for DR distribution

The harmonic mean indicates that the typical hydraulic head values occurring within the generated time series data as a result of change in S and change DR are in the ranges of 0m – 4000m and 0m – 1650m, respectively. These are given in **Figure 4** and **Figure 5**, respectively.

The SD appears to decrease for the S distribution and increase for the DR distribution over time as depicted in **Figure 6** and **Figure 7**, respectively. This essentially means that DR has a greater deviation from the mean, and therefore a greater extent of uncertainty and error. Moreover, the DR becomes significantly high after approximately 30 days, which means that the vulnerability to error is greater after 30 days.

The graphs generated for skewness (**Figure 8** and **Figure 9**) both indicate the data are not normal for both S and DR distributions. Moreover, when skewness is positive, it yields larger error, and thus a greater uncertainty in the data. This is significant for the DR distribution, and more so after about 30 days. This supports what was previously said regarding DR having a greater extent of uncertainty and error.

As mentioned, a higher kurtosis is associated with a presence of outliers. As a result, the distribution generated for S (**Figure 10**) indicates that a shorter time period in hydraulic head changes would have less uncertainty as compared to a longer period of time. This is because as time increases for this parameter, the value of kurtosis increases (presence of outliers increase). This means that as time increases, the error associated with S increases as well. On the other hand, the value of kurtosis is high already at a much earlier time for the distribution given for DR (**Figure 11**). Ultimately, this means that both S and DR yields data that is not normal, but DR is the parameter associated with greater error and uncertainty.

This statistical analysis infers that although both S and DR have associated error and so too uncertainty, it is DR that yields greater amount of error. This makes sense when looking at the initial plots (**Figure 2** and **Figure 3**) because DR variation depicted an even greater change in hydraulic head over time, in comparison to S variation.

With that being said, it is imperative to correctly distinguish between aquifer systems when using literature as a means for a value of S, if no field work can be done. Even in the case where field work or laboratory work can be done to estimate S, critical consideration should be given to the reliability of the estimates as they have a tendency to be disputed (discussed in section 2.4.3). On the other hand, the aforementioned also suggests that incorrect DR estimates would yield inadequate changes in hydraulic head; and this would ultimately yield unreliable recharge estimates. More importantly, this means that sufficient consideration should be given to estimates of transmissivity, flow type, and flow path length, as these information are needed to estimate DR. These considerations should be done with much thought for a groundwater system associated with significant heterogeneity, because this phenomenon causes one aquifer system's hydraulic properties to differ significantly from an adjacent aquifer system, and/or even within the same aquifer system. These considerations should be made because an inaccurate S or DR estimate would yield significant error in the eventual recharge estimate.

In this study, this problem is addressed by developing a new groundwater recharge estimation model which accounts for the effect of heterogeneity, viscoelasticity and the memory effect within a geological formation. To add, uncertainties are also accounted for later on in this study through mathematical formulations. This is given in Chapter 6 which follows.

CHAPTER 6: NEW APPROACH TO GROUNDWATER RECHARGE ESTIMATION

As indicated in the beginning of this study, the aim of the study focuses on developing a new approach to groundwater recharge estimation by accounting for the concepts of heterogeneity and viscoelasticity, using fractional differentiation. Subsequently, this chapter provides the application of the various fractional derivatives, and thus shows the derivation of the corresponding exact solutions. In addition, the numerical simulations for each exact solution is given along with an interpretation thereof.

To add, this work forms part of: **Spannenberg, J; & Atangana, A., 2016. New non-linear model of groundwater recharge: Inclusion of memory, heterogeneity and visco-elasticity. *Journal of Hydrology and Earth System Sciences*.**

6.1. MATHEMATICAL FORMULATION FOR GROUNDWATER RECHARGE MODEL WITH POWER LAW

$$S\,{}_0^cD_t^\alpha h(t) = R - \frac{h(t)}{DR} \tag{20.1}$$

Where

$$S\,{}_0^cD_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial h}{\partial \tau}(\tau)\mathrm{d}\tau \tag{20.2}$$

The above equation has a memory effect as well as the capability of describing the change in hydraulic head globally. In addition, the fractional order introduced here accounts for heterogeneity of a geological formation.

The following solution to the equation (20.1) is obtained by applying the Laplace transform on both sides.

$$\mathcal{L}\left(S\,{}_0^cD_t^\alpha h(t)\right)(P) = \mathcal{L}(R)(P) - \mathcal{L}\left(\frac{h(t)}{DR}\right) \tag{20.3}$$

$$S\left[P^\alpha \tilde{h}(P) - P^{\alpha-1}h(0)\right] = \frac{R}{P} - \frac{\tilde{h}(P)}{DR} \tag{20.4}$$

$$\tilde{h}(\mathrm{P})\left[\mathrm{SP}^\alpha + \frac{1}{\mathrm{DR}}\right] = \mathrm{SP}^{\alpha-1}\mathrm{h}(0) + \frac{\mathrm{R}}{\mathrm{P}} \tag{20.5}$$

$$\tilde{h}(\mathrm{P}) = \frac{\mathrm{SP}^{\alpha-1}\mathrm{h}(0) + \frac{\mathrm{R}}{\mathrm{P}}}{\mathrm{SP}^\alpha + \frac{1}{\mathrm{DR}}} \tag{20.6}$$

$$\tilde{h}(\mathrm{P}) = \frac{\mathrm{SP}^{\alpha-1}\mathrm{h}(0)}{\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}} + \frac{\mathrm{R}}{\mathrm{P}\left[\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}\right]} \tag{20.7}$$

Now applying the inverse Laplace transformation, the following is obtained:

$$\tilde{h}(\mathrm{t}) = \mathcal{L}^{-1}\left(\frac{\mathrm{SP}^{\alpha-1}\mathrm{h}(0)}{\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}}\right) + \mathcal{L}^{-1}\left(\frac{\mathrm{R}}{\mathrm{P}\left[\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}\right]}\right) \tag{20.8}$$

$$= \mathcal{L}^{-1}\left(\frac{\mathrm{P}^{\alpha-1}\mathrm{h}(0)}{\mathrm{P}^{\alpha}+\frac{1}{\mathrm{DR}\alpha}}\right) + \mathcal{L}^{-1}\left(\frac{\mathrm{R}}{\mathrm{P}\left[\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}\right]}\right) \tag{20.9}$$

$$= \mathrm{h}(0)\mathrm{E}\left(-\frac{1}{\mathrm{DR}\alpha}\mathrm{t}^{\alpha}\right) + \mathcal{L}^{-1}\left(\frac{\mathrm{R}}{\mathrm{P}\left[\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}\right]}\right) \tag{20.10}$$

Let $\mathcal{L}^{-1}\left(\frac{\mathrm{R}}{\mathrm{P}}\right) = \mathrm{R}$, $\mathcal{L}^{-1}\left(\frac{1}{\mathrm{P}^{\alpha}+\frac{1}{\mathrm{DRS}}}\right) = \mathrm{E}_{\alpha}\left(-\frac{1}{\mathrm{DRS}}\mathrm{t}\right)$

Using the convolution theorem the following is obtained:

$$\mathcal{L}^{-1}\left(\frac{\mathrm{R}}{\mathrm{P}\left[\mathrm{SP}^{\alpha}+\frac{1}{\mathrm{DR}}\right]}\right) = \frac{\mathrm{R}}{\mathrm{S}}\int_0^{\mathrm{t}} \mathrm{E}_{\alpha}\left[-\frac{1}{\mathrm{DRS}}(\mathrm{t}-\tau)\right]\mathrm{d}\tau \tag{20.11}$$

Therefore the exact solution of the new groundwater equation is the following:

$$h(\mathrm{t}) = h(0)\mathrm{E}_{\alpha}\left(-\frac{1}{\mathrm{DR}\alpha}\mathrm{t}^{\alpha}\right) + \frac{\mathrm{R}}{\mathrm{S}}\int_0^{\mathrm{t}} E_{\alpha}\left[-\frac{1}{\mathrm{DRS}}\ (\mathrm{t}-\tau)\right] \tag{20.12}$$

Let $y = t - \tau$.

$$h(\mathrm{t}) = h(0)\mathrm{E}_{\alpha}\left(-\frac{\mathrm{t}^{\alpha}}{\mathrm{DR}\alpha}\right) + \frac{\mathrm{R}}{\mathrm{S}}\int_0^{\mathrm{t}} E_{\alpha}\left[-\frac{1}{\mathrm{DRS}}\mathrm{y}\right]d\tau \tag{20.13}$$

6.2. NUMERICAL SOLUTION TO THE RECHARGE MODEL WITH POWER LAW

In this section, the numerical approximation of the used differential operation for the differentiation namely, the Caputo derivative with fractional order is firstly given.

Let n be a natural number greater than 1, then:

$${}_0^cD_t^{\alpha} f(t_n) = \frac{1}{\Gamma(1-\alpha)}\int_0^{tn}(t_n-\tau)^{-\alpha}\frac{d}{d\tau}f(\tau)d\tau \tag{20.14}$$

Applying the Crank-Nicolson Solution:

$$ {}_{0}^{c}D_{t}^{\alpha} f(t_n) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t_n} (t_n - \tau)^{-\alpha} \frac{f(\Delta\tau+\tau)-f(\tau)}{\Delta\tau} d\tau \tag{20.15} $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{f(\mathrm{t_{k+1}})-f(t_k)}{\Delta t} (t_n - \tau)^{\alpha} d\tau \tag{20.16} $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{f(\mathrm{t_{k+1}})-f(t_k)}{\Delta t} \int_{0}^{t_n} (t_n - \tau)^{\alpha} d\tau \tag{20.17} $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{f(\mathrm{t_{k+1}})-f(t_k)}{\Delta t} \int_{t_n-t_k}^{t_n-t_{k+1}} -Y^{-\alpha} dY \tag{20.18} $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{f(\mathrm{t_{k+1}})-f(t_k)}{\Delta t} \left(\int_{t_n-t_k}^{t_n-t_{k+1}} - \frac{Y^{-\alpha+1}}{1-\alpha} \right) \tag{20.19} $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{f(\mathrm{t_{k+1}})-f(t_k)}{\Delta t} \{(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha} \} \tag{20.20} $$

$$ = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{f(\mathrm{t_{k+1}})-f(t_k)}{\Delta t} (\Delta t)^{1-\alpha} \{(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\} \tag{20.21} $$

Replacing (20.21) in (20.1), the following is obtained:

$$ S\left[\frac{1}{\Gamma(1-\alpha)}\right] \sum_{k=0}^{n} \frac{h(\mathrm{t_{k+1}})-h(t_k)}{\Delta t} (\Delta t)^{1-\alpha} \{(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\} = R - \frac{h(\mathrm{t_{n+1}})-h(n)}{2} \tag{20.22} $$

Let $\mathrm{a} = \frac{\mathrm{S}}{\Gamma(1-\alpha)} (\Delta \mathrm{t})^{-\alpha}$, $\Phi_{\alpha}^{\mathrm{n,k}} = (\mathrm{n}-\mathrm{k})^{1-\alpha} - (\mathrm{n}-\mathrm{k}-1)^{1-\alpha}$

$$ \sum_{k=0}^{n} \{h(\mathrm{t_{k+1}}) - h(t_k)\} a\, \Phi_{\alpha}^{\mathrm{n,k}} = R - \frac{h(\mathrm{t_{n+1}})-h(n)}{2} \tag{20.23} $$

$$ \{h(\mathrm{t_{n+1}}) - h(n)\} a \Phi_{\alpha}^{\mathrm{n,n}} + \sum_{\mathrm{k=0}}^{\mathrm{n-1}} \{ \mathrm{h}(\mathrm{t_{k+1}}) - h(t_k)\} a \Phi_{\alpha}^{\mathrm{n-1,k}} = R - \frac{h(\mathrm{t_{n+1}})-h(n)}{2} \tag{20.24} $$

Rearrange and the following is obtained:

$$ \left\{a\Phi_{\alpha}^{\mathrm{n,n}} + \frac{1}{2}\right\} h(\mathrm{t_{n+1}}) = \left\{a\Phi_{\alpha}^{\mathrm{n,n}} - \frac{1}{2}\right\} h(t_n) + \sum_{k=0}^{n-1} \{h(\mathrm{t_{k+1}}) - h(t_k)\} \Phi_{\alpha}^{\mathrm{n-1,k}} + \mathrm{R} \tag{20.25} $$

6.3. MATHEMATICAL FORMULATION FOR GROUNDWATER RECHARGE MODEL WITH EXPONENTIAL DECAY LAW:

In this section, the analysis of the groundwater recharge model is presented using the law of exponential decay. This model could be used for those geological formations within which the recharge process follows the law of exponential decay. In this section, the modified recharge equation is given as:

$$S\,{}_{0}^{CF}D_t^{\alpha}h(t) = R - \frac{h(t)}{DR} \tag{21.1}$$

Where ${}_{0}^{CF}D_t^{\alpha}$is the well known Caputo-Fabrizio operator which is defined as:

$${}_{0}^{CF}D_t^{\alpha}h(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \frac{d}{dt}h(t)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau \tag{21.2}$$

Based on the literature review (refer to Chapter 4), this operator was introduced due to the necessity of employing the behaviour of classical viscoelastic material; and in the case of this study, the viscoelastic material represents the geological formation through which recharge occurs.

The exact solution of the modified model will be obtained using the Laplace transform. Thus by applying the Laplace transform both sides, the following is obtained:

$$S\mathcal{L}\left({}_{0}^{CF}D_t^{\alpha}h(t)\right) = \mathcal{L}\left(R - \frac{h(t)}{DR}\right) \tag{21.3}$$

Nevertheless, the Laplace transform of the Caputo-Fabrizio derivative is given as:

$$\mathcal{L}\left({}_{0}^{CF}D_t^{\alpha}h(t)\right) = \frac{P\tilde{h}(P)-h(0)}{P+\alpha(1-P)} \tag{21.4}$$

Replacing this in (20.3), the following is obtained:

$$S\frac{P\tilde{h}(P)-h(0)}{P+\alpha(1-P)} = \frac{R}{P} - \frac{\tilde{h}(P)}{DR} \tag{21.5}$$

$$\tilde{h}(P)\left[\frac{SP}{P+\alpha(1-P)} - \frac{1}{DR}\right] = \frac{Sh(0)}{P+\alpha(1-P)} + \frac{R}{P} \tag{21.6}$$

$$\tilde{h}(P) = \frac{\frac{Sh(0)}{P+\alpha(1-P)}+\frac{R}{P}}{\frac{SP}{P+\alpha(1-P)}-\frac{1}{DR}} \tag{21.7}$$

$$\tilde{h}(P) = \frac{\frac{Sh(0)+PR+\alpha(1-P)R}{P(P+\alpha(1-P))}}{\frac{SP-P-\alpha(1-P)}{DR(P+\alpha(1-P))}} \tag{21.8}$$

$$= \frac{[PSh(0)+PR+\alpha(1-P)R]DR}{P[SP-P-\alpha(1-P)} \tag{21.10}$$

$$= \frac{P[Sh(0)+R-\alpha R]DR+\alpha RDR}{P[P(S-1+\alpha)-\alpha]} \tag{21.11}$$

$$= \frac{(Sh(0)+R-\alpha R)DR}{P-\frac{\alpha}{S-1+\alpha}} \frac{1}{S-1+\alpha} + \frac{\alpha RDR}{P[P(S-1+\alpha)-\alpha]} \tag{21.12}$$

The exact solution is obtained by taking the inverse Laplace on both sides and also using the convolution theorem:

$$h(t) = \frac{DR(Sh(0)+R-\alpha R)}{S+\alpha-1} exp\left[\frac{\alpha}{\alpha-1+S} t\right] + \frac{\alpha RDR}{S+\alpha-1} \int_0^t exp\left[\frac{\alpha(t-\tau)}{\alpha-1+\alpha}\right] d\tau \tag{21.13}$$

It is important to note that when α = 1, the solution of the classical model is obtained.

6.4. NUMERICAL SOLUTION TO THE RECHARGE MODEL WITH EXPONENTIAL DECAY LAW

To accommodate for the researchers working in numerical analysis, the finite approximation of the Caputo-Fabrizio derivative is presented:

If $n \geq 1$, then

$${}_{0}^{CF}D_t^{\alpha} f(t_n) = \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \frac{d}{d\tau} f(\tau)\, exp\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)\right] d\tau \tag{21.14}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{f(t_{k+1})+f(t_k)}{\Delta t} exp\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)\right] d\tau \tag{21.15}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \frac{f(t_{k+1})+f(t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} exp\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)\right] d\tau \tag{21.16}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \frac{f(t_{k+1})+f(t_k)}{\Delta t} \int_{t_n-t_{k+1}}^{t_n-t_k} exp\left[-\frac{\alpha}{1-\alpha} y\right] dy \tag{21.17}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \frac{f(t_{k+1})+f(t_k)}{\Delta t} \frac{1-\alpha}{\alpha} exp\, \Big|_{t_k-t_{k+1}}^{t_n-t_k} \tag{21.18}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \frac{f(t_{k+1})+f(t_k)}{\Delta t} \frac{1-\alpha}{\alpha} exp\, \Big|_{t_k-t_{k+1}}^{t_n-t_k} \tag{21.19}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \frac{f(t_{k+1})+f(t_k)}{\Delta t} \left\{ exp\left[\frac{\alpha}{1-\alpha}(t_n-t_k)\right] - exp\left[\frac{\alpha}{1-\alpha}(t_n-t_{k+1})\right] \right\} \tag{21.20}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{n} \frac{f(t_{k+1})+f(t_k)}{\Delta t} \Phi_{n,k}^{\alpha} \tag{21.21}$$

Where $\Phi_{n,k}^{\alpha} = exp\left[-\frac{\alpha}{1-\alpha}(t_n - t_k)\right] - exp[-\frac{\alpha}{1-\alpha}(t_k - t_{k+1})]$

Replacing the first approximation in the original equation, the following is obtained:

$$S\left[\frac{M(\alpha)}{\alpha}\sum_{k=0}^{n}\frac{h(t_{k+1})+h(t_k)}{\Delta t}\Phi_{\mathrm{n,k}}^{\alpha}\right] = R - \frac{h(t_{n+1})+h(t_n)}{2} \tag{21.22}$$

To obtain the recursive formula, the above equation is reformulate as follows:

Let $a_1 = \frac{M(\alpha)S}{\alpha\Delta t}$, then

$$h(t_{n+1})\left[a_1\Phi_{\mathrm{n,n}}^{\alpha} + \frac{1}{2}\right] = h(t_n)\left[a_1\Phi_{\mathrm{n,k}}^{\alpha} - \frac{1}{2}\right] + a_1\sum_{k=0}^{n-1}\{h(t_{k+1}) - h(t_k)\}\Phi_{\mathrm{n-1,k}}^{\alpha} + R \tag{21.23}$$

The above is the numerical solution of the recharge equation with the exponential decay law.

6.5. MODELLING GROUNDWATER RECHARGE WITH THE GENERALISED EXPONENTIAL DECAY LAW

It is important to note that the geological formations through which recharge takes place, is sometimes very complex and can be a combination of both heterogeneity and viscoelasticity. In this situation neither the power law nor the exponential decay law can be used to portray the change in hydraulic head for a given percentage of recharge, thus a more suitable operator for differentiation must be used. Recently and for the purpose of extending the limitation to the power law and exponential decay law, a new operation of differentiation was introduced and used in many fields of science and engineering to model a few real world problems with great success. The new operation is given as:

$$^{ABC}_{0}D_t^{\alpha}f(t) = \frac{AB(\alpha)}{1-\alpha}\int_0^t \frac{d}{d\tau}f(\tau)\,E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right]d\tau \tag{22.1}$$

Thus replacing the time classical derivative by the new operation of differentiation, the following modified model is obtained:

$$S^{ABC}_{0}D_t^{\alpha}h(t) = R - \frac{h(t)}{DR} \tag{22.2}$$

This model takes into account the recharge of groundwater in a medium with different layers, heterogeneity and viscoelasticity.

Next, the exact solution is derived, using once the Laplace transform:

$$S\left[\frac{AB(\alpha)}{1-\alpha}\frac{P^{\alpha}\tilde{h}(P)-P^{\alpha-1}h(0)}{P^{\alpha}+\frac{\alpha}{1-\alpha}}\right]=\frac{R}{P}-\frac{\tilde{h}(P)}{DR} \tag{22.3}$$

$$\tilde{h}(P)\left[\frac{SAB(\alpha)}{1-\alpha}\frac{P^{\alpha}}{P^{\alpha}+\frac{\alpha}{1-\alpha}}+\frac{1}{DR}\right]=\frac{SAB(\alpha)}{1-\alpha}\frac{P^{\alpha-1}h(0)}{P^{\alpha}+\frac{\alpha}{1-\alpha}}+\frac{R}{P} \tag{22.4}$$

$$\tilde{h}(P)=\frac{\Phi(\alpha)\frac{P^{\alpha-1}}{P^{\alpha}+\Phi,(\alpha)}+\frac{R}{P}}{\Phi(\alpha)\frac{P^{\alpha}}{P^{\alpha}+\Phi,(\alpha)}+\frac{1}{DR}} \tag{22.5}$$

$$=\frac{\Phi(\alpha)P^{\alpha-1}+P^{\alpha}R+\Phi(\alpha)}{P[P^{\alpha}+\Phi,(\alpha)]}\cdot\frac{DR\left(P^{\alpha}\Phi,(\alpha)\right)}{DR\Phi(\alpha)P^{\alpha}+P^{\alpha}+\Phi(\alpha)} \tag{22.6}$$

$$=\frac{[\Phi(\alpha)P^{\alpha-1}+P^{\alpha}R+\Phi,(\alpha)]DR}{P[DR\Phi(\alpha)P^{\alpha}+P^{\alpha}+\Phi,(\alpha)]} \tag{22.7}$$

$$=\frac{P^{\alpha-1}DR(\Phi(\alpha)+R)}{P^{\alpha}+\frac{\Phi,(\alpha)}{\Phi(\alpha)DR+1}}\cdot\frac{1}{\Phi(\alpha)DR+1}+\frac{\Phi,R}{P\left[P^{\alpha}+\frac{\Phi,(\alpha)}{\Phi(\alpha)DR+1}\right]}\cdot\frac{1}{\Phi(\alpha)DR+1} \tag{22.8}$$

Thus using the inverse Laplace transform and the convolution theorem for Laplace transform, the following is obtained:

$$h(t)=\frac{DR(\Phi(\alpha)+R)}{DR\Phi(\alpha)+1}E_{\alpha}\left[-\frac{\Phi,(\alpha)}{\Phi(\alpha)DR+1}t^{\alpha}\right]+\frac{\Phi,R}{\Phi(\alpha)DR+1}\cdot\int_0^t E_{\alpha}\left[-\frac{\Phi,(\alpha)}{\Phi(\alpha)DR+1}(t-\tau)\right]d\tau \tag{22.9}$$

Here $\Phi(\alpha)=\frac{AB(\alpha)h(0)}{1-\alpha}$, $\Phi(\alpha)=\frac{\alpha}{1-\alpha}$

The numerical solution of the model with the generalised Mittag-Leffler Law is now presented. To do this, first a presentation of the approximation of the Atangana-Baleanu derivative with fractional order let n be a positive natural number, and the time period is sub-devised as follows:

$t_o \leq t_1 \leq t_2 \ldots \leq t_{n-1} \leq t_n, \Delta t = t_k - t_{k-1}$

Thus,

$${}^{ABC}_{0}D_t^{\alpha}h(t_n)=\frac{AB(\alpha)}{1-\alpha}\int_0^{t_n}\frac{d}{d\tau}h(\tau)\,E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)^{\alpha}\right]d\tau \tag{22.10}$$

$$=\frac{AB(\alpha)}{1-\alpha}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\frac{h(t_{k+1})-h(t_k)}{\Delta t}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)^{\alpha}\right]d\tau \tag{22.11}$$

$$=\frac{AB(\alpha)}{1-\alpha}\sum_{k=0}^{n}\frac{h(t_{k+1})-h(t_k)}{\Delta t}\delta_{n,k}^{\alpha} \tag{22.12}$$

Where

$$\delta_{n,k}^{\alpha} = \int_{t_k}^{t_{k+1}} E_\alpha\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)^\alpha\right] d\tau$$

$$\delta_{n,k}^{\alpha} = (t_n - t_{k+1})E_{\alpha,2}\left[\frac{\alpha}{1-\alpha}(t_n - t_{k+1})\right] - (t_n - t_k)E_{\alpha,2}\left[\frac{\alpha}{1-\alpha}(t_n - t_k)\right]$$

Therefore replacing this in the main equation, (22.13) is obtained:

$$S\left[\frac{AB(\alpha)}{1-\alpha}\sum_{k=0}^{n}\frac{h(t_{k+1})-h(t_k)}{\Delta t}S_{n,k}^{\alpha}\right] = R - \frac{h(t_{n+1})-h(t_n)}{DR} \quad (22.13)$$

6.6. NUMERICAL SIMULATION AND INTERPRETATION

In this section, the numerical solution presented in previous section to show the graphical representation of the groundwater recharge model for different values of the fractional differentiation is given. The recursive formulae used in these simulations are equation (22.13), (21.23) and (20.25). The parameters used in the simulations are $DR = 200$, $R = 40\ \%$ and $S = 0.002$. The numerical simulations are depicted in **Figure 11**, **12**, **13**, **14** and **15**.

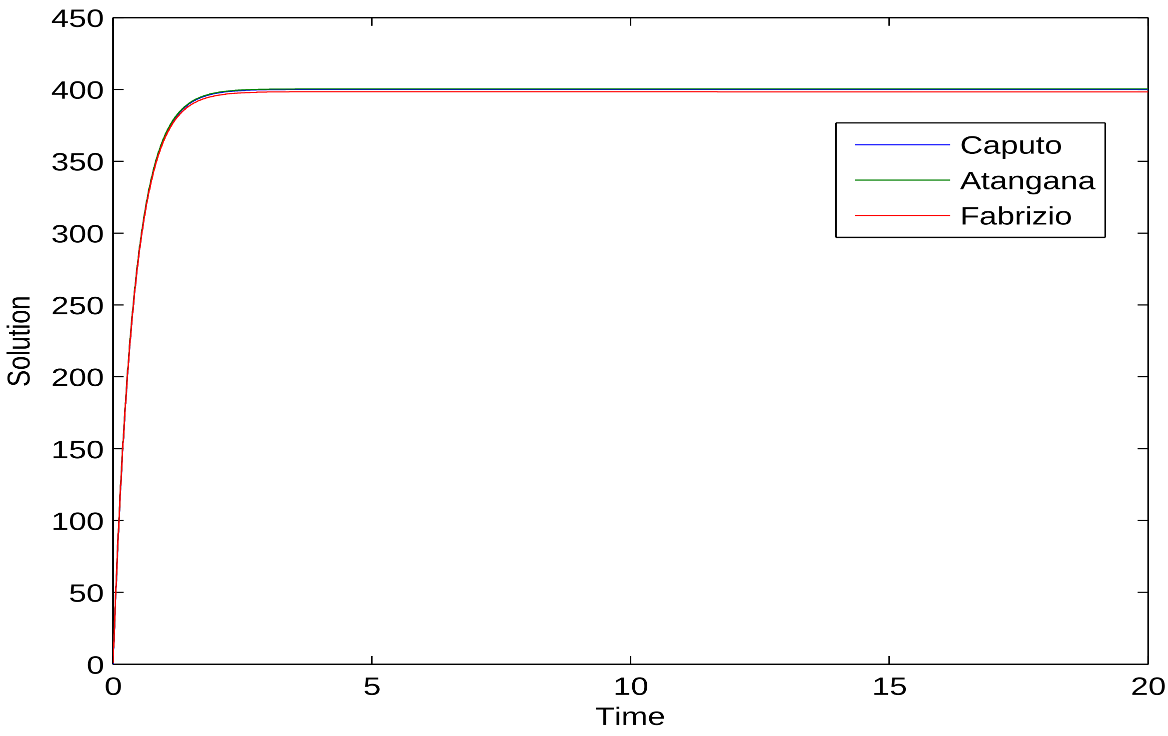

Figure 12: Numerical simulation for α=1

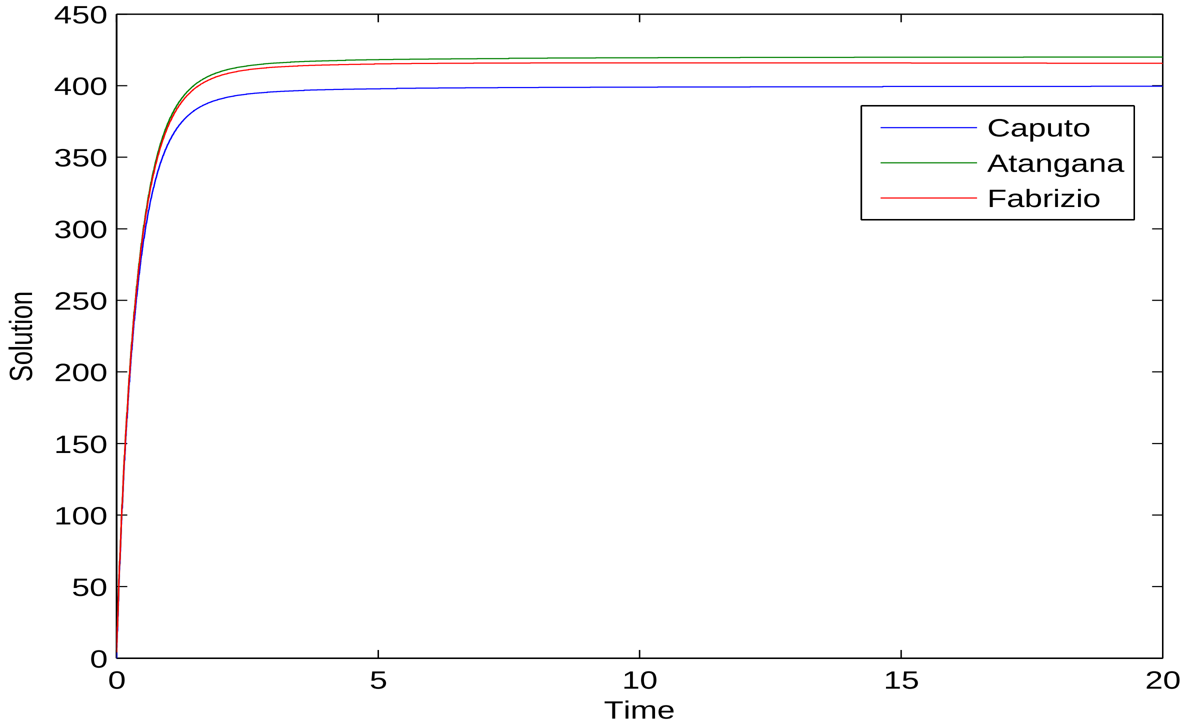

Figure 13: Numerical simulation for α=0.95

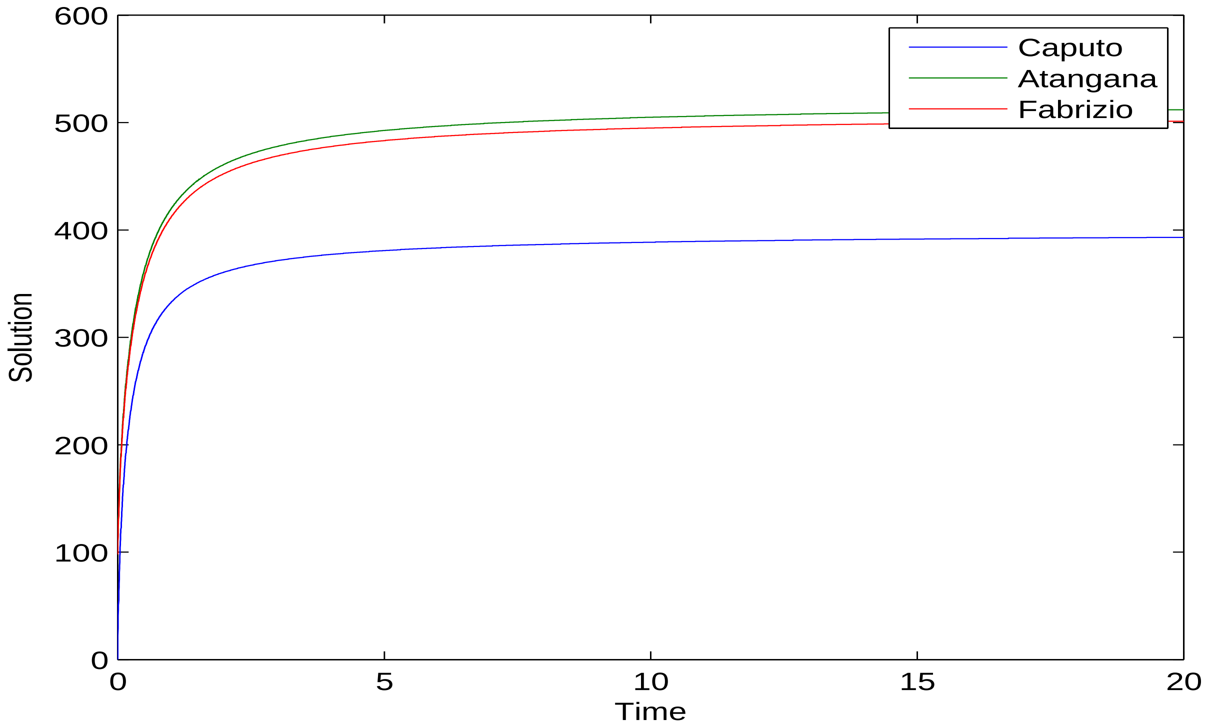

Figure 14: Numerical simulation for α=0.85

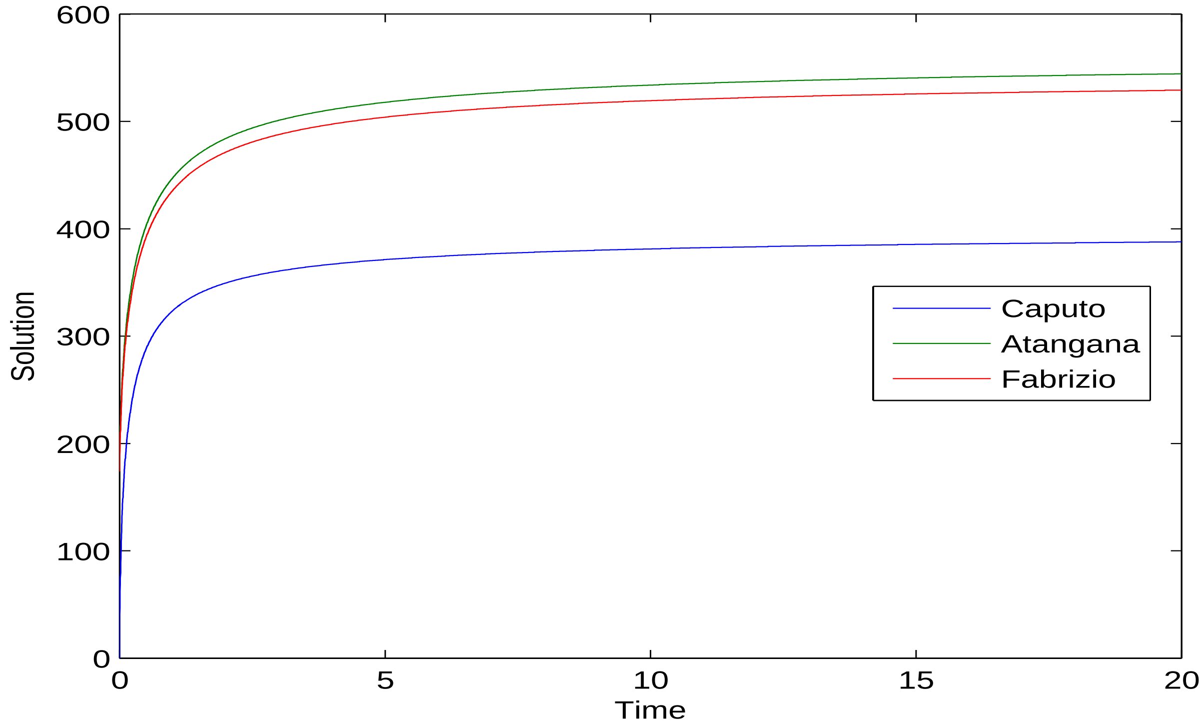

Figure 15: Numerical simulation for α=0.6

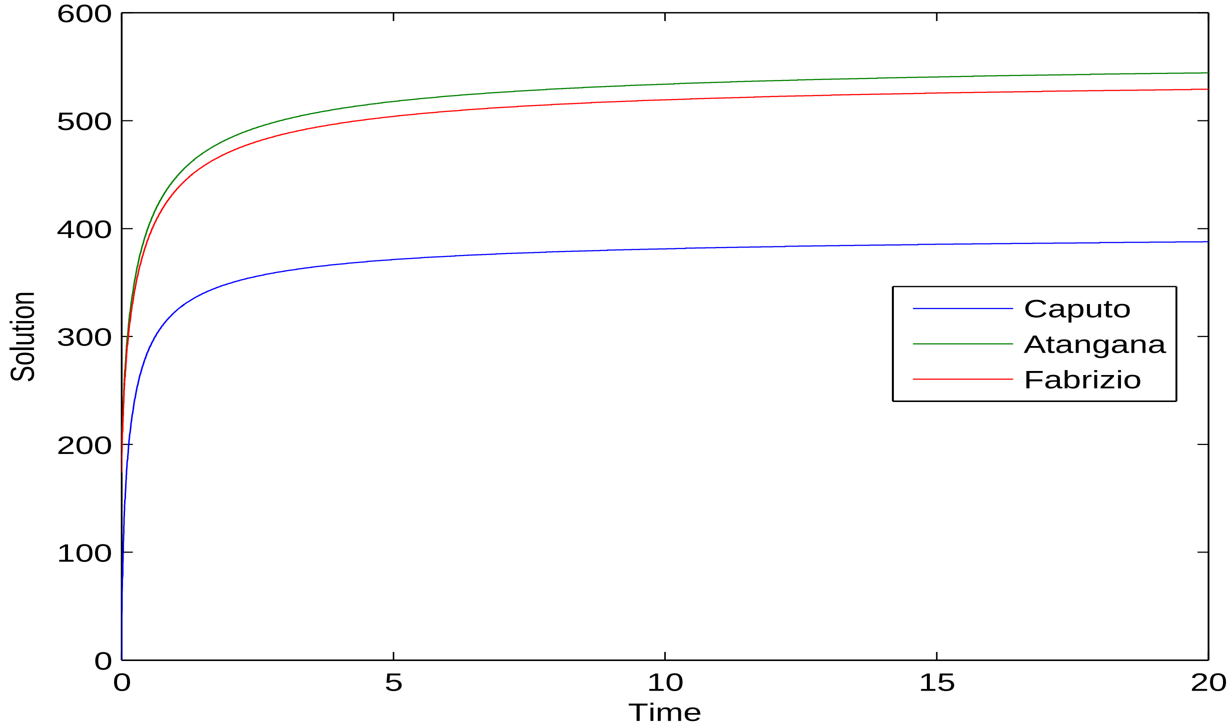

Figure 16: Numerical simulation for α=0.4

Upon doing numerical simulations for each of these solutions, it can be said that the results depict the behaviour of a particular real world problem within which a groundwater system's hydraulic head changes over time. The above graphical representations depict how the change in hydraulic head varies for each simulated solution. When $\alpha = 1$, the solutions depict a similar distribution of hydraulic head over time. However, when α decreases by 0.05, the hydraulic head distribution for the Caputo simulation deviates slightly from the Caputo-Fabrizio and Atangana-Baleanu. These two latter solutions only show a minor deviation from each other for $\alpha = 0.95$. When α decreases even more to 0.85, the Caputo solution deviates even more from the other solutions. This can be attributed to the fact that the Caputo derivative does not account for viscoelasticity. Furthermore, when α decreases to 0.6 and 0.4, the Caputo-Fabrizio solution starts deviating slightly more from the Atangana-Baleanu solution. This can be attributed to the fact that the Caputo-Fabrizio solution does not account for complex systems having both heterogeneity and viscoelasticity, in a way that the Atangana-Baleanu derivative does. Since a decrease in α causes the Caputo and Caputo-Fabrizio solutions to vary in its hydraulic head distribution over time, it is assumed that groundwater recharge within heterogeneous and viscoelastic geological formations is well described using the concept of differentiation with the generalized Mittag-Leffler law or the Atangana-Baleanu derivative. To add, groundwater recharge occurring through elastic geological formations can be modelled using the Caputo and Caputo-Fabrizio derivatives.

6.7. APPLICATION OF THE ATANGANA DERIVATIVE WITH MEMORY

As indicated in Chapter 2 and Chapter 3, accurate recharge estimation relies on the certainty in the factors affecting recharge, as well as the parameters used within a recharge equation. Uncertainty could cause one to be left with over/under-estimations. Furthermore, literature indicates there is usually several different sources of uncertainty in a model's output. This can be uncertainty related to the lack of knowledge on model inputs, inaccurate model parameters, and structural uncertainty associated with mathematical formulations governing the model. Furthermore, the study of functions may be used to address this uncertainty in models. This is done by explicitly incorporating uncertainty into mathematical equations governing a model, by means of a function of both time and space (Atangana, 2014). This section of this dissertation introduces the Atangana derivative with memory as a means of accounting for uncertainty in the recharge equation.

In mathematics, a dynamic scheme is a tuple (D, h, B) with D as a manifold that could be either a locally Banach space or Euclidean space, B the domain for time which is a set non-negative real, and h is an evolution rule $t \rightarrow f^t$ the range is of course a diffeomorphism of a manifold to itself.

6.7.1. Definition of the derivative

Let D be a dynamic system with domain B (time or space), let u be a positively defined function called the uncertainty function of D, within the domain B, then if h $\in D$*, then the Atangana derivative with memory of a function h denoted by* $U^u(f)$ *is defined as:*

$$U^{u(t)}\big(h(t)\big) = \big(1 - u(t)\big)h(t) + u(t)h(t) \tag{23.1}$$

If $u = 1$ the first derivative is recovered (the local derivative), if $u = 0$, the initial function is recovered. This follows the primary law of derivative.

6.7.2. Properties of the Atangana derivative with memory

Addition:

$$U^{u(t)}\big(ah(t) + bf(t)\big) = aU^{u(t)}\big(h(t)\big) + bU^{u(t)}\big(f(t)\big) \tag{23.2}$$

Multiplication:

$$U^{u(t)}\big(fg(t)\big) = f(t)U^{u(t)}\big(g(t)\big) + u(t)g(t)f(t) \tag{23.3}$$

Using the definition of Atangana derivative with memory, the following is obtained:

$$U^{u(t)}\big(fg(t)\big) = \big(1 - u(t)\big)f(t)g(t) + u(t)\big(f(t)f(t)\big)$$

$$U^{u(t)}\big(fg(t)\big) = \big(1 - u(t)\big)f(t)g(t) + u(t)[f(t)f(t)g(t) + g(t)f(t)]$$

$$U^{u(t)}\big(fg(t)\big) = f(t)U^{u(t)}\big(g(t)\big) + u(t)g(t)f(t) \tag{23.4}$$

Division:

$$U^{u(t)}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)U^{u(t)}f(t) - u(t)f(t)g(t)}{g^2(t)} \tag{23.5}$$

Lipchitz condition: Let $f(t)$ and $g(t)$ be two functions then

$$\left\|U^{u(t)}\big(g(t)\big) - U^{u(t)}\big(f(t)\big)\right\| \leq \left\|\big(1 - u(t)\big)g(t) + u(t)g(t) - \big(1 - u(t)\big)f(t) - u(t)f(t)\right\| \leq |1 - u(t)|\|g(t) - f(t)\| + |u(t)|\|g(t) - f(t)\| \leq a\|g(t) - f(t)\| +$$

$$b\|g(t)-f(t)\| \leq a\|g(t)-f(t)\| + b\alpha\|g(t)-f(t)\| \leq H\|g(t)-f(t)\| \tag{23.6}$$

The above shows that the Atangana derivative with memory possesses the Lipchitz condition.

6.7.3. Application to the selected groundwater recharge equation

This section focuses on modifying the selected groundwater recharge equation governing the EARTH model by incorporating the effect of heterogeneity, variability and uncertainty in the equation. Considering a scenario where the groundwater system is of homogenous, uniform aquifer properties and rainfall recharge, the addition of water to the water table may be estimated using the selected equation. This equation is given by (23.7). As indicated, use of this equation has been given in many studies (refer to Chapter 3).

$$S\frac{dh(t)}{dt} = R - \frac{h(t)}{DR} \tag{23.7}$$

Where S is specific yield; $\frac{dh}{dt}$ is change in water level over a period of time; R is recharge; h is groundwater level; and DR is drainage resistance which is a site specific parameters.

The above equation does not consider the effect of heterogeneity, variability and uncertainties in a groundwater system. To expand, the equation describes a model using constant S and DR with change in head to obtain a certain percentage of recharge. However, constant S and DR is only appropriate provided the geological system within recharge takes place is homogeneous, uniform and non-variable. Therefore, for inclusion of these properties into the recharge equation, the ordinary derivative in (23.7) is replaced by the uncertain function equation in (23.1). In doing so, the following is obtained:

$$\big(1-u(t)\big)h(t) + u(t)h'(t)S = R - \frac{h(t)}{DR} \tag{23.8}$$

$$u(t)h'(t) = \frac{R}{S} - \frac{h(t)}{DRS} - \big(1-u(t)\big)h(t) \tag{23.9}$$

$$u(t)h'(t) = \frac{R}{S} - \frac{h(t)}{DRS} - 1h(t) + u(t)h(t) \tag{23.10}$$

$$u(t)h'(t) = \frac{R}{S} - h(t)\left(\frac{1}{DRS} + 1 - u(t)\right) \tag{23.11}$$

$$u(t)h'(t) = \frac{R}{S} - \left(\frac{1}{DRS} + 1 - u(t)\right)h(t) \tag{23.12}$$

$$h'(t) = \frac{R}{u(t)S} - \frac{\left(\frac{1}{DRS}+1-u(t)\right)}{u(t)} h(t) \tag{23.13}$$

$$h'(t) = R_1(t) - \frac{\frac{1}{\mathrm{S}}+\mathrm{DR}-\mathrm{DRu(t)}}{DRu(t)} h(t) \tag{23.14}$$

$$h'(t) = R_1(t) - \frac{h(t)}{\frac{\frac{DRu(t)}{}}{\frac{1}{\mathrm{S}}+\mathrm{DR}-\mathrm{DRu(t)}}} \tag{23.15}$$

$$h'(t) = R_1(t) - \frac{h(t)}{DR_1(t)} \tag{23.16}$$

Now the groundwater recharge equation has non-constant parameters. To expand, both the contribution of recharge and drainage resistance are now a function of time. Physically, the function $R_1(t)$ is the recharge percentage at a specific time “t”, and $DR_1(t)$ is the drainage resistance at the time “t”.

Remark: if the uncertain function $u(t) = 1$, the classical equation with constant parameters is recovered. Essentially, the modified equation is non-linear, and this means it is more appropriate for modelling real world groundwater recharge problems. Furthermore, since the new equation has non-constant parameters, then the equation cannot be solved using the Laplace transform. Therefore, in this case the classical method for ordinary differential equations is used, to obtain:

$$h(t) = \frac{\int_0^t \exp[\int P(t)]\cdot g(t)dt+c}{\exp[\int P(t)dt]} \tag{23.17}$$

Where $g(t) = R_1(t)$ and $P(t) = \frac{1}{\frac{DRu(t)}{\frac{1}{\mathrm{S}}+\mathrm{DR}-\mathrm{DRu(t)}}}$

So the exact solution is given by:

$$h(t) = \frac{\int exp[\int P(t)]\cdot R_1 dt+c}{exp\left[\int \frac{\frac{1}{SDR}+1-u(t)}{DRu(t)} dt\right]} \tag{23.18}$$

One can select the uncertain function to be the Error function defined as:

$$erf(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt \tag{23.19}$$

The uncertain function must be selected in a way that at $t = 0,\ u(0) = 0$ at $t = T$

$u(T) = 1$.

Nevertheless, if the uncertain function is complicated in a way that $DR_1(t)$ is not integrable, one can use an iterative or numerical method.

6.7.4. Iterative method for the new model

In this section the non-linear groundwater recharge model is solved using an iterative method. To achieve this, the integral is applied on (23.16) to obtain:

$$h(t) = \int_0^t \left(R_1(\tau) - \frac{h(\tau)}{DR_1(\tau)} \right) d\tau \tag{23.20}$$

The following iterative formula is considered:

$$h_{n+1}(t) = \int_0^t \left(R_1(t) - \frac{h_n(t)}{DR_1(t)} \right) d\tau + h(0) \tag{23.21}$$

The above iterative formula can be used to generate the solution to the new model, provided the initial condition is known.

Now the stability of the used method is presented:

Let Γ be a function defined as:

$$\Gamma\big(h(t)\big) = \int_0^t \left(R_1(\tau) - \frac{h(\tau)}{DR_1(\tau)} \right) d\tau \tag{23.22}$$

Let h_1 and h_2 be two different functions, then:

$$\|\Gamma h_1 - \Gamma h_2\| = \left\| \int_0^t R_1(\tau) - R_1(\tau) - \frac{h_1(\tau) - h_2(\tau)}{DR_1(\tau)} d\tau \right\| \le \int_0^t \frac{\|h_1(\tau) - h_2(\tau)\|}{\|DR_1(\tau)\|} d\tau \le \frac{1}{M} \|h_1(t) - h_2(t)\| T \le \frac{T}{M} \|h_1(t) - h_2(t)\| \tag{23.23}$$

So Γ possesses the Lipchitz conditions. Now the following iterative formula is defined via Γ:

$\Gamma(h_n) = h_{n+1}$, then $n \epsilon N$

$$\|h_{n+1} - h_n\| = \|\Gamma(h_n) - \Gamma(h_{n-1})\| \le \frac{T}{M} \| h_n - h_{n-1}\| \le \frac{T}{M} \|\Gamma(h_{n-1}) - \Gamma(h_{n-2})\| \left(\frac{T}{M}\right)^2 \|h_{n-1} - h_{n-2}\| \le \left(\frac{T}{M}\right)^n \|h_1 - h_0\| \tag{23.24}$$

A selection of $M > T$ is made, so that $\lim_{n\to 0}\left(\frac{T}{M}\right)^n \to 0$

Under the above conditions, it is concluded that $(h_n)_{n\epsilon N}$ the Cauchy sequence in Banach space therefore converge. Thus, one can find a function $\tilde{h}(t)$in a way that $\lim_{n\to n} h_n(t) = \tilde{h}(t)$

In this case,

$$\lim_{n\to 0}\Gamma(h_n) = \lim_{n\to\infty} h_{n+1} \to \Gamma(\tilde{h}) = \tilde{h}$$

Thus $\tilde{h}$ is a fixed point of Γ

6.7.5. Numerical solution

In the section, the numerical solution using the Crank-Nicolson scheme is presented.

$$\frac{h^{j+1}-h^j}{\Delta t} = R_1^j - \frac{1}{DR_1^j}\left(\frac{h^{j+1}+h^j}{2}\right) \tag{23.25}$$

Rearranging, the following is obtained:

$$\left(\frac{1}{\Delta t} + \frac{1}{2DR_1^j}\right)h^{j+1} = \left(\frac{1}{\Delta t} - \frac{1}{2DR_1^j}\right)h^j + R_1^j \tag{23.26}$$

The above numerical solution can be used to generate the simulation for different functions $u(t)$.

The new non-linear model can further be generalised by replacing the local differential operator with a non-linear differential operator such as ${}_0^{ABR}D_t^\alpha$, ${}_0^{CFR}D_t^\alpha$ or ${}_0^{RL}D_t^\alpha$.

By replacing $\frac{d}{dt}$ by ${}_0^{RL}D_t^\alpha$, the following is obtained:

$${}_0^{RL}D_t^\alpha h(t) = R_1(t) - \frac{h(t)}{DR_1(t)} \tag{23.27}$$

The above model has the ability to account for elasticity of a geological formation as well as the memory effect.

By replacing $\frac{d}{dt}$ by${}_0^{CFR}D_t^\alpha$, the following is obtained:

$${}_0^{CFR}D_t^\alpha h(t) = R_1(t) - \frac{h(t)}{DR_1(t)} \tag{23.28}$$

The above model has the ability to account for heterogeneity as well as the memory effect.

By replacing $\frac{d}{dt}$ by ${}^{ABR}_{0}D_t^{\alpha}$, the following is obtained:

$$ {}^{ABR}_{0}D_t^{\alpha}h(t) = R_1(t) - \frac{h(t)}{DR_1(t)} \quad (23.27)$$

The above model has the ability to account for heterogeneity, memory and viscoelasticity.

These three models will be solved numerically, and to do this, the numerical approximation of the three non-local operators are presented.

Firstly, ${}^{RL}_{0}D_t^{\alpha}$ is presented:

By definition:

$$ {}^{RL}_{0}D_t^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t h(t)\,(x-\tau)^{-\alpha}d\tau \quad (23.28)$$

$$ = \frac{d}{dt}F(t) \quad (23.29)$$

According to Crank-Nicolson this can be given for $n \geq 1$ as:

$$ {}^{RL}_{0}D_t^{\alpha}h(t) = \left[\frac{F(t_{n-1})-F(t_n)}{\Delta t}\right] \quad (23.30)$$

Where,

$$ F(t_{n-1}) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{n+1}} h(\tau)\,(t_{n+1}-\tau)^{-\alpha}d\tau \quad (23.31)$$

$$ = \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} h(t_k)\,(t_{n+1}-\tau)^{-\alpha}d\tau \quad (23.32)$$

$$ = \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n} h(t_k)\int_{t_k}^{t_{k+1}}(t_{n+1}-\tau)^{-\alpha}\,d\tau \quad (23.33)$$

Let $y = t_{n+1}-\tau, \quad dy = d\tau, \quad y = t_{n+1}-t_{k+1}, \quad y = t_{n+1}-t_k$

$$ F(t_{n+1}) = \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n} h(t_n)\int_{t_{n+1}-t_{k+1}}^{t_{n+1}-t_k} Y^{-\alpha}\,dY \quad (23.34)$$

$$ = \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n} h(t_n)\left(-\frac{Y^{1-\alpha}}{1-\alpha}\right)\Big|_{t_{n+1}-t_{k+1}}^{t_{n+1}-t_k} \quad (23.35)$$

$$ = \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\sum_{k=0}^{n} h(t_n)\left\{(t_{n+1}-t_{k+1})^{1-\alpha}-(t_{n+1}-t_k)^{1-\alpha}\right\} \quad (23.36)$$

$$ = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n} h(t_n)\left\{(t_{n+1}-t_{k+1})^{1-\alpha}-(t_{n+1}-t_k)^{1-\alpha}\right\} \quad (23.37)$$

Thus replacing $n+1$ by n, the following is obtained:

$$ F(t_n) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n-1} h(t_n)\left\{(t_n-t_{k+1})^{1-\alpha}-(t_n-t_k)^{1-\alpha}\right\} \quad (23.38)$$

Thus,

$$ {}^{RL}_{0}D_t^{\alpha}h(t_n) = \frac{1}{\Delta t\Gamma(\alpha)}\Big\{\sum_{k=0}^{n} h(t_n)\,\{(t_{n+1}-t_{k+1})^{1-\alpha}-(t_{n+1}-t_k)^{1-\alpha}\} + \sum_{k=0}^{n-1} h(t_n)\,\{(t_n-t_{k+1})^{1-\alpha}-(t_n-t_k)^{1-\alpha}\}\Big\} \tag{23.39} $$

Using the same derivation for ${}^{ABR}_{0}D_t^{\alpha}h(t)$ the following is obtained:

$$ {}^{ABR}_{0}D_t^{\alpha}h(t) = \frac{AB(\alpha)}{\Delta t(1-\alpha)}\{\sum_{k=0}^{n} h(t_n)\Phi_n^{\alpha,1} - \sum_{k=0}^{n-1} h(t_n)\Phi_n^{\alpha,2}\} \tag{23.40} $$

Where,

$$ \Phi_n^{\alpha,1} = (t_{n+1}-t_{k+1})E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-t_{k+1})^{\alpha}\right] + (t_{n+1}-t_k)E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-t_k)^{\alpha}\right] \tag{23.41} $$

and

$$ \Phi_n^{\alpha,2} = (t_{n+1}-t_{k+1})E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(t_n-t_{k+1})^{\alpha}\right] + (t_n-t_k)E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(t_n-t_k)^{\alpha}\right] \tag{23.42} $$

Using the same derivation for ${}^{CFR}_{0}D_t^{\alpha}h(t)$ the following is obtained:

$$ {}^{CFR}_{0}D_t^{\alpha}h(t) = \frac{M(\alpha)}{\Delta t(\alpha)}\{\sum_{k=0}^{n} h(t_k)\,\Phi_\alpha^k - \sum_{k=0}^{n-1} h(t_k)\,\Phi_{\alpha,1}^k\} \tag{23.43} $$

Where,

$$ \Phi_\alpha^k = exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-t_{k+1})\right] - exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-t_k)\right] \tag{23.44} $$

and

$$ \Phi_{\alpha,1}^k = exp\left[-\frac{\alpha}{1-\alpha}(t_n-t_{k+1})\right] - exp\left[-\frac{\alpha}{1-\alpha}(t_n-t_k)\right] \tag{23.45} $$

6.7.6. Numerical simulation with local differentiation

As indicated in the previous section, the modified recharge equation accounts for an uncertain function. This section presents the numerical simulation which graphically signifies the effect of the uncertain function $u[t]$ on the distribution of the hydraulic head. For these simulations, theoretical $S = 0.002, DR = 200$, and $R = 40\%$ were used as constant values in order to signify the effect of $u[t]$. Here $u[t] = 1$ and $u[t] = Erf[t]$ were considered and depicted in **Figure 17** and **Figure 18**, respectively. The change in $u[t]$ clearly depicts the effect of the uncertain function on the distribution of hydraulic head. This means the uncertain function has a significant effect on hydraulic head distribution, which ultimately means $u[t]$ has a significant effect on groundwater recharge. Ultimately, this section indicates that by incorporating uncertainties (doubt) into the selected groundwater recharge equation as a function of time and space, there is a new and different approach for modelling groundwater recharge.

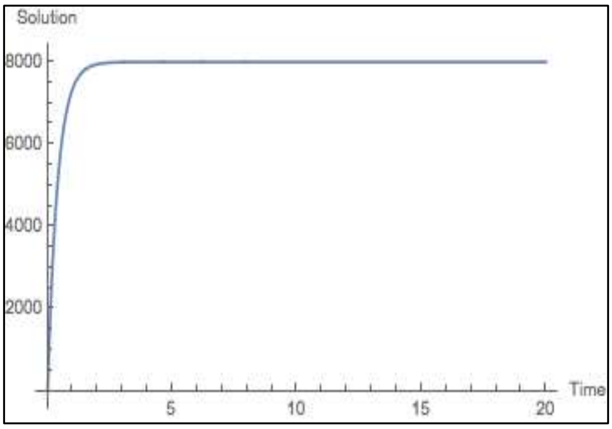

Figure 17: Numerical simulation for local derivative with u[t] =1

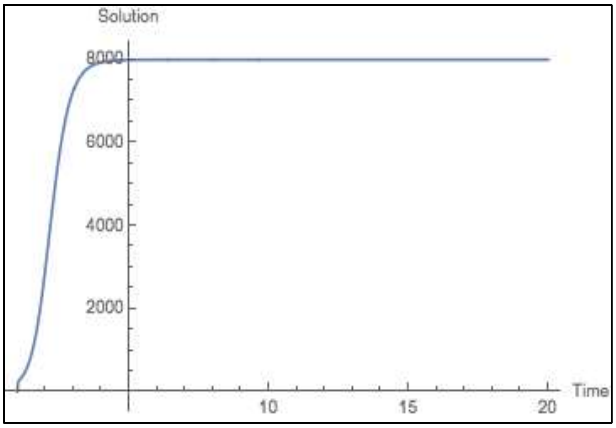

Figure 18: Numerical simulation for local derivative with u[t] =Erf[t]

CHAPTER 7: CONCLUSION

The aim of this study was to develop a new approach to recharge investigations by a new trend of differentiation due to the limitations of spatial and temporal variation in recharge. In addressing this aim, the uncertainty analysis revealed that minor changes in both storativity (S) and drainage resistance (DR) yields a considerable change in hydraulic head over time. This means that if field measurements, laboratory analysis and/or analytical models yield questionable data/information, the eventual recharge estimate will not be reliable. The statistical analysis revealed that both S and DR yields uncertainty in their distribution of hydraulic head over time. Although both parameters yielded uncertainty, DR appeared to have a greater extent of uncertainty and therefore error among the two. This means that when using this equation for recharge, considerable thought should be put into the accuracy of these parameters. Furthermore, the fractional differentiation and the numerical analysis of each exact solution infer three models suiting a different kind of geological formation. The model based on the generalised Mittag-Leffler law (Atangana-Baleanu fractional derivative) is more suitable for all classifications of geological formations, and these include homogeneous, heterogeneous, and viscoelastic subsurfaces. The model based on the exponential decay law (Caputo-Fabrizio) is suitable for heterogeneous subsurfaces, and finally the model based on the power law (Caputo fractional derivative) is suitable for elastic, homogeneous subsurfaces. Additionally, the Eton approach was used to incorporate an uncertain function, and results for this modification yields a linear equation which can be associated with non-constant parameters. Numerical analysis revealed that the distribution in hydraulic head is significantly dependent on this uncertain function. Ultimately, this approach to recharge estimation reveals fractional differentiation is yet again successfully used to model real world scenarios. In addition, it can be concluded that a model incorporating heterogeneity, viscoelasticity and the memory effect will generate a new understanding to groundwater recharge occurrence.

REFERENCES

Adams, S., Titus, R. & Xu, Y., 2004. *Groundwater Recharge Assessment of the Basement Aquifers of Central Namaqualand*, Bellville, South Africa: Water Research Commission.

Ali, F., Saqib, M., Khan, I. & Sheikh, N. A., 2016. Application of Caputo-Fabrizio derivatives to MHD free convection flow of generalized Walters'-B fluid model. *Eur. Phys. J. Plus.*, 131(377), pp. 1-10.

Alkahtani, B. S. T. & Atangana, A., 2016. Modeling the potential energy field caused by mass density distribution with Eton approach. *Open Phys*., 14, p. 106–113.

Alqahtani, R. T., 2016. Atangana-Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer. *J. Nonlinear Sci. Appl.*, Volume 9, pp. 3647-3654.

Antonellini, M. & Mollema, P. N., 2016. *Heterogeneities in fractured aquifers: Examples from outcrops and implications for fluid flow modelling*, Bologna, Italy: University of Bologna, Department of Biological, Geological and Environmental Sciences.

Atangana, A., 2014. A Generalized Advection Dispersion Equation. *Journal of Earth System Science*. 123(1), pp. 101-108.

Atangana, A., 2015. *Derivative with a New Parameter: Theory, Methods and Applications*, Bloemfontein, South Africa: Academic Press.

Atangana, A. & Alkahtani, B. S. T., 2015. Analysis of the Keller–Segel Model with a Fractional Derivative without Singular Kernel. *Entropy*, Volume 17, pp. 4439-4453.

Atangana, A. & Alqahtani, R. T., 2016. Numerical approximation of the space-time Caputo-Fabrizio fractional derivative andapplication to groundwater pollution equation. *Advances in Difference Equations*, Volume 156, pp. 1-13.

Atangana, A. & Koca, I., 2016. Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. *Chaos, Solitons and Fractals*, Volume 89, p. 447–454.

Atangana, A. & Van Tonder, G., 2014. *Stochastic Risk and Uncertainty Analysis for Shale Gas Extraction in the Karoo Basin of South Africa*, Bloemfontein, South Africa: Hindawi Publishing Corporation.

Baalousha, H., 2005. Using the CRD Method for Quantification of Groundwater Recharge in the Gaza Strip, Palestine. *Environ Geol*, Volume 48, pp. 889-900.

Baalousha, H., 2007. *Application of Automatic Differentiation in Groundwater Sensitivity Analysis*. MODSIM07 - Land, Water and Environmental Management: Integrated Systems for Sustainability, Proceedings, pp. 2728-2734.

Baalousha, H., 2009. Using Orthogonal Array Sampling to Cope with Uncertainty in Groundwater Problems. *Ground Water*, 47(5), pp. 709-713.

Baalousha, H. M., 2016. Using Monte Carlo Simulation to Estimate Natural Groundwater Recharge in Qatar. Model. *Earth Syst. Environ*, 2(87), pp. 1-7.

Baleanu, D., 2013. *Advances on Fractional Dynamics of Complex Systems*, Salve Regina University, Newport, United States

Bhattacharya, A. K., 2010. Artificial Groundwater Recharge With Special Reference to India. *International Journal of Recent Research and Applied Studies*, 4(2), pp. 215-221.

Bird, R., Lightfoot, E., Stewart, W., 2002. *Transport Phenomena*, Ney York, United States: John Wiley and Sons.

Bogena, H., Kunkel, R., Montzka, C. & Wendland, F., 2005. Uncertainties in the Simulation of Groundwater Recharge at Different Scales. *Advances in Geosciences*, Volume 5, pp. 25-30.

Borana, R. N., Pradhan, V. H. & Mehta, M. N., 2013. Numerical Solution of Boussinesq Equation Arising in One-Dimensional Infiltration Phenomenon by Using Finite Difference Method. *International Journal of Research in Engineering and Technology*, 2(8), pp. 202-209.

Bredehoeft, J. D., Neuzil, C. E. & Milly, P. C. D., 1983. *Regional Flow in the Dakota Aquifer: A Study of the Role of Confining Layers*. Alexandra, United States: U.S Geological Survey.

Bredenkamp, D. B., Van Tonder, G. J. & Van Rensburg, H. J., 1995. *Manual on Quantitative Estimation of Groundwater Recharge and Aquifer Storativity*, Pretoria, South Africa: Water Research Commission.

Bugan, R. D. H., Jovanovic, N. Z. & Dzikiti, S., 2011. *Groundwater Recharge: Accurately Representing Evapotranspiration*, Stellenbosch, South Africa: Council for Scientific and Industrial Research.

Callahan, T. P., 2011. *Non-Newtonian Fluid Injection into Granular Media. School of Civil and Environmental Engineering*, Atlanta, United States: Georgia Institute of Technology.

Cherkauer, D. S., 2004. Quantifying Groundwater Recharge at Multiple Scales Using PRMS and GIS. *Ground Water*, 42(1), pp. 97-110.

Cherry, A. J., 2000. A multi-tracer estimation of groundwater recharge in a glaciofluvial aquifer in southeastern Manitoba., Ottawa, Canada: *Ottawa-Carleton Geoscience Centre and University of Ottawa.*

Ciesielski, M. & Leszczynski, J., 2003. *Numerical simulations of anomalous diffusion*, Czestochowa, Poland: Institute of Mathematics and Computer Science, Technical University of Czestochowa.

Ci-qun, L., 1985. Nonsteady Flow of Groundwater in Aquifer Systems with Viscoelastic Properties. *Applied Mathematics and Mechanics*, 7(11), pp. 1027-1033.

Ci-qun, L., 1987. Unsteady flow of ground-water in an aquifer system with viscoelastic properties. *Transport in Porous Media*, 2(5), pp. 455-463.

Conrad, J., Nel, J. & Wentzel, J., 2007. *The Challenges and Implications of Assessing Groundwater Recharge: A Case Study – Northern Sandveld*, Western Cape, South Africa, Pretoria, South Africa: Department of Water Affairs and Forestry.

Cooper, J. D., 1980. *Measurement of Moisture Fluxes in Unsaturated Soil in Thetford Forest*, Norfolk and Suffolk, England: Institute of Hydrology.

Corapcioglu, Y., 1976. *Mathematical Modelling of Leaky Aquifers with Rheological Properties*, Ankara, Turkey: Middle East Technical University, pp. 191-200.

Dages, C. *et al*., 2009. Estimating the Role of a Ditch Network in Groundwater Recharge in a Mediterranean Catchment Using a Water Balance Approach. *Journal of Hydrology*, Volume 375, pp. 498-512.

De Vries, J. J., Selaolo, E. T. & Beekman, H. E., 2000. Groundwater Recharge in the Kalahari, with Reference to Paleo-Hydrologic Conditions. *Journal of Hydrology*, Volume 238, pp. 110-123.

De Vries, J. J. & Simmers, I., 2002. Groundwater recharge: an overview of processes and challenges. *Hydrogeology Journal* (2002) 10:5–17. pp. 1-13.

Delin, D. N., Healy, R. W., Lorenz, D. & Nimmo, J. R., 2007. *Comparison of Local to Regional-Scale Estimates of Ground-Water Recharge in Minnesota, USA*, Minnesota, United States: U.S. Geological Survey.

Dripps, W., Anderson, M. P. & Potter, K. W., 2001. *Temporal and Spatial Variability of Groundwater Recharge*, Madison, United States: University of Wisconsin Water Resources Institute.

DWAF, 2006. *Groundwater Resource Assessment II – Task 3aE Recharge*, Pretoria, South Africa: Department of Water Affairs and Forestry, pp. 1-85.

Dzieliński, A., Sarwas, G. & Sierociuk, D., 2011. Comparison and validation of integer and fractional order ultracapacitor models. *Advances in Difference Equations*, Volume 11, pp. 1-15.

Fan, J., Oestergaard, K. T., Guyot, A. & Lockington, D. A., 2014. Estimating Groundwater Recharge and Evapotranspiration from Water Table Fluctuations under Three Vegetation Covers in a Coastal Sandy Aquifer of Subtropical Australia. *Journal of Hydrology*, Volume 519, pp. 1120-1129.

Freeze, R. A. & Cherry, J. A., 1979. *Groundwater*, Englewood Cliffs, United States: Prentice-Hall, Inc.

Gebreyohannes, H. G., 2008. *Groundwater Recharge Modelling A case study in the Central Veluwe, The Netherlands*, Enschede, The Netherlands: International Institute for Geo-information Science and Earth Observation.

Gómez-Aguilar, J. F., Escobar-Jiménez, R. F., López-López, M. G. & Alvarado-Martínez, V. M., 2016. Atangana-Baleanu fractional derivative applied to electromagnetic waves in dielectric media. *Journal of Electromagnetic Waves and Applications*, pp. 1-17.

Gomo, M., G, S. & Van Tonder , G. J., 2012. Investigation of Groundwater Recharge and Stable Isotopic Characteristics of an Alluvial Channel. *Hydrology Current Research*, 1(12), p. 7.

Gong, H., Pan, Y. & Xu, Y., 2012. Spatio-temporal Varaition of Groundwater Recharge in Response to Variability in Precipitation, Land Use and Soil in Yanqing Basi, Beijing, China. *Hydrogeology Journal*, Volume 20, pp. 1331-1340.

Gribovszki, Z., Szilágyi, J. & Kalicz, P., 2010. Diurnal fluctuations in shallow groundwater levels and streamflow rates and their interpretation – A review. *Journal of Hydrology*, Volume 385, pp. 371 - 383.

Harter, T., 1994. *Unconditional and Conditional Simulation of Flow and Transport in Heterogeneous, Variably Saturated Porous Media*, Arizona, United States: The University of Arizona.

Harter, T. & Rollins, L., 2008. *Watersheds, Groundwater and Drinking Water: A Practical Guide*, California, United States: UCANR Publications.

Hashemi, H., Berndtsson, R., Kompani-Zare, M. & Persson, M., 2013. Natural vs. Artificial Groundwater Recharge, Quantification through Inverse Modelling. *Hydrology and Earth System Sciences*, Volume 17, pp. 637-650.

Healy, R. W., 2010. *Estimating Groundwater Recharge*, Cambridge, England: Cambridge University Press..

Hillenmeyer, M., 2005. *Applied Statistics*. Stanford, United States: Stanford University.

Khebchareon, M., 2012. Crank-Nicolson Finite Element for 2-D Groundwater Flow, Advection-Dispersion and Interphase Mass Transfer: I. Model Development. International *Journal of Numerical Analysis and Modeling.*, 3(2), pp. 109-125.

Krásný, J. & Sharp, J. M., 2007. *Groundwater in Fractured Rocks: selected papers from the Groundwater in Fractured Rocks International Conference*, Prague 2003: Taylor & Francis/Balkema, AK Leiden, The Netherlands..

Kreith, F. *et al*., 1999. *Mechanical Engineering Handbook. In: Fluid Mechanic. Boca Raton*, United States: CRC Press LLC, pp. 1-208.

Kresic, N., 2007. *Hydrogeology and Groundwater Modelling*, 2nd edition, New York, United States: CRC Press.

Laderkarl, U. L. *et al*., 2005. Groundwater Recharge and Evapotranspiration for two Natural Ecosystems Covered with Oak and Heather. *Journal of Hydrology*, Volume 300, pp. 76-99.

Lerner, D. N., Issar, A. S. & Simmers, I., 1990. Groundwater Recharge. A Guide to Understanding and Estimating Natural Recharge. *International Contributions to Hydrogeology*, Volume 8, p. 42.

LeVeque, R. J., 2005. *Finite Difference Methods for Differential Equations*, St. Louis, Missouri, United States: University of Washington.

Liu, Y. & Zhang, C., 1993. A Comparitive Study of the Calculation Methods for Recharge of Rainfall Seepage to Groundwater Water in Plain Area. *Ground Water*, 31(1), pp. 12-18.

Mackay, D. J. C., 1998. *Introduction to Monte Carlo Methods*, Cambridge, United Kingdom: Cambridge University.

Manghi, F., Mortazavu, B., Crother, C. & Hamdi, M. R., 2009. Estimating Regional Groundwater Recharge Using a Hydrological Budget Method. *Water Resour Manage*, Volume 23, p. 2475–2489.

Misstear, B. D. R., 2000. *Groundwater Recharge Assessment: a Key Component of River Basin Management*, Dublin, Ireland: Department of Civil, Structural & Environmental Engineering, Trinity College Dublin.

Miyazaki, T., Ibrahimi, M. K. & Nishimura, T., 2012. Shallow Groundwater Dynamics Controlled by Lisse and Reverse Wieringermeer Effects. *Journal of Sustainable Watershed Science & Management*, 1(12), pp. 36-45.

Nanjaro, G., Cruz-Fuentes, T., del Carmen Cabrera , M. & Custodio, E., 2015. Estimating Natural Recharge by Means of Chloride Mass Balance in a Volcanic Aquifer: Northeastern Gran Canaria (Canary Islands, Spain). *Water*, 7(6), 2555-2574.

Nolan, B. T. *et al*., 2006. *Factors Influencing Groundwater Recharge in the Eastern United States*, Reston, United States: Elsevier.

Nyende, J., van Tonder, G. & Vermeulen, D., 2013. Application of Isotopes and Recharge Analysis in Investigating Surface Water and Groundwater in Fractured Aquifer under Influence of Climate Variability. *J Earth Sci Clim Change*, 4(4), p. 148.

Patankar, S., 1980. *Numerical Heat Transfer and Fluid Flow*, Saint Paul, United States: CRC Press.

Prathapar, S. A. & Sides, R. D., 1993. *A Practical Guide for Estimating Recharge from Water Table Hydrographs*, Canberra, Australia: Commonwealth Scientific and Industrial Research Organisation.

Rushton, K. R., 2004. *Groundwater Hydrology: Conceptual and Computational Models*, Chichester, England: John Wiley & Sons Ltd.

Scanlon, B. R. & Dutton, A., 2003. *Groundwater Recharge in Texas, Lawrence*, Texas: Kansas Geological Survey.

Scanlon, B. *et al*., 2011. *Estimation of Groundwater Recharge to the Gulf Coast Aquifer in Texas*, Sioux Falls, United States: U.S Geological Survey,

Scanlon, B. R. *et al*., 2006. *Global Synthesis of Groundwater Recharge in Semiarid and Arid Regions*, Texas, United States: John Wiley & Sons, Ltd.

Seiler, K. P. & Gat, J. R., 2007. *Groundwater Recharge from Run-off, Infiltration and Percolation*. Volume 55 ed. Dordrecht, The Netherlands: Springer.

Shaban, A., Khawlie, M. & Abdallah, C., 2005. Use of Remote Sensing and GIS to Determine Recharge Potential Zones: The Case of Occidental Lebanon. *Hydrogeology Journal*, Volume 14, pp. 433-443.

Shahraiyni, H. T. & Ataie-Ashtiani, B., 2012. Mathematical Forms and Numerical Schemes for the Solution of Unsaturated Flow Equations. *Journal of Irrigation and Drainage Engineering*, pp. 63-72.

Shamsudduha, M., Taylor, R. G., Ahmed, K. M. & Zahid, A., 2010. The Impact of Intensive Groundwater Abstraction on Recharge to a Shallow Regional Aquifer System: Evidence from Bangladesh. *Hydrogeology Journal*, Volume 19, pp. 901-916.

Simmers, I., 1987. *Estimation of Natural Groundwater Recharge*, Antalya, Turkey: D. Reidel Publishing Company.

Simmers, I., 2005. *Understanding Water in a Dry Environment: IAH International Contributions to Hydrogeology 23*. Print ed. Amsterdam, The Netherlands: Taylor & Francis.

Simmers, L., 2013. *Estimation of Natural Groundwater Recharge*, Amsterdam, The Netherlands: Springer.

Singhai, B. B. & Gupta, R. P., 1999. *Applied Hydrogeology of Fractured Rocks,* 1st ed, Roorkee, India: Springer Science & Business Media.

Singhal, B. B. S. & Gupta, R. P., 2010. *Applied Hydrogeology of Fractured Rocks,* 2nd ed, Dordrecht, Heidelberg, London, New York: Springer.

Singh, S., Kaushal, S., Siag, M. & Yadav, A., 2005. Studies on Groundwater Recharge through Surface Drains. *Water SA*, 31(2), pp. 151-156.

Sochi, T., 2009. *Single-Phase Flow of Non-Newtonian Fluids in Porous Media*, London, England: University College London.

Somaratne, N. & Smettem, R. J., 2013. *Theory of the Generaliszed Chloride Mass Balance Method for Recharge Estimation in Groundwater Basins Characterised by Point and Diffuse Recharge*, Crawley, Western Australia: Copernicus Publications.

Sun, X. *et al*., 2013. Application of the rainfall infiltration breakthrough (RIB) model for groundwater recharge estimation in west coastal South Africa. *Water SA*, 39(2), pp. 221-230.

Tegeri, G. G., 2009. *Groundwater Resource Assessment of the Aynalem Wellfield through Transient Flow Modelling (Mekele, Ethiopia)*, Enschede, The Netherlands: International Institute for Geo-information Science and Earth Observation.

Toure, A., Diekkruger, B. & Mariko, A., 2016. Impact of Climate Change on Groundwater Resources in the Klela Basin, Southern Mali. *Hydrology*, 3(17), pp. 1-17.

USDA, 1967. *Groundwater Recharge,* United States: Department of Agriculture.

Van Rooyen, J. M., 2014. *A Methodology to Quantify the Risks of Urbanisation on Groundwater Systems in South Africa,* Potchefstroom: North-West University.

Van Tonder, G. & Xu, Y., 2000. *A Guide for Estimation of Groundwater Recharge in South Africa*, Bloemfontein, South Africa: University of the Freestate.

Van Wyk, E., 2011. *Estimation of Episodic Groundwater Recharge in Semi-Arid Fractured Hard Rock Aquifers*, Bloemfontein: Department of Geohydrology, University of the Freestate.

Varni, M., Comas, R., Weinzettel, P. & Dietrich, S., 2013. Application of the Water Table Fluctuation Method to Characterize Groundwater Recharge in the Pampa Plain, Argentina. *Hydrological Sciences Journal*, 58(7), pp. 1444-1455.

Vincent, J., 2012. *Chapter One: Basic Elasticity and Viscoelasticity,* 3rd ed. Structural Biomaterials. Princeton, United States: Princeton University Press, pp. 1-28.

Wang, B. *et al.*, 2008. Estimating Groundwater Recharge in Hebei Plain China under Varying Land Use Practices Using Tritium and Bromide Tracers. *Journal of Hydrology*, Volume 356, pp. 209-222.

WRC, 2015. Drought management – Strengthening our knowledge armoury. Water Wheel Magazine, November/December, p. 4.

Wu, Y., 2005. *Groundwater Recharge Estimation in Table Mountain Group Aquifer Systems with a Case Study of Kammanassie Area*, Cape Town, South Africa: The University of the Western Cape.

Wyss, G. D. & Jorgensen, K. H., 1998. *A User's Guide to LHS: Sandia's Latin Hypercube Sampling Software*, Albuquerque, United States: Risk Assessment and Systems Modeling Department, Sandia National Laboratories.

Xu, Y. & Beekman, H. E., 2003. *Groundwater Recharge Estimation in Southern Africa*, Cape Town, South Africa: United Nations Educational, Scientific and Cultural Organization (UNESCO).

Yadav, A., Sonje, A., Mathur, P. & Jain, D. A., 2012. A Review of Artificial Groundwater Recharge. International *Journal of Pharma and Bio Sciences*, 3(3), pp. 304-311.

Yeh, H. F., Lee, C. H., Hsu, K. C. & Chaunge, P. H., 2008. GIS for the Assessment of the Groundwater Recharge Potential Zone. *Environ Geol*, Issue 58, pp. 185-195.

Zhen-min, Z., 2011. Application of Monte Carlo Method in Recharge Calculation of Underground Water Resources. *Procedia Engineering*, Volume 23, pp. 316-319.