PARAMETRIC AND NONPARAMETRIC BAYESIAN STATISTICAL INFERENCE IN ANIMAL SCIENCE

by

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It is imperative that I begin by thanking Prof Abrie for setting such a patient and supportive example for me that I try to emulate. Shared visions of utilizing Bayesian Inference in Animal Breeding turned our ideas into the present PhD thesis. I know that the road to success is neither easy nor straight...

"There are curves called referees...
Loops called uncertainty...
Speed bumps called time...
You will have flats called disappointments...
BUT
If you have a driver called God...
Insurance called Faith...
Caution lights called Family and Friends...
An engine called Perseverance and...
Spares called Determination and Persistence,
You will make it to a place called Success!"

Indeed, I know now that nothing in the world can take the place of persistence. Talent will not; nothing is more common than unsuccessful men with talent. Genius will not; unrewarded genius is almost a proverb. Education will not; the world is full of educated derelicts;

Determination and Persistence alone are Omnipotent.

In Memory of my late Mom
(1942-1993)

CONTENTS

CHAPTER 1

«Bayesian Statistics and Animal Breeding Theory»

Introductory words: *Harville, (1990), (see also Gianola, (1990)) stated "A more extensive use of Bayesian ideas by animal breeders and other practitioners is desirable and is more feasible from a computational standpoint than commonly thought. The Bayesian approach can be used to devise prediction procedures that are more sensible – from both a Bayesian and Frequentist perspective – than those in current use". The Bayesian approach is also conceptually more appealing than the Classical approach.*

1.1 Prologue

Animal breeding theory deals with the formulation and validation of mathematical (primarily statistical) models aimed at developing procedures for selecting and mating individuals so that performance is optimal in some sense. The primary goal of such selection experiments in animal breeding is to identify animals to use for producing the next generation of progeny in order to maximize genetic progress with respect to traits of interest. Examples of traits that have been subject to such selection experiments are milk yield in diary cows, rapid weight gain in pigs, and weaning weights in lambs.

A bold improvement in genetic evaluation occurred when mixed model methodology was introduced. Familial relationships between sires enhanced the accuracy at which breeding values were estimated because the effective separation of genetic merit from environment effects became possible. In animal breeding experiments, the observed trait values, or phenotypes, are modeled as the sum of a number of effects, including individual breeding values. Also, the breeding values are modeled as correlated random effects, with the correlation arising due to known genetic relationships. To maximize future progress of a population, the goal is to identify the animals with the highest breeding values.

Genetic evaluation of South African flocks of sheep started with the analysis of the experimental Merino flock at Klerefontein near Carnavon. This was followed by single flock evaluation as part of a variety of postgraduate studies and the evaluation of progeny groups of rams for the industry (Van Wyk, 1992). The Dormer sheep stud, started at the Elsenburg College of Agriculture in 1940, also represents such a flock for animal breeding experiments. The main object in developing the Dormer was the establishment of a mutton sheep breed which would be well adapted to the conditions prevailing in the Western Cape (winter rainfall) and which could produce the desired type of ram for crossbreeding purposes (Swart 1967).

Only a small example from this Dormer stud data was used. It was not our attention to reanalyze the data from a genetic point of view; rather, we used it to illustrate how the Bayesian approach and Gibbs sampling could be applied to real animal breeding problems. This data form an integrated part of the statistical methods introduced in the thesis and can be found in APPENDIX B.

1.2 The Mixed Model Methodology

1.2.1 Background

The mixed model methodology was first developed for animal genetics and breeding research. In resent years, however, the mixed model has also been introduced in variety of other disciplines (e.g. sociology and education) to analyze experiments with more complex data structures. These mixed models are also called ***Hierarchical Models, Random Effects Models or Variance Components Models***. As the mixed model methodology is heavily based on matrix notation, it is important that a clear notation is used in the development of the theory in the present thesis.

1.2.2 Notation

§ A matrix is always put in a bold letter, and a vector in a bold underlined letter, e.g. $\mathbf{Y}$ is the matrix of observations, whereas $\underline{\mathbf{Y}}$ presents the vector of observations. Greek letters are used for the fixed and random effect vectors and the letters are not underlined, e.g. β is a vector of fixed effects, and γ the vector of random effects.

§ The transpose of a matrix $\mathbf{X}$ or vector $\underline{\mathbf{Y}}$ is denoted by $\mathbf{X}'$ or $\underline{\mathbf{Y}}'$, respectively.

§ The inverse of a matrix $\mathbf{X}$ is denoted by $\mathbf{X}^{-1}$.

§ The generalized inverse of a matrix $\mathbf{X}$ is denoted by $\mathbf{X}^{-}$.

The mixed linear model postulates that the observable random vector $\underline{\mathbf{Y}}$ is a linear combination of the fixed effects and random effects plus a random error (residual). In its simplest form the univariate mixed linear model can be written in matrix notation as

$$\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon. \tag{1.1}$$

$\underline{\mathbf{Y}}$ ($n \times 1$) is a vector of observed values for the trait on which selection is desired. Only a single trait per animal is considered for most of the analysis, although the model and analysis described can be modified to accommodate multiple traits.

β ($p \times 1$) is a vector of fixed effects uniquely defined so that the corresponding design matrix $\mathbf{X}$ ($n \times p$) has full column rank, p. Loosely speaking, a fixed effect, in a Bayesian sense, is a random variable on which prior knowledge is diffuse or vague, i.e. a priori the investigator is indifferent to its likely value.

Furthermore, γ ($q \times 1$) is a vector of unobservable random effects with $\gamma \sim N\left(\underline{\mathbf{0}}, \mathbf{A}\sigma_\gamma^2\right)$ and design matrix $\mathbf{Z}$ ($n \times q$). σ_γ^2 is an unknown scalar and $\mathbf{A}$ ($q \times q$) is called a relationship (genetic covariance) matrix. Its elements reflect the genetic relationship among the sires. The random effects in the present case, are the breeding values, which account for the variation in $\underline{\mathbf{Y}}$ due to genetic merit.

Note that, in the case of a ***Sire Model***, a breeding value refers to a sire's value as a parent in a breeding program, and it is a measure of the animal's progeny performance relative to the mean value of its breed. Genetic evaluation is heavily dependent on the genetic correlation among individuals, both for higher accuracy and for unbiased results. Therefore, the genetic relationship among individuals is of fundamental importance in the prediction of breeding values.

For the unobservable vector of random errors $\varepsilon\,(n \times 1)$, statistical independent of γ, it is common to assume independent normal distributions with mean vector $\underline{0}$ and variance-covariance matrix $\sigma_\varepsilon^2 I_n$ $\mathbf{I}_n$. $\mathbf{I}_n$ represents an $n \times n$ identity matrix and σ_ε^2 an unknown scalar.

As mentioned, σ_γ^2 and σ_ε^2 are unknown scalar-value parameters called variance components.

1.3 The Classical Solution

Data from animal breeding experiments are commonly analyzed using a mixed linear model in order to estimate or predict the breeding values of individual animals. When the values of the variance components of the model are not known, the ***Classical Approach*** to the problem of predicting linear combinations of the different effects has been to estimate the variance components and to proceed thereafter as if these estimates were the true values of the components.

Patterson and Thompson (1971) developed a method to derive unbiased estimates of the unknown variance components based on the maximum likelihood principle; called the ***Restricted Maximum Likelihood Estimation (REML)***. This method is based on the likelihood of a vector whose components are independent linear combinations of the observations. The basic idea is to end up with a random vector that contains all the information on the variance components but no longer contains information on the fixed effects parameters. However, there are several problems with this (classical) approach.

1. The properties of the predictors are hard to assess. This is particularly the case when estimates of variance components are substituted for their true values.

2. When the values of the variance components are estimated from the data, the sampling errors are generally not taken into account in the subsequent analysis. Therefore, the variance of the prediction error will generally be underestimated.

3. Depending upon the size and characteristics of the data, point estimates of the variance components can be highly variable. For certain values of the components estimates, the predictors obtained by substituting these values in the "Best Linear Unbiased Predictor" are intuitively unappealing.

The classical frequentist solution of the mixed linear model (1.1) can be obtained from Henderson's mixed model equations. Henderson, in Henderson *et al.* (1959), developed a set of equations that simultaneously yielded best linear unbiased predictors of the random effects and best linear unbiased estimators of the fixed effects. They were derived by maximizing the joint density of $\underline{\mathbf{Y}}$ and γ, i.e.

$$\begin{aligned} f(\underline{\mathbf{Y}},\gamma) &= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}\gamma)'(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}\gamma)\right\} \\ &\times \left(\frac{1}{2\pi\sigma_\gamma^2}\right)^{\frac{q}{2}} |\mathbf{A}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\gamma^2}\gamma'\mathbf{A}^{-1}\gamma\right\}. \end{aligned} \tag{1.2}$$

Equating to zero the partial derivation of (1.2) with respect to elements, first of β and then of γ give the mixed model equations, written more compactly as

$$\begin{bmatrix} \mathbf{X'X} & \mathbf{X'Z} \\ \mathbf{Z'X} & \mathbf{Z'Z}+\frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\mathbf{A}^{-1} \end{bmatrix} \begin{bmatrix} \hat\beta \\ \hat\gamma \end{bmatrix} = \begin{bmatrix} \mathbf{X'}\underline{\mathbf{Y}} \\ \mathbf{Z'}\underline{\mathbf{Y}} \end{bmatrix}, \tag{1.3}$$

where $\hat{\beta}$ and $\hat{\gamma}$ denote the solutions of β and γ. By substituting the REML estimates of σ_ε^2 and σ_γ^2 in equation (1.3) give the classical frequentist solution to the mixed linear model.

Hence, the objective is now to propose the ***Bayesian Approach*** as a conceptional strategy to solve problems arising in animal breeding theory, to illustrate how well known results can be retrieved from the Bayesian perspective, and to suggest possible areas of research in which ***Bayesian Approach*** and ***Mixed Linear Model Methodology*** can lead to fruitful results.

1.4 The Traditional Bayesian Solution

Harville (1990), (see also Gianola, (1990)) stated, "A more extensive use of Bayesian ideas by animal breeders and other practitioners is desirable and is more feasible from a computational standpoint than commonly thought. The Bayesian approach can be used to devise prediction procedures that are more sensible – from both a Bayesian and Frequentist perspective – than those in current use". The Bayesian approach is also conceptually more appealing than the Classical approach with the following advantages:

1. The Bayesian practitioner does not need to commit himself to a point estimate of the variance components in order to obtain a point predictor for the variables of interest, and credibility intervals can easily be obtained.

2. Uncertainty about the true values of the variance components is formally incorporated into the analysis through the choice of an appropriate prior distribution.

3. Given the data, the prior information about the unknown parameters, and a well-defined loss function, there exists an optimal Bayes predictor.

4. All the available information about the random variable to be predicted is contained in the posterior distribution of the random variable. Therefore, the practitioner can base all of his inferences on this distribution.

5. The Bayesian approach is conceptually more appealing than the classical approach.

Critics of the Bayesian approach have most often cited the following points:

1. The Bayesian practitioner must formally express his prior beliefs about the unknown parameters in the form of a probability distribution.

2. The Bayesian methodology is computer intensive. In many situations, integrations in several dimensions are required to obtain the required posterior distributions.

These might have been valid criticisms in the past but by using (a) Non-Informative priors like Jeffreys and Reference priors and (b) Numerical integration techniques like Markov Chain Monte Carlo Methods and more specifically Gibbs Sampling, these problems can be overcome.

When analyzing the mixed linear model (or any model) using a Bayesian approach, it only matters whether a specified quantity is observable or not. In equation (1.1), $\underline{\mathbf{Y}}$, $\mathbf{X}$ and $\mathbf{Z}$ are observable whilst, β and γ are unobservable. No further classifications are necessary.

In the classical approach to analysis of data using a mixed linear model the distinctions of fixed versus random, known versus unknown, parameter versus statistic, are all-important. These classifications dictate the type of estimation and inference that are possible. In Bayesian modeling we treat β, γ and all the variance components in the same way: they are unobservable.

In many Bayesian problems, marginal posterior distributions are often needed to make appropriate inferences. However, due to the complexity of the joint posterior distribution it is impossible to obtain these marginal densities analytically and because of the many unknowns, very difficult to calculate numerically. Instead, a Markov Chain Monte Carlo (MCMC) method, called ***Gibbs Sampling***, will be implemented to estimate the marginal posterior densities of the different parameters.

Recently due to the work by Gelfand and Smith (1990), Gelfand *et al.* (1990), Carlin *et al.* (1992) and Gelfand *et al.* (1992), the ***Gibbs Sampler*** has been shown as an useful tool for applied Bayesian inference in a broad variety of statistical problems. The Gibbs sampler is implicit in the work of Hastings (1970) and made popular in the image-processing context by Geman and Geman (1984).

The Gibbs sampler is an adaptive Monte Carlo integration technique. The typical objective of the sampler is to collect a sufficiently large enough number of parameter realizations from conditional posterior densities in order to obtain accurate estimates of the marginal posterior densities. The principle requirement of the sampler is that all conditional densities must be available in the sense that random variables can be generated from them. Once the marginal densities are obtained, it is easy to calculate summary statistics from the posterior distributions.

The method is of great appeal on account of its simple logical foundation and reasonable ease of implementation. The next section elaborates the role of the sampler in relating conditional and marginal distributions from animal breeding theory.

1.5 Prior Distributions

For modelling the hierarchy, the distribution of ε gives the sampling distribution, which, in classical statistics, is the distribution of the data conditional on all the parameters. In a Bayesian analysis this distribution is called the likelihood function and it is always the first stage in a traditional Bayesian analysis with prior distributions relegated to other stages.

From (1.1) it follows that the conditional distribution that generates the data (likelihood function) is

$$\underline{\mathbf{Y}} \mid \beta, \gamma, \sigma_\varepsilon^2 \sim N_n\left(\mathbf{X}\beta + \mathbf{Z}\gamma, \mathbf{I}_n\sigma_\varepsilon^2\right) \tag{1.4}$$

where $\mathbf{I}_n$ represents an $n \times n$ identity matrix and $N(\mu, \Sigma)$ denotes the n-dimensional multivariate normal distribution with mean vector μ and variance-covariance matrix Σ.

An integral part of Bayesian analysis is now the assignment of a prior distribution to the unknown parameters in the model. The information contained in the prior distribution is combined with the information supplied by the data, through the likelihood function (if it is known), into the conditional posterior distribution of the parameters given the data, which is known as the posterior distribution. All inferences about the model parameters are based on the posterior distribution. In the above model, *"flat"* or uniform prior distributions are assigned to the vector of fixed effects and error variance, as to represent lack of prior knowledge.

Therefore

$$p(\beta, \sigma_\varepsilon^2) = p(\beta)\, p(\sigma_\varepsilon^2) \propto \text{constant}. \tag{1.5}$$

Further, the prior distribution of the vector γ is given by

$$\gamma \mid \mathbf{A}, \sigma_\gamma^2 \sim N_q(\underline{\mathbf{0}}, \mathbf{A}\sigma_\gamma^2). \tag{1.6}$$

As mentioned in the case of the sire model the elements of the numerator relationship matrix **A** describe the covariance of the sires due to shared genes, and γ is the vector of breeding values. Also

$$p(\sigma_\gamma^2) \propto \text{constant}. \tag{1.7}$$

1.6 Joint and Full Conditional Posterior Distributions

The joint posterior distribution of the unknowns (β, γ, σ_ε^2, σ_γ^2) is proportional to the product of the likelihood function, and the joint prior distribution is given by

$$p(\beta,\gamma,\sigma_\varepsilon^2,\sigma_\gamma^2 \mid \mathbf{D}) \propto \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}\gamma\right)'\left(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}\gamma\right)\right\}$$

$$\times\left(\frac{1}{\sigma_\gamma^2}\right)^{\frac{q}{2}} \exp\left\{-\frac{1}{2\sigma_\gamma^2}\gamma'\mathbf{A}^{-1}\gamma\right\}, \qquad (1.8)$$

where $\mathbf{D} = (\underline{\mathbf{Y}}, \mathbf{X})$ denotes the data.

The required full conditional for the fixed effects, is multivariate normal:

$$\beta \mid \gamma, \sigma_\varepsilon^2, \sigma_\gamma^2, \mathbf{D} \sim N_p(\hat{\beta}, (\mathbf{X}'\mathbf{X})^{-1}\sigma_\varepsilon^2), \qquad (1.9)$$

where $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\underline{\mathbf{Y}} - \mathbf{Z}\gamma)$. Note that this distribution does not depend on σ_γ^2.

The conditional distribution of γ is also multivariate normal:

$$\gamma \mid \beta, \sigma_\varepsilon^2, \sigma_\gamma^2, \mathbf{D} \sim N_q\left\{\gamma^*, \left(\mathbf{Z}'\mathbf{Z} + \mathbf{A}^{-1}\frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\right)^{-1}\sigma_\varepsilon^2\right\} \qquad (1.10)$$

where

$$\gamma^* = \left(\mathbf{Z}'\mathbf{Z} + \mathbf{A}^{-1}\frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\right)^{-1}\mathbf{Z}'(\underline{\mathbf{Y}} - \mathbf{X}\beta).$$

For the variance components, the conditionals are

$$p(\sigma_\varepsilon^2 \mid \beta,\gamma,\sigma_\gamma^2,\mathbf{D}) = K_\varepsilon\left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}\gamma)'(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}\gamma)\right\}$$

$$\sigma_\varepsilon^2 > 0,$$

$$(1.11)$$

an Inverse Gamma density where

$$K_\varepsilon = \left\{ \frac{(\underline{\mathbf{Y}} - \mathbf{X}\beta - \mathbf{Z}\gamma)'(\underline{\mathbf{Y}} - \mathbf{X}\beta - \mathbf{Z}\gamma)}{2} \right\}^{\frac{n-2}{2}} \frac{1}{\Gamma(\frac{n-2}{2})},$$

and

$$p(\sigma_\gamma^2 \mid \beta, \gamma, \sigma_\varepsilon^2, \mathbf{D}) = K_\gamma \left(\frac{1}{\sigma_\gamma^2} \right)^{\frac{q}{2}} \exp\left\{ -\frac{1}{2\sigma_\gamma^2} \gamma' \mathbf{A}^{-1} \gamma \right\} \qquad \sigma_\gamma^2 > 0 \tag{1.12}$$

also an Inverse Gamma where

$$K_\gamma = \left\{ \frac{\gamma' \mathbf{A}^{-1} \gamma}{2} \right\}^{\frac{q-2}{2}} \frac{1}{\Gamma\left(\frac{q-2}{2}\right)}.$$

Moreover, animal breeders are often interested in the posterior distributions of functions of variance components like the intraclass correlation coefficient, $\rho = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$ and the variance ratio $v = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$. The conditional posterior distributions of these parameters can be obtained by making use of transformation of random variables. For example, the conditional posterior distribution of ρ can be obtained by using the transformation of $\sigma_\gamma^2 \to \rho$ in the conditional posterior density of σ_γ^2, equation (1.12).

Since the Jacobian of the transformation, $\dfrac{\partial \sigma_\gamma^2}{\partial \rho} = \dfrac{\sigma_\varepsilon^2}{(1-\rho)^2}$, equation (1.12) can be written as

$$p(\rho \mid \beta, \gamma, \sigma_\varepsilon^2, \mathbf{D}) = K_\gamma \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{q}{2}-1} \left(\frac{1}{\rho}\right)^{\frac{q}{2}} (1-\rho)^{\frac{q-4}{2}} \exp\left\{-\frac{(1-\rho)}{2\sigma_\varepsilon^2}\gamma' \mathbf{A}^{-1}\gamma\right\} \qquad (1.13)$$

$$0 < \rho < 1,$$

and for the variance ratio,

$$p(v \mid \beta, \gamma, \sigma_\varepsilon^2, \mathbf{D}) = K_\gamma \left(\frac{1}{v\sigma_\varepsilon^2}\right)^{\frac{q}{2}} \exp\left\{-\frac{1}{2v\sigma_\varepsilon^2}\gamma' \mathbf{A}^{-1}\gamma\right\} \qquad (1.14)$$

$$0 < v < 1.$$

It is clear from equations (1.9) – (1.14) that account of the genetic covariance matrix has been taken into the conditional posterior distributions.

1.7 The Gibbs Sampler

1.7.1 Background

The Gibbs sampler enjoyed an initial surge popularity starting in 1984 with Geman and Geman, who studied image-processing models. Gelfand and Smith (1990) then put the sampler in a new light, revealing its potential in a wide variety of conventional statistical problems. It is characterized by always using full conditionals, however, other sets of conditionals may also be used, sets which are also sufficient to determine joint distributions. The ultimate value of the Gibbs sampler lies in its practical potential. Now that the groundwork has been laid in pioneering research work, the present research is focused on exploring and expanding the Gibbs sampler using mixed linear model methodology to animal breeding problems.

The Gibbs sampler is a technique for generating random variables from a marginal distribution indirectly, without having to calculate the density. In that which follows, it is easy to see that the Gibbs sampler is iterative and based only on elementary properties of Markov chains. In this respect, there are two issues of concern: convergence and uniqueness. However, Geman and Geman (1984) showed that under mild regularity conditions, the Gibbs sampler converges uniquely to the appropriate marginal distributions. Casella and George (1990) discuss numerical means to accelerate convergence. Another way of speeding up convergence is to integrate out analytically some nuisance parameters from the joint posterior distribution before running the Gibbs sampler.

1.7.2 Illustrating the Gibbs Sampler

Suppose we are given a joint density $f(x,y_1,y_2,\dots,y_n)$ and are interested in obtaining characteristics of the marginal density

$$f(x) = \int\dots\int f(x,y_1,y_2,\dots,y_n)dy_1d_2\dots dy_n \qquad (1.14)$$

such as the mean or variance. Probably the most natural and straightforward approach would be to calculate $f(x)$ and use it to obtain the desired characteristic. However, there are many cases where the integration in (1.14) is extremely difficult to perform, either analytically or numerically. In such cases the Gibbs sampler provides an alternative method for obtaining $f(x)$, i.e. to generate a Markov chain of random variables (also called a "Gibbs sequence") that converge to the distribution of interest $f(x)$.

Rather than compute or approximate $f(x)$ directly, the Gibbs sampler allows us effectively to generate a sample $X_1, X_2, \ldots, X_m \sim f(x)$ *without requiring* $f(x)$. By simulating a large enough sample, the mean, variance, or any other characteristic of $f(x)$ can be calculated to the desired degree of accuracy. It is important to note that, in effect, the end results of any calculations, although based on simulations, are the population quantities. Thus by taking a large enough sample, any population characteristic, even the density itself, can be obtained by averaging the final conditional densities from each Gibbs sequence. These estimates are called Rao-Blackwell estimates (Gelfand & Smith, 1990). An alternative form of estimating the marginal posterior densities is by obtaining kernel density estimators; however, the Rao-Blackwell estimates are more accurate.

To understand the working of the Gibbs sampler, we explore it in the three-variable case. The initial values $Y_1^{(0)} = y_1^{(0)}$, $Y_2^{(0)} = y_2^{(0)}$ and $Y_3^{(0)} = y_3^{(0)}$ are specified and the rest of the Gibbs sequence of random variables is obtained iteratively by alternately generating values in the following way:

Draw

$$x^{(1)} \text{ from } f(\,x\,/\,y_1^{(0)}, y_2^{(0)}, y_3^{(0)}\,)$$

then

$$y_1^{(1)} \text{ from } f(\,y_1\,/\,x^{(1)}, y_2^{(0)}, y_3^{(0)}\,),$$

also draw

$$y_2^{(1)} \text{ from } f(\,y_2\,/\,x^{(1)}, y_1^{(1)}, y_3^{(0)}\,)$$

and

$$y_3^{(1)} \text{ from } f(\,y_3\,/\,x^{(1)}, y_1^{(1)}, y_2^{(1)}\,).$$

This completes one iteration of the scheme. Thus, at the k^{th} iteration we draw

$$x^{(k)}. \text{ from } f(\,x\,/\,y_1^{(k\text{-}1)}, y_2^{(k\text{-}1)}, y_3^{(k\text{-}1)}\,)$$

then

$$y_1^{(k)} \text{ from } f(\,y_1\,/\,x^{(k)}, y_2^{(k\text{-}1)}, y_3^{(k\text{-}1)}\,)$$

then

$$y_2^{(k)} \text{ from } f(\,y_2\,/\,x^{(k)}, y_1^{(k)}, y_3^{(k\text{-}1)}\,)$$

then

$$y_3^{(k)} \text{ from } f(\,y_3\,/\,x^{(k)}, y_1^{(k)}, y_2^{(k)}\,).$$

Geman and Geman (1984) have shown that under fairly general conditions, the distribution of $x^{(k)}$ converges to $f(x)$ (the true marginal distribution of x) as k nears infinity. Thus, the value $x^{(k)}$ can be regarded as a simulated observation from $f(x)$ if k is large enough. By repeating the Gibbs sequence m times, the Gibbs sampler generates m observations

$$X_1^{(k)}, \ldots, X_m^{(k)}.$$

If the repetitions are independent, using predetermined initial values $y_1^{(0)}$, $y_2^{(0)}$ and $y_3^{(0)}$ for each sequence, the final values will be independent. Thus, by simulating a large enough sample, characteristics such as the mean and variance of $f(x)$ can be determined to the desired degree of accuracy (Van der Merwe & Botha, 1993). Characteristics of $f(y_1)$, $f(y_2)$ and $f(y_3)$ can be obtained in a similar way. It is important to remember that to generate m random variables with approximate density $f(x)$, we have to generate $(2k) \times m$ random variables, where k is the length of each Gibbs sequence (Casella & George, 1990).

In the light of the aforementioned, the Gibbs sampler can thus be thought of as a practical implementation of the fact that knowledge of the conditional distributions is sufficient to determine a joint distribution, if it exists. In the Markov Chain Monte Carlo procedure and more specifically Gibbs sampling, we construct a stochastic process that has the desired posterior distribution as its stationary distribution and then simulate the process. Standard routines are used to generate random numbers from these required distributions. Selective algorithms of the Gibbs sampler are given in APPENDIX A.

In the case of the mixed linear model, we begin with a set of arbitrary starting values for the model parameters, $\beta^{(0)}$, $\gamma^{(0)}$, $\sigma_\varepsilon^{2(0)}$, $\sigma_\gamma^{2\,(0)}$ and then successively generate values from the conditional posterior distribution of each of the parameters, conditioning on the most recently generated values of the other parameters of each step.

The Gibbs sampler for $p(\beta,\ \gamma\ , \sigma_\varepsilon^2, \sigma_\gamma^2 \mid \mathbf{D})$ is as follows:

(0) Select starting values for $\gamma^{(0)}, \sigma_\varepsilon^{2\,(0)}, \sigma_\gamma^{2\,(0)}$. Set $i = 0$.

(1) Sample $\beta^{(i+1)}$ from (1.9),

(2) Sample $\sigma_\varepsilon^{2\,(i+1)}$ from (1.11),

(3) Sample $\gamma^{(i+1)}$ from (1.10),

(4) Sample $\sigma_\gamma^{2\,(i+1)}$ from (1.12),

(5) Set $i=i+1$ and return to (1).

MATLAB software has been developed to generate the samples that enabled us to obtain the marginal posterior densities for the model parameter, using the Gibbs sampler. The full conditional posteriors were updated after every iteration. We ran multiple chains, i.e. m=101 000 of the Gibbs sampler to obtain draws from the posterior distributions of the model parameters given the data. The first 1 000 draws of each chain were discarded, and then every 10^{th} draw was saved. By saving every 10^{th} draw, the chain yielded a posterior sample of 1 000 approximately uncorrelated draws. All posterior analyses were based on these 1 000 draws, giving us a full Bayesian solution to all the mixed linear model parameters.

1.8 An Example

1.8.1 The Data

This section describes the analysis of data from the Dormer Sheep Stud started at the Elsenburg College of Agriculture near Stellenbosch. The main objective in developing the Dormer was the establishment of a mutton sheep breed which would be well adapted to the conditions prevailing in the Western Cape and which could produce the desired type of ram for crossbreeding purposes.

The origin of the Dormer Sheep breed can be traced back to December 1940 when four imported Dorset Horn rams were each mated to fourteen registered and thirty-five grade German (presently S.A. Mutton) Merino ewes. This was a direct consequence of a comprehensive series of crossbreeding studies carried out at the Elsenburg Agricultural College from 1927 over a period in excess of ten years. After the initial cross only the two rams with the best progeny results were used in the next breeding season (December 1941). Each was mated to 20 registered German Merino ewes. As no further crossbreeding between these parental breeds were practiced after 1941, only two first-cross rams and 77 first-cross ewe lambs served as basic material for further development of the new breed. It could therefore be concluded that the Dormer originated from a small number of animals. From the parental side it descends from only four Dorset Horn rams and because of selection only 31 registered and 40 high-grade German Merino ewes eventually contributed to the development of the Dormer breed. Although the Dormer sheep stud originated from a small number of animals, it can be assumed that, being a cross between two unrelated breeds; the inbreeding coefficients of the base animals were zero (Van Wyk, 1992).

Sheep used in the analysis were born in the period 1943 - 1950. Single sire mating was practiced with 25 to 30 ewes allocated to each ram. A spring breeding season (6 weeks duration) was used throughout the study. A total of n = 879 weaning weight records, from the progeny of q = 17 sires were available after editing, and p =17 fixed effects were included in the final model. The data can be observed in APPENDIX B. The REML estimates were obtained by using the MTDFREML program developed by Boldman *et al.* (1995).

The mixed linear model used for this data structure, is the sire model of section (1.2), $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$, where $\underline{\mathbf{Y}}$ (879 × 1) vector of weaning weights. In this application, $\mathbf{X}$ is a (879 × 17) design matrix of regressors, with one column corresponding to the *overall mean* weaning weight, seven columns corresponding to the *season of birth* effects, six to the *age of dam* effects, one to the *sex of lambs* effects, and two final columns corresponding to the *birth status* effects.

$\beta(17 \times 1)$ is the vector of ***fixed effects***. Using this notation, β_0 is the average weaning weight of female lambs born in 1950 if the age of the dam is 8 years or older, and the birth status "triplets". β_1 is the difference in average weaning weight between lambs born in 1943 and those born in 1950. β_2 is the difference in average weaning weight between lambs born in 1944 and those born in 1950. Whilst β_7 is the difference in average weaning weight between lambs born in 1949 and lambs born in 1950, β_8 measures the difference in average weaning weight of lambs with dams 2 years of age and those with dams 8 years and older of age. Further, β_{13} is the difference in average weaning weight of lambs with dams 7 years of age and those with dams 8 years and older of age. The difference in average weaning weight between male and female lambs is measured by β_{14} and β_{15} measures the difference in average weaning weight between single births and triplets. Finally, β_{16} is the difference in average weaning weight between twins and triplets.

The design matrix **Z** (879×17) is a matrix identifying the random effects. Note that γ is a (17×1) vector of random effects consisting of the breeding values for the 17 sires for which the data are observed.

1.8.2 Analysis of Variance Components

For the classical analysis, the estimates of the variance components are found by maximizing the likelihood function as developed by Patterson and Thompson (1971). Given these estimates, the ***Best Linear Unbiased Predictions (BLUBs)*** for γ and β are then obtained by solving Henderson's mixed model equations (equation (1.3)). Posterior means, and modes of the Traditional Bayesian analysis, 95% credibility intervals, and the REML estimates (along with standard errors) are summarized in Table 1.1. The REML estimates of σ_ε^2 and σ_γ^2 are more similar to the modes of the posterior distributions than the means. This is because the REML estimate represents the mode of the marginal likelihood and thus might be better compared to the mode of the posterior distribution.

Table 1.1 REML and Traditional Bayesian Estimates (posterior values) for the Variance Components, along with 95% Credibility Intervals.

Parameter	REML	Traditional Bayes		95% Credibility Interval
		Mean	Mode	
σ_ε^2	21.1096	21.2595	21.2619	19.3130 ; 23.2531
σ_γ^2	3.08	4.9239	3.01	1.2461 ; 12.1343

Using the posterior densities for σ_ε^2 and σ_γ^2, the marginal posterior densities are estimated as the average of the conditional posterior densities, obtained from the Gibbs sampler, and are depicted in Figures 1.1 and 1.2. Note that the distribution in Figure 1.2 is quite skew, resulting in a difference between the posterior mean and posterior mode (quantities will not coincide).

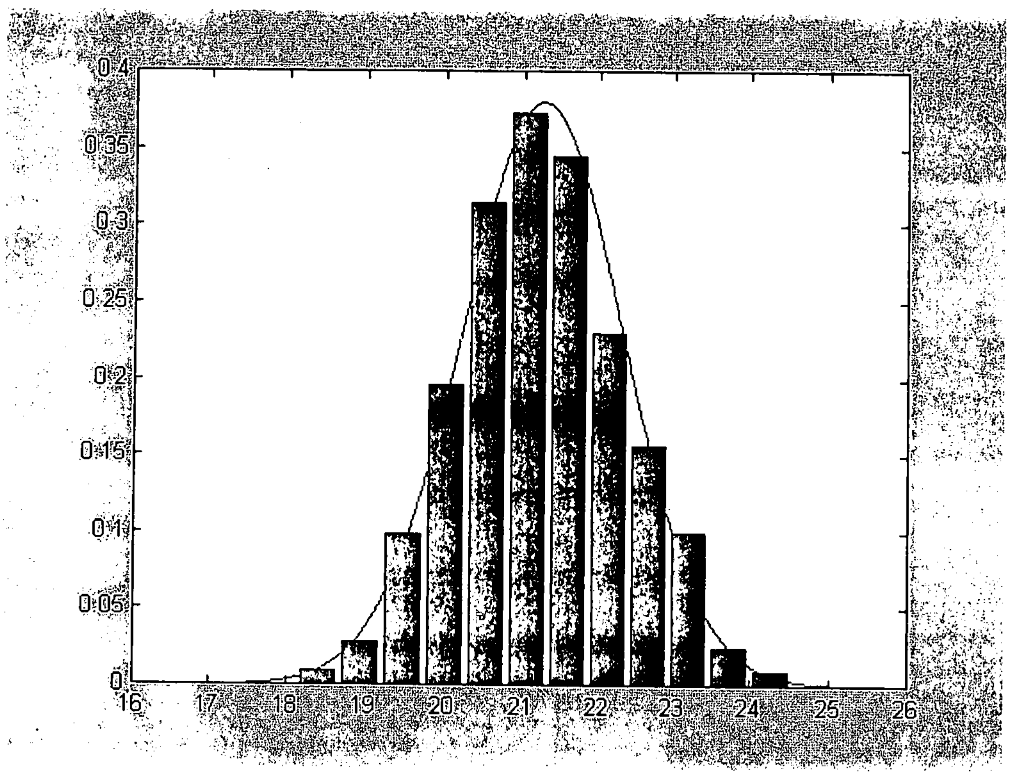

Figure 1.1 Histogram and Estimated Marginal Posterior Density of the Variance Component, σ_ε^2 (Error variance).

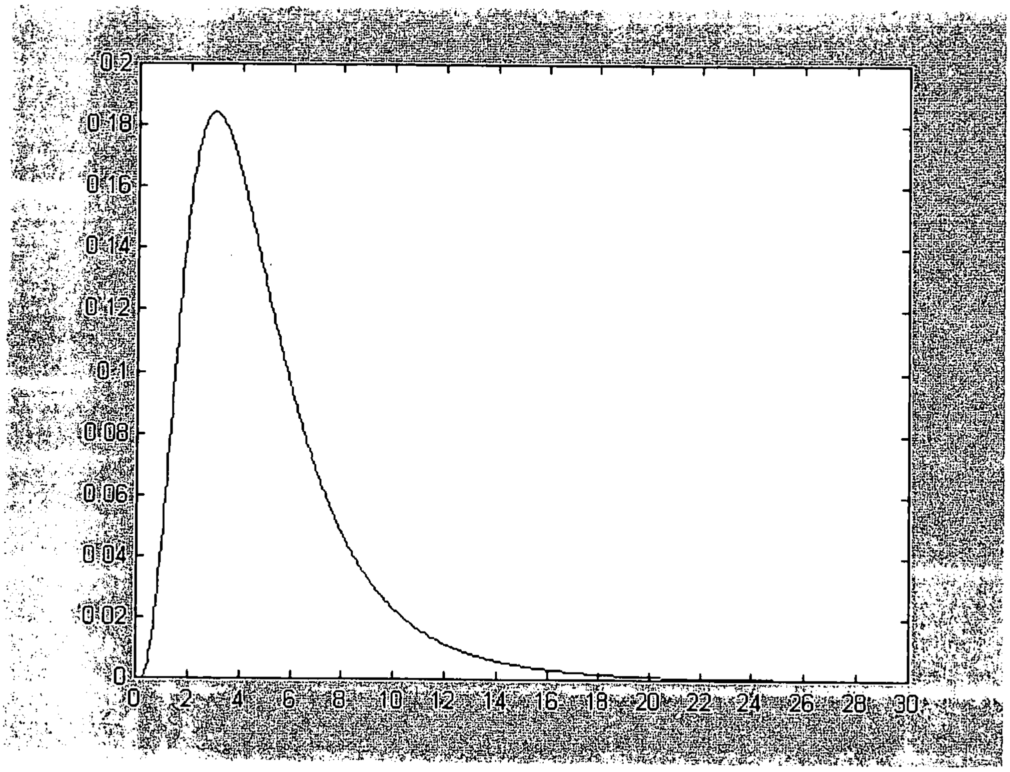

Figure 1.2 Estimated Marginal Posterior Density of the Variance Component σ_γ^2 (Sire variance).

The posterior means and modes of the Traditional Bayesian analysis and 95% credibility intervals of functions of variance components like the intraclass correlation coefficient, $\rho = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, and the variance ratio, $v = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ are summarized in Table 1.2. It is evident from this table that the credibility interval for the intraclass correlation coefficient does not contain 0.5. This result corresponds well to the statement made by Wang *et al.* (1992) namely that from a genetic point of view, an intraclass correlation coefficient of 0.5 is not possible in a sire model.

Table 1.2 Traditional Bayesian Estimates of Functions of the Variance Components, along with 95% Credibility Intervals.

Parameter	REML	Traditional Bayes		95% Credibility Interval
		Mean	Mode	
ρ	0.127	0.1789	0.133	0.0550 ; 0.3658
ν	0.146	0.2326	0.140	0.0582 ; 0.5768

The posterior distributions of these functions are illustrated in Figure 1.3 below.

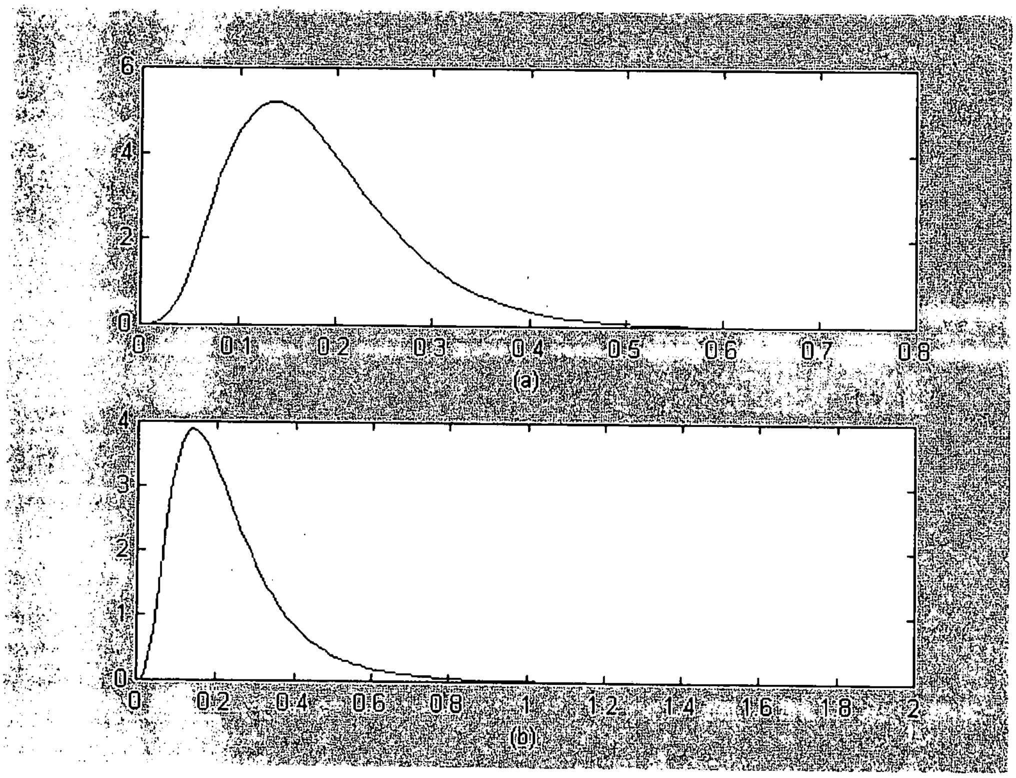

Figure 1.3 The Estimated Marginal Posterior Density of the (a) Intraclass Correlation Coefficient, $\rho = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, and the (b) Variance Ratio, $\nu = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$.

We can conclude that a Bayesian approach to variance component estimation has several practical advantages over a classical approach.

Firstly, although the estimate for a variance component is always positive, the REML estimate's asymptotic distribution can generate interval estimates that include negative values. This potentially embarrassing phenomenon is often overlooked in the discussions of likelihood-based methods. An interval estimate such as a highest posterior density region will not include negative values.

Secondly, highest posterior density regions are never empty, whereas confidence intervals for the ratio of two variances can be empty. One can also report the whole of the posterior probability distribution, not just a single number, and report some measure of posterior precision. Finally, classical estimators generally have intractable sampling distributions and standard errors are hard to calculate (Van der Merwe & Botha, 1993)

1.8.3 Analysis of Random Effects

Table 1.3 contains the BLUPs (with the variance components fixed at the REML estimates) and posterior means of the random effects (breeding values) for the 17 sires, along with the posterior ranks of each animal based on its breeding value and 95% credibility intervals. To put these numbers in perspective, progeny from sire 3 (ranked 1st according to its Trad. Bayes and REML estimates) with an estimated breeding value of 3.4858 will therefore have an estimated average weaning weight of 3.4858 kilogram more than the progeny from the rest of the sires. The progeny from sire 10 (ranked 17th according to its Trad. Bayes and REML estimates) on the other hand with an estimated breeding value of -1.7983 will have an estimated average weaning weight of 1.7983 kilogram less than the average weaning weight of lambs from the rest of the sires.

Further inspection of the credibility intervals in Table 1.3 shows that the lower bound of the 95% credibility interval for the breeding value of sire 3 is 1.2143 whilst the upper bound is 6.1194. Since this interval does not contain zero, we can be reasonable sure that the average weaning weight of lambs born from sire 3 will be between 1.21 and 6.12 kilogram more than the average weaning weight of lambs born from the other sires. Furthermore, by comparing the 95% credibility intervals of the breeding values in Table 1.3 it is clear that the upper limit of the interval in the case of sire 10 is smaller than the lower limit of the corresponding interval for sire 3. By implication this means that sire 10 will never (very seldom) produce progeny with greater weaning weights than sire 3.

Table 1.3 Estimated Breeding Values for 17 Sires from the Elsenburg Dormer Stud, Posterior Rankings, 95% Credibility Intervals, and Standard Errors of REML Estimates.

Sire ID	Trad. Bayes	Rank	95% Credibility Interval	REML	Rank	SE's
41037	0.7350	3	-1.4728 ; 3.4395	0.5781	3	1.06
41004	0.2478	6	-1.6531 ; 2.6977	0.1396	6	0.92
41019	3.4858	1	1.2143 ; 6.1194	3.33	1	0.99
43002	-1.1985	14	-3.7586 ; 1.3778	-1.181	14	1.18
44170	-0.0930	7	-2.7340 ; 2.7585	-0.17	7	1.18
44174	-0.6524	10	-3.9471 ; 2.3055	-0.5694	10	1.34
44042	-1.3053	15	-3.6029 ; 0.9157	-1.2565	16	0.95
45070	-1.1460	13	-3.6855 ; 1.1319	-0.9631	13	0.93
45135	-0.5301	9	-3.2348 ; 1.9326	-0.5371	9	1.1
46015	-1.7983	17	-4.4578 ; 0.3174	-1.7092	17	0.96
46037	-0.8524	11	-3.2205 ; 1.4669	-0.8423	11	0.91
48014	-1.0059	12	-3.6639 ; 1.2705	-0.9537	12	0.97
48052	-0.4208	8	-2.8863 ; 1.9739	-0.3019	8	1
48148	-1.4307	16	-4.0524 ; 0.9945	-1.256	15	1.1
49053	0.5309	4	-2.6618 ; 3.7327	0.463	4	1.31
49134	0.9219	2	-2.0470 ; 4.1511	0.795	2	1.34
49046	0.4395	5	-2.9563 ; 3.7575	0.4059	5	1.41

It is evident from the table that the Traditional Bayes estimated are quite close to the REML estimates. This is not surprising to us, since as showed by Harville, (1974) (see also Searle, Casella and McCulloch, (1992)) that when uniform or "flat" priors are assigned to the vector of fixed effects and variance components, the modes of the marginal posterior distributions are very close to the REML estimates.

If on the other hand proper priors were assigned to the unknown parameters and if the sample size was quite small, the differences between Bayesian and non-Bayesian results could have been quite substantial. The assignment of a proper prior to a specific parameter must however be justifiable from a practical point of view.

The marginal posterior densities for the breeding values of sire 3 and 10, and the difference in breeding values for these two sires, are displayed in Figures 1.4 – 1.6 respectively.

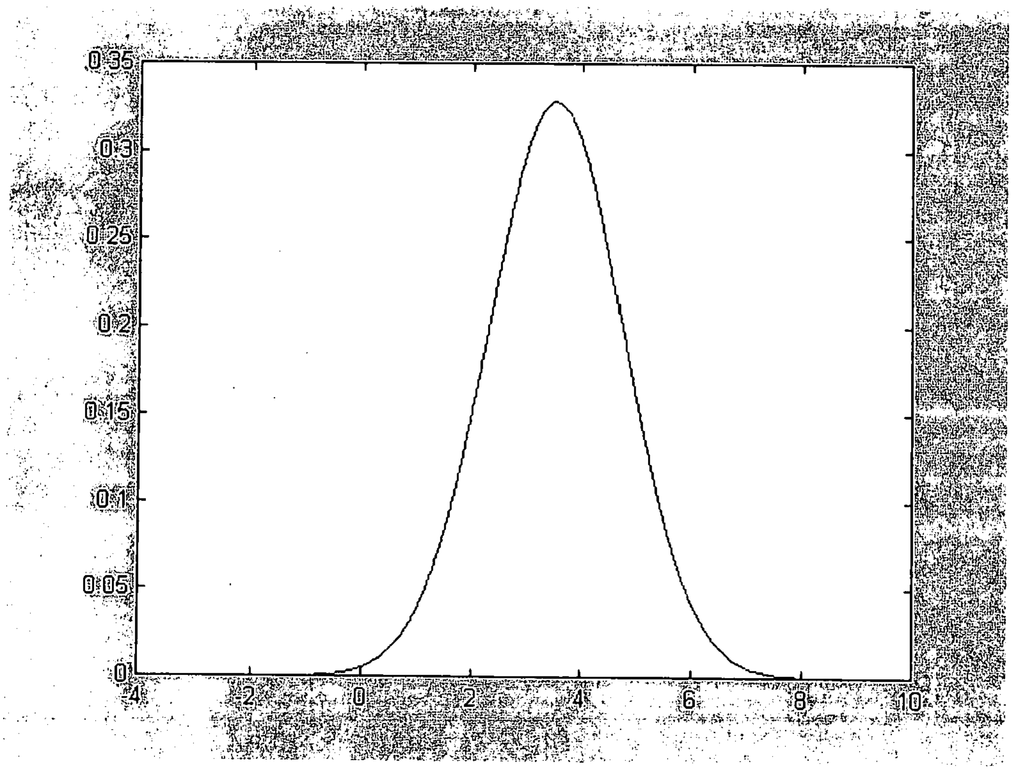

Figure 1.4 The Estimated Marginal Posterior Density of the Breeding Value for Sire 3 (γ_3), ID41019.

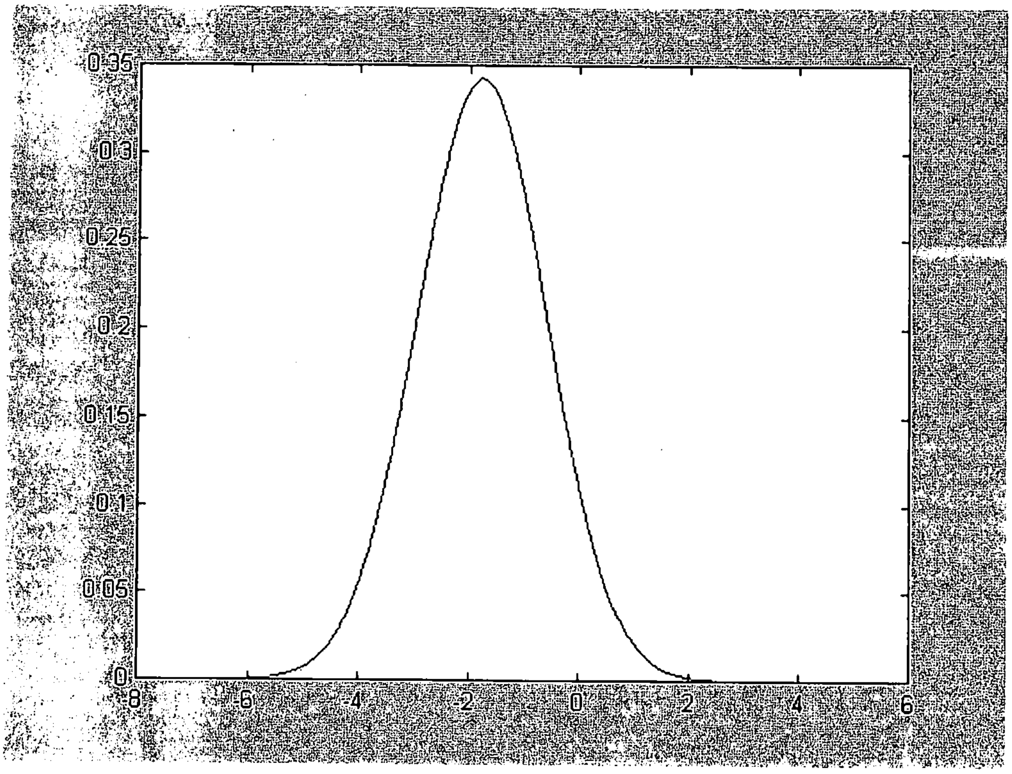

Figure 1.5 The Estimated Marginal Posterior Density of the Breeding Value for Sire 10 (γ_{10}), ID46015.

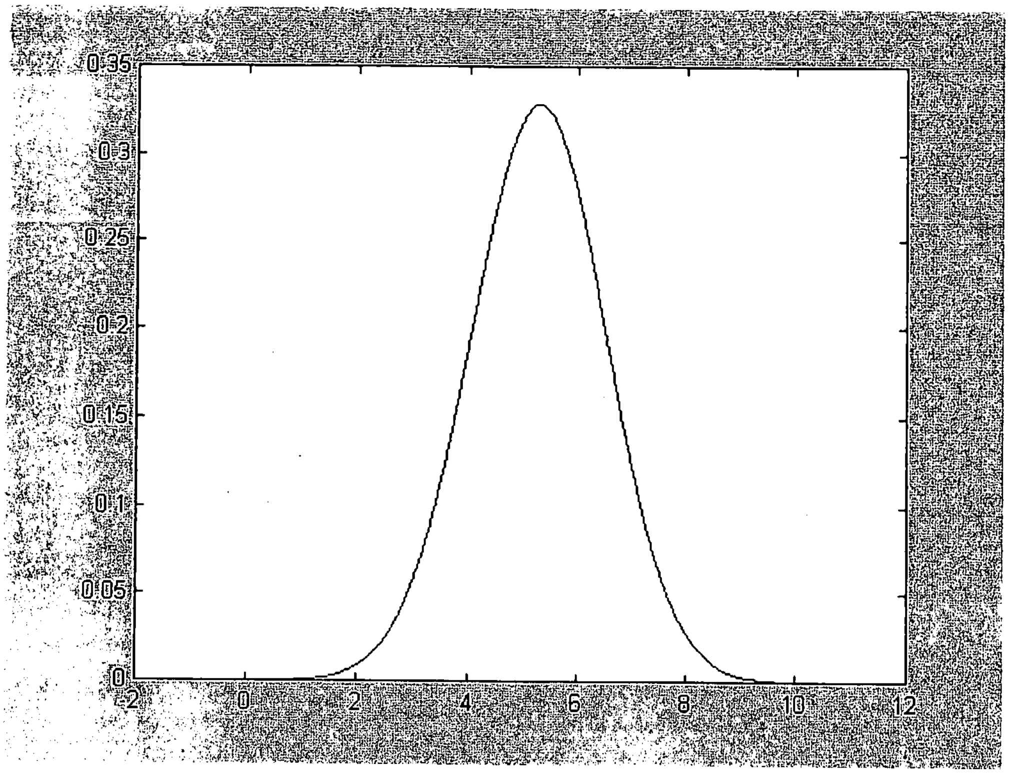

Figure 1.6 The Estimated Marginal Posterior Density of the Difference in Breeding Value between Sire 3 and Sire 10 ($\gamma_3 - \gamma_{10}$).

A key difference between REML/BLUP predictions and Bayesian inference is the treatment of the variance components. To obtain the BLUP estimates, the variance components are fixed at a single value, ignoring uncertainty associated with estimating their values. The Bayesian analysis incorporates this uncertainty by averaging over the plausible values of the variance components, making it a more feasible method of analysis, since these components are very important in evaluating the breeding potential of the sires in the model.

1.8.4 Analysis of Fixed Effects

Duchateau *et al.* (1998) stated that the emphasis in breeding experiments is on the variance components and on the prediction of particular random effects, but estimation of the fixed effects is also important, thus Table 1.4 summarizes the estimated fixed effect for the mixed linear model given the data along with selected joint marginal posterior densities presented in Figures 1.8 – 1.12 (obtained from the Gibbs sampler).

Table 1.4 Estimated Values of Selected Fixed Effects, 95% Credibility Intervals, and REML Estimates.

Fixed Effect	Trad. Bayes	95% Credibility Interval	REML
β_0	22.9655	19.2315 ; 26.9031	21.50
β_7	5.3523	4.1515 ; 6.5310	5.25
β_{14}	3.6690	2.9835 ; 4.3353	3.54
β_{15}	9.4874	7.1923 ; 11.7688	9.35
β_{16}	2.9621	0.6574 ; 5.2308	2.88

As described in section (1.9.1), β_0 is the average weaning weight of female lambs born in 1950 if the age of the dam is 8 years or older, and the birth status "triplets". β_7 measures the expected difference in average weaning weight between lambs born in 1949 and lambs born in 1950. It is therefore clear that lambs born in 1949 had an average weaning weight of 5.3523 kilogram more than lambs born in 1950.

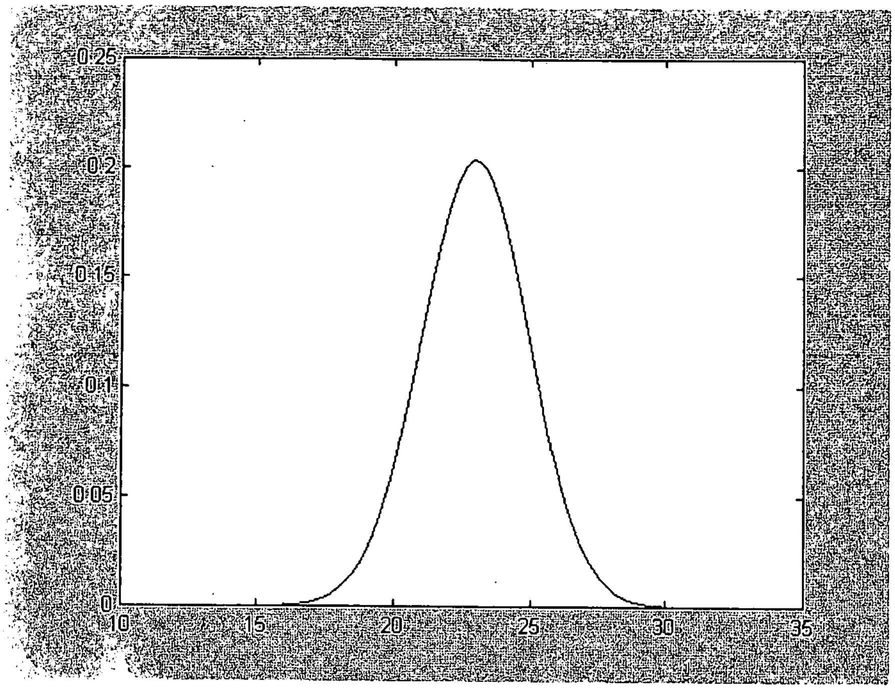

Figure 1.8 Estimated Marginal Posterior Density of β_0, the expected average weaning weight of female lambs born in 1950 if the age of the dam is 8 years or older, and the birth status "triplets".

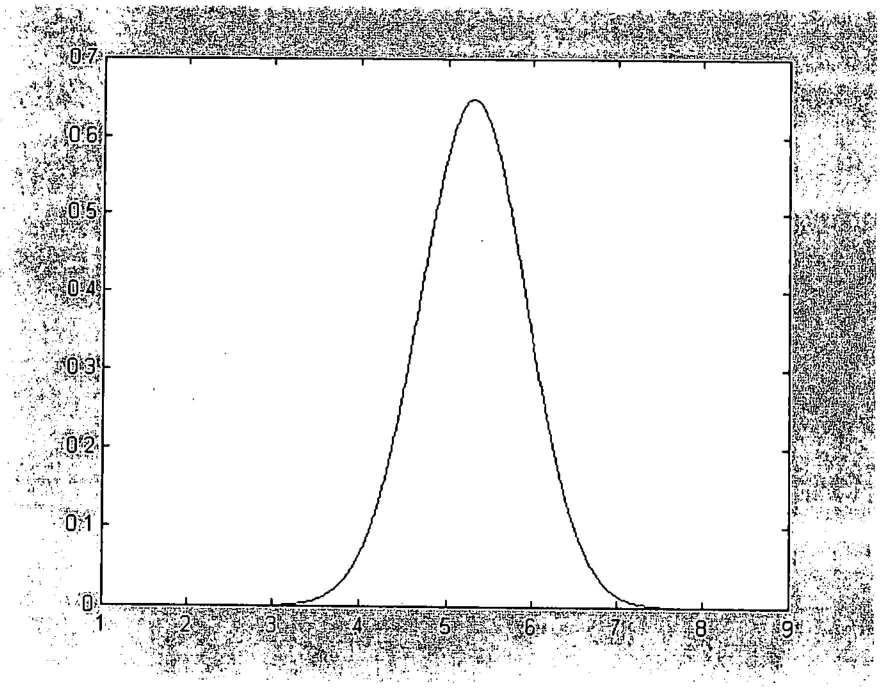

Figure 1.9 Estimated Marginal Posterior Density of β_7, the expected difference in average weaning weight between lambs born in 1949 and in 1950.

β_{14} measures the expected difference in average weaning weight between male and female lambs. It can therefore be concluded that male lambs will have an average weaning weight of 3.6690 kilogram more than female lambs. From the 95% credibility interval it can be seen that the difference in average weaning weight between male and female lambs can be as large as 4.3353 kilogram.

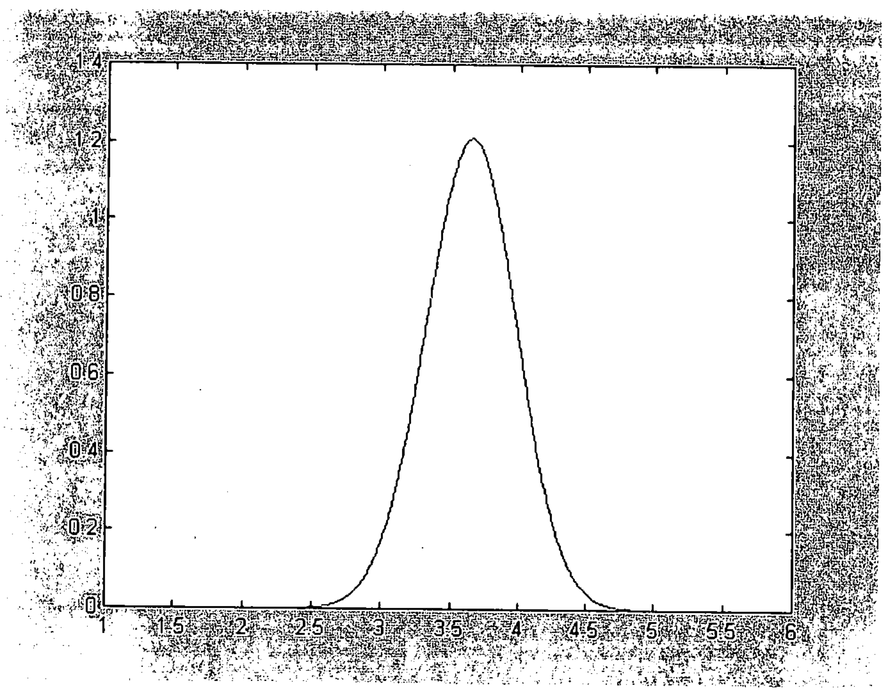

Figure 1.10 Estimated Marginal Posterior Density of β_{14}, the expected difference in average weaning weight between male and female lambs.

β_{15} measures the expected difference in average weaning weight between single births and triplets. It can therefore be expected that single births will have an average weaning weight of 9.4874 kilogram more than triplets.

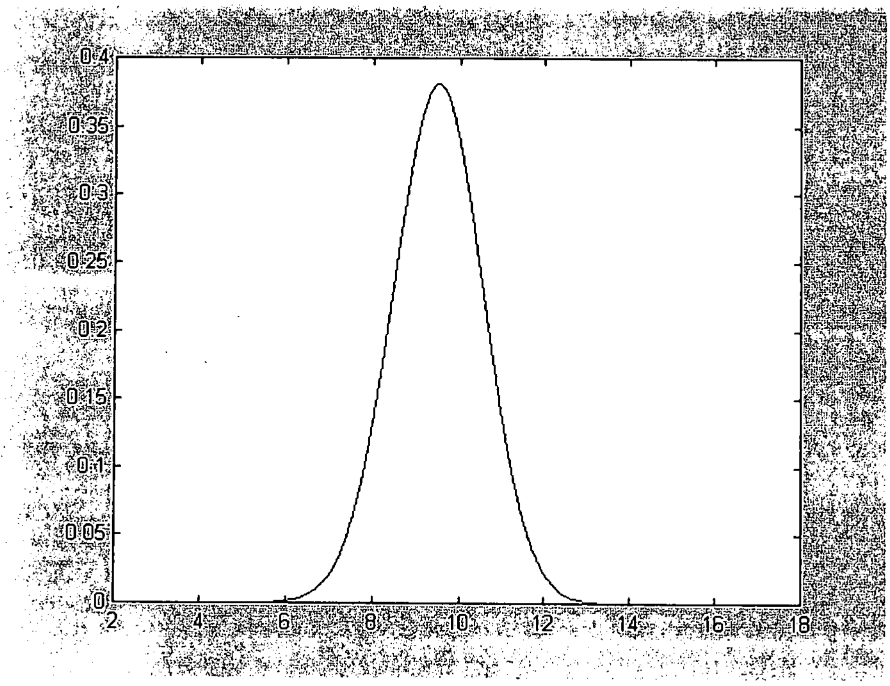

Figure 1.11 Estimated Marginal Posterior Density of β_{15}, the expected difference in average weaning weight between single births and triplets.

β_{16} measures the expected difference in average weaning weight between a pare of twins at birth and triplets. Therefore, a pair of twins at birth will have an average weaning weight of 2.9621 kilogram more than triplets. It is therefore clear that birth status can dramatically affect the expected weaning weight of a sire's progeny, thus affecting its breeding value. According to van Wyk (1992) single born lambs constitute only 36.02% of all lambs born whilst twins and triplets make up 59.93% and 4.05% respectively.

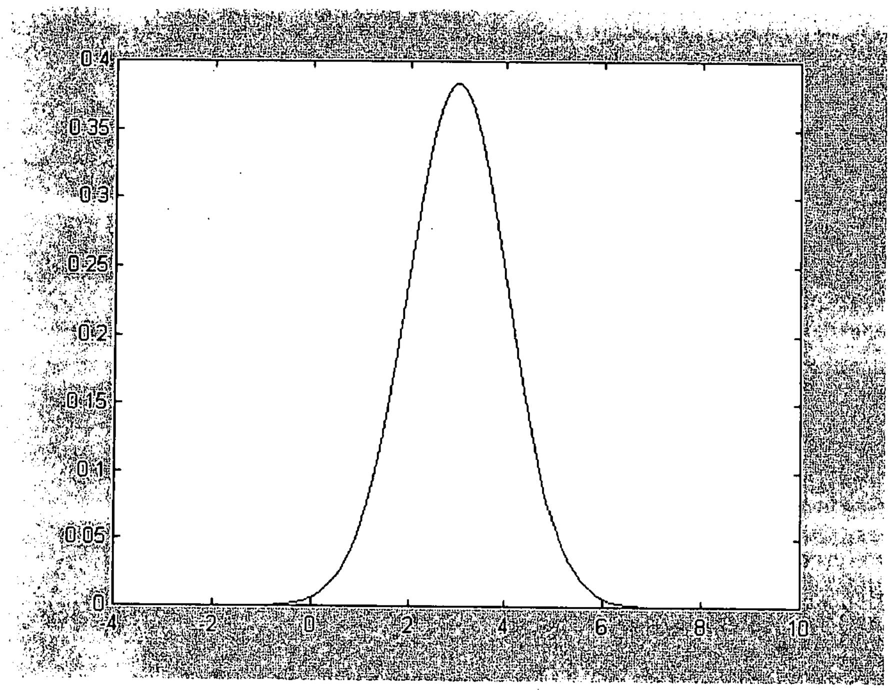

Figure 1.12 Estimated Marginal Posterior Density of β_{16}, the expected difference in average weaning weight between a pare of twins at birth and triplets.

One appealing future of the Gibbs simulation approach to the Bayesian data analysis is that we obtain an approximate sample from the joint distribution of all the unknown parameters given the data. This sample provides adequate information to estimate any quantity of interest.

1.9 Chapter Summary

The present chapter illustrated an extension of the Gibbs sampler to solve problems arising in animal breeding theory. Formulae were derived and presented to implement the Gibbs sampler in a more general mixed linear model. With this extension, a full Bayesian solution to the problem of inference about variance components, functions thereof, and random effects in such a mixed linear model was possible. Once the marginal densities were obtained from the Gibbs sampler, it was easy to calculate summary statistics from the posterior distributions, e.g. posterior means, modes and credibility intervals. Moreover, as mentioned before, the similarities between the Bayesian and REML estimates were not surprising to us because of the assignment of uniform or "flat" priors to the vector of fixed effects and variance components. If on the other hand proper priors were assigned to the unknown parameters and if the sample size was quite small, the differences between Bayesian and non-Bayesian results could have been more substantial.

Arguing from a Bayesian viewpoint, the Gibbs sampling turned an analytically intractable multidimensional integration problem into a feasible numerical one, and is conceptually more appealing than the classical approach.

Since the Gibbs sampler is now established in animal breeding problems, the objective of the thesis will be to extent some selected issues regarding The Mixed Linear Model in animal breeding, e.g. The Bayesian Method of Moments (BMOM) approach to the full Bayesian solution, Reference -, Probability-Matching -, and Dirichlet Process Priors for the random effects.

© Parts of this chapter have been published in the South African Statistical Journal.
(See Van der Merwe *et al.* 2000)

CHAPTER 2

«Bayesian Method of Moments»

Introductory words: *After reviewing the purposes and basic principles of the BMOM approach previously presented and applied by Zellner and co-workers to multiple and multivariate regression models as well as simultaneous equation problems, a new application of the approach is presented. In this section the BMOM procedure is extended to the Mixed Linear Model with illustrative examples from the animal breeding theory.*

2.1 Prologue

On the BMOM and the capability of comparing BMOM and Traditional Bayes models, Barnard (1997) has written:

"And above all any method is welcome which, unlike nonparametrics, remains fully quantifiable without paying obeisance to the model which one knows is false. And your proposal to compare BMOM results with a model based one should achieve the best of both worlds."

In addition, Laskey (1997) comments:

"When prior knowledge about the form of the likelihood function is extremely weak, standard Bayesian analysis can be 'brittle' in the sense of (1) producing absurd conclusions given not-obvious-absurd inputs and (2) being extremely sensitive to minor variation in inputs. On the other hand, BMOM gives good

answers for the questions it addresses while not purporting to go beyond the information that is really there in the prior and the data."

Another view of BMOM is provided by Soofi (1997) in the following words:

"I consider the BMOM as an ingenious contribution to the entire field of statistics. The BMOM is elegant and easily applicable because it is free from the strong UNVERIFIABLE assumptions that we usually make just in order to enable us to handle a problem."

(see Zellner *et al.* (1999) for quotes)

In the traditional likelihood and Bayesian approaches, it is usually assumed that enough information is available to formulate a likelihood function and, in the Bayesian approach, a prior density for the parameters of the selected likelihood function. However, if not enough information is available to specify a form for the likelihood function, then clearly there will be problems in both the traditional likelihood and Bayesian approaches. In situations like this, some resort to non-likelihood based methods is proposed, e.g. the Bayesian Method of Moments (BMOM), first introduced by Arnold Zellner in 1994. Given the data, BMOM enables researchers to compute post data densities for parameters and future observations if the form of the likelihood function is unknown. The BMOM approach provides a solution to the famous inverse problem proposed by Bayes (1763) and hence the name Bayesian Method of Moments.

As illustrated in Chapter 1, an essential element of the Bayesian approach is Bayes' theorem, also referred to in the literature as the principle of inverse probability. In problems involving "inverse probability" we have given data and from the information in the data try to infer what random process generated them. On the other hand, in problems of "direct probability" we know the random process, including values of its parameters and from this knowledge make probability statements about outcomes or data produced by the known random process. Problems of statistical estimation are thus seen to be problems of "inverse probability", whereas many gambling problems are problems in direct probability.

In the BMOM approach the posterior and predictive moments, based on a few relatively weak assumptions are used to obtain maximum entropy densities for the parameters, realized error terms and future values of the variables. Shannon (1948) defines entropy (or uncertainty) as

$$W = -\int p(y) \log p(y)\, dy \tag{2.1}$$

where $p(y)$ is a probability density function. Maximizing W subject to various side conditions is well known in the literature as a method for deriving the forms of minimal information distributions. Shannon (1948) has also indicated how maximum entropy (ME) distributions can be derived by a straightforward application of calculus of variation techniques. In particular he has shown that the ME distribution that maximizes entropy subject to a normalization condition is just the uniform distribution. By adding additional side conditions given the first two moments of the distribution are imposed, the ME distribution is the normal distribution. On the other hand if just the side conditions on the zeroth and first moments are utilized, the maxent density is an exponential density. For discussion and application of maximum entropy, see for example Jaynes (1982, 1988); Shore and Johnson (1980); Cover and Thomas (1991); Zellner and Highfield (1988) and Zellner (1997).

In the sections to come, the theory and results derived by Zellner (1997) will be extended to the mixed linear model, with an appropriate example from an animal breeding experiment. We will also discuss coherent procedures of updating BMOM maxent post-data densities for parameters and future observations.

2.2 Review of the BMOM approach

In Table 2.1, the inputs and outputs of the Traditional Bayesian (TB) and the BMOM approaches are summarized. In both approaches, given the data is an important input along with an entertained model for the given data, say the mixed linear model. In the TB approach sampling assumptions for the data or the model's error terms are introduced in order to obtain a likelihood function. This likelihood function and the prior density for its parameters are inputs to Bayes' theorem and the outputs are posterior and predictive densities for parameters, realized error terms and future observations.

It is evident that in the BMOM approach no sampling assumptions about the given observed data are made. Rather certain assumptions are made about the realized error terms' properties. Given these assumptions, posterior moments of the parameters are derived that incorporate the information in the given data. These moments are then used as side conditions in the derivation of maxent probability density functions for the parameters, realized error terms and future observations.

Table 2.1 Inputs and Outputs of Traditional Bayesian and BMOM Approaches.

A. Traditional Bayesian Approach

INPUTS		OUTPUTS	
1.	Data, **D**	1.	Posterior Density
2.	Prior information, I_1	2.	Predictive Density
3.	Sampling Assumptions	3.	Point & Interval Estimates
4.	Data Density & Likelihood Function	4.	Point and Interval Predictions
5.	Prior Density		
6.	Bayes' Theorem		

B. BMOM Approach

INPUTS		OUTPUTS
1.	Data, **D**	SAME AS ABOVE
2.	Prior Information, I_2	
3.	Mathematical Form of the model	
4.	Moments of parameters and future values	
5.	Maxent Principle	

2.3 Extension of the BMOM to the Mixed Linear Model

In section 1.2 the mixed linear model in its simples form was defined as

$$\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon . \tag{2.2}$$

In the introductory paragraph it was stated that the BMOM approach is particular useful where there is difficulty in formulating an appropriate likelihood function. Without a likelihood function, it is not possible to pursue traditional likelihood and Bayesian approaches to estimation and hypothesis testing. In the next section only the mathematical form of the model as defined in (2.2) will be used, i.e. no specific distribution will be assigned to the vector ε. The likelihood function will therefore be considered as unknown. This is different from the assumption in the previous sections. Let us for the time being assume that γ is given, (2.2) can then be written as

$$\underline{\mathbf{Y}} - \mathbf{Z}\gamma = \mathbf{X}\beta + \varepsilon \tag{2.3}$$

i.e.

$$\underline{\mathbf{Y}}^* = \mathbf{X}\beta + \varepsilon . \tag{2.4}$$

For given γ, we will take $\underline{\mathbf{Y}}^*$ as our new dependent variable. Equation (2.3) is now the usual multiple regression model. To assume that the model in (2.3) is adequate implies, among other things, that there are no systematic elements in the realized error term vector ε, correlated with variables in **X**. This assumption is formalized as Assumption 1 in the BMOM approach as follows:

Assumption 1:

$$\mathbf{X}' E(\varepsilon \mid \mathbf{D}, \gamma) = 0$$

where $E(\varepsilon \mid \boldsymbol{D}, \gamma)$ denotes the post-data mean of the realized error vector ε, given the data and γ; that is, the given, unknown values of the elements of the realized error vector are considered subjectively random, just as in Bayesian analysis of the realized error terms (Chaloner & Brant, 1988; Chaloner, 1994; Zellner & Moulton, 1985). Thus, the assumption indicates that the columns of X are orthogonal to the vector $E(\varepsilon \mid \mathbf{D}, \gamma)$. Further, from (2.4) by taking the posterior expectation, it follows that

$$\underline{\mathbf{Y}}^* = \mathbf{X} E(\beta \mid \mathbf{D}, \gamma) + E(\varepsilon \mid \mathbf{D}, \gamma)$$

and Assumption 1 implies that the observation vector $\underline{\mathbf{Y}}^*$ is the sum of two orthogonal vectors. Note also since the first column of $\mathbf{X}$ is a $n \times 1$ vector of ones, denoted by $\underline{l}$, we have from the assumption,

$$\underline{l}' E(\varepsilon \mid \mathbf{D}, \gamma) = 0 \qquad (2.5)$$

or

$$E\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\right) = E(\bar{\varepsilon} \mid \mathbf{D}, \gamma) = 0\,. \qquad (2.6)$$

Thus, given that we assume that we have an appropriate form and an adequate number of terms included in (2.4), the expectation of the mean of the realized error terms is assumed equal to zero, i.e. there is no systematic component in the realized error vector.

Proof:

If we multiply both sides of (2.4) by $(\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{X}'$, we obtain

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}^{*} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon\,. \tag{2.7}$$

Now take the post-data expectation of both sides in (2.7), noting that $E(\beta \mid \mathbf{D}) = \hat{\beta}$, we have

$$\hat{\beta} = E(\beta \mid \mathbf{D},\gamma) + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\varepsilon \mid \mathbf{D},\gamma) \tag{2.8}$$

and from Assumption 1

$$E(\beta \mid \mathbf{D},\gamma) = \hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}^{*}. \tag{2.9}$$

That is, the post-data expectation of the regression coefficient vector is equal to the least squares estimate. Further, the post data mean of the realized error vector in (2.4) is

$$E(\varepsilon \mid \boldsymbol{D},\ \gamma) = \underline{\mathbf{Y}}^{*} - \mathbf{X}\hat{\beta} = \hat{\varepsilon} \tag{2.10}$$

where $\hat{\varepsilon}$ is the least squares residual vector that satisfies $\mathbf{X}'\hat{\varepsilon} = 0$. Note also that from (2.7) and (2.10),

$$
\begin{aligned}
\varepsilon - \hat{\varepsilon} &= \underline{\mathbf{Y}}^* - \mathbf{X}\beta - (\underline{\mathbf{Y}}^* - \mathbf{X}\hat{\beta}) \\
&= \underline{\mathbf{Y}}^* - \mathbf{X}\beta - \{\underline{\mathbf{Y}}^* - \mathbf{X}(\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon)\} \\
&= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon \\
&= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\varepsilon - \hat{\varepsilon})
\end{aligned}
\tag{2.11}
$$

where the last step follows from the orthogonality condition mentioned above, $\mathbf{X}'\hat{\varepsilon} = 0$. We can thus write

$$
\begin{aligned}
Var(\varepsilon \mid \mathbf{D},\gamma) &= E\{(\varepsilon - \hat{\varepsilon})(\varepsilon - \hat{\varepsilon})' \mid \mathbf{D},\gamma\} \\
&= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E\{(\varepsilon - \hat{\varepsilon})(\varepsilon - \hat{\varepsilon})' \mid \mathbf{D},\gamma\}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'
\end{aligned}
\tag{2.12}
$$

which defines a functional equation that the post-data covariance matrix for ε, $Var(\varepsilon \mid \mathbf{D},\gamma)$ must satisfy. Since there are only p free elements of ε in the n equations in (2.4), $Var(\varepsilon \mid \mathbf{D},\gamma)$ must be of rank p. Thus we introduce the following assumption that fixes the form of the realized error vector up to a multiplicative positive scalar multiplier.

Assumption 2:

$$Var(\varepsilon \mid \sigma_\varepsilon^2, \mathbf{D},\gamma) = \sigma_\varepsilon^2 \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

where σ_ε^2 is a variance parameter to be defined below. We use Assumption 2 to evaluate the post-data covariance matrix of β as follows

$$
\begin{aligned}
Var(\beta \mid \sigma_\varepsilon^2, \mathbf{D},\gamma) &= E\{(\beta - \hat{\beta})(\beta - \hat{\beta})' \mid \mathbf{D},\gamma\} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E\{(\beta - \hat{\beta})(\beta - \hat{\beta}' \mid \mathbf{D},\gamma\}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma_\varepsilon^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}
\tag{2.13}
$$

where $\sigma_\varepsilon^2 = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 = \frac{1}{n}\varepsilon'\varepsilon$.

It is seen that the parameter σ_ε^2 is represented as an average of the sum of squared deviations of the realized error terms from their expected mean of zero, $E(\bar{\varepsilon} \mid \boldsymbol{D}, \gamma) = 0$ which follows from Assumption 1.

Proof:

Using the definition of σ_ε^2 we have

$$
\begin{aligned}
E(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) &= E\frac{1}{n}(\varepsilon'\varepsilon \mid \mathbf{D},\gamma) \\
&= E\frac{1}{n}\{(\underline{\mathbf{Y}}^* - \mathbf{X}\beta)'(\underline{\mathbf{Y}}^* - \mathbf{X}\beta) \mid \mathbf{D},\gamma\} \\
&= E\frac{1}{n}\left[\left\{(\underline{\mathbf{Y}}^* - \mathbf{X}\hat{\beta}) - (\mathbf{X}\beta - \mathbf{X}\hat{\beta})\right\}'\left\{(\underline{\mathbf{Y}}^* - \mathbf{X}\hat{\beta}) - (\mathbf{X}\beta - \mathbf{X}\hat{\beta})\right\} \mid \mathbf{D},\gamma\right] \\
&= E\frac{1}{n}\left[\left\{(\underline{\mathbf{Y}}^* - \mathbf{X}\hat{\beta})'(\underline{\mathbf{Y}}^* - \mathbf{X}\hat{\beta}) - 2(\underline{\mathbf{Y}}^* - \mathbf{X}\hat{\beta})'(\mathbf{X}\beta - \mathbf{X}\hat{\beta}) + (\beta - \hat{\beta})(\mathbf{X}'\mathbf{X})(\beta - \hat{\beta})\right\} \mid \mathbf{D},\gamma\right]
\end{aligned}
$$

Since the middle term is equal to zero, $\underline{\mathbf{Y}}^{*\prime}\mathbf{X} - \hat{\beta}'\mathbf{X}'\mathbf{X} = 0$, it follows that

$$
\begin{aligned}
E(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) &= \frac{1}{n}\left[\hat{\varepsilon}'\hat{\varepsilon} + E\left\{(\beta - \hat{\beta})'(\mathbf{X}'\mathbf{X})(\beta - \hat{\beta}) \mid \mathbf{D},\gamma\right\}\right] \\
&= \frac{1}{n}\left\{\hat{\varepsilon}'\hat{\varepsilon} + pE(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma)\right\} \\
&= \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-p}.
\end{aligned}
$$

That is, $E(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) = \dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{n-p} = s^2$. (2.14)

Note that this post-data expectation differs from the post-data mean of σ_ε^2 in a diffuse prior-normal likelihood traditional Bayesian approach (TB), namely $E_{TB}(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) = \dfrac{ks^2}{k-2}$. For small values of $k = n - p$, the last expression is much larger than s^2. As pointed out in Zellner (1996), Tobias and Zellner (1999) and mentioned in the introduction, the proper maxent density for β given σ_ε^2, $\mathbf{D}$ and γ can now be derived from the above assumptions.

Corollary 1: *The proper maxent density for σ_ε^2 using the first moment of σ_ε^2 given in (2.14) is an exponential density,*

$$h_e(\sigma_\varepsilon^2 \mid, \mathbf{D},\gamma) = \frac{1}{s^2}\exp\left\{-\frac{\sigma_\varepsilon^2}{s^2}\right\} \qquad 0 < \sigma_\varepsilon^2 < \infty \tag{2.15}$$

*which will be called **BMOM 1**.*

Corollary 2: *The proper maxent density for β given σ_ε^2, D and γ, using the first two moments is a normal distribution*

$$f_N(\beta \mid \sigma_\varepsilon^2, \mathbf{D},\gamma) \sim N\left[\hat{\beta}, (\mathbf{X}'\mathbf{X})^{-1}\sigma_\varepsilon^2\right]. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$p(\beta,\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) = f_N(\beta \mid \sigma_\varepsilon^2, \mathbf{D},\gamma)\, h_e(\sigma_\varepsilon^2 \mid, \mathbf{D},\gamma). \tag{2.17}$$

Also, as shown in Tobias and Zellner (1999) and Zellner (1997), higher order post-data moments of σ_ε^2 can be evaluated and used as moment side conditions in deriving maxent densities.

Corollary 3: *The proper maxent density for σ_ε^2 using the first four moments of σ_ε^2 can be approximated by a Pearson density. This approach will be called **BMOM 2***

By extending the method of Tobias and Zellner (1999), these higher order post-data moments of σ_ε^2 can be obtained in the following way. From the definition of σ_ε^2 it follows that

$$\begin{aligned}\sigma_\varepsilon^2 &= \frac{1}{n}\varepsilon'\varepsilon = \frac{1}{n}\left(ks^2 + (\beta - \hat{\beta}'(\mathbf{X'X})(\beta - \hat{\beta})\right) \\ &= \frac{1}{n}\left(ks^2 + \sigma_\varepsilon^2 Q\right)\end{aligned} \tag{2.18}$$

and from (2.16) it follows that $Q = \frac{1}{\sigma_\varepsilon^2}(\beta - \hat{\beta})'(\mathbf{X'X})(\beta - \hat{\beta})$ has a chi-square density with p degrees of freedom.

As shown by Tobias and Zellner (1979) this fact can be employed to evaluate the moments of σ_ε^2 as illustrated below. From (2.18) it follows that

$$\sigma_\varepsilon^{2j} = \frac{1}{n^j}\left(ks^2 + \sigma_\varepsilon^2 Q\right)^j \qquad \text{for } j = 1,2,\ldots\ldots \tag{2.19}$$

By using the binomial expansion and known moments of χ_p^2, the following recursive formulae can be derived:

$$E(\sigma_\varepsilon^{2j} \mid \mathbf{D},\gamma) = \frac{(ks^2)^j + \sum_{i=1}^{j-1}\binom{j}{i}(ks^2)^{j-i} E(\sigma_\varepsilon^{2i} \mid \mathbf{D})\left[p(p+2)......(p+2(i-1))\right]}{n^j - \left[p(p+2)......(p+2(j-1))\right]}. \quad (2.20)$$

Thus, from expression (2.20) the first two moments above zero and the variance of σ_ε^2 are given by

$$E(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) = s^2,$$

$$E(\sigma_\varepsilon^4 \mid \mathbf{D},\gamma) = \frac{s^4(k^2+2kp)}{n^2 - p(p+2)},$$

$$Var(\sigma_\varepsilon^2 \mid \mathbf{D},\gamma) = s^4\left(\frac{2p}{n^2 - p(p+2)}\right).$$

(2.21)

A similar approach as just described can be used to obtain the post-data densities of γ and σ_γ^2. For the mixed linear model defined in (2.2) we can also assume that β instead of γ is known. Equation (2.2) can then be written as

$$\underline{\mathbf{Y}} - \mathbf{X}\beta = \mathbf{Z}\gamma + \varepsilon$$

i.e.

$$\underline{\widetilde{\mathbf{Y}}} = \mathbf{Z}\gamma + \varepsilon. \quad (2.22)$$

Using the same arguments as given in (2.7) – (2.16) it follows that the maxent density for γ is normal with mean, $\hat{\gamma} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\tilde{\underline{\mathbf{Y}}}$ and variance $\tilde{\sigma}_\gamma^2 = (\mathbf{Z}'\mathbf{Z})^{-1}\sigma_\varepsilon^2$. To implement the normal prior for γ (which is an integral part of mixed linear model analysis) the likelihood of γ given σ_ε^2 and β is considered to be proportional to the maxent density.

Multiplying the likelihood with the prior (1.6) gives

$$\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\gamma-\hat{\gamma})'(\mathbf{Z}'\mathbf{Z})(\gamma-\hat{\gamma})\right\}\times\exp\left\{-\frac{1}{2\sigma_\gamma^2}\gamma'\mathbf{A}^{-1}\gamma\right\} \tag{2.23}$$

which implies that the posterior distribution of γ is normal with density

$$\begin{aligned} &P_N(\gamma \mid \beta,\sigma_\varepsilon^2,\sigma_\gamma^2,\mathbf{D} \sim N_q\left\{\tilde{\gamma},\left(\mathbf{Z}'\mathbf{Z}+\mathbf{A}^{-1}\frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\right)^{-1}\sigma_\varepsilon^2\right\}. \\ &\text{where } \tilde{\gamma} = \left(\mathbf{Z}'\mathbf{Z}+\mathbf{A}^{-1}\frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\right)^{-1}\mathbf{Z}'(\underline{\mathbf{Y}}-\mathbf{X}\beta).\end{aligned} \tag{2.24}$$

Equation (2.24) is identical to the conditional posterior derived in the traditional Bayesian case (equation (1.10)). Also, the conditional posterior density for σ_γ^2 in the BMOM case is identical to equation (1.12), this follows from the normal prior density (equation (1.6)).

Finally, we note that the post-data moments for σ_ε^2 can be employed to compute post-data densities for the realized error terms and functions of the realized errors that are often useful for diagnostic purposes as has been recognized in the traditional Bayesian analysis; see Chapter 1.

Having derived a range of post-data densities for the mixed linear model and indicating how BMOM analysis can be performed, we now turn to implement the Gibbs sampler to obtain the posterior densities for the model parameters.

2.4 The Gibbs Sampler

The Gibbs sampler is once again employed to obtain finite sample post-data parameter densities as described for the traditional Bayesian approach (Chapter 1) with one exemption that in the BMOM case, σ_ε^2 would be sampled from different maxent densities, i.e.

- $h_e(\sigma_\varepsilon^2 |, \mathbf{D}, \gamma) = \dfrac{1}{s^2}\exp\left\{-\dfrac{\sigma_\varepsilon^2}{s^2}\right\} \qquad 0 < \sigma_\varepsilon^2 < \infty$ (see also equation (2.16))

for the BMOM 1, using one moment, and for BMOM 2 using the recursive equation

- $E(\sigma_\varepsilon^{2j} \mid \mathbf{D}, \gamma) = \dfrac{(ks^2)^j + \sum_{i=1}^{j-1}\binom{j}{i}(ks^2)^{j-i}E(\sigma_\varepsilon^{2i} \mid \mathbf{D})[p(p+2)......(p+2(i-1))]}{n^j - [p(p+2)......(p+2(j-1))]}$

(see also equation (2.20))

and a Pearson curve approximation with four moments.

Thus, the Gibbs sampler for $p(\beta,\ \gamma, \sigma_\varepsilon^2, \sigma_\gamma^2 \mid \mathbf{D})$ is:

(0) Select starting values for $\gamma^{(0)}, \sigma_\varepsilon^{2\,(0)}, \sigma_\gamma^{2\,(0)}$. Set $i = 0$.

(1) Sample $\beta^{(i-1)}$ from (1.9),

(2) Sample $\sigma_\varepsilon^{2\,(i-1)}$ from (2.16; 2.20 or other densities),

(3) Sample $\gamma^{(i-1)}$ from (1.10),

(4) Sample $\sigma_\gamma^{2\,(i-1)}$ from (1.12),

(5) Set $i=i+1$ and return to (1).

2.5 Another Bayesian Method of Moments Approach for the Mixed Linear Model

In this section another approach to the BMOM analysis will be given which is "more distribution free" or "less likelihood" than the previous one.

As in section 2.3 the proper maxent densities of β and σ_ε^2 will be obtained by using Assumptions 1 and 2. These densities are given in equations (2.15) and (2.16). Higher order post-data moments of σ_ε^2 can also be calculated and used as moment side conditions in deriving other maxent densities. The derivations that follow will however be different from those given in equations (2.22) and (2.24).

Substitute starting values $\gamma^{(0)}$ and $\beta^{(0)}$ in equation (2.2), to calculate

$$\varepsilon^{(0)} = \underline{\mathbf{Y}} - \mathbf{X}\beta^{(0)} - \mathbf{Z}\gamma^{(0)}. \tag{2.25}$$

Also for given $\gamma^{(0)}$, draw $\sigma_\varepsilon^{2(1)}$ and $\beta^{(1)}$ from (2.15) and (2.16). A new γ which will be called

$$\gamma_{post}^{(1)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left(\underline{\mathbf{Y}} - \mathbf{X}\beta^{(1)} - \varepsilon^{(0)}\right) \tag{2.26}$$

can now be calculated as well as

$$\sigma_{\gamma}^{2(1)} = \frac{1}{q}\left(\gamma_{post}^{(1)\,\prime}\mathbf{A}^{-1}\gamma_{post}^{(1)}\right) \tag{2.27}$$

where "post" means "posterior". To implement the normal prior assumption for γ (equation (1.6)) draw a $\gamma_{post}^{(1)}$ from the normal distribution $N\left(\underline{0}, \sigma_{\gamma}^{2(1)}\mathbf{A}\right)$ and calculate

$$\varepsilon^{(1)} = \underline{\mathbf{Y}} - \mathbf{X}\beta^{(1)} - \mathbf{Z}\gamma_{prior}^{(1)} \tag{2.28}$$

to complete the first iteration. After k iterations in which the conditional distributions were updated at each iteration, the Gibbs sampler has generated the values $\beta^{(k)}$, $\gamma_{post}^{(k)}$, $\sigma_{\varepsilon}^{2(k)}$ and $\sigma_{\gamma}^{2(k)}$. The process is then repeated m times.

For our practical problem the BMOM posterior densities using in this section and those derived from the previous section were for all purposes the same. It is therefore clear that in the case of the BMOM analysis the posterior moments, based on a few relatively weak assumptions can be used to obtain post data densities (maximum entropy densities) for parameters and realized error terms without the use of the likelihood function or prior density. The assumption of no prior information has as consequence different types of derivations (post data densities) that differ from those obtained using the traditional Bayesian approach where prior information was assigned to the unknown parameters β, γ, σ_{ε}^{2} and σ_{γ}^{2}.

As mentioned no prior densities are necessary for the BMOM procedure but if some prior information is available it can be built into the BMOM procedure. Since the assumption of a normal prior for the random effects γ is an integral part of the mixed linear model, the prior density $\gamma \sim N(\underline{0}, \sigma_\gamma^2 \mathbf{A})$ is also used in the BMOM analysis.

2.6 An Example

2.6.1 The Data

Consider the Dormer sheep stud of Elsenburg (see section 1.8.1). Recall that the sheep used in the analysis were born in the period 1943 - 1950. A total of $n = 879$ weaning weight records, from the progeny of $q = 17$ sires were available after editing, and $p = 17$ fixed effects were included in the final model.

The mixed linear model used for this data structure, is the sire model of section (1.2), $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$, where $\underline{\mathbf{Y}}$ (879 × 1) vector of weaning weights. $\beta(17 \times 1)$ is the vector of fixed effects, and $\mathbf{X}$ a (879 × 17) design matrix of regressors, with one column corresponding to the *overall mean* weaning weight, seven columns corresponding to the *season of birth* effects, six to the *age of dam* effects, one to the *sex of lambs* effects, and two final columns corresponding to the *birth status* effects. Furthermore, $\mathbf{Z}$ is a (879 × 17) matrix identifying the (17 × 1) vector of random effects γ consisting of the breeding values for the 17 sires for which the data are observed.

Finally, ε is an unobservable vector of random residuals (879×1) such that the distribution of ε is assumed to be independent normal with mean vector $\underline{\mathbf{0}}$ and variance-covariance matrix $\sigma_\varepsilon^2\, \mathbf{I_n}$. $\mathbf{I_n}$ represents an identity matrix (879×879).

MATLAB software has once again been developed to generate the samples that enabled us to obtain the finite sample post-data parameter densities, using the Gibbs sampler. The full conditional posteriors are updated after every iteration. The first 1 000 draws of each chain are discarded, and then every 10^{th} draw is saved. By saving every 10^{th} draw, the chain yielded a posterior sample of 1 000 approximately uncorrelated draws. All posterior analyses are based on these m = 1 000 draws.

2.6.2 Analysis of Variance Components

Posterior modes of the Traditional Bayesian analysis from Chapter 1, post-data estimates obtained from the BMOM approach, drawing first from an exponential distribution (BMOM 1) and then from a Pearson Type 4 curve (BMOM 2), as well as the 95% credibility intervals for the variance components are summarized in Table 2.1. Functions of the variance components (ρ and v) are given in Table 2.2. The post-data densities for the variance components are provided in Figures 2.1 and 2.2, and for ρ and v in Figure 2.3.

Table 2.1 Traditional Bayesian Estimates (posterior modes) and Estimates from the BMOM Analysis of the Variance Components, along with 95% Credibility Intervals.

Parameters	Trad. Bayes	BMOM 1	95 % Credibility Interval (BMOM1)	BMOM 2	95 % Credibility Interval (BMOM2)
σ_ε^2	21.2595	21.1125	0.0000 ; 64.5875	20.2873	19.9979 ; 20.6214
σ_γ^2	3.01	3.86	1.5651 ; 14.3427	3.16	1.2464 ; 13.1652

From Table 2.1 it is evident that there is not much difference between the traditional Bayesian estimates and the BMOM estimates for the error variance, σ_ε^2. However, the 95% credibility interval for σ_ε^2 in the case of the BMOM1 differs substantially from the corresponding interval for BMOM 2 (and the Traditional Bayes). This is as expected because the exponential density is quite skewed. By imposing additional side conditions in the case of BMOM 2 and thus reducing entropy, the Pearson Type 4 curve is more informative than the exponential density. Note that the distribution in Figure 2.2 is quite skewed, resulting once again in a difference between the posterior means and posterior modes and discrepancies in credibility intervals.

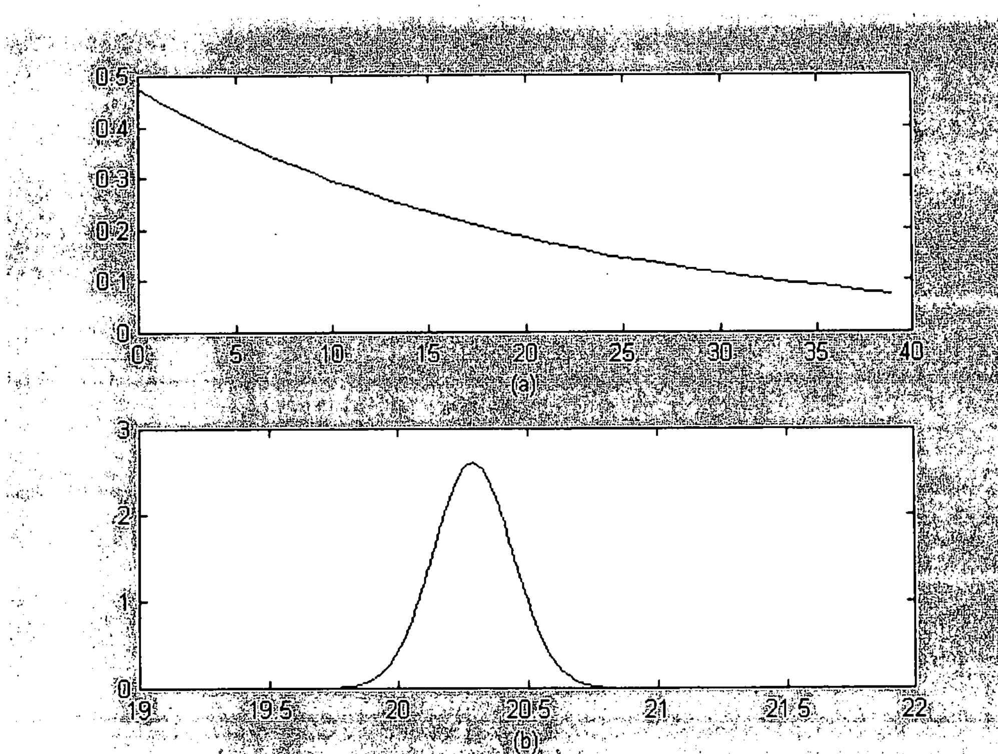

Figure 2.1 Estimated Marginal Post-data Densities for σ_ε^2 in the case of BMOM 1 (a) and BMOM 2 (b). Note that BMOM 1 is a proper maxent density (exponential) which has mean $E(\sigma_\varepsilon^2) = s^2$, while BMOM 2 is a proper Pearson type 4 curve density.

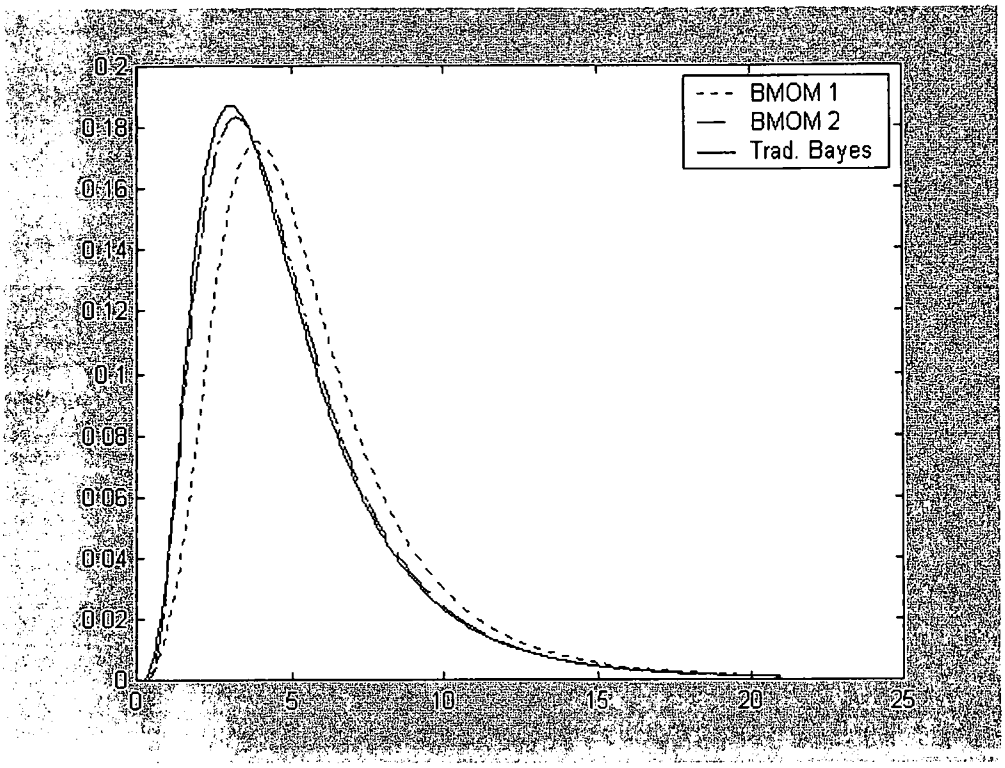

Figure 2.2 Estimated Marginal Post-data Densities for σ_γ^2 in the case of BMOM 1 (short dashed line), Mean = 5.6778; BMOM 2 (long dashed line), Mean = 5.0361 and the Traditional Bayesian Density (solid line), Mean = 4.9239.

We also observed the same type of parameter behavior for functions of the variance components in Table 2.2.

Table 2.2 Traditional Bayesian Estimates (posterior modes) and Estimates from the BMOM Analysis of Functions of the Variance Components, along with 95% Credibility Intervals.

Parameters	**Trad. Bayes**	**BMOM 1**	**95 % Credibility Interval (BMOM1)**	**BMOM 2**	**95 % Credibility Interval (BMOM2)**
ρ	0.133	0.102	0.0417 ; 0.8351	0.155	0.0577 ; 0.3944
v	0.140	0.082	0.0435 ; 10.2976	0.162	0.0613 ; 0.6513

Except for BMOM 1, the 95% credibility interval for the intraclass correlation coefficient does not contain 0.5. This result corresponds to the statement made by Wang, *et* al. (1993), namely that from a genetic point of view, an intraclass correlation of 0.5 is not possible in a sire model. Moreover, the 95% credibility of ρ in the case of BMOM 1 differs substantially from the corresponding intervals for BMOM 2 and Traditional Bayes. This was expected because the exponential density (the maxent using only one moment) is quite skewed. In practice usually two or more moments are available. The credibility intervals for BMOM 2 and Traditional Bayes on the other hand are for all purposes the same. Indeed, if proper priors were assigned to the variance components, and if the sample size was quite small, the difference between the BMOM and Traditional Bayes results could have been quite substantial. The assignment of a proper prior to the variance components must however be justifiable from a practical point of view. In some animal breeding experiments for example it is known that $\sigma_\gamma^2 < \frac{1}{3}\sigma_\varepsilon^2$. This information can, if necessary, be used to formulate a proper prior on the interval $\left[0; \frac{1}{3}\right]$ for the variance ratio $v = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$.

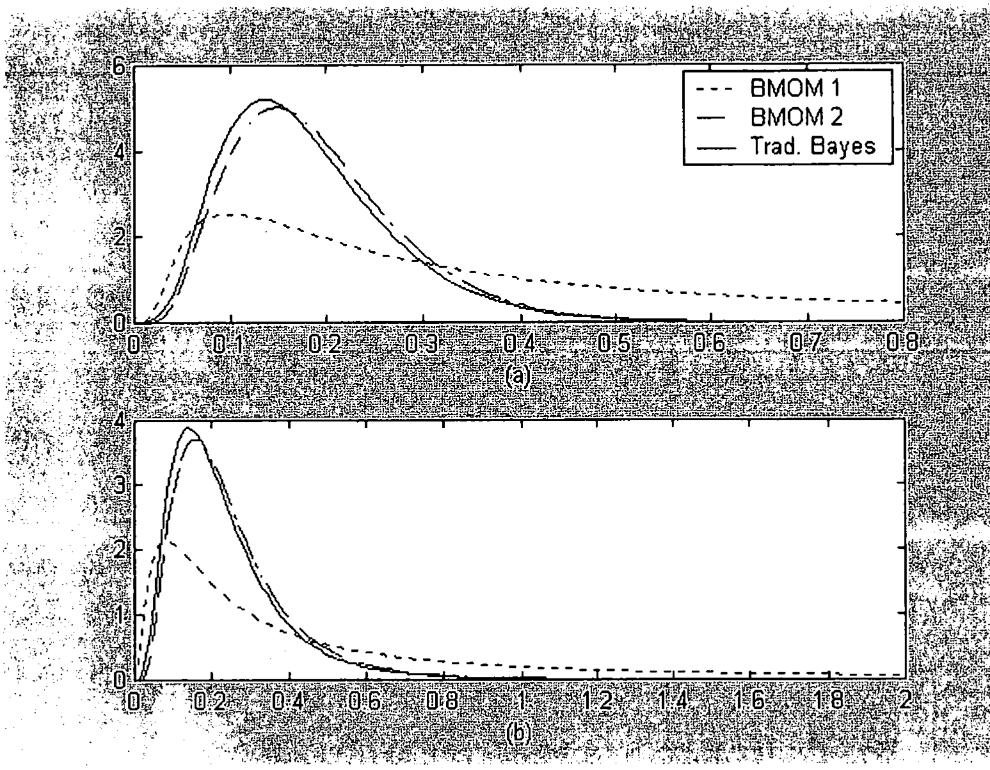

Figure 2.3 The Estimated Marginal Post-data Density of the (a) Intraclass Correlation Coefficient, $\rho = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, and the (b) Variance Ratio, $v = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$.

2.6.3 Analysis of Random Effects

The posterior distributions of the random effects can be obtained directly from the Gibbs sampler. Table 2.3 contains the post-data means and corresponding post-data rankings based on the mean values of the random effects (breeding values) for the 17 sires. It is evident from the table that the estimates using the different procedures are quite close to each other. The Traditional Bayes and BMOM 2 estimates are for all practical purposes the same.

Rather than to comment on results for all 17 sires, we will focus our discussion on the two animals ranked highest using the Traditional Bayes and BMOM analysis.

Table 2.3 Estimated Breeding Values for 17 Sires from the Elsenburg Dormer Stud, and Post-data Rankings using BMOM and Traditional Bayesian approaches. REML estimates along with Standard Errors are also included.

Sire ID	Trad Bayes	Rank	BMOM 1	Rank	BMOM 2	Rank	REML	Rank	SE's
41037	0.7350	3	0.8889	3	0.7098	3	0.5781	3	1.06
41004	0.2478	6	0.3397	6	0.2415	6	0.1396	6	0.92
41019	3.4858	1	3.6370	1	3.4931	1	3.3300	1	0.99
43002	-1.1985	14	-1.3089	14	-1.246	14	-1.1810	14	1.18
44170	-0.0930	7	-0.0387	7	-0.0943	7	-0.1700	7	1.18
44174	-0.6524	10	-0.7942	10	-0.6847	10	-0.5694	10	1.34
44042	-1.3053	15	-1.3338	15	-1.356	15	-1.2565	16	0.95
45070	-1.1460	13	-1.1793	13	-1.1833	13	-0.9631	13	0.93
45135	-0.5301	9	-0.5069	9	-0.5984	9	-0.5371	9	1.10
46015	-1.7983	17	-1.8758	17	-1.8861	17	-1.7092	17	0.96
46037	-0.8524	11	-0.8960	11	-0.9098	11	-0.8423	11	0.91
48014	-1.0059	12	-1.1001	12	-1.0759	12	-0.9537	12	0.97
48052	-0.4208	8	-0.4541	8	-0.4708	8	-0.3019	8	1.00
48148	-1.4307	16	-1.5019	16	-1.475	16	-1.2560	15	1.10
49053	0.5309	4	0.8152	4	0.4479	4	0.4630	4	1.31
49134	0.9219	2	1.3176	2	0.9641	2	0.7950	2	1.34
49046	0.4395	5	0.6804	5	0.3379	5	0.4059	5	1.41

The post-data densities for the top three sires and the sire ranked lowest (ID46015) are included in Figures 2.4 – 2.7

Consider the results of the BMOM 2 analysis for the discussion. As might be expected, the best two sires from the two analysis overlap, with the progeny from Sire 3 ranked 1st according to its Traditional Bayes, BMOM and REML estimates. With an estimated breeding value of 3.4931, the progeny from this sire will therefore have an estimated average weaning weight of 3.49 kilogram more than the progeny from the rest of the sires. Also, the progeny from Sire 16 (ID49134 and ranked 2nd), with an estimated breeding value of 0.7950 will have an estimated average weaning weight of 0.8 kilogram more than the average weaning weight of lambs from the rest of the sires. The rest of the estimates can be interpreted in the same fashion.

Another appealing feature of the proposed simulation approaches to BMOM and Traditional Bayes data analysis is that there are only minor disagreements on post-data rankings in the next fifteen sires. In comparing the REML and BMOM estimates, the only difference in posterior rankings is reported for Sire 7 (ID44042) and Sire 14 (ID48148). In the REML analysis, Sire 17 is ranked 16th and Sire 14 ranked 15th, whereas a visa versa ranking is evident from the Traditional Bayes and BMOM analysis. Once again this is not surprising to us, since as showed by Harville, (1974) (see also Searle, Casella and McCulloch, (1992)) that when uniform or "flat" priors are assigned to the vector of fixed effects and variance components and normal priors for the random effects, the modes of the marginal posterior distributions are very close to the Traditional Bayes estimates.

Thus, this sample provides adequate information to estimate any other quantities of interest, e.g. we can also address the question of how well we can detect the best animal, as well as probability distributions of rank positions for the top sires in the stud. Indeed, the estimates in Table 2.3 indicate that there is minor uncertainty about the exact breeding value of individual sires, and it likely indicates considerable certainty about the best selection.

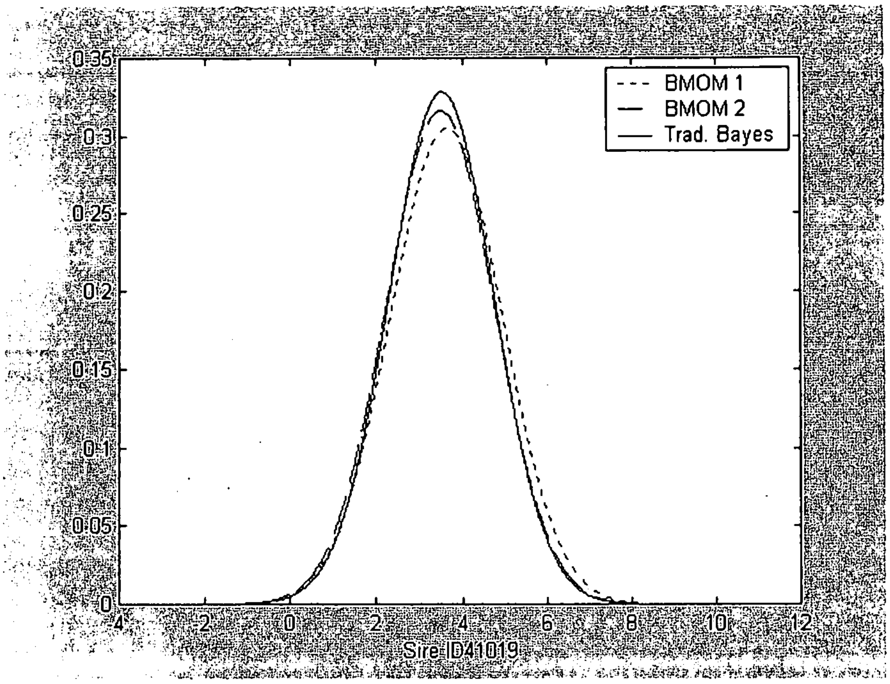

Figure 2.4 Estimated Marginal Post-data Densities for Sire 3 (ID41019) ranked 1st according to Traditional Bayes, BMOM and REML Estimates.

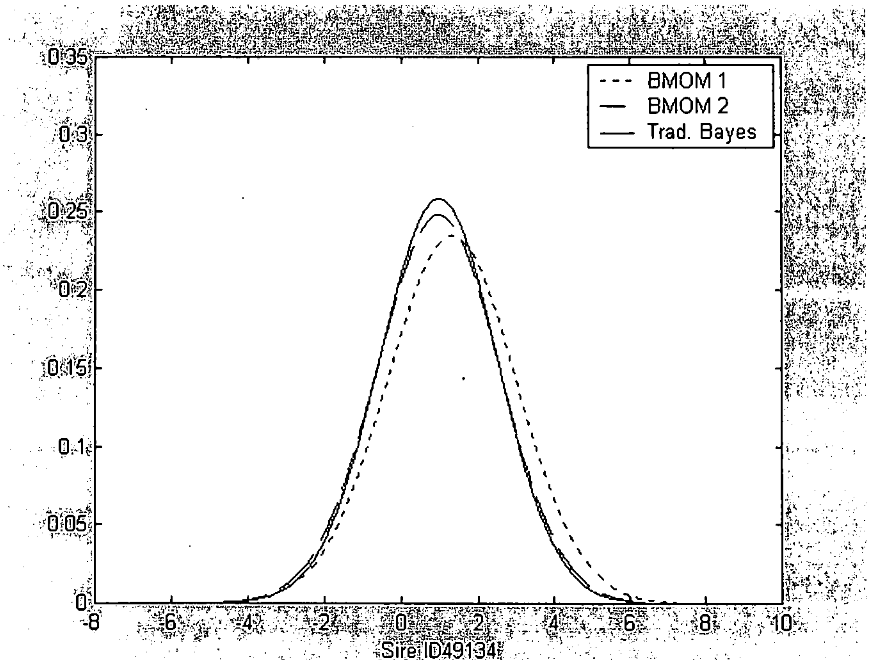

Figure 2.5 Estimated Marginal Post-data Densities for Sire 16 (ID49134) ranked 2nd according to Traditional Bayes, BMOM and REML Estimates.

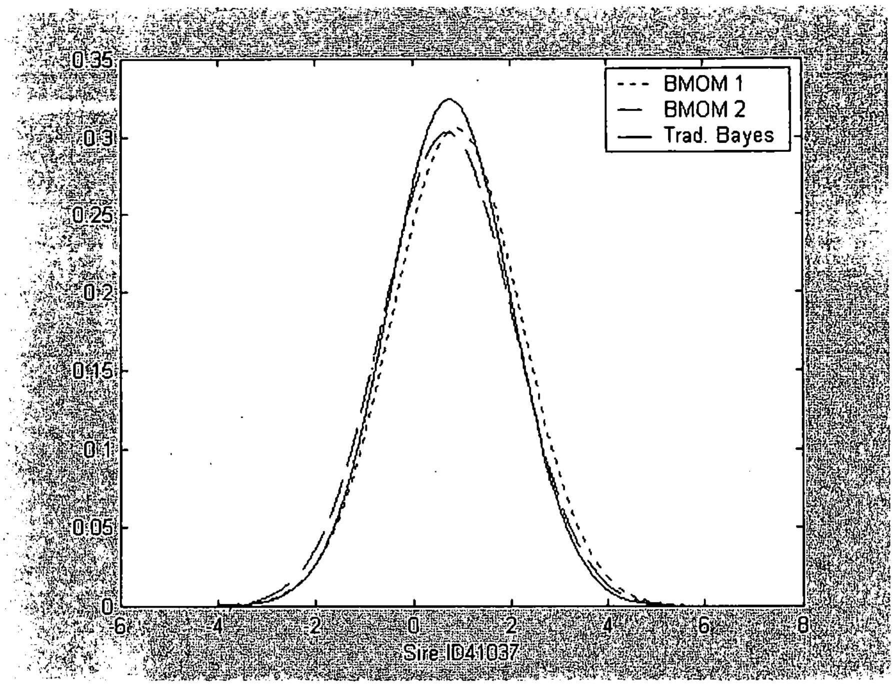

Figure 2.6 Estimated Marginal Post-data Densities for Sire 1 (ID41037) ranked 3rd according to Traditional Bayes, BMOM and REML Estimates.

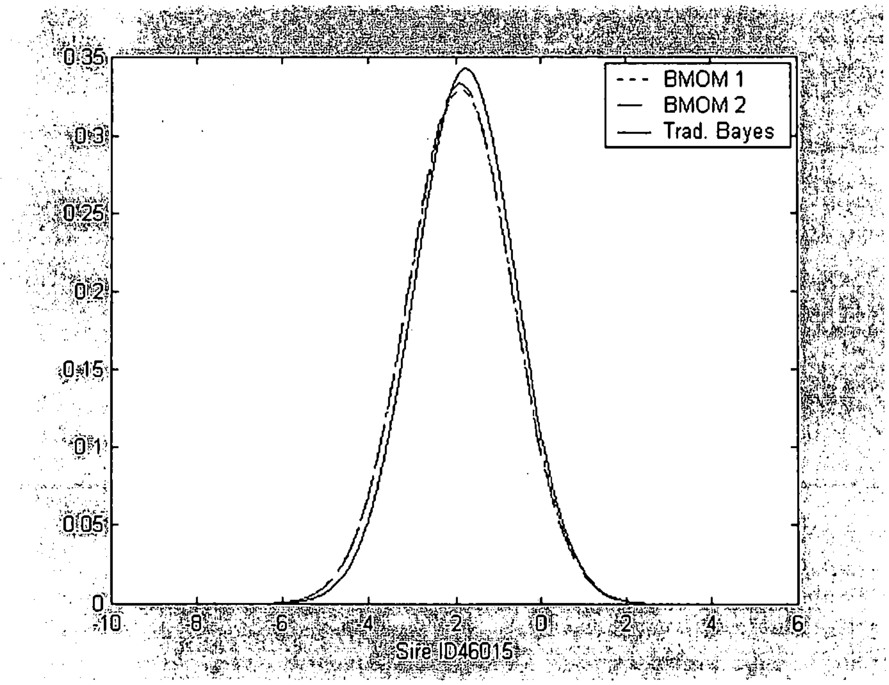

Figure 2.7 Estimated Marginal Post-data Densities for Sire 10 (ID46015) ranked 17th according to Traditional Bayes, BMOM and REML Estimates.

2.6.4 Analysis of Fixed Effects

The object of interest in this present section may not only be the values of the fixed effects, Table 2.4a – c, but also the post-data densities thereof. With respect to the values of the estimates, we have previously demonstrated how these values must be interpreted numerically. Following the results of the Traditional Bayes and BMOM analysis, we now focus upon the post-data densities of the fixed effects that are depicted in Figures 2.8 – 2.11.

Table 2.4 Traditional Bayesian Estimates (a), Estimates from the BMOM 1 (b) and Estimated from the BMOM 2 analysis (c).

(a)

Fixed Effect	**Trad. Bayes**	**95% Credibility Interval**
β_0	22.9655	19.2315 ; 26.9031
β_7	5.3523	4.1515 ; 6.5310
β_{14}	3.6690	2.9835 ; 4.3353
β_{15}	9.4874	7.1923 ; 11.7688
β_{16}	2.9621	0.6574 ; 5.2308

(b)

Fixed Effect	**BMOM 1**	**95% Credibility Interval**
β_0	23.0226	19.1496 ; 27.0333
β_7	5.2995	4.1210 ; 6.3710
β_{14}	3.6757	3.0873 ; 4.3134
β_{15}	9.5505	7.5584 ; 11.6362
β_{16}	3.0469	1.0836 ; 5.0166

(c)

Fixed Effect	**BMOM 2**	**95% Credibility Interval**
β_0	22.947	19.3359 ; 26.6482
β_7	5.3037	4.1175 ; 6.4561
β_{14}	3.6854	3.0037 ; 4.2923
β_{15}	9.4580	7.2886 ; 11.5086
β_{16}	2.9574	0.7831 ; 5.0853

An inspection of the post-data densities of the selected fixed effects shows that the difference between traditional Bayesian and BMOM results can be quite substantial, especially in the case of BMOM 1. Since no likelihood was assumed for the BMOM analysis, these post-data densities depended on the form of the maxent densities that in turn depended on the number of moments used. We observed that by imposing only one side condition (exponential density with one moment), the post-data density for BMOM 1 is more "spiked" than for traditional Bayes or BMOM 2. The double exponential effect of the marginal post-data densities of the fixed effects in the case of BMOM 1 can easily be recognized from Figures 2.8 – 2.11 below.

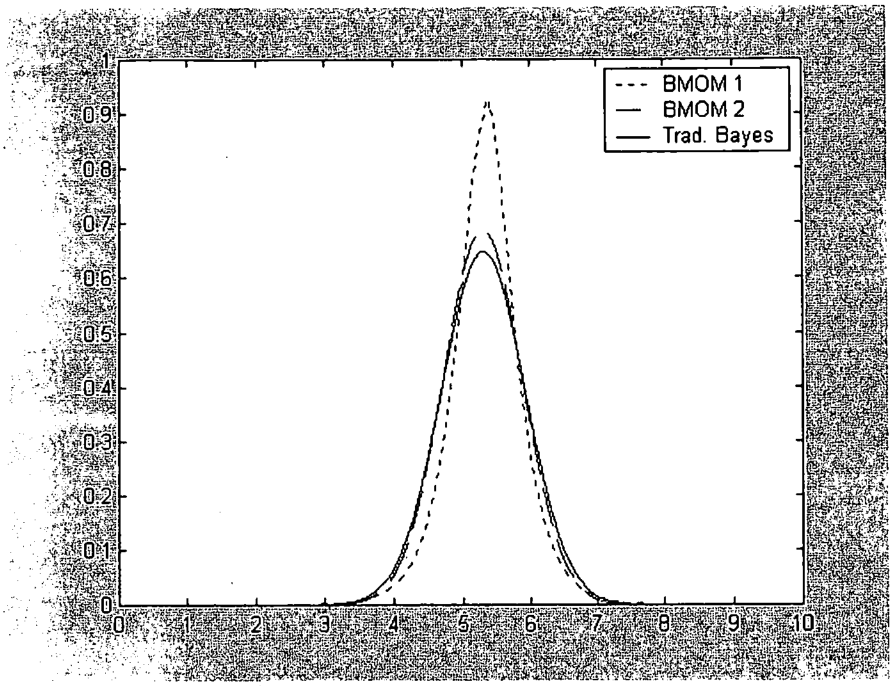

Figure 2.8 Estimated Marginal Post-data Densities for the Expected Difference in Average Weaning Weight between lambs born in 1949 and in 1950 as measured by β_7.

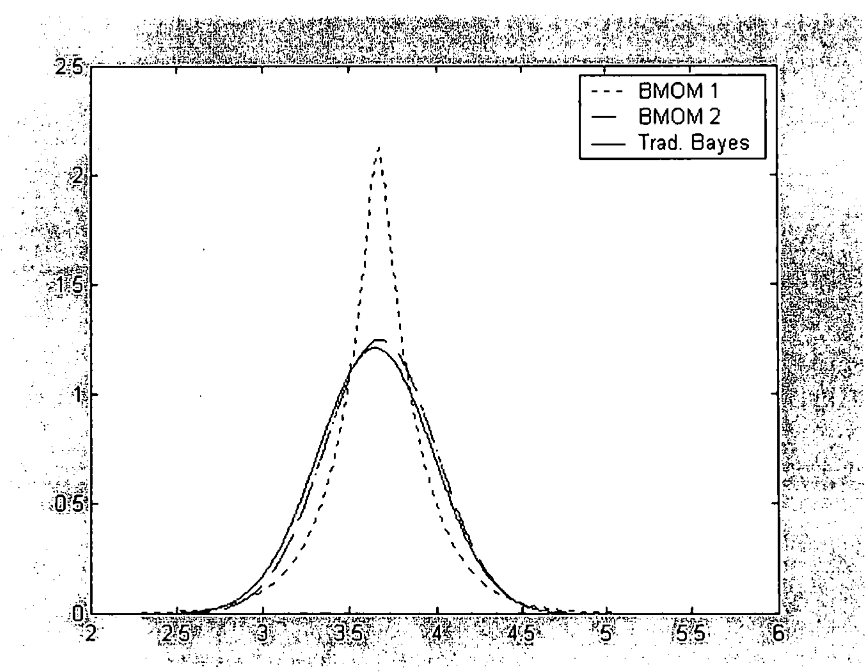

Figure 2.9 Estimated Marginal Post-data Densities for the Expected Difference in Average Weaning Weight between male and female lambs as measured by β_{14}.

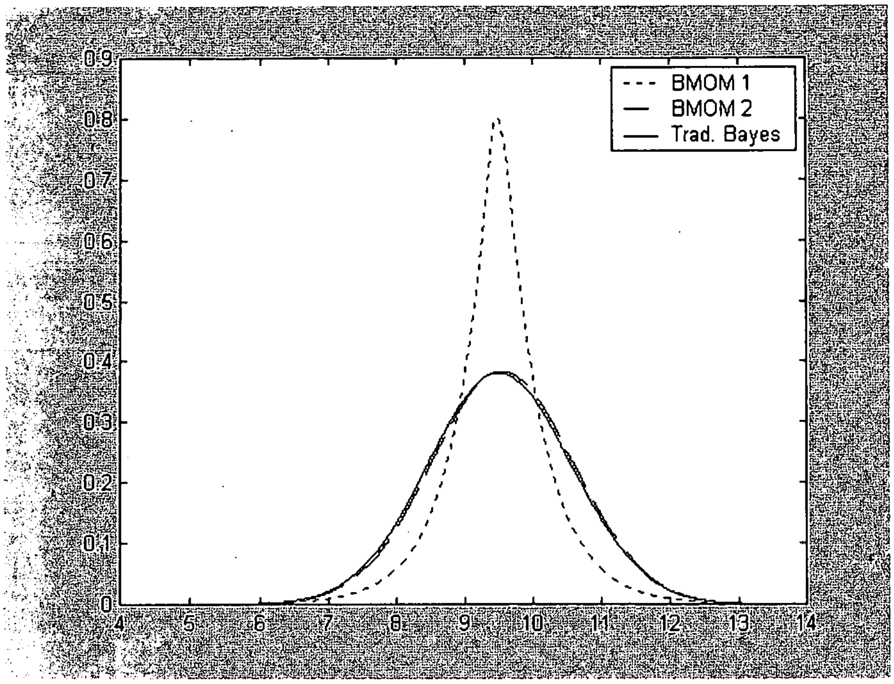

Figure 2.10 Estimated Marginal Post-data Densities for the Expected Difference in Average Weaning Weight between single births and triplets as measured by β_{15} .

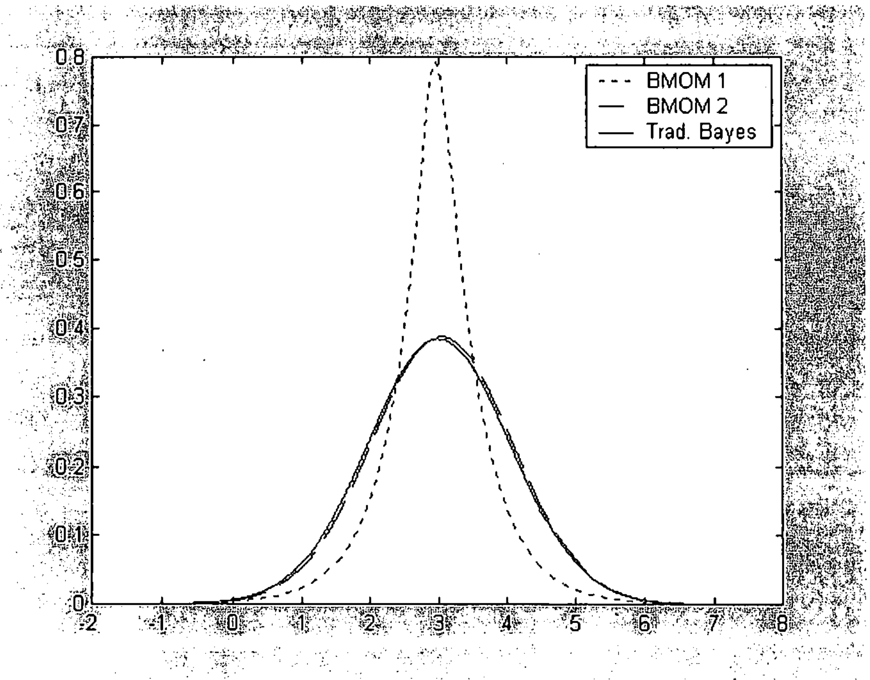

Figure 2.11 Estimated Marginal Post-data Densities for the Expected Difference in Average Weaning Weight between a pare of twins at birth and triplets as measured by β_{16}.

2.7 Chapter Summary

In this chapter we have indicated how to apply the Bayesian Method of Moments procedure in the analysis of the mixed linear model when information in not available to formulate a likelihood function. On introducing and proving simple assumptions relating to the moments of the realized error terms and the future, as yet unobserved error terms, we derived post-data moments of parameters and future values of the dependent variable. Using these moments as side conditions, proper maxent densities for the model parameters were derived and could easily be computed for the Dormer data set.

Further, it was evident that in the proposed BMOM approach, no sampling assumptions about the given observed data were made. Rather certain assumptions were made about the realized error terms' properties. Given these assumptions, posterior moments of the parameters were derived that incorporated the information in the given data. These moments were then used as side conditions in the derivation of maxent probability density functions for the parameters, realized error terms and future observations.

It was also shown that in the computed example, where use is made of the Gibbs sampler to compute finite sample post-data parameter densities, some BMOM maxent densities are very similar to the traditional Bayesian densities, whilst others are not. As mentioned several times before, this is expected, since as showed by Harville, (1974) (see also Searle, Casella and McCulloch, (1992)) that when uniform or "flat" priors are assigned to the vector of fixed effects and variance components and normal priors for the random effects, the modes of the marginal posterior distributions are very close to the Traditional Bayes estimates.

From the aforementioned, it should be appreciated that the BMOM approach yields useful inverse inferences without using assumed likelihood functions, prior densities for their parameters and Bayes' theorem. Hence, it is the case that the BMOM techniques extended in the present thesis to the mixed linear model provide valuable and significant solutions in applying traditional likelihood or Bayesian analysis in animal breeding problems.

"Finally, the BMOM is elegant and easily applicable because it is free from the strong UNVERIFIABLE assumptions that we usually make just in order to enable us to handle a problem."

© Parts of this chapter have been published in the South African Statistical Journal.
(See Van der Merwe *et al.* 2000)

© Parts of this chapter have been accepted for publication in the *'Collection of Refereed Articles' – ISBA2000*[1].
(See Van der Merwe and Pretorius 2001)

[1] International Society for Bayesian Analysis

CHAPTER 3

«The Dirichlet Process»

Introductory words: *It is very important to accurately model the distribution of the random effects when predictions of future observations from a given subject are desired. From the Bayesian perspective, inferential interest in the present chapter focuses on the posterior distribution of the random effects. Allowing distributions other than the normal for the random effects may more accurately model our prior beliefs, or it may allow us to better express our uncertainty about the true distribution of the random effects.*

3.1 Prologue

In his 1972 review of Bayesian statistics, Dennis Lindley identified as a success story for Bayesian ideas the advances made in problems of many parameters and the growth of what is now referred to as ***Bayesian Hierarchical Modeling***. He also identified non-parametrics as an area notable for lack of Bayesian progress, bemoaning the fact that non-parametric statistics was a 'subject about which the Bayesian method is embarrassingly silent'. However, it is undoubtedly the case that the wide application of hierarchical models is one of the major success stories of modern Bayesian statistics since the early nineties, with tremendous growth and substantial contributions via Markov chain simulations on problems usually referred to as Non-Parametric Density Estimation. Simultaneously, these computational methods allow development and application of data and prior models that significantly extend the scope for closer representation of real-world problems.

Mixture priors, especially Dirichlet Mixtures have opened the way to serious Bayesian developments in (so-called) ***Non-parametric Modeling*** and ***Density Estimation***. It is my purpose in this chapter to exhibit the Mixed Linear Model for non-parametric modeling and density estimation, to show how posterior computations via Gibbs sampling simulations can be routinely applied, and to provide illustrative examples from animal breeding problems. There has been some work towards this end in the classical setting. In the Bayesian paradigm, it has been accomplished for the repeated measures (West, Muller, & Escobar, 1994) and for the randomized complete block design (Bush & MacEachern, 1996).

We provide a general framework for Bayesian analysis of mixed linear models in which a non-parametric Dirichlet process prior is specified for the random effects. Only recently have tools allowing Bayesian analysis to become computationally feasible; here we provide a detailed exploration of an animal breeding application of interest (see also Kleinman and Ibrahim, (1998)).

3.2 The Classical Perspective

From the classical perspective, the distribution of the random effects has an important effect on some quantities of interest. Changing the distribution of the random effects will change the estimated random effect for each individual. This point is important because there are many applications in which an estimate of the random effect itself is desired. For example, in Tsiatis, DeGruttola, and Wulfsohn (1995), the estimated random effects are used alternatively as covariates themselves in a Cox regression model or to create values for time-varying covariates in such a model.

Similarly, Mori, Woodworth, and Woolson (1992), De Gruttola and Tu (1994), and Wu and Carroll (1988) all present complex models in which the random effects are both estimated and used in predicting other pieces of the model. In such applications, unbiased estimation of the random effects is crucial and the assumption of normality may introduce bias (Kleinmann & Ibrahim, 1998).

Classical non-parametric and semi-parametric methods have a measure of popularity, e.g. the Kaplan-Meier estimator, kernel density estimation, and Cox regression. No population distributional assumptions are made in any of these cases, except for the proportional hazards assumption in the case of Cox regression. We argue that a state of no knowledge at all is hardly, if ever, realistic: we would typically at least have some ideas concerning location and spread. Such information can be incorporated into a Bayesian non-parametric prior.

3.3 The Bayesian Perspective

From the Bayesian perspective, inferential interest focuses on the posterior distribution of the random and fixed effects. Allowing distributions other than the normal for the random effects, may more accurately model our prior beliefs, or it may allow us to better express out uncertainty about the true distribution of the random effects. It is also very important to accurately model the distribution of the random effects when prediction for a future observation from a given subject is desired. Another situation in which it would be desirable to relax the assumption of normality is when inference is to be made about the distribution of the random effects itself.

Another attraction of our approach is that it allows exact Bayesian inference, even in small sample sizes. This is accomplished through the use of the Gibbs sampler. Computational tools are developed and demonstrated how the Gibbs sampler can be implemented for the mixed linear model. It is also showed how to make Bayesian inference for all of the model parameters in the model.

3.4 The Mixture of Dirichlet Process (MDP)

3.4.1 Background

MDP models have become increasingly popular for modeling when conventional parametric models would impose unreasonably stiff constraints on the distributional assumptions. Examples include empirical Bayes problems (Escobar, 1994), non-parametric regression (Müller, Erkanli & West, 1996), density estimation (Escobar & West, 1995; Gasparini, 1996), hierarchical modeling (MacEachern, 1994; West, Müller & Escobar, 1994) etc. Despite this large variety of applications, the core of MDP models can basically be thought of as a simple Bayes model given by the likelihood and prior.

Mixture of Dirichlet process priors can be of great importance in animal breeding experiments especially in the case of undeclared preferential treatment of animals. As long as genetic evaluation systems lack information about such preferential treatment, predictions of breeding values of favored animals obtained with mixed Gaussian models are inflated whereas those of other animals are deflated. So far, this problem has not yet been solved satisfactorily. Prof. Gianola, well-known animal breeder, suggested a "robust" mixed effect linear model based on the t – distribution for "preferential treatment problems". The t – distribution however does not cover departure form symmetry whilst the Dirichlet process prior will be able to do so.

3.4.2 The Model Structure

As mentioned before, an appropriate mixed linear model for a problem arising from animal breeding experiments is given by

$$\underline{\mathbf{y}}_i = \mathbf{X}_i\beta + \mathbf{z}_i\gamma_i + \varepsilon_i \qquad (3.1)$$

$\underline{\mathbf{y}}_i$ is a $n_i \times 1$ vector of weaning weights for the progeny of the i^{th} sire; $\beta\,(p \times 1)$ is a vector of fixed effects uniquely defined so that the corresponding design matrix $\mathbf{X}_i\ (n_i \times p)$ has full column rank, p. Also, $\mathbf{z}_i$ is a vector of n_i elements 1, γ_i is the unobservable random effect of sire i, and for the unobservable vector of random residuals, $\varepsilon_i\ (n_i \times 1)$, it is common to assume a multivariate normal distribution with mean vector $\underline{\mathbf{0}}$ and variance-covariance matrix $\sigma_\varepsilon^2 I_{n_i}$ where I_{n_i} represents a $n_i \times n_i$ identity matrix and σ_ε^2 an unknown scalar (error variance) .

However, for γ_i $(q \times 1)$, the vector of unobservable random effects which is usually taken to be normally distributed with mean zero, the normal prior is replaced with a non-parametric prior, followed by a Dirichlet process prior on the general distribution. In the section to come, it is illustrated how to apply the Mixture of Dirichlet Process Prior to the mixed linear model.

3.4.3 The Dirichlet Process Prior in the case of the Mixed Linear Model

As in Kleinman and Ibrahim (1998) we will present a mixed linear model for which the random effects have a non-parametric distribution. The non-parametric Bayesian approach for the random effects is to specify a prior distribution on the space of all possible distribution functions. This prior for the mixed linear model is applied to the general prior of the distribution of the random effects. This can be accomplished with a Dirichlet process prior distribution. This means that the usual normal prior on the random effects is replaced with a non-parametric prior, followed by a Dirichlet prior on the general distribution. The foundation of this technology is discussed in Ferguson (1973), where the Dirichlet process and its usefulness as a prior distribution are discussed. The practical application of such models, using Gibbs sampling, has been pioneered by several researchers, e.g. Doss (1994), MacEachern (1994), Escobar (1996), Lui (1996) and West *et al.* (1994).

Assume that G is sampled from a Dirichlet process with parameter G_0 and M, where G_0 is a probability measure and M is a positive real constant, i.e.

$$G \sim DP(M \cdot G_0) \tag{3.2}$$

The parameter G_0, often called the base prior, is a location parameter for the Dirichlet process prior and it approximates the true non-parametric shape of G. Thus, it is the best guess at what G is believed to be and $E(G) = G_0$. The role of G_0 for the Dirichlet process prior is similar to the role that the median and mean play in the typical prior distribution; it is the location parameter. It is thus our best guess of where the true values are. Therefore, if there are prior subjective beliefs, prior expert opinions, or theoretical considerations that G belongs to a small, finite set of possible distributions, then the prior distribution of G_0 should have support on this set. If the set of distributions is not finite, then a finite subset of "typical" distributions that belong to this set could be chosen that represents the larger set (Escobar, 1994). Because the algorithms developed in the present thesis will average over the posterior distribution of G_0, a natural smoothing occurs. Also, this Dirichlet process prior is a third-stage prior in a hierarchical Bayesian structure. When estimating normal means, it is common to assume that G is a normal distribution with unknown mean and variance. We will let G_0 be a normal distribution and use the data to estimate the model parameters.

The parameter M, a type of dispersion parameter for the Dirichlet process prior, is a measure of the strength in the belief that G is G_0. Although it may be hard to quantify, M is a positive scalar that is related to how "clumpy" the data are (often called a precision parameter). Clumpy data occur when the different sires are concentrated into a few clusters. In practice it is difficult to select appropriate values for this parameter. Instead, it is suggested to place a prior distribution on this parameter, and simulate it given the data. West (1992) assumed that $M \sim Ga(a,b)$ a gamma prior with $a > 0$ and scale $b > 0$. We may extend this idea to include a reference prior (uniform for $log(M)$) by letting $a \to 0$ and $b \to 0$. In the final section of the chapter we use the latter, which means that $p(M) \propto M^{-1}$ and $\mathrm{M} > 0$.

To simplify the use of the Dirichlet process prior, note that when G is integrated over its prior distribution, the sequence of γ_i follows a general Polya urn scheme; that is

$$\gamma_i \sim G_0 \qquad (i = 1,\dots,q), \tag{3.3}$$

$$\gamma_q \mid \gamma_1,\dots,\gamma_{q-1} \begin{cases} = \gamma_j & \textit{with probability} \quad \dfrac{1}{M+q-1} \\ \sim G_0 & \textit{with probability} \quad \dfrac{M}{M+q-1} \end{cases} . \tag{3.4}$$

From (3.4) it is easy to sample a sequence of $\gamma_1, \gamma_2, \ldots, \gamma_q$ given G_0 and M. There are two special cases in which the mixture of Dirichlet process models lead to the fully parametric case. As $M \to \infty$, $G \to G_0$ so that the base prior is the prior distribution of the random effects. Also, if $\gamma_i \equiv \gamma_j$ for all i, the same is true. When G is fully parametric, the joint posterior can easily be found. For the implementation of the Dirichlet process prior, the different γ_i 's must be considered separately. We find that the conditional posterior of γ_i is given by

$$
\begin{aligned}
&p(\gamma_i \mid \beta, \sigma_\varepsilon^2, \sigma_\gamma^2, \gamma_{-i}, M) \propto \sum_{j \neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i\beta + \mathbf{z}_i\gamma_j, \sigma_\varepsilon^2 I_{n_i}) \cdot \delta_{\gamma_j} \\
&+ \left\{ M \int_{-\infty}^{\infty} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i\beta + \mathbf{z}_i\gamma_i, \sigma_\varepsilon^2 I_{n_i}) \phi(\gamma_i \mid 0, \sigma_\gamma^2) d\gamma_i \right\} \qquad (3.5) \\
&\times \phi(\gamma_i \mid 0, \sigma_\gamma^2) p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j)
\end{aligned}
$$

where $p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j) = \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}\beta + \mathbf{z}_i\gamma_i, \sigma_\varepsilon^2 I_{n_i})$ and $\phi(\cdot \mid \mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2. Also, γ_{-i} denotes the vector of random effects for the sires excluding sire i and δ_s is a degenerated distribution with point mass at s.

Consider the integral

$$
\begin{aligned}
\Lambda_i &= M \int_{-\infty}^{\infty} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i\beta + \mathbf{z}_i\gamma_i, \sigma_\varepsilon^2 I_{n_i}) \phi(\gamma_i \mid 0, \sigma_\gamma^2) d\gamma_i \\
&= M \int_{-\infty}^{\infty} \left(\frac{1}{2\sigma_\varepsilon^2 \pi} \right)^{\frac{n_i}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} (\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma_i)'(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma_i) \right\} \times \qquad (3.6) \\
&\left(\frac{1}{2\sigma_\gamma^2 \pi} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\gamma_i^2}{2\sigma_\gamma^2} \right\} d\gamma_i .
\end{aligned}
$$

The exponent of the integral is $\frac{1}{\sigma_\varepsilon^2}\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma_i\right)\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma_i\right) + \frac{\gamma_i^2}{\sigma_\gamma^2}$. This can be written as

$$\frac{1}{\sigma_\varepsilon^2}\left(\underline{\tilde{\mathbf{y}}}_i - \mathbf{z}_i\gamma_i\right)\left(\underline{\tilde{\mathbf{y}}}_i - \mathbf{z}_i\gamma_i\right) + \frac{\gamma_i^2}{\sigma_\gamma^2}$$

$$= \frac{1}{\sigma_\varepsilon^2}\underline{\tilde{\mathbf{y}}}_i'\underline{\tilde{\mathbf{y}}}_i - 2\frac{1}{\sigma_\varepsilon^2}\underline{\tilde{\mathbf{y}}}_i'\mathbf{z}_i\gamma_i + \frac{1}{\sigma_\varepsilon^2}\gamma_i^2\mathbf{z}_i'\mathbf{z}_i + \frac{\gamma_i^2}{\sigma_\gamma^2}$$

where

$$\underline{\tilde{\mathbf{y}}}_i = \underline{\mathbf{y}}_i - \mathbf{X}_i\beta.$$

Following usual algebra routes, i.e. completing the square with respect to γ_i, we find

$$\Lambda_i = M(2\pi)^{-\frac{1}{2}(1+n_i)}\left(\sigma_\gamma^2\right)^{-\frac{1}{2}}\left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}}\exp\left\{\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta)'\Phi_i(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta)\right\}\left[\frac{1}{\sigma_\varepsilon^2}(\mathbf{z}_i'\mathbf{z}_i) + \frac{1}{\sigma_\gamma^2}\right]^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}$$

$$\Lambda_i = M(2\pi)^{-\frac{n_i}{2}}\left(\sigma_\gamma^2\right)^{-\frac{1}{2}}\Omega_i^{\frac{1}{2}}\left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta)'\Psi_i(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta)\right\} \qquad (3.7)$$

where

$$\Omega_i = \left[\frac{1}{\sigma_\varepsilon^2}(\mathbf{z}_i'\mathbf{z}_i) + \frac{1}{\sigma_\gamma^2}\right]^{-1} \quad \text{and} \quad \Psi_i = \left(\frac{1}{\sigma_\varepsilon^2}\mathbf{z}_i\Omega_i\mathbf{z}_i' - I_{n_i}\right).$$

Thus, after implementing the results in (3.7) into (3.6), we get

$$p(\gamma_i \mid \beta,\sigma_\varepsilon^2,\sigma_\gamma^2,\gamma_{-i},M) \propto \left(\sum_{j=1} \left(\sigma_\varepsilon^2\right)^{\frac{n_i}{2}} \exp\left\{ \frac{1}{2\sigma_\varepsilon^2} \left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma_i\right)' \left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma_i\right) \right\} \delta_{\gamma_j} \right) + M\,\Omega_i^{\frac{1}{2}} \left(\sigma_\gamma^2\right)^{-\frac{1}{2}} \left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}} \times \phi(\gamma_i \mid 0,\sigma_\gamma^2)\, p(\underline{\mathbf{y}}_i \mid \gamma_i,\beta,\sigma_\varepsilon^2,\underline{\mathbf{y}}_j). \tag{3.8}$$

In the above specification, each summand in the conditional posterior of γ_i is separated into two elements. The first element is a mixing probability, and the second is a distribution to be mixed. So with probability proportional to

$$\frac{\phi(\underline{\mathbf{y}}_i \mid X_i\beta + z_i\gamma_j;\sigma_\varepsilon^2 I_{n_i})}{\Lambda_i + \sum_{j=1;j\neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid X_i\beta + z_i\gamma_j;\sigma_\varepsilon^2 I_{n_i})} \tag{3.9}$$

we select γ_i from distribution δ_{γ_j}, which means that we set $\gamma_i = \gamma_j$. Also, with probability proportional to

$$\frac{\Lambda_i}{\Lambda_i + \sum_{j=1;j\neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid X_i\beta + \mathbf{z}_i\gamma_j;\sigma_\varepsilon^2 I_{n_i})} \tag{3.10}$$

we select γ_i from

$$p(\gamma_i \mid \beta,\sigma_\varepsilon^2,\sigma_\gamma^2,\underline{\mathbf{y}}_i) \propto \phi(\gamma_i \mid 0,\sigma_\gamma^2)\, p(\underline{\mathbf{y}}_i \mid \gamma_i,\beta,\sigma_\varepsilon^2,\underline{\mathbf{y}}_j). \tag{3.11}$$

Thus, we sample γ_i from its full conditional posterior,

$$p(\gamma_i / \beta, \sigma_\gamma^2, \sigma_\varepsilon^2, \underline{\mathbf{y}}_i) = N\left\{(\mathbf{z}_i'\mathbf{z}_i + \frac{\sigma_\varepsilon^2}{\sigma_\gamma^2})^{-1}\mathbf{z}_i'(\underline{\mathbf{y}}_i - X_i\beta);(\mathbf{z}_i'\mathbf{z}_i + \frac{\sigma_\varepsilon^2}{\sigma_\gamma^2})^{-1}\sigma_\varepsilon^2\right\}. \quad (3.12)$$

This results in a mixture distribution where one piece is a normal distribution and all of the others are point masses. There is some plausible intuition behind this above mixture scheme. If the breeding value, γ_i of sire i has a relatively large residual using sire j's breeding value, then γ_j is relatively less likely to be chosen as the breeding value of sire i. Conversely, if the breeding value of sire i has a relatively small residual using sire j's breeding value, then the random effect γ_j is relatively more likely to be chosen as the breeding value of sire i. On the other hand, the greater the residual for sire i, the greater the probability that sire i will get a new value from $p(\cdot\,,\cdot)$ in (3.12).

This scheme results in what MacEachern (1994) calls a cluster structure among the different sires. This cluster structure partitions the q different sires into k groups, where $0 < k \leq q$. Thus, all the sires in a specific cluster will have identical breeding values and sires in different clusters will have different breeding values. This may sound farfetched, but since the Gibbs sampler is repeated several times, the algorithm leads to reduced variation and hence faster convergence of the estimated random effects to their true values. The average of the simulated values for each breeding value is then computed, thus every sire will have its own unique breeding value.

The fully conditional posterior density for each of the other unknowns is obtained by regarding all other parameters in the joint posterior as known.

3.4.4 The Uniform Prior for β and σ_ε^2

The full conditionals for β and σ_ε^2 in the non-parametric model (3.1) are the same as in the parametric model. Thus, an uniform prior distribution is assigned to β and σ_ε^2 as to represent lack of prior knowledge about the vector of fixed effects and error variance. Therefore

$$p(\beta, \sigma_\varepsilon^2) = p(\beta)\, p(\sigma_\varepsilon^2) \propto \text{constant}. \tag{3.13}$$

The required full conditional for the fixed effects, is multivariate normal:

$$\beta \mid \gamma_i, \sigma_\varepsilon^2, \underline{\mathbf{y}}_i, \sim N_p \left\{ \hat{\beta}, \left(\sum_{i=1}^{q} (\mathbf{X}_i'\mathbf{X}_i) \right)^{-1} \sigma_\varepsilon^2 \right\} \tag{3.14}$$

where $\hat{\beta} = \left(\sum_{i=1}^{q} (\mathbf{X}_i'\mathbf{X}_i) \right)^{-1} \sum_{i=1}^{q} \mathbf{X}_i'(\underline{\mathbf{y}}_i - \mathbf{z}_i\gamma_i)$.

For the variance component, σ_ε^2 the conditional posterior is

$$p(\sigma_\varepsilon^2 \mid \beta, \gamma, \underline{\mathbf{y}}) = K_\varepsilon \prod_{i=1}^{q} \left(\frac{1}{\sigma_\varepsilon^2} \right)^{\frac{n_i}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} (\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma)'(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{z}_i\gamma) \right\}$$
$$\sigma_\varepsilon^2 > 0 \tag{3.15}$$

an Inverse Gamma density where

$$K_{\varepsilon} = \left\{ \frac{\sum_{i=1}^{q} (\underline{\mathbf{y}}_i - \mathbf{X}_i \beta - \mathbf{z}_i \gamma)'(\underline{\mathbf{y}}_i - \mathbf{X}_i \beta - \mathbf{z}_i \gamma)}{2} \right\}^{\frac{n-2}{2}} \frac{1}{\Gamma(\frac{n-2}{2})}.$$

Also, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_q)$, $\underline{\mathbf{y}} = (\underline{\mathbf{y}}_1', \underline{\mathbf{y}}_2', \dots, \underline{\mathbf{y}}_q')'$ and $\sum_{i=1}^{q} n_i = n$, the sample size.

3.4.5 Prior for σ_γ^2

Typically, the variance σ_γ^2 in the base measure of the Dirichlet process in (3.2) is unknown and therefore a suitable prior distribution must be specified for it. Note that, once this has been accomplished the base measure is no longer marginally normal. For convenience, suppose

$$p(\sigma_\gamma^2) \propto \text{constant}$$

to present lack of prior knowledge about σ_γ^2. After choosing random effects for each of the sires, the sires will be grouped into clusters (groups) in which the sires have equal γ_i's. That is, after selecting a new γ_i for each sire i in the sample, there will be some number k, $0 < k \le q$, of unique values among the random γ_i's. Denote these unique values by λ_l, $l=1,\dots,k$. Additionally let l represent the set of sires with common random effect λ_l.

Note that knowing the random effects is equivalent to knowing k, all of the λ_l's and the cluster membership l. Then for the purposes of calculating the full conditional of σ_γ^2, the λ_l are k independent observations from $N(0, \sigma_\gamma^2)$.

Thus

$$p(\sigma_\gamma^2 \mid \lambda, \underline{y}) = K_\lambda \left(\frac{1}{\sigma_\gamma^2}\right)^{\frac{k}{2}} \exp\left\{-\frac{1}{2\sigma_\gamma^2}\lambda'\lambda\right\} \qquad \sigma_\gamma^2 > 0 \tag{3.16}$$

an Inverse Gamma density where

$$K_\lambda = \left\{\frac{\lambda'\lambda}{2}\right\}^{\frac{k-2}{2}} \frac{1}{\Gamma\left(\frac{k-2}{2}\right)}$$

and

$$\lambda = [\lambda_1 \lambda_2 \ldots \lambda_k]'.$$

Bush and MacEachern (1996) and Kleinman and Ibrahim (1998) recommended one additional piece of the model as an aid to convergence for the Gibbs sampler. To speed mixing over the entire parameter space, they suggest moving around the λ's after determining how the γ_i's are grouped. Thus, in addition, a conditional posterior density is derived for the λ's, i.e.

$$p(\lambda_l \mid \beta, \sigma_\varepsilon^2, \sigma_\gamma^2, \underline{y}) \propto \phi(\lambda_l \mid 0, \sigma_\gamma^2) \prod_{i\in l} p(\underline{y}_i \mid \beta, \sigma_\varepsilon^2) \tag{3.17}$$

which implies that

$$\lambda_l \mid \beta, \sigma_\varepsilon^2, \sigma_\gamma^2, \underline{\mathbf{y}} \sim N\left\{\tilde{\tilde{\gamma}}_l, \left(\sum_{i\in l} \mathbf{z}_i{}'\mathbf{z}_i + \frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\right)^{-1} \sigma_\varepsilon^2\right\} \tag{3.18}$$

where

$$\tilde{\tilde{\gamma}}_l = \left(\sum_{i\in l} \mathbf{z}_i{}'\mathbf{z}_i + \frac{\sigma_\varepsilon^2}{\sigma_\gamma^2}\right)^{-1} \left(\sum_{i\in l} \mathbf{z}_i{}'\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta\right)\right). \tag{3.19}$$

This additional piece is now incorporated into the final Gibbs sampler. Before the Gibbs sampler is presented, we first address the simulation of the precision parameter, M. In the preceding section we assumed that this parameter for the Dirichlet process prior was fixed. In practice it is difficult to select appropriate values for M. Instead, a prior distribution is placed on M and a posterior distribution is derived. Because this parameter has an important influence on the estimation, special care is going into the selection of a broad range of values for M and into the simulation thereof.

3.4.6 Simulation of the Precision Parameter M

When defining a Dirichlet process prior, recall that M represents the weight of our believe that G is the distribution of G_0. This parameter thus determines the prior distribution of k, the number of additional normal components in the mixture, and is a critical smoothing parameter of the mixed linear model. M is also, as mentioned before, related to how "clumpy" the data are. When there are only a few clusters among the sires in the model, the estimate of the normal means from the Dirichlet process prior will be similar to the non-parametric Bayes estimator. When there are almost q (random effects) different clusters, the estimator from the Dirichlet process prior will be similar to the parametric Bayes estimator. Thus, the parameter M adjusts this estimator to behave like either a parametric estimator, which uses the data in a global manner, or a non-parametric estimator, which uses the data in a local manner. In Antoniak (1974) it is shown that the prior distribution of k, the number of clusters, may be written as

$$p(k \mid M, q) = c_q(k) q! M^k \frac{\Gamma(M)}{\Gamma(M+q)} \qquad k = 1, 2, \ldots, q \tag{3.20}$$

and $c_q(k) = p(k \mid M = 1, q)$, not involving M. West (1992) mentioned that if required, the factors $c_q(k)$ can easily be computed using recurrence formulae for Stirling numbers. It is also shown that the conditional posterior distribution of M is given by

$$p(M \mid k, \beta, \gamma, \sigma_\varepsilon^2, \sigma_\gamma^2, \underline{\mathrm{y}}) = p(M \mid k) \propto p(M) p(k \mid M) \tag{3.21}$$

where $p(M)$ is the prior and the likelihood function is defined in (3.20). West (1992) also assumed $M \sim Ga(a,b)$, a Gamma prior with shape $a > 0$ and scale $b > 0$ (which we may extend to include a reference prior, Uniform for log(M), by letting $a \to 0$ and $b \to 0$. In this section we will use the latter, which means that

$$p(M) \propto M^{-1} \qquad M > 0. \tag{3.22}$$

Equation (3.21) can be expressed as a mixture of two gamma posteriors, and the conditional distribution of the mixing parameter, x given M and k is a simple beta. This can be illustrated as follows. For $M > 0$, the gamma functions in (3.20) can be written as

$$\frac{\Gamma(M)}{\Gamma(M+q)} = \frac{(M+q)Be(M+1,q)}{M\Gamma(q)} \tag{3.23}$$

where $Be(\cdot,\cdot)$ is the usual beta function. Then in (3.21) and for any $k=1,2,...,q$ the posterior of M for k, is

$$\begin{aligned} p(M \mid k) &\propto p(M)M^{k-1}(M+q)Be(M+1,q) \\ &\propto M^{k-2}(M+q)\int_0^1 x^M(1-x)^{q-1}dx, \end{aligned} \tag{3.24}$$

using the definition of the beta function.

From (3.24) it also follows that the joint posterior density of M and x is

$$p(M,x \mid k) \propto M^{k-2}(M+q)x^M(1-x)^{q-1}, \qquad 0<M,\ 0<\mathrm{x}<1$$

and the conditional posteriors $p(M \mid x,k)$ and $p(x \mid M,k)$ can be determined as follows.

Firstly

$$p(M \mid x,k) \propto M^{k-2}(M+q)\exp\{-M(\log(x))\},$$

$$\propto M^{k-1}\exp\{-M(\log(x))\}+qM^{k-2}\exp\{-M(-\log(x)\} \quad M>0 \tag{3.25}$$

which reduces easily to a mixture of two gamma densities, viz.

$$M \mid x,k \sim \pi_x Ga(k,-\log(x))+(1-\pi_x)Ga(k-1,-\log(x)) \tag{3.26}$$

with weights π_x defined by $\dfrac{\pi_x}{1-\pi_x}=\dfrac{k-1}{q(-\log(x))}$. Also note that $\log(x)=\log_e(x)=ln(x)$.

Secondly

$$p(x \mid M,k) \propto x^{M}(1-x)^{q-1} \qquad 0<x<1 \tag{3.27}$$

so that $x \mid M,k \sim Be(M+1,q)$, a beta distribution with mean $\dfrac{M+1}{M+q+1}$.

It should now be clear how M could be sampled at each stage of the simulation. Hence, at each Gibbs iteration, the currently sampled values of M and k allow us to draw a new value of M by first sampling an x value from the simple beta distribution in (3.27), conditional on M and k, both fixed at the most recent values; then M is sampled from the mixture of gammas in (3.26) based on the same k and the x value just generated. On completion of the simulation, we will have a series of sampled values of M, k, x and all the other parameters. Note that only the sampled values k and x are needed in estimating the posterior $p(M \mid \underline{\mathbf{y}})$ via the usual Monte Carlo average of conditional posteriors, viz.

$$p(M \mid \underline{\mathbf{y}}) \cong N^{-1}\sum_{i=1}^{N} p(M \mid x^{(i)},k^{(i)}) \tag{3.28}$$

where the summands are simply the conditional gamma mixtures in (3.26).

To calculate the posterior distribution of all the model parameters, we developed an important Gibbs sampler algorithm for simulation. On completion of the simulation, we will have a series of sampled values for all the model parameters. The next section illustrates an application of the most recent developed Gibbs sampler for an animal breeding experiment.

3.4.7 The Gibbs Sampler

Markov chain Monte Carlo methods, particularly Gibbs sampling, are now very often used in the thesis and once again the model described in paragraph 3.4.2 can be implemented through this sampling technique. As usual in Gibbs sampling, we identify collections of complete conditional posterior distributions that determine the marginal posteriors for all the parameters. Hence, the Gibbs sampler for $p(\beta,\sigma_\varepsilon^2,\gamma,\sigma_\gamma^2,M\mid\underline{\mathbf{y}})$ can be described as follows.

(0) Select starting values for $\gamma^{(i)}$ and $\sigma_\varepsilon^{2(i)}$. Set $i = 0$

(1) Sample $\beta^{(i+1)}$ from $p(\beta\mid\gamma^{(i)},\sigma_\varepsilon^{2(i)},\underline{\mathbf{y}})$ according to (3.14)

(2) Sample $\sigma_\varepsilon^{2(i+1)}$ from $p(\sigma_\varepsilon^2\mid\beta^{(i+1)},\gamma^{(i)},\underline{\mathbf{y}})$ according to (3.15)

(3.1) Sample $\gamma_1^{(i+1)}$ from $p(\gamma_1\mid\beta^{(i+1)},\sigma_\varepsilon^{2(i+1)},\sigma_\gamma^2,\gamma_{-1}^{(i)},M,\underline{\mathbf{y}})$ according to (3.9) or (3.10)

(3.2) ...

(3.q) Sample $\gamma_q^{(i+1)}$ from $p(\gamma_q\mid\beta^{(i+1)},\sigma_\varepsilon^{2(i+1)},\sigma_\gamma^2,\gamma_{-q}^{(i)},M,\underline{\mathbf{y}})$ according to (3.9) or (3.10)

(4.1) Sample $\lambda_1^{(i+1)}$ from $p(\lambda_1\mid\beta^{(i+1)},\sigma_\varepsilon^{2(i+1)},\sigma_\gamma^{2(i0},\underline{\mathbf{y}})$ according to (3.18)

(4.2) ...

(4.k) Sample $\lambda_k^{(i+1)}$ from $p(\lambda_k\mid\beta^{(i+1)},\sigma_\varepsilon^{2(i+1)},\sigma_\gamma^{2(i0},\underline{\mathbf{y}})$ according to (3.18)

(5) Sample σ_γ^2 from $p(\sigma_\gamma^2\mid\lambda^{(i+1)},\underline{\mathbf{y}})$ according to (3.16)

(6.1) Sample $x^{(i+1)}$ from $p(x\mid M^{(i)},k^{(i)})$ according to (3.27)

(6.2) Sample $M^{(i+1)}$ from $p(M\mid x^{(i+1)},k^{(i)})$ according to (3.26)

(7) Set $i=i+1$ and return to (1)

3.5 An Example

3.5.1 The Data

The example used for illustrative purposes are based on an experiment undertaken at the International Livestock Research Institute (ILRI) at the University of Nairobi, Kenya in the early 90's (Duchateau, *et al.*, 1998). The data are shown in APPENDIX C. The goal of the research was to select for improved Helminth resistance in sheep.

The female sheep used in the experiment are from three different breeds, whereas the males are from two breeds. In each of the six crosses, there are at least 25 and at most 42 different sires, and each sire within a crossbreed has on average offspring of 6.4 lambs. The weaning weight is measured for each lamb. The age at which lambs are weaned may differ from animal to animal, and therefore, a variable expressing the age of the animal at weaning is included as a fixed effect as well as the sex of the lambs. Finally, the sires are included as random effects.

Although the same sire is mated to ewes from different breeds, the sire nested in breed is taken as a single random effect γ_i and it is assumed that these random effects are independent. A total of n = 1277 weaning weight records, from the progeny of q = 200 sires are available after editing, and *breed*, *sex* and *age* are included as fixed effects in the final model.

The mixed linear model used for this data structure, is the sire model of section (1.2), $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$, where $\underline{\mathbf{Y}}$ (1277 × 1) vector of weaning weights. β (8 × 1) is the vector of fixed effects, and the design matrix **Z**, a (1277 × 200) matrix identifying the (200 × 1) vector of random effects consisting of the breeding values for the 200 sires for which the data is observed, **X** a (1277 × 8) matrix of full rank.

MATLAB software has once again been developed to generate the samples that enable us to obtain the finite sample post-data parameter densities, using the Gibbs sampler. The full conditional posteriors are updated after every iteration. The first 1 000 draws of each chain are discarded, and then every 10^{th} draw is saved. By saving every 10^{th} draw, the chain yielded a posterior sample of 1 000 approximately uncorrelated draws. All posterior analyses are based on these m = 1 000 draws.

3.5.2 Analysis of Variance Components

Posterior modes of the Traditional Bayesian analysis, 95% credibility intervals, and the REML estimates are summarized in Table 3.1. Note that the REML point estimates of σ_ε^2 and σ_γ^2 and the posterior modes obtained from the Gibbs sampler (Traditional Bayes) do not differ much. This was because we assigned uniform or "flat" priors to the vector of fixed effects and variance components, and a normal prior to the vector of random effects (Harville, 1974; Searle, Casella & McCulloch, 1992).

Table 3.1 REML and Traditional Bayesian Estimates (posterior modes) of the Variance Components, along with 95% Credibility Intervals.

Parameters	REML	Trad. Bayes	95 % Credibility Interval
σ_ε^2	4.8639	4.8885	4.4617 ; 5.3059
σ_γ^2	0.6802	0.7211	0.4312 ; 1.0496

In practice it is difficult to select appropriate values for the parameter, M. Recall that M is a positive scalar that is related to how "clumpy" the data are (often called a precision parameter), and clumpy data occur when the different sires are concentrated into a few clusters. Because the parameter value has an important influence on the estimation, a broad range of possible fixed values for M is chosen, i.e. $M = 5$, 50, 100 and 1000. Furthermore, a prior distribution is placed on M and values from the estimated posterior distribution of M are used in the simulations. The results are summarized in Tables 3.2 and 3.3 below.

Table 3.2 Posterior Estimates of the Error Variance Component, σ_ε^2 (different values of M), along with 95% Credibility Intervals.

M	Posterior Mode	95 % Credibility Interval
5	4.9183	4.4788 ; 5.3612
50	4.8907	4.4649 ; 5.3286
100	4.8615	4.4192 ; 5.2731
1000	4.8743	4.4480 ; 5.3107
Sim M	4.8667	4.4527 ; 5.2810

Table 3.2 Posterior Estimates of the Model Variance Component, σ_γ^2 (different values of M), along with 95% Credibility Intervals.

M	Posterior Mode	Posterior Mean	95 % Credibility Interval
5	0.6310	0.8835	0.4213 ; 1.7733
50	0.6090	0.8659	0.4407 ; 1.0276
100	0.6290	0.8859	0.4407 ; 1.0276
1000	0.7275	0.7320	0.4213 ; 1.0476
Sim M	0.6340	0.6532	0.3790 ; 1.0026

For large values of M, the "Dirichlet" estimates coincide with the Traditional Bayes and REML estimates. In these cases there are almost 200 different clusters/groups among the different sires, resulting in a similar behaviour of the estimates from the Dirichlet process prior and the Traditional Bayes procedure.

Using the posterior densities for σ_ε^2 and σ_γ^2, the marginal posterior densities are estimated as the average of the posterior densities and are displayed in Figures 3.1 and 3.2. Also, the distributions in Figure 3.2 are quite skew, resulting in a difference between the posterior means and posterior modes. The density for $M = 100$ is omitted from figure 3.1 since the estimated marginal density for this value of M and the density obtained from the Traditional Bayes analysis are basically the same.

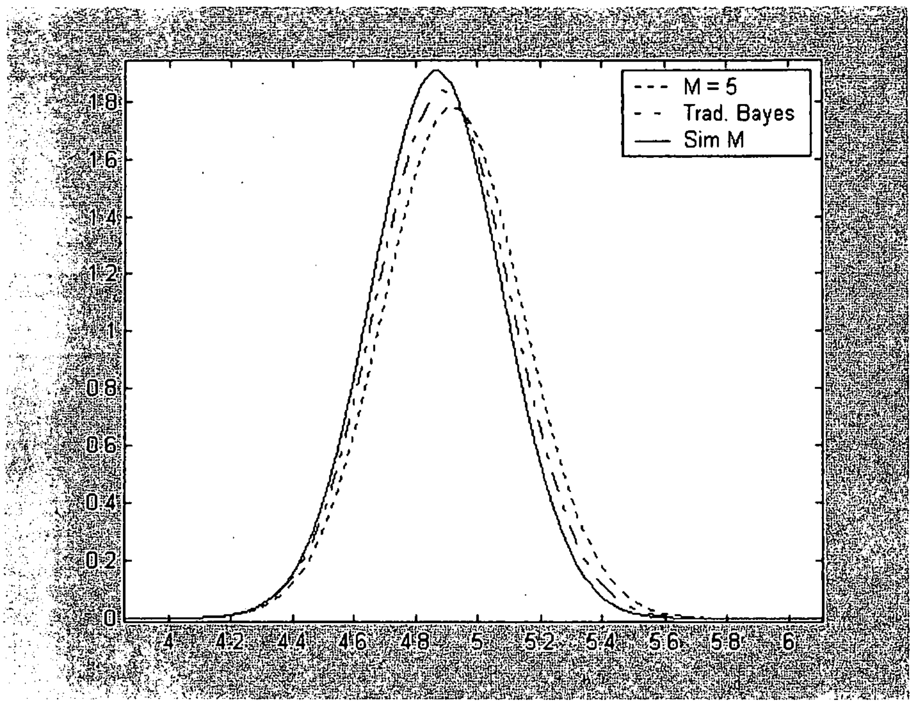

Figure 3.1 Estimated Marginal Posterior Densities of the Variance Component, σ_ε^2 for different cases of *M*, (Sim M); Traditional Bayes and when *M* is fixed at $M = 5$.

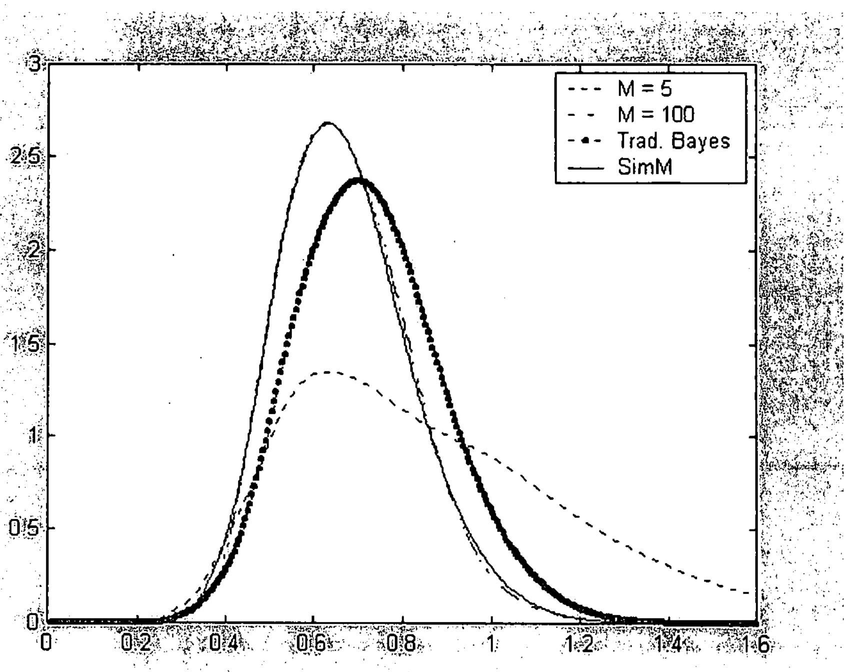

Figure 3.2 Estimated Marginal Posterior Densities of the Variance Component σ_γ^2 .

The similarity of the σ_ε^2 results in Tables 3.1 and 3.2 is an indication, for this data set, that the results are not sensitive to the choice of the precision parameter, *M*. From the marginal posterior densities of σ_ε^2 in Figure 3.1 the effect of changing *M* on the distribution of the random effects is also minimal. These densities are virtually identical, and it is clear that there is no uncertainty about the exact location and height in the different densities. However, the posterior density for σ_γ^2 when *M* is fixed at 5 (Figure 3.2) shows some uncertainty in the shape of the density. This can be expected because M is part of the prior for the random effects and will therefore influence σ_γ^2 much more than σ_ε^2. This may also be related to the large variation in the importance sampling weights for smaller values of *M*. The other densities have similar shapes with only noticeable shifts in the posterior modes.

The posterior means and modes of the Traditional Bayesian analysis; different values for *M*, and 95% credibility intervals of functions of variance components like the intraclass correlation coefficient, $\rho = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, and the variance ratio, $v = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ are summarized in Tables 3.3 and 3.4. Once again it is evident from this table that the credibility interval for the intraclass correlation coefficient does not contain 0.5. This result corresponds well with the statement made by Wang *et al.* (1992) namely that from a genetic point of view, an intraclass correlation coefficient of 0.5 is not possible in a sire model.

Using again the conditional posterior densities for these functions of the variance components, the marginal posterior densities are estimated as the average of the conditional posterior densities and are displayed in Figures 3.3 and 3.4.

Table 3.3 Estimates of $\rho = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, the Intraclass Correlation Coefficient for different cases of *M*, Simulated *M* and Traditional Bayes Results, along with 95% Credibility Intervals.

M	Posterior Mode	Posterior Mean	95% Credibility Interval
5	0.114	0.1497	0.0765 ; 0.2677
100	0.115	0.1189	0.0696 ; 0.1706
Sim M	0.120	0.1179	0.0710 ; 0.1723
Trad. Bayes	0.130	0.1282	0.0779 ; 0.1807

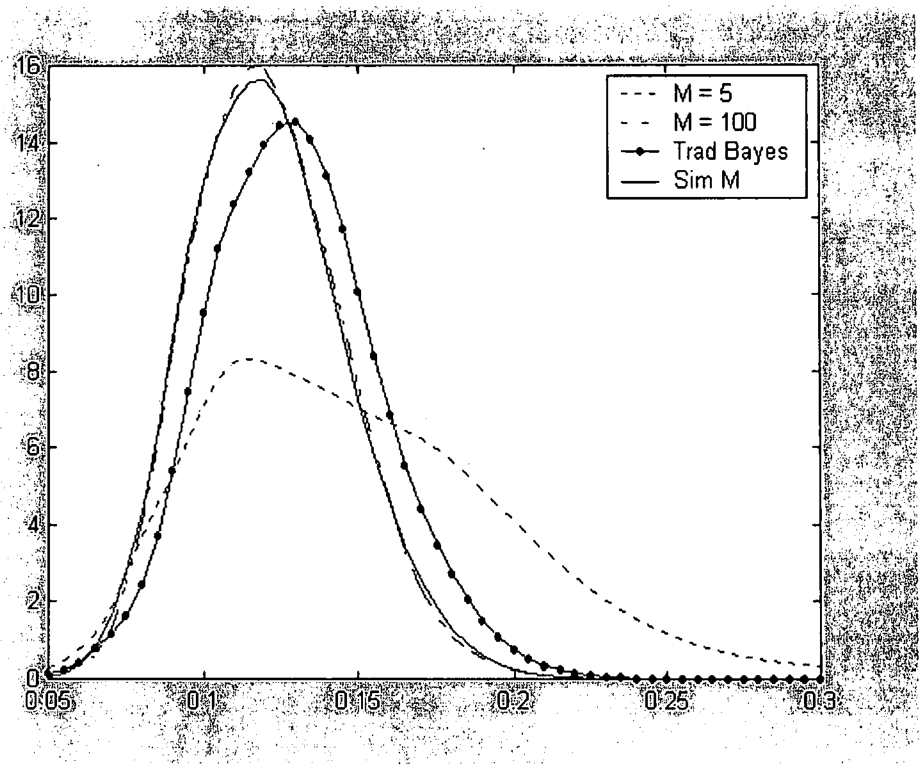

Figure 3.3 The Estimated Marginal Posterior Density of the Intraclass Correlation Coefficient, $\rho = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$.

Table 3.4 Estimates of $\nu = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$, a Function of the Variance Components, along with 95% Credibility Intervals.

M	Posterior Mode	Posterior Mean	95% Credibility Interval
5	0.130	0.1803	0.0829 ; 0.3655
100	0.135	0.1359	0.0748 ; 0.2057
Sim M	0.135	0.1347	0.0765 ; 0.2082
Trad. Bayes	0.145	0.1481	0.0844 ; 0.2205

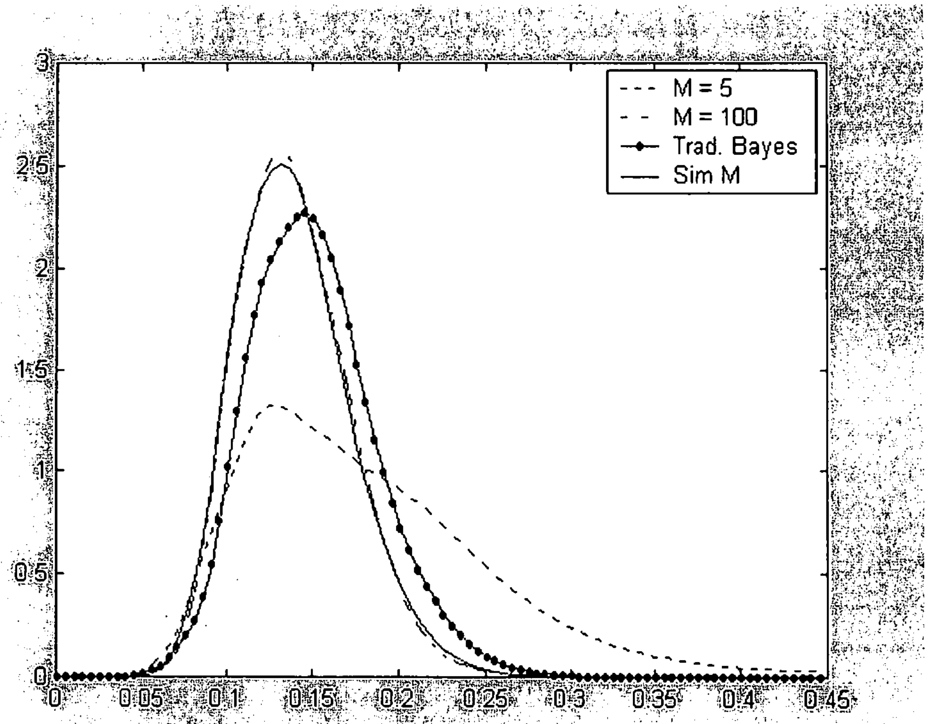

Figure 3.4 The Estimated Marginal Posterior Density of the Variance Ratio, $\nu = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$.

The same conclusion can be drawn for the posterior distributions of the intraclass correlation coefficient, ρ and variance ratio, v (which are functions of the variance components) as for the variance components, i.e. more or less identical marginal posterior densities except for small values of M.

Another commonly derived statistic, the heritability (h^2) of the trait, which is also function of the two variance components is calculated and reported in Table 3.5. This statistic describes the proportion of the total variation in the environment of the study attributable to genetics. In this formula $h^2 = \dfrac{4\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, σ_γ^2 is multiplied by 4 in the numerator to account for the fact that lambs from the same sire are half siblings and the sire accounts for half of the inherited genetic component, and $\sigma_\gamma^2 + \sigma_\varepsilon^2$ is the phenotypic variance. The higher the heritability, which lies between 0 and 1, the higher the proportion of the total variation that can be assumed to be genetic in origin.

Table 3.4 REML and Bayesian Estimates of $h^2 = \dfrac{4\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$, the heritability of the trait.

REML	Trad Bayes	M = 5	M = 100	Sim M
0.49	0.5127	0.4756	0.5179	0.5097

Some caution is needed in variance component problems in genetics. For example, some genetic models dictate bounds for a particular variable. If one employs the Sire model, as in the case of the present thesis, the intraclass correlation coefficient must lie inside the [0, $\frac{1}{4}$] interval, because heritability is between 0 and 1. This implies that the variance ratio $\nu = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ is between $[0, \frac{1}{3}]$, and that $0 \le \sigma_\gamma^2 \le \frac{\sigma_\varepsilon^2}{3}$. These profound concerns are evident in the above example, except for the results of the variance components and functions thereof when M is set to small values ($M = 5$), but this is due to some uncertainty about the real distribution of the posterior densities.

3.5.3 Analysis of Random Effects

As mentioned before (see section 3.4) the non-parametric Bayesian approach for the random effects is to specify a prior distribution on the space of all possible distribution functions. This prior for the mixed linear model is applied for the general prior of the distribution of the random effects. This can be accomplished with a Dirichlet process prior distribution. This means that the usual normal prior on the random effects is replaced with a non-parametric prior, followed by a Dirichlet prior with precision parameter M, on the general distribution. Because this precision parameter value has an important influence on the estimation, a broad range of possible fixed values for M is chosen ($M = 5$, 50, 100 and 1000) to reflect small, moderate and large departures from normality in the case of the random effects. Furthermore, a prior distribution is placed on M and the conditional posterior distribution of M becomes part of the Gibbs sampler.

Rather than report results for all 200 sires, we focus our discussion on the first 10 sires in the analysis. The tables referred to in this section contain only these 10 animals. The rest of the results are summarized in APPENDIX E. Since animal breeders might be interested in the breeding value of specific sires in order to determine which sires should be retained for future selection, we used these breeding values to find the REML-, Traditional Bayes-, and Dirichlet process ranks of the different sires. The results are summarized in Table 3.4 and 3.5, along with 95% credibility intervals. Moreover, the REML estimates represent the mode of the marginal likelihood and thus might be better compared to the modes of the posterior distributions of the random effects. These modes (breeding values) are provided in the aforementioned tables.

Table 3.4 REML Estimates and their Standard Errors (SE's), Traditional Bayes Estimates along with 95% Credibility Intervals and Posterior Rankings of the first 10 of the 200 Sires.

SIRE_ID	REML Estimate	SE	Posterior Rank	Trad. Bayes Estimate	95% Credibility Interval	Posterior Rank
1971	-0.1061	0.6654	9	-0.1024	-1.4569 ; 1.1898	9
1972	0.5241	0.5349	5	0.5689	-0.4500 ; 1.5455	5
1973	0.3888	0.6399	6	0.4464	-0.9447 ; 1.6482	6
1974	1.9339	0.5486	1	1.9627	0.7862 ; 3.1311	1
1980	0.9299	0.5975	3	0.9635	-0.2615 ; 2.1269	4
1991	0.3611	0.6396	7	0.3741	-1.0761 ; 1.6095	7
1999	0.9266	0.6654	4	0.9939	-0.2873 ; 2.2231	3
4907	0.2289	0.5790	8	0.2676	-0.8033 ; 1.4348	8
4908	1.6614	0.4921	2	1.6662	0.5431 ; 2.8049	2
4909	-0.7628	0.6653	10	-0.7645	-2.1227 ; 0.4956	10

Note that the REML estimates and the posterior means (see Pretorius and Van der Merwe, 2000; APPENDIX E) obtained from the Gibbs sampler for $M = 5$ are quite different. There are two factors to keep in mind when examining this difference. Firstly, there is considerable uncertainty about the distribution and central values of the breeding values when $M = 5$. Secondly, as mentioned earlier, the REML estimate of the breeding value is more similar to the mode of the posterior distribution than the mean. Figures 3.5 – 3.16 show the posterior distributions of the first 3 sires in the data set, where it is evident that the posterior modes obtained from the Gibbs sampler are in fact similar to the REML estimates of the breeding values.

Table 3.5 Dirichlet Process Estimates for different values of M, along with 95% Credibility Intervals and Posterior Rankings of the first 10 of the 200 Sires.

SIRE_ID	M = 5 Modes	95% Credibility Interval	Posterior Rank	Sim M Modes	95% Credibility Interval	Posterior Rank
1971	-0.0125	-1.3874 ; 1.2378	9	-0.1097	-1.4280 ; 1.1601	9
1972	0.2510	-1.3396 ; 1.4029	7	0.5081	-0.5712 ; 1.5426	5
1973	0.3580	-1.3874 ; 1.5452	4	0.3934	-0.9254 ; 1.7685	6
1974	1.4265	0.3051 ; 2.6513	1	1.9026	0.8062 ; 3.1085	1
1980	1.0350	-1.2272 ; 2.1293	3	0.9193	-0.2371 ; 2.1947	3
1991	0.3420	-1.3396 ; 1.4029	5	0.3438	-0.8887 ; 1.6423	7
1999	0.3224	-1.2685 ; 2.4430	6	0.8936	-0.4606 ; 2.2582	4
4907	0.1910	-1.3874 ; 1.2378	8	0.2293	-0.8966 ; 1.3925	8
4908	1.1420	0.2435 ; 2.6513	2	1.6490	0.6079 ; 2.7037	2
4909	-1.2825	-1.8916 ; 0.3827	10	-0.7711	-1.9938 ; 0.4418	10

Uncertainty in the values of the breeding values when M equals 5 is indicated in the large posterior credibility intervals. Thus a wide range of values for the breeding values is quite possible. However, as might be expected, the best sires from the REML, Traditional Bayes, and Dirichlet process analyses (Sim M) overlap, with a minor disagreement between rankings of SireID1980 and SireID1999 (ranked *visa versa* in the Traditional Bayes analysis).

Beyond the top three sires (according to the Dirichlet process when M was fixed at 5), there are major differences in the order of the best to the worst sire. This is because of the great deal of uncertainty about the values of the breeding values. This uncertainty is also reflected in the wide credibility intervals. Note that the estimated breeding values take account of the variability in the sire variance, σ_γ^2 depicted in Figure 3.2.

Taking account of this variability can be important in evaluating the breeding potential of the animals. This is a clear example of why it is important to correctly model the distribution of the random effects; very different results may be obtained as a result of changing the precision parameter, M.

To further illustrate the uncertainty in some of the posterior densities, we have calculated the marginal posterior densities of the breeding values for the first 3 sires in Figures 3.5 – 3.16. (SireID1972, SireID1973 and SireID1974). In these figures we plotted: the densities when M is fixed at 5 and 50, the densities obtained from the Traditional Bayes analysis, and densities when M is simulated from a mixture of distributions in the Dirichlet process, given the data. We noted that these posterior densities have many features of interest.

The next section is devoted to the overriding influence of the precision parameter M on the posterior densities of the 200 breeding values of the different sires. M is, as mentioned before, related to how "clumpy" the data are. When there are only a few clusters among the sires in the model, the estimate of the normal means from the Dirichlet process prior will be similar to the non-parametric Bayes estimator. When there are almost 200 different clusters, the estimator from the Dirichlet process prior will be similar to the parametric Bayes estimator.

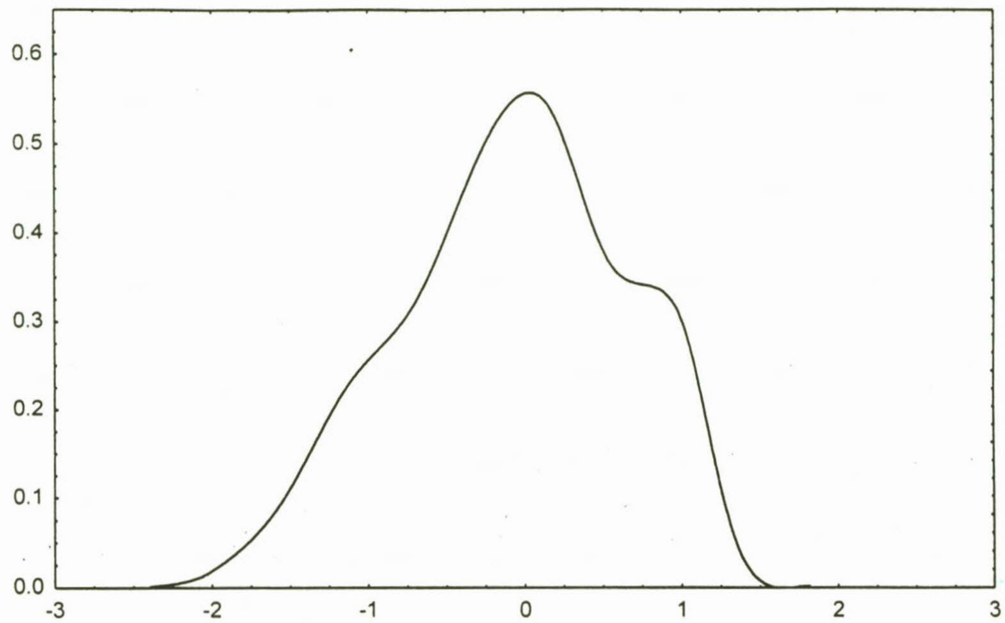

Figure 3.5 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1971 (γ_1) when $M = 5$ in the Dirichlet Process.

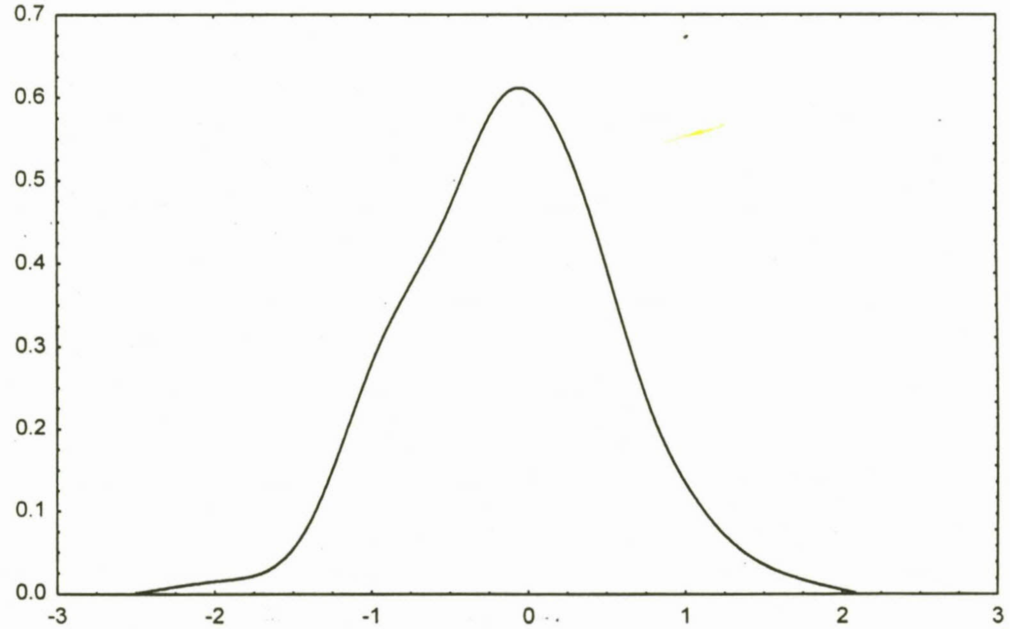

Figure 3.6 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1971 (γ_1) when $M = 50$ in the Dirichlet Process.

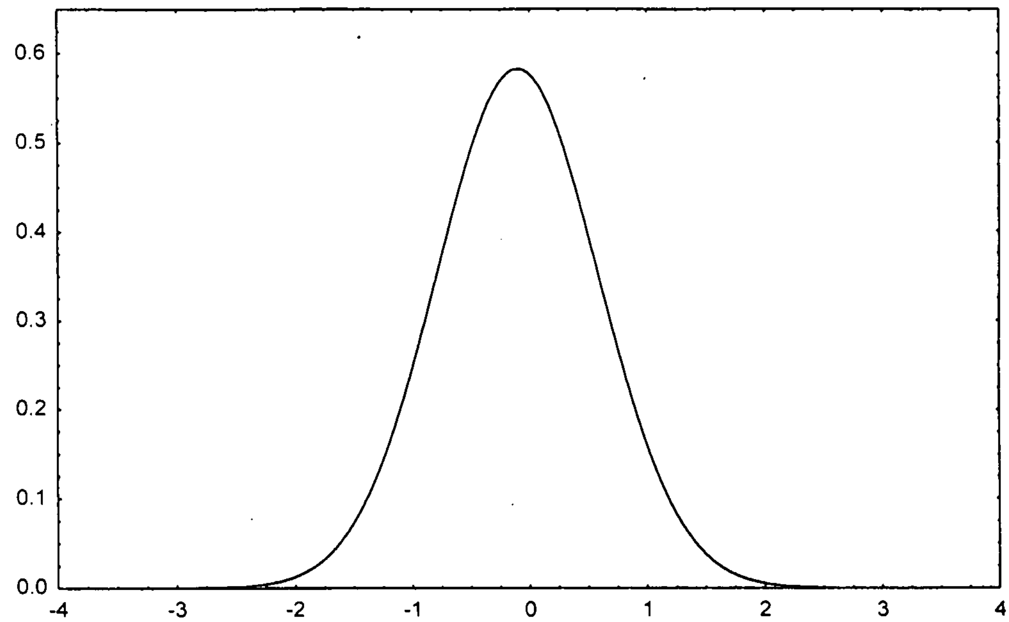

Figure 3.7 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1971 (γ_1) from the Traditional Bayes Analysis.

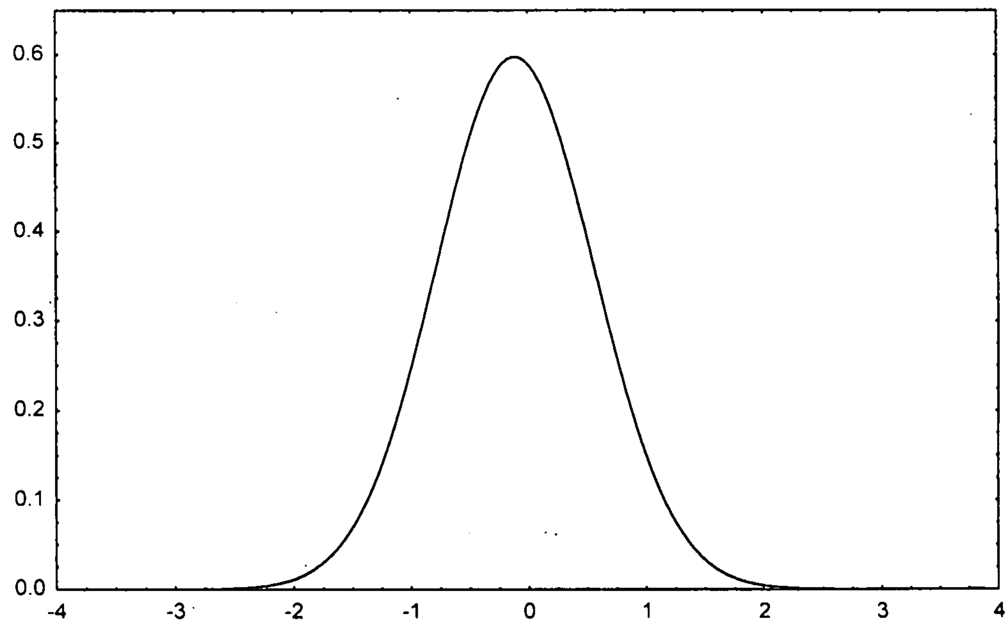

Figure 3.8 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1971 (γ_1) when M is Simulated in the Dirichlet Process.

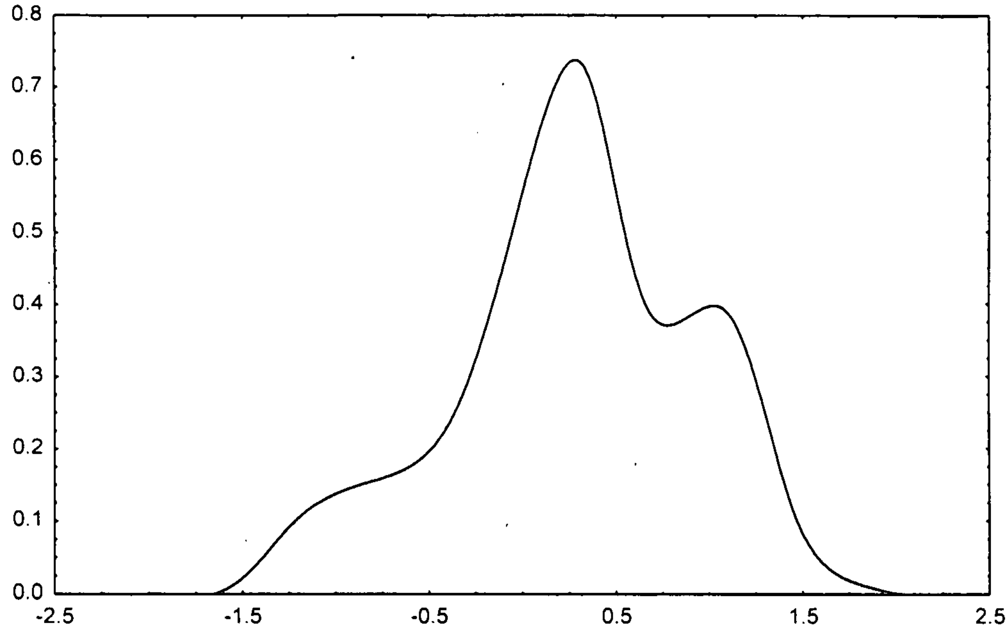

Figure 3.9 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1972 (γ_2) when $M = 5$ in the Dirichlet Process.

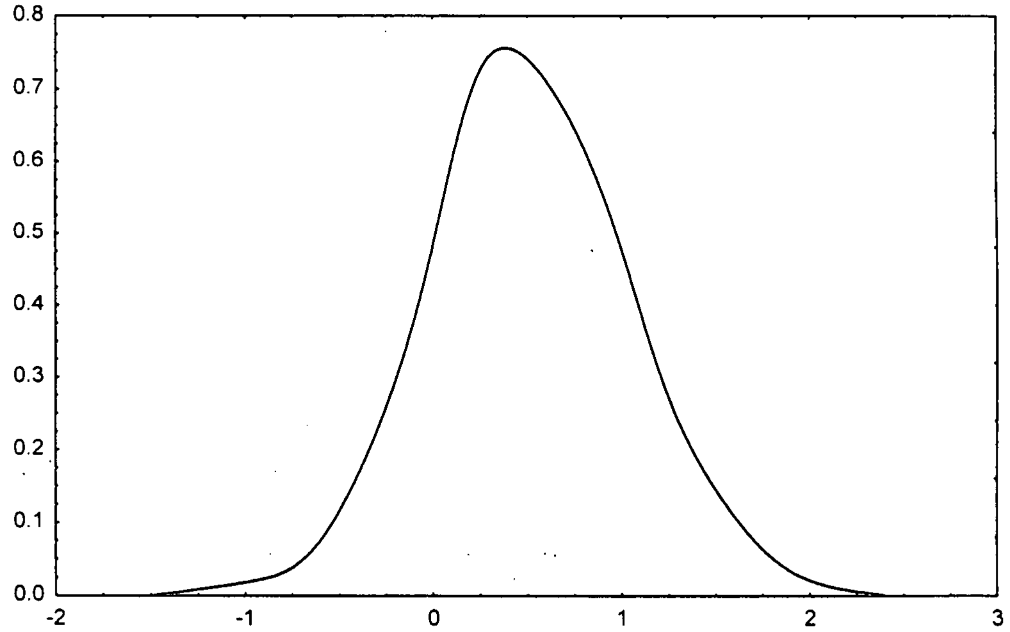

Figure 3.10 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1972 (γ_2) when $M = 50$ in the Dirichlet Process.

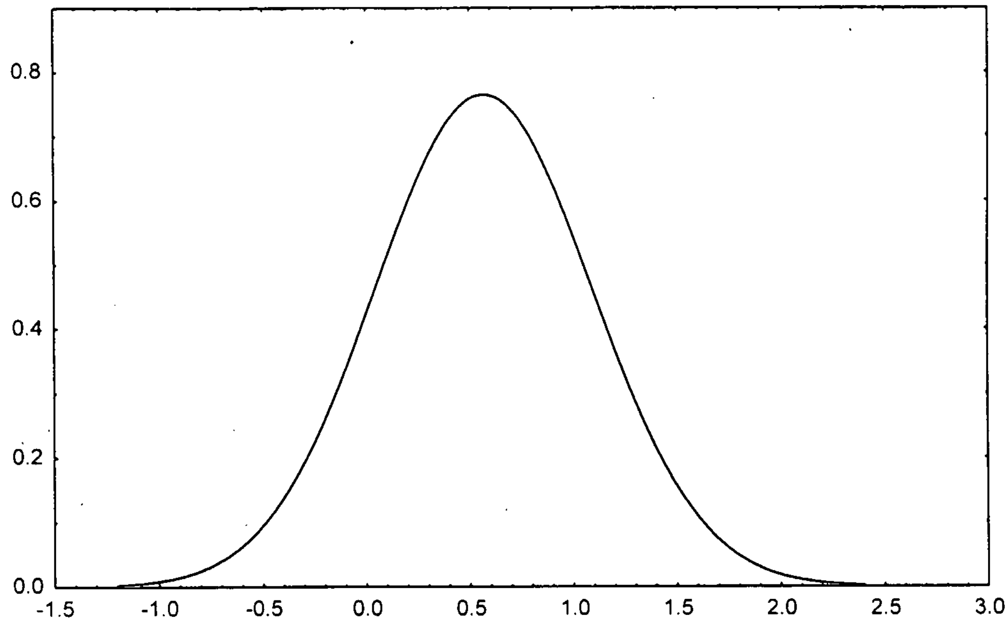

Figure 3.11 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1972 (γ_2) from the Traditional Bayes Analysis.

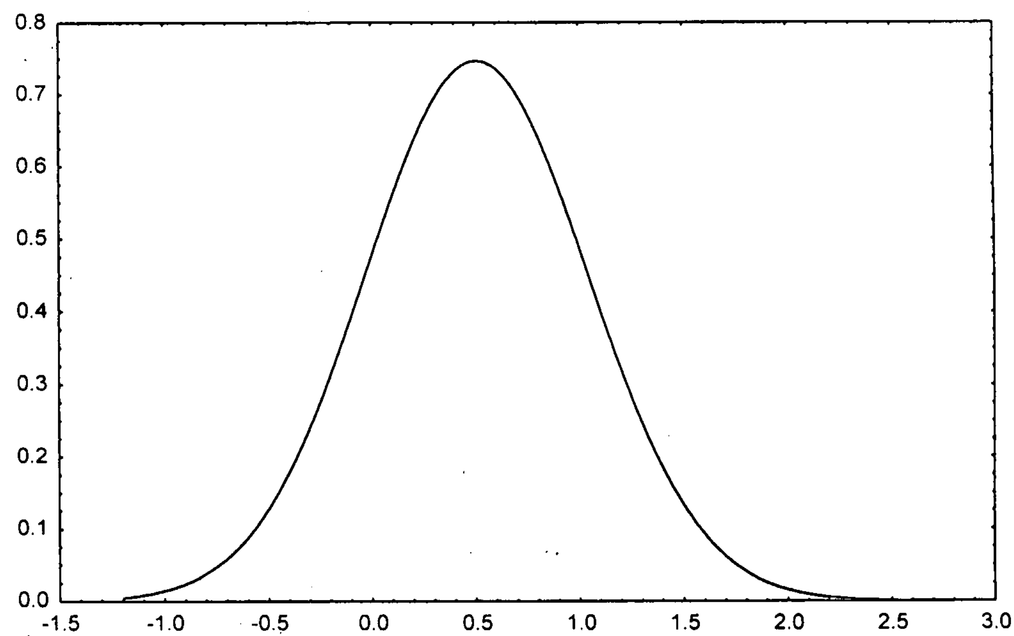

Figure 3.12 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1972 (γ_2) when M is simulated in the Dirichlet Process.

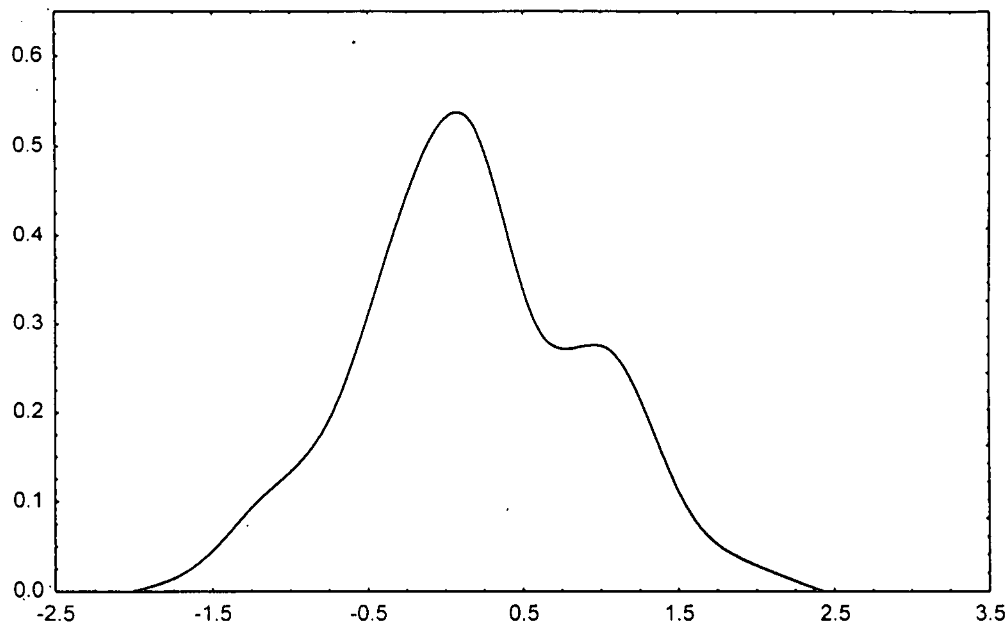

Figure 3.13 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1973 (γ_3) when $M = 5$ in the Dirichlet Process.

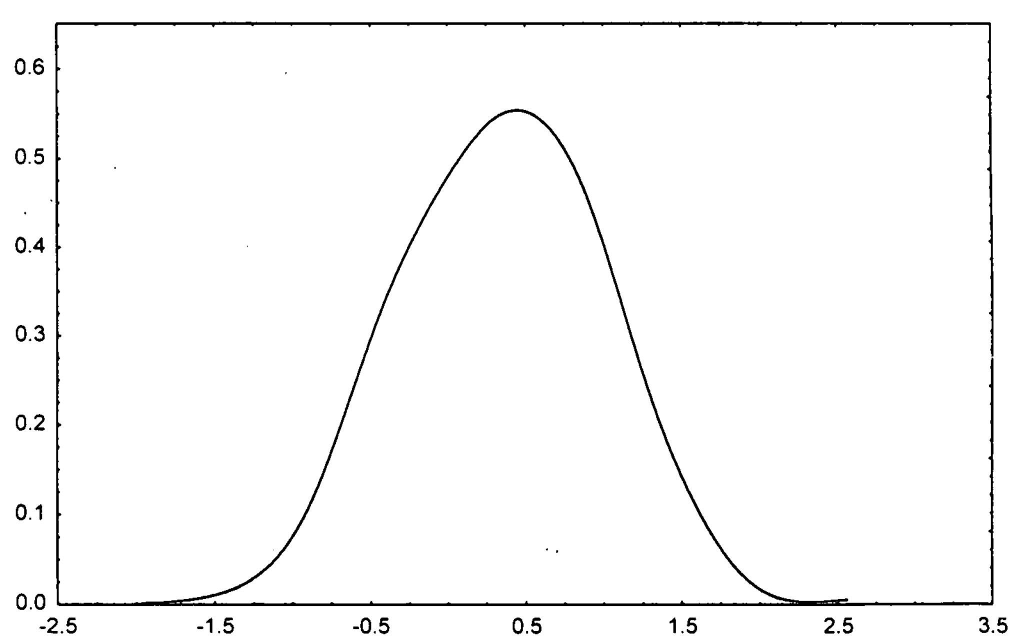

Figure 3.14 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1973 (γ_3) when $M = 50$ in the Dirichlet Process.

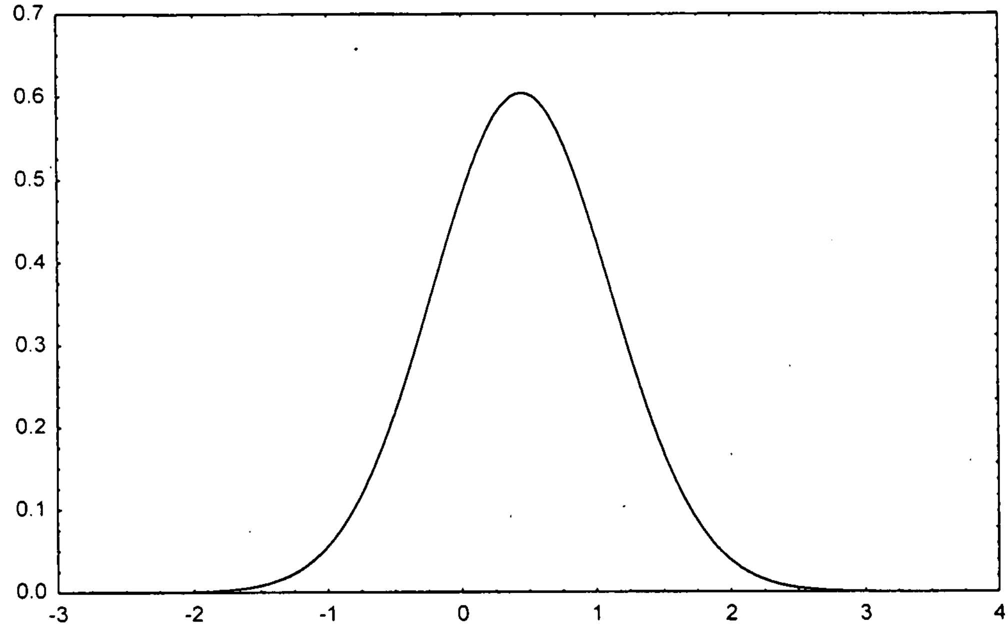

Figure 3.15 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1973 (γ_3) from the Traditional Bayes Analysis.

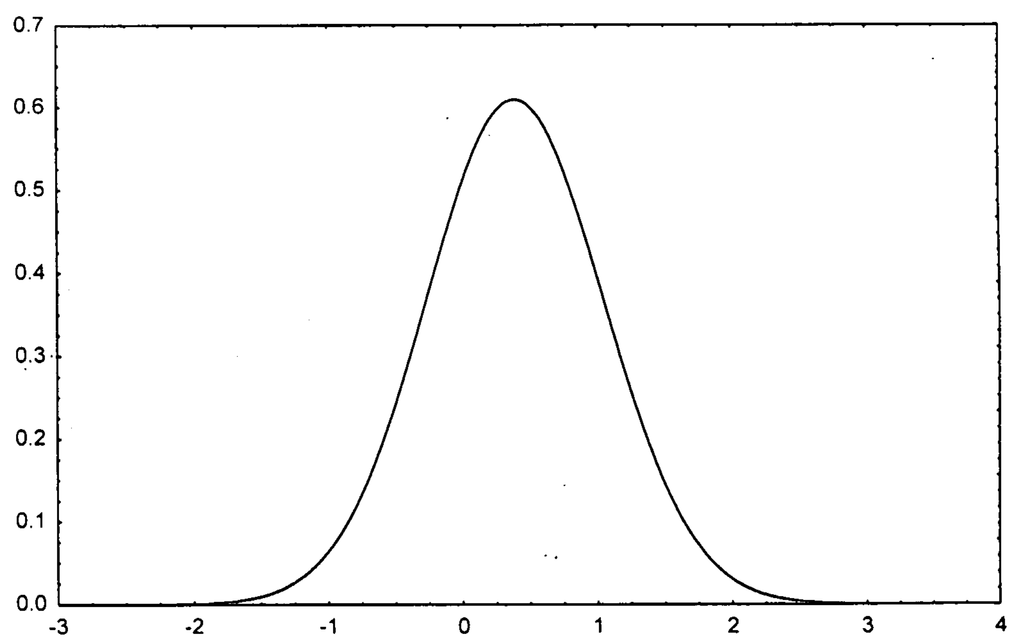

Figure 3.16 The Estimated Marginal Posterior Density of the Breeding Value for Sire ID1973 (γ_3) when M is Simulated in the Dirichlet Process.

Unlike the densities of the fixed effects, the values of M have a large effect on the posterior densities of the random effects, since the random effects are directly affected by the relaxation of the normal assumption when M is set equal to 5 and 50. While for $M = 50$ it is clear that there are some uncertainty about the shape and boundaries of the densities. This is even more pronounced with $M = 5$, and may be related to the larger variation in the importance sampling weights for smaller M. We also note from these first two densities that as M increases, the shapes of the densities tend to become more bell-shaped and symmetrical. Thus, a value of $M = 5$ reflects a large departure from normality in the posterior density of the breeding value.

For smaller values of M the sires are grouped into less clusters, with the average number of clusters, $\widetilde{k} = 124$ when $M = 5$. Furthermore, a value of $M = 50$ reflects a moderate to small departure from normality, with $\widetilde{k} = 145$. Thus, for small values of M, the estimates of the normal means from the Dirichlet process prior are similar to the values of the non-parametric Bayes estimates.

However, when $M = 1000$ the estimator from the Dirichlet process prior is similar to the parametric Bayes estimator and the density reveals no departure from normality (Figures 3.7, 3.11 and 3.15), with $\widetilde{k} = 196$. The estimated marginal posterior density for M given $\widetilde{k} = 196$ is presented in Figure 3.17.

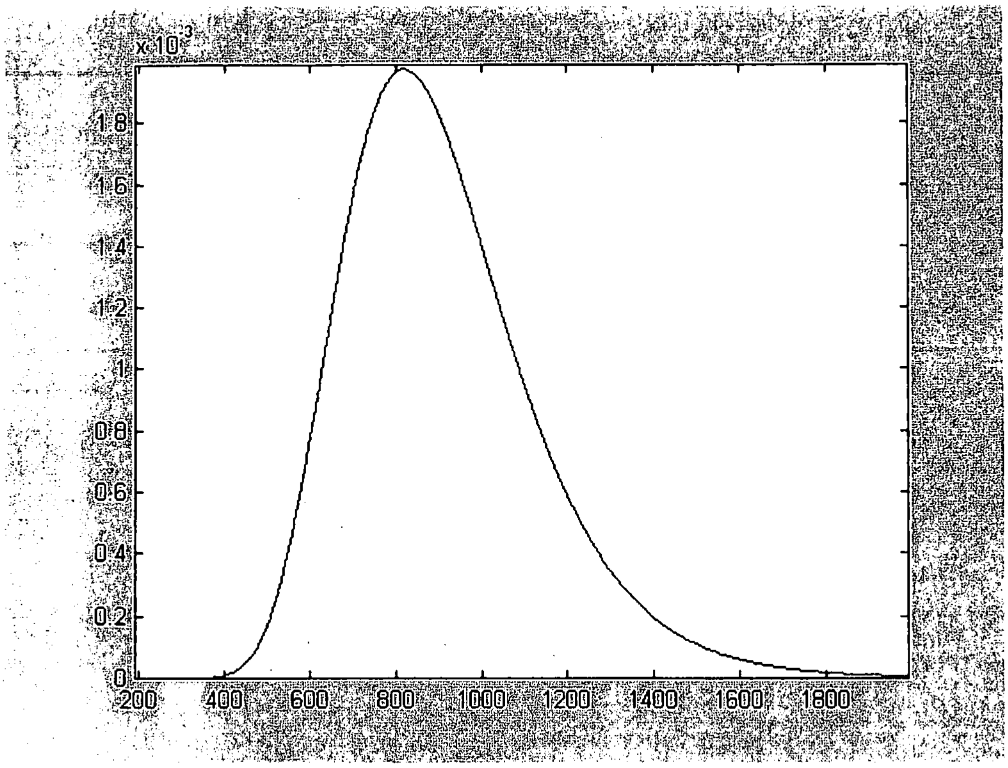

Figure 3.17 Estimated Marginal Posterior Density of M with $\widetilde{k} = 196$, and Posterior Mode $M_0 = 820$.

When M is simulated, given the data, the average number of clusters, $\widetilde{k} = 140$ with $\widetilde{M} = 220$. The estimated marginal posterior density and the unconditional marginal posterior density for the simulated M values are displayed in Figures 3.18 and 3.19. Finally, the observed histograms for the number of clusters, $\widetilde{k}$ for different values of M are presented in Figures 3.20 – 3.22.

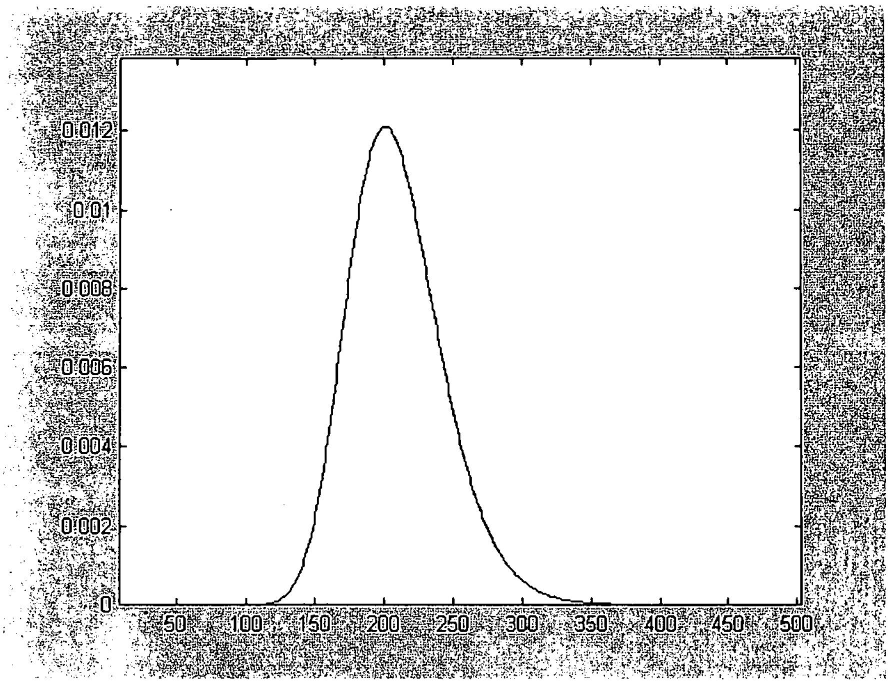

Figure 3.18 Estimated Marginal Posterior Density of M with $\widetilde{k} = 140$, $\widetilde{M} = 220$ and Posterior Mode $M_0 = 200$.

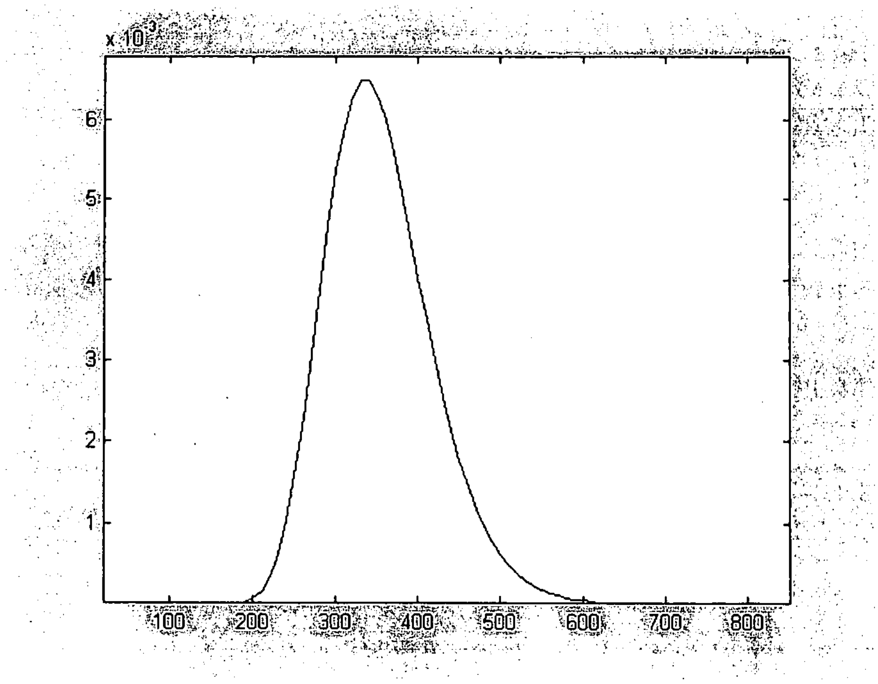

Figure 3.19 Estimated Unconditional Marginal Posterior Density of M with $\widetilde{k} = 140$, $\widetilde{M} = 220$ and Posterior Mode $M_0 = 320$.

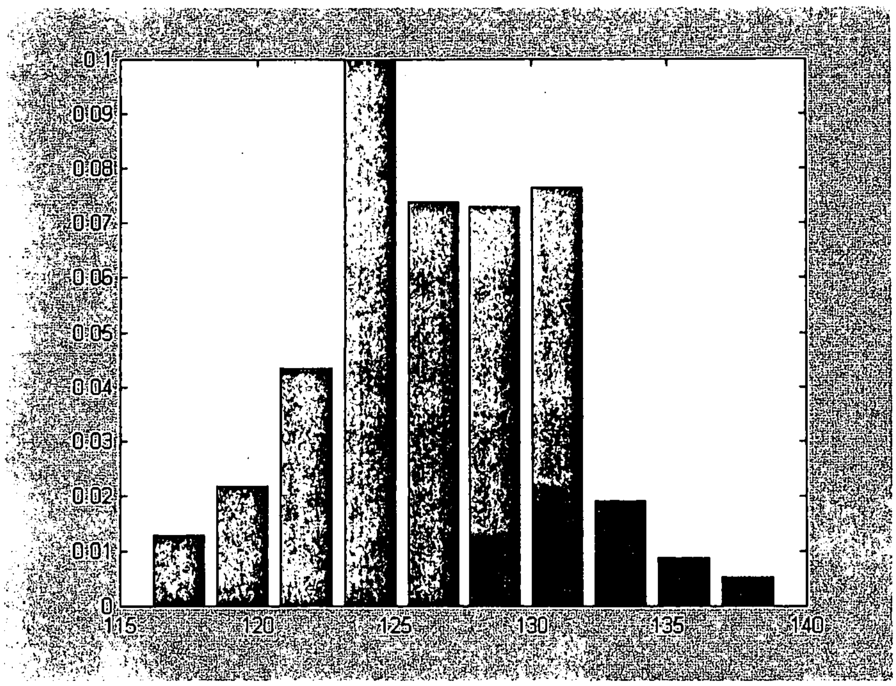

Figure 3.20 Observed Histogram for the Number of Clusters, $\tilde{k}$ when $M = 5$;

$$\tilde{k} = 124$$

Figure 3.21 Observed Histogram for the Number of Clusters, $\tilde{k}$ when $M = 1000$;

$$\tilde{k} = 196$$

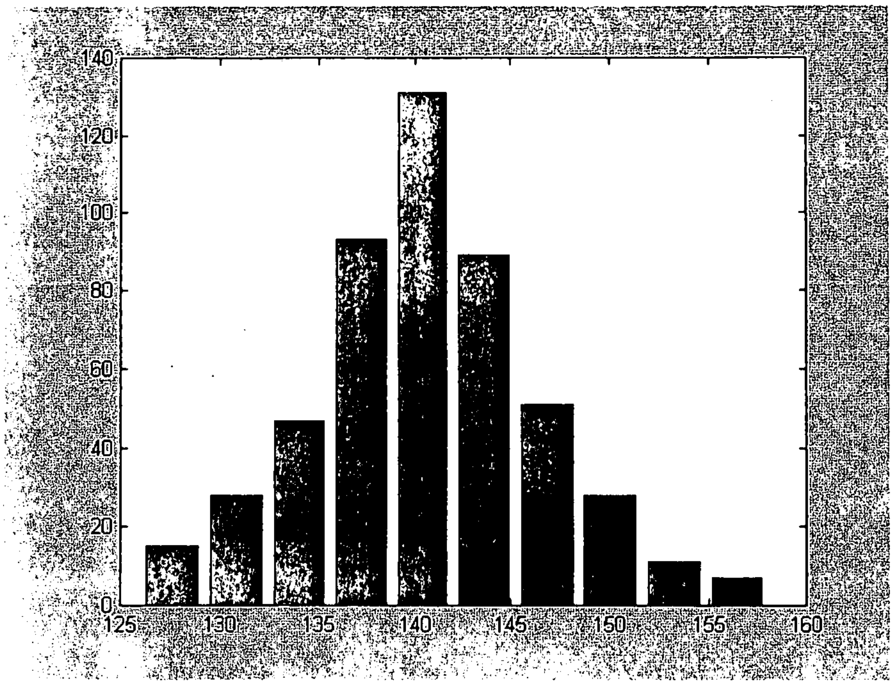

Figure 3.22 Observed Histogram for the Number of Clusters, $\tilde{k}$ when M is Simulated from a Mixture of Distributions, given the Data; $\tilde{k} = 140$.

A substantial statistical issue that remains to be tackled is the great discrepancy between pictures of the posterior densities of the random effects as the value of M changes. Indeed, if data are very sparse and not very clumpy, then non-parametric maximum likelihood methods may not work very well. But if the data are very clumpy, with modes that are spread out, then standard parametric Bayes methods do not work very well, and non-parametric Bayes methods work quite well. The question can be asked "why non-parametrics?" According to Walker *et al.*, (1999), the answer depends on the particular problem and the procedure under consideration, but many, if not most statisticians appear to feel that it is desirable in many contexts to make fewer assumptions about the underlying population from which the data are obtained than are required from a parametric analysis. Also, the mixture of Dirichlet process priors will cover departures from symmetry and cases where the assumptions of unimodality do not hold for the random effects in the mixed linear model.

3.5.4 Analysis of Fixed Effects

The estimates for the different fixed effects are summarized in Tables 3.6 and 3.7. The results obtained when M is fixed at 5 and 500, along with 95% credibility intervals are given in Table 3.6. The REML estimates, estimates from the Traditional Bayes analysis, and estimates when M is simulated from the data, are presented in Table 3.7. From these tables it is clear that the changing of the values of M have a minor effect on the posterior estimates of the fixed effects. These minor differences are attributable to the little mass of the Dirichlet process prior on the fixed effects.

To put these estimates into perspective, we focus our discussion only on the results of the analysis when M is simulated given the data (Sim. M column). As might be expected, there is a significant effect of sex with female lambs weighing on average 0.7057 kg ($CI_{0.95}$ = [0.4575 ; 0.9523]) less at weaning than males. Furthermore, there is a significant effect of age at weaning with weaning weights increasing by 0.0464 kg per daily increase in age ($CI_{0.95}$ = [0.0400 ; 0.0532]). There are also significant differences among breeds, with breeds 1 – 4 having significant higher weaning weights than breeds 5 – 6.

Table 3.6 Estimated Values of the Fixed Effects for M = 5 and 500, as well as the 95% Credibility Intervals.

	M = 5	95% Credibility Interval	M = 500	95% Credibility Interval
β_0 (Intercept)	4.7406	5.3418 ; 6.1000	4.8115	3.5054 ; 6.0282
β_1 (Breed 1)	1.5575	0.7430 ; 2.3669	1.5898	0.6819 ; 2.2147
β_2 (Breed 2)	1.4068	0.5691 ; 2.1645	1.4941	0.7420 ; 2.2386
β_3 (Breed 3)	1.3466	0.6641 ; 2.0364	1.3960	0.6406 ; 2.1626
β_4 (Breed 4)	0.9581	0.2810 ; 1.5885	0.9337	0.1968 ; 1.6439
β_5 (Breed 5)	-0.4066	-1.2002 ; 0.3532	-0.3256	-1.2729 ; 0.4002
β_6 (Sex)	0.699	0.4375 ; 0.9576	0.7092	0.4755 ; 0.9681
β_7 (Age)	0.0461	0.0399 ; 0.0529	0.0457	0.0398 ; 0.0526

Table 3.7 REML estimates and SE's, Traditional Bayes Estimates with 95% Credibility Intervals, and Dirichlet Process Prior Estimates when M is simulated given the data.

	REML	SE for REML	Trad. Bayes	95% Credibility Interval	Sim M	95% Credibility Interval
β_0 (Intercept)	5.0030	0.537	4.3439	3.2871 ; 5.3149	4.3267	3.2806 ; 5.4281
β_1 (Breed 1)	1.6820	0.376	1.6706	0.9035 ; 2.3790	1.6760	0.8870 ; 2.4189
β_2 (Breed 2)	1.5150	0.383	1.5017	0.8259 ; 2.2473	1.5043	0.7582 ; 2.2397
β_3 (Breed 3)	1.4360	0.357	1.4326	0.7821 ; 2.0731	1.4267	0.7189 ; 2.1200
β_4 (Breed 4)	0.9480	0.357	0.9278	0.2524 ; 1.5857	0.9381	0.2249 ; 1.6617
β_5 (Breed 5)	-0.4060	0.393	-0.4258	-1.2394 ; 0.3043	-0.4079	-1.1983 ; 0.3194
β_6 (Sex)	0.7040	0.128	0.7046	0.4648 ; 0.9461	0.7057	0.4575 ; 0.9523
β_7 (Age)	0.0465	0.003	0.0463	0.0398 ; 0.0529	0.0464	0.0400 ; 0.0532

The estimated marginal posterior densities are now been calculated and depicted in Figures 3.23 – 3.30. Note once again that the posterior densities are only calculated for the fixed effects when M is simulated, given the data. These figures also show the minor effect of changing M on the posterior distributions of the fixed effects. Unlike the densities for the random effects, the plots are, as expected, bell-shaped and symmetrical since the fixed effects are not directly affected by the relaxation of the normal assumptions (when M is small).

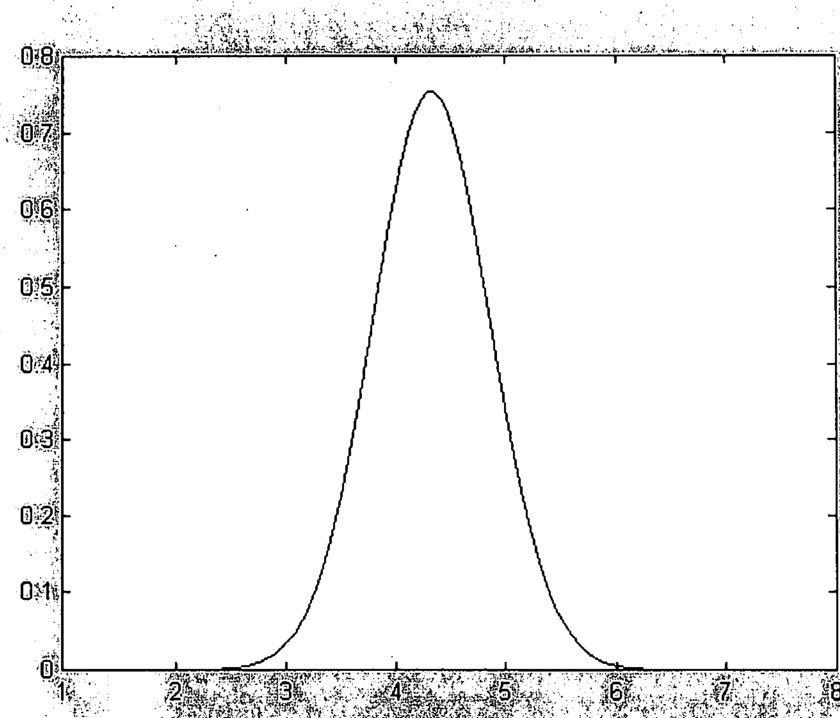

Figure 3.23 Estimated Marginal Posterior Density of β_0, the Intercept .

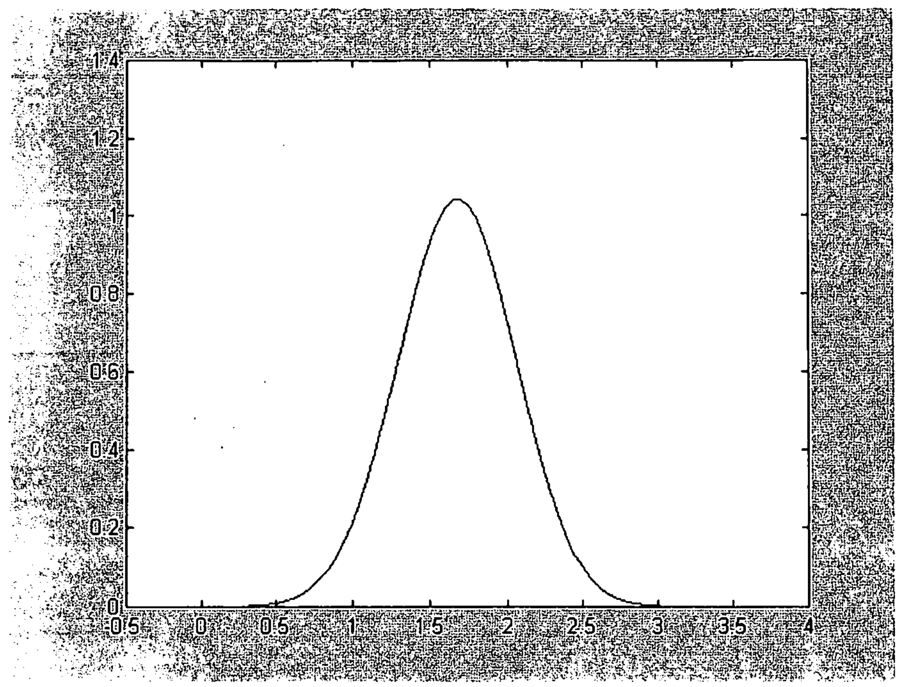

Figure 3.24 Estimated Marginal Posterior Density of β_1, the Expected Difference in Average Weaning Weight between lambs of Breed 1 and Breed 6.

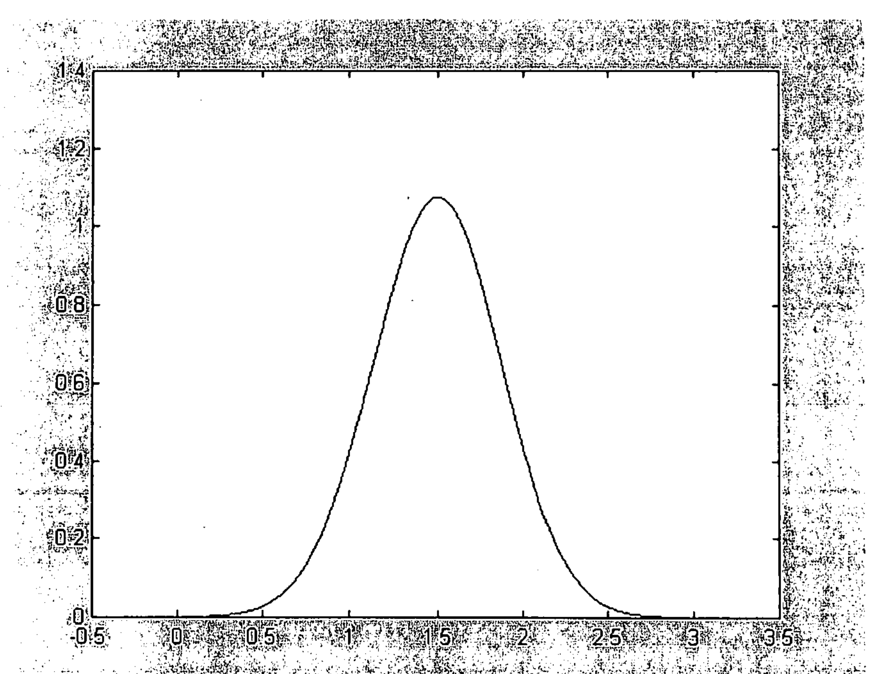

Figure 3.25 Estimated Marginal Posterior Density of β_2, the Expected Difference in Average Weaning Weight between lambs of Breed 2 and Breed 6.

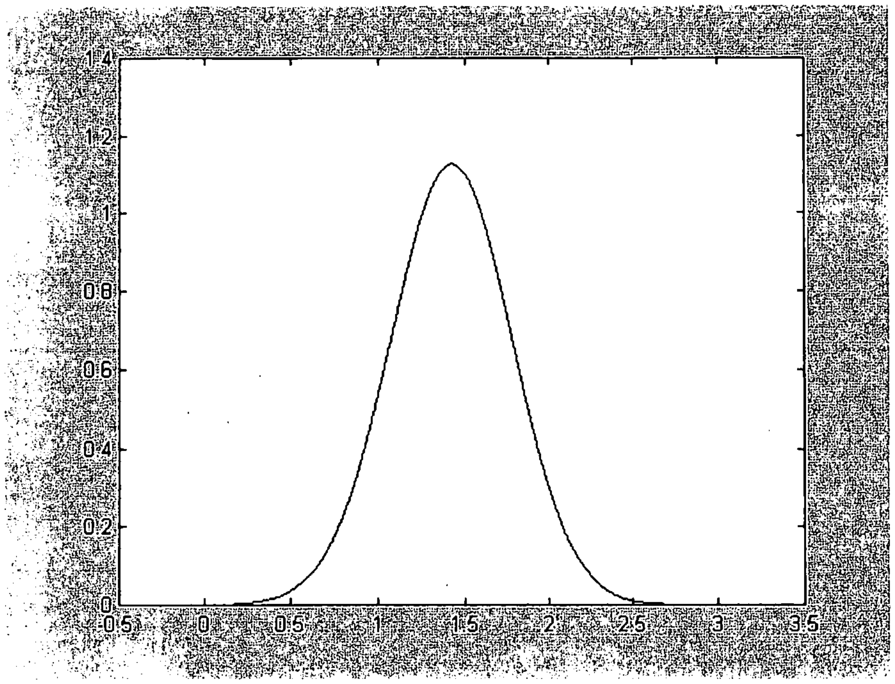

Figure 3.26 Estimated Marginal Posterior Density of β_3, the Expected Difference in Average Weaning Weight between lambs of Breed 3 and Breed 6.

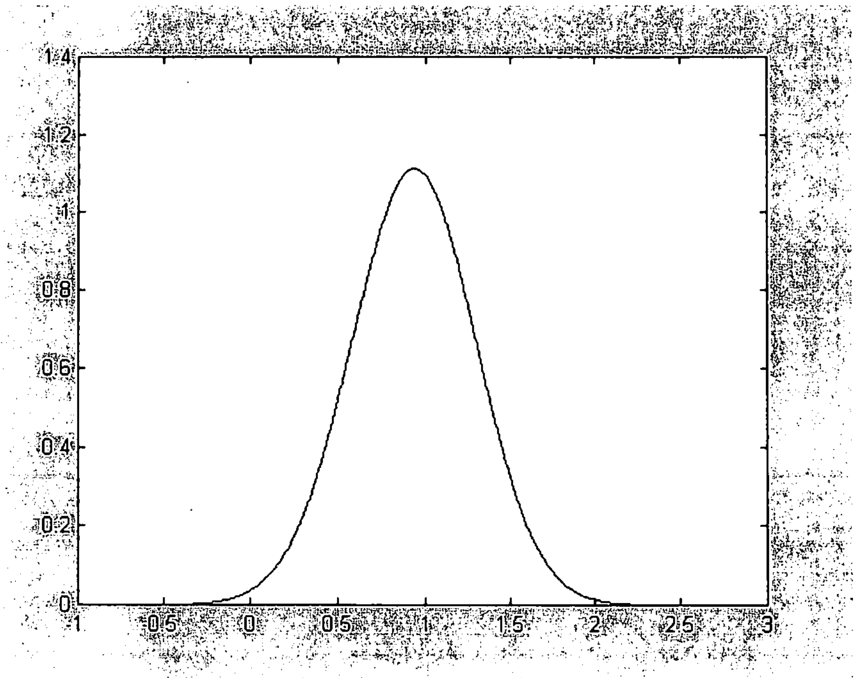

Figure 3.27 Estimated Marginal Posterior Density of β_4, the Expected Difference in Average Weaning Weight between lambs of Breed 4 and Breed 6.

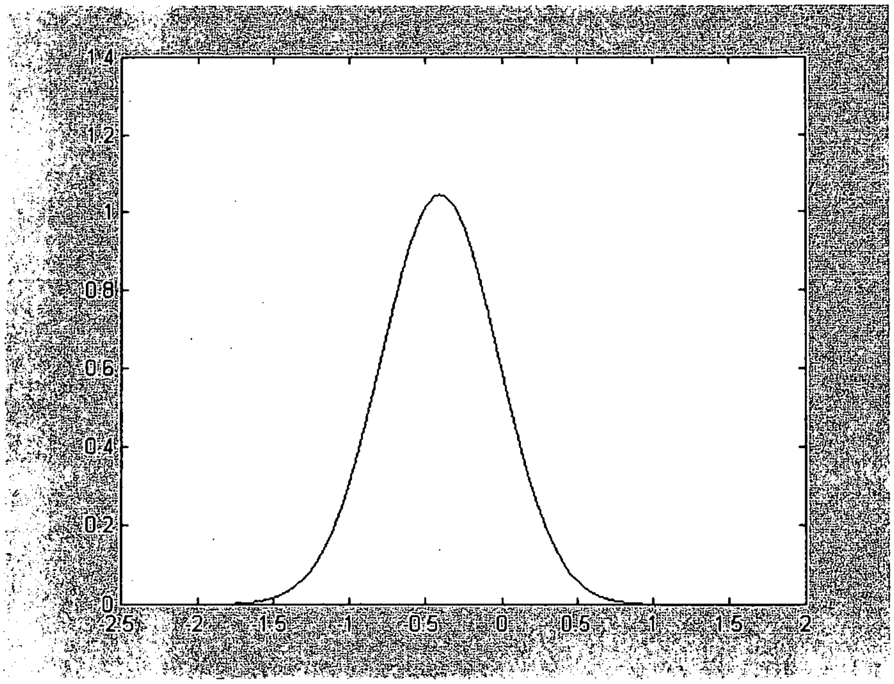

Figure 3.28 Estimated Marginal Posterior Density of β_5, the Expected Difference in Average Weaning Weight between lambs of Breed 5 and Breed 6.

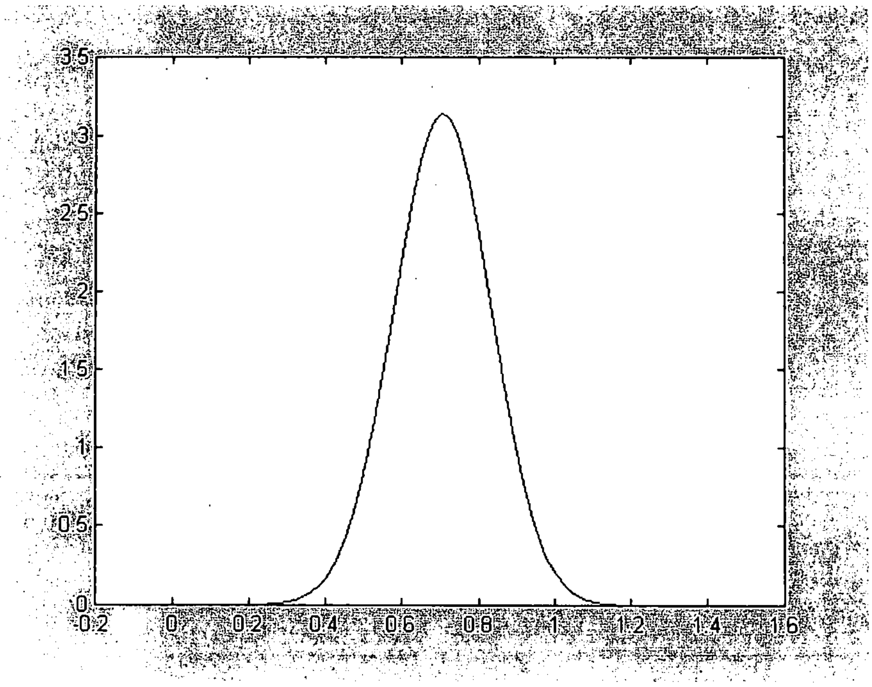

Figure 3.29 Estimated Marginal Posterior Density of β_6, the Expected Difference in Average Weaning Weight between Male and Female lambs.

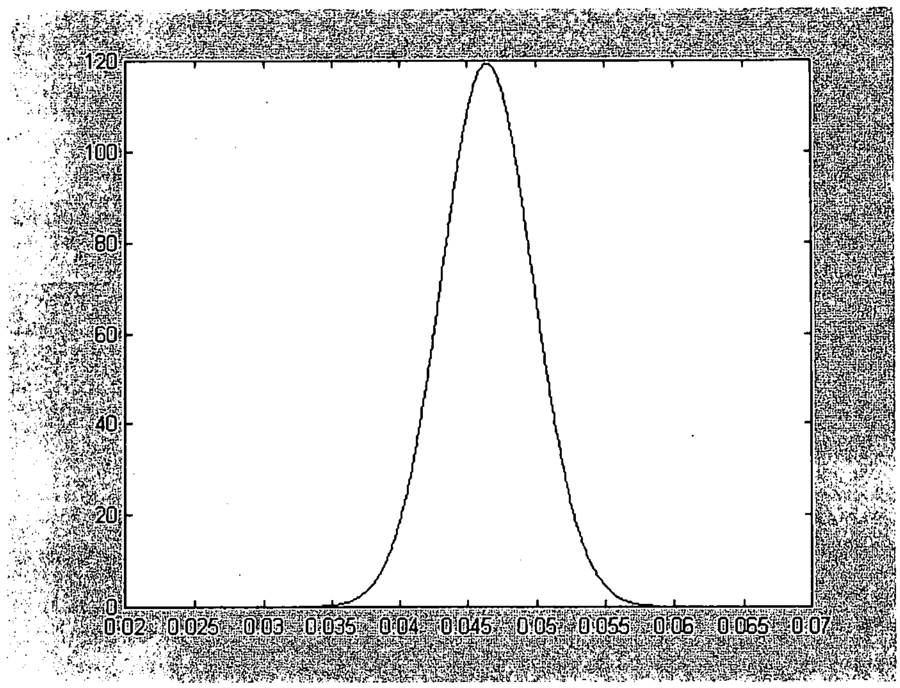

Figure 3.30 Estimated Marginal Posterior Density of β_7, the Average Increase in Weaning Weight per Daily Increase in Age.

3.6 An Experimental Design – Model Validation

Much of the current research focused on the distributional properties of Bayesian models compared to classical models. At present one would suggest the results of these models should be compared by means of partial F-tests, residual analysis, cross validation (data-splitting) or tests of overall model adequacy. However, model validation involves an assessment of how the fitted models will perform in practice, i.e. how successful it will be when applied to new or future data. An experiment will thus be conducted where all the genetic parameters in the mixed linear model are known, and the dependent variable will be built. These parameters, using REML and Bayesian methods (Dirichlet process) will then be estimated.

Again, consider the following mixed linear model

$$\underline{\mathbf{y}}_i = \mathbf{X}_i\beta + \mathbf{z}_i\gamma_i + \varepsilon_i$$

where $\varepsilon_i \sim N(0, \sigma^2_\varepsilon I_n)$ and for which the random effects have a non-parametric Dirichlet process prior distribution, i.e. $\gamma_i \sim G$ where $G \sim DP(M \cdot G_o)$. The parameters of the Dirichlet process are $G_o = N(0, \sigma^2_\gamma)$, a probability measure, and $M = 100$. The values for the variance components are $\sigma^2_\varepsilon = 4.88$ and $\sigma^2_\gamma = 0.7211$. The only fixed effect in the model is sex ($\beta_1 = 0.705$). Thus, the male lambs weigh on average 0.705 kg more than female lambs. A total of 200 sires are added as random sire effects to the model. For each sire, 10 male and 10 female weaning weights are generated using the Dirichlet process.

Table 3.8 reports the estimated variance components used for the experimental data set. A small difference between the estimates and the actual values of these variance components are observed, indicating that the Bayesian approach using the Gibbs sampler is certainly valuable and worthwhile in the context of animal breeding and selection.

Table 3.9 contains the estimated breeding values of the first 10 sires in the data set along with their rankings. The second column (EXP) in the table is the actual breeding values of the sires. The fourth and sixth columns contain the results when a Dirichlet process is implemented in the Gibbs sampler. For the fourth column, the precision parameter M is set equal to the true value, i.e. 100, whereas the six column contains results when this parameter is simulated given the data

Table 3.8 Estimated Variance Components for the Experimental Data using the Dirichlet Process Prior and REML Analysis.

	MDP, M = 100	MDP, Sim M	REML
σ^2_γ	0.765	0.768	0.772
σ^2_ε	4.932	4.935	4.929

Note that the true values for the variance components are, for $\sigma^2_\varepsilon = 4.88$ and $\sigma^2_\gamma = 0.7211$.

Table 3.9 Estimated Breeding Values of 10 of the 200 Sires for the Experimental Data along with their Posterior Rankings.

Rank	EXP	Sire_ID	MDP, $M = 100$	Sire_ID	MDP, Sim M	Sire_ID	REML	Sire_ID
1	1.4702	9	1.413	9	1.4188	9	1.44597	9
2	1.3975	2	1.3183	2	1.3239	2	1.32166	2
3	0.381	5	0.913	10	0.8727	10	0.87709	10
4	0.2997	1	0.2711	5	0.256	5	0.26953	6
5	0.146	10	0.2381	6	0.2142	6	0.2679	5
6	-0.082	6	-0.0503	7	-0.1292	7	-0.02923	7
7	-0.2163	8	-0.1544	3	-0.2231	3	-0.15932	3
8	-0.2835	11	-0.2535	1	-0.2281	1	-0.16895	1
9	-0.3977	4	-0.2812	11	-0.3324	11	-0.26275	11
10	-0.4936	7	-0.4713	8	-0.4889	8	-0.44905	8

From the table we can show that the Dirichlet process in Bayesian inference regarding breeding experiments is a very promising method. According to the experimental data, it is known that sires 9,2,5,1 and 10 are ranked as the five best sires in the model. The Dirichlet process ranked sires 9,2,10,5 and 6 as the best animals. This is an 80% success rate in the ranking procedure.

Sires 9,2,10,6 and 5 are also ranked as the best sires by the REML analysis. In the next section dealing with the model adequacy, the *SSE* (sum square errors) is calculated and reported in Table 3.10 below:

Table 3.10 The Calculated Sum Square Errors for the different Analysis.

	REML	**MDP, M = 100**	**MDP, Sim M**
SSE	46.145	46.44	44.833

From the results, it is believed that the Bayesian non-parametrics, using the Gibbs sampler, have as much to offer as the REML analysis. Since the posterior densities resulting from the Gibbs sampler can easily be used to construct confidence intervals for the model parameters, the potential mathematical consequences of the toolkit that is explored here in the world of the animal breeder is evident.

3.7 Chapter Summary

The important contribution of this chapter revolves around the non-parametric modelling of the random effects. We have applied a general technique for Bayesian non-parametrics to this important class of models, the mixed linear model for animal breeding experiments.

Our technique involved specifying a non-parametric prior for the distribution of the random effects and a Dirichlet process prior on the space of prior distributions for that non-parametric prior. The mixed linear model was then fitted with a Gibbs sampler, which turned an analytical intractable multidimensional integration problem into a feasible numerical one, overcoming most of the computational difficulties usually experience with the Dirichlet process. This proposed procedure also represents a new application of the mixture of Dirichlet process model to problems arising from animal breeding experiments. The application to and discussion of the breeding experiment from Kenya is helpful for understanding the importance and utility of the Dirichlet process, and inference for all the mixed linear model parameters.

As far as non-parametric *versus* parametric analysis are concerned, in relatively 'well-behaved' cases, where a parametric analysis would have coped, we typically obtain similar forms of posterior inference, particularly posterior modes, but with appropriately greater range of uncertainty in posterior means, as indicated in the case when the precision parameter is relatively small. When the appropriate form of the posterior should be 'badly behaved', the non-parametric analyses will reflect this, whereas most parametric analyses would not reveal this fact.

However, as mentioned before, a substantial statistical issue that still remains to be tackled is the great discrepancy between resulting posterior densities of the random effects as the value of the precision parameter, M changes. The work in this area is ongoing and needs a careful understanding, especially where inferences may be sensitive to the distributional assumption on the random effects.

We believe that Bayesian non-parametrics have much to offer, and can be applied to a wide range of statistical procedures. As far as Bayesian *versus* Classical approaches are concerned, we note the very real advantage of being able to input broad prior ideas of characteristics such as location, scale and shape. Moreover, the much richer and more tractable forms of inference that are presented as a consequence of the Gibbs simulation-based approach to computation are quite profound.

© Parts of this chapter (simulation study) have been published in the South African Journal of Animal Science.
(See Pretorius & Van der Merwe, 2000)

© Parts of this chapter (Dirichlet process results) are submitted and *in press* in *Genetics, Selection and Evolution.*
(See Van der Merwe and Pretorius, 2000 (*in press*))

© Parts of this chapter (Dirichlet process results) have been accepted for publication in *'Collection of Refereed Articles' – ISBA2000*[1].
(See Pretorius and Van der Merwe, 2001)

[1] International Society for Bayesian Analysis

CHAPTER 4

«The Dirichlet Process in Veterinary Medicine Research»

Introductory words: *In the present thesis, parallel developments of mixed linear models have also taken place in veterinary medicine research. The aim of the present chapter is to expand the Dirichlet process prior to a veterinary medicine problem in which the observations are correlated with each other.*

4.1 Prologue

As mentioned in the previous chapter, Mixture priors, especially Dirichlet Mixtures have opened the way to serious Bayesian developments in Non-parametric Modeling and Density Estimation. Moreover, mixtures of Dirichlet process models (MDP) have become increasingly popular for modeling when conventional parametric models would impose unreasonably stiff constraints on the distributional assumptions. There has been some work towards this end in the classical setting, however in the Bayesian paradigm, non-parametric modeling is still very scant and introductory.

As mentioned in Chapter 3, from the Bayesian perspective, inferential interest focuses on the posterior distribution of the random and fixed effects. Allowing distributions other than the normal for the random effects may more accurately model our prior beliefs, or it may allow us to better express out uncertainty about the true distribution of the random effects. However, one major question arising in Bayesian analysis concerns the sensitivity of the results to the chosen prior

In the next sections, we describe how the MDP model can be applied to the mixed linear model. We show the full conditional distributions and how the Gibbs sampling can be implemented for both the conjugate (section 4.3) and non-conjugate case (section 4.4). Indeed, if we assume the sampling distribution for $\underline{\mathbf{y}}_i$ to be normal, then a normal base measure for the random effects completes a conjugate MDP model. If on the other hand, the base measure is specified as a multivariate t – distribution (i.e. a scale mixture of the multivariate normal distribution), then the base measure for the random effects completes a non-conjugate MDP model.

We provide a detailed exploration of a veterinary medicine application of interest in an experiment where the mixed linear model is appropriate. Finally, we present an extension of the work to a non-conjugate mixture of Dirichlet process model, and compare the results of this analysis to that of the conjugate MDP model.

4.2 The Experiment and Model Structure

An important assumption in the use of mixed linear models is that the observations are independent from each other. In many practical situations this assumption does not hold. The most common situation is where different measurements are taken on the same individual, leading to what is known as a repeated measures design. Often these measurements are taken periodically over time. Alternatively, observations may be spatially collected in which case those closest together may be most alike. To demonstrate how the mixed linear model can be used for repeated measures design, the following experiment is analyzed.

The aim of the study was to see whether there are differences in the change in PCV between the two breeds of cattle, N'Dama and Boran, following a trypanosome infection[1]. A variable often measured to evaluate the severity of the diseases is packed cell volume (PCV), which is the percentage of the volume of the blood serum taken up by the red blood cells. Low PCV corresponds to anaemia and can indicate infection with the disease.

Depending on the design of the experiment, different models could be fitted to the data, but it will be shown that the mixed linear model framework provides a unified way to investigate the changes over time of PCV in the case of the two different breeds. Moreover, the Gibbs sampler developed in the previous chapter can also be useful here to show how posterior computations via Gibbs sampling simulations can be routinely applied to the experiment. The data are shown in APPENDIX D.

The appropriate mixed linear model for the experiment is given by

$$\underline{\mathbf{y}}_i = \mathbf{X}_i\beta + \mathbf{Z}_i\gamma_i + \varepsilon_i \tag{4.1}$$

$\underline{\mathbf{y}}_i$ is a $n_i \times 1$ vector of PCV measurements for the i^{th} animal; $\beta\,(p \times 1)$ is a vector of uniquely defined fixed effects and that the corresponding design matrix $\mathbf{X}_i$ is $(n_i \times p)$. For the present example, we fit the average slopes and intercepts for the two breeds as the fixed effects.

[1] Parasitic disease transmitted by Tse-tse flies

Also, $\mathbf{Z}_i$ ($n_i \times v$) is a matrix of covariates for the $v \times 1$ vector of random effects. Further, γ_i is the unobservable random effect, and for the unobservable vector of random errors, ε_i ($n_i \times 1$), it is common to assume independent normal distributions.

For γ_i ($v \times 1$), the vector of unobservable random effects which is usually taken to be normally distributed, the normal prior is replaced with a non-parametric prior, followed by a Dirichlet process prior on the general distribution. For the present example, it is also assumed that each animal has a random slope and intercept, and that this random slope and intercept are normally distributed. Thus, the intercept and slope parameters describing the linear relationship between PCV and breed contain both fixed and random effects.

Considering the above model structure, the covariates for the fixed effects are

x_0 : Intercept

x_1 : Time in days

$$x_2 : \begin{cases} 1 \text{ if N'Dama breed} \\ 0 \text{ if Boran breed} \end{cases}$$

x_3 : $x_1\, x_2$ - the time by treatment interaction,

and the covariates for the random effects are (with $v = 2$)

Z_0 : Intercept

Z_1 : Time in days

Further

γ_{0i} = Intercept for the i^{th} individual

γ_{1i} = Slope for the i^{th} individual,

and

β_0 = Intercept for PCV measurements on Boran breed,

β_1 = Slope for PCV measurements over time on Boran breed,

β_2 = Difference between the intercepts of the N'Dama and Boran breeds, and

β_3 = Difference between the slopes of the N'Dama and Boran breeds.

Therefore we have

$$\mathbf{X}_1\beta = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 1 & 9 & 0 & 0 \\ 1 & 14 & 0 & 0 \\ 1 & 17 & 0 & 0 \\ 1 & 18 & 0 & 0 \\ 1 & 21 & 0 & 0 \\ 1 & 23 & 0 & 0 \\ 1 & 25 & 0 & 0 \\ 1 & 29 & 0 & 0 \\ 1 & 31 & 0 & 0 \\ 1 & 35 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \quad ; \quad \mathbf{Z}_i\gamma_i = \begin{bmatrix} Z_1 & Z_2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 7 \\ 1 & 9 \\ 1 & 14 \\ 1 & 17 \\ 1 & 18 \\ 1 & 21 \\ 1 & 23 \\ 1 & 25 \\ 1 & 29 \\ 1 & 31 \\ 1 & 35 \end{bmatrix} \begin{bmatrix} \gamma_{0i} \\ \gamma_{1i} \end{bmatrix}$$

with $\mathbf{X}_1 = \mathbf{X}_2 = \ldots = \mathbf{X}_6$, and

$$
\mathbf{X}_7\beta = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 4 \\ 1 & 7 & 1 & 7 \\ 1 & 9 & 1 & 9 \\ 1 & 14 & 1 & 14 \\ 1 & 17 & 1 & 17 \\ 1 & 18 & 1 & 18 \\ 1 & 21 & 1 & 21 \\ 1 & 23 & 1 & 23 \\ 1 & 25 & 1 & 25 \\ 1 & 29 & 1 & 29 \\ 1 & 31 & 1 & 31 \\ 1 & 35 & 1 & 35 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}
$$

with $\mathbf{X}_7 = \mathbf{X}_8 = \ldots = \mathbf{X}_{12}$.

Further

$$
\underline{\mathbf{y}}_1 = \begin{bmatrix} 36.2 \\ 35.9 \\ . \\ . \\ . \\ . \\ 21.3 \\ 17.8 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{y}}_7 = \begin{bmatrix} 30.4 \\ 33.0 \\ . \\ . \\ . \\ . \\ 24.5 \\ 22.6 \end{bmatrix}.
$$

As in the previous chapter we will present a mixed linear model for which the vector of random effects have a non-parametric distribution. The non-parametric Bayesian approach for the random effects is to specify a prior distribution on the space of all possible distribution functions. This prior for the mixed linear model is applied to the general prior of the distribution of the random effects.

Indeed, as shown before, this can be accomplished with a Dirichlet process prior distribution. This means that the usual normal prior on the random effects is replaced with a non-parametric prior, followed by a Dirichlet prior on the general distribution.

In the next section the required conditional posterior distributions for the model parameters are given. Note that although the detailed derivations of these distributions can be seen in Chapter 3, some extra derivations will be given in the present chapter. Only minor changes are made to the Gibbs sampler because of the correlated observations (repeated design).

4.3 Priors and Conditional Posterior Distributions for the Conjugate MDP Model

4.3.1 The Uniform Prior for β and σ_ε^2

The full conditionals for β and σ_ε^2 for the conjugate MDP model in the case of repeated measures, are the same as in Chapter 3, i.e. an uniform prior distribution for both β and σ_ε^2 as to represent lack of prior knowledge about the vector of fixed effects and error variance. Therefore

$$p(\beta, \sigma_\varepsilon^2) = p(\beta)\, p(\sigma_\varepsilon^2) \propto \text{constant} \tag{4.2}$$

The required full conditional for the fixed effects, is multivariate normal:

$$\beta \mid \gamma_i, \sigma_\varepsilon^2, \underline{\mathbf{y}}_i, \sim N_p \left\{ \hat{\beta}, \left(\sum_{i=1}^{q} (\mathbf{X}_i{}'\mathbf{X}_i) \right)^{-1} \sigma_\varepsilon^2 \right\} \tag{4.3}$$

where $\hat{\beta} = \left(\sum_{i=1}^{q} (\mathbf{X}_i{}'\mathbf{X}_i) \right)^{-1} \sum_{i=1}^{q} \mathbf{X}_i{}'(\underline{\mathbf{y}}_i - \mathbf{Z}_i\gamma_i)$.

For the variance component, σ_ε^2 the conditional is

$$p(\sigma_\varepsilon^2 \mid \beta, \gamma, \underline{\mathbf{y}}\ ) = K_\varepsilon \prod_{i=1}^{q} \left(\frac{1}{\sigma_\varepsilon^2} \right)^{\frac{n_i}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} (\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i)'(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i) \right\}$$

$$\sigma_\varepsilon^2 > 0 \tag{4.4}$$

an Inverse Gamma density where

$$K_\varepsilon = \left\{ \frac{\sum_{i=1}^{q} (\underline{\mathbf{y}} - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i)'(\underline{\mathbf{y}} - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i)}{2} \right\}^{\frac{n-2}{2}} \frac{1}{\Gamma(\frac{n-2}{2})}.$$

Also, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$, $\underline{\mathbf{y}} = (\underline{\mathbf{y}}_1{}', \underline{\mathbf{y}}_2{}', \ldots, \underline{\mathbf{y}}_q{}')'$, $\sum_{i=1}^{q} n_i = n$ the sample size, and $\gamma_i = \begin{bmatrix} \gamma_{0i} \\ \gamma_{1i} \end{bmatrix}$

where γ_{0i} = Intercept for the i^{th} individual and γ_{1i} = Slope for the i^{th} individual, as defined before.

4.3.2 Prior for D

The variance covariance matrix D in the base measure of the Dirichlet process is unknown and therefore a suitable prior distribution must be specified for it, i.e.

$$p(\boldsymbol{D}) \propto \text{constant}$$

to present lack of prior knowledge about D. After choosing random effects for each subject, the different subjects will be grouped into clusters (groups) in which the subjects have equal γ_i's (equal intercepts and equal slopes). That is, after selecting a new γ_i for each subject i in the sample, there will be some number ξ, $0 < \xi \le q$, of unique values among the random γ_i's. Denote these unique values by λ_l, $l=1 \ldots \xi$. Additionally let l represent the set of subjects with common random effect λ_l.

Note that knowing the random effects is equivalent to knowing ξ, all of the γ_i's and the cluster membership l. Then for the purpose of calculating the full conditional of D, the λ_l are ξ independent observations from $N(0,\boldsymbol{D})$.

Thus

$$p(\boldsymbol{D} \mid \lambda, \underline{\mathrm{y}}) \propto |\boldsymbol{D}|^{-\frac{\xi}{2}} \exp\left\{-\frac{1}{2}\sum_{l=1}^{\xi} \lambda'_l \boldsymbol{D}^{-1} \lambda_l\right\} \qquad (4.5)$$

an Inverse Wishart distribution where

$$\lambda = [\lambda_1 \lambda_2 \ldots \lambda_\xi]'.$$

See Chapter 3 for more details on the additional piece of the model to be added as an aid to convergence for the Gibbs sampler (equations (3.17) with $\sigma_\gamma^2 = \boldsymbol{D}$). This additional piece is incorporated into the final Gibbs sampler. The simulation procedure of the precision parameter, M remains the same as in Chapter 3, and an algorithm (1.3) for simulating from a Wishart distribution is given in APPENDIX A. Simulation from the Wishart distribution can also easily be done by using the algorithm of Odell and Feiveson (1966).

4.3.3 Dirichlet Process Prior for γ_i

As mentioned in the previous chapter the mixture of Dirichlet Process Prior is simplified in practice by the Polya urn representation, using the fact that marginally, the γ_i are distributed as the base measure along with the added property that $p(\gamma_i = \gamma_j,\ i \neq j) > 0$.

Therefore

$$\gamma_i \sim G_0 \qquad (i = 1, \ldots, q), \tag{4.6}$$

$$\gamma_q \mid \gamma_1, \ldots, \gamma_{q-1} \begin{cases} = \gamma_j & \text{with probability} \quad \dfrac{1}{M+q-1} \\ \sim G_0 & \text{with probability} \quad \dfrac{M}{M+q-1} \end{cases} \tag{4.7}$$

We find that the conditional posterior of γ_i is given by.

$$p(\gamma_i \mid \beta, \sigma_\varepsilon^2, \boldsymbol{D}, \gamma_{-i}, M) \propto \sum_{j \neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i\beta + \mathbf{Z}_i\gamma_j, \sigma_\varepsilon^2 I_{n_i}) \cdot \delta_{\gamma_j}$$
$$+ \left\{ M \int_{-\infty}^{\infty} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i\beta + \mathbf{Z}_i\gamma_i, \sigma_\varepsilon^2 I_{n_i}) \phi(\gamma_i \mid 0, \boldsymbol{D}) d\gamma_i \right\} \tag{4.8}$$
$$\times \phi(\gamma_i \mid 0, \boldsymbol{D}) p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j)$$

where $p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j) = \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}\beta + \mathbf{Z}_i\gamma_i, \sigma_\varepsilon^2 I_{n_i})$ and $\phi(\cdot \mid \mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2 . Also, γ_{-i} denotes the vector of random effects for the subjects excluding subject i and δ_s is a degenerate distribution with point mass at s.

Consider again the integral

$$\Lambda_i = M \int_{-\infty}^{\infty} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i\beta + \mathbf{Z}_i\gamma_i, \sigma_\varepsilon^2 I_{n_i}) \phi(\gamma_i \mid 0, \boldsymbol{D}) d\gamma_i$$
$$= M \int_{-\infty}^{\infty} \left(\frac{1}{2\sigma_\varepsilon^2 \pi} \right)^{\frac{n_i}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} (\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i)'(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i) \right\} \times$$
$$\left(\frac{1}{2\pi} \right)^{\frac{v}{2}} |\boldsymbol{D}|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \gamma_i' \boldsymbol{D}^{-1} \gamma_i \right\} d\gamma_i. \tag{4.9}$$

Following the usual algebraic routes, i.e. completing the square with respect to γ_i, it follows from (4.9) that

$$\Lambda_i = M(2\pi)^{-\frac{n_i}{2}} |\boldsymbol{D}|^{-\frac{1}{2}} |\Omega_i|^{\frac{1}{2}} \left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}} \exp\left\{ \frac{1}{2\sigma_\varepsilon^2} (\underline{\mathbf{y}}_i - \mathbf{X}_i\beta)' \Psi_i (\underline{\mathbf{y}}_i - \mathbf{X}_i\beta) \right\} \tag{4.10}$$

where

$$\Omega_i = \left[\frac{1}{\sigma_\varepsilon^2} (\mathbf{Z}_i'\mathbf{Z}_i) + \boldsymbol{D}^{-1} \right]^{-1} \quad \text{and} \quad \Psi_i = \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{Z}_i \Omega_i \mathbf{Z}_i' - I_{n_i} \right).$$

Proof:

The exponent of the integral is $\frac{1}{\sigma_\varepsilon^2}\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i\right)'\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i\right) + \gamma_i'\boldsymbol{D}^{-1}\gamma_i$. This can be written as

$$\frac{1}{\sigma_\varepsilon^2}\left(\underline{\tilde{\mathbf{y}}}_i - \mathbf{Z}_i\gamma_i\right)'\left(\underline{\tilde{\mathbf{y}}}_i - \mathbf{Z}_i\gamma_i\right) + \gamma_i'\boldsymbol{D}^{-1}\gamma_i$$

$$= \frac{1}{\sigma_\varepsilon^2}\underline{\tilde{\mathbf{y}}}_i'\underline{\tilde{\mathbf{y}}}_i - 2\frac{1}{\sigma_\varepsilon^2}\underline{\tilde{\mathbf{y}}}_i'\mathbf{Z}_i\gamma_i + \frac{1}{\sigma_\varepsilon^2}\gamma_i^2\mathbf{Z}_i'\mathbf{Z}_i + \gamma_i'\boldsymbol{D}^{-1}\gamma_i$$

where

$$\underline{\tilde{\mathbf{y}}}_i = \underline{\mathbf{y}}_i - \mathbf{X}_i\beta.$$

Following the usual algebraic routes, i.e. completing the square with respect to γ_i where γ_i is a $\nu \times 1$ vector, it follows from (4.9) that

$$\frac{1}{\sigma_\varepsilon^2}\gamma_i'\mathbf{Z}_i'\mathbf{Z}_i\gamma_i + \gamma_i'\boldsymbol{D}^{-1}\gamma_i - 2\frac{1}{\sigma_\varepsilon^2}\underline{\tilde{\mathbf{y}}}_i'\mathbf{Z}_i\gamma_i =$$

$$\left(\gamma_i - \left\{\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i'\mathbf{Z}_i + \boldsymbol{D}^{-1}\right\}^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i'\underline{\tilde{\mathbf{y}}}_i\right)'\left\{\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i'\mathbf{Z}_i) + \boldsymbol{D}^{-1}\right\}\left(\gamma_i - \left\{\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i'\mathbf{Z}_i + \boldsymbol{D}^{-1}\right\}^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i'\underline{\tilde{\mathbf{y}}}_i\right)$$
$$-\left(\frac{1}{\sigma_\varepsilon^2}\right)^2\underline{\tilde{\mathbf{y}}}_i'\mathbf{Z}_i\left\{\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i'\mathbf{Z}_i) + \boldsymbol{D}^{-1}\right\}^{-1}\mathbf{Z}_i'\underline{\tilde{\mathbf{y}}}_i.$$

Hence, the exponent $\frac{1}{\sigma_\varepsilon^2}\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i\right)'\left(\underline{\mathbf{y}}_i - \mathbf{X}_i\beta - \mathbf{Z}_i\gamma_i\right) + \gamma_i'\boldsymbol{D}^{-1}\gamma_i$ can be written as

$$\frac{1}{\sigma_\varepsilon^2}\underline{\tilde{\mathbf{y}}}_i{}'\underline{\tilde{\mathbf{y}}}_i - \left(\frac{1}{\sigma_\varepsilon^2}\right)^2 \underline{\tilde{\mathbf{y}}}_i{}'\mathbf{Z}_i\left\{\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\boldsymbol{D}^{-1}\right\}^{-1}\mathbf{Z}_i{}'\underline{\tilde{\mathbf{y}}}_i +$$

$$\left(\gamma_i - \left\{\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\mathbf{Z}_i+\boldsymbol{D}^{-1}\right\}^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\underline{\tilde{\mathbf{y}}}_i\right)'\left\{\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\boldsymbol{D}^{-1}\right\}\left(\gamma_i - \left\{\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\mathbf{Z}_i+\boldsymbol{D}^{-1}\right\}^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\underline{\tilde{\mathbf{y}}}_i\right)$$

From the above expression it follows,

$$\Lambda_i = M(2\pi)^{-\frac{1}{2}(v+n_i)}|\boldsymbol{D}|^{-\frac{1}{2}}\left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}}\exp\left\{\frac{1}{2\sigma_\varepsilon^2}\tilde{\mathbf{y}}_i{}'\left[\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i\left\{\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\boldsymbol{D}^{-1}\right\}^{-1}\mathbf{Z}_i{}'-I_{n_i}\right]\tilde{\mathbf{y}}_i\right\}$$

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\gamma_i - \left\{\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\mathbf{Z}_i+\boldsymbol{D}^{-1}\right\}^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\underline{\tilde{\mathbf{y}}}_i\right)'\left\{\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\boldsymbol{D}^{-1}\right\}\times\right.$$
$$\left.\left(\gamma_i - \left\{\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\mathbf{Z}_i+\boldsymbol{D}^{-1}\right\}^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i{}'\underline{\tilde{\mathbf{y}}}_i\right)\right\}d\gamma_i$$

and we find

$$\Lambda_i = M(2\pi)^{-\frac{1}{2}(v+n_i)}|\boldsymbol{D}|^{-\frac{1}{2}}\left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}}\exp\left\{\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{y}}_i-\mathbf{X}_i\beta)'\Psi_i(\underline{\mathbf{y}}_i-\mathbf{X}_i\beta)\right\}\left[\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\boldsymbol{D}^{-1}\right]^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}.$$

Therefore

$$\Lambda_i = M(2\pi)^{-\frac{n_i}{2}}|\boldsymbol{D}|^{-\frac{1}{2}}\Omega_i^{\frac{1}{2}}\left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}}\exp\left\{\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{y}}_i-\mathbf{X}_i\beta)'\Psi_i(\underline{\mathbf{y}}_i-\mathbf{X}_i\beta)\right\}$$

where

$$\Omega_i = \left[\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\boldsymbol{D}^{-1}\right]^{-1} \quad \text{and} \quad \Psi_i = \left(\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i\Omega_i\mathbf{Z}_i{}'-I_{n_i}\right).$$

Thus, as explained in Chapter 3, each summand in the conditional posterior distribution of γ_i (equation (4.8)) is separated into two elements. The first element is a mixing probability, and the second is a distribution to be mixed. So with probability

$$\frac{\phi(\underline{\mathbf{y}}_i \mid X_i\beta + \mathbf{Z}_i\gamma_j; \sigma_\varepsilon^2 I_{n_i})}{\Lambda_i + \sum_{j=1; j\neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid X_i\beta + \mathbf{Z}_i\gamma_j; \sigma_\varepsilon^2 I_{n_i})} \tag{4.11}$$

we select from distribution δ_{γ_j}, which means that we set $\gamma_i = \gamma_j$. Also, with probability

$$\frac{\Lambda_i}{\Lambda_i + \sum_{j=1; j\neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid X_i\beta + \mathbf{Z}_i\gamma_j; \sigma_\varepsilon^2 I_{n_i})} \tag{4.12}$$

we select from

$$p(\gamma_i \mid \beta, \sigma_\varepsilon^2, \boldsymbol{D}, \underline{\mathbf{y}}_i) \propto \phi(\gamma_i \mid 0, \boldsymbol{D}) p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j), \tag{4.13}$$

meaning we sample γ_i from its full conditional,

$$p(\gamma_i / \beta, \boldsymbol{D}, \sigma_\varepsilon^2, \underline{\mathbf{y}}_i) = N\left\{(\mathbf{Z}_i'\mathbf{Z}_i + \sigma_\varepsilon^2 \boldsymbol{D}^{-1})^{-1}\mathbf{Z}_i'(\underline{\mathbf{y}}_i - X_i\beta); (\mathbf{Z}_i'\mathbf{Z}_i + \sigma_\varepsilon^2 \boldsymbol{D}^{-1})^{-1}\sigma_\varepsilon^2\right. \tag{4.14}$$

Before applying the conjugate MDP model to a veterinary medicine experiment, we first turn to the non-conjugate MDP model, and show how to apply this model structure to our mixed linear model.

4.4 Priors and Conditional Posterior Distributions for the Non-Conjugate MDP Model (Modified MDP Model)

As mentioned before, samples of the γ_i from a modified distribution can be obtained by specifying the base measure as a multivariate normal scale mixture. The multivariate t – distribution can be obtained as a scale mixture of the multivariate normal distribution as follows. If

$$\begin{aligned} x \mid \eta &\sim N_p(\mu, \eta^{-1}\Sigma), \\ \eta &\sim ga(\frac{s}{2}, \frac{s}{2}), \end{aligned} \tag{4.15}$$

then the marginal distribution of x is $St_p(s, \mu, \Sigma)$, where $St_p(s, \mu, \Sigma)$ is a p – dimensional Student t – distribution with s degrees of freedom, mean μ, and dispersion matrix Σ. Also ga = *Gamma* distribution. From a sampling perspective, sampling from

$$\eta \sim ga(\frac{s}{2}, \frac{s}{2})$$

and then from

$$x \mid \eta \sim N_p(\mu, \eta^{-1}\Sigma)$$

is equivalent to sampling from $St_p(s, \mu, \Sigma)$. Thus, marginally, the distribution of x is multivariate t, with η being integrated out. This representation is often used in the Gibbs sampling literature (Kleinman & Ibrahim, 1998; also see Wakefield *et al.*, 1995 and the references therein). Note that with a common η for all i, we generate dependent samples for the γ_i's. To obtain independent samples for the γ_i's, we must specify a separate η_i for each γ_i and take the η_i's to be i.i.d. *gamma* variates. The specifications for the modified MDP model are as follows.

4.4.1 The Uniform Prior for β and σ_ε^2

The full conditionals for β and σ_ε^2 for the multivariate t model (non-conjugate MDP model) are the same as in the conjugate MDP model, i.e. a uniform prior distribution for both β and σ_ε^2 as to represent lack of prior knowledge about the vector of fixed effects and error variance (see equations (3.13) and (4.2)). Thus the required full conditional for the fixed effects, β is again multivariate normal (see equations (3.14) and (4.3)), and for the variance component, σ_ε^2 an Inverse Gamma density (see equations (3.15) and (4.4)).

4.4.2 Prior and Conditional Posterior for η

From the discussion in section (4.4) we will specify for η_i the prior

$\eta \sim ga(\frac{\rho}{2}, \frac{\rho}{2})$, i.e.

$$p(\eta) = \frac{(\rho)^{\frac{\rho}{2}} \eta^{\frac{\rho}{2}-1} \exp\left\{-\frac{\rho\eta}{2}\right\}}{2^{\frac{\rho}{2}} \Gamma\left(\frac{\rho}{2}\right)}. \tag{4.16}$$

In the example that follows, we will use $\rho = 4$, which means that instead of using a normal prior as base measure, we will use a t – distribution with 4 degrees of freedom for G_0.

The conditional posterior distribution for η is given by

$$p(\eta \mid \gamma_i, \beta, \sigma_\varepsilon^2, \boldsymbol{D}, \underline{\mathbf{y}}) = K \frac{(\rho)^{\frac{\rho}{2}} \eta^{\frac{\rho}{2}-1} \exp\left\{-\frac{\rho\eta}{2}\right\}}{2^{\frac{\rho}{2}} \Gamma\left(\frac{\rho}{2}\right)} \times \left|\eta^{-1}\boldsymbol{D}\right|^{-\frac{\xi}{2}} \exp\left\{-\frac{1}{2}\sum_{l=1}^{\xi} \lambda_l' \left(\eta \boldsymbol{D}^{-1}\right) \lambda_l\right\}$$

(4.17)

which leads to

$$K\eta^{\frac{(vk+\rho)}{2}-1} \exp\left\{-\frac{\eta}{2}\left(\rho + \sum_{l=1}^{\xi} \lambda_l' \boldsymbol{D}^{-1} \lambda_l\right)\right\}, \quad (4.18)$$

a Gamma distribution. Hence, from a sampling perspective, $\tilde{\eta} \sim \chi^2_{vk+\rho}$, i.e. η is sampled from a Chi-square distribution with $vk+\rho$ degrees of freedom and η can be calculated as

$$\eta = \frac{\tilde{\eta}}{\left(\rho + \sum_{l=1}^{\xi} \lambda_l' \boldsymbol{D}^{-1} \lambda_l\right)}. \quad (4.19)$$

Equations (4.17), (4.18) and (4.19) also follow from the fact that after selecting a new γ_i for each subject i in the sample, there will be some number ξ, $0 < \xi \leq q$, of unique values λ_l, $l = 1, \ldots, \xi$ among the random γ_i's (see also section (4.3.2) for more details on ξ). Thus, for the modified MDP model an additional piece is added to the Gibbs sampler. Also, note that we use a somewhat simpler generation of the multivariate t than is usually found in applications of the Gibbs sampler.

4.4.3 Prior for D

As in the normal case, section (4.3.2) we specify for $\boldsymbol{D}$ the prior

$$p(\boldsymbol{D}) \propto \text{constant}$$

to present lack of prior knowledge for $\boldsymbol{D}$. For given η the posterior is therefore given by

$$\begin{aligned} p(\boldsymbol{D} \mid \lambda, \underline{y}, \eta) &\propto \eta^{-1}\boldsymbol{D}|^{-\frac{\xi}{2}} \exp\left\{-\frac{1}{2}\sum_{l=1}^{\xi} \lambda'_l \, \eta \boldsymbol{D}^{-1} \lambda_l\right\} \\ &\propto \eta^{\frac{\nu\xi}{2}} |\boldsymbol{D}|^{-\frac{\xi}{2}} \exp\left\{-\frac{1}{2}\boldsymbol{D}^{-1}\sum_{l=1}^{\xi} \eta(\lambda_l \lambda_l')\right\}. \end{aligned} \tag{4.20}$$

where $\lambda = [\lambda_1 \lambda_2 ... \lambda_\xi]'$.

Further, as shown in Chapter 3, to speed mixing over the entire parameter space, it is suggested to move around the λ's after determining how the γ_i's are grouped. Thus, in addition, a posterior density is derived for the λ's, i.e.

$$p(\lambda_l \mid \beta, \sigma_\varepsilon^2, \eta, \boldsymbol{D}, \underline{\mathbf{y}}) \propto \phi(\lambda_l \mid 0, \eta^{-1}\boldsymbol{D}) \prod_{i \in l} p(\underline{\mathbf{y}}_i \mid \beta, \sigma_\varepsilon^2) \tag{4.21}$$

which implies that

$$\lambda_l \mid \beta, \sigma_\varepsilon^2, \eta, \boldsymbol{D}, \underline{\mathbf{y}} \sim N\left\{\tilde{\tilde{\gamma}}_l^{\eta}, \left(\sum_{i \in l} \mathbf{Z}_i'\mathbf{Z}_i + \eta \boldsymbol{D}^{-1}\sigma_\varepsilon^2\right)^{-1} \sigma_\varepsilon^2\right\} \tag{4.22}$$

where

$$\tilde{\gamma}_l^{\eta} = \left(\sum_{i \in l} \mathbf{Z}_i{}' \mathbf{Z}_i + \eta \boldsymbol{D}^{-1} \sigma_\varepsilon^2 \right)^{-1} \left(\sum_{i \in l} \mathbf{Z}_i{}' \left(\underline{\mathbf{y}}_i - \mathbf{X}_i \beta \right) \right). \tag{4.23}$$

4.4.4 Dirichlet Process Prior for γ_i

From the discussion of the conjugate MDP model in section (4.3), we find that he conditional posterior of γ_i is given by

$$\begin{aligned} p(\gamma_i \mid \beta, \sigma_\varepsilon^2, \eta, \boldsymbol{D}, \gamma_{-i}, M) &\propto \sum_{j \neq i}^{q} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i \beta + \mathbf{Z}_i \gamma_j, \sigma_\varepsilon^2 I_{n_i}) \cdot \delta_{\gamma_j} \\ &+ \left\{ M \int_{-\infty}^{\infty} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i \beta + \mathbf{Z}_i \gamma_i, \sigma_\varepsilon^2 I_{n_i}) \phi(\gamma_i \mid 0, \eta^{-1} \boldsymbol{D}) d\gamma_i \right\} \\ &\times \phi(\gamma_i \mid 0, \eta^{-1} \boldsymbol{D}) p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j) \end{aligned} \tag{4.24}$$

where $p(\underline{\mathbf{y}}_i \mid \gamma_i, \beta, \sigma_\varepsilon^2, \underline{\mathbf{y}}_j) = \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}\beta + \mathbf{Z}_i \gamma_i, \sigma_\varepsilon^2 I_{n_i})$ and $\phi(\cdot \mid \mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2 . Also, γ_{-i} denotes the vector of random effects for the subjects excluding subject i and δ_s is a degenerate distribution with point mass at s. Consider again the integral

$$\begin{aligned} \Lambda_i^{\eta} &= M \int_{-\infty}^{\infty} \phi(\underline{\mathbf{y}}_i \mid \mathbf{X}_i \beta + \mathbf{Z}_i \gamma_i, \sigma_\varepsilon^2 I_{n_i}) \phi(\gamma_i \mid 0, \eta^{-1} \boldsymbol{D}) d\gamma_i \\ &= M \int_{-\infty}^{\infty} \left(\frac{1}{2\sigma_\varepsilon^2 \pi} \right)^{\frac{n_i}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} (\underline{\mathbf{y}}_i - \mathbf{X}_i \beta - \mathbf{Z}_i \gamma_i)' (\underline{\mathbf{y}}_i - \mathbf{X}_i \beta - \mathbf{Z}_i \gamma_i) \right\} \times \\ &\quad \left(\frac{1}{2\pi} \right)^{\frac{v}{2}} \left| \eta^{-1} \boldsymbol{D} \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \gamma_i{}' \eta^{-1} \boldsymbol{D}^{-1} \gamma_i \right\} d\gamma_i . \end{aligned} \tag{4.25}$$

Completing the square with respect to γ_i where γ_i is a $v \times 1$ vector, it follows from (4.25) that

$$\Lambda_i^\eta \quad = M(2\pi)^{-\frac{n_i}{2}}\left|\eta^{-1}\boldsymbol{D}\right|^{-\frac{1}{2}}\left|\Omega_i^\eta\right|^{\frac{1}{2}}\left(\sigma_\varepsilon^2\right)^{-\frac{n_i}{2}}\exp\left\{\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{y}}_i-\mathbf{X}_i\beta)'\Psi_i^\eta(\underline{\mathbf{y}}_i-\mathbf{X}_i\beta)\right\} \tag{4.26}$$

where

$$\Omega_i^\eta = \left[\frac{1}{\sigma_\varepsilon^2}(\mathbf{Z}_i{}'\mathbf{Z}_i)+\eta\boldsymbol{D}^{-1}\right]^{-1} \quad \text{and} \quad \Psi_i^\eta = \left(\frac{1}{\sigma_\varepsilon^2}\mathbf{Z}_i\Omega_i^\eta\mathbf{Z}_i{}'-I_{n_i}\right).$$

Thus, as explained in Chapter 3, each summand in the conditional posterior distribution of γ_i (equation 4.24) is separated into two elements. The first element is a mixing probability, and the second is a distribution to be mixed. So with probability

$$\frac{\phi(\underline{\mathbf{y}}_i \mid X_i\beta+\mathbf{Z}_i\gamma_j;\sigma_\varepsilon^2 I_{n_i})}{\Lambda_i^\eta+\sum_{j=1;j\neq i}^{q}\phi(\underline{\mathbf{y}}_i \mid X_i\beta+\mathbf{Z}_i\gamma_j;\sigma_\varepsilon^2 I_{n_i})} \tag{4.27}$$

we select from distribution δ_{γ_j}, which means that we set $\gamma_i = \gamma_j$. Also, with probability

$$\frac{\Lambda_i^\eta}{\Lambda_i^\eta+\sum_{j=1;j\neq i}^{q}\phi(\underline{\mathbf{y}}_i \mid X_i\beta+\mathbf{Z}_i\gamma_j;\sigma_\varepsilon^2 I_{n_i})} \tag{4.28}$$

we select from

$$p(\gamma_i \mid \beta,\sigma_\varepsilon^2,\eta,\boldsymbol{D},\underline{\mathbf{y}}_i) \propto \phi(\gamma_i \mid 0,\eta^{-1}\boldsymbol{D})p(\underline{\mathbf{y}}_i \mid \gamma_i,\beta,\sigma_\varepsilon^2,\underline{\mathbf{y}}_j), \tag{4.29}$$

meaning we sample γ_i from its full conditional,

$$p(\gamma_i / \beta, \sigma_\varepsilon^2, \eta, \boldsymbol{D}, \underline{\mathbf{y}}_i) = N\{(\mathbf{Z}_i'\mathbf{Z}_i + \sigma_\varepsilon^2 \eta \boldsymbol{D}^{-1})^{-1} \mathbf{Z}_i'(\underline{\mathbf{y}}_i - X_i\beta); (\mathbf{Z}_i'\mathbf{Z}_i + \sigma_\varepsilon^2 \eta \boldsymbol{D}^{-1})^{-1} \sigma_\varepsilon^2\}. \tag{4.30}$$

Since all the necessary conditional posterior densities are derived and given, we illustrate our methodology with an experiment from veterinary medicine research.

4.5. A Veterinary Medicine Example

As mentioned in the introductory sections of the chapter, the aim of the study was to see whether there are differences in the change in PCV between the two breeds following a trypanosomal infection. The estimates are obtained from the Gibbs sampler, in which the full conditional posteriors are updated after every iteration. All posterior analyses are based on 1 000 draws, giving us a full non-parametric Bayesian solution to all the mixed linear model parameters.

4.5.1 REML Solution

The following REML results are taken from Duchateau, *et al.* (1998). The variance components of the random effects can be obtained from Table 1. The first line in this table, named '*d*(1,1)', corresponds to the variability of the intercepts within breed, the second line, named '*d*(2,1)', to the covariance between the intercept and the slope, and the third line, '*d*(2,2)', to the variability of the slopes within breed, and 'Residual' to σ_ε^2. The variance of the intercepts is the largest component and much larger than the variance of the slopes. There is a negative covariance of –0.2551 between the slope and the intercepts, implying that the higher the initial PCV the greater it's subsequent fall.

The intercept and slope parameters describing the linear relationship between PCV and breed contain both fixed and random effects. Average fixed intercepts and slopes are assumed for each breed, and each animal has a random slope and intercept.

Table 4.1 Covariance Parameter Estimates obtained from the REML Analysis.

Cov Parm	Estimate
d(1,1)	12.4646
d(2,1)	-0.2551
d(2,2)	0.0058
σ_ε^2	4.3531

The average linear relationship for each breed can be obtained from Table 4.2 below.

Table 4.2 REML Solution for the Estimates of the Fixed Effects.

Effect	Breed	Estimate	SE	DF	t	Pr > |t|
Intercept		35.06	1.501	10	23.35	0.0001
Breed	BO	-0.842	2.123	144	-0.4	0.692
Breed	ND	0	.	.	.	.
Time*Breed	BO	-0.413	0.0374	144	-11.04	0.0001
Time*Breed	ND	-0.276	0.0375	144	-7.37	0.0001

From this table, the linear regression equation for the two breeds can be obtained as

BO: $PCV = 34.00 - 0.413t$

ND: $PCV = 35.06 - 0.276t$

Both the intercept and slope of the fitted relationship is thus different from animal to animal. Individual regression lines for each animal can be obtained from Table 4.3.

Table 4.3 Individual Intercepts and Slopes for the Different Animals (random effects) obtained from the REML Analysis.

Effect	Anim_ID	Estimate	SE	DF	t	Pr > |t|
Intercept	BO241	-1.836	1.638	144	-1.12	0.264
Time	BO241	0.0563	0.0417	144	1.35	0.179
Intercept	BO322	-4.014	1.638	144	-2.45	0.0154
Time	BO322	0.084	0.0417	144	2.03	0.044
Intercept	BO326	-3.53	1.638	144	-2.16	0.032
Time	BO326	0.0676	0.0417	144	1.62	0.107
Intercept	BO209	4.698	1.638	144	2.87	0.0047
Time	BO209	-0.095	0.0417	144	-2.28	0.024
Intercept	BO37	0.8398	1.638	144	0.51	0.608
Time	BO37	-0.0359	0.0417	144	-0.86	0.39
Intercept	BO1	3.841	1.638	144	2.35	0.02
Time	BO1	-0.0777	0.0417	144	-1.86	0.064
Intercept	ND60	-2.1658	1.638	144	-1.32	0.188
Time	ND60	0.0338	0.0417	144	0.81	0.418
Intercept	ND66	3.3628	1.638	144	2.05	0.041
Time	ND66	-0.0512	0.0417	144	-1.23	0.222
Intercept	ND72	-2.688	1.638	144	-1.64	0.102
Time	ND72	0.057	0.0417	144	1.37	0.174
Intercept	ND73	0.2487	1.638	144	0.15	0.879
Time	ND73	-0.0248	0.0417	144	-0.60	0.552
Intercept	ND74	-2.8312	1.638	144	-1.73	0.086
Time	ND74	0.06	0.0417	144	1.44	0.152
Intercept	ND75	4.073	1.638	144	2.49	0.014
Time	ND75	-0.0749	0.0417	144	-1.80	0.0747

For instance, for the first animal, BO241, the fitted relationship is

$$\begin{aligned} \text{BO241: PCV} &= (34.22 - 1.836) + (-0.413 + 0.0563)\,t \\ &= 32.38 - 0.357\,t \end{aligned}$$

These regression lines are determined for each animal and are presented in Figure 4.1. This figure demonstrates that PCV tend to decrease more rapidly the higher the initial PCV. This illustrates the negative covariance between slope and intercept observed in the fitting of the model and Table 4.1.

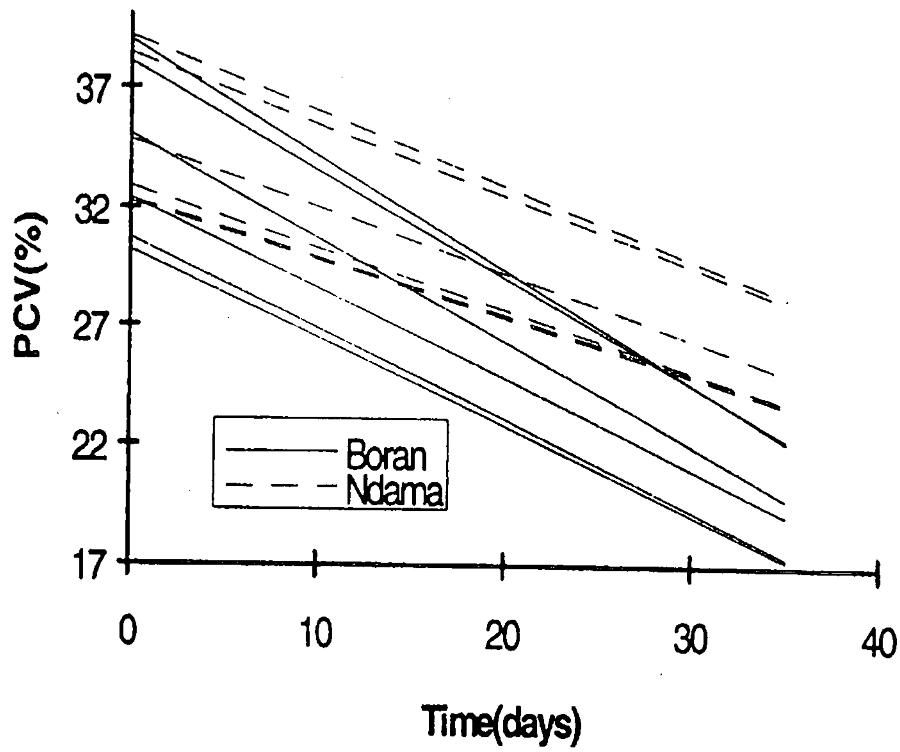

Figure 4.1 Change of PCV in Time for Individual Animals based on the REML Analysis. (This figure is taken from Duchateau *et al.* (1998)).

4.5.2 Non-Parametric Bayesian Solution

The covariance parameter estimates (modes of the posterior distributions) for the conjugate MDP model and the modified MDP model are displayed in Table 4.4 below. The reason for determining the posterior modes and not the posterior means is because the REML estimate is more similar to the mode of the posterior distribution than the mean. Also, it is well known that the REML estimate is the mode of the marginal likelihood.

Table 4.4 Covariance Parameter Estimates (modes of the posterior distributions) from the Conjugate MDP and Non-conjugate MDP Model as obtained by the Non-parametric Bayesian Analysis.

Cov Parm	Conjugate MDP Estimate	95% Credibility Interval	Non-Conjugate MDP Estimate	95% Credibility Interval
D(1,1)	13.50	7.4122 ; 209.5653	15.75	5.2152 ; 225.2366
D(2,1)	-0.28	-4.6765 ; -0.0643	-0.31	-5.4513 ; -0.0788
D(2,2)	0.0076	0.0024 ; 0.1198	0.01	0.0008 ; 0.1357
σ_ε^2	4.610	3.8070 ; 7.172	4.690	3.7915 ; 8.007

The main difference between the results of the conjugate MDP model and the non-conjugate MDP model is that the 95% credibility intervals for the variance components are wider under the modified base measure (non-conjugate MDP model). This is to be expected, as the variance covariance matrix ***D*** is directly affected by the relaxation of the normal assumption. Moreover, the similarity of the results for the σ_ε^2 indicates that this variance parameter is not sensitive to the choice of one of these two base measures. Furthermore, as evident from the above tables it is also clear that the Bayesian estimates coincide well with the REML estimates. The observed histogram for σ_ε^2 from the conjugate MDP model is given in Figure 4.2. Using the conditional posterior densities for $d(1,1)$ and $d(2,2)$ the marginal posterior densities are estimated as the average of the posterior densities and are displayed in Figures 4.3 – 4.4. The attenuation of the width of the 95% credibility intervals is also evident on Figures 4.3 and 4.4.

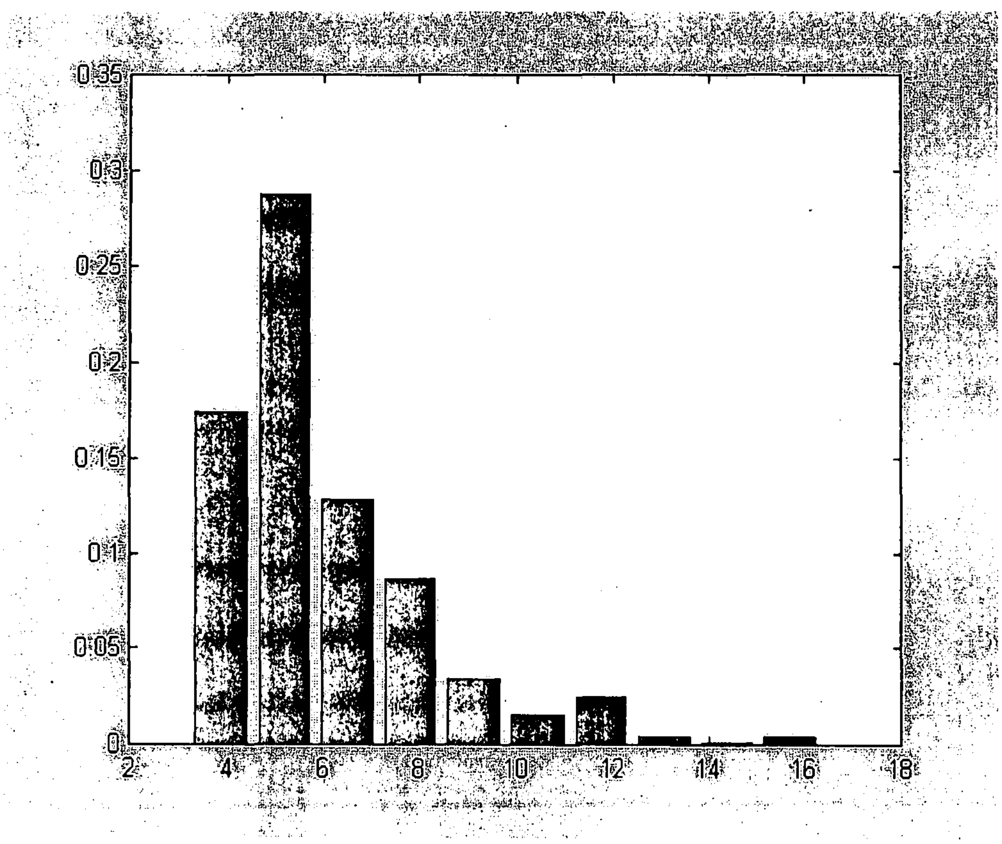

Figure 4.2 Histogram of the Posterior Distribution of σ_ε^2 (Error Variance) with Mean = 6.032 and Mode = 4.610 (Conjugate MDP Model).

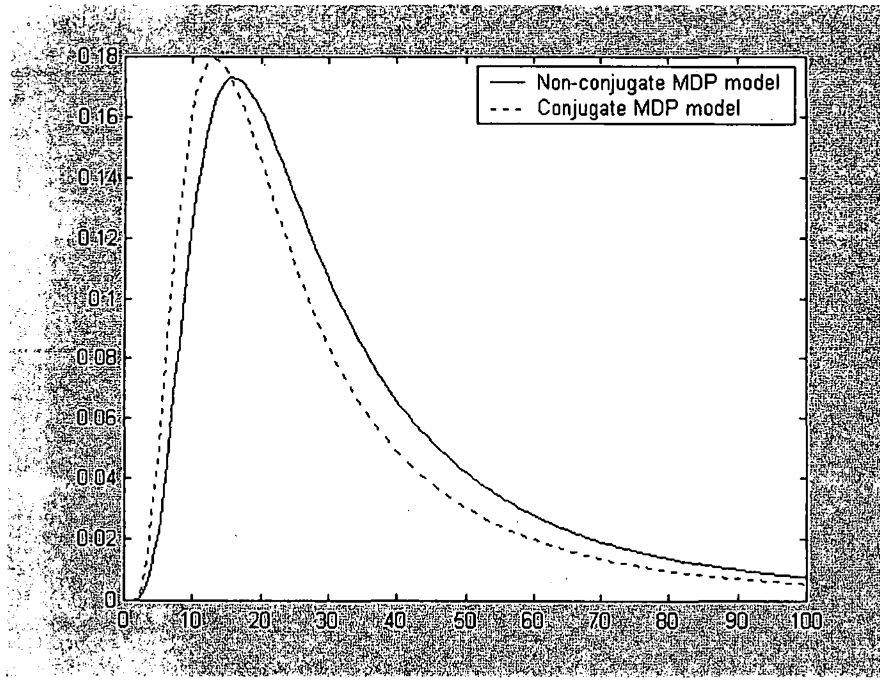

Figure 4.3 Posterior Density of the Intercept Variance: $p(d_{11}|$ D) for the Conjugate MDP Model, Mode = 13.50; and for the Non-conjugate MDP Model, Mode = 15.75.

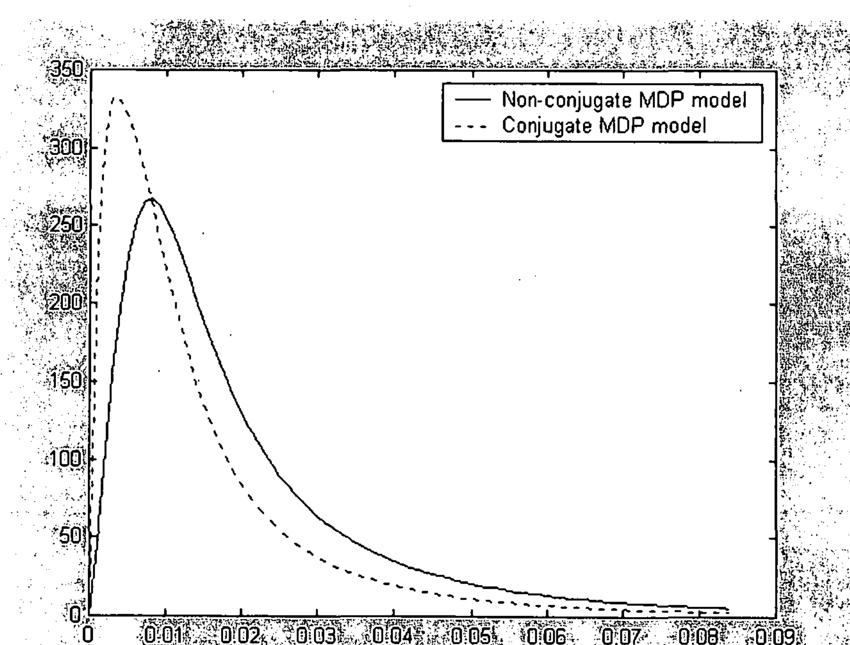

Figure 4.4 Posterior Density of the Slope Variance: $p(d_{22}|$ **D**) for the Conjugate MDP Model, Mode = 0.0076; and for the Non-conjugate MDP Model, Mode = 0.01.

Only the results of the conjugate MDP model will be reported in the next sections. The average linear relationship for each breed can be obtained from Table 4.5 along with 95% credibility intervals. According to the proposed model, β_0 is the intercept of the Boran breed and β_1 the slope of the Boran breed. β_2 measures the difference between intercepts for N'Dama and Boran breeds, and β_3 the difference between slopes for these two breeds.

Table 4.5 Bayesian Solution and 95% Credibility Intervals for the Estimates of the Fixed Effects from the Conjugate MDP Model.

Effect	Breed	Estimate	95% Credibility Interval
β_0	BO	35.397	30.9278 ; 39.2172
β_1	BO	-0.43	-0.5492 ; -0.3325
β_2	ND-BO	0.9851	-4.0721 ; 5.9692
β_3	ND-BO	0.1345	-0.0019 ; 0.2641

If one considers β_3, the difference in the rate of decrease of PCV measurement between N'Dama and Boran breeds it follows from the 95% credibility interval (-0.0019 ; 0.2641) that there is no difference in the rate since this interval includes zero. However, a 90% credibility interval does not include zero meaning that there is a significant difference in the rate of decrease of PCV measurements between the two breeds at a 0.1 level of significance. This conclusion can also be drawn from Figure 4.8.

Further, also from this table, the linear regression equation for the two breeds can be obtained as

BO: $PCV = 35.39 - 0.43\,t \quad \{PCV = \beta_0 + \beta_1\,t\}$

ND: $PCV = 36.38 - 0.30\,t \quad \{PCV = (\beta_0 + \beta_2) + (\beta_1 + \beta_3)\,t\}$

Both the intercept and slope of the fitted relationship are again different from animal to animal. Using the conditional posterior densities for the above parameters and Gibbs sampling, the marginal posterior densities are then estimated and displayed in Figures 4.5 – 4.8.

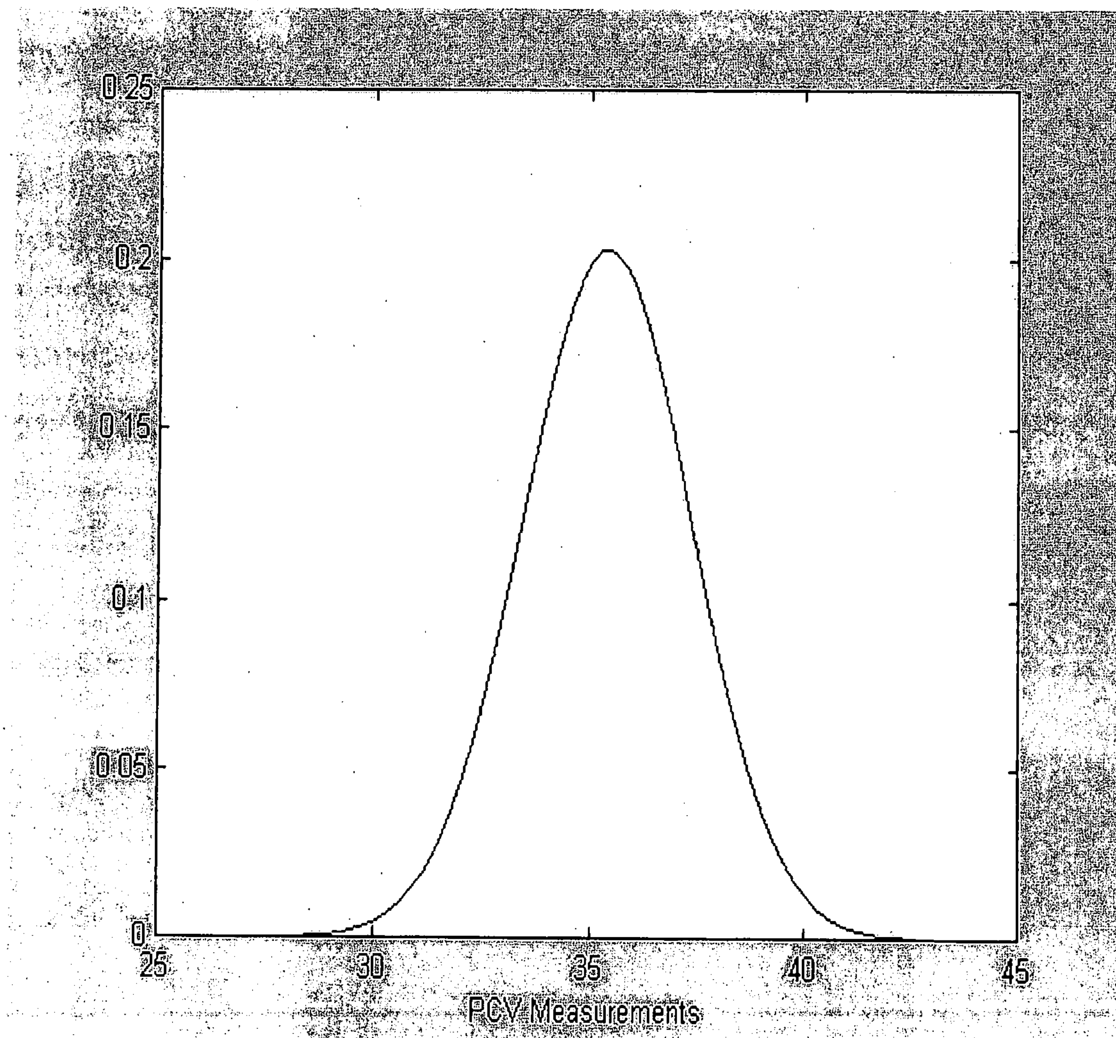

Figure 4.5 Posterior Density of the Intercept for Boran: $p(\beta_0 \mid \mathbf{D})$, Mean = 35.3971.

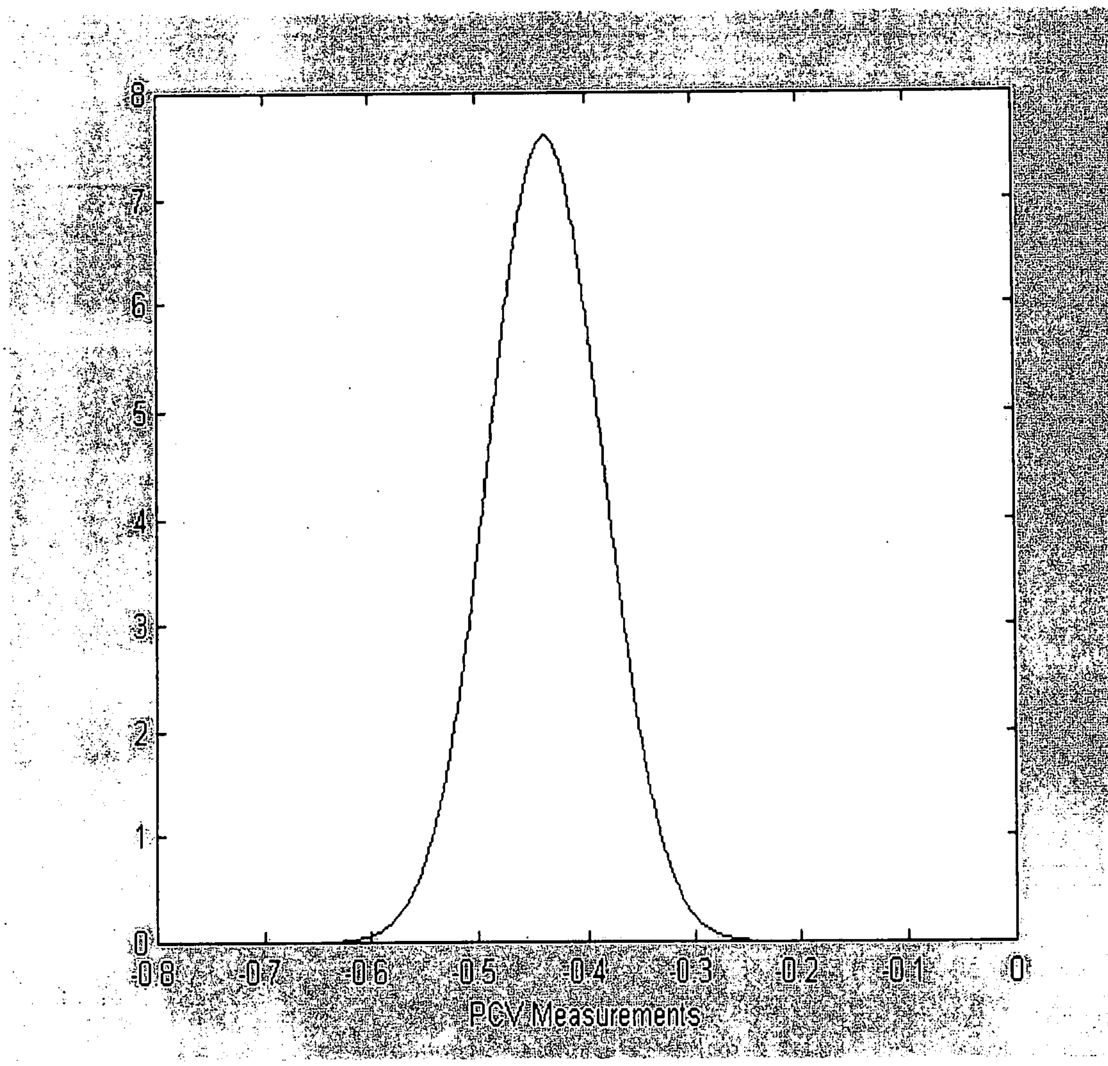

Figure 4.6 Posterior Density of the Slope for Boran: $p(\beta_1 \mid \mathbf{D})$, Mean = -0.4373.

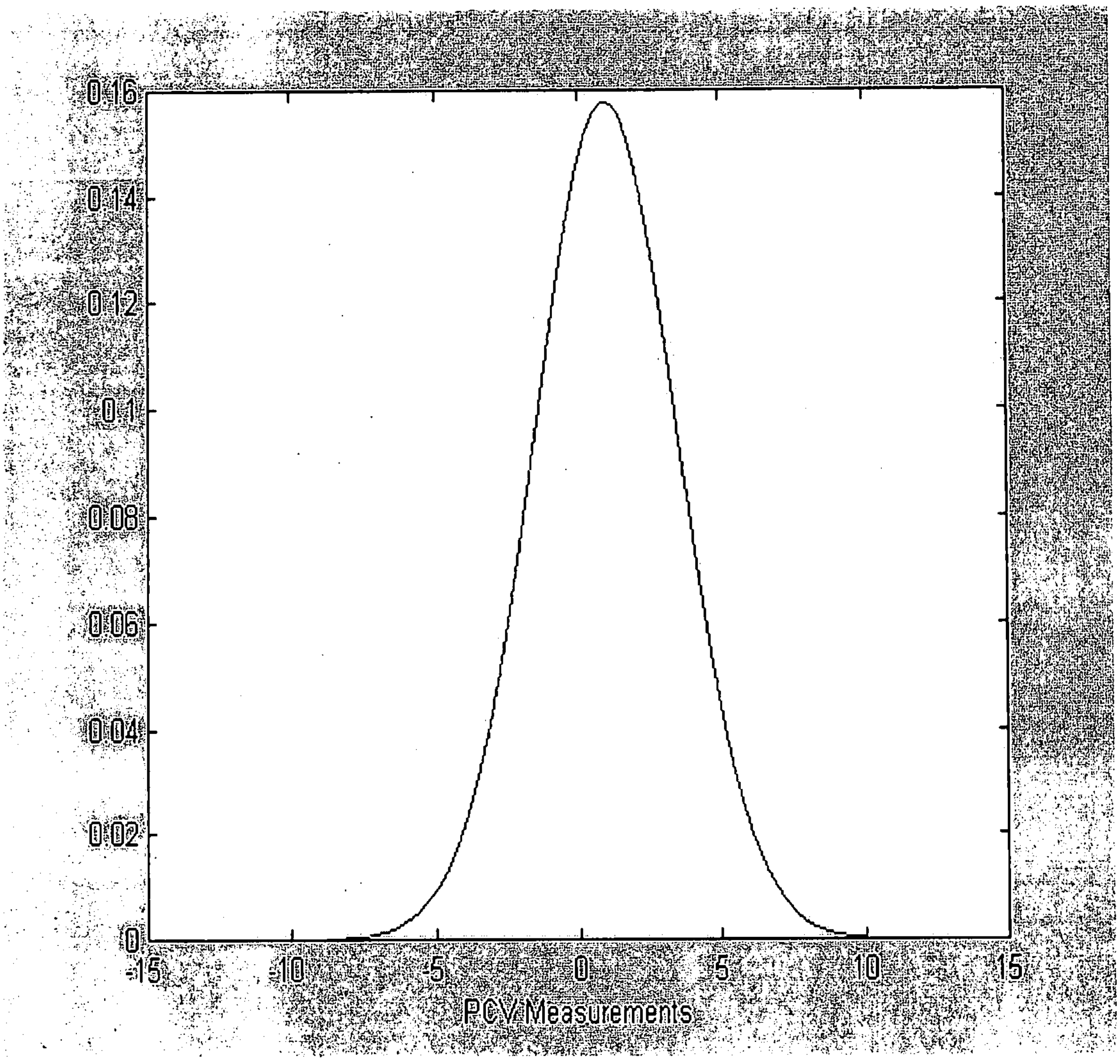

Figure 4.7 Posterior Density of the Difference between Intercepts for N'Dama and Boran Breeds: $p(\beta_2|\,\mathbf{D})$, Mean = 0.9851.

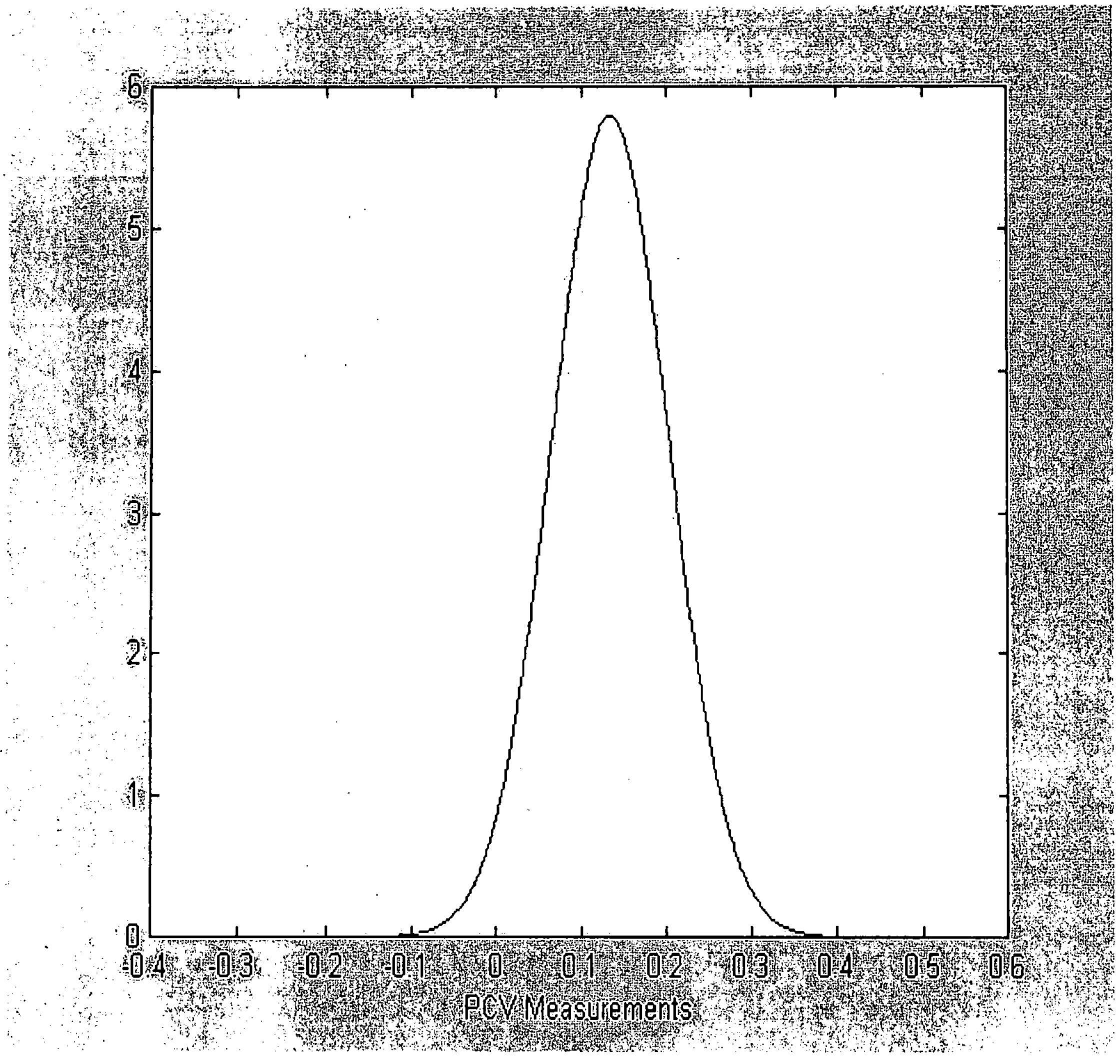

Figure 4.8 Posterior Density of the Difference in the Rate of Decrease of PCV Measurement between N'Dama and Boran Breeds: $p(\beta_3|\ \mathbf{D})$, Mean = 0.1345.

Individual regression lines for each animal can be obtained from Table 6.

Table 6 Individual Intercepts and Slopes for the Different Animals (random effects) obtained from the Dirichlet Process.

Effect	Anim_ID	Estimate
Intercept	BO241	-2.6158
Time	BO241	0.0759
Intercept	BO322	-4.5198
Time	BO322	0.0982
Intercept	BO326	-4.2742
Time	BO326	0.0857
Intercept	BO209	3.3598
Time	BO209	-0.0685
Intercept	BO37	-0.1336
Time	BO37	-0.0246
Intercept	BO1	1.2138
Time	BO1	-0.0235
Intercept	ND60	-3.1529
Time	ND60	0.0512
Intercept	ND66	1.8883
Time	ND66	-0.0194
Intercept	ND72	-3.5944
Time	ND72	0.0748
Intercept	ND73	-0.9109
Time	ND73	-0.0127
Intercept	ND74	-3.7435
Time	ND74	0.0793
Intercept	ND75	1.694
Time	ND75	-0.0223

As in the REML analysis, these regression lines can be determined for each animal and are presented in Figure 4.9. This figure also complements the REML results, i.e. that PCV tends to decrease more rapidly the higher the initial PCV, and once again illustrates the negative covariance between slope and intercept observed in the fitting of the model. Moreover, there seems to be a difference between the N'Dama and Boran breeds.

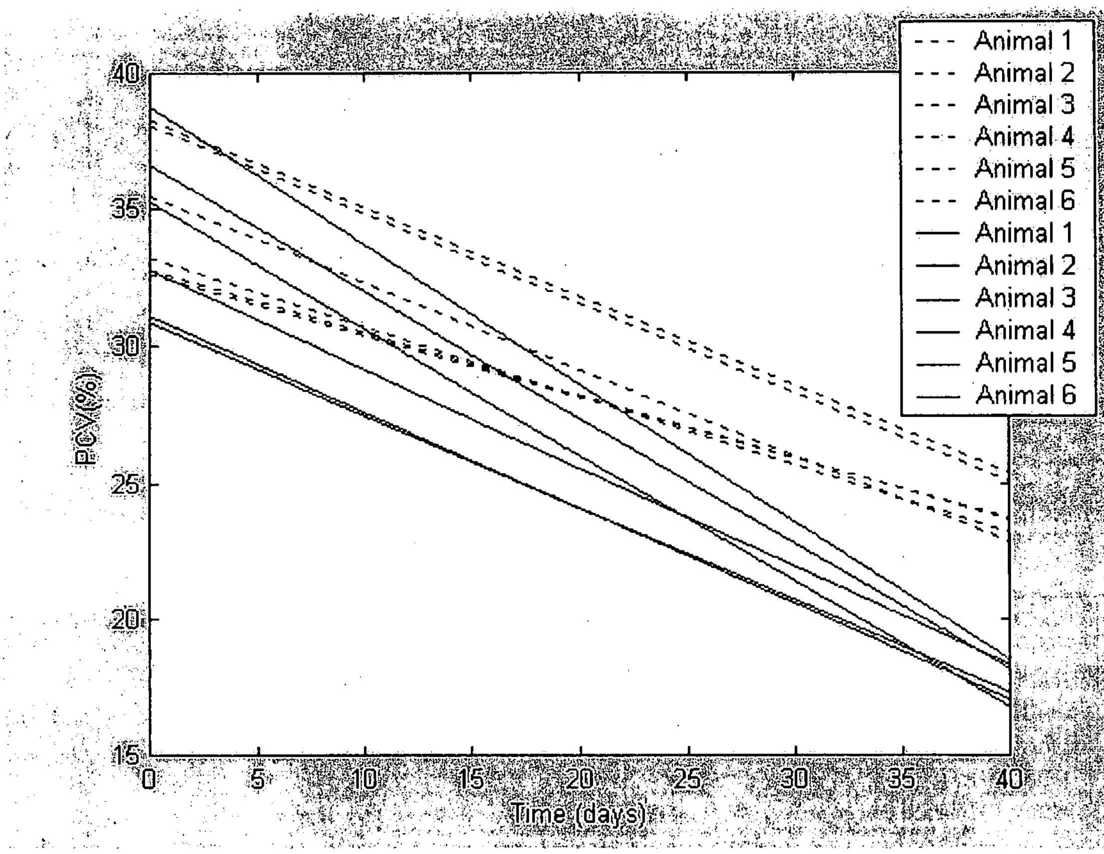

Figure 4.9 Change of PCV in Time for Individual Animals based on the Bayesian Non-Parametric Mixed Model. (Dashed = N'Dama; Solid = Boran). The lines are numbered from top to bottom.

Let us now turn to the important parameter of the Dirichlet process, *M*. Recall that the parameter *M*, a type of dispersion parameter for the Dirichlet process prior, is a measure of the strength in the belief that G is G_0. Although it may be hard to quantify, *M* is a positive scalar that is related to how "clumpy" the data are (often called a precision parameter). Clumpy data occur when the different subjects are concentrated into a few clusters. In practice it is difficult to select appropriate values for this parameter. Instead, it is suggested to place a prior distribution on this parameter, and simulate it given the data. West (1992), assumed that $M \sim Ga(a,b)$ a gamma prior with $a > 0$ and scale $b > 0$. We may extend this idea to include a reference prior (uniform for $log(M)$) for the repeated measure design by letting $a \to 0$ and $b \to 0$.

Moreover, when defining a Dirichlet process prior, recall that *M* determines the prior distribution of ξ, the number of additional normal components in the mixture, and it is a critical smoothing parameter of the mixed linear model. When there are only a few clusters among the animals in the model, the estimate of the normal means from the Dirichlet process prior will be similar to the non-parametric Bayes estimator, and when there are almost q = 12 (total number of random effects) different clusters, the estimator from the Dirichlet process prior will be similar to the parametric Bayes estimator.

When M is simulated, given the data, the average number of clusters, $\bar{\xi} = 9$ with mode $M_0 = 16.10$. The estimated marginal posterior density and the unconditional marginal posterior density for the simulated *M* values are displayed in Figures 4.10 and 4.11.

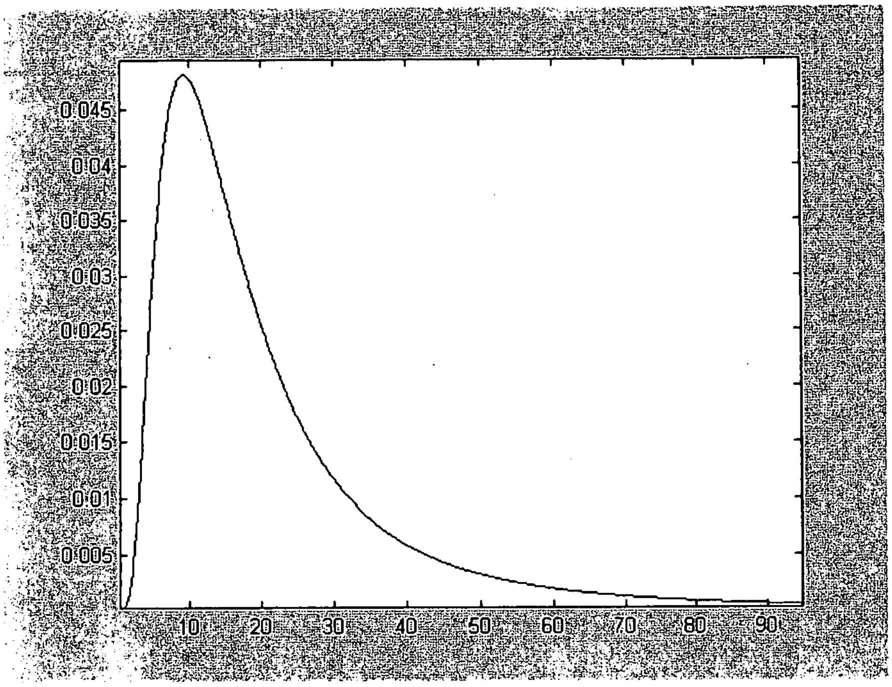

Figure 4.10 Estimated Marginal Posterior Density of M with $\bar{\xi} = 9$, and Posterior Mode $M_0 = 10.0$.

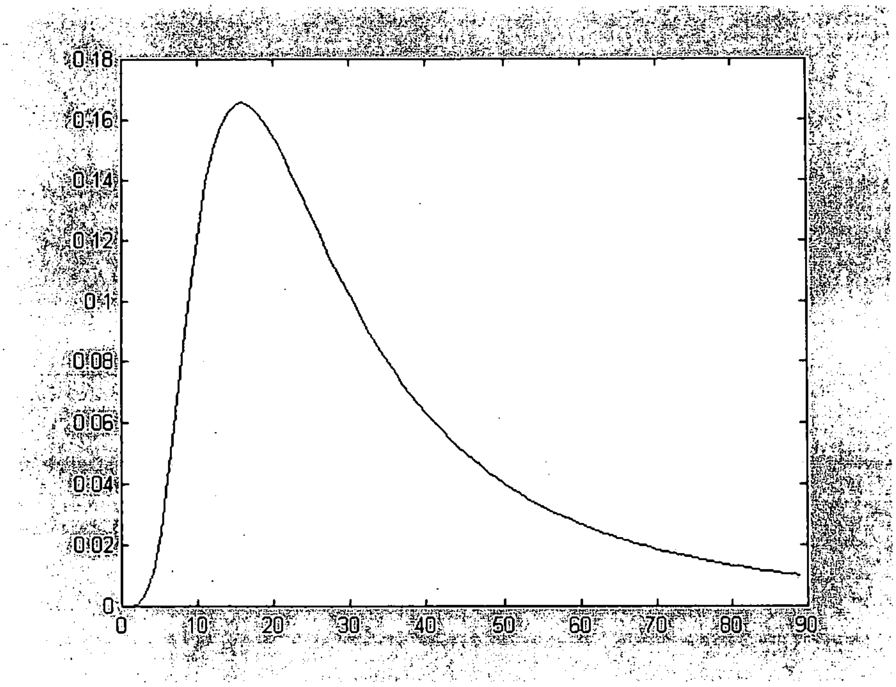

Figure 4.11 Estimated Unconditional Marginal Posterior Density of M with Posterior Mode $M_0 = 16.10$.

Finally, the observed histogram for the number of clusters, ξ for the simulated values of M is presented in Figure 4.12.

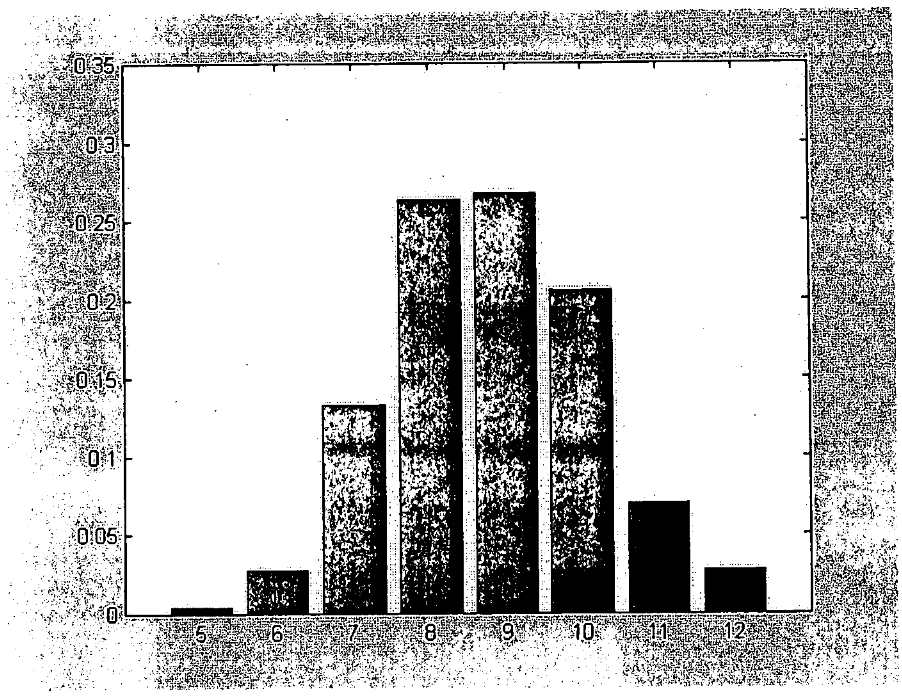

Figure 4.12 Observed Histogram for the Number of Clusters, ξ when M is simulated;

$$\bar{\xi} = 9$$

According to the above figure, the different animals for the two breeds are grouped into 9 different clusters. Figure 4.9 also supports this conclusion. In this figure it is clear that the lines for animals 4,5 and 6 of the N'Dama breed are very similar. Hence these three animals are grouped into one cluster, having the same slope and intercept. Moreover, the same conclusion can also be drawn for animals 5 and 6 from the Boran breed. Thus, with these different animals forming two groups, we have an average of 9 groups/clusters among the twelve different animals.

4.6 Chapter Summary

In this chapter, we have applied a general technique for Bayesian non-parametrics to the mixed linear model. Our technique involved specifying a non-parametric prior for the distribution of the random effects and a Dirichlet process prior on the space of prior distributions for the non-parametric prior. Moreover, we also present the use of a modified MDP model. The resulting model was fitted with a Gibbs sampler. The modified MDP model is a new generalization, as is the computational imputation of this model.

The application to and discussion of an interesting data set from veterinary medicine research was helpful for understanding the importance and utility of these two MDP models, and the Bayesian mixed linear model framework provided a unified way to investigate the changes over time of PCV in the two different breeds. Future work suggested by researchers (Kleinman & Ibrahim, 1998), includes allowing a different η_i for each i in the modified base measure and perhaps more complicated base measures. The MDP model for the random effects would be particularly useful in these kinds of models since they depend heavily on the random effects, which are greatly affected by the MDP model.

CHAPTER 5

«Reference and Probability-Matching Priors»

Introductory words: *Besides the intrinsic interest of developing good non-informative priors for the variance components problem (mixed linear model), a number of theoretically interesting issues arise in application of the proposed procedures. For example, in animal breeding experiments, interest may be in making inferences about ratios of variance components or functions thereof, rather than about individual variance components themselves. This important aspect is explored in the present chapter.*

5.1 Prologue

Box and Tiao (1973) wrote: "The sampling theory approach to the variance component problem encounters a number of snags. These have bothered statisticians for many years, as is evident by the great variety of attempts which have been made to resolve the problems". Determination of reasonable non-informative priors in multiparameter problems is not easy; common non-informative priors, such as Jeffrey's prior, can have features that have an unexpectedly dramatic effect on the posterior. In recognition of this problem, Bernardo (1979), proposed the ***Reference Prior*** approach to the development of non-informative priors, the key feature of which was a possible dependence of the reference prior on specification of parameters of interest and nuisance parameters.

In this chapter the reference prior of Berger and Bernardo (1992) is derived for the mixed linear model and the solution depends on the ordering of the parameters and how the parameter vector is divided into sub-vectors. In spite of these difficulties ere is growing evidence, mainly through examples that reference priors provide "sensible" answers from a Bayesian point of view and some more limited evidence that frequentist properties of inference from reference posteriors are asymptotically "reasonable". We will also examine whether the reference priors satisfy the probability-matching criterion.

5.2 The Mixed Linear Model

From section 1.2, we again have the mixed linear model, which postulates that the observable random vector $\underline{\mathbf{Y}}$ is a linear combination of the fixed effects and random effects plus a random error term. In its simplest form the univariate mixed linear model can be written in matrix notation as

$$\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon . \tag{5.1}$$

As before, $\underline{\mathbf{Y}}$ ($n \times 1$) is a vector of observed values for the trait on which selection is desired, β ($p \times 1$) is a vector of fixed effects uniquely defined so that the corresponding design matrix $\mathbf{X}$ ($n \times p$) has full column rank, p. Also, γ ($q \times 1$) is a vector of unobservable random effects with $\gamma \sim N\left(\underline{\mathbf{0}}, \mathbf{A}\sigma_\gamma^2\right)$ and design matrix $\mathbf{Z}$ ($n \times q$). σ_γ^2 is an unknown scalar and $\mathbf{A}$ ($q \times q$) is called a relationship (genetic covariance) matrix.

For the unobservable vector of random errors terms, ε ($n \times 1$), it is common to assume independent normal distributions with mean vector $\underline{\mathbf{0}}$ and variance-covariance matrix $\sigma_\varepsilon^2 I_n$ where I_n represents a $n \times n$ identity matrix and σ_ε^2 an unknown scalar.

As before, σ_γ^2 and σ_ε^2 are the variance components.

Lemma 1

In model (5.1) γ is a random parameter vector whilst β, σ_ε^2 and σ_γ^2 are population parameters. Therefore the likelihood function depends only on β, σ_ε^2 and σ_γ^2.

Since $\underline{\mathbf{Y}} \mid \beta, \gamma, \sigma_\varepsilon^2 \sim N(\mathbf{X}\beta + \mathbf{Z}\gamma, \sigma_\varepsilon^2 I_n)$ and $\gamma \sim N(\underline{\mathbf{0}}, \mathbf{A}\sigma_\gamma^2)$ it follows that the marginal distribution of $\underline{\mathbf{Y}}$ is

$$\underline{\mathbf{Y}} \mid \beta, \sigma_\varepsilon^2, \sigma_\gamma^2 \sim N\left(\mathbf{X}\beta, \sigma_\gamma^2 \mathbf{ZAZ'} + \sigma_\varepsilon^2 I_n\right). \tag{5.2}$$

Proof:

Since

$\underline{\mathbf{Y}} \mid \beta, \gamma, \sigma_\varepsilon^2 \sim N(\mathbf{X}\beta + \mathbf{Z}\gamma, \sigma_\varepsilon^2 I_n)$, we have

$$E(\underline{\mathbf{Y}} \mid \beta, \gamma, \sigma_\varepsilon^2) = \mathbf{X}\beta + \mathbf{Z}\gamma \quad \text{and}$$

$$E(\underline{\mathbf{Y}} \mid \beta, \sigma_\varepsilon^2) = \mathbf{X}\beta \quad \text{because} \tag{5.3}$$

$$\gamma \sim N\left(\underline{\mathbf{0}}, I_q\right). \tag{5.4}$$

Further,

$$Var(\underline{\mathbf{Y}} \mid \beta,\gamma,\sigma_\varepsilon^2) = \sigma_\varepsilon^2 I_n$$

$$E(\underline{\mathbf{Y}}\underline{\mathbf{Y}}' \mid \beta,\gamma,\sigma_\varepsilon^2) = Var(\underline{\mathbf{Y}} \mid \beta,\gamma,\sigma_\varepsilon^2) + E(\underline{\mathbf{Y}} \mid \beta,\gamma,\sigma_\varepsilon^2)E(\underline{\mathbf{Y}}' \mid \beta,\gamma,\sigma_\varepsilon^2)$$

$$= \sigma_\varepsilon^2 I_n + (\mathbf{X}\beta + \mathbf{Z}\gamma)(\mathbf{X}\beta + \mathbf{Z}\gamma)' \quad \text{and}$$

$$= \sigma_\varepsilon^2 I_n + \mathbf{X}\beta\beta'\mathbf{X}' + \mathbf{Z}\gamma\beta'\mathbf{X}' + \mathbf{X}\beta\gamma'\mathbf{Z}' + \mathbf{Z}\gamma\gamma'\mathbf{Z}'. \tag{5.5}$$

Therefore,

$E(\underline{\mathbf{Y}}\underline{\mathbf{Y}}' \mid \beta,\sigma_\varepsilon^2) \quad = \sigma_\varepsilon^2 I_n + \mathbf{X}\beta\beta'\mathbf{X}' + \sigma_\gamma^2\mathbf{ZAZ}'$ which follows from the fact that

$E(\gamma) = 0,$

and

$$E(\gamma\gamma') = E(\gamma\gamma') - E(\gamma)E(\gamma') = Var(\gamma) = \sigma_\gamma^2\mathbf{A}.$$

Also

$$Var(\underline{\mathbf{Y}} \mid \beta,\sigma_\varepsilon^2) \quad = E(\underline{\mathbf{Y}}\underline{\mathbf{Y}}' \mid \beta,\sigma_\varepsilon^2) - E(\underline{\mathbf{Y}} \mid \beta,\sigma_\varepsilon^2)E(\underline{\mathbf{Y}}' \mid \beta,\sigma_\varepsilon^2), \tag{5.6}$$

$$= \sigma_\varepsilon^2 I_n + \mathbf{X}\beta\beta'\mathbf{X}' + \sigma_\gamma^2\mathbf{ZAZ}' - \mathbf{X}\beta\beta'\mathbf{X}'$$

$$= \sigma_\varepsilon^2 I_n + \sigma_\gamma^2\mathbf{ZAZ}'. \tag{5.7}$$

Thus, we have from (5.3) and (5.7) that

$$\underline{\mathbf{Y}} \mid \beta,\sigma_\varepsilon^2,\sigma_\gamma^2 \sim N\left(\mathbf{X}\beta, \sigma_\gamma^2\mathbf{ZAZ}' + \sigma_\varepsilon^2 I_n\right).$$

Therefore, the integrated likelihood function ignoring the constant (see also Chen (1994)) is given by

$$L(\beta,\sigma_\varepsilon^2,\sigma_\gamma^2 \mid \underline{\mathbf{Y}}) \propto |\sigma_\gamma^2\mathbf{ZAZ'}+\sigma_\varepsilon^2 I_n|^{-\frac{1}{2}} \times \exp\left\{-\frac{1}{2}(\underline{\mathbf{Y}}-\mathbf{X}\beta)'(\sigma_\gamma^2\mathbf{ZAZ'}+\sigma_\varepsilon^2 I_n)^{-1}(\underline{\mathbf{Y}}-\mathbf{X}\beta)\right\}. \tag{5.8}$$

5.3 Reference and Probability-Matching Priors

5.3.1 Background

Prior distributions are needed to complete the Bayesian specification of the mixed linear model. In the following section the reference prior algorithm of Berger and Bernardo will be used to obtain the reference prior. The prior is motivated by an asymptotic argument i.e. of maximizing asymptotic missing information. In the case of a scalar parameter the reference prior is the Jeffreys prior, which does have the feature of providing accurate frequentist inference. In the multiparameter setting the situation is much less clear. The reference prior algorithm is relatively complicated and as mentioned the solution depends on the ordering of the parameters and how the parameter vector is divided into sub-vectors.

5.3.2 The Fisher Information Matrix

In the case of the mixed linear model, we are concerned with inferences of β, σ_ε^2 and σ_γ^2. This is a typical situation where reference priors had been shown to be very promising (Yang & Chen, 1995). It is sometimes reasonable to consider the parameters (β, σ_ε^2 , v), where $v=\dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ rather than (β, σ_ε^2 ,σ_γ^2) (Box & Tiao, 1973). The reason for this is that the between group variance parameter,

σ_γ^2 can always be written as a function of v in Henderson's mixed model equations (Ye, 1994). Using v instead of σ_γ^2 will facilitate the calculations considerable.

From (5.8) the log-likelihood is obtained:

$$l = \log_e L(\beta, v, \sigma_\varepsilon^2) = -\frac{n}{2}\log_e(\sigma_\varepsilon^2) - \frac{1}{2}\log_e |v\mathbf{ZAZ}' + I_n|$$
$$-\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-1}(v\mathbf{ZAZ}' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta). \tag{5.9}$$

As in the case of the Jeffreys prior, the reference prior method is derived from the Fisher information matrix. To obtain the Fisher information matrix the expected values of the second derivatives must be calculated, i.e.

$E\left(\dfrac{\partial^2 l}{\partial(\sigma_\varepsilon^2)^2}\right)$, $E\left(\dfrac{\partial^2 l}{\partial\sigma_\varepsilon^2 \partial v}\right)$, $E\left(\dfrac{\partial^2 l}{\partial v^2}\right)$ and $E\left(\dfrac{\partial^2 l}{\partial\beta(\partial\beta)'}\right)$ for example, must be obtained.

Thus,

$$\frac{\partial l}{\partial\sigma_\varepsilon^2} = -\frac{n}{2}\left(\frac{1}{\sigma_\varepsilon^2}\right) + \frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-2}(v\mathbf{ZAZ}' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta) \tag{5.10}$$

and

$$\frac{\partial^2 l}{\partial(\sigma_\varepsilon^2)^2} = \frac{n}{2}\left(\frac{1}{\sigma_\varepsilon^2}\right)^2 - \frac{2}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-3}(v\mathbf{ZAZ}' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta). \tag{5.11}$$

Taking the expectation of 5.11) with respect to $\underline{\mathbf{Y}}$, it follows that

$$
\begin{aligned}
E\left(\frac{\partial^2 l}{\partial(\sigma_\varepsilon^2)^2}\right) &= \frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2} - tr\left(\sigma_\varepsilon^2\right)^{-3}(v\mathbf{ZAZ'}+I_n)^{-1}\left(\sigma_\varepsilon^2\right)(v\mathbf{ZAZ'}+I_n) \\
&= \frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2} - \left(\sigma_\varepsilon^2\right)^{-2} trI_n \\
&= -\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2},
\end{aligned}
$$

and

$$
-E\left(\frac{\partial^2 l}{\partial(\sigma_\varepsilon^2)^2}\right) = \frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2} = \frac{1}{2}I\left(\sigma_\varepsilon^2\right). \tag{5.12}
$$

Next, differentiate l with respect to σ_ε^2 and v

$$
\frac{\partial l}{\partial \sigma_\varepsilon^2} = -\frac{n}{2}\left(\frac{1}{\sigma_\varepsilon^2}\right) + \frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-2}(v\mathbf{ZAZ'}+I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta) \tag{5.13}
$$

and

$$
\frac{\partial^2 l}{\partial \sigma_\varepsilon^2 \partial v} = -\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'\left(\sigma_\varepsilon^2\right)^{-2}(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta). \tag{5.14}
$$

(See Searle, Casella and McCulloch, (1992) p 456)

If we now take the expectation of (5.14), we have

$$
\begin{aligned}
-E\left(\frac{\partial l}{\partial\sigma_\varepsilon^2\partial v}\right) &= \frac{1}{2}tr\left(\sigma_\varepsilon^2\right)^{-2}(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}(\sigma_\varepsilon^2)(v\mathbf{ZAZ'}+I_n)\\
&= \frac{1}{2}tr\left(\sigma_\varepsilon^2\right)^{-1}(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\\
&= \frac{1}{2}I(\sigma_\varepsilon^2, v).
\end{aligned}
\tag{5.15}
$$

Differentiate twice with respect to v and consider l in two parts, first

$$l_1 = -\frac{1}{2}\log_e|v\mathbf{ZAZ'}+I_n|,$$

then

$$
\begin{aligned}
\frac{\partial l_1}{\partial v} &= -\frac{1}{2}tr\left\{\left(v\mathbf{ZAZ'}+I_n\right)^{-1}\frac{\partial\left(v\mathbf{ZAZ'}+I_n\right)}{\partial v}\right\}\\
&= -\frac{1}{2}tr\left\{\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 l_1}{\partial v^2} &= \frac{1}{2}tr\left\{\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right\}\\
&= \frac{1}{2}tr\left\{\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right\}^2.
\end{aligned}
\tag{5.16}
$$

Therefore, from (5.16) it follows that

$$-E\left(\frac{\partial^2 l_1}{\partial v^2}\right) = \frac{1}{2}tr\left\{\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right\}^2. \tag{5.17}$$

Consider now the second part of l, i.e.

$$l_2 = -\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-1}(v\mathbf{ZAZ'}+I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)$$

and

$$\frac{\partial l_2}{\partial v} = \frac{1}{2}(\sigma_\varepsilon^2)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta),$$

$$\begin{aligned}\frac{\partial^2 l_2}{\partial v^2} = &-\frac{1}{2}(\sigma_\varepsilon^2)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}\\ &\times\frac{\partial\{(v\mathbf{ZAZ'}+I_n)(\mathbf{ZAZ'})^{-1}(v\mathbf{ZAZ'}+I_n)\}}{\partial v}\\ &\times(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)\end{aligned} \tag{5.18}$$

$$\begin{aligned}= &-\frac{1}{2}(\sigma_\varepsilon^2)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}\\ &\times 2(v\mathbf{ZAZ'}+I_n)^{-1}(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})(v\mathbf{ZAZ'}+I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta).\end{aligned}$$

Thus, taking the expectation of the above $\frac{\partial^2 l_2}{\partial v^2}$ we have

$$-E\left(\frac{\partial^2 l_2}{\partial v^2}\right) = tr\{(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2. \tag{5.19}$$

If we now combine the results in (5.17) and (5.19), we have

$$\begin{aligned}-E\left(\frac{\partial^2 l}{\partial v^2}\right) &= -E\left(\frac{\partial^2 l_1}{\partial v^2} + \frac{\partial^2 l_2}{\partial v^2}\right)\\ &= \frac{1}{2}tr\{(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2\\ &= \frac{1}{2}I(v).\end{aligned} \tag{5.20}$$

Finally, if we differentiate l with respect to β, we have

$$I(\beta)=\left[\frac{\partial(\mathbf{X}\beta)}{\partial\beta}\right]\left(\sigma_\varepsilon^2\right)^{-1}(v\mathbf{ZAZ}'+I_n)^{-1}\left[\frac{\partial(\mathbf{X}\beta)}{\partial\beta}\right]' \\ =\left(\sigma_\varepsilon^2\right)^{-1}\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}. \tag{5.21}$$

The expected values of the other second order derivates are equal to zero. The Fisher information matrix is therefore given by

$$I(\beta,\sigma_\varepsilon^2,v)=\begin{bmatrix} I(\beta) & \underline{0} & \underline{0} \\ \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2) & \frac{1}{2}I(\sigma_\varepsilon^2,v) \\ \underline{0}' & \frac{1}{2}I(v,\sigma_\varepsilon^2) & \frac{1}{2}I(v) \end{bmatrix}. \tag{5.22}$$

Although the Jeffreys prior is invariant under reparameterization and as mentioned, has been proven to be a successful non-informative prior for one-dimensional parameter problems, Jeffreys himself had noticed difficulties in multi-dimensional parameter problems, especially when nuisance parameters are present. In the following section the reference priors for β, σ_ε^2 and v are derived. Note that the reference priors depend on the "group ordering" of the parameters.

5.3.3 Reference Prior for β, σ_ε^2 and v

Berger and Bernardo suggested that one allows multiple groups "ordered" in terms of inferential importance, with the reference prior being determined through a succession of analyses for the implied conditional problems. They particularly recommended the reference prior based on having each parameter in its own group, i.e. having each conditional reference prior be only one-dimensional. Notations such as $\{\beta, \sigma_\varepsilon^2, v\}$ will be used to specify the groups and the importance of the parameters; $\{\beta, \sigma_\varepsilon^2, v\}$ means that there are three groups, with β being the most important, and v the least.

Lemma 2

For the mixed linear model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$ the reference prior for the group ordering $\{\beta, \sigma_\varepsilon^2, v\}$ is given by

$$\pi_R(\beta, \sigma_\varepsilon^2, v) \propto \sigma_\varepsilon^{-2}\left\{tr\left[\left(v(\mathbf{ZAZ'}) + I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}}. \tag{5.23}$$

Note that only the reference prior for the ordering $\{\beta, \sigma_\varepsilon^2, v\}$ will be derived, since the reference priors for the orderings, $\{\sigma_\varepsilon^2, \beta, v\}$ and $\{\sigma_\varepsilon^2, v, \beta\}$ can be computed in a similar fashion. These priors are the same as the one for $\{\beta, \sigma_\varepsilon^2, v\}$ given in (5.23).

Proof:

Following the notations in Berger and Bernardo (1992), (see also Yang and Chen, (1995) and Ye, (1994)) the functions h_j, which are needed to calculate the reference prior for the group ordering $\{\beta, \sigma_\varepsilon^2, v\}$ are obtained from $I(\beta, \sigma_\varepsilon^2, v)$, see equation (5.22):

$$h_1 = \left| \quad I(\beta) - \begin{bmatrix}\underline{0} & \underline{0}\end{bmatrix} \begin{bmatrix} \frac{1}{2}I(\sigma_\varepsilon^2) & \frac{1}{2}I(\sigma_\varepsilon^2, v) \\ \frac{1}{2}I(\sigma_\varepsilon^2, v) & \frac{1}{2}I(v) \end{bmatrix}^{-1} \begin{bmatrix}\underline{0}' \\ \underline{0}'\end{bmatrix} \quad \right|.$$

Thus,

$$h_1 = |I(\beta)| = \left|\left(\sigma_\varepsilon^2\right)^{-1} \mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right|. \tag{5.24}$$

To obtain h_2, calculate the following matrix

$$\begin{bmatrix} I(\beta) & \underline{0} \\ \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2) \end{bmatrix} - \left\{\frac{1}{2}I(v)\right\}^{-1} \begin{bmatrix} \underline{0} \\ \frac{1}{2}I(\sigma_\varepsilon^2, v) \end{bmatrix} \begin{bmatrix} \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2, v) \end{bmatrix}$$

and consider the (2,2) element, then

$$\begin{aligned} h_2 &= \left|\frac{1}{2}I(\sigma_\varepsilon^2) - \frac{1}{2}\frac{I^2(\sigma_\varepsilon^2, v)}{I(v)}\right| \\ &= \frac{1}{2}\left\{\frac{n}{(\sigma_\varepsilon^2)^2} - \frac{\left\{tr\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\left(\mathbf{ZAZ}'\right)\right\}^2}{(\sigma_\varepsilon^2)^2}\right\} \\ &= \frac{1}{2}(\sigma_\varepsilon^2)^{-2}\left\{n - \left\{tr\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\left(\mathbf{ZAZ}'\right)\right\}^2\right\}. \end{aligned} \tag{5.25}$$

Also

$$h_3 = \left|\frac{1}{2}I(v)\right| = \frac{1}{2}tr\left\{\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\left(\mathbf{ZAZ}'\right)\right\}^2. \tag{5.26}$$

The conditional prior of v given β and σ_ε^2 is

$$\pi\left(v \mid \beta, \sigma_\varepsilon^2\right) \propto |h_3|^{\frac{1}{2}} = \left\{tr\left[\left(v\left(\mathbf{ZAZ}'\right) + I_n\right)^{-1}\left(\mathbf{ZAZ}'\right)\right]^2\right\}^{\frac{1}{2}}. \tag{5.27}$$

Hence, $E\left[\log_e |h_2| \mid \beta, \sigma_\varepsilon^2\right] = -\log_e \sigma_\varepsilon^4 + c$. So the conditional prior of v and σ_ε^2 given β is

$$\pi\ (\sigma_\varepsilon^2, v \mid \beta) \propto \sigma_\varepsilon^{-2} \left\{ tr\left[\left(v(\mathbf{ZAZ'}) + I_n \right)^{-1} (\mathbf{ZAZ'}) \right]^2 \right\}^{\frac{1}{2}}.$$

It implies that $E\left[\log_e |h_1| \mid \beta\right] = c$ and therefore, for the mixed linear model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$ the reference prior for the group ordering $\{\beta, \sigma_\varepsilon^2, v\}$ is given by

$$\pi_R(\beta, \sigma_\varepsilon^2, v) \propto \sigma_\varepsilon^{-2} \left\{ tr\left[\left(v(\mathbf{ZAZ'}) + I_n \right)^{-1} (\mathbf{ZAZ'}) \right]^2 \right\}^{\frac{1}{2}}.$$

Lemma 3

For the mixed linear model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$ the reference prior for the group ordering $\{\beta, v, \sigma_\varepsilon^2\}$ is given by

$$\pi_R(\beta, v, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2} \left\{ tr\left[\left(v\mathbf{ZAZ'} + I_n \right)^{-1} (\mathbf{ZAZ'}) \right]^2 - \frac{1}{n} \left[tr\left(v\mathbf{ZAZ'} + I_n \right)^{-1} (\mathbf{ZAZ'}) \right]^2 \right\}^{\frac{1}{2}}. \tag{5.28}$$

As will be proved later, this prior is also a probability-matching prior for v.

Proof:

Once again, the functions h_j (j=1,2,3), which are needed to calculate the reference prior for the group ordering $\{\beta, v, \sigma_\varepsilon^2\}$, are obtained. The Fisher information matrix for this ordering is given by

$$I(\beta,\sigma_\varepsilon^2,v)=\begin{bmatrix} I(\beta) & \underline{0} & \underline{0} \\ \underline{0}' & \frac{1}{2}I(v) & \frac{1}{2}I(v,\sigma_\varepsilon^2) \\ \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2,v) & \frac{1}{2}I(\sigma_\varepsilon^2) \end{bmatrix}. \tag{5.29}$$

Also,

$$h_1=\left|I(\beta)\right|. \tag{5.30}$$

From the Fisher information matrix in (5.29), h_2 follows by calculating the following matrix

$$\begin{bmatrix} I(\beta) & \underline{0} \\ \underline{0}' & \frac{1}{2}I(v) \end{bmatrix}-\left\{\frac{1}{2}I(\sigma_\varepsilon^2)\right\}^{-1}\begin{bmatrix} \underline{0} \\ \frac{1}{2}I(\sigma_\varepsilon^2,v) \end{bmatrix}\begin{bmatrix} \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2,v) \end{bmatrix}$$

and the (2,2) element is

$$\begin{aligned} h_2 &= \left|\frac{1}{2}I(v)-\frac{1}{2}\frac{I^2(\sigma_\varepsilon^2,v)}{I(\sigma_\varepsilon^2)}\right| \\ &= \frac{1}{2}\left\{tr\{(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2-\frac{(\sigma_\varepsilon^2)^2\{tr(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2}{(\sigma_\varepsilon^2)^2 n}\right\} \\ &= \frac{1}{2}\left\{tr\{(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2-\frac{1}{n}\{tr(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2\right\}. \end{aligned} \tag{5.31}$$

Also

$$h_3=\left|\frac{1}{2}I(\sigma_\varepsilon^2)\right|=\frac{n}{2}(\sigma_\varepsilon^2)^{-2}. \tag{5.32}$$

Now

$\pi(\beta)\propto(h_1)^{\frac{1}{2}}=1$ because it does not contain β

$$\pi(v \mid \beta) \propto (h_2)^{\frac{1}{2}} = \left\{ tr\left\{ (v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\right\}^2 - \frac{1}{n}\left\{ tr(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\right\}^2 \right\}^{\frac{1}{2}},$$

and

$$\pi(\sigma_\varepsilon^2 \mid v, \beta) \propto (h_3)^{\frac{1}{2}} = (\sigma_\varepsilon^2)^{-1}.$$

Therefore we have

$$\begin{aligned}\pi_R(\beta, v, \sigma_\varepsilon^2) &= \pi(\beta)\pi(v \mid \beta)\pi(\sigma_\varepsilon^2 \mid v, \beta) \\ &\propto \sigma_\varepsilon^{-2}\left\{ tr\left[(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\right]^2 - \frac{1}{n}\left[tr(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\right]^2 \right\}^{\frac{1}{2}}.\end{aligned} \qquad (5.33)$$

Interesting, the priors in Lemmas 2 and 3 are not dependent on any limits of the compact subsets, nor do they yield improper posterior distributions. Therefore, it is more convenient to deal for example with the reparameterization $\{\beta, \sigma_\varepsilon^2, v\}$, providing of course that σ_γ^2 is not of independent interest. Note also that (5.33) is also the reference prior for the group orderings $\{v, \beta, \sigma_\varepsilon^2\}$ and $\{v, \sigma_\varepsilon^2, \beta\}$. Additionally, one can also consider the following priors for the different group orderings, namely

- $\pi(\sigma_\varepsilon^2, v) =$ constant, and
- $\pi(\sigma_\varepsilon^2, v) \propto \sigma_\varepsilon^2$

In the second case we have an uniform or "flat" prior on σ_ε^2 and σ_γ^2, in other words

- $\pi(\sigma_\varepsilon^2, \sigma_\gamma^2) =$ constant.

5.3.4 Reference Prior for the Intraclass Correlation Coefficient, ρ

There are two main reasons for considering intraclass correlation coefficients instead of variance components. First of all, the interest might center on them due to the nature of the study. Animal breeding provides an excellent example, where making inferences about heritability requires modeling the intraclass correlation coefficients. However, even if that is not the case, it may be easier to elicit priors using intraclass correlation coefficients since they are defined on the unit hypercube, as apposed to variance components which take values on the product span of the positive half of the real line. For more details see Gönen (2000).

Frequentists analysis of data often results in a negative estimate of the variance component as well as a confidence interval for the intraclass correlation coefficient that covers the entire parameter span. In contrast the Bayesian method produces sensible answers with different prior densities. Hence these methods and likelihood based methods leave a lot to be desired, a void which Bayesian analysis can fill and as mentioned before a Bayesian approach is intuitively appealing as well, since the parameter space is naturally restricted.

The reference prior for the intraclass correlation coefficient $\rho = \dfrac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$ can also be derived for different group orderings of ρ, β and σ_ε^2. Only the reference prior for the group ordering $\{\rho, \beta, \sigma_\varepsilon^2\}$ will be derived since, as mentioned before, the reference priors for the other group orderings can be computed in a similar way.

Lemma 4

For the mixed linear model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$ the reference prior for the group ordering $\{\rho, \beta, \sigma_\varepsilon^2\}$ is given by

$$\pi_R(\rho,\beta,\sigma_\varepsilon^2) \propto \frac{\sigma_\varepsilon^{-2}}{(1-\rho)^2}\left\{tr\left[\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2 - \frac{1}{n}\left[tr\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}} \tag{5.34}$$

Proof:

According to Ye (1994), the Fisher information matrix $I(\rho, \beta, \sigma_\varepsilon^2)$ can be derived from $I(v, \beta, \sigma_\varepsilon^2)$ by making the transformation

$$I(\rho,\beta,\sigma_\varepsilon^2) = \mathbf{P}'I(v,\beta,\sigma_\varepsilon^2)\mathbf{P} \tag{5.35}$$

where $v = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2} = \dfrac{\rho}{1-\rho}$ and

$$\mathbf{P} = \frac{\partial(v,\beta,\sigma_\varepsilon^2)}{\partial(\rho,\beta,\sigma_\varepsilon^2)} = \begin{bmatrix} \dfrac{1}{(1-\rho)^2} & \underline{0}' & \underline{0} \\ \underline{0} & I & \underline{0} \\ \underline{0} & \underline{0} & I \end{bmatrix}.$$

From equation (5.35) it follows that

$$I(\rho,\beta,\sigma_\varepsilon^2)=\begin{bmatrix}\frac{1}{2}I(\rho) & \underline{0}' & \frac{1}{2}I(\rho,\sigma_\varepsilon^2)\\ \underline{0} & I(\beta) & \underline{0}\\ \frac{1}{2}I(\sigma_\varepsilon^2,\rho) & \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2)\end{bmatrix} \tag{5.36}$$

where

$$\frac{1}{2}I(\rho)=\frac{1}{2(1-\rho)^4}tr\left[(\mathbf{ZAZ'})\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}\right]^2,$$

$$\frac{1}{2}I(\rho,\sigma_\varepsilon^2)=\frac{1}{2}I(\sigma_\varepsilon^2,\rho)=\frac{1}{2\sigma_\varepsilon^2(1-\rho)^2}tr\left[(\mathbf{ZAZ'})\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}\right],$$

$$\frac{1}{2}I(\sigma_\varepsilon^2)=\frac{1}{2}n(\sigma_\varepsilon^2)^{-2}$$

and

$$I(\beta)=(\sigma_\varepsilon^2)^{-1}\mathbf{X}'\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}\mathbf{X}.$$

From (5.36) it follows that

$$\begin{aligned}h_1&=\left|\frac{1}{2}I(\rho)-\begin{bmatrix}\underline{0}' & \frac{1}{2}I(\rho,\sigma_\varepsilon^2)\end{bmatrix}\begin{bmatrix}I^{-1}(\beta) & \underline{0}\\ \underline{0}' & 2I^{-2}(\sigma_\varepsilon^2)\end{bmatrix}\begin{bmatrix}\underline{0}\\ \frac{1}{2}I(\rho,\sigma_\varepsilon^2)\end{bmatrix}\right|\\&=\frac{1}{2}I(\rho)-\frac{1}{2}\frac{I^2(\rho,\sigma_\varepsilon^2)}{I(\sigma_\varepsilon^2)}\\&=\frac{1}{2(1-\rho)^4}tr\left[(\mathbf{ZAZ'})\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}\right]^2-\frac{1}{2n(1-\rho)^4}\left\{tr(\mathbf{ZAZ'})\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}\right\}^2.\end{aligned}$$

Calculate the following matrix

$$\begin{bmatrix} \frac{1}{2}I(\rho) & \underline{0}' \\ \underline{0} & I(\beta) \end{bmatrix} - 2I^{-1}(\sigma_\varepsilon^2)\begin{bmatrix} \frac{1}{2}I(\rho,\sigma_\varepsilon^2) \\ \underline{0} \end{bmatrix}\begin{bmatrix} \frac{1}{2}I(\rho,\sigma_\varepsilon^2) & \underline{0}' \end{bmatrix}$$

and consider the (2,2) element of this matrix, then

$h_2 = I(\beta)$, and

$$h_3 = \frac{1}{2}I(\sigma_\varepsilon^2) = \frac{1}{2}n(\sigma_\varepsilon^2)^{-2}.$$

Now

$$\pi(\rho) \propto (h_1)^{\frac{1}{2}} = \frac{1}{(1-\rho)^2}\left[tr\left\{(\mathbf{ZAZ}')\left(\frac{\rho}{1-\rho}\mathbf{ZAZ}'+I_n\right)^{-1}\right\}^2 - \frac{1}{n}\left\{tr(\mathbf{ZAZ}')\left(\frac{\rho}{1-\rho}\mathbf{ZAZ}'+I_n\right)^{-1}\right\}^2\right]^{\frac{1}{2}}$$

and

$\pi(\beta \mid \rho) \propto (h_2)^{\frac{1}{2}} = 1$ because it does not contain β, and

$$\pi(\sigma_\varepsilon^2 \mid \rho,\beta) \propto (h_3)^{\frac{1}{2}} = \sigma_\varepsilon^{-2}.$$

Therefore we have

$$\pi_R(\rho,\beta,\sigma_\varepsilon^2) = \pi(\rho)\pi(\beta \mid \rho)\pi(\sigma_\varepsilon^2 \mid \rho,\beta)$$

and thus the reference prior for the group ordering $\{\rho, \beta, \sigma_\varepsilon^2\}$ is given by

$$\pi_R(\rho,\beta,\sigma_\varepsilon^2) \propto \frac{\sigma_\varepsilon^{-2}}{(1-\rho)^2}\left\{tr\left[\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2 - \frac{1}{n}\left[tr\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}}$$

Corollary 1

Equation (5.34) is also the reference prior for the group ordering $\{\beta, \rho, \sigma_\varepsilon^2\}$ and $\{\rho, \sigma_\varepsilon^2, \beta\}$. Also, in a similarly way, it follows that the reference prior for the group orderings $\{\beta, \sigma_\varepsilon^2, \rho\}$, $\{\sigma_\varepsilon^2, \beta, \rho\}$ and $\{\sigma_\varepsilon^2, \rho, \beta\}$ is given by

$$\pi_R(\beta,\sigma_\varepsilon^2,\rho) \propto \frac{\sigma_\varepsilon^{-2}}{(1-\rho)^2}\left\{tr\left[\left(\frac{\rho}{1-\rho}\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}}. \tag{5.37}$$

5.3.5 Probability-Matching Priors

In this section we will derive the probability-matching priors for the parameters of the mixed linear model and see whether they include some of the reference priors developed in the previous sections.

Recently Datta and Ghosh (1995) derived the differential equation that a prior must satisfy if the posterior probability of an one-sided credibility interval for a parametric function and its frequentist probability agree up to $0(n^{-1})$ where n is the sample size. They proved that the agreement between the posterior probability and the frequentist probability holds if and only if

$$\sum_{\alpha=1}^{m} \frac{\partial}{\partial \theta_{\alpha}} \{\eta(\theta)\pi(\theta)\} = 0 \tag{5.38}$$

where $\pi(\theta)$ is the probability-matching prior for θ the vector of unknown parameters.

Also

$$\Delta_t = \left[\frac{\partial}{\partial \theta_1} t(\theta), \ldots, \frac{\partial}{\partial \theta_m} t(\theta) \right]' \tag{5.39}$$

and

$$\eta(\theta) = \frac{I^{-1}(\theta)\Delta_t(\theta)}{\sqrt{\Delta_t'(\theta) I^{-1}(\theta)\Delta_t(\theta)}} = \left[\eta_1(\theta), \ldots, \eta_m(\theta)\right]'. \tag{5.40}$$

It is clear that $\eta'(\theta)I(\theta)\eta(\theta) = 1$ for all θ where $I^{-1}(\theta)$ is the inverse of $I(\theta)$, the Fisher information matrix of θ and $t(\theta)$ the parameter of interest. The following lemma can now be stated.

Lemma 5

For the mixed linear model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$ the probability-matching prior for $v = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ is given by

$$\pi_M(v, \beta, \sigma_\varepsilon^2) = \sigma_\varepsilon^{-2}\left[tr\left\{(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\right\}^2 - \frac{1}{n}\left\{tr(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\right\}^2\right]^{\frac{1}{2}}. \tag{5.41}$$

Proof:

From the equation (5.22) it follows that the inverse if the Fisher information matrix is given by

$$I^{-1}(\beta, \sigma_\varepsilon^2, v) = \begin{bmatrix} I^{-1}(\beta) & \underline{0} & \underline{0} \\ \underline{0}' & \dfrac{\frac{1}{2}I(v)}{\mathbf{H}} & \dfrac{-\frac{1}{2}I(\sigma_\varepsilon^2, v)}{\mathbf{H}} \\ \underline{0}' & \dfrac{-\frac{1}{2}I(v, \sigma_\varepsilon^2)}{\mathbf{H}} & \dfrac{\frac{1}{2}I(\sigma_\varepsilon^2)}{\mathbf{H}} \end{bmatrix}.$$

where $\mathbf{H} = \frac{1}{4}I(\sigma_\varepsilon^2)I(v) - \frac{1}{4}I^2(\sigma_\varepsilon^2, v)$.

Now

$$t(\theta) = v, \quad \frac{\partial t(\theta)}{\partial \beta} = \underline{0}, \quad \frac{\partial t(\theta)}{\partial \sigma_\varepsilon^2} = 0 \quad \text{and} \quad \frac{\partial t(\theta)}{\partial (v)} = 1$$

$$\Delta'_t(\theta) = \begin{bmatrix}\underline{0}' & \underline{0} & 1\end{bmatrix}$$

and

$$I^{-1}(\theta)\Delta_t(\theta) = \begin{bmatrix} \underline{0}' & \dfrac{-\frac{1}{2}I(\sigma_\varepsilon^2, v)}{\mathbf{H}} & \dfrac{\frac{1}{2}I(\sigma_\varepsilon^2)}{\mathbf{H}} \end{bmatrix}'.$$

Also $\Delta'_t(\hat{\theta})I^{-1}(\theta)\Delta(\theta) = \dfrac{\frac{1}{2}I(\sigma_\varepsilon^2)}{\mathbf{H}}$ and

$$\eta(\theta) = \frac{I^{-1}(\theta)\Delta_t(\theta)}{\sqrt{\Delta'_t(\theta)I^{-1}(\theta)\Delta_t(\theta)}} = \begin{bmatrix} \underline{0}' & \dfrac{-\frac{1}{2}I(\sigma_\varepsilon^2, v)}{\mathbf{H}^{\frac{1}{2}}\sqrt{\frac{1}{2}I(\sigma_\varepsilon^2, v)}} & \dfrac{\sqrt{\frac{1}{2}I(\sigma_\varepsilon^2)}}{\mathbf{H}^{\frac{1}{2}}} \end{bmatrix}' = \begin{bmatrix} \eta_1(\theta) \\ \eta_2(\theta) \\ \eta_3(\theta) \end{bmatrix}.$$

The prior $\pi_M(\theta) = \pi_M(\beta, \sigma_\varepsilon^2, v)$ must be chosen in such a manner that the differential equation (5.38) is satisfied, i.e. that

$$\frac{\partial\eta_1(\theta)}{\partial\beta}\pi(\theta) + \frac{\partial\eta_2(\theta)}{\partial\sigma_\varepsilon^2}\pi(\theta) + \frac{\partial\eta_3(\theta)}{\partial v}\pi(\theta) = 0. \tag{5.42}$$

Hence, if we choose $\pi(\theta) = \pi_M(\theta) = \pi_M(\beta, \sigma_\varepsilon^2, v) = \mathbf{H}^{\frac{1}{2}}$ then (5.42) is true and

$$\begin{aligned} \pi_M(\beta, \sigma_\varepsilon^2, v) &= \left[I(\sigma_\varepsilon^2)I(v) - I^2(\sigma_\varepsilon^2, v)\right]^{\frac{1}{2}} \\ &= \left[\frac{n}{2(\sigma_\varepsilon^2)^2}\frac{1}{2}tr\left\{\left(v\mathbf{ZAZ}+I_n\right)^{-1}(\mathbf{ZAZ})\right\}^2 - \left(\frac{1}{2}\right)^2(\sigma_\varepsilon^2)^{-2}\left\{tr\left(v\mathbf{ZAZ}+I_n\right)^{-1}(\mathbf{ZAZ})\right\}^2\right]^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\pi_M(v,\beta,\sigma_\varepsilon^2)=\sigma_\varepsilon^{-2}\left[tr\{(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2-\frac{1}{n}\{tr(v\mathbf{ZAZ'}+I_n)^{-1}(\mathbf{ZAZ'})\}^2\right]^{\frac{1}{2}}. \tag{5.43}$$

Corollary 2

Since equation (5.43) is exactly the same as (5.28) it follows that the probability-matching prior $\pi_M(v,\beta,\sigma_\varepsilon^2)$ is also a reference prior for the group orderings $\{\beta, v, \sigma_\varepsilon^2\}$, $\{v, \beta, \sigma_\varepsilon^2\}$ and $\{v, \sigma_\varepsilon^2, \beta\}$.

A very important property of probability-matching priors is that, unlike uniform priors, they always remain invariant under any one-to-one transformation of parameters. Therefore, since $v=\dfrac{\rho}{1-\rho}$ and $\dfrac{\partial v}{\partial\rho}=\dfrac{1}{(1-\rho)^2}$ the probability-matching prior for the intraclass correlation coefficient, ρ, i.e. $\pi_M(\rho,\beta,\sigma_\varepsilon^2)$ is given by (5.37).

5.4 Reference Posterior Distributions

5.4.1 Background

Applications of reference priors in the case of the mixed linear model with two variance components complicate the Gibbs sampling procedure somewhat. However, for more than two variance components the algorithm becomes quite difficult to apply. The reason for this is that the conditional posterior distribution of σ_γ^2 or $v = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ in our case is not the well-known Inverse Gamma density anymore, but an unknown distribution from which it is difficult to simulate.

One way to overcome this problem is to replace the Gibbs sampling procedure as discussed in Chapter 1 with ordinary Monte Carlo simulations. Therefore, instead of using the well-known conditional posterior densities $p(\beta \mid \gamma, \sigma_\varepsilon^2, \sigma_\gamma^2, \underline{\mathbf{Y}})$, $p(\gamma \mid \beta, \sigma_\varepsilon^2, \sigma_\gamma^2, \underline{\mathbf{Y}})$, $p(\sigma_\varepsilon^2 \mid \gamma, \sigma_\gamma^2, \beta, \underline{\mathbf{Y}})$ and $p(\sigma_\gamma^2 \mid \beta, \sigma_\varepsilon^2, \gamma, \underline{\mathbf{Y}})$, posterior densities of the form $p(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}})p(\sigma_\varepsilon^2 \mid v, \underline{\mathbf{Y}})p(v \mid \underline{\mathbf{Y}})$ and $p(\gamma \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}})p(\sigma_\varepsilon^2 \mid v, \underline{\mathbf{Y}})p(v \mid \underline{\mathbf{Y}})$ will be used to simulate joint and marginal posterior densities.

From Searle *et al.* (1992) (p 359) it follows after some algebraic manipulations that

$$\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}} \sim N\left\{E\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right), Var\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right)\right\} \tag{5.44}$$

where

$$E\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right) = \left[\mathbf{X}'\left(v\mathbf{ZAZ}' + I_n\right)^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'\left(v\mathbf{ZAZ}' + I_n\right)^{-1}\underline{\mathbf{Y}}, \tag{5.45}$$

and

$$Var\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right) = \sigma_\varepsilon^2\left\{\mathbf{X}'\left(v\mathbf{ZAZ}' + I_n\right)^{-1}\mathbf{X}\right\}^{-1}. \tag{5.46}$$

Also from Searle *et al.*, (1992) it can be shown that

$$\gamma \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}} \sim N\left\{E\left(\gamma \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right); Var\left(\gamma \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right)\right\} \tag{5.47}$$

where

$$E\left(\gamma \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right) = v\mathbf{AZ}'\left(v\mathbf{ZAZ}' + I_n\right)^{-1}\left\{\underline{\mathbf{Y}} - E\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right)\right\} \tag{5.48}$$

and

$$Var\left(\gamma \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right) = \sigma_\varepsilon^2\left\{v\mathbf{A} - v\mathbf{AZ}'\left[\left(v\mathbf{ZAZ}' + I_n\right)^{-1} - \left(v\mathbf{ZAZ}' + I_n\right)^{-1}\mathbf{X}\left(\mathbf{X}'\left(v\mathbf{ZAZ}' + I_n\right)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\left(v\mathbf{ZAZ}' + I_n\right)^{-1}\right]\mathbf{ZA}v\right\}. \tag{5.49}$$

$E\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right)$ is defined in equation (5.45).

Further, to facilitate the calculations we will make use of the fact that

$$\left(v\mathbf{ZAZ}' + I_n\right)^{-1} = I_n - \mathbf{ZG}^{-1}\mathbf{Z}'$$

where $\mathbf{G} = v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z}$.

Therefore, instead of calculating the inverse of a n – dimensional matrix, the results can be obtained from the inverse of a q – dimensional matrix. Equations (5.44 – 5.49) are true whether the uniform ("flat"), or reference priors are used in the simulation procedure because the parameters β or γ do not occur in the priors.

5.4.2 Calculation of the Posterior Density $p(\sigma_\varepsilon^2, v \mid \underline{\mathbf{Y}})$

From equation (5.8) (see also equation (5.9)) it follows that the likelihood function

$$L(\beta,\sigma_\varepsilon^2,v) \propto \left(\sigma_\varepsilon^2\right)^{-\frac{n}{2}} \mid v\mathbf{ZAZ}'+I_n \mid^{-\frac{1}{2}} \times \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{Y}}-\mathbf{X}\beta)'(v\mathbf{ZAZ}'+I_n)^{-1}(\underline{\mathbf{Y}}-\mathbf{X}\beta)\right\}$$

and the marginal likelihood

$$L\left(\sigma_\varepsilon^2,v\right)=\int_\beta L\left(\beta,\sigma_\varepsilon^2,v\right)d\beta .$$

By completing the square with respect to β it follows that

$$L\left(\sigma_\varepsilon^2,v\right)=\left(\sigma_\varepsilon^2\right)^{-\frac{1}{2}(n-p)}\left|\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right|^{-\frac{1}{2}}\left|v\mathbf{ZAZ}'+I_n\right|^{-\frac{1}{2}} \times \exp\left\{-\frac{\underline{\mathbf{Y}}'}{2\sigma_\varepsilon^2}\left[\left(v\mathbf{ZAZ}'+I_n\right)^{-1} -\right.\right.$$

$$\left.\left.\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\left(\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\right]\underline{\mathbf{Y}}\right. . \tag{5.50}$$

By multiplying $L\left(\sigma_\varepsilon^2,v\right)$ with the prior $\pi\left(\sigma_\varepsilon^2,v\right)$, the joint posterior

$$p\left(\sigma_\varepsilon^2,v \mid \underline{\mathbf{Y}}\right)=p\left(\sigma_\varepsilon^2 \mid v,\underline{\mathbf{Y}}\right)p\left(v \mid \underline{\mathbf{Y}}\right)$$

follows.

5.4.3 Calculation of the Posterior Density $p(\sigma_\varepsilon^2 \mid v, \underline{\mathbf{Y}})$

In the case of the reference priors

$$p(\sigma_\varepsilon^2 \mid v, \underline{\mathbf{Y}}) = K_1 \left(\sigma_\varepsilon^2\right)^{-\frac{1}{2}\{n-p+2\}} \times \exp\left\{-\frac{\underline{\mathbf{Y}}'}{2\sigma_\varepsilon^2}\left[\left(v\mathbf{ZAZ}'+I_n\right)^{-1} - \left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\left(\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\right]\underline{\mathbf{Y}}\right\}, \tag{5.51}$$

an Inverse Gamma density where

$$K_1 = \left\{\underline{\mathbf{Y}}'\left[\left(v\mathbf{ZAZ}'+I_n\right)^{-1} - \left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\left(\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\right]\underline{\mathbf{Y}}\right\}^{\frac{n-p}{2}} \frac{2^{-\left(\frac{n-p}{2}\right)}}{\Gamma\left\{\frac{1}{2}(n-p)\right\}}. \tag{5.52}$$

The posterior distribution $p\left(\sigma_\varepsilon^2 \mid v, \underline{\mathbf{Y}}\right)$ using $\pi\left(\sigma_\varepsilon^2, v\right) =$ constant (uniform prior on σ_ε^2 and v) can easily be derived by substituting $p-2$ for p in (5.51). On the other hand, if a uniform prior is used on σ_ε^2 and σ_γ^2, then $p\left(\sigma_\varepsilon^2 \mid v, \underline{\mathbf{Y}}\right)$ can also be obtained from (5.51) by substituting $p-2$ for $p+2$.

5.4.4 Calculation of the Posterior Density $p(v \mid \underline{\mathbf{Y}})$

(I) For the reference prior (see also equation (5.23))

$$\pi_R(\beta, \sigma_\varepsilon^2, v) \propto \sigma_\varepsilon^{-2}\left\{tr\left[\left(v(\mathbf{ZAZ'}) + I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}}$$

the reference posterior density of v is given by

$$p_1(v \mid \underline{\mathbf{Y}}) \propto \left|\mathbf{X'}\left(v\mathbf{ZAZ'}+I_n\right)^{-1}\mathbf{X}\right|^{-\frac{1}{2}}\left|v\mathbf{ZAZ'}+I_n\right|^{-\frac{1}{2}} \times \left\{\underline{\mathbf{Y}}'\left[\left(v\mathbf{ZAZ'}+I_n\right)^{-1} - \left(v\mathbf{ZAZ'}+I_n\right)^{-1}\mathbf{X}\left(\mathbf{X'}\left(v\mathbf{ZAZ'}+I_n\right)^{-1}\mathbf{X}\right)^{-1}\mathbf{X'}\left(v\mathbf{ZAZ'}+I_n\right)^{-1}\right]\underline{\mathbf{Y}}\right\}^{-\frac{1}{2}(n-p)} \times \left\{tr\left[\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}}. \tag{5.53}$$

(II) For the reference prior (see also equation (5.28))

$$\pi_R(v, \beta, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2}\left\{tr\left[\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2 - \frac{1}{n}\left[tr\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}}$$

the reference posterior density is given by

$$p_2(v \mid \underline{\mathbf{Y}}) \propto \left|\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right|^{-\frac{1}{2}} \left|v\mathbf{ZAZ}'+I_n\right|^{-\frac{1}{2}} \times \left\{\underline{\mathbf{Y}}'\left[(v\mathbf{ZAZ}'+I_n)^{-1} - (v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\left(\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\right]\underline{\mathbf{Y}}\right\}^{-\frac{1}{2}(n-p)} \times \left\{tr\left[(v\mathbf{ZAZ}'+I_n)^{-1}(\mathbf{ZAZ}')\right]^2 - \frac{1}{n}\left[tr(v\mathbf{ZAZ}'+I_n)^{-1}(\mathbf{ZAZ}')\right]^2\right\}^{\frac{1}{2}}. \tag{5.54}$$

(III) For the reference prior $\pi_C\left(v,\beta,\sigma_\varepsilon^2\right) \propto$ constant, the posterior density is given by

$$p_C(v,\mid \underline{\mathbf{Y}}) \propto \left|\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right|^{-\frac{1}{2}} \left|v\mathbf{ZAZ}'+I_n\right|^{-\frac{1}{2}} \times \left\{\underline{\mathbf{Y}}'\left[(v\mathbf{ZAZ}'+I_n)^{-1} - (v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\left(\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\right]\underline{\mathbf{Y}}\right\}^{-\frac{1}{2}(n-p-2)}. \tag{5.55}$$

(IV): For the prior $\pi_K\left(\beta,\sigma_\gamma^2,\sigma_\varepsilon^2\right) \propto$ constant, the posterior density is given by

$$p_K(v,\mid \underline{\mathbf{Y}}) = \left|\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right|^{-\frac{1}{2}} \left|v\mathbf{ZAZ}'+I_n\right|^{-\frac{1}{2}} \times \left\{\underline{\mathbf{Y}}'\left[(v\mathbf{ZAZ}'+I_n)^{-1} - (v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\left(\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\right]\underline{\mathbf{Y}}\right\}^{-\frac{1}{2}(n-p-4)}. \tag{5.56}$$

5.4.5 Posterior Distribution of $\theta = \mathbf{X}\beta + \mathbf{Z}\gamma$ and Predictive Distribution of $y_f = x_f'\beta + z_f'\gamma + \varepsilon_f$

In this section the posterior density of $\theta = \mathbf{X}\beta + \mathbf{Z}\gamma$ as well as the predictive density of a future observation $y_f = x_f'\beta + z_f'\gamma + \varepsilon_f$ will be derived. Here y_f is (1×1), x_f is $(p\times 1)$, z_f is $(q\times 1)$ and $\varepsilon_f \sim N(0,\sigma_\varepsilon^2)$. These densities will enable us to estimate or predict the weaning weights of lambs for the different sires. Also the year, age of dam, sex of the lamb and birth status effects will be taken into account. However, the following lemma must first be proved.

Lemma 6

The conditional posterior mean and variance of $\theta = \mathbf{X}\beta + \mathbf{Z}\gamma$ are given by

$$E(\theta \mid v,\sigma_\varepsilon^2,\underline{\mathbf{Y}}) = \mathbf{X}E(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}) + v\mathbf{ZAZ}'(v\mathbf{ZAZ}'+I_n)^{-1}\{\underline{\mathbf{Y}} - \mathbf{X}E(\beta \mid v,\sigma_\varepsilon^2,\underline{\mathbf{Y}})\}, \tag{5.57}$$

where

$$E(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}) = \left(\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'(v\mathbf{ZAZ}'+I_n)^{-1}\underline{\mathbf{Y}},$$

and

$$\begin{aligned} Var(\theta \mid v,\sigma_\varepsilon^2,\underline{\mathbf{Y}}) = {} & \sigma_\varepsilon^2\mathbf{Z}'(v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' + \left[I_n - v\mathbf{ZAZ}'(v\mathbf{ZAZ}'+I_n)^{-1}\right]\mathbf{X}\times \\ & Var(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}})\mathbf{X}'\left[I_n - v\mathbf{ZAZ}'(v\mathbf{ZAZ}'+I_n)^{-1}\right], \end{aligned} \tag{5.58}$$

where

$$Var\left(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}\right) = \sigma_\varepsilon^2 \left(\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right)^{-1}.$$

Proof:

Equation (5.57) follows from equations (5.45) and (5.48).

Further,

$$Var\left(\theta \mid \beta, \underline{\mathbf{Y}}, \sigma_\varepsilon^2, v\right) = Var\left\{\left(\mathbf{X}\beta + \mathbf{Z}\gamma \mid \beta, \underline{\mathbf{Y}}, \sigma_\varepsilon^2, v\right)\right.$$

$$= Var\left\{\mathbf{Z}\gamma \mid \beta, \underline{\mathbf{Y}}, \sigma_\varepsilon^2,\right\} = \sigma_\varepsilon^2 \mathbf{Z}\left(v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'$$

his follows from Searle, *et al.*, (1992), p 357, equation (40).

Also, the unconditional variance of θ is given by

$$Var\left(\theta \mid \underline{\mathbf{Y}}, \sigma_\varepsilon^2, v\right) = E_\beta\left\{\sigma_\varepsilon^2 \mathbf{Z}\left(v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'\right\} + Var_\beta\left\{E\left(\theta \mid v, \sigma_\varepsilon^2, \beta, \underline{\mathbf{Y}}\right)\right.$$

$$= \sigma_\varepsilon^2 \mathbf{Z}\left(v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}' + Var\left\{I_n - v\mathbf{ZAZ}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\right\}\mathbf{X}\beta$$

$$= \sigma_\varepsilon^2 \mathbf{Z}\left(v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}' + \left\{I_n - v\mathbf{ZAZ}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\right\}\mathbf{X}\,Var\left(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}\right) \times$$

$$\mathbf{X}'\left\{I_n - v\mathbf{ZAZ}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\right\}',$$

(5.58)

where

$$Var\left(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}\right) = \sigma_\varepsilon^2\left(\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right)^{-1}.$$

Corollary 3

Since $\theta = \mathbf{X}\beta + \mathbf{Z}\gamma$ is a linear combination of normally distributed random variables, the posterior distribution of θ is also normal for given v and σ_ε^2. Therefore

$$\theta \mid \underline{\mathbf{Y}}, v, \sigma_\varepsilon^2 \sim N\left\{E\left(\theta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right), Var\left(\theta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right)\right. . \quad (5.59)$$

Further, the conditional predictive density of a future observation

$$y_f = x_f'\beta + z_f'\gamma + \varepsilon_f,$$

is normal with mean

$$E\left(y_f \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}\right) = x_f' E\left(\beta \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}\right) + v z_f' \mathbf{AZ}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\left\{\underline{\mathbf{Y}} - \mathbf{X}E\left(\beta \mid v, \sigma_\varepsilon^2, \underline{\mathbf{Y}}\right)\right\}, \quad (5.60)$$

and variance

$$Var\left(y_f \mid \sigma_\varepsilon^2, v, \underline{\mathbf{Y}}\right) = \sigma_\varepsilon^2\left\{1 + z_f'\left(v^{-1}\mathbf{A}^{-1} + \mathbf{Z}'\mathbf{Z}\right)^{-1} z_f + \left\{x_f' - v z_f' \mathbf{AZ}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right\} \times \right.$$

$$\left.\left\{\mathbf{X}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right\}^{-1}\left\{x_f' - v z_f' \mathbf{AZ}'\left(v\mathbf{ZAZ}'+I_n\right)^{-1}\mathbf{X}\right\}\right\}.$$

The unconditional posterior predictive distribution of an unobserved observation $p(y_f \mid \underline{\mathbf{Y}})$, has therefore two components of uncertainty: (1) the fundamental variability of the model, represented by the variance σ_ε^2 in y_f not accounted for by $x_f^{'}\beta$ and $z_f^{'}\gamma$; and (2) the posterior uncertainty in $\beta, \gamma, \sigma_\varepsilon^2$ and σ_γ^2 (or v) due to the finite sample size of $\underline{\mathbf{Y}}$. According to Geisser (1975) the prediction of observables or potential observables is of much greater relevance than the estimation of what are often artificial constructed parameters.

5.5 An Example

5.5.1 The Data

Consider again the Dormer sheep stud of Elsenburg (see section (1.8.1)). A total of n = 879 weaning weight records, from the progeny of q = 17 sires were available after editing, and p = 17 fixed effects were included in the final model.

As before the mixed linear model used for this data structure, is the sire model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}\gamma + \varepsilon$, where $\underline{\mathbf{Y}}$ (879 × 1) vector of weaning weights. β(17 × 1) is the vector of fixed effects, $\mathbf{X}$ a (879 × 17) incident matrix, and the design matrix $\mathbf{Z}$, a (879 × 17) matrix identifying the (17 × 1) vector of random effects including the breeding values for the 17 sires for which the data are observed.

In this section estimation of fixed and random effects as well as prediction of future weaning weights will be obtained for the Dormer sheep stud using reference and uniform priors.

5.5.2 Estimation and Prediction using Uniform and Reference Priors

In the example that follows, the following reference priors will be considered for the model parameters. For the group ordering $\{\beta,\ \sigma_\varepsilon^2,\ v\}$:

$$\text{Ref1:}\quad \pi_R(\beta,\sigma_\varepsilon^2,v)\propto\sigma_\varepsilon^{-2}\left\{tr\left[\left(v(\mathbf{ZAZ'})+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}},$$

$$\text{Ref2:}\quad \pi_R(\beta,v,\sigma_\varepsilon^2)\propto\sigma_\varepsilon^{-2}\left\{tr\left[\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2-\frac{1}{n}\left[tr\left(v\mathbf{ZAZ'}+I_n\right)^{-1}(\mathbf{ZAZ'})\right]^2\right\}^{\frac{1}{2}},$$

and uniform prior in section (IV) page 192.

5.5.3 Analysis of Variance Components

Posterior modes for the uniform prior, 95% credibility intervals, and the estimates obtained when the different reference priors (Reference estimates) are used, are summarized in Table 5.1.

Table 5.1 Point – and Reference Estimates (posterior modes) and 95% Credibility Intervals for the Variance Components.

Analysis	**Estimate**	**95% Credibility Interval**
Uniform	21.2620	19.2579 ; 23.3612
Ref1	21.2028	19.2887 ; 23.3191
Ref2	21.2015	19.2847 ; 23.2233

Using the conditional posterior densities for σ_ε^2 , and Monte Carlo simulations the marginal posterior densities are estimated as the average of the posterior densities and are displayed in Figure 5.1. Considering the results in the above table and the figure below, we conclude that for our practical problem the posterior densities using the first reference prior (Ref1), those derived from the second reference prior (Ref2) and when the uniform prior (Uniform) is used, are for all purposes the same. This comes as no surprise since the posterior density of the error variance component σ_ε^2 is not directly influenced by the different reference priors.

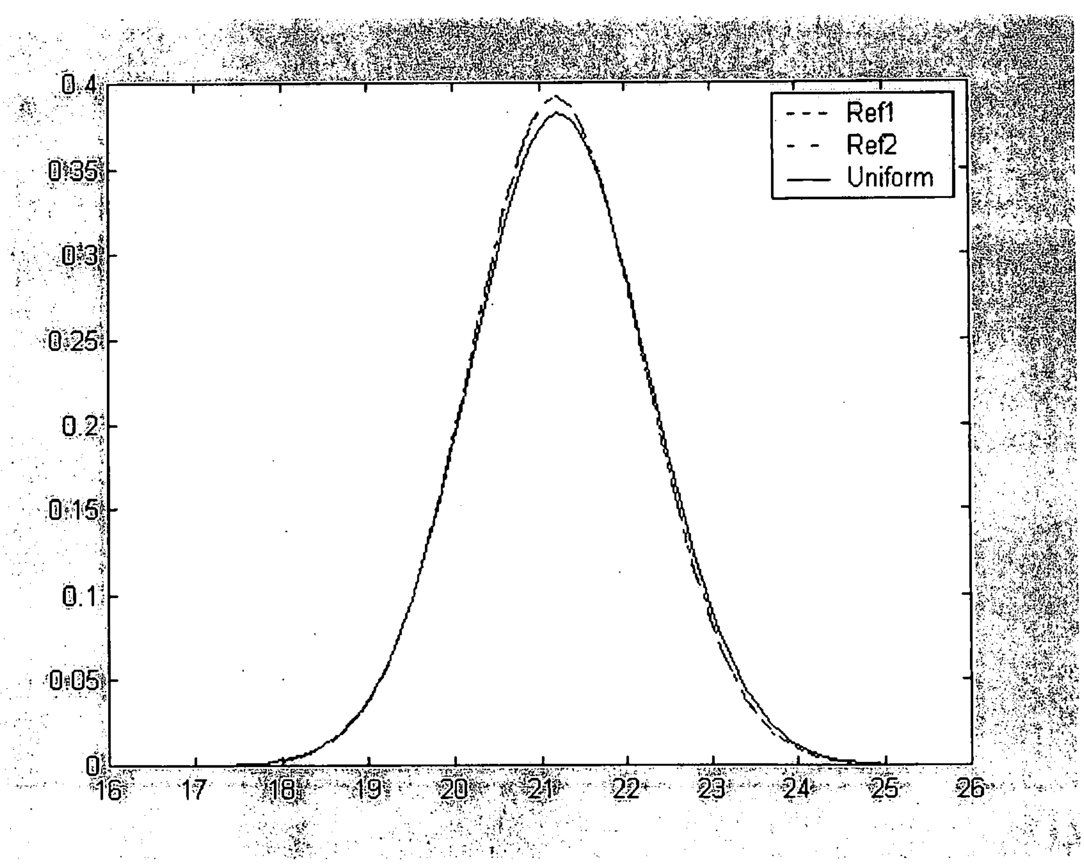

Figure 5.1 Estimated Marginal Posterior Densities of the Variance Component, σ_ε^2 .

However, as mentioned in paragraph (5.3.4) researchers are often more interested in functions of the variance components. This is due to the nature of their study, e.g. in breeding experiments, animal breeders are making inferences about heritability, which requires modeling the intraclass correlation coefficients.

Table 5.2 reports the estimates for the intraclass correlation coefficient, $\rho = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}$ and the variance ratio $\nu = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$. The 95% credibility intervals are also reported in this table. Note that the reference priors for the group ordering, $\{\rho, \beta, \sigma_\varepsilon^2\}$ and $\{\beta, \sigma_\varepsilon^2, \rho\}$ are used in the analysis and will be called *Ref1* and *Ref2*. The results yield that the estimates from the two reference prior analysis are very much the same. Hence, the main result of the different reference priors on the sire variance is attenuation of the width of the 95% credibility intervals. Note that these intervals are wider under the uniform prior than the different reference priors.

Also, the 95% credibility intervals for the intraclass correlation coefficients do not contain 0.5. As mentioned before, this result corresponds well to the statement made by Wang *et al.*, (1992) namely that from a genetic point of view, an intraclass correlation of 0.5 is not possible in a sire model. The marginal posterior densities are also estimated and displayed in Figures 5.2 and 5.3. The wider intervals are quite evident in the shape of the marginal posterior densities displayed in the figures.

Table 5.2 Estimates (posterior modes) using the Uniform and Reference Priors of Functions of the Variance Components, along with 95% Credibility Intervals

Parameters	Uniform	Ref1	95 % Credibility Interval	Ref2	95 % Credibility Interval
ρ	0.133	0.120	0.0338 ; 0.3031	0.115	0.0340 ; 0.3030
ν	0.140	0.120	0.0350; 0.4350	0.110	0.0355 ; 0.4350

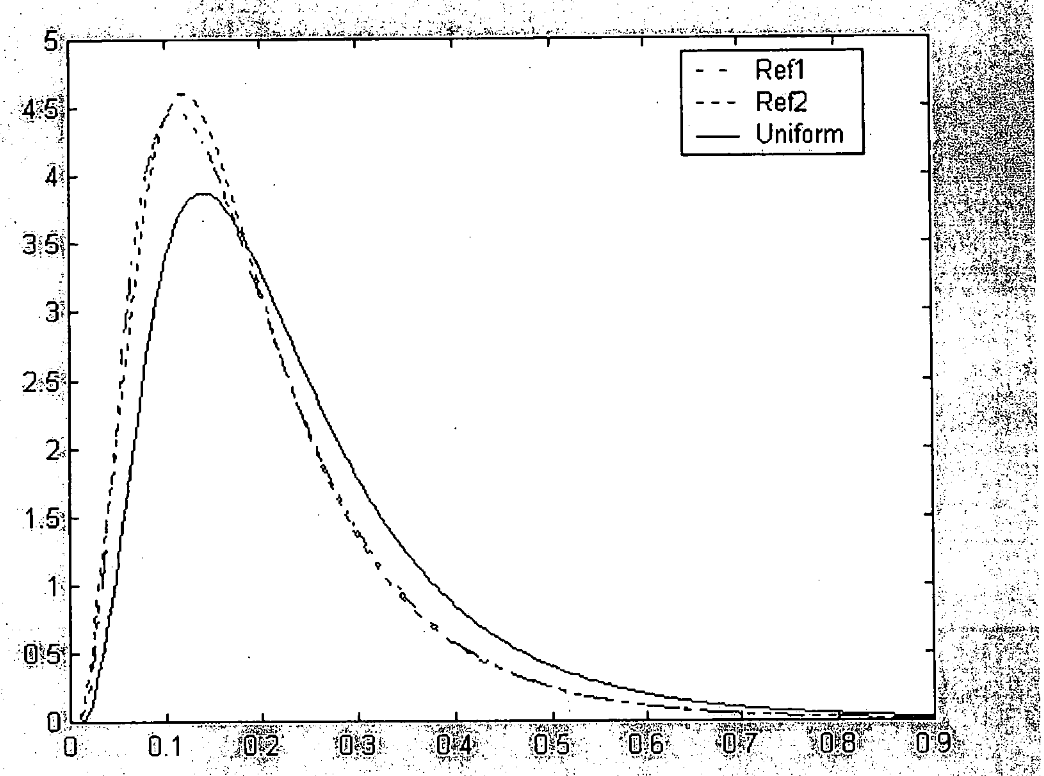

Figure 5.2 The Estimated Marginal Posterior Density of the Variance Ratio, $\nu = \dfrac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$.

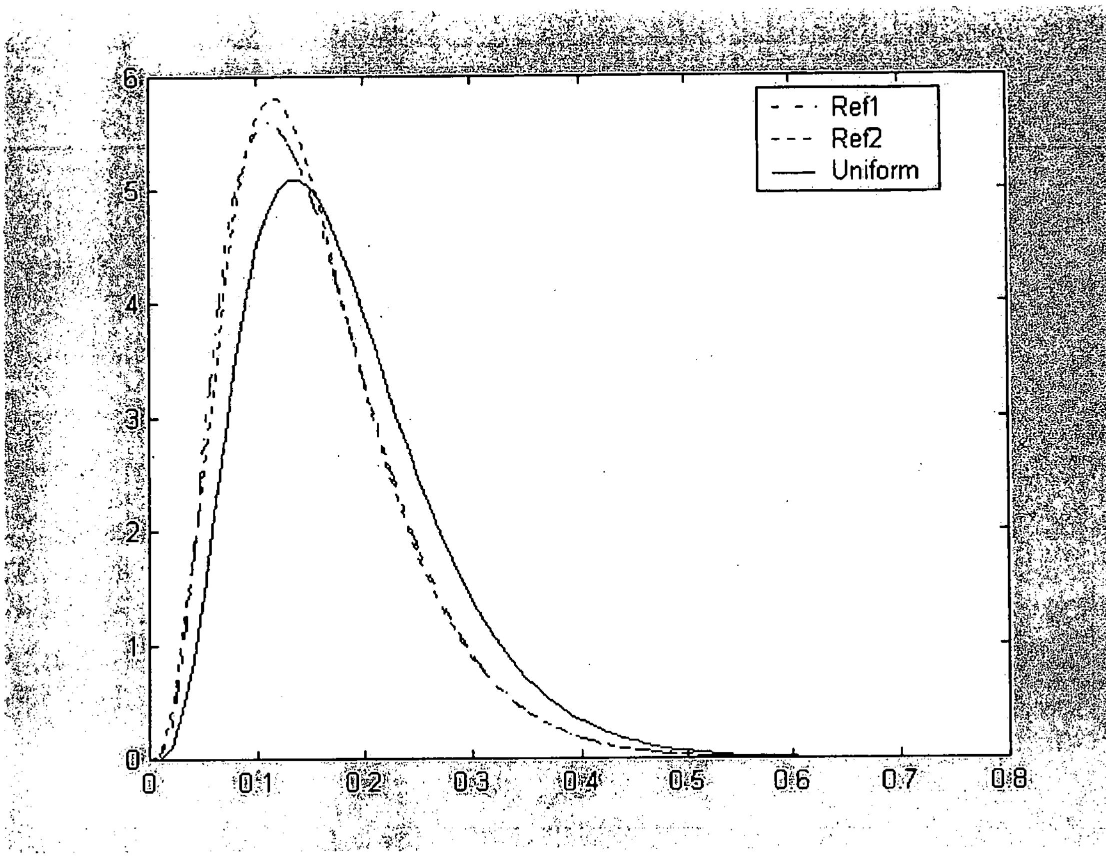

Figure 5.3 The Estimated Marginal Posterior Density of the Intraclass Correlation Coefficient,

$$\rho = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \sigma_\varepsilon^2}.$$

5.5.4 Analysis of Random Effects

In Table 5.3 the estimated breeding values (posterior modes) of the 17 sires and the 95% credibility intervals are given. The marginal posterior densities for the breeding values of sire 3 and 10 are displayed in Figures 5.4 and 5.5 respectively. In all three analyses, the results of the estimates are very much the same, with minor differences in the width of the 95% credibility intervals.

Table 5.3 Estimated Breeding Values for 17 Sires from the Elsenburg Dormer Stud, and 95% Credibility Intervals.

Sire ID	Ref1	95% Credibility Interval	Ref2	95% Credibility Interval
41037	0.6538	-1.3640 ; 3.0539	0.6575	-1.4208 ; 2.9119
41004	0.1900	-1.6150 ; 2.3719	0.1906	-1.6480 ; 2.3881
41019	3.3282	1.3131 ; 5.7251	3.3314	1.3208 ; 5.8563
43002	-1.1093	-3.5516 ; 1.3940	-1.1046	-3.5189 ; 1.2828
44170	-0.0857	-2.5505 ; 2.4474	-0.0926	-2.5478 ; 2.3214
44174	-0.5709	-3.2919 ; 2.0173	-0.5661	-3.3822 ; 2.1433
44042	-1.1895	-3.2556 ; 0.7401	-1.1858	-3.1829 ; 0.6891
45070	-1.0288	-3.2473 ; 0.7452	-1.0189	-3.1385 ; 0.8123
45135	-0.5068	-2.8742 ; 1.8211	-0.499	-2.8643 ; 1.7676
46015	-1.6490	-3.9451 ; 0.2502	-1.6401	-3.7808 ; 0.2536
46037	-0.7021	-2.8284 ; 1.0831	-0.6912	-2.7557 ; 1.1339
48014	-0.8675	-3.0533 ; 1.1596	-0.8557	-3.0533 ; 1.0817
48052	-0.3462	-2.6278 ; 1.7172	-0.3328	-2.5345 ; 1.7411
48148	-1.2589	-3.7277 ; 1.0333	-1.2445	-3.5498 ; 0.9884
49053	0.4203	-2.5061 ; 3.5098	0.4302	-2.4457 ; 3.4350
49134	0.7776	-1.9565 ; 3.6495	0.8125	-1.9465 ; 3.8679
49046	0.4152	-2.6140 ; 3.2126	0.4230	-2.6800 ; 3.6002

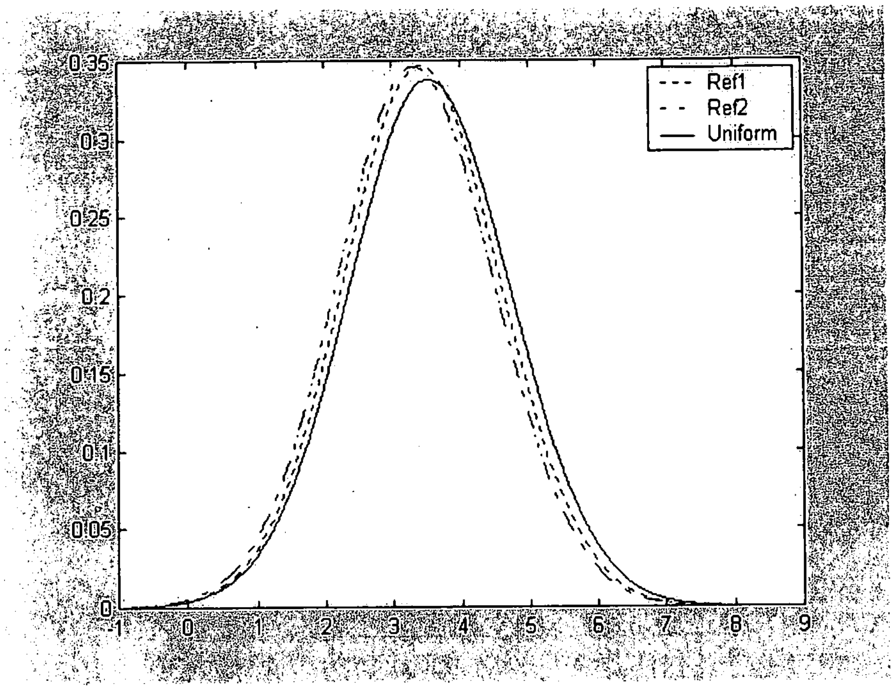

Figure 5.4 The Estimated Marginal Posterior Density of the Breeding Value for Sire 3 (ID41019) (γ_3).

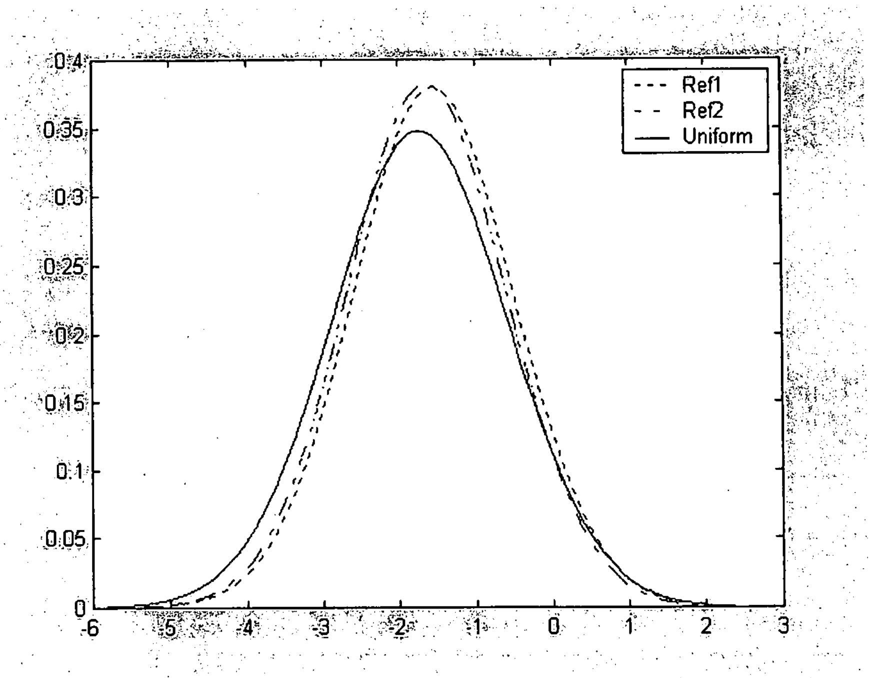

Figure 5.5 The Estimated Marginal Posterior Density of the Breeding Value for Sire 10 (ID46015) (γ_{10}).

The similarity of the results indicates that, for this example, the results are not that sensitive to the choice of either the two reference priors or the uniform or "flat" prior. However, the application to and discussion of the example is helpful for understanding the implementation of the different priors via the Gibbs sampler. For smaller sample sizes the differences can be quite substantial (large).

5.5.5 Analysis of Fixed Effects

Table 5.4 summarizes the estimated fixed effects. Selected parameters in the case of the mixed linear model with corresponding joint marginal posterior densities are presented in Figures 5.6 – 5.10.

Table 5.4 Estimated Values of Selected Fixed Effects, 95% Credibility Intervals, and Reference Estimates.

Parameter	**Ref1**	**95% Credibility Interval**	**Ref2**	**95% Credibility Interval**
β_0	22.8796	18.9515 ; 26.5691	22.8927	18.9515 ; 26.5691
β_7	5.2692	4.1121 ; 6.3856	5.2680	4.1003 ; 6.3796
β_{14}	3.6501	3.0255 ; 4.2836	3.6494	3.0256 ; 4.2851
β_{15}	9.5337	7.3747 ; 11.6265	9.5340	7.4161 ; 11.6158
β_{16}	3.0345	0.8671 ; 5.1672	3.0339	0.8988 ; 5.1642

As expected, the estimates of the fixed effects using the different priors are for all practical purposes the same, and in particular the results from the two reference priors. There is one factors to keep in mind when examining this similarity in the results, i.e. that the different priors do not directly influence the posterior densities of the fixed effects to the same extend as in the case of the random effects.

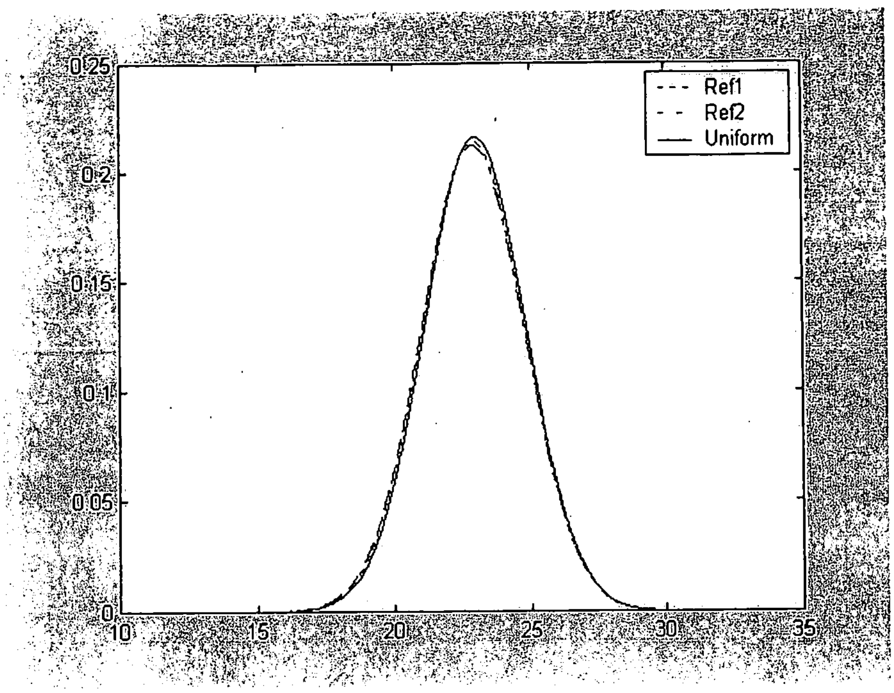

Figure 5.6 Estimated Marginal Posterior Density of β_0, the average weaning weight of female lambs born in 1950 if the age of the dam is 8 years or older, and the birth status "triplets".

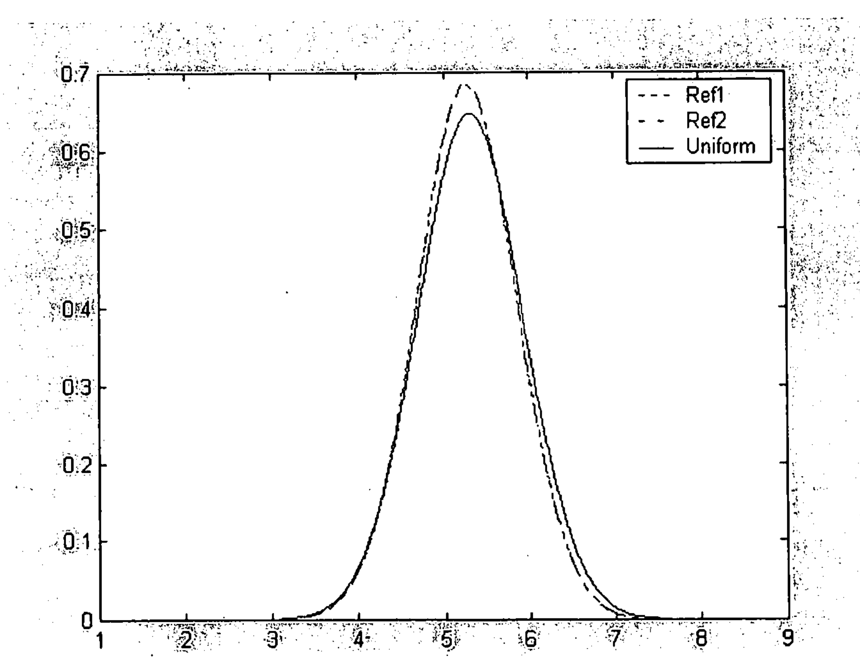

Figure 5.7 Estimated Marginal Posterior Density of β_7, the expected difference in average weaning weight between lambs born in 1949 and in 1950.

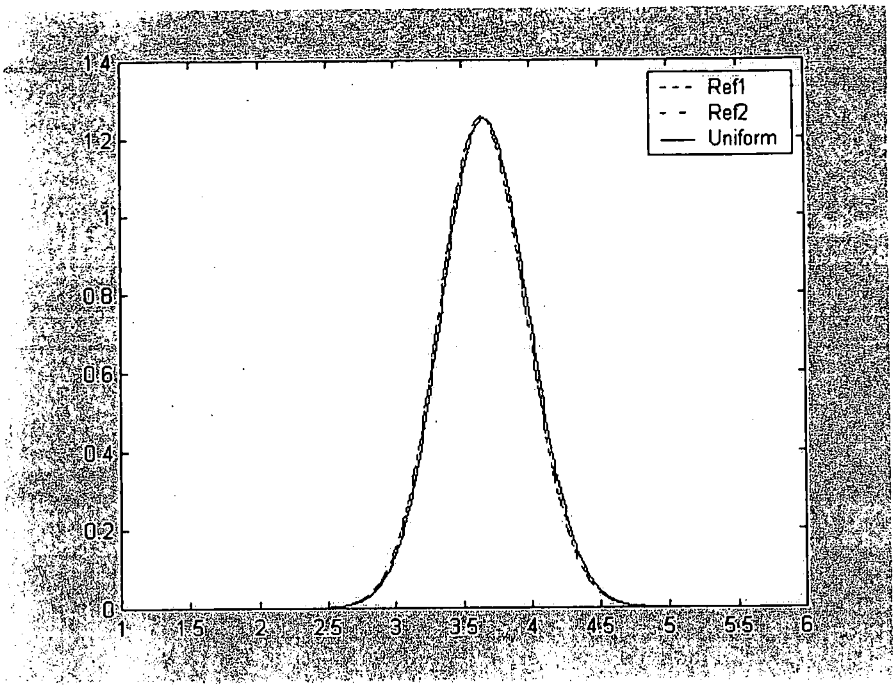

Figure 5.8 Estimated Marginal Posterior Density of β_{14} , the expected difference in average weaning weight between male and female lambs.

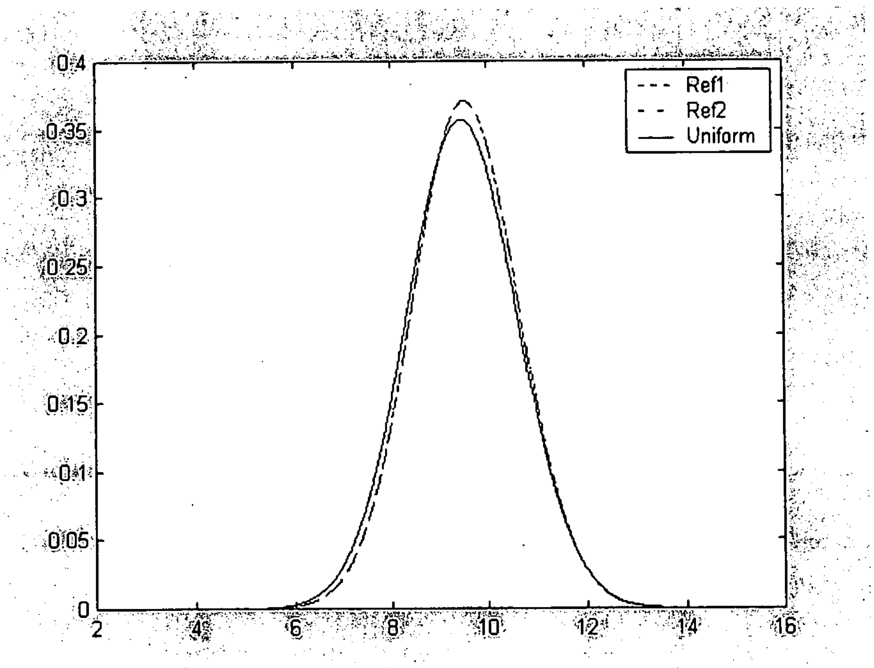

Figure 5.9 Estimated Marginal Posterior Density of β_{15} , the expected difference in average weaning weight between single births and triplets.

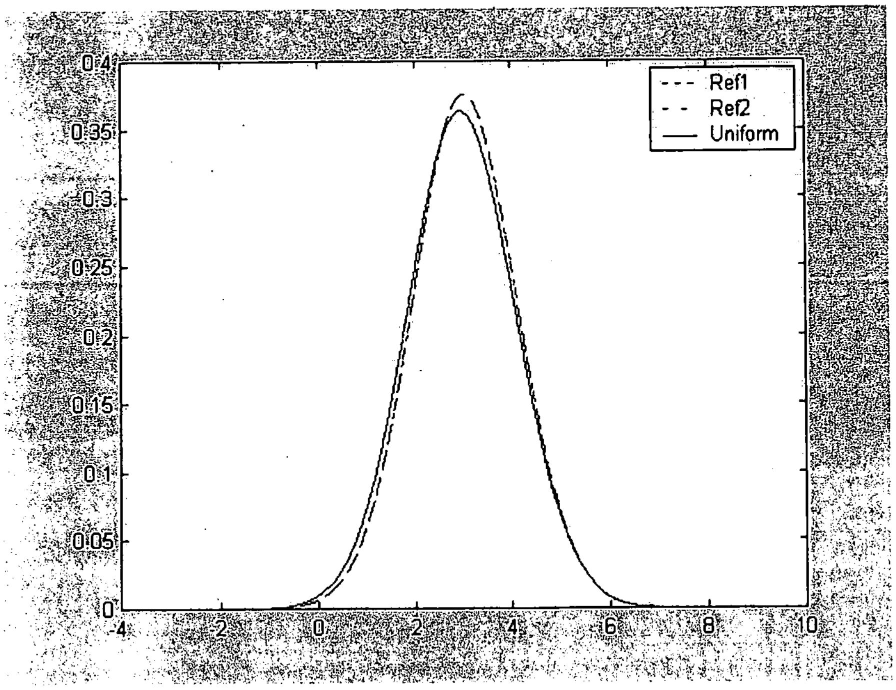

Figure 5.10 Estimated Marginal Posterior Density of β_{16} , the expected difference in average weaning weight between a pare of twins at birth and triplets.

5.5.6 Predictive Density of a Future Observation

The densities derived in Lemma 6 and Corollary 3 will now enable us to estimate or predict the weaning weights of lambs for the different sires. Also the year, age of dam, sex of the lamb and birth status effects will be taken into account. Using the notation in Chapter 1, the weaning weight of a female lamb born in 1950 if the age of the dam is 8 years or older, and the birth status "triplets", will be predicted. Figure 5.11 displays such a predictive density if Sire 15 (ID49053) is used to impregnate the dam.

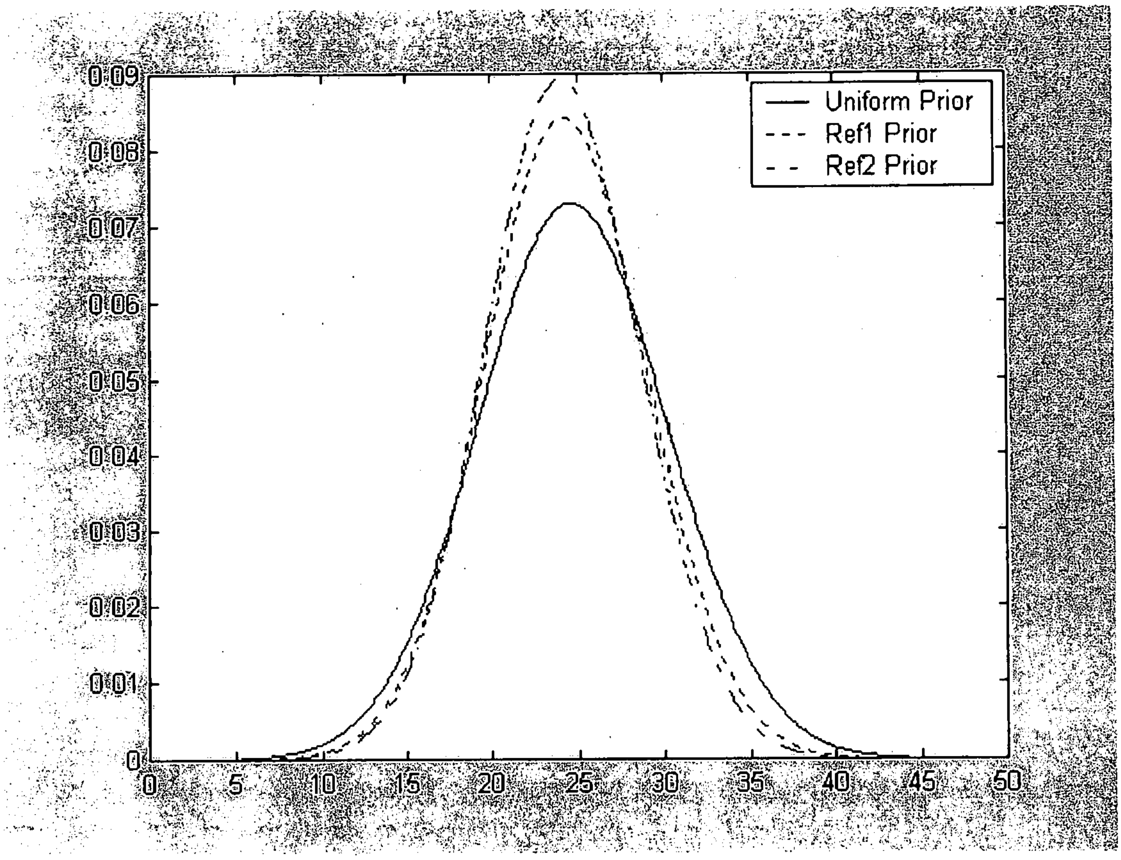

Figure 5.11 Predictive Density of the Expected Weaning Weight of a Female Lamb born in 1950 if the Age of the Dam is 8 years or older, and the Birth Status "Triplets". The Posterior Modes are Uniform = 25.10; Ref1 = 23.90; Ref2 = 23.85.

The respective 95% Bayesian predictive intervals for the three analyses under the different prior specification are Uniform = [12.56 ; 37.01], Ref1 = [15.98 ; 34.17] and Ref2 = [16.01 ; 34.12] respectively. Thus, the analyses for our Dormer data result in predictive densities that are close to each other which can be observed both in the predictive density and in the posterior modes and credibility intervals. In the derivations of the reference priors it was mentioned that the second reference prior (Ref2) is also a probability-matching prior, which means that its credibility intervals will have the correct coverage probability from a frequentist point of view.

5.6 Priors for the Mixed Linear Model in the Case of Three Variance Components

5.6.1 Reference Prior for the Three Variance Components

In this section, the reference priors for the mixed linear model in the case of three variance components will be derived. The application to and discussion of an example conforming to a mixed linear model with unbalanced data concludes the chapter. A uniform prior ("flat") and a proper prior (also see Theobald, Fivat and Thompson, (1997) for more details) are considered for the example. The Gibbs sampler is once again used to implement the different priors and to obtain marginal posterior densities for the different variance components.

Consider the following mixed linear model:

$$\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}_1\gamma_1 + \mathbf{Z}_2\gamma_2 + \varepsilon \tag{5.61}$$

where $\underline{\mathbf{Y}}(n\times 1)$, $\mathbf{X}(\mathrm{n}\times p)$, $\beta(p\times 1)$, $\mathbf{Z}_i(n\times q_i)$ and $\gamma_i(q_i\times 1)$ where $i=1,2$.

Also,

$$\gamma_i \sim N\left(\underline{\mathbf{0}}, \mathbf{A}_i\sigma^2_{\gamma_i}\right) \text{ and } \varepsilon \sim N\left(\underline{\mathbf{0}}, I_n\sigma^2_{\varepsilon}\right).$$

Therefore as before, the likelihood function is given by

$$L(\beta,\sigma^2_\varepsilon,\sigma^2_{\gamma_1},\sigma^2_{\gamma_2}) \propto \left|\sum_{i=1}^{2}\sigma^2_{\gamma_i}\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_1{}' + \sigma^2_\varepsilon I_n\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\underline{\mathbf{Y}}-\mathbf{X}\beta\right)'\left(\sum_{i=1}^{2}\sigma^2_{\gamma_i}\mathbf{Z}_i\mathbf{A}\mathbf{Z}_i{}'+\sigma^2_\varepsilon I_n\right)^{-1}\left(\underline{\mathbf{Y}}-\mathbf{X}\beta\right)\right\}. \tag{5.62}$$

As in (5.) we will use the transformation $v_i = \frac{\sigma_{\gamma_i}}{\sigma_\varepsilon^2}$ $(i = 1,2)$, then

$$l = \log_e L(\beta, \sigma_\varepsilon^2, v_1, v_2) = -\frac{n}{2}\log_e(\sigma_\varepsilon^2) - \frac{1}{2}\log_e \left| \sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n \right|$$

$$-\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-1}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta). \tag{5.63}$$

Here as in the case of the previous sections, the reference priors will be derived from the Fisher information matrix. To obtain the Fisher information matrix, the expected values of the second order derivatives must be calculated. These can be found as follows:

Differentiate equation (5.63) twice with respect to β, then

$$I(\beta) = \left(\sigma_\varepsilon^2\right)^{-1} \mathbf{X}' \left(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n \right) \mathbf{X}. \tag{5.64}$$

Differentiate equation (5.63) twice with respect to σ_ε^2:

$$\frac{\partial l}{\partial \sigma_\varepsilon^2} = -\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-1} + \frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-2}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta),$$

and

$$\frac{\partial^2 l}{\partial \left(\sigma_\varepsilon^2\right)^2} = -\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2} + \frac{2}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-3}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta).$$

Therefore

$$E\left(\frac{\partial^2 l}{\partial\left(\sigma_\varepsilon^2\right)^2}\right)=\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2}-tr(\sigma_\varepsilon^2)^{-3}(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}(\sigma_\varepsilon^2)\times(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)$$

$$=\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2}-\left(\sigma_\varepsilon^2\right)^{-2}trI_n=-\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-2}=-\frac{1}{2}I\left(\sigma_\varepsilon^2\right).$$

(5.65)

Hence, differentiate equation (5.63) with respect to σ_ε^2 and v_i:

$$\frac{\partial l}{\partial\sigma_\varepsilon^2}=-\frac{n}{2}\left(\sigma_\varepsilon^2\right)^{-1}+\frac{1}{2}(\underline{\mathbf{Y}}-\mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-2}(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}(\underline{\mathbf{Y}}-\mathbf{X}\beta)$$

$$\frac{\partial^2 l}{\partial\sigma_\varepsilon^2\partial v_i}=-\frac{1}{2}(\underline{\mathbf{Y}}-\mathbf{X}\beta)'\left(\sigma_\varepsilon^2\right)^{-2}(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}\left(\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}\right)\times(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}(\underline{\mathbf{Y}}-\mathbf{X}\beta)$$

Thus, taking the expected value, we have that

$$E\left(\frac{\partial^2 l}{\partial\sigma_\varepsilon^2\partial v_i}\right)=-\frac{1}{2}tr\left(\sigma_\varepsilon^2\right)^{-2}(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}\left(\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}\right)\times$$

$$(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}\sigma_\varepsilon^2(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)$$

$$=-\frac{1}{2}tr\left(\sigma_\varepsilon^2\right)^{-2}(\sum_{i=1}^{2}v_i\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}+I_n)^{-1}\left(\mathbf{Z}_i\mathbf{A}_i\mathbf{Z}_i^{'}\right)$$

$$=-\frac{1}{2}I\left(\sigma_\varepsilon^2,v_i\right)\quad(i=1,2).$$

(5.66)

If we differentiate equation (5.63) twice with respect to v_i:

Consider first $l_1 = -\frac{1}{2}\log_c\left|(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)\right|$, then

$$\frac{\partial l_1}{\partial v_i} = -\frac{1}{2} tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1} \frac{\partial(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)}{\partial v_i}\right\}$$

$$= -\frac{1}{2} tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'}\right)\right\} \text{ and}$$

$$\frac{\partial^2 l_1}{\partial v_i^2} = \frac{1}{2} tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'}\right)(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'}\right)\right\}.$$

$$E\left(\frac{\partial^2 l_1}{\partial v_i^2}\right) = \frac{1}{2} tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'}\right)\right\}^2.$$

Now consider

$$l_2 = -\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'(\sigma_\varepsilon^2)^{-1}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)$$

where

$$\frac{\partial l_2}{\partial v_i} = \frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'\left(\sigma_\varepsilon^2\right)^{-1}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'}\right) \times (v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i^{'} + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)$$

and

$$\frac{\partial^2 l_2}{\partial v_i^2} = -\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'\left(\sigma_\varepsilon^2\right)^{-1}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right)(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1} \times$$

$$\frac{\partial\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right)^{-1}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)\right\}}{\partial v_i} \times$$

$$(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right)(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta)$$

$$= -\frac{1}{2}(\underline{\mathbf{Y}} - \mathbf{X}\beta)'\left(\sigma_\varepsilon^2\right)^{-1}(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right)(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1} \times$$

$$2\left\{\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n\right\} \times \left\{\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n\right\}^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right) \times$$

$$(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}(\underline{\mathbf{Y}} - \mathbf{X}\beta).$$

Thus

$$E\left\{\frac{\partial^2 l_2}{\partial v_i^2}\right\} = -tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right)\right\}^2.$$

Combining the above results, we have

$$E\left\{\frac{\partial^2 l}{\partial v_i^2}\right\} = E\left\{\frac{\partial^2 l_1}{\partial v_i^2} + \frac{\partial^2 l_2}{\partial v_i^2}\right\}$$

$$= -\frac{1}{2} tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i'\right)\right\}^2$$

$$:= -\frac{1}{2} I(v_i). \tag{5.67}$$

Finally,

$$E\left(\frac{\partial^2 l}{\partial v_1 \partial v_2}\right) = -\frac{1}{2} tr\left\{(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_1 \mathbf{A}_1 \mathbf{Z}_1'\right)(\sum_{i=1}^{2} v_i \mathbf{Z}_i \mathbf{A}_i \mathbf{Z}_i' + I_n)^{-1}\left(\mathbf{Z}_2 \mathbf{A}_2 \mathbf{Z}_2'\right)\right\}$$
$$= -\frac{1}{2} I(v_1, v_2).$$

(5.68)

As before the expected values of the other second order derivatives are equal to zero. The Fisher information matrix is therefore given by

$$I(\beta, \sigma_\varepsilon^2, v_1, v_2) = \begin{bmatrix} I(\beta) & \underline{0} & \underline{0} & \underline{0} \\ \underline{0}' & \frac{1}{2} I(\sigma_\varepsilon^2) & \frac{1}{2} I(\sigma_\varepsilon^2, v_1) & \frac{1}{2} I(\sigma_\varepsilon^2, v_2) \\ \underline{0}' & \frac{1}{2} I(v_1, \sigma_\varepsilon^2) & \frac{1}{2} I(v_1) & \frac{1}{2} I(v_1, v_2) \\ \underline{0}' & \frac{1}{2} I(v_2, \sigma_\varepsilon^2) & \frac{1}{2} I(v_2, v_1) & \frac{1}{2} I(v_2) \end{bmatrix}. \quad (5.69)$$

The following lemma can now be stated:

Lemma 7

For the mixed linear model $\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}_1\gamma_1 + \mathbf{Z}_2\gamma_2 + \varepsilon$ the reference prior for the group ordering $\left\{\beta, \sigma_\varepsilon^2, (v_1, v_2)\right\}$ is given by

$$\pi_R\left\{\beta, \sigma_\varepsilon^2, (v_1, v_2)\right\} = \left(\sigma_\varepsilon^2\right)^{-1} \left\{I(v_1) I(v_2) - I^2(v_1, v_2)\right\}^{\frac{1}{2}}$$

where $I(v_1)$, $I(v_2)$ and $I(v_1, v_2)$ are defined in equations (5.67) and (5.68).

Proof:

The reference prior for the group ordering $\{\beta, \sigma_\varepsilon^2, (v_1, v_2)\}$ is obtained from the Fisher information matrix $I(\beta, \sigma_\varepsilon^2, v_1, v_2)$ (equation 5.69) by calculating the functions h_j (j = 1,2,3). Now

$$h_1 = I(\beta) - \begin{bmatrix}\underline{0} & \underline{0} & \underline{0}\end{bmatrix}\begin{bmatrix} \frac{1}{2}I(\sigma_\varepsilon^2) & \frac{1}{2}I(\sigma_\varepsilon^2, v_1) & \frac{1}{2}I(\sigma_\varepsilon^2, v_2) \\ \frac{1}{2}I(v_1, \sigma_\varepsilon^2) & \frac{1}{2}I(v_1) & \frac{1}{2}I(v_1, v_2) \\ \frac{1}{2}I(v_2, \sigma_\varepsilon^2) & \frac{1}{2}I(v_2, v_1) & \frac{1}{2}I(v_2) \end{bmatrix}\begin{bmatrix}\underline{0}' \\ \underline{0}' \\ \underline{0}'\end{bmatrix}$$
$$= I(\beta)$$
$$\pi(\beta) = \text{constant}.$$

To calculate h_2, consider the matrix

$$\begin{bmatrix} I(\beta) & \underline{0} \\ \underline{0}' & \frac{1}{2}I(\sigma_\varepsilon^2) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{1}{2}I(\sigma_\varepsilon^2, v_1) & \frac{1}{2}I(\sigma_\varepsilon^2, v_2) \end{bmatrix}\begin{bmatrix} \frac{1}{2}I(v_1) & \frac{1}{2}I(v_1, v_2) \\ \frac{1}{2}I(v_2, v_1) & \frac{1}{2}I(v_2) \end{bmatrix}^{-1}\begin{bmatrix} 0 & \frac{1}{2}I(\sigma_\varepsilon^2, v_1) \\ 0 & \frac{1}{2}I(\sigma_\varepsilon^2, v_2) \end{bmatrix},$$

then

$$h_2 = \frac{1}{2}I(\sigma_\varepsilon^2) - \begin{bmatrix} \frac{1}{2}I(\sigma_\varepsilon^2, v_1) & \frac{1}{2}I(\sigma_\varepsilon^2, v_2) \end{bmatrix}\begin{bmatrix} \frac{1}{2}I(v_1) & \frac{1}{2}I(v_1, v_2) \\ \frac{1}{2}I(v_2, v_1) & \frac{1}{2}I(v_2) \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}I(\sigma_\varepsilon^2, v_1) \\ \frac{1}{2}I(\sigma_\varepsilon^2, v_2) \end{bmatrix}$$
$$= \left(\sigma_\varepsilon^2\right)^{-2}$$

multiply with terms that do not contain σ_ε^{-2} , and therefore

$$\pi(\sigma_\varepsilon^2 \mid \beta) \propto \sigma_\varepsilon^{-2}.$$

Further

$$h_3 = \begin{vmatrix} \frac{1}{2} I(v_1) & \frac{1}{2} I(v_1, v_2) \\ \frac{1}{2} I(v_2, v_1) & \frac{1}{2} I(v_2) \end{vmatrix} = \frac{1}{4} I(v_1) I(v_2) - \frac{1}{4} I^2(v_1, v_2),$$

and

$$\pi(v_1, v_2 \mid \beta, \sigma_\varepsilon^2) \propto h_3^{\frac{1}{2}} = \{I(v_1) I(v_2) - I^2(v_1, v_2)\}^{\frac{1}{2}}.$$

The reference prior for the group ordering $\{\beta, \sigma_\varepsilon^2, (v_1, v_2)\}$ is thus given by

$$\pi_R\{\beta, \sigma_\varepsilon^2, (v_1, v_2)\} = (\sigma_\varepsilon^2)^{-1} \{I(v_1) I(v_2) - I^2(v_1, v_2)\}^{\frac{1}{2}}.$$

5.6.2 Proper Prior for the Variance Components (Theobald *et al.* (1997))

Consider the following proper priors for the different variance components, i.e.

(I) $$p_p(\sigma_\varepsilon^2 \mid v_\varepsilon, k_\varepsilon) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(v_\varepsilon + 2)} \exp\left\{-\frac{v_\varepsilon k_\varepsilon}{2\sigma_\varepsilon^2}\right\}, \quad i = 1,2. \qquad (5.70)$$

$$\sigma_\varepsilon^2 > 0$$

where k_ε^{-2} can be interpreted as the prior expectation of σ_ε^{-2}.

(II) $$p_p\left(\sigma^2_{\gamma_i} \mid v_{\gamma_i}, k_{\gamma_i}\right) \propto \left(\sigma^2_{\gamma_i}\right)^{-\frac{1}{2}(v_{\gamma_i}+2)} \exp\left\{-\frac{v_{\gamma_i} k_{\gamma_i}}{2\sigma^2_{\gamma_i}}\right\}, \qquad i = 1,2. \tag{5.71}$$

where, as before, $k^{-2}_{\gamma_i}$ can be interpreted as the prior expectation of $\sigma^{-2}_{\gamma_i}$.

5.7 Joint and Conditional Posterior Densities for the Mixed Linear Model in the Case of Three Variance Components

The joint posterior distribution of $p\left(\beta, \gamma_1, \gamma_2, \sigma^2_{\gamma_1}, \sigma^2_{\gamma_2}, \sigma^2_\varepsilon \mid \underline{\mathbf{Y}}\right)$ is given by

$$\left(\sigma^2_\varepsilon\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2_\varepsilon}\left(\underline{\mathbf{Y}} - \mathbf{X}\beta - \mathbf{Z}_1\gamma_1 - \mathbf{Z}_2\gamma_2\right)'\left(\underline{\mathbf{Y}} - \mathbf{X}\beta - \mathbf{Z}_1\gamma_1 - \mathbf{Z}_2\gamma_2\right)\right\} \times$$

$$\left(\sigma^2_{\gamma_1}\right)^{-\frac{q_1}{2}} \exp\left\{-\frac{1}{2\sigma^2_{\gamma_1}}\gamma_1'\gamma_1\right\} \times \left(\sigma^2_{\gamma_2}\right)^{-\frac{q_2}{2}} \exp\left\{-\frac{1}{2\sigma^2_{\gamma_2}}\gamma_2'\gamma_2\right\}. \tag{5.72}$$

(I) Uniform Prior

The conditional posterior distributions for the variance components when the uniform or "flat" prior is used, i.e.

$$p_u\left(\beta, \sigma^2_{\gamma_1}, \sigma^2_{\gamma_2}, \sigma^2_\varepsilon\right) = \text{constant}$$

are given by

$$p_u(\sigma_\varepsilon^2 \mid \beta,\gamma_1,\gamma_2,\sigma_{\gamma_1}^2,\sigma_{\gamma_i}^2,\underline{\mathbf{Y}}) =$$

$$K_\varepsilon\left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)'(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)\right\}$$

$$\sigma_\varepsilon^2>0, \tag{5.73}$$

an Inverse Gamma density where

$$K_\varepsilon=\left\{\frac{(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)'(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)}{2}\right\}^{\frac{n-2}{2}}\frac{1}{\Gamma(\frac{n-2}{2})},$$

and

$$p_u(\sigma_{\gamma_i}^2 \mid \beta,\gamma_i,\sigma_\varepsilon^2,\underline{\mathbf{Y}})=K_{\gamma_i}\left(\frac{1}{\sigma_{\gamma_i}^2}\right)^{\frac{q_i}{2}}\exp\left\{-\frac{1}{2\sigma_{\gamma_i}^2}\gamma_i'\gamma_i\right\}$$

$$\sigma_{\gamma_i}^2>0 \tag{5.74}$$

also an Inverse Gamma where $i=1,2$ and

$$K_{\gamma_i}=\left\{\frac{\gamma_i'\gamma_i}{2}\right\}^{\frac{q_i-2}{2}}\frac{1}{\Gamma\left(\frac{q_i-2}{2}\right)}.$$

Further, for the proper priors in equations (5.70) and (5.71) we have

(II) Proper Prior

The conditional posterior distributions for the variance components σ_ε^2 and $\sigma_{\gamma_i}^2$ where i = 1,2, are given by

$$p_p(\sigma_\varepsilon^2 \mid \beta,\gamma_1,\gamma_2,\sigma_{\gamma_1}^2,\sigma_{\gamma_i}^2,\underline{\mathbf{Y}}) =$$

$$K_\varepsilon\left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{(n+v_\varepsilon+2)}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left[(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)'(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)+v_\varepsilon k_\varepsilon\right]\right\}$$

$$\sigma_\varepsilon^2 > 0, \tag{5.75}$$

an Inverse Gamma density where

$$K_\varepsilon = \left\{\frac{(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)'(\underline{\mathbf{Y}}-\mathbf{X}\beta-\mathbf{Z}_1\gamma_1-\mathbf{Z}_2\gamma_2)+v_\varepsilon k_\varepsilon}{2}\right\}^{\frac{n+v_\varepsilon}{2}} \frac{1}{\Gamma(\frac{n+v_\varepsilon}{2})},$$

and finally

$$p_p(\sigma_{\gamma_i}^2 \mid \beta,\gamma_i,\sigma_\varepsilon^2,\underline{\mathbf{Y}}) = K_{\gamma_i}\left(\frac{1}{\sigma_{\gamma_i}^2}\right)^{\frac{q_i+v_{\gamma_i}+2}{2}} \exp\left\{-\frac{1}{2\sigma_{\gamma_i}^2}\left[\gamma_i'\gamma_i+v_{\gamma_i}k_{v_i}\right]\right\}$$

$$\sigma_{\gamma_i}^2 > 0 \tag{5.76}$$

also an Inverse Gamma where

$$K_{\gamma_i} = \left\{\frac{\gamma_i'\gamma_i}{2}\right\}^{\frac{q_i+v_{\gamma_i}}{2}} \frac{1}{\Gamma\left(\frac{q_i+v_{\gamma_i}}{2}\right)}.$$

The conditional posterior densities for the random and fixed effects are give in Chapter 1 where $\mathbf{Z}\gamma$ is replaced with $\mathbf{Z}_1\gamma_1 - \mathbf{Z}_2\gamma_2$.

5.8 An Example

5.8.1 The Data

As an example conforming to a model of random effects only, with unbalanced data, consider age-adjusted milk production records (305 days) obtained in the same year and herd from cows whose sires and dams were considered randomly representative of a large population. The example consists of 44 production records (in kg) of full-sib daughters and is shown in APPENDIX F. The model for this example is (see also equation (5.61)),

$$\underline{\mathbf{Y}} = \mathbf{X}\beta + \mathbf{Z}_1\gamma_1 + \mathbf{Z}_2\gamma_2 + \varepsilon$$

where $\underline{\mathbf{Y}}(44\times 1)$, $\mathbf{X}(44\times 1)$, $\beta(1\times 1)$, $\mathbf{Z}_i(44\times q_i)$ and $\gamma_i(q_i\times 1)$ where $i = 1,2$.

Further,

$$\gamma_i \sim N\left(\underline{\mathbf{0}}, I_i\sigma^2_{\gamma_i}\right),$$

$$\varepsilon \sim N\left(\underline{\mathbf{0}}, I_n\sigma^2_{\varepsilon}\right),$$

$$q_1 = 4 \text{ and } q_2 = 20.$$

Thus, the study includes 4 sires, 20 dams and 44 milk production records from the daughters. Also, there is only one fixed effect included in the final model. For further details see Gill (1978).

5.8.2 Analysis of Variance Components

Bayesian analyses of the data set are given using uniform priors and proper priors from Theobald *et al.*, (1997) (see equations (5.70) and (5.71)) with

$$v_\varepsilon = v_\gamma = 1,\ k_{\gamma_1} = 151380,\ k_{\gamma_2} = 126735 \text{ and } k_\varepsilon = 859997$$

the variance components where $k_{\gamma_1}, k_{\gamma_2}$ and k_ε are the ANOVA estimates for $\sigma^2_{\gamma_1}, \sigma^2_{\gamma_2}$ and σ^2_ε respectively.

The prior distribution for the variance components is proper and leads to a proper joint posterior distribution, but the small values chosen for v_ε and v_γ correspond to a very dispersed distribution; this is intended to reveal any problems with convergence of the Gibbs sampler algorithm. The sample data provide little information about the variance components, as the number of observations is only 44, the number of sires 4, and the number of dames 20.

Table 5.5 contains the posterior modes of the estimates obtained under the uniform or "flat" prior specification, and the posterior modes of the estimates obtained using the proper priors for the variance components. The respective 95% credibility intervals are also displayed in the table, as well as the ANOVA estimates and ANOVA-based 95% confidence intervals based on a Safterthwaite (1946) approximation to the distribution of a linear combination of χ^2 random variables (see Gill (1978)). This table reveals little difference between the implementations in the marginal posterior modes for the model variance, σ_ε^2, except for a tendency of the 95% credibility interval to be wider under the uniform prior specification. This is observed both in the table and the marginal posterior density of σ_ε^2, displayed in Figure 5.12. The estimated marginal posterior densities of $\sigma_{\gamma_1}^2$ and $\sigma_{\gamma_2}^2$ under the different prior specifications are also calculated and shown in Figures 5.13 and 5.14.

Table 5.5 Marginal Posterior Modes of the Variance Component under Uniform and Proper specifications and 95% Credibility Intervals. Also, ANOVA Estimates and 95% Approximated Confidence Intervals. Note that σ_γ^2 (sires) = $\sigma_{\gamma_1}^2$, and σ_γ^2 (Dams) = $\sigma_{\gamma_2}^2$.

Parameter	ANOVA Estimates	95% Confidence Intervals	Uniform Estimates	95% Credibility Intervals	Proper Estimates	95% Credibility Intervals
σ_ε^2	859 997	524 335 ; 1 664 376	830 010	533 060 ; 1 481 9000	801 200	551 240 ; 1 432 500
σ_γ^2(sires)	151 380	28 812 ; 144 751 070	174 126	60 207 ; 104 020 011	140 250	30 145 ; 13 793 000
σ_γ^2(dams)	126 735	16 683 ; 83 814 925	195 148	9 878 ; 11 100 200	87 850	12 910 ; 728 040

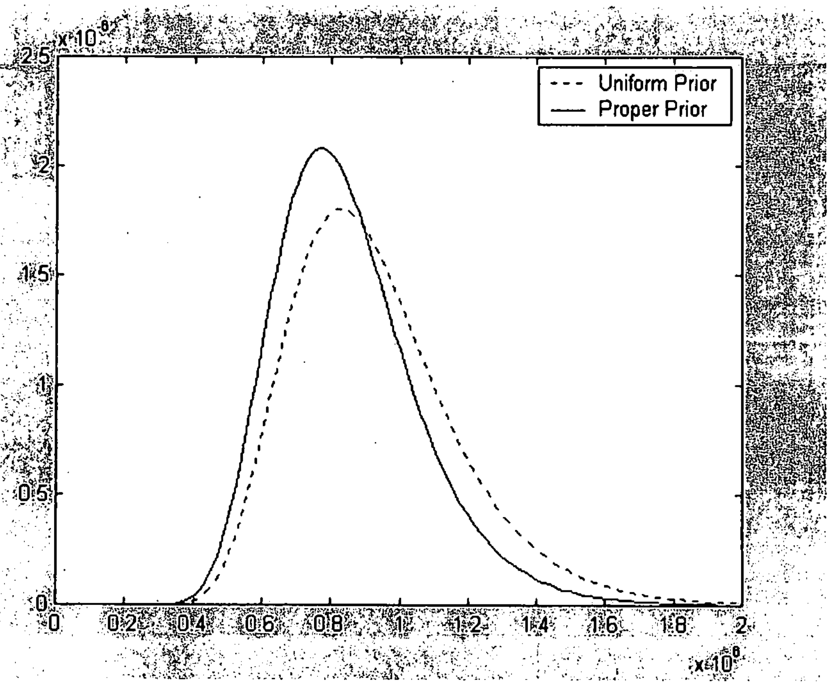

Figure 5.12 Estimated Marginal Posterior Densities of σ_{ε}^{2} using Uniform and Proper Priors for this Variance Component.

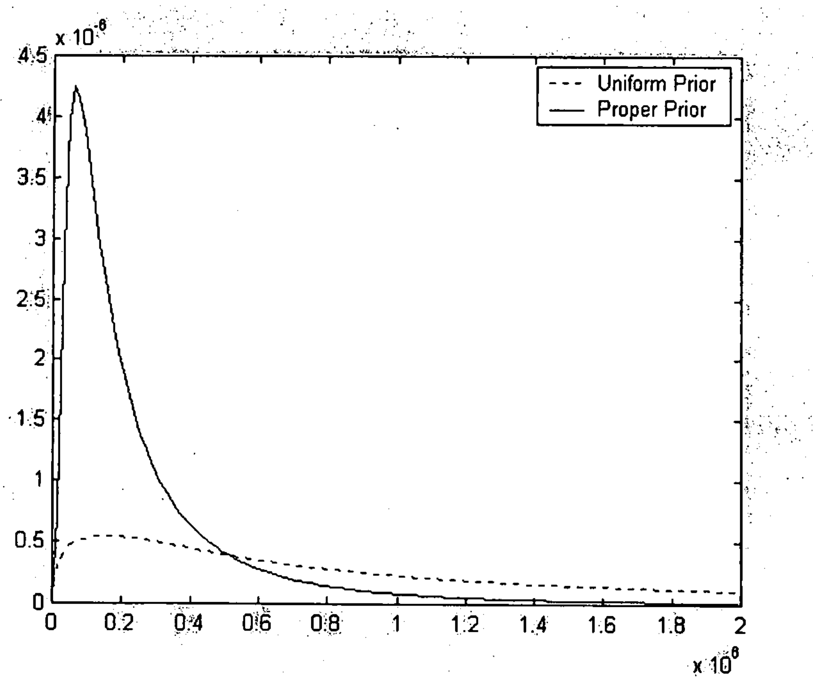

Figure 5.13 Estimated Marginal Posterior Densities of $\sigma_{\gamma_1}^{2}$ using Uniform and Proper Priors for this Variance Component.

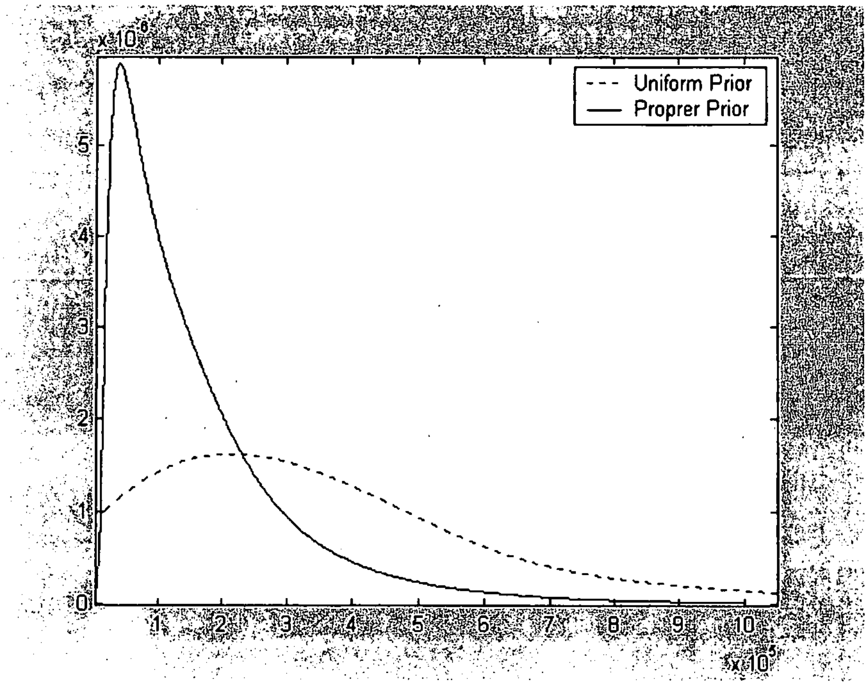

Figure 5.14 Estimated Marginal Posterior Densities of $\sigma^2_{\gamma_2}$ using Uniform and Proper Priors for this Variance Component.

These figures reveal quite a difference in the marginal posterior densities of $\sigma^2_{\gamma_1}$ and $\sigma^2_{\gamma_2}$ under the uniform prior and proper prior specifications. Not only are there differences between the posterior modes, but also the length and values covered in the 95% credibility intervals differ quite substantially. The proper prior also results in posterior densities with more mass close to zero, whereas the posterior densities using the uniform prior yield much more uncertainty about the true posterior distribution of the variance components.

From Table 5.5 it is also clear that the 95% confidence intervals obtained from the χ^2 approximations are in general wider than the corresponding Bayesian intervals. According to Hamada and Weerahandi (2000) the coverage of the ANOVA-based confidence intervals could in certain cases be drastically different than the nominal values.

5.9 Chapter Summary

For many Bayesians finding an appropriate prior distribution, when faced with a specific decision problem, can be quite difficult. It is often recommended to choose conjugate priors whenever possible because they are so computationally convenient. However, this is quite a limited family and there are many instances where they should not be used.

Another possibility is to approximate our prior beliefs by developing non-informative priors especially for the parameters of the mixed linear model via the ***Reference Prior*** algorithm. At the very least, this algorithm can be thought of as a method for generating interesting candidate non-informative priors, either for sensitivity studies of for investigation of their performances. As stated in Bernardo (1979) the motivation and idea for the reference prior is basically to choose the prior, which in a certain asymptotic sense maximizes the information in the posterior that is provided by the data. From the different lemmas it is evident that the group orderings of the model parameters are very important since different orderings will frequently result in different reference priors. This dependence of the reference prior on the group chosen and their ordering is unavoidable. Berger and Bernardo (1992) stated that many examples exist which illustrate that no single non-informative prior will work well for all functions of a given high dimensional parameter. As mentioned and more fully discussed in Berger and Bernardo (1989b) it is suggested to use the reference prior corresponding to single element groups, with the group ordered according to the inferential importance of the parameters. That different orderings of the nuisance parameters can yield different answers even has positive aspects; one can then conduct a sensitivity study over the choice if the non-informative prior. Whilst our feeling is that study of performance of reference priors is certainly to be encouraged, we have found it to be generally high satisfactory. Indeed, we would feel reasonably confident in using them in situations in which further study is impossible.

CHAPTER 6

«Conclusion and Summary»

6.1 Conclusion

Arguing from a Bayesian viewpoint, the ***Gibbs Sampling Algorithms*** presented in the thesis turned analytically intractable multidimensional integration problems arising from animal breeding theory, into feasible and appealing numerical problems. It is clear from the analyses that the Bayesian practitioner does not need to commit him to a point estimate of the variance components in order to obtain a point predictor for the variables of interest. Also, all the available information about the random variables to be predicted is contained in the posterior distributions of the random variables. Therefore, the practitioner can base all of his inferences on these distributions.

We believe that BMOM and Bayesian Non-parametrics have much to offer. In the case of BMOM, if not enough information is available to specify a form for the likelihood function, then clearly there will be problems in both the traditional likelihood and Bayesian approaches. In situations like this, some resort to non-likelihood based methods is proposed, e.g. the Bayesian Method of Moments (BMOM), first introduced by Arnold Zellner. Given the data, BMOM then enables researchers to compute post data densities for parameters and future observations if the form of the likelihood function is unknown, and provides a solution to the famous inverse problem proposed by Bayes (1763).

As far as Bayesian parametric *versus* nonparametric analyses are concerned, in relatively 'well-behaved' cases, where a parametric analysis would have coped, we typically obtain similar forms of posterior inference, particularly posterior densities. When the appropriate forms of posterior inference should be 'badly behaved' the nonparametric analysis will reflect this, whereas most parametric analyses would not reveal this fact.

A further question that arises is the ever known "...Bayesian or Classical...?" We now note the very real advantage of being able to input broad prior ideas of different characteristics such as location, scale and shape. The much richer and more tractable forms of inference that are presented as a consequence of the Gibbs simulation-based approach to computation are quite profound and significant.

Finally, it is evident that a full Bayesian solution to the problem of inference about variance components, functions thereof, and random effects in the ***Mixed Linear Model*** is possible, and contribute significantly to the theory of animal breeding.

With that in mind, we would like to conclude this thesis with the inspiring words of Daniel Gianola (1986):

> *"...In navigating through the waters of prediction of breeding values,*
> *estimation of genetic parameters and of inferences about populations*
> *undergoing selection or assortative mating, we found that the Bayesian*
> *inference brought us to familiar harbors or to new exiting lands.*
> *However, a great deal of exploration remains ahead..."*

6.2 Summary

Chapter 1 illustrated an extension of the Gibbs sampler to solve problems arising in animal breeding theory. Formulae were derived and presented to implement the Gibbs sampler where-after marginal densities, posterior means, modes and credibility intervals were obtained from the Gibbs sampler.

In the Bayesian Method of Moment chapter we have illustrated how this approach, based on a few relatively weak assumptions, is used to obtain maximum entropy densities, realized error terms and future values of the parameters for the mixed linear model. Given the data, it enables researchers to compute post data densities for parameters and future observations if the form of the likelihood function is unknown. On introducing and proving simple assumptions relating to the moments of the realized error terms and the future, as yet unobserved error terms, we derived post-data moments of parameters and future values of the dependent variable. Using these moments as side conditions, proper maxent densities for the model parameters were derived and could easily be computed. It was also shown that in the computed example, where use was made of the Gibbs sampler to compute finite sample post-data parameter densities, some BMOM maxent densities were very similar to the traditional Bayesian densities, whilst others were not.

It should be appreciated that the BMOM approach yielded useful inverse inferences without using assumed likelihood functions, prior densities for their parameters and Bayes' theorem, also it was the case that the BMOM techniques extended in the present thesis to the mixed linear model provided valuable and significant solutions in applying traditional likelihood or Bayesian analysis in animal breeding problems.

The important contribution of Chapter 3 and 4 revolved around the nonparametric modeling of the random effects. We have applied a general technique for Bayesian nonparametrics to this important class of models, the mixed linear model for animal breeding experiments. Our technique involved specifying a nonparametric prior for the distribution of the random effects and a Dirichlet process prior on the space of prior distributions for that nonparametric prior. The mixed linear model was then fitted with a Gibbs sampler, which turned an analytical intractable multidimensional integration problem into a feasible numerical one, overcoming most of the computational difficulties usually experience with the Dirichlet process.

This proposed procedure also represented a new application of the mixture of Dirichlet process model to problems arising from animal breeding experiment. The application to and discussion of the breeding experiment from Kenya was helpful for understanding the importance and utility of the Dirichlet process, and inference for all the mixed linear model parameters. However, as mentioned before, a substantial statistical issue that still remains to be tackled is the great discrepancy between resulting posterior densities of the random effects as the value of the precision parameter, M changes. We believe that Bayesian nonparametrics have much to offer, and can be applied to a wide range of statistical procedures. In addition to the Dirichlet Process Prior, we will look in the future at other nonparametric priors like the Pòlya tree priors and Bernoulli trips.

Whilst our feeling in the final chapter was that study of performance of non-informative was certainly to be encouraged, we have found the group reference priors to generally be high satisfactory, and felt reasonably confident in using them in situations in which further study was impossible. Results from the different theorems yielded that the group orderings of the mixed model parameters are very important since different orderings will frequently result in different reference priors. This dependence of the reference prior on the group chosen and their ordering was unavoidable. Our motivation and idea for the reference prior was basically to choose the prior, which in a certain asymptotic sense maximized the information in the posterior that was provided by the data.

The thesis has surveyed a range of current research in the area of Bayesian parametric and nonparametric inference in animal science. The work is ongoing and several problems remain unresolved. In particular, more work is required in the following areas: a full Bayesian nonparametric analysis involving covariate information; multivariate priors based on stochastic processes; multivariate error models involving Pòlya trees; developing exchangeable processes to cover a larger class of problems and nonparametric sensitivity analysis.

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APPENDIX A

«Selective Algorithms for the Gibbs Sampler»

1.1 Algorithm for the Traditional Bayes Analysis

```
%%Load data with X, Y and Z matrices

%%Initialize the different variables
   N      =  the sample size;
   SimTot=  the simulation total;
   Gibbs =  save every ..th sampled value in the Gibbs sampler;

   q   =  the number of random effects;
   p   =  the number of fixed effects;

%%The Random Effects

   UVec=[0.13957685
         3.32999015
         0.57818853
         ...
         ...
         0.79500881];  A qx1 vector of starting values for the Gibbs
                         sampler (Random effects)

%%Calculate initial values to be used, e.g.

   InvXtX=inv(X'*X);
   Yster=Y-Z*UVec;
   BetaHat=invXtX*(X'*Yster);
   e=Yster-X*BetaHat;


%%The Gibbs Sampler
for i=1:SimTot

   for j=1:Gibbs
```

%%Simulate σ_ε^2 from an Inverse Gamma

```
%%Specify the degree of freedom.  Note that a Gamma distribution is the
%%sum of df squared random numbers, i.e.
```

```
   df=N-2;
   x=(randn(df,1))^2;
   v=sum(Xx);

%%Calculate

   numvec=(Y-X*BetaVec-Z*UVec)'*(Y-X*BetaVec-Z*UVec);
   O2e=numvec/v;

%%where BetaVec is the px1 vector of fixed effects

%%Simulate BetaVec from a Normal distribution.
%%Calculate

   muB=BetaHat;
   sigmaB=sqrtm(invXtX*o2e);

   BetaVec=sigmaB*randn(p,1)+muB;

%%Simulate σ_γ^2 from an Inverse Gamma distribution
%%Calculate

   df=q-2;
   x=(randn(df,1))^2;
   v=sum(Xx);
   Ainv=inv(A);

   o2u=(UVec'*Ainv*UVec)/v;

%%Simulate UVec from a Normal distribution
%%Calculate

   muU=[inv((Z'*Z+(o2e/o2u)*Ainv))]* [Z'*(Y-X*BetaVec)];
   sigmaU=sqrtm(o2e*inv(Z'*Z +  (o2e/o2u) *  Ainv));
   x=randn(q,1);

   UVec=sigmaU*x+muU;

   End %% Update the different model parameters

END  %% Program


%%Calculate and display the averages of the different model parameters

%%SAVE results.
```

1.2 Algorithm for Simulating a γ_i with a certain Probability (see Dirichlet process)

```
%%Specify the vector with the different probabilities
   y1=ProbVec  %%e.g. ProbVec = [0.25 0.25 0.45 0.05] with sum(ProbVec)=1

%%Specify the vector to choose from
   sires=   %%e.g. [1 2 3 4], thus sire i can be set equal to sire 2 with
            %%probability 0.25, sire 3 with probability 0.45 etc.

%%Then calculate

   Oppv= sum(1*y1);
   y1=y1./Oppv;
   CumSumY=cumsum(y1')';
   kwh=1*csY;
   tr=rand(1,1);
   kla=(kwh-tr);
   kl=abs(kwh-tr);
   klein=min(abs(kwh-tr));
   IN=find(kl==klein);
   s=size(IN);
   if s(1,1)>1
      IN=IN(1,1);
   end
   kleina=kla(IN);
   if kleina < 0
    ID=IN+1;
   else
    ID=IN;
   end

   pp=sires(ID);


%%Thus, sire i will have the same breeding value as the sire in the
%%position ID of the vector 'sires'.
```

1.3 Algorithm for Simulating the Variance-covariance Matrix D from an Inverse Wishart

%%First, simulate A from a Wishart $\left(\left(\sum \lambda_l \lambda_l'\right)^{-1}, k\text{-}v\text{-}1, v\right)$ distribution

%%Simulate $\underline{l}_1, \underline{l}_2, \ldots, \underline{l}_{k-v-1}$ form $N_v(\underline{0}, I_v)$, i.e.(k-v-1) vectors each of
%%order (vx1) from a multivariate normal distribution

```
loop=k-v-1

H=[];
for q=1:loop
   H=[H randn(v,1)];
end
```

%%Define H = $[\underline{l}_1, \underline{l}_2, \ldots, \underline{l}_{k-v-1}]$, then calculate $\mathbf{A} = \left(\sum \lambda_l \lambda_l'\right)^{-\frac{1}{2}} \mathbf{HH'} \left(\sum \lambda_l \lambda_l'\right)^{-\frac{1}{2}}$

```
for i=0:k
   sta=i*2+1;
   sto=(i+1)*2;
   u=UVec(sta:sto);
   utu=u*u';
   UtU=UtU+utu;
end

mat=sqrtm(inv(UtU));
A=mat*H*H'*mat;

D=inv(A);
```

%%Then D = A^{-1} is from an Inverse Wishart distribution. Simulation from the Wishart distribution can also easily be done by using the algorithm of Odell and Feiveson (1966) in A Numerical procedure to generate a sample Covariance matrix, *Journal of American Statistical Association*, 61, 198 - 203.

1.4 Algorithm for Simulating the Precision Parameter, *M* (see Dirichlet process)

```
%%This algorithm illustrates the simulation of the parameter M and mixing
%%parameter X in the Dirichlet process prior.  Gr is the number of
%%groups/clusters for the analysis

%% Initialize starting values for X

XVar=0.5;

%%Simulate two values Z1 and Z2 from two Chi-square distributions where
%%df= degrees of freedom i.e.

%%Simulate u
df=2*GR;
x=(randn(1,df))^2;
u=sum(x);
Z1=u/(-2*log(Xvar));

%%Simulate v
df=2*(GR-1);
x=(randn(1,df))^2;
v=sum(x);
Z2=v/(-2*log(Xvar));

%%Set the value of M to any of Z1 or Z2 with probability 0.5 for each
%%See the algorithm in 1.2


%%Simulate another two values Z3 and Z4 from two Chi-square distributions,
%%given M

df=2*(M+1);
x=(randn(1,df))^2;
v=sum(x);
Z3=v/(2*(M+1));

df=2*k;    %%Number of sires k=200;
x=(randn(1,df))^2;
v=sum(x);
Z4=v/(2*k);

%%Calculate the following

y=Z3/Z4;


X1=y*(M+1)/k;
X2=1+(y*(M+1)/k);

XVar=X1/X2;
```

1.5 Algorithm for Plotting the Unconditional Posterior Density of the Precision Parameter, *M* (see Dirichlet process)

```
%%Load the results

%%Specify the interval for the parameter

M=[10:1000];

%%Calculate the value of the conditional posterior distribution at each M
%%SimTot is the number of simulations, and Gr the vector of resulting
%%groups/clusters from the analysis


for j=1:SimTot
   k=Gr(1,j);

   for i=1:1
      m=M(1,i);
      y=(m^(k/2)*(m+q)*beta(m+1,q))*(m^(k/2-2));
      Y=[Y y];
   end

YY=YY+Y;
end

%%Calculate the marginal posterior as the average of all the conditional
%%posterior densities

Y=YY/SimTot;

plot(M,Y);
```

1.6 Setup Algorithm for an Animal Breeding Experiment (Chapter 3, § 3.6)

```
%%Initialize the specifications of the experiment, i.e. the number of
%%observations to sample for each sire for each fixed effect

n=20;

%%Per sire, i.e. n/2 male and n/2 female if sex is included as a fixed
%%effect

%%Number of Sires,
k=200;

%%Total number of observations,
NN=n*k;


%%Specify the true values of the variance components and fixed
%%effect(s), β_i
```

σ_ε^2=4.88; σ_γ^2=0.7211; β_0 = 0.705,

```
%%Construct a vector Y (Nx1) of observations distributed N(0,1)
Y=randn(NN,1);

%%Simulate the different random effects according to the Polya Urn scheme
%%and add it to the vectors Y of observations (see Chapter 3, §3.4.3)


%%Add a random residual to each observation

for i=1:k
   e=randn(n,1)*sqrt(σ_ε^2);
   Y=[Y (U(i,1)+e)];
end


%%Add an overall mean to each observation

Y=Y+12;

%%Add the effect of the fixed effect to each observation

Y(1:(n/2),:)=Y(1:(n/2),:)+0.705;

%%Construct the different matrices corresponding to the experiment, i.e. X
%%and Z


%%SAVE the experimental data.
```

APPENDIX B.

In this APPENDIX we present the Dormer stud sample used in estimating the breeding values and variance components. It refers to 879 weaning weight records from the progeny of 17 sires from the Elsenburg Dormer sheep stud near Stellenbosch. The sheep used in the analysis were born in the period 1943 – 1950. The animal ID, sire ID, year (season of birth), age of dam, sex birth status and weaning weight are presented in this table.

Animal ID refers to the ID of the lamb that was born.

Sire ID and dam ID refers to the parents of the lamb.

Year (season of birth) refers to the year in which the lamb was born. Here 1943 is denoted by 43 and 1944 by 44, ect.

Age of dam refers to the age of the dam used in producing the progeny. Here age 2 is denoted by 2 and age 3 is denoted by 3, ect.

Sex refers to the sex of the lamb. Here male is denoted by 1 and female by 2.

Birth status refers to the birth status of the lamb. Here single births are denoted by 1, twins are denoted by 2 and triplets are denoted by 3.

Weaning weight refers to the weaning weight of the lambs in kilogram.

ANIMAL ID	SIRE ID	DAM ID	SEASON	AGE OF DAM	SEX	BIRTH STATUS	WEANING WEIGHTS
43002	41037	41076	43	2	1	1	34.2
43003	41037	41130	43	2	2	1	33.1
43004	41037	41029	43	2	1	1	40.1
43005	41004	41134	43	2	1	1	32
43006	41037	41096	43	2	1	2	33.6
43008	41037	41007	43	2	1	1	38.5
43009	41037	41167	43	2	1	1	37.9
43010	41037	41031	43	2	2	1	37.8
43011	41004	41040	43	2	1	1	30.6
43012	41004	41165	43	2	2	2	21
43013	41004	41165	43	2	1	2	27.1
43014	41037	41028	43	2	2	1	38.2
43015	41004	41001	43	2	1	1	35.9
43018	41004	41023	43	2	2	1	28.3
43019	41004	41104	43	2	1	2	25.2
43020	41004	41104	43	2	2	2	25.2
43021	41004	41112	43	2	1	1	36
43022	41004	41181	43	2	2	2	26.8
43023	41004	41181	43	2	1	2	26.8
43024	41004	41187	43	2	2	1	36.3
43025	41004	41068	43	2	2	1	31.8
43026	41004	41061	43	2	2	2	26.7
43027	41004	41061	43	2	1	2	27.4
43028	41004	41033	43	2	1	1	36.5
43029	41037	41051	43	2	2	1	34.6
43030	41004	41034	43	2	1	1	32.1
43032	41004	4105	43	2	2	2	29.1
43033	41004	4105	43	2	2	2	25.9
43035	41037	41087	43	2	1	1	37.1
43037	41004	41046	43	2	2	1	27.9
43038	41004	41074	43	2	1	2	36.2
43039	41004	41074	43	2	1	2	30.7
43043	41037	41018	43	2	1	1	37.9
43044	41004	41171	43	2	1	1	44.8
43045	41004	41047	43	2	2	1	39.3
43049	41004	41080	43	2	2	1	24.7
43053	41037	41172	43	2	2	1	33.6
43058	41037	41069	43	2	1	1	41.9
43061	41037	41041	43	2	1	1	37.6
43062	41004	41074	43	2	1	2	38.2
43063	41004	41074	43	2	2	2	30.1

43064	41037	41139	43	2	1	1	36.6
43067	41004	41163	43	2	1	2	29.6
43068	41004	41163	43	2	1	2	29.6
43070	41004	41175	43	2	2	1	34.9
43071	41037	41042	43	2	1	1	34
43072	41037	41015	43	2	2	1	35.3
43077	41037	41115	43	2	1	1	30.6
43081	41037	41093	43	2	2	1	35.5
43083	41037	41053	43	2	2	1	34.5
43084	41004	41030	43	2	2	1	33.8
43085	41004	41013	43	2	1	1	41.1
43086	41004	41010	43	2	2	1	34.2
43087	41037	41053	43	2	2	1	34.5
43088	41037	41058	43	2	2	1	32.1
43089	41004	41055	43	2	2	1	36.6
43093	41004	41023	43	2	1	1	37.7
43094	41004	41077	43	2	1	2	34.5
43095	41004	41077	43	2	2	2	23.6
43098	41037	41088	43	2	2	1	35
43100	41037	41096	43	2	2	2	27.7
43101	41004	41050	43	2	1	1	39
43102	41037	41138	43	2	1	1	39.4
43104	41037	41151	43	2	1	1	36.4
43110	41037	41044	43	2	2	2	31.8
43111	41037	41060	43	2	2	1	21.4
43117	41037	41057	43	2	2	1	31.8
43118	41037	41035	43	2	1	1	38.2
43136	41004	41174	43	2	2	1	30.2
44002	41037	41040	44	3	2	1	29.8
44003	41004	41130	44	3	2	1	26.6
44007	41037	41096	44	3	1	2	34.4
44012	41004	41068	44	3	1	1	41.7
44013	41004	41003	44	3	2	1	37.1
44022	41004	41171	44	3	2	2	36.1
44026	41004	42080	44	2	2	1	35
44027	41037	41031	44	3	2	1	33.5
44028	41037	41008	44	3	2	1	38
44033	41004	41025	44	3	1	2	36.9
44034	41004	41025	44	3	1	2	33.8
44040	41004	41165	44	3	2	1	35
44041	41037	42071	44	2	1	1	35.1
44042	41004	41187	44	3	2	1	37.7
44043	41004	41187	44	3	2	2	29.4
44050	41004	41144	44	3	2	1	32.8
44052	41037	41029	44	3	1	2	33.9
44053	41037	41029	44	3	1	2	31.5

44054	41004	41135	44	3	1	2	30.5
44055	41004	41135	44	3	1	2	33
44063	41004	41181	44	3	1	2	42.5
44064	41004	41047	44	3	2	2	30
44066	41004	41047	44	3	1	2	27.2
44072	41037	41167	44	3	2	1	37.3
44076	41037	41029	44	3	1	2	29.4
44077	41037	41029	44	3	2	2	28.8
44082	41004	41015	44	3	2	1	36.4
44083	41004	41087	44	3	2	1	38.7
44084	41004	41035	44	3	1	1	38.1
44093	41004	41051	44	3	2	2	30
44094	41004	41051	44	3	2	2	29.1
44096	41037	41096	44	3	1	2	32.2
44098	41037	41105	44	3	2	2	34.1
44110	41037	41099	44	3	2	1	37.4
44115	41004	41023	44	3	2	1	32.8
44121	41004	42071	44	2	2	2	23.1
44122	41004	42071	44	2	2	2	19.7
44123	41004	41122	44	3	1	1	34.6
44125	41037	41058	44	3	1	1	39.8
44127	41004	41042	44	3	1	2	26.4
44128	41004	41042	44	3	1	2	30.3
44129	41004	41139	44	3	1	1	37.6
44130	41004	41172	44	3	2	1	35.5
44134	41037	42115	44	2	2	1	32
44147	41004	42060	44	2	2	2	35.8
44149	41004	41061	44	2	1	1	41
44157	41037	41028	44	3	1	2	33.6
44158	41037	41028	44	3	2	2	29.3
44162	41004	41034	44	3	1	2	30.3
44163	41004	41034	44	3	2	2	25.9
44165	41037	41033	44	3	2	2	29.6
44166	41037	41001	44	3	2	1	38.6
44167	41004	41074	44	3	1	2	26.1
44168	41004	41074	44	3	2	3	20.7
44169	41004	41136	44	3	1	1	38.6
44170	41004	41077	44	3	1	1	34.8
44172	41037	42065	44	3	1	2	23.4
44173	41037	41063	44	3	2	2	22.8
44174	41004	41010	44	3	1	1	33.2
44178	41004	41115	44	3	1	1	38.5
44179	41037	42071	44	2	1	1	39.8
44183	41004	41083	44	3	1	1	32.6
44184	41037	41023	44	3	2	1	35.2
44191	41037	42065	44	2	1	1	29

44192	41004	41043	44	3	1	1	32.1
44198	41004	41138	44	3	2	1	38.9
44204	41037	41093	44	3	1	1	38.8
44205	41037	41041	44	3	2	1	37.9
44210	41004	42069	44	2	1	2	37.8
44212	41037	41174	44	3	1	1	31.2
44213	41004	41005	44	3	2	1	31.8
44217	41004	41175	44	3	2	1	35.9
44220	41037	41104	44	3	2	2	30.8
44222	41037	41013	44	3	2	1	38.9
44224	41037	41055	44	3	2	2	34.4
44228	41037	41018	44	3	2	1	35
44230	41037	41131	44	3	2	1	37.2
44232	41004	41053	44	3	2	1	36.8
44236	41037	42062	44	2	2	1	34.7
44244	41037	42071	44	2	1	2	30.4
44245	41037	42071	44	2	2	2	27.8
44250	41037	41057	44	3	2	1	38.9
44252	41037	41050	44	3	2	2	33
44253	41037	41050	44	3	2	2	31.2
45001	41004	41015	45	4	2	2	23.4
45002	41004	41015	45	4	1	2	28.5
45003	41004	41122	45	4	2	2	31.9
45004	41004	41122	45	4	1	2	34.4
45005	43002	43136	45	2	1	1	38.4
45006	43002	43029	45	2	1	1	35.5
45007	41019	41040	45	4	1	1	41
45008	41004	43003	45	2	1	1	36.9
45009	43002	43045	45	2	1	1	27.5
45010	41019	41057	45	4	1	2	30.4
45011	41019	41057	45	4	2	2	30
45015	41004	41010	45	4	2	1	39.9
45019	41004	43100	45	2	1	2	33.4
45020	41004	43100	45	2	2	2	23.9
45022	43002	43024	45	2	2	2	21.2
45026	43002	43095	45	2	2	1	30.8
45027	43002	43049	45	2	2	1	28.8
45031	41019	42071	45	3	2	1	33.5
45032	43002	43026	45	2	2	1	20.3
45033	43002	43063	45	2	1	2	32.6
45034	43002	43063	45	2	2	2	24.9
45036	41019	42115	45	3	1	2	44.3
45040	41019	41028	45	4	1	1	48.1
45042	41019	41031	45	4	2	2	37.8
45044	43002	43033	45	2	1	1	39.9
45045	41004	41135	45	4	1	2	31.6

45046	41004	41135	45	4	2	2	25.5
45047	41004	41013	45	4	1	1	47.6
45048	43002	43089	45	2	2	2	28.4
45049	43002	43089	45	2	2	2	29
45050	41019	41105	45	4	1	2	41.3
45051	41019	41105	45	4	2	2	38
45053	41004	42069	45	3	2	2	28.6
45054	41004	42069	45	3	2	2	31.8
45055	41004	41025	45	4	2	2	35
45056	41004	41025	45	4	2	2	35.3
45059	41019	41008	45	4	2	1	35.8
45061	41019	4100	45	4	2	2	34.1
45062	41019	4100	45	4	2	2	26.9
45063	41019	41041	45	4	2	1	40.6
45064	41004	41138	45	4	1	1	51.5
45066	41004	41069	45	4	1	1	39.5
45069	41004	43088	45	2	2	2	23.2
45070	41004	41044	45	4	1	2	46.3
45071	41004	41044	45	4	2	2	31.1
45072	43002	43018	45	2	1	1	36.7
45073	43002	43084	45	2	2	1	35.3
45074	41004	42071	45	3	2	2	26.6
45075	41004	42071	45	3	1	2	26.2
45076	41004	41034	45	4	1	2	37.4
45077	41004	41034	45	4	2	2	34.6
45081	41019	41018	45	4	1	1	38
45082	41004	41051	45	4	2	2	26.2
45083	41004	41051	45	4	1	2	30.6
45084	41004	41074	45	4	2	3	22.2
45085	41004	41074	45	4	1	3	30
45086	41004	41074	45	4	1	3	29.2
45094	41004	41172	45	4	2	1	40
45096	41004	42087	45	3	2	1	36.3
45098	41019	41001	45	4	2	1	47.2
45101	41019	41167	45	4	1	1	49.3
45102	41019	42065	45	3	2	1	40.8
45103	43002	43086	45	2	2	1	36.4
45111	41004	41083	45	4	1	2	38.9
45112	41004	41083	45	4	2	2	31.2
45113	43002	43072	45	2	2	2	28.2
45114	43002	43072	45	2	1	2	34.4
45115	41004	43014	45	2	2	1	33.9
45116	41019	41093	45	4	2	1	41.2
45118	41004	43081	45	2	1	2	30.8
45119	41004	43081	45	2	2	2	26.2
45121	41004	41136	45	4	2	2	27.1

45122	41004	41136	45	4	1	2	30.8
45123	41019	41055	45	4	1	1	49.6
45126	41004	41144	45	4	1	1	47.5
45127	41019	42062	45	3	1	2	42.5
45129	41019	42062	45	3	2	2	39.3
45132	41019	41133	45	4	2	1	41.5
45133	41004	41047	45	4	1	2	40.8
45134	41004	41047	45	4	2	2	27.3
45135	43002	43117	45	2	1	1	34.3
45153	41004	41043	45	4	2	1	42.6
45155	43002	43037	45	2	1	2	39.3
45156	41019	41023	45	4	2	1	38.6
45190	41004	41171	45	4	2	1	47.4
45205	41004	42060	45	3	1	1	35.2
45207	41004	41087	45	4	2	1	32.8
45208	41004	42080	45	3	2	1	42.1
45211	41004	41115	45	4	2	1	46.5
46001	41019	41040	46	5	1	1	41.7
46002	41004	41044	46	5	1	2	35.2
46003	41004	41044	46	5	1	2	28.4
46004	41004	41122	46	5	2	1	37.8
46005	41004	41015	46	5	2	2	26.8
46006	41004	41015	46	5	2	2	29.3
46007	41019	41057	46	5	1	1	39.6
46008	41004	41135	46	5	2	1	37.6
46009	44170	43084	46	3	2	2	29.2
46010	44170	43084	46	3	2	2	26.3
46012	41004	41139	46	5	1	2	32.2
46013	41004	41139	46	5	2	2	30.1
46014	41004	41043	46	5	1	1	41.9
46015	41004	42087	46	4	1	1	41.1
46020	41004	41034	46	5	2	2	22.4
46021	41004	41034	46	5	1	2	32.3
46022	44170	43018	46	3	2	1	31.3
46023	41004	41069	46	5	2	2	29.4
46024	41004	41069	46	5	2	2	30.3
46025	41004	41138	46	5	2	1	33.2
46026	41019	42071	46	4	2	1	30.4
46031	41019	41167	46	5	1	1	47.6
46032	41004	41172	46	5	1	1	21.5
46033	41004	41171	46	5	1	2	30.6
46034	41004	41171	46	5	2	2	24.7
46037	44170	44228	46	2	1	1	40.1
46040	44170	43045	46	3	1	1	31.9
46041	41019	41033	46	5	1	2	35.8
46042	41019	41033	46	5	1	2	26.1

46044	41004	42080	46	4	1	1	40
46045	41019	42115	46	4	1	2	34.9
46046	41019	42115	46	4	2	2	32.4
46047	41004	42060	46	4	2	1	37.3
46048	41019	41041	46	5	2	2	35.8
46050	44170	43033	46	3	1	1	44.1
46054	44170	44158	46	2	2	1	34.6
46055	44170	44098	46	2	1	1	26.4
46057	41004	41051	46	5	1	1	28
46058	41004	41010	46	5	1	1	23.6
46063	44170	44072	46	2	1	1	27.7
46064	41004	41061	46	5	2	2	32.2
46065	41004	41061	46	5	1	2	33.6
46070	41004	4100	46	5	1	2	20.9
46072	41004	41144	46	5	2	1	36.2
46073	41004	41025	46	5	2	1	36.5
46076	41019	41008	46	5	1	2	38
46086	44170	44184	46	2	1	1	26.1
46087	41004	41136	46	5	2	2	23.8
46088	41004	41136	46	5	1	2	33.1
46095	41004	41074	46	5	2	2	24
46096	41004	41074	46	5	1	2	29
46097	41004	41083	46	5	1	2	26.6
46098	41004	41083	46	5	1	2	28
46102	41019	42062	46	4	2	1	33.6
46125	41019	42065	46	4	1	1	42.7
46131	41019	41023	46	5	1	2	30.6
46132	41019	41023	46	5	1	2	31.7
46133	41019	41031	46	5	2	2	31.3
46134	41019	41031	46	5	2	2	31.4
46135	44170	44224	46	2	1	1	34.7
46136	44170	44222	46	2	1	1	36.6
46137	44170	44165	46	2	1	1	23.2
46147	41019	41093	46	5	2	1	36.5
46149	41004	41047	46	5	1	1	38.8
46163	44170	44205	46	2	2	1	34.6
46170	41004	42074	46	4	2	1	30.5
46175	41004	41115	46	5	2	1	29.2
46177	41019	41001	46	5	2	1	44.5
46181	44174	44166	46	2	1	2	28
46182	44174	44166	46	2	2	2	24
46189	44174	43086	46	3	1	1	23.9
46198	44042	44083	46	2	1	1	26.9
46199	44042	43110	46	3	1	1	31.1
46205	44042	43100	46	3	2	1	27.8
46206	44042	44013	46	2	1	1	33.6

46208	44174	43063	46	3	1	2	22.7
46209	44174	43063	46	3	2	2	23
46210	44042	44063	46	2	2	1	23.1
46214	44042	43081	46	3	2	2	22.6
46215	44042	43081	46	3	2	2	23.5
46217	44042	44082	46	2	1	1	29.1
46222	44042	44121	46	2	1	1	29.8
46224	44174	44134	46	2	2	1	38.3
46227	44042	44252	46	2	1	1	28.6
46228	44042	44093	46	2	2	1	31
47001	45070	45073	47	2	2	2	21.8
47002	45070	45073	47	2	1	2	24.6
47003	45070	45049	47	2	1	2	23.2
47004	45135	45211	47	2	1	2	22.1
47006	41019	41040	47	6	1	1	30.9
47007	44042	43003	47	4	1	1	29.2
47008	45135	45054	47	2	2	2	20.9
47009	45135	45054	47	2	1	2	24.7
47010	45135	43018	47	4	1	2	26
47011	45135	44205	47	3	1	2	26.3
47012	45135	44205	47	3	1	2	28.5
47014	45135	43033	47	4	2	1	29.4
47015	44042	44013	47	3	2	1	28.6
47016	41004	41025	47	6	1	2	37.1
47017	45135	44222	47	3	2	2	17.6
47018	45135	44222	47	3	1	2	28.1
47019	45135	44134	47	3	1	1	38.5
47020	45135	44098	47	3	1	1	33.4
47022	45070	45115	47	2	2	1	29.3
47023	41019	41031	47	6	1	2	32.2
47024	41019	41031	47	6	2	2	27.6
47026	45135	44158	47	3	2	1	28.6
47028	41004	41010	47	6	1	1	35
47029	41004	41015	47	6	2	1	25.4
47030	41004	41015	47	6	1	1	23
47033	45135	45077	47	2	1	1	35.4
47034	44042	44217	47	3	2	1	25
47035	45135	45082	47	2	1	2	21
47036	45135	45082	47	2	1	2	22.2
47037	41004	4100	47	6	1	1	37.4
47040	45135	44165	47	3	1	1	32.4
47042	41004	41047	47	6	1	2	36.2
47043	44042	44082	47	3	2	1	28.2
47049	41004	41135	47	6	1	2	28.5
47051	45135	45053	47	2	1	1	30.4
47055	44042	43110	47	4	1	2	24

47056	44042	43110	47	4	2	2	24.1
47060	44042	44063	47	3	2	2	17.2
47064	41004	42060	47	5	1	1	33.3
47067	41004	42087	47	5	1	2	33
47068	41004	42087	47	5	2	2	23.2
47074	44042	44213	47	3	2	1	27.6
47076	41019	41008	47	6	2	2	25
47077	41019	41008	47	6	2	2	29.8
47078	45135	45121	47	2	2	2	11.5
47080	45135	45190	47	2	1	2	26.8
47083	45135	44184	47	3	2	1	25.6
47087	45135	45015	47	2	2	1	25.7
47091	41004	41083	47	6	1	1	27.1
47092	41004	41051	47	6	2	2	16.4
47094	44042	44022	47	3	2	2	18.2
47095	44042	44022	47	3	2	2	22.8
47096	44042	45102	47	2	1	2	24.2
47097	44042	45102	47	2	2	2	23.2
47098	41019	41033	47	6	1	1	37.7
47101	41004	42071	47	5	2	2	17.1
47102	41004	42071	47	5	1	2	21.1
47105	45135	44072	47	3	2	1	26.8
47108	41004	41138	47	6	1	2	28.1
47109	41004	41138	47	6	2	2	26.2
47111	41004	45026	47	2	1	2	12.4
47112	41019	42071	47	5	2	2	21.2
47113	41019	42071	47	5	2	2	21
47114	45135	44224	47	3	1	2	26.4
47115	45135	44224	47	3	2	2	23.9
47117	41004	41144	47	6	1	1	28.4
47118	41004	41136	47	6	1	2	27.9
47119	41004	41136	47	6	2	2	22.3
47120	45135	45055	47	2	2	1	26.9
47121	41004	41043	47	6	1	2	38.8
47123	41019	41028	47	6	1	2	28.2
47124	41019	41028	47	6	2	2	31.1
47126	41019	41023	47	6	1	2	22.7
47127	41019	41023	47	6	1	2	26.6
47130	45135	43037	47	4	2	2	22.6
47131	45135	43037	47	4	1	2	30.8
47132	44042	45098	47	2	2	1	30.8
47133	44042	45098	47	2	2	1	30.8
47134	41019	42065	47	5	1	2	38
47135	41019	41057	47	6	1	2	31.4
47136	44042	45061	47	2	1	1	33.3
47139	45135	43086	47	4	1	1	28.3

47140	44042	43029	47	4	2	2	24.9
47141	44042	43029	47	4	2	2	25.2
47147	45135	43063	47	4	2	2	23.9
47148	45135	43063	47	4	2	2	21.6
47150	45135	45056	47	2	2	2	22.1
47151	45135	45056	47	2	2	2	18.7
47161	44042	44253	47	3	1	2	31.9
47167	45070	45048	47	2	2	2	25.2
47169	45070	45048	47	2	1	2	22.1
47170	45135	43012	47	4	1	1	30.6
47174	45135	45094	47	2	2	1	34
47184	44042	43100	47	4	1	1	22.3
47188	45135	43024	47	4	2	1	27.3
47189	44042	44252	47	3	1	1	32.9
47204	41004	41044	47	6	1	1	32.6
47208	41019	41041	47	6	2	2	30.6
47210	44042	45063	47	2	1	1	23
47211	44042	44064	47	3	1	1	32.6
47212	45135	43020	47	4	2	1	24.9
47213	45135	44027	47	3	1	1	32.2
47215	41004	41061	47	6	2	2	26.1
47216	41004	41061	47	6	1	2	26
48001	46015	45049	48	3	2	2	22.1
48002	46015	45049	48	3	2	2	25.4
48003	41004	41031	48	7	2	2	22.6
48004	41004	41031	48	7	1	2	33.6
48006	44042	41040	48	7	1	2	33.8
48007	41004	41144	48	7	1	2	24.4
48008	41004	41144	48	7	2	2	24.6
48009	46037	44083	48	4	1	1	39.5
48010	46037	45054	48	3	2	1	35.2
48011	44042	45115	48	3	1	2	20.9
48012	44042	45115	48	3	1	2	29.1
48013	41004	42080	48	6	2	1	42.7
48014	45070	45116	48	3	1	1	42.5
48015	46037	44022	48	4	1	1	39.3
48016	46037	46170	48	2	1	1	33.6
48017	46015	44205	48	4	2	2	27.7
48018	46015	44205	48	4	1	2	32.4
48023	44042	41028	48	7	1	2	37.4
48027	41004	46134	48	2	1	1	40
48028	41004	41122	48	7	1	2	28.9
48029	41004	41122	48	7	2	2	23.8
48030	41004	46177	48	2	2	2	31.1
48031	41004	46177	48	2	2	2	30.9
48032	41004	45036	48	3	2	2	29.9

48034	41004	41001	48	7	2	2	32.9
48035	41004	41001	48	7	2	2	32.8
48036	41004	41010	48	7	2	1	39.9
48039	46015	45027	48	3	1	1	32.1
48042	46037	45211	48	3	2	2	27.1
48043	46037	45211	48	3	1	2	30.4
48045	45070	46006	48	2	1	1	47.3
48046	46015	46064	48	2	1	1	42.6
48047	45070	45063	48	3	2	2	35.6
48049	46037	45082	48	3	1	2	20
48050	46037	45082	48	3	2	2	27.2
48051	41004	46048	48	2	2	1	27.7
48052	44042	41074	48	7	1	3	25.6
48054	44042	41074	48	7	2	3	22.5
48056	45070	43033	48	5	2	2	26.2
48057	45070	43033	48	5	2	2	29.6
48058	41004	42060	48	6	2	1	37
48060	45070	44121	48	4	2	1	38.1
48062	46037	46076	48	2	1	1	45.3
48064	46015	46004	48	2	1	2	25.2
48065	46015	46004	48	2	2	2	25.5
48066	44042	45073	48	3	2	2	29.5
48067	44042	45073	48	3	2	2	27.6
48068	46015	46175	48	2	2	2	25
48069	46015	46175	48	2	2	2	26.1
48070	44042	45102	48	3	1	2	27.8
48071	44042	45102	48	3	2	2	29.6
48072	45070	43084	48	5	1	1	39.3
48076	44042	43063	48	5	1	2	25.4
48077	44042	43063	48	5	1	2	25.8
48078	44042	41093	48	7	1	2	23.3
48079	44042	41093	48	7	2	2	25.4
48080	41004	42065	48	6	1	1	45.7
48082	41004	41044	48	7	1	1	43
48084	41004	41057	48	7	1	1	44
48085	46037	45077	48	3	1	1	46.1
48086	46037	45056	48	3	1	1	35.2
48087	44042	43029	48	5	1	2	33.6
48088	44042	43029	48	5	1	2	28.1
48089	44042	44082	48	4	2	1	36.6
48090	46015	44072	48	4	2	1	38.3
48091	46037	45003	48	3	1	1	47.2
48092	45070	46005	48	2	1	1	39.8
48093	46037	43086	48	5	2	1	38.9
48095	44042	44253	48	4	1	2	38.7
48097	45070	43014	48	5	2	1	38.5

48098	46037	44115	48	4	1	1	44.5
48100	44042	41023	48	7	2	2	36.4
48102	44042	44163	48	4	1	2	30.3
48103	44042	44163	48	4	2	2	24.4
48104	45070	43100	48	5	1	1	36
48107	46037	44217	48	4	1	1	35
48110	46037	43089	48	5	1	3	33.2
48111	46037	43089	48	5	1	3	19.7
48112	46037	43089	48	5	2	3	21.8
48113	45070	46209	48	2	1	2	19.3
48114	45070	46209	48	2	2	2	20.7
48116	45070	43026	48	5	1	1	36.8
48117	45070	45094	48	3	2	1	31.1
48118	46015	44098	48	4	2	2	28.8
48119	46015	44098	48	4	1	2	29.2
48120	46015	46008	48	2	1	2	23.3
48121	46015	46008	48	2	2	2	21.7
48122	44042	41105	48	7	1	2	34.7
48123	44042	41105	48	7	1	2	31.1
48124	45070	45156	48	3	2	2	20.9
48125	45070	45156	48	3	1	2	25.7
48126	46015	46072	48	2	2	2	25.2
48127	46015	46072	48	2	1	2	27.6
48129	45070	43020	48	5	2	1	27.8
48130	46037	44064	48	4	1	1	37.5
48131	46037	45015	48	3	2	2	24.2
48132	46037	45015	48	3	2	2	25.5
48133	46015	44228	48	4	1	2	30.9
48134	46015	44228	48	4	2	2	25.3
48136	46037	46047	48	2	1	1	33.2
48137	44042	43072	48	5	1	2	29.6
48138	44042	43072	48	5	1	2	31.5
48139	44042	41083	48	7	2	1	28
48140	46015	44184	48	4	1	1	37.6
48141	46015	44224	48	4	1	2	29.7
48142	46015	44224	48	4	1	2	29
48143	46015	46023	48	2	2	2	24.6
48144	46015	46023	48	2	2	2	23.6
48145	41004	46147	48	2	2	1	36.4
48146	45070	43003	48	5	1	1	32.9
48147	46037	44013	48	4	2	1	33.6
48150	45070	45153	48	3	2	2	30.8
48151	45070	45153	48	3	1	2	28.4
48152	45070	45061	48	3	2	1	28.2
48155	45070	46102	48	2	2	1	30.4
48156	44042	43083	48	5	1	1	37.5

48158	46015	46025	48	2	2	1	32.9
48159	45070	43018	48	5	2	1	23.8
48160	44042	43110	48	5	2	2	26.4
48161	44042	43110	48	5	1	2	33.1
48162	45070	41047	48	7	2	1	31.6
48164	46037	44213	48	4	1	1	39.8
48166	46015	4622	48	2	2	2	23
48167	46015	4622	48	2	2	2	25.2
48170	41004	41043	48	7	1	2	39.7
48171	44042	44252	48	4	2	2	31
48172	44042	44252	48	4	1	2	33.1
48173	46015	46205	48	2	2	1	27.3
48174	45070	46026	48	2	2	1	29.1
48175	45070	45103	48	3	1	1	26.7
48177	46037	45208	48	3	2	1	32.2
48179	44042	44158	48	4	2	1	37.5
48181	44042	41033	48	7	2	1	33.5
48184	45070	43025	48	5	1	1	31.3
48185	46037	45053	48	3	1	1	35.9
48186	46037	44220	48	4	2	1	35
48187	45070	43024	48	5	2	1	33.8
48188	45070	46224	48	2	2	1	33.7
48204	46037	45096	48	3	2	1	22.1
48205	41004	41135	48	7	2	2	30.6
48206	41004	41135	48	7	2	2	28.5
48218	45070	45132	48	3	2	1	35.6
49001	45070	47208	49	2	1	2	34.3
49002	45070	45027	49	4	2	2	28.1
49003	45070	45027	49	4	2	2	27
49004	48014	47076	49	2	2	2	18
49005	46037	43089	49	6	2	2	25.1
49006	46037	43089	49	6	2	2	29
49008	46015	44064	49	5	2	1	31
49011	48014	47022	49	2	2	1	35.6
49012	48014	47148	49	2	1	1	38.8
49013	48014	47092	49	2	2	1	36.2
49014	46015	44217	49	5	2	1	33.7
49017	45070	45102	49	4	2	2	26.6
49018	45070	45102	49	4	2	2	27.4
49019	48052	45103	49	4	2	1	39.8
49020	46037	43086	49	6	2	1	39
49021	46037	44184	49	5	2	1	37.7
49022	45070	44224	49	5	2	1	41.9
49023	46037	44082	49	5	2	1	34
49024	46037	45054	49	4	1	2	36.3
49025	46037	45054	49	4	2	2	29.6

49026	48014	47034	49	2	1	1	41.9
49029	45070	41031	49	8	2	2	17.3
49030	45070	41031	49	8	2	2	23.6
49032	46037	46013	49	3	1	2	30.6
49033	45070	46147	49	3	2	2	39.5
49034	45070	46147	49	3	2	2	28.9
49035	48014	47102	49	2	2	1	34.8
49036	45070	45156	49	4	2	2	29.7
49037	45070	45156	49	4	1	2	16.8
49038	48052	47008	49	2	1	1	39.8
49039	46037	43063	49	6	1	2	28
49040	46037	43063	49	6	1	2	26.5
49041	46037	44098	49	5	2	1	37.2
49042	45070	44072	49	5	2	1	42
49043	45070	45116	49	4	1	2	41.4
49044	45070	45116	49	4	1	2	35.9
49045	48052	47029	49	2	1	1	32.3
49046	48052	47068	49	2	1	2	38.3
49047	45070	46048	49	3	2	2	33.5
49048	45070	46048	49	3	2	2	34.5
49049	46037	44121	49	5	2	1	38.8
49050	46037	47151	49	2	2	1	37.6
49051	48052	41093	49	8	1	2	32.6
49052	45070	41093	49	8	2	2	24
49053	48014	47141	49	2	1	2	50.3
49054	48014	47141	49	2	2	2	22.8
49055	45070	41047	49	8	2	2	36.8
49056	45070	41047	49	8	2	2	33.2
49057	48014	46010	49	3	1	1	39.3
49058	46037	43084	49	6	1	1	44.9
49059	45070	46177	49	3	1	1	45.7
49060	46037	45004	49	4	2	2	38.6
49062	48052	45073	49	4	1	1	37.8
49063	48014	47074	49	2	2	1	35.6
49064	48014	44228	49	5	1	1	42.9
49065	48014	47112	49	2	1	1	43.8
49067	45070	41033	49	8	2	1	37
49068	46015	45115	49	4	1	3	33.8
49069	46015	45115	49	4	2	3	27.1
49070	46015	45115	49	4	2	3	28
49071	48014	47147	49	2	2	2	29.8
49072	48014	47147	49	2	2	2	34.4
49073	48052	47124	49	2	1	2	36.5
49074	48052	47124	49	2	2	2	30.4
49075	45070	46134	49	3	1	1	49.4
49076	48014	46073	49	3	1	1	44.6

49077	46015	46072	49	3	2	1	34.7
49078	45070	44252	49	5	2	2	41
49079	45070	44252	49	5	1	2	36.4
49080	46037	45056	49	4	2	2	32.4
49081	46037	45056	49	4	1	2	37.9
49082	46037	44115	49	5	2	1	43.6
49083	48014	46005	49	3	1	2	35
49084	48014	46005	49	3	1	2	29.2
49085	46037	42065	49	7	2	2	40.8
49088	45070	45063	49	4	1	2	36.8
49089	45070	45063	49	4	2	2	30.5
49090	46015	46175	49	3	2	2	29.3
49091	46015	46175	49	3	1	2	34.4
49092	46037	45055	49	4	2	1	37.2
49094	48014	47043	49	2	2	1	36.8
49095	46015	46004	49	3	2	1	38.8
49097	46037	44022	49	5	1	1	46
49099	45070	46026	49	3	2	2	30.7
49100	45070	46026	49	3	2	2	29.9
49101	46015	45077	49	4	2	1	38.5
49103	48052	44158	49	5	2	1	42
49104	48052	47174	49	2	1	1	43.5
49106	46037	45015	49	4	1	2	33.3
49107	46037	45015	49	4	1	2	33.9
49108	46037	46006	49	3	1	2	44.5
49109	46037	46006	49	3	2	2	30.1
49110	48052	47119	49	2	1	2	35.3
49111	48052	47119	49	2	2	2	27.8
49112	46015	43100	49	6	1	1	32.1
49113	48014	47094	49	2	2	2	26.7
49115	45070	41055	49	8	2	1	45
49117	46015	45211	49	4	2	1	37
49118	46015	43014	49	6	2	1	35.1
49119	48052	47215	49	2	1	1	31.8
49121	46015	43003	49	6	2	1	32.6
49122	46015	43029	49	6	1	1	45
49123	48052	47080	49	2	2	1	32.2
49125	46037	44163	49	5	1	1	41.8
49126	46015	43020	49	6	1	1	36.5
49127	46037	46023	49	3	1	2	37
49128	46037	46023	49	3	2	2	28.6
49129	46015	44220	49	5	2	1	40.5
49132	45070	42060	49	7	1	1	43.1
49134	46037	45082	49	4	1	1	43.9
49135	46015	44013	49	5	2	1	39.1
49136	48014	47056	49	2	2	1	35.5

49139	45070	44253	49	5	1	3	40.5
49141	46015	4622	49	3	1	2	35.1
49142	46015	4622	49	3	2	2	30.9
49143	45070	41083	49	8	2	2	27.4
49144	45070	41083	49	8	1	2	33
49145	46037	45153	49	4	2	2	33.1
49146	46037	45153	49	4	2	2	33.5
49149	46015	46025	49	3	1	2	38.7
49150	46015	46025	49	3	1	2	23.6
49151	48014	46009	49	3	1	2	27.1
49152	48014	46009	49	3	2	2	27.3
49154	46015	43110	49	6	2	1	33.2
49155	46015	44205	49	5	2	1	36.3
49156	48014	47113	49	2	1	2	28.3
49157	48014	47113	49	2	2	2	28.1
49158	48014	47132	49	2	1	2	35.4
49159	48014	47132	49	2	1	2	34.2
49160	46037	45053	49	4	2	2	27.3
49161	46037	45053	49	4	2	2	31.4
49162	45070	41028	49	8	1	2	37.8
49163	45070	41028	49	8	1	2	35.1
49167	45070	41043	49	8	1	2	28.5
49168	45070	41043	49	8	1	2	29.7
49169	46015	46064	49	3	2	1	43.4
49171	46015	43072	49	6	1	1	44.4
49172	46015	47120	49	2	2	1	39.1
49175	48052	47167	49	2	1	1	28
49177	46037	45208	49	4	1	1	45.8
49183	48052	47001	49	2	1	1	38.4
49187	48014	47188	49	2	2	2	26
49188	48014	47188	49	2	2	2	19.2
49193	46037	44213	49	5	1	1	47.1
49196	48014	47014	49	2	1	1	41.6
49197	46015	46205	49	3	2	1	35.7
49198	46037	43033	49	6	1	1	39
49201	46015	47115	49	2	2	1	39
49202	46015	47130	49	2	2	1	32.7
49206	46015	43026	49	6	1	1	45.1
49208	48052	47087	49	2	2	1	35
49212	45070	45132	49	4	2	1	39.7
49218	46037	46224	49	3	1	3	37
49219	46037	46224	49	3	2	3	23.4
49220	46037	46224	49	3	2	3	33
49225	48014	47140	49	2	1	3	40.7
49227	48014	47015	49	2	1	3	26.2
49228	46037	46024	49	6	2	2	33.4

49229	46037	43024	49	6	2	2	34.9
49231	45070	41057	49	8	1	1	45.7
50005	48014	45208	50	5	1	2	21.1
50006	48014	45208	50	5	2	2	32.1
50007	48014	45103	50	5	1	1	36.5
50010	45070	42060	50	8	2	1	33.5
50011	48140	47008	50	3	2	3	22.2
50012	48140	47008	50	3	2	3	25.3
50014	48014	48118	50	2	1	2	21.9
50015	48014	48118	50	2	2	2	25.1
50018	48014	43033	50	7	1	2	25.3
50019	48014	43033	50	7	2	2	22.1
50021	48140	47087	50	3	2	1	20.3
50023	48140	48160	50	2	1	2	29.9
50024	48140	48160	50	2	2	2	25.2
50027	48140	47141	50	3	2	1	28.6
50028	48014	47140	50	3	2	1	34.2
50030	48014	48144	50	2	1	1	33.1
50031	48140	47080	50	3	1	2	29.1
50032	48140	47080	50	3	2	2	21.2
50033	48140	48089	50	2	1	1	29.4
50034	48014	47056	50	3	2	1	31.1
50036	45070	42065	50	8	2	1	31.6
50040	49053	48054	50	2	2	2	36.1
50042	48140	48079	50	2	1	1	38
50043	48014	43089	50	7	1	2	27.4
50044	48014	43089	50	7	2	2	26.5
50045	48014	44064	50	6	2	1	29
50047	48140	47148	50	3	1	2	29.3
50048	48140	47148	50	3	2	2	21.2
50052	48140	43014	50	7	1	1	38.3
50053	48140	47105	50	3	1	1	37.8
50054	45070	44098	50	6	1	1	34.7
50055	48014	45056	50	5	1	1	35.4
50056	46037	46170	50	4	2	1	22.9
50057	46037	44013	50	6	1	2	32.8
50058	46037	44013	50	6	1	2	25.9
50060	46037	45003	50	5	2	1	36.4
50061	48014	48001	50	2	1	2	31.1
50062	48014	48001	50	2	1	2	27.6
50063	48014	45094	50	5	2	1	37.1
50065	45070	45132	50	5	2	1	37.1
50066	46037	46006	50	4	2	2	31.9
50067	46037	46006	50	4	2	2	28.1
50068	46037	45116	50	5	2	1	31.6
50069	46037	45077	50	5	2	1	33.3

50070	48140	47130	50	3	2	1	34.1
50072	49053	48165	50	2	2	2	20.5
50074	45070	44205	50	6	2	1	34.1
50075	45070	44072	50	6	2	1	35.7
50076	46037	44022	50	6	2	1	30.6
50077	48014	48068	50	2	2	1	28.7
50078	48052	44217	50	6	2	1	30.4
50079	49053	48058	50	2	1	1	36.7
50080	48052	48060	50	2	1	1	34.5
50081	48014	46205	50	4	2	1	34.4
50083	45070	47077	50	3	2	1	15.8
50084	46037	43018	50	7	2	1	26.3
50087	48014	48134	50	2	2	1	30.2
50088	49134	48036	50	2	1	2	33.6
50089	49134	48036	50	2	2	2	24.5
50090	48014	47097	50	3	2	2	23.4
50091	48014	47097	50	3	2	2	15.4
50093	46037	46004	50	4	1	2	30.4
50094	46037	46004	50	4	1	2	35.6
50095	48052	48047	50	2	2	1	32.6
50096	48014	48143	50	2	2	2	29.1
50098	45070	47112	50	3	1	1	37.2
50099	48014	47132	50	3	1	2	31.8
50100	48014	47132	50	3	2	2	30.6
50101	48140	48171	50	2	1	2	20.9
50102	48140	48171	50	2	2	2	24.9
50103	48052	48162	50	2	2	2	21.2
50104	48052	48162	50	2	2	2	22.5
50105	48014	47043	50	3	2	1	28.2
50106	48140	43110	50	7	2	1	18.9
50108	46037	47215	50	3	2	1	31.1
50109	49053	48158	50	2	1	2	29.6
50110	49053	48158	50	2	2	2	23.1
50111	48014	4622	50	4	1	2	30.5
50112	48014	4622	50	4	2	2	32.9
50113	49053	48166	50	2	1	2	33.3
50114	49053	48166	50	2	2	2	26.7
50115	46037	45015	50	5	2	1	32
50116	48140	47113	50	3	2	1	24.7
50117	46037	46064	50	4	2	1	37.3
50118	48014	46009	50	4	2	1	34.5
50119	49134	48030	50	2	1	1	40
50120	48014	48090	50	2	1	2	21.7
50121	48014	48090	50	2	2	2	28.3
50122	46037	45153	50	5	2	2	29.7
50123	46037	45153	50	5	2	2	23.1

50124	48052	46026	50	4	1	2	31.7
50125	48052	46026	50	4	2	2	27.6
50128	46037	43003	50	7	2	2	23
50129	46037	46073	50	4	2	2	20.7
50130	46037	46073	50	4	1	2	25.4
50131	46037	46025	50	4	1	1	36.1
50132	48140	44224	50	6	2	1	33.5
50133	48052	48002	50	2	1	1	31.6
50134	46037	43086	50	7	1	2	32.2
50135	46037	43086	50	7	1	2	36.5
50138	48014	48069	50	2	2	1	25.6
50139	49134	48179	50	2	2	1	40.8
50140	48140	47115	50	3	2	2	35.9
50142	46037	45211	50	5	2	1	30
50143	49134	48003	50	2	1	2	28.1
50144	49134	48003	50	2	2	2	21.1
50145	48014	48121	50	2	1	2	28.1
50146	48014	48121	50	2	2	2	25.6
50149	48140	48139	50	2	2	1	25.9
50150	46037	44121	50	6	2	1	32.4
50151	49053	48051	50	2	1	1	34.1
50152	46037	44163	50	6	2	2	20.5
50153	46037	44163	50	6	2	2	19.6
50154	48052	48017	50	2	2	2	17.2
50155	48052	48017	50	2	2	2	21.6
50158	46037	44082	50	6	1	2	28.7
50159	46037	44083	50	6	2	2	23.9
50160	49046	48050	50	2	2	1	34.4
50163	48140	48100	50	2	2	1	35.3
50164	48052	46048	50	4	1	2	38.7
50167	46037	46175	50	4	2	1	31.1
50168	46037	47119	50	3	2	1	28.9
50169	48052	48056	50	2	2	2	26.7
50170	48052	48056	50	2	2	2	23.1
50171	48140	47014	50	3	2	1	30
50172	48140	48067	50	2	2	1	35.1
50173	48140	46224	50	4	2	1	37.4
50174	48140	48177	50	2	1	1	32.8
50175	49046	48167	50	2	2	2	27.6
50176	49046	48167	50	2	2	2	25.7
50178	48052	44228	50	6	1	2	33
50179	48052	44228	50	6	2	2	32.9
50185	49046	48173	50	2	1	1	34.2
50186	48052	44115	50	6	2	1	38.6
50188	48052	47022	50	3	2	1	28.3
50191	46037	43026	50	7	1	1	36.7

50192	48052	47174	50	3	2	2	25.1
50193	48052	47174	50	3	2	2	31.9
50194	48014	44184	50	6	2	2	26.9
50195	48014	44184	50	6	2	2	25.7
50196	48052	43024	50	7	1	1	38.9
50197	48052	48187	50	2	2	1	27.3
50200	48052	44213	50	6	1	1	50.6
50201	48052	43072	50	7	1	1	42.7
50202	46037	41028	50	9	1	2	36.6
50203	46037	41028	50	9	1	2	30.1

APPENDIX C

The example used for illustrative purposes are based on an experiment undertaken at the International Livestock Research Institute (ILRI) at the University of Nairobi, Kenya in the early 90's (Duchateau, *et al.*, 1998). **The goal of the research was to select for improved helminth resistance in sheep.**

The female sheep used in the experiment are from three different breeds (br), whereas the males are from two breeds. In each of the six crosses, there are at least 25 and at most 42 different sires, and each sire within a crossbreed has on average offspring of 6.4 lambs. The weaning weight is measured for each lamb (YWW).

Although the same sire is mated to ewes from different breeds, the sire nested in breed is taken as a single random effect and it is assumed that these random effects are independent. A total of n = 1277 weaning weight records, from the progeny of q = 200 sires are available after editing, and *breed*, *sex* and *age* are included as fixed effects in the final model.

								Br1	Br2	Br3	Br4	Br5	Br6	Age
Breed	**Sire_ID**	**Sex(B6)**	**Age**	**Y(WW)**	**Sex**	**Sire_Nr**	**B0**	**B1**	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**
1	1971	F	145	11.2	0	1	1	1	0	0	0	0	0	145
1	1971	F	140	15.4	0	1	1	1	0	0	0	0	0	140
1	1971	F	140	10.9	0	1	1	1	0	0	0	0	0	140
1	1971	M	122	11.4	1	1	1	1	0	0	0	0	1	122
1	1972	F	152	16	0	2	1	1	0	0	0	0	0	152
1	1972	M	151	13.2	1	2	1	1	0	0	0	0	1	151
1	1972	F	146	14.9	0	2	1	1	0	0	0	0	0	146
1	1972	F	139	7.9	0	2	1	1	0	0	0	0	0	139
1	1972	M	132	15.7	1	2	1	1	0	0	0	0	1	132
1	1972	F	131	13.1	0	2	1	1	0	0	0	0	0	131
1	1972	F	128	12.5	0	2	1	1	0	0	0	0	0	128
1	1972	M	124	12.2	1	2	1	1	0	0	0	0	1	124
1	1972	M	122	14.6	1	2	1	1	0	0	0	0	1	122
1	1972	F	115	13.6	0	2	1	1	0	0	0	0	0	115
1	1972	M	110	12.6	1	2	1	1	0	0	0	0	1	110
1	1973	F	157	11.9	0	3	1	1	0	0	0	0	0	157
1	1973	M	157	20	1	3	1	1	0	0	0	0	1	157
1	1973	M	156	17.1	1	3	1	1	0	0	0	0	1	156
1	1973	M	143	10.8	1	3	1	1	0	0	0	0	1	143
1	1973	M	136	12.5	1	3	1	1	0	0	0	0	1	136
1	1974	M	159	15.8	1	4	1	1	0	0	0	0	1	159
1	1974	F	156	12.8	0	4	1	1	0	0	0	0	0	156
1	1974	F	156	20.2	0	4	1	1	0	0	0	0	0	156
1	1974	M	155	18.7	1	4	1	1	0	0	0	0	1	155
1	1974	M	144	18.5	1	4	1	1	0	0	0	0	1	144
1	1974	M	143	18.8	1	4	1	1	0	0	0	0	1	143
1	1974	F	136	11.1	0	4	1	1	0	0	0	0	0	136
1	1974	M	135	16.8	1	4	1	1	0	0	0	0	1	135
1	1974	M	134	17.5	1	4	1	1	0	0	0	0	1	134
1	1974	F	126	14.2	0	4	1	1	0	0	0	0	0	126
1	1980	M	175	18.5	1	5	1	1	0	0	0	0	1	175
1	1980	M	155	17.9	1	5	1	1	0	0	0	0	1	155
1	1980	M	151	12.7	1	5	1	1	0	0	0	0	1	151
1	1980	F	148	13.4	0	5	1	1	0	0	0	0	0	148
1	1980	M	145	12.7	1	5	1	1	0	0	0	0	1	145
1	1980	M	141	19	1	5	1	1	0	0	0	0	1	141
1	1980	M	124	13.4	1	5	1	1	0	0	0	0	1	124
1	1991	F	134	10.6	0	6	1	1	0	0	0	0	0	134
1	1991	M	128	13.1	1	6	1	1	0	0	0	0	1	128
1	1991	F	121	14.3	0	6	1	1	0	0	0	0	0	121
1	1991	M	112	15.6	1	6	1	1	0	0	0	0	1	112
1	1991	M	98	10.4	1	6	1	1	0	0	0	0	1	98

1	1999	M	149	14	1	7	1	1	0	0	0	0	1	149
1	1999	F	144	18.3	0	7	1	1	0	0	0	0	0	144
1	1999	M	141	17.3	1	7	1	1	0	0	0	0	1	141
1	1999	M	134	13.2	1	7	1	1	0	0	0	0	1	134
1	4907	M	155	18	1	8	1	1	0	0	0	0	1	155
1	4907	F	153	13.1	0	8	1	1	0	0	0	0	0	153
1	4907	F	152	13.1	0	8	1	1	0	0	0	0	0	152
1	4907	M	152	9.5	1	8	1	1	0	0	0	0	1	152
1	4907	M	135	13.8	1	8	1	1	0	0	0	0	1	135
1	4907	M	132	15.7	1	8	1	1	0	0	0	0	1	132
1	4907	M	129	12.6	1	8	1	1	0	0	0	0	1	129
1	4907	M	115	12	1	8	1	1	0	0	0	0	1	115
1	4908	F	162	19	0	9	1	1	0	0	0	0	0	162
1	4908	M	148	15.3	1	9	1	1	0	0	0	0	1	148
1	4908	M	147	18.8	1	9	1	1	0	0	0	0	1	147
1	4908	F	139	15.1	0	9	1	1	0	0	0	0	0	139
1	4908	M	134	16.1	1	9	1	1	0	0	0	0	1	134
1	4908	F	118	13.2	0	9	1	1	0	0	0	0	0	118
1	4908	M	132	14.2	1	9	1	1	0	0	0	0	1	132
1	4908	F	126	13.5	0	9	1	1	0	0	0	0	0	126
1	4908	M	125	13.8	1	9	1	1	0	0	0	0	1	125
1	4908	F	125	14.2	0	9	1	1	0	0	0	0	0	125
1	4908	F	122	18.9	0	9	1	1	0	0	0	0	0	122
1	4908	F	122	12.7	0	9	1	1	0	0	0	0	0	122
1	4908	M	94	11.4	1	9	1	1	0	0	0	0	1	94
1	4908	F	169	12.8	0	9	1	1	0	0	0	0	0	169
1	4908	M	156	16.4	1	9	1	1	0	0	0	0	1	156
1	4909	F	145	6.3	0	10	1	1	0	0	0	0	0	145
1	4909	M	140	11.5	1	10	1	1	0	0	0	0	1	140
1	4909	F	124	13.2	0	10	1	1	0	0	0	0	0	124
1	4909	F	117	9.6	0	10	1	1	0	0	0	0	0	117
1	4910	F	157	12.1	0	11	1	1	0	0	0	0	0	157
1	4910	F	154	13	0	11	1	1	0	0	0	0	0	154
1	4910	M	150	13.2	1	11	1	1	0	0	0	0	1	150
1	4910	F	143	19.1	0	11	1	1	0	0	0	0	0	143
1	4910	F	121	12.3	0	11	1	1	0	0	0	0	0	121
1	4910	F	129	9.3	0	11	1	1	0	0	0	0	0	129
1	4910	F	129	14.3	0	11	1	1	0	0	0	0	0	129
1	4910	F	127	15	0	11	1	1	0	0	0	0	0	127
1	4910	M	121	11.4	1	11	1	1	0	0	0	0	1	121
1	4910	F	115	12.4	0	11	1	1	0	0	0	0	0	115
1	4910	M	102	14.3	1	11	1	1	0	0	0	0	1	102
1	4910	F	84	10.2	0	11	1	1	0	0	0	0	0	84
1	4910	M	97	9.1	1	11	1	1	0	0	0	0	1	97
1	4911	M	158	17.3	1	12	1	1	0	0	0	0	1	158
1	4911	F	156	17.3	0	12	1	1	0	0	0	0	0	156

1	4911	M	155	12.4	1	12	1	1	0	0	0	0	1	155
1	4911	M	147	11.5	1	12	1	1	0	0	0	0	1	147
1	4911	M	147	16.5	1	12	1	1	0	0	0	0	1	147
1	4911	M	144	16.9	1	12	1	1	0	0	0	0	1	144
1	4911	F	138	12	0	12	1	1	0	0	0	0	0	138
1	4911	M	137	13.3	1	12	1	1	0	0	0	0	1	137
1	4912	F	155	17	0	13	1	1	0	0	0	0	0	155
1	4912	M	144	10.8	1	13	1	1	0	0	0	0	1	144
1	4912	F	140	8.7	0	13	1	1	0	0	0	0	0	140
1	4912	F	137	11.3	0	13	1	1	0	0	0	0	0	137
1	4915	F	128	10.1	0	14	1	1	0	0	0	0	0	128
1	4915	F	121	14.2	0	14	1	1	0	0	0	0	0	121
1	4915	M	113	15.6	1	14	1	1	0	0	0	0	1	113
1	4915	M	101	9.6	1	14	1	1	0	0	0	0	1	101
1	4916	F	112	10.4	0	15	1	1	0	0	0	0	0	112
1	4916	M	98	10.2	1	15	1	1	0	0	0	0	1	98
1	4916	M	117	10.1	1	15	1	1	0	0	0	0	1	117
1	4916	F	109	13.2	0	15	1	1	0	0	0	0	0	109
1	5001	F	119	7	0	16	1	1	0	0	0	0	0	119
1	5001	F	172	15	0	16	1	1	0	0	0	0	0	172
1	5002	M	113	13.4	1	17	1	1	0	0	0	0	1	113
1	5002	F	107	11.7	0	17	1	1	0	0	0	0	0	107
1	5002	F	100	7.9	0	17	1	1	0	0	0	0	0	100
1	5002	F	99	10.5	0	17	1	1	0	0	0	0	0	99
1	5002	M	152	9.2	1	17	1	1	0	0	0	0	1	152
1	5003	M	134	15.1	1	18	1	1	0	0	0	0	1	134
1	5003	F	127	12.4	0	18	1	1	0	0	0	0	0	127
1	5004	M	111	15.5	1	19	1	1	0	0	0	0	1	111
1	5004	M	110	13.2	1	19	1	1	0	0	0	0	1	110
1	5004	F	108	10.4	0	19	1	1	0	0	0	0	0	108
1	5004	F	107	10.7	0	19	1	1	0	0	0	0	0	107
1	5004	F	106	14.8	0	19	1	1	0	0	0	0	0	106
1	5005	M	130	13.7	1	20	1	1	0	0	0	0	1	130
1	5005	M	104	11.6	1	20	1	1	0	0	0	0	1	104
1	5005	F	97	9.5	0	20	1	1	0	0	0	0	0	97
1	5005	F	88	15.7	0	20	1	1	0	0	0	0	0	88
1	5005	F	163	9.3	0	20	1	1	0	0	0	0	0	163
1	5007	M	173	11.1	1	21	1	1	0	0	0	0	1	173
1	5007	F	166	12.5	0	21	1	1	0	0	0	0	0	166
1	5007	M	163	16.5	1	21	1	1	0	0	0	0	1	163
1	5007	F	121	6.7	0	21	1	1	0	0	0	0	0	121
1	5007	M	115	10.1	1	21	1	1	0	0	0	0	1	115
1	5008	F	175	13.8	0	22	1	1	0	0	0	0	0	175
1	5008	F	158	10.8	0	22	1	1	0	0	0	0	0	158
1	5008	M	150	8.1	1	22	1	1	0	0	0	0	1	150
1	5008	M	111	14.7	1	22	1	1	0	0	0	0	1	111

1	5009	M	115	13.4	1	23	1	1	0	0	0	0	1	115
1	5009	M	110	9.8	1	23	1	1	0	0	0	0	1	110
1	5010	F	169	13.5	0	24	1	1	0	0	0	0	0	169
1	5010	M	143	15.6	1	24	1	1	0	0	0	0	1	143
1	5010	F	140	11.6	0	24	1	1	0	0	0	0	0	140
1	5011	M	175	10.5	1	25	1	1	0	0	0	0	1	175
1	5011	F	166	11.9	0	25	1	1	0	0	0	0	0	166
1	5011	F	112	10.9	0	25	1	1	0	0	0	0	0	112
1	5012	M	168	17	1	26	1	1	0	0	0	0	1	168
1	5012	M	156	12.6	1	26	1	1	0	0	0	0	1	156
1	5013	M	172	13.7	1	27	1	1	0	0	0	0	1	172
1	5013	F	168	13.5	0	27	1	1	0	0	0	0	0	168
1	5013	F	157	11.5	0	27	1	1	0	0	0	0	0	157
1	5071	F	116	9.7	0	28	1	1	0	0	0	0	0	116
1	5071	F	110	12.7	0	28	1	1	0	0	0	0	0	110
1	5071	M	133	16	1	28	1	1	0	0	0	0	1	133
1	5071	F	132	13.1	0	28	1	1	0	0	0	0	0	132
1	5073	M	90	8.9	1	29	1	1	0	0	0	0	1	90
1	5076	M	117	11	1	30	1	1	0	0	0	0	1	117
1	5205	F	108	7.5	0	31	1	1	0	0	0	0	0	108
1	5324	F	136	6.2	0	32	1	1	0	0	0	0	0	136
1	5326	F	93	8.5	0	33	1	1	0	0	0	0	0	93
1	5326	M	135	10.2	1	33	1	1	0	0	0	0	1	135
1	5328	M	126	13.9	1	34	1	1	0	0	0	0	1	126
1	5328	F	121	10.7	0	34	1	1	0	0	0	0	0	121
1	5328	M	118	7.4	1	34	1	1	0	0	0	0	1	118
1	5328	F	138	14	0	34	1	1	0	0	0	0	0	138
1	5328	M	135	12.8	1	34	1	1	0	0	0	0	1	135
1	5329	F	122	12.4	0	35	1	1	0	0	0	0	0	122
1	5329	F	98	6.2	0	35	1	1	0	0	0	0	0	98
1	5329	M	137	12.6	1	35	1	1	0	0	0	0	1	137
1	5329	F	132	11.3	0	35	1	1	0	0	0	0	0	132
1	5329	M	132	9.1	1	35	1	1	0	0	0	0	1	132
1	5329	F	128	13	0	35	1	1	0	0	0	0	0	128
1	5330	F	122	8.8	0	36	1	1	0	0	0	0	0	122
1	5330	M	141	8.3	1	36	1	1	0	0	0	0	1	141
1	5330	F	139	7.9	0	36	1	1	0	0	0	0	0	139
1	5330	F	131	8.3	0	36	1	1	0	0	0	0	0	131
1	5330	M	115	10.5	1	36	1	1	0	0	0	0	1	115
1	5337	F	129	10.3	0	37	1	1	0	0	0	0	0	129
1	5337	M	127	12.5	1	37	1	1	0	0	0	0	1	127
1	5337	M	116	10.9	1	37	1	1	0	0	0	0	1	116
1	5337	F	115	7.8	0	37	1	1	0	0	0	0	0	115
1	5338	M	121	9.2	1	38	1	1	0	0	0	0	1	121
1	5338	F	139	11	0	38	1	1	0	0	0	0	0	139
1	5338	M	112	10.1	1	38	1	1	0	0	0	0	1	112

2	1975	F	147	11.2	0	39	1	0	1	0	0	0	0	147
2	1975	M	145	12.8	1	39	1	0	1	0	0	0	1	145
2	1975	M	136	12.5	1	39	1	0	1	0	0	0	1	136
2	1975	M	134	12.4	1	39	1	0	1	0	0	0	1	134
2	1975	M	133	16.6	1	39	1	0	1	0	0	0	1	133
2	1975	F	126	8.9	0	39	1	0	1	0	0	0	0	126
2	1976	M	154	15.5	1	40	1	0	1	0	0	0	1	154
2	1976	F	147	9.5	0	40	1	0	1	0	0	0	0	147
2	1976	F	142	15.3	0	40	1	0	1	0	0	0	0	142
2	1976	M	139	16.2	1	40	1	0	1	0	0	0	1	139
2	1976	F	123	11.7	0	40	1	0	1	0	0	0	0	123
2	1979	M	153	16.3	1	41	1	0	1	0	0	0	1	153
2	1979	F	153	15.9	0	41	1	0	1	0	0	0	0	153
2	1979	F	152	18.7	0	41	1	0	1	0	0	0	0	152
2	1979	F	148	15.7	0	41	1	0	1	0	0	0	0	148
2	1979	F	146	13.1	0	41	1	0	1	0	0	0	0	146
2	1979	M	146	15	1	41	1	0	1	0	0	0	1	146
2	1979	F	132	12.5	0	41	1	0	1	0	0	0	0	132
2	1979	F	120	11	0	41	1	0	1	0	0	0	0	120
2	1979	M	111	10.3	1	41	1	0	1	0	0	0	1	111
2	1979	M	109	10.8	1	41	1	0	1	0	0	0	1	109
2	1981	M	157	15.6	1	42	1	0	1	0	0	0	1	157
2	1981	F	153	16.2	0	42	1	0	1	0	0	0	0	153
2	1981	F	153	11.6	0	42	1	0	1	0	0	0	0	153
2	1981	M	140	11	1	42	1	0	1	0	0	0	1	140
2	1981	M	136	12.2	1	42	1	0	1	0	0	0	1	136
2	1981	M	135	11.9	1	42	1	0	1	0	0	0	1	135
2	1981	M	131	12.5	1	42	1	0	1	0	0	0	1	131
2	1981	M	117	14.5	1	42	1	0	1	0	0	0	1	117
2	1982	M	130	8.1	1	43	1	0	1	0	0	0	1	130
2	1982	M	125	14.6	1	43	1	0	1	0	0	0	1	125
2	1982	M	118	10.6	1	43	1	0	1	0	0	0	1	118
2	1982	F	84	10.1	0	43	1	0	1	0	0	0	0	84
2	1983	M	156	18.1	1	44	1	0	1	0	0	0	1	156
2	1983	F	156	17.1	0	44	1	0	1	0	0	0	0	156
2	1983	F	153	15.9	0	44	1	0	1	0	0	0	0	153
2	1983	F	152	15	0	44	1	0	1	0	0	0	0	152
2	1983	F	147	17.2	0	44	1	0	1	0	0	0	0	147
2	1984	F	151	16.9	0	45	1	0	1	0	0	0	0	151
2	1984	F	150	12.1	0	45	1	0	1	0	0	0	0	150
2	1984	F	147	8.2	0	45	1	0	1	0	0	0	0	147
2	1984	M	145	15.2	1	45	1	0	1	0	0	0	1	145
2	1984	M	141	15.3	1	45	1	0	1	0	0	0	1	141
2	1984	F	135	13.5	0	45	1	0	1	0	0	0	0	135
2	1986	M	124	15.7	1	46	1	0	1	0	0	0	1	124
2	1986	M	121	9.9	1	46	1	0	1	0	0	0	1	121

2	1988	F	145	13.4	0	47	1	0	1	0	0	0	0	145
2	1988	F	141	16.9	0	47	1	0	1	0	0	0	0	141
2	1988	M	139	17.1	1	47	1	0	1	0	0	0	1	139
2	1988	F	137	13.2	0	47	1	0	1	0	0	0	0	137
2	4901	F	153	17.2	0	48	1	0	1	0	0	0	0	153
2	4901	F	139	15.7	0	48	1	0	1	0	0	0	0	139
2	4901	F	139	10.8	0	48	1	0	1	0	0	0	0	139
2	4901	M	139	9.8	1	48	1	0	1	0	0	0	1	139
2	4901	F	135	13.5	0	48	1	0	1	0	0	0	0	135
2	4901	M	131	13.3	1	48	1	0	1	0	0	0	1	131
2	4901	M	123	15.8	1	48	1	0	1	0	0	0	1	123
2	4901	M	122	14.8	1	48	1	0	1	0	0	0	1	122
2	4901	F	120	10.7	0	48	1	0	1	0	0	0	0	120
2	4901	M	81	11.2	1	48	1	0	1	0	0	0	1	81
2	4902	M	148	17.5	1	49	1	0	1	0	0	0	1	148
2	4902	M	147	14.5	1	49	1	0	1	0	0	0	1	147
2	4902	M	146	17.1	1	49	1	0	1	0	0	0	1	146
2	4902	M	145	14.4	1	49	1	0	1	0	0	0	1	145
2	4902	M	139	8.8	1	49	1	0	1	0	0	0	1	139
2	4902	M	139	15.5	1	49	1	0	1	0	0	0	1	139
2	4902	M	132	14.3	1	49	1	0	1	0	0	0	1	132
2	4902	M	125	9.3	1	49	1	0	1	0	0	0	1	125
2	4902	F	119	12	0	49	1	0	1	0	0	0	0	119
2	4902	M	116	10.4	1	49	1	0	1	0	0	0	1	116
2	4903	F	150	15.4	0	50	1	0	1	0	0	0	0	150
2	4903	F	146	12.1	0	50	1	0	1	0	0	0	0	146
2	4903	M	144	15.5	1	50	1	0	1	0	0	0	1	144
2	4903	F	141	16.8	0	50	1	0	1	0	0	0	0	141
2	4903	F	137	13.9	0	50	1	0	1	0	0	0	0	137
2	4903	F	135	12.7	0	50	1	0	1	0	0	0	0	135
2	4903	M	125	13	1	50	1	0	1	0	0	0	1	125
2	4903	M	115	8.5	1	50	1	0	1	0	0	0	1	115
2	4903	M	114	7.1	1	50	1	0	1	0	0	0	1	114
2	4903	F	113	8.5	0	50	1	0	1	0	0	0	0	113
2	4903	M	107	12.2	1	50	1	0	1	0	0	0	1	107
2	4903	M	100	12.3	1	50	1	0	1	0	0	0	1	100
2	4905	M	155	17.2	1	51	1	0	1	0	0	0	1	155
2	4905	M	149	16.4	1	51	1	0	1	0	0	0	1	149
2	4905	M	143	10.1	1	51	1	0	1	0	0	0	1	143
2	4905	M	138	10.2	1	51	1	0	1	0	0	0	1	138
2	4905	M	134	15.3	1	51	1	0	1	0	0	0	1	134
2	4905	F	125	10.9	0	51	1	0	1	0	0	0	0	125
2	4905	M	127	13.1	1	51	1	0	1	0	0	0	1	127
2	4905	M	121	13.7	1	51	1	0	1	0	0	0	1	121
2	4905	M	117	7.6	1	51	1	0	1	0	0	0	1	117
2	4905	F	117	14	0	51	1	0	1	0	0	0	0	117

2	4905	F	114	15.2	0	51	1	0	1	0	0	0	0	114
2	4905	F	161	13	0	51	1	0	1	0	0	0	0	161
2	4905	F	150	10	0	51	1	0	1	0	0	0	0	150
2	4905	F	149	12.1	0	51	1	0	1	0	0	0	0	149
2	4905	F	146	8.3	0	51	1	0	1	0	0	0	0	146
2	4906	F	145	12	0	52	1	0	1	0	0	0	0	145
2	4906	M	143	16.2	1	52	1	0	1	0	0	0	1	143
2	4906	F	140	13.7	0	52	1	0	1	0	0	0	0	140
2	4906	F	138	15.9	0	52	1	0	1	0	0	0	0	138
2	4906	M	115	11.9	1	52	1	0	1	0	0	0	1	115
2	4906	M	112	13.8	1	52	1	0	1	0	0	0	1	112
2	4906	F	117	10.7	0	52	1	0	1	0	0	0	0	117
2	4906	M	107	14.2	1	52	1	0	1	0	0	0	1	107
2	4906	M	96	12.2	1	52	1	0	1	0	0	0	1	96
2	4906	F	163	14.7	0	52	1	0	1	0	0	0	0	163
2	4906	M	161	11.5	1	52	1	0	1	0	0	0	1	161
2	4906	F	151	11.7	0	52	1	0	1	0	0	0	0	151
2	4906	F	148	13	0	52	1	0	1	0	0	0	0	148
2	4906	M	146	15.4	1	52	1	0	1	0	0	0	1	146
2	4913	F	130	14.1	0	53	1	0	1	0	0	0	0	130
2	4913	F	120	11.7	0	53	1	0	1	0	0	0	0	120
2	4913	F	117	11.6	0	53	1	0	1	0	0	0	0	117
2	4913	F	111	10.3	0	53	1	0	1	0	0	0	0	111
2	4914	M	129	14.3	1	54	1	0	1	0	0	0	1	129
2	4914	F	126	6.8	0	54	1	0	1	0	0	0	0	126
2	4914	F	117	10.5	0	54	1	0	1	0	0	0	0	117
2	4914	M	168	15.1	1	54	1	0	1	0	0	0	1	168
2	4914	F	164	10.5	0	54	1	0	1	0	0	0	0	164
2	4914	F	151	7	0	54	1	0	1	0	0	0	0	151
2	4914	F	150	10.1	0	54	1	0	1	0	0	0	0	150
2	4914	F	122	10.1	0	54	1	0	1	0	0	0	0	122
2	4918	M	129	18.8	1	55	1	0	1	0	0	0	1	129
2	4918	M	110	11.2	1	55	1	0	1	0	0	0	1	110
2	4918	M	107	12.8	1	55	1	0	1	0	0	0	1	107
2	4919	M	170	10.6	1	56	1	0	1	0	0	0	1	170
2	4919	M	168	12.2	1	56	1	0	1	0	0	0	1	168
2	4919	F	165	10.5	0	56	1	0	1	0	0	0	0	165
2	4919	F	161	11.7	0	56	1	0	1	0	0	0	0	161
2	4921	M	118	9.7	1	57	1	0	1	0	0	0	1	118
2	4921	F	113	13.6	0	57	1	0	1	0	0	0	0	113
2	4921	F	112	9	0	57	1	0	1	0	0	0	0	112
2	4921	M	108	7.3	1	57	1	0	1	0	0	0	1	108
2	4921	M	97	13.5	1	57	1	0	1	0	0	0	1	97
2	4921	M	168	12	1	57	1	0	1	0	0	0	1	168
2	4921	F	159	13.9	0	57	1	0	1	0	0	0	0	159
2	4921	F	159	10.3	0	57	1	0	1	0	0	0	0	159

Appendix C: Dataset from Duchateau, et al., 1998; International Livestock Research Institute (ILRI), Kenya

2	4921	M	156	11.3	1	57	1	0	1	0	0	0	1	156
2	4923	M	126	11.7	1	58	1	0	1	0	0	0	1	126
2	4923	M	113	14.5	1	58	1	0	1	0	0	0	1	113
2	4923	M	106	12.7	1	58	1	0	1	0	0	0	1	106
2	4923	F	102	10.7	0	58	1	0	1	0	0	0	0	102
2	4923	M	168	11.8	1	58	1	0	1	0	0	0	1	168
2	4923	M	163	14.5	1	58	1	0	1	0	0	0	1	163
2	4923	M	159	10.6	1	58	1	0	1	0	0	0	1	159
2	5015	M	167	14	1	59	1	0	1	0	0	0	1	167
2	5015	M	166	14.8	1	59	1	0	1	0	0	0	1	166
2	5015	F	161	12.1	0	59	1	0	1	0	0	0	0	161
2	5015	M	139	12.6	1	59	1	0	1	0	0	0	1	139
2	5015	M	128	14.9	1	59	1	0	1	0	0	0	1	128
2	5016	F	175	11.6	0	60	1	0	1	0	0	0	0	175
2	5016	M	161	11.1	1	60	1	0	1	0	0	0	1	161
2	5016	M	160	13	1	60	1	0	1	0	0	0	1	160
2	5016	M	156	12	1	60	1	0	1	0	0	0	1	156
2	5016	F	147	13.2	0	60	1	0	1	0	0	0	0	147
2	5016	M	102	7.4	1	60	1	0	1	0	0	0	1	102
2	5017	F	171	11.1	0	61	1	0	1	0	0	0	0	171
2	5017	M	162	16.6	1	61	1	0	1	0	0	0	1	162
2	5017	F	161	12	0	61	1	0	1	0	0	0	0	161
2	5017	M	156	10.4	1	61	1	0	1	0	0	0	1	156
2	5017	F	117	6.1	0	61	1	0	1	0	0	0	0	117
2	5017	F	126	6.5	0	61	1	0	1	0	0	0	0	126
2	5018	M	166	12.3	1	62	1	0	1	0	0	0	1	166
2	5018	F	113	11.1	0	62	1	0	1	0	0	0	0	113
2	5019	F	168	17.2	0	63	1	0	1	0	0	0	0	168
2	5019	M	162	13.6	1	63	1	0	1	0	0	0	1	162
2	5019	F	159	10.5	0	63	1	0	1	0	0	0	0	159
2	5019	M	159	9.2	1	63	1	0	1	0	0	0	1	159
2	5019	F	134	11.9	0	63	1	0	1	0	0	0	0	134
2	5019	M	118	8.7	1	63	1	0	1	0	0	0	1	118
2	5019	F	98	11.3	0	63	1	0	1	0	0	0	0	98
2	5020	F	164	12.5	0	64	1	0	1	0	0	0	0	164
2	5020	M	161	16.9	1	64	1	0	1	0	0	0	1	161
2	5020	M	159	10.5	1	64	1	0	1	0	0	0	1	159
2	5020	M	154	13.3	1	64	1	0	1	0	0	0	1	154
2	5020	F	132	11.4	0	64	1	0	1	0	0	0	0	132
2	5204	F	109	9	0	65	1	0	1	0	0	0	0	109
2	5207	F	121	10.2	0	66	1	0	1	0	0	0	0	121
2	5331	F	121	11.3	0	67	1	0	1	0	0	0	0	121
2	5334	F	125	9.3	0	68	1	0	1	0	0	0	0	125
2	5336	F	129	12.8	0	69	1	0	1	0	0	0	0	129
2	5336	M	122	8.9	1	69	1	0	1	0	0	0	1	122
3	1971	M	160	14.5	1	70	1	0	0	1	0	0	1	160

3	1971	F	159	12.1	0	70	1	0	0	1	0	0	0	159
3	1971	M	154	13.9	1	70	1	0	0	1	0	0	1	154
3	1971	F	151	10.5	0	70	1	0	0	1	0	0	0	151
3	1971	F	148	11.7	0	70	1	0	0	1	0	0	0	148
3	1971	F	148	12.6	0	70	1	0	0	1	0	0	0	148
3	1971	M	147	8.7	1	70	1	0	0	1	0	0	1	147
3	1971	F	144	6.5	0	70	1	0	0	1	0	0	0	144
3	1971	M	139	17.2	1	70	1	0	0	1	0	0	1	139
3	1972	M	152	18.6	1	71	1	0	0	1	0	0	1	152
3	1972	M	149	15.2	1	71	1	0	0	1	0	0	1	149
3	1972	F	149	16	0	71	1	0	0	1	0	0	0	149
3	1972	M	148	16.1	1	71	1	0	0	1	0	0	1	148
3	1972	F	145	8.3	0	71	1	0	0	1	0	0	0	145
3	1972	M	143	15.7	1	71	1	0	0	1	0	0	1	143
3	1972	F	126	16.7	0	71	1	0	0	1	0	0	0	126
3	1972	F	122	10.5	0	71	1	0	0	1	0	0	0	122
3	1972	F	122	18.1	0	71	1	0	0	1	0	0	0	122
3	1972	F	119	8.8	0	71	1	0	0	1	0	0	0	119
3	1972	F	114	12.8	0	71	1	0	0	1	0	0	0	114
3	1972	F	110	8.8	0	71	1	0	0	1	0	0	0	110
3	1972	F	107	11.7	0	71	1	0	0	1	0	0	0	107
3	1973	M	152	12.5	1	72	1	0	0	1	0	0	1	152
3	1973	F	150	15.4	0	72	1	0	0	1	0	0	0	150
3	1973	M	149	21.2	1	72	1	0	0	1	0	0	1	149
3	1973	F	144	9.9	0	72	1	0	0	1	0	0	0	144
3	1973	F	140	13.3	0	72	1	0	0	1	0	0	0	140
3	1973	F	127	14.4	0	72	1	0	0	1	0	0	0	127
3	1974	F	159	11.6	0	73	1	0	0	1	0	0	0	159
3	1974	F	152	15.2	0	73	1	0	0	1	0	0	0	152
3	1974	M	141	11.5	1	73	1	0	0	1	0	0	1	141
3	1974	M	140	12.1	1	73	1	0	0	1	0	0	1	140
3	1980	M	158	15.2	1	74	1	0	0	1	0	0	1	158
3	1980	F	153	8.8	0	74	1	0	0	1	0	0	0	153
3	1980	M	151	18.6	1	74	1	0	0	1	0	0	1	151
3	1980	M	147	16.3	1	74	1	0	0	1	0	0	1	147
3	1980	M	138	14.2	1	74	1	0	0	1	0	0	1	138
3	1980	M	126	9.2	1	74	1	0	0	1	0	0	1	126
3	1991	M	128	15.4	1	75	1	0	0	1	0	0	1	128
3	1991	F	122	15.1	0	75	1	0	0	1	0	0	0	122
3	1991	F	118	11.8	0	75	1	0	0	1	0	0	0	118
3	1991	F	113	8.4	0	75	1	0	0	1	0	0	0	113
3	1991	F	92	10.9	0	75	1	0	0	1	0	0	0	92
3	1999	M	155	14.7	1	76	1	0	0	1	0	0	1	155
3	1999	F	136	11.5	0	76	1	0	0	1	0	0	0	136
3	1999	M	136	16	1	76	1	0	0	1	0	0	1	136
3	1999	F	124	11.7	0	76	1	0	0	1	0	0	0	124

3	4907	M	148	11.7	1	77	1	0	0	1	0	0	1	148
3	4907	F	146	11.6	0	77	1	0	0	1	0	0	0	146
3	4907	M	131	12.8	1	77	1	0	0	1	0	0	1	131
3	4907	M	127	13.9	1	77	1	0	0	1	0	0	1	127
3	4907	M	132	11.6	1	77	1	0	0	1	0	0	1	132
3	4907	F	124	13.7	0	77	1	0	0	1	0	0	0	124
3	4907	F	120	13.8	0	77	1	0	0	1	0	0	0	120
3	4907	M	112	9.9	1	77	1	0	0	1	0	0	1	112
3	4907	M	108	11.7	1	77	1	0	0	1	0	0	1	108
3	4907	F	106	12.4	0	77	1	0	0	1	0	0	0	106
3	4908	M	143	15	1	78	1	0	0	1	0	0	1	143
3	4908	F	139	12.6	0	78	1	0	0	1	0	0	0	139
3	4908	F	138	15.4	0	78	1	0	0	1	0	0	0	138
3	4908	F	133	15.9	0	78	1	0	0	1	0	0	0	133
3	4908	M	128	11.3	1	78	1	0	0	1	0	0	1	128
3	4908	F	132	12.6	0	78	1	0	0	1	0	0	0	132
3	4908	M	129	12.4	1	78	1	0	0	1	0	0	1	129
3	4908	M	129	15.4	1	78	1	0	0	1	0	0	1	129
3	4908	F	118	12.1	0	78	1	0	0	1	0	0	0	118
3	4908	M	91	15.4	1	78	1	0	0	1	0	0	1	91
3	4908	M	173	13.2	1	78	1	0	0	1	0	0	1	173
3	4908	F	170	13.9	0	78	1	0	0	1	0	0	0	170
3	4908	M	159	12.9	1	78	1	0	0	1	0	0	1	159
3	4908	F	158	15.2	0	78	1	0	0	1	0	0	0	158
3	4908	F	146	12.1	0	78	1	0	0	1	0	0	0	146
3	4908	F	136	10.1	0	78	1	0	0	1	0	0	0	136
3	4909	F	153	14.9	0	79	1	0	0	1	0	0	0	153
3	4909	F	152	17.3	0	79	1	0	0	1	0	0	0	152
3	4909	M	152	15.6	1	79	1	0	0	1	0	0	1	152
3	4909	F	152	16.7	0	79	1	0	0	1	0	0	0	152
3	4909	F	149	8.7	0	79	1	0	0	1	0	0	0	149
3	4909	F	148	11.5	0	79	1	0	0	1	0	0	0	148
3	4909	F	144	12.6	0	79	1	0	0	1	0	0	0	144
3	4909	M	144	14.3	1	79	1	0	0	1	0	0	1	144
3	4909	F	138	13.5	0	79	1	0	0	1	0	0	0	138
3	4910	F	149	14.9	0	80	1	0	0	1	0	0	0	149
3	4910	M	140	13.4	1	80	1	0	0	1	0	0	1	140
3	4910	F	140	11.9	0	80	1	0	0	1	0	0	0	140
3	4910	F	128	13.1	0	80	1	0	0	1	0	0	0	128
3	4910	F	125	12.5	0	80	1	0	0	1	0	0	0	125
3	4910	M	124	9.2	1	80	1	0	0	1	0	0	1	124
3	4910	M	124	7	1	80	1	0	0	1	0	0	1	124
3	4910	M	119	14.3	1	80	1	0	0	1	0	0	1	119
3	4910	M	117	14	1	80	1	0	0	1	0	0	1	117
3	4910	M	106	11.6	1	80	1	0	0	1	0	0	1	106
3	4910	M	99	10.3	1	80	1	0	0	1	0	0	1	99

3	4910	F	93	7.3	0	80	1	0	0	1	0	0	0	93
3	4911	M	155	18.1	1	81	1	0	0	1	0	0	1	155
3	4911	M	155	12.9	1	81	1	0	0	1	0	0	1	155
3	4911	M	155	17	1	81	1	0	0	1	0	0	1	155
3	4911	F	150	16.7	0	81	1	0	0	1	0	0	0	150
3	4911	F	149	12.7	0	81	1	0	0	1	0	0	0	149
3	4912	F	156	16.8	0	82	1	0	0	1	0	0	0	156
3	4912	M	155	13.9	1	82	1	0	0	1	0	0	1	155
3	4912	M	155	17.9	1	82	1	0	0	1	0	0	1	155
3	4912	F	153	14.3	0	82	1	0	0	1	0	0	0	153
3	4912	M	150	11.4	1	82	1	0	0	1	0	0	1	150
3	4912	M	144	10.5	1	82	1	0	0	1	0	0	1	144
3	4912	F	144	11.9	0	82	1	0	0	1	0	0	0	144
3	4912	F	143	12.8	0	82	1	0	0	1	0	0	0	143
3	4912	F	140	16	0	82	1	0	0	1	0	0	0	140
3	4912	F	126	12.3	0	82	1	0	0	1	0	0	0	126
3	4912	M	115	15.1	1	82	1	0	0	1	0	0	1	115
3	4915	F	119	8.4	0	83	1	0	0	1	0	0	0	119
3	4915	M	112	8.1	1	83	1	0	0	1	0	0	1	112
3	4915	F	110	12.5	0	83	1	0	0	1	0	0	0	110
3	4915	F	120	10.3	0	83	1	0	0	1	0	0	0	120
3	4915	F	109	11.9	0	83	1	0	0	1	0	0	0	109
3	4915	F	99	7.6	0	83	1	0	0	1	0	0	0	99
3	4916	M	127	16.4	1	84	1	0	0	1	0	0	1	127
3	4916	F	126	12.8	0	84	1	0	0	1	0	0	0	126
3	4916	F	125	6.8	0	84	1	0	0	1	0	0	0	125
3	4916	F	121	12.7	0	84	1	0	0	1	0	0	0	121
3	4916	M	114	12.3	1	84	1	0	0	1	0	0	1	114
3	4916	M	111	10.7	1	84	1	0	0	1	0	0	1	111
3	4916	M	99	10.7	1	84	1	0	0	1	0	0	1	99
3	4916	M	98	12.2	1	84	1	0	0	1	0	0	1	98
3	4916	M	113	12	1	84	1	0	0	1	0	0	1	113
3	4916	M	110	6.7	1	84	1	0	0	1	0	0	1	110
3	4916	F	87	12.2	0	84	1	0	0	1	0	0	0	87
3	5001	F	167	15.5	0	85	1	0	0	1	0	0	0	167
3	5001	F	166	15.6	0	85	1	0	0	1	0	0	0	166
3	5001	M	165	13.8	1	85	1	0	0	1	0	0	1	165
3	5001	M	159	12.2	1	85	1	0	0	1	0	0	1	159
3	5001	F	154	11.2	0	85	1	0	0	1	0	0	0	154
3	5002	M	120	11.9	1	86	1	0	0	1	0	0	1	120
3	5002	F	111	15.2	0	86	1	0	0	1	0	0	0	111
3	5002	F	109	11	0	86	1	0	0	1	0	0	0	109
3	5002	F	104	12.8	0	86	1	0	0	1	0	0	0	104
3	5002	M	102	10.6	1	86	1	0	0	1	0	0	1	102
3	5002	M	97	12.1	1	86	1	0	0	1	0	0	1	97
3	5002	M	97	9.9	1	86	1	0	0	1	0	0	1	97

3	5002	M	172	11.9	1	86	1	0	0	1	0	0	1	172
3	5002	M	170	14.8	1	86	1	0	0	1	0	0	1	170
3	5002	F	166	12.9	0	86	1	0	0	1	0	0	0	166
3	5003	F	131	16.7	0	87	1	0	0	1	0	0	0	131
3	5003	M	126	14.9	1	87	1	0	0	1	0	0	1	126
3	5003	F	119	12.6	0	87	1	0	0	1	0	0	0	119
3	5003	M	95	13.5	1	87	1	0	0	1	0	0	1	95
3	5003	M	92	11.7	1	87	1	0	0	1	0	0	1	92
3	5003	F	83	11.3	0	87	1	0	0	1	0	0	0	83
3	5003	F	170	15.1	0	87	1	0	0	1	0	0	0	170
3	5003	M	166	14.6	1	87	1	0	0	1	0	0	1	166
3	5003	M	159	10.3	1	87	1	0	0	1	0	0	1	159
3	5003	F	151	13.6	0	87	1	0	0	1	0	0	0	151
3	5004	F	114	15.2	0	88	1	0	0	1	0	0	0	114
3	5004	F	111	6.4	0	88	1	0	0	1	0	0	0	111
3	5004	M	110	12.7	1	88	1	0	0	1	0	0	1	110
3	5004	F	106	8.6	0	88	1	0	0	1	0	0	0	106
3	5004	F	105	11.3	0	88	1	0	0	1	0	0	0	105
3	5004	M	94	12	1	88	1	0	0	1	0	0	1	94
3	5005	F	118	12.4	0	89	1	0	0	1	0	0	0	118
3	5005	F	108	16.4	0	89	1	0	0	1	0	0	0	108
3	5005	M	102	9.9	1	89	1	0	0	1	0	0	1	102
3	5005	F	96	12.6	0	89	1	0	0	1	0	0	0	96
3	5005	F	83	12.9	0	89	1	0	0	1	0	0	0	83
3	5005	F	83	11.8	0	89	1	0	0	1	0	0	0	83
3	5005	F	171	14	0	89	1	0	0	1	0	0	0	171
3	5005	M	163	16.5	1	89	1	0	0	1	0	0	1	163
3	5005	M	162	13.7	1	89	1	0	0	1	0	0	1	162
3	5005	F	159	9.2	0	89	1	0	0	1	0	0	0	159
3	5005	M	138	12.2	1	89	1	0	0	1	0	0	1	138
3	5006	M	124	12	1	90	1	0	0	1	0	0	1	124
3	5007	M	170	7.6	1	91	1	0	0	1	0	0	1	170
3	5007	F	169	15.2	0	91	1	0	0	1	0	0	0	169
3	5007	F	161	13.2	0	91	1	0	0	1	0	0	0	161
3	5007	F	157	12	0	91	1	0	0	1	0	0	0	157
3	5007	M	151	14.1	1	91	1	0	0	1	0	0	1	151
3	5007	M	141	10.8	1	91	1	0	0	1	0	0	1	141
3	5007	M	113	9.5	1	91	1	0	0	1	0	0	1	113
3	5007	M	110	8.5	1	91	1	0	0	1	0	0	1	110
3	5007	M	107	12.6	1	91	1	0	0	1	0	0	1	107
3	5008	F	170	14.8	0	92	1	0	0	1	0	0	0	170
3	5008	M	164	16.1	1	92	1	0	0	1	0	0	1	164
3	5008	F	162	12.3	0	92	1	0	0	1	0	0	0	162
3	5008	M	160	14	1	92	1	0	0	1	0	0	1	160
3	5008	F	110	10.5	0	92	1	0	0	1	0	0	0	110
3	5008	M	99	7.5	1	92	1	0	0	1	0	0	1	99

3	5008	M	96	12.1	1	92	1	0	0	1	0	0	1	96
3	5008	M	92	7.4	1	92	1	0	0	1	0	0	1	92
3	5009	M	172	17.2	1	93	1	0	0	1	0	0	1	172
3	5009	F	168	10.6	0	93	1	0	0	1	0	0	0	168
3	5009	M	165	13.6	1	93	1	0	0	1	0	0	1	165
3	5009	M	161	10.2	1	93	1	0	0	1	0	0	1	161
3	5009	M	115	8.6	1	93	1	0	0	1	0	0	1	115
3	5009	F	109	14.3	0	93	1	0	0	1	0	0	0	109
3	5009	M	98	8.9	1	93	1	0	0	1	0	0	1	98
3	5009	F	93	9	0	93	1	0	0	1	0	0	0	93
3	5010	F	165	15.2	0	94	1	0	0	1	0	0	0	165
3	5010	M	163	12.4	1	94	1	0	0	1	0	0	1	163
3	5010	F	160	15	0	94	1	0	0	1	0	0	0	160
3	5011	F	172	12.4	0	95	1	0	0	1	0	0	0	172
3	5011	F	168	13.4	0	95	1	0	0	1	0	0	0	168
3	5011	F	162	11.4	0	95	1	0	0	1	0	0	0	162
3	5011	M	159	12.2	1	95	1	0	0	1	0	0	1	159
3	5011	F	159	13.3	0	95	1	0	0	1	0	0	0	159
3	5011	M	158	14.6	1	95	1	0	0	1	0	0	1	158
3	5011	F	146	11.6	0	95	1	0	0	1	0	0	0	146
3	5011	F	117	14.1	0	95	1	0	0	1	0	0	0	117
3	5011	M	115	11.5	1	95	1	0	0	1	0	0	1	115
3	5011	F	114	7.8	0	95	1	0	0	1	0	0	0	114
3	5011	M	111	9.9	1	95	1	0	0	1	0	0	1	111
3	5011	M	111	9.2	1	95	1	0	0	1	0	0	1	111
3	5012	M	169	17.6	1	96	1	0	0	1	0	0	1	169
3	5012	F	166	14.5	0	96	1	0	0	1	0	0	0	166
3	5012	F	159	13.1	0	96	1	0	0	1	0	0	0	159
3	5013	M	168	11.6	1	97	1	0	0	1	0	0	1	168
3	5013	F	162	10.7	0	97	1	0	0	1	0	0	0	162
3	5013	M	154	13.1	1	97	1	0	0	1	0	0	1	154
3	5013	M	150	9.4	1	97	1	0	0	1	0	0	1	150
3	5013	F	148	11.9	0	97	1	0	0	1	0	0	0	148
3	5071	F	111	10.3	0	98	1	0	0	1	0	0	0	111
3	5071	F	104	12.8	0	98	1	0	0	1	0	0	0	104
3	5071	F	96	12.5	0	98	1	0	0	1	0	0	0	96
3	5071	F	93	5.8	0	98	1	0	0	1	0	0	0	93
3	5071	F	85	10.2	0	98	1	0	0	1	0	0	0	85
3	5071	F	132	10.7	0	98	1	0	0	1	0	0	0	132
3	5071	F	125	9.4	0	98	1	0	0	1	0	0	0	125
3	5071	F	122	11.7	0	98	1	0	0	1	0	0	0	122
3	5071	M	121	11.7	1	98	1	0	0	1	0	0	1	121
3	5071	F	120	14	0	98	1	0	0	1	0	0	0	120
3	5073	M	118	11.8	1	99	1	0	0	1	0	0	1	118
3	5076	F	105	11.7	0	100	1	0	0	1	0	0	0	105
3	5205	F	121	9.3	0	101	1	0	0	1	0	0	0	121

3	5205	M	121	11.7	1	101	1	0	0	1	0	0	1	121
3	5205	M	111	13.8	1	101	1	0	0	1	0	0	1	111
3	5205	M	104	7	1	101	1	0	0	1	0	0	1	104
3	5206	F	120	12.1	0	102	1	0	0	1	0	0	0	120
3	5206	F	120	10.2	0	102	1	0	0	1	0	0	0	120
3	5206	M	117	15.7	1	102	1	0	0	1	0	0	1	117
3	5206	M	116	11	1	102	1	0	0	1	0	0	1	116
3	5206	M	109	9.6	1	102	1	0	0	1	0	0	1	109
3	5206	F	105	9.1	0	102	1	0	0	1	0	0	0	105
3	5206	M	102	12.9	1	102	1	0	0	1	0	0	1	102
3	5321	F	139	10.9	0	103	1	0	0	1	0	0	0	139
3	5321	M	136	15.3	1	103	1	0	0	1	0	0	1	136
3	5322	M	121	9.1	1	104	1	0	0	1	0	0	1	121
3	5322	M	121	10.1	1	104	1	0	0	1	0	0	1	121
3	5322	M	119	11.4	1	104	1	0	0	1	0	0	1	119
3	5322	M	117	11.7	1	104	1	0	0	1	0	0	1	117
3	5322	F	113	7.7	0	104	1	0	0	1	0	0	0	113
3	5322	F	103	8.7	0	104	1	0	0	1	0	0	0	103
3	5324	F	110	11.7	0	105	1	0	0	1	0	0	0	110
3	5324	F	106	11.8	0	105	1	0	0	1	0	0	0	106
3	5326	M	116	12	1	106	1	0	0	1	0	0	1	116
3	5326	M	132	9.2	1	106	1	0	0	1	0	0	1	132
3	5326	M	127	11	1	106	1	0	0	1	0	0	1	127
3	5328	M	129	11.3	1	107	1	0	0	1	0	0	1	129
3	5328	M	127	12.5	1	107	1	0	0	1	0	0	1	127
3	5328	F	121	13.4	0	107	1	0	0	1	0	0	0	121
3	5328	M	117	8.4	1	107	1	0	0	1	0	0	1	117
3	5328	M	113	7	1	107	1	0	0	1	0	0	1	113
3	5328	M	143	13.6	1	107	1	0	0	1	0	0	1	143
3	5328	M	127	11.1	1	107	1	0	0	1	0	0	1	127
3	5329	M	103	8.1	1	108	1	0	0	1	0	0	1	103
3	5329	F	100	11.5	0	108	1	0	0	1	0	0	0	100
3	5329	F	100	8.3	0	108	1	0	0	1	0	0	0	100
3	5329	M	99	7.4	1	108	1	0	0	1	0	0	1	99
3	5329	F	94	11.1	0	108	1	0	0	1	0	0	0	94
3	5329	F	144	5.5	0	108	1	0	0	1	0	0	0	144
3	5329	F	140	13.4	0	108	1	0	0	1	0	0	0	140
3	5329	F	136	13.4	0	108	1	0	0	1	0	0	0	136
3	5329	M	133	12	1	108	1	0	0	1	0	0	1	133
3	5329	F	120	9.1	0	108	1	0	0	1	0	0	0	120
3	5330	F	129	10.2	0	109	1	0	0	1	0	0	0	129
3	5330	M	128	8	1	109	1	0	0	1	0	0	1	128
3	5330	M	120	6.7	1	109	1	0	0	1	0	0	1	120
3	5330	M	116	12.1	1	109	1	0	0	1	0	0	1	116
3	5330	M	137	12.2	1	109	1	0	0	1	0	0	1	137
3	5330	F	137	9.7	0	109	1	0	0	1	0	0	0	137

3	5330	F	135	10.7	0	109	1	0	0	1	0	0	0	135
3	5330	F	134	13.4	0	109	1	0	0	1	0	0	0	134
3	5330	F	129	6.9	0	109	1	0	0	1	0	0	0	129
3	5337	M	134	8.7	1	110	1	0	0	1	0	0	1	134
3	5337	F	119	16.2	0	110	1	0	0	1	0	0	0	119
3	5337	F	117	11.2	0	110	1	0	0	1	0	0	0	117
3	5337	F	121	13	0	110	1	0	0	1	0	0	0	121
3	5337	F	116	9.8	0	110	1	0	0	1	0	0	0	116
3	5338	F	129	10.7	0	111	1	0	0	1	0	0	0	129
3	5338	M	121	10.6	1	111	1	0	0	1	0	0	1	121
3	5338	M	121	9.7	1	111	1	0	0	1	0	0	1	121
3	5338	M	117	9.2	1	111	1	0	0	1	0	0	1	117
3	5338	M	116	15.1	1	111	1	0	0	1	0	0	1	116
3	5338	M	115	8.5	1	111	1	0	0	1	0	0	1	115
3	5338	F	143	12.8	0	111	1	0	0	1	0	0	0	143
3	5338	M	142	13.4	1	111	1	0	0	1	0	0	1	142
3	5338	F	141	10.3	0	111	1	0	0	1	0	0	0	141
3	5338	M	139	15.1	1	111	1	0	0	1	0	0	1	139
3	5338	F	136	9.5	0	111	1	0	0	1	0	0	0	136
3	5338	M	123	12.7	1	111	1	0	0	1	0	0	1	123
4	1975	F	153	14.4	0	112	1	0	0	0	1	0	0	153
4	1975	F	153	15.9	0	112	1	0	0	0	1	0	0	153
4	1975	M	151	12.4	1	112	1	0	0	0	1	0	1	151
4	1975	M	142	16.4	1	112	1	0	0	0	1	0	1	142
4	1975	M	142	14.1	1	112	1	0	0	0	1	0	1	142
4	1975	M	141	14.8	1	112	1	0	0	0	1	0	1	141
4	1975	F	141	15	0	112	1	0	0	0	1	0	0	141
4	1975	M	128	11.9	1	112	1	0	0	0	1	0	1	128
4	1975	M	127	12.8	1	112	1	0	0	0	1	0	1	127
4	1975	F	126	14.9	0	112	1	0	0	0	1	0	0	126
4	1975	F	125	9	0	112	1	0	0	0	1	0	0	125
4	1975	M	121	13.5	1	112	1	0	0	0	1	0	1	121
4	1975	M	114	5.9	1	112	1	0	0	0	1	0	1	114
4	1976	F	151	12.7	0	113	1	0	0	0	1	0	0	151
4	1976	M	151	14.9	1	113	1	0	0	0	1	0	1	151
4	1976	M	151	10.7	1	113	1	0	0	0	1	0	1	151
4	1976	M	146	17.8	1	113	1	0	0	0	1	0	1	146
4	1976	M	145	15	1	113	1	0	0	0	1	0	1	145
4	1976	M	144	17	1	113	1	0	0	0	1	0	1	144
4	1979	M	154	15.5	1	114	1	0	0	0	1	0	1	154
4	1979	M	149	8.8	1	114	1	0	0	0	1	0	1	149
4	1979	M	147	6.6	1	114	1	0	0	0	1	0	1	147
4	1979	F	145	12.7	0	114	1	0	0	0	1	0	0	145
4	1979	F	144	13.2	0	114	1	0	0	0	1	0	0	144
4	1979	M	143	14.5	1	114	1	0	0	0	1	0	1	143
4	1979	M	135	12.8	1	114	1	0	0	0	1	0	1	135

4	1979	M	132	10	1	114	1	0	0	0	1	0	1	132
4	1979	M	123	9.1	1	114	1	0	0	0	1	0	1	123
4	1979	M	114	11.3	1	114	1	0	0	0	1	0	1	114
4	1979	M	119	12	1	114	1	0	0	0	1	0	1	119
4	1979	F	116	12.5	0	114	1	0	0	0	1	0	0	116
4	1979	M	99	8.5	1	114	1	0	0	0	1	0	1	99
4	1981	M	154	18.2	1	115	1	0	0	0	1	0	1	154
4	1981	F	148	9	0	115	1	0	0	0	1	0	0	148
4	1981	F	147	12.3	0	115	1	0	0	0	1	0	0	147
4	1981	F	145	13.4	0	115	1	0	0	0	1	0	0	145
4	1981	F	144	15	0	115	1	0	0	0	1	0	0	144
4	1981	M	136	11.9	1	115	1	0	0	0	1	0	1	136
4	1982	F	121	10.3	0	116	1	0	0	0	1	0	0	121
4	1983	F	156	12.7	0	117	1	0	0	0	1	0	0	156
4	1983	F	144	12	0	117	1	0	0	0	1	0	0	144
4	1983	F	143	11.7	0	117	1	0	0	0	1	0	0	143
4	1983	M	142	11.6	1	117	1	0	0	0	1	0	1	142
4	1983	M	136	12.8	1	117	1	0	0	0	1	0	1	136
4	1983	M	133	11.6	1	117	1	0	0	0	1	0	1	133
4	1983	F	120	13.2	0	117	1	0	0	0	1	0	0	120
4	1984	M	150	15.2	1	118	1	0	0	0	1	0	1	150
4	1984	M	147	16.3	1	118	1	0	0	0	1	0	1	147
4	1984	F	147	11.3	0	118	1	0	0	0	1	0	0	147
4	1984	F	146	12.5	0	118	1	0	0	0	1	0	0	146
4	1984	M	145	16.3	1	118	1	0	0	0	1	0	1	145
4	1984	F	140	17.8	0	118	1	0	0	0	1	0	0	140
4	1984	F	139	14.1	0	118	1	0	0	0	1	0	0	139
4	1984	F	133	6.8	0	118	1	0	0	0	1	0	0	133
4	1986	M	128	11.9	1	119	1	0	0	0	1	0	1	128
4	1986	M	126	16.5	1	119	1	0	0	0	1	0	1	126
4	1986	M	123	11.5	1	119	1	0	0	0	1	0	1	123
4	1986	M	122	12.6	1	119	1	0	0	0	1	0	1	122
4	1986	F	119	10.8	0	119	1	0	0	0	1	0	0	119
4	1988	F	156	14.9	0	120	1	0	0	0	1	0	0	156
4	1988	F	153	11.5	0	120	1	0	0	0	1	0	0	153
4	1988	F	150	12.4	0	120	1	0	0	0	1	0	0	150
4	1988	F	148	11.2	0	120	1	0	0	0	1	0	0	148
4	1988	F	143	12.3	0	120	1	0	0	0	1	0	0	143
4	1988	F	138	13.5	0	120	1	0	0	0	1	0	0	138
4	1988	M	131	10.3	1	120	1	0	0	0	1	0	1	131
4	1988	F	127	6.5	0	120	1	0	0	0	1	0	0	127
4	1988	M	127	9.9	1	120	1	0	0	0	1	0	1	127
4	4901	M	155	14.4	1	121	1	0	0	0	1	0	1	155
4	4901	M	151	15.1	1	121	1	0	0	0	1	0	1	151
4	4901	M	150	16.9	1	121	1	0	0	0	1	0	1	150
4	4901	F	141	13.3	0	121	1	0	0	0	1	0	0	141

4	4901	M	137	13.6	1	121	1	0	0	0	1	0	1	137
4	4901	F	133	12.8	0	121	1	0	0	0	1	0	0	133
4	4901	F	127	10.5	0	121	1	0	0	0	1	0	0	127
4	4901	M	130	12.6	1	121	1	0	0	0	1	0	1	130
4	4901	F	130	13.3	0	121	1	0	0	0	1	0	0	130
4	4901	M	128	16.8	1	121	1	0	0	0	1	0	1	128
4	4901	F	128	7.6	0	121	1	0	0	0	1	0	0	128
4	4901	M	125	10.1	1	121	1	0	0	0	1	0	1	125
4	4901	F	119	12.8	0	121	1	0	0	0	1	0	0	119
4	4901	M	113	13	1	121	1	0	0	0	1	0	1	113
4	4902	F	150	14	0	122	1	0	0	0	1	0	0	150
4	4902	M	146	10.6	1	122	1	0	0	0	1	0	1	146
4	4902	F	146	11.7	0	122	1	0	0	0	1	0	0	146
4	4902	F	138	11.2	0	122	1	0	0	0	1	0	0	138
4	4902	F	137	12	0	122	1	0	0	0	1	0	0	137
4	4902	M	137	12.8	1	122	1	0	0	0	1	0	1	137
4	4902	F	134	12.8	0	122	1	0	0	0	1	0	0	134
4	4902	F	127	12.5	0	122	1	0	0	0	1	0	0	127
4	4902	M	124	7.6	1	122	1	0	0	0	1	0	1	124
4	4902	M	123	15.2	1	122	1	0	0	0	1	0	1	123
4	4902	M	113	10	1	122	1	0	0	0	1	0	1	113
4	4902	M	113	13.1	1	122	1	0	0	0	1	0	1	113
4	4902	F	142	6.6	0	122	1	0	0	0	1	0	0	142
4	4902	M	140	12.5	1	122	1	0	0	0	1	0	1	140
4	4902	F	134	10.5	0	122	1	0	0	0	1	0	0	134
4	4902	F	131	9.1	0	122	1	0	0	0	1	0	0	131
4	4902	M	127	13.4	1	122	1	0	0	0	1	0	1	127
4	4902	F	127	11.3	0	122	1	0	0	0	1	0	0	127
4	4902	M	125	10.3	1	122	1	0	0	0	1	0	1	125
4	4902	M	118	12.2	1	122	1	0	0	0	1	0	1	118
4	4902	M	109	11.9	1	122	1	0	0	0	1	0	1	109
4	4903	M	154	15.4	1	123	1	0	0	0	1	0	1	154
4	4903	M	148	13.8	1	123	1	0	0	0	1	0	1	148
4	4903	M	147	16.3	1	123	1	0	0	0	1	0	1	147
4	4903	M	145	14.9	1	123	1	0	0	0	1	0	1	145
4	4903	M	145	13.1	1	123	1	0	0	0	1	0	1	145
4	4903	F	142	13.6	0	123	1	0	0	0	1	0	0	142
4	4903	M	123	14	1	123	1	0	0	0	1	0	1	123
4	4903	F	121	12.6	0	123	1	0	0	0	1	0	0	121
4	4903	F	118	11.3	0	123	1	0	0	0	1	0	0	118
4	4903	F	116	13.1	0	123	1	0	0	0	1	0	0	116
4	4903	F	112	13.2	0	123	1	0	0	0	1	0	0	112
4	4905	M	157	6.6	1	124	1	0	0	0	1	0	1	157
4	4905	F	151	12.6	0	124	1	0	0	0	1	0	0	151
4	4905	F	148	9.5	0	124	1	0	0	0	1	0	0	148
4	4905	M	148	14.5	1	124	1	0	0	0	1	0	1	148

4	4905	M	146	14.2	1	124	1	0	0	0	1	0	1	146
4	4905	M	142	11.6	1	124	1	0	0	0	1	0	1	142
4	4905	M	140	11.6	1	124	1	0	0	0	1	0	1	140
4	4905	F	123	11.6	0	124	1	0	0	0	1	0	0	123
4	4905	F	117	15.4	0	124	1	0	0	0	1	0	0	117
4	4905	M	106	10.3	1	124	1	0	0	0	1	0	1	106
4	4905	M	173	14.2	1	124	1	0	0	0	1	0	1	173
4	4905	M	168	12.6	1	124	1	0	0	0	1	0	1	168
4	4905	F	165	10.4	0	124	1	0	0	0	1	0	0	165
4	4905	M	165	13.1	1	124	1	0	0	0	1	0	1	165
4	4905	M	159	13.4	1	124	1	0	0	0	1	0	1	159
4	4905	M	157	14.3	1	124	1	0	0	0	1	0	1	157
4	4905	F	154	12.8	0	124	1	0	0	0	1	0	0	154
4	4905	F	153	14	0	124	1	0	0	0	1	0	0	153
4	4906	M	153	14.4	1	125	1	0	0	0	1	0	1	153
4	4906	M	144	11.2	1	125	1	0	0	0	1	0	1	144
4	4906	F	144	13.1	0	125	1	0	0	0	1	0	0	144
4	4906	M	138	10.1	1	125	1	0	0	0	1	0	1	138
4	4906	F	136	13.9	0	125	1	0	0	0	1	0	0	136
4	4906	F	136	13.4	0	125	1	0	0	0	1	0	0	136
4	4906	M	132	12.7	1	125	1	0	0	0	1	0	1	132
4	4906	M	129	13.1	1	125	1	0	0	0	1	0	1	129
4	4906	F	121	12.8	0	125	1	0	0	0	1	0	0	121
4	4906	M	83	11.4	1	125	1	0	0	0	1	0	1	83
4	4906	F	167	14.3	0	125	1	0	0	0	1	0	0	167
4	4906	M	165	16.3	1	125	1	0	0	0	1	0	1	165
4	4906	F	148	10.8	0	125	1	0	0	0	1	0	0	148
4	4906	F	139	9.4	0	125	1	0	0	0	1	0	0	139
4	4906	M	137	9.7	1	125	1	0	0	0	1	0	1	137
4	4913	M	124	14	1	126	1	0	0	0	1	0	1	124
4	4913	M	124	12.4	1	126	1	0	0	0	1	0	1	124
4	4913	M	116	12.3	1	126	1	0	0	0	1	0	1	116
4	4913	M	116	11.1	1	126	1	0	0	0	1	0	1	116
4	4913	M	114	9.9	1	126	1	0	0	0	1	0	1	114
4	4913	M	113	15.1	1	126	1	0	0	0	1	0	1	113
4	4913	F	106	8.6	0	126	1	0	0	0	1	0	0	106
4	4913	M	100	10.1	1	126	1	0	0	0	1	0	1	100
4	4914	F	127	13.4	0	127	1	0	0	0	1	0	0	127
4	4914	M	124	16.8	1	127	1	0	0	0	1	0	1	124
4	4914	F	123	14	0	127	1	0	0	0	1	0	0	123
4	4914	F	119	15.9	0	127	1	0	0	0	1	0	0	119
4	4914	M	112	10.2	1	127	1	0	0	0	1	0	1	112
4	4914	F	173	14.7	0	127	1	0	0	0	1	0	0	173
4	4914	M	168	17	1	127	1	0	0	0	1	0	1	168
4	4914	F	162	11.8	0	127	1	0	0	0	1	0	0	162
4	4914	M	159	10.3	1	127	1	0	0	0	1	0	1	159

4	4914	M	157	12	1	127	1	0	0	0	1	0	1	157
4	4914	M	155	15.1	1	127	1	0	0	0	1	0	1	155
4	4914	F	154	12.6	0	127	1	0	0	0	1	0	0	154
4	4914	F	124	7.6	0	127	1	0	0	0	1	0	0	124
4	4914	M	123	12.4	1	127	1	0	0	0	1	0	1	123
4	4914	M	122	12.7	1	127	1	0	0	0	1	0	1	122
4	4914	M	119	11.1	1	127	1	0	0	0	1	0	1	119
4	4914	M	118	12.4	1	127	1	0	0	0	1	0	1	118
4	4914	M	117	11.6	1	127	1	0	0	0	1	0	1	117
4	4914	F	114	7.7	0	127	1	0	0	0	1	0	0	114
4	4914	F	143	13.3	0	127	1	0	0	0	1	0	0	143
4	4914	F	128	11.1	0	127	1	0	0	0	1	0	0	128
4	4914	M	126	11.2	1	127	1	0	0	0	1	0	1	126
4	4918	F	114	12.7	0	128	1	0	0	0	1	0	0	114
4	4918	F	111	9.6	0	128	1	0	0	0	1	0	0	111
4	4918	M	108	14.1	1	128	1	0	0	0	1	0	1	108
4	4918	M	106	9.7	1	128	1	0	0	0	1	0	1	106
4	4918	M	106	10.7	1	128	1	0	0	0	1	0	1	106
4	4918	F	104	9.8	0	128	1	0	0	0	1	0	0	104
4	4918	F	96	10.3	0	128	1	0	0	0	1	0	0	96
4	4918	F	120	13	0	128	1	0	0	0	1	0	0	120
4	4918	M	119	12.1	1	128	1	0	0	0	1	0	1	119
4	4918	F	110	11.8	0	128	1	0	0	0	1	0	0	110
4	4918	M	101	7.2	1	128	1	0	0	0	1	0	1	101
4	4918	F	98	9.4	0	128	1	0	0	0	1	0	0	98
4	4919	F	168	12.2	0	129	1	0	0	0	1	0	0	168
4	4919	F	167	10.6	0	129	1	0	0	0	1	0	0	167
4	4919	F	167	13.2	0	129	1	0	0	0	1	0	0	167
4	4919	F	164	10.8	0	129	1	0	0	0	1	0	0	164
4	4919	F	155	10.2	0	129	1	0	0	0	1	0	0	155
4	4919	F	154	9	0	129	1	0	0	0	1	0	0	154
4	4919	M	151	9.1	1	129	1	0	0	0	1	0	1	151
4	4919	F	121	8.7	0	129	1	0	0	0	1	0	0	121
4	4919	F	114	9.3	0	129	1	0	0	0	1	0	0	114
4	4919	M	103	9.7	1	129	1	0	0	0	1	0	1	103
4	4919	F	133	8.8	0	129	1	0	0	0	1	0	0	133
4	4919	M	129	12.3	1	129	1	0	0	0	1	0	1	129
4	4919	M	117	9.1	1	129	1	0	0	0	1	0	1	117
4	4919	M	116	13.2	1	129	1	0	0	0	1	0	1	116
4	4919	F	144	8.8	0	129	1	0	0	0	1	0	0	144
4	4919	F	141	13.9	0	129	1	0	0	0	1	0	0	141
4	4919	F	136	13	0	129	1	0	0	0	1	0	0	136
4	4919	M	133	9.8	1	129	1	0	0	0	1	0	1	133
4	4919	F	130	11.8	0	129	1	0	0	0	1	0	0	130
4	4919	F	129	11.6	0	129	1	0	0	0	1	0	0	129
4	4919	M	115	7.6	1	129	1	0	0	0	1	0	1	115

4	4921	F	129	9.7	0	130	1	0	0	0	1	0	0	129
4	4921	F	121	9.1	0	130	1	0	0	0	1	0	0	121
4	4921	M	119	8.7	1	130	1	0	0	0	1	0	1	119
4	4921	M	117	10	1	130	1	0	0	0	1	0	1	117
4	4921	F	114	9.3	0	130	1	0	0	0	1	0	0	114
4	4921	M	109	13.3	1	130	1	0	0	0	1	0	1	109
4	4921	M	107	12.3	1	130	1	0	0	0	1	0	1	107
4	4921	M	106	12.1	1	130	1	0	0	0	1	0	1	106
4	4921	M	93	11.5	1	130	1	0	0	0	1	0	1	93
4	4921	M	171	10.7	1	130	1	0	0	0	1	0	1	171
4	4921	F	168	13.1	0	130	1	0	0	0	1	0	0	168
4	4921	F	160	10.8	0	130	1	0	0	0	1	0	0	160
4	4921	M	160	10.3	1	130	1	0	0	0	1	0	1	160
4	4921	M	155	11.4	1	130	1	0	0	0	1	0	1	155
4	4921	M	155	11.6	1	130	1	0	0	0	1	0	1	155
4	4923	M	128	12.9	1	131	1	0	0	0	1	0	1	128
4	4923	F	122	10.9	0	131	1	0	0	0	1	0	0	122
4	4923	M	120	11.8	1	131	1	0	0	0	1	0	1	120
4	4923	M	119	13.8	1	131	1	0	0	0	1	0	1	119
4	4923	F	98	10.4	0	131	1	0	0	0	1	0	0	98
4	4923	M	93	9.4	1	131	1	0	0	0	1	0	1	93
4	4923	M	172	16.1	1	131	1	0	0	0	1	0	1	172
4	4923	F	165	13.5	0	131	1	0	0	0	1	0	0	165
4	4923	M	164	16	1	131	1	0	0	0	1	0	1	164
4	4923	F	162	13.5	0	131	1	0	0	0	1	0	0	162
4	4923	M	159	9.7	1	131	1	0	0	0	1	0	1	159
4	4923	M	159	13	1	131	1	0	0	0	1	0	1	159
4	4923	F	156	10.7	0	131	1	0	0	0	1	0	0	156
4	4923	M	154	12.3	1	131	1	0	0	0	1	0	1	154
4	5015	M	174	16.1	1	132	1	0	0	0	1	0	1	174
4	5015	F	164	9.7	0	132	1	0	0	0	1	0	0	164
4	5015	F	158	12.8	0	132	1	0	0	0	1	0	0	158
4	5015	M	155	13.5	1	132	1	0	0	0	1	0	1	155
4	5015	F	154	12.1	0	132	1	0	0	0	1	0	0	154
4	5015	M	151	12.5	1	132	1	0	0	0	1	0	1	151
4	5015	M	146	16	1	132	1	0	0	0	1	0	1	146
4	5015	F	139	6	0	132	1	0	0	0	1	0	0	139
4	5015	M	135	10	1	132	1	0	0	0	1	0	1	135
4	5015	F	122	10.7	0	132	1	0	0	0	1	0	0	122
4	5015	F	120	13.4	0	132	1	0	0	0	1	0	0	120
4	5016	M	172	11.5	1	133	1	0	0	0	1	0	1	172
4	5016	F	168	10.5	0	133	1	0	0	0	1	0	0	168
4	5016	M	168	12.7	1	133	1	0	0	0	1	0	1	168
4	5016	M	163	13.5	1	133	1	0	0	0	1	0	1	163
4	5016	M	119	6.1	1	133	1	0	0	0	1	0	1	119
4	5016	F	118	9.3	0	133	1	0	0	0	1	0	0	118

4	5016	M	117	13.7	1	133	1	0	0	0	1	0	1	117
4	5016	F	108	8.5	0	133	1	0	0	0	1	0	0	108
4	5016	F	104	12.3	0	133	1	0	0	0	1	0	0	104
4	5017	F	163	13.8	0	134	1	0	0	0	1	0	0	163
4	5017	F	162	13.5	0	134	1	0	0	0	1	0	0	162
4	5017	F	161	11.5	0	134	1	0	0	0	1	0	0	161
4	5017	F	157	12	0	134	1	0	0	0	1	0	0	157
4	5017	M	152	14.2	1	134	1	0	0	0	1	0	1	152
4	5017	F	151	13.1	0	134	1	0	0	0	1	0	0	151
4	5017	F	108	11.5	0	134	1	0	0	0	1	0	0	108
4	5017	M	106	8.3	1	134	1	0	0	0	1	0	1	106
4	5017	F	142	9.4	0	134	1	0	0	0	1	0	0	142
4	5017	M	142	15.1	1	134	1	0	0	0	1	0	1	142
4	5017	M	136	9.3	1	134	1	0	0	0	1	0	1	136
4	5017	F	133	10.5	0	134	1	0	0	0	1	0	0	133
4	5017	M	117	10.2	1	134	1	0	0	0	1	0	1	117
4	5017	M	100	12.9	1	134	1	0	0	0	1	0	1	100
4	5018	M	167	16	1	135	1	0	0	0	1	0	1	167
4	5018	F	166	14.2	0	135	1	0	0	0	1	0	0	166
4	5018	F	164	8.6	0	135	1	0	0	0	1	0	0	164
4	5018	F	162	13	0	135	1	0	0	0	1	0	0	162
4	5018	M	158	12.1	1	135	1	0	0	0	1	0	1	158
4	5018	F	146	14.5	0	135	1	0	0	0	1	0	0	146
4	5018	F	115	12.1	0	135	1	0	0	0	1	0	0	115
4	5018	F	114	7.8	0	135	1	0	0	0	1	0	0	114
4	5018	F	110	11.9	0	135	1	0	0	0	1	0	0	110
4	5018	F	110	12	0	135	1	0	0	0	1	0	0	110
4	5018	M	98	8.5	1	135	1	0	0	0	1	0	1	98
4	5019	F	168	12.7	0	136	1	0	0	0	1	0	0	168
4	5019	F	166	14.6	0	136	1	0	0	0	1	0	0	166
4	5019	F	166	13.5	0	136	1	0	0	0	1	0	0	166
4	5019	M	152	7.5	1	136	1	0	0	0	1	0	1	152
4	5019	F	146	12.7	0	136	1	0	0	0	1	0	0	146
4	5019	M	140	14.2	1	136	1	0	0	0	1	0	1	140
4	5019	M	139	10.4	1	136	1	0	0	0	1	0	1	139
4	5019	F	121	13.6	0	136	1	0	0	0	1	0	0	121
4	5019	F	114	9.1	0	136	1	0	0	0	1	0	0	114
4	5019	F	109	10.4	0	136	1	0	0	0	1	0	0	109
4	5019	F	108	7.8	0	136	1	0	0	0	1	0	0	108
4	5019	M	108	7.9	1	136	1	0	0	0	1	0	1	108
4	5019	M	140	14.5	1	136	1	0	0	0	1	0	1	140
4	5019	F	129	9.3	0	136	1	0	0	0	1	0	0	129
4	5019	F	129	9.5	0	136	1	0	0	0	1	0	0	129
4	5020	F	167	16.1	0	137	1	0	0	0	1	0	0	167
4	5020	M	167	12.5	1	137	1	0	0	0	1	0	1	167
4	5020	M	165	12.6	1	137	1	0	0	0	1	0	1	165

4	5020	F	165	14.5	0	137	1	0	0	0	1	0	0	165
4	5020	M	162	12.7	1	137	1	0	0	0	1	0	1	162
4	5020	F	157	11.7	0	137	1	0	0	0	1	0	0	157
4	5020	F	153	10.8	0	137	1	0	0	0	1	0	0	153
4	5020	M	152	12.1	1	137	1	0	0	0	1	0	1	152
4	5020	F	118	13.2	0	137	1	0	0	0	1	0	0	118
4	5020	M	106	7.6	1	137	1	0	0	0	1	0	1	106
4	5020	F	102	8.5	0	137	1	0	0	0	1	0	0	102
4	5020	F	102	10.8	0	137	1	0	0	0	1	0	0	102
4	5020	M	140	14.4	1	137	1	0	0	0	1	0	1	140
4	5020	M	137	11.4	1	137	1	0	0	0	1	0	1	137
4	5020	F	133	11.3	0	137	1	0	0	0	1	0	0	133
4	5020	M	132	8.9	1	137	1	0	0	0	1	0	1	132
4	5020	F	132	11.1	0	137	1	0	0	0	1	0	0	132
4	5020	M	113	8.7	1	137	1	0	0	0	1	0	1	113
4	5204	M	118	12.2	1	138	1	0	0	0	1	0	1	118
4	5204	M	115	10.2	1	138	1	0	0	0	1	0	1	115
4	5204	M	113	10.1	1	138	1	0	0	0	1	0	1	113
4	5204	F	110	9	0	138	1	0	0	0	1	0	0	110
4	5204	F	103	6.2	0	138	1	0	0	0	1	0	0	103
4	5207	M	116	9	1	139	1	0	0	0	1	0	1	116
4	5207	F	114	11.6	0	139	1	0	0	0	1	0	0	114
4	5207	M	110	10.1	1	139	1	0	0	0	1	0	1	110
4	5207	M	108	8.6	1	139	1	0	0	0	1	0	1	108
4	5208	M	110	12.1	1	140	1	0	0	0	1	0	1	110
4	5208	M	110	9.4	1	140	1	0	0	0	1	0	1	110
4	5208	F	106	8.3	0	140	1	0	0	0	1	0	0	106
4	5208	F	105	7.6	0	140	1	0	0	0	1	0	0	105
4	5331	M	128	13.5	1	141	1	0	0	0	1	0	1	128
4	5331	M	121	9.8	1	141	1	0	0	0	1	0	1	121
4	5331	F	120	9.5	0	141	1	0	0	0	1	0	0	120
4	5331	F	118	8.9	0	141	1	0	0	0	1	0	0	118
4	5331	M	114	7.7	1	141	1	0	0	0	1	0	1	114
4	5331	M	96	11	1	141	1	0	0	0	1	0	1	96
4	5331	F	116	13.2	0	141	1	0	0	0	1	0	0	116
4	5331	M	115	8.7	1	141	1	0	0	0	1	0	1	115
4	5331	F	114	10.4	0	141	1	0	0	0	1	0	0	114
4	5331	M	108	12.2	1	141	1	0	0	0	1	0	1	108
4	5331	F	97	11.2	0	141	1	0	0	0	1	0	0	97
4	5332	M	129	11.8	1	142	1	0	0	0	1	0	1	129
4	5332	M	122	10.1	1	142	1	0	0	0	1	0	1	122
4	5332	M	119	9.9	1	142	1	0	0	0	1	0	1	119
4	5332	F	118	8.2	0	142	1	0	0	0	1	0	0	118
4	5332	F	116	11.2	0	142	1	0	0	0	1	0	0	116
4	5332	F	112	8.9	0	142	1	0	0	0	1	0	0	112
4	5332	M	140	11.4	1	142	1	0	0	0	1	0	1	140

4	5332	M	135	13.2	1	142	1	0	0	0	1	0	1	135
4	5332	F	134	13.5	0	142	1	0	0	0	1	0	0	134
4	5332	F	126	10.4	0	142	1	0	0	0	1	0	0	126
4	5332	M	113	8.4	1	142	1	0	0	0	1	0	1	113
4	5333	M	125	11	1	143	1	0	0	0	1	0	1	125
4	5333	F	122	11.6	0	143	1	0	0	0	1	0	0	122
4	5333	M	121	10.7	1	143	1	0	0	0	1	0	1	121
4	5333	M	119	7.8	1	143	1	0	0	0	1	0	1	119
4	5333	F	110	7.9	0	143	1	0	0	0	1	0	0	110
4	5333	F	108	10	0	143	1	0	0	0	1	0	0	108
4	5333	F	107	9.9	0	143	1	0	0	0	1	0	0	107
4	5334	M	121	9.1	1	144	1	0	0	0	1	0	1	121
4	5334	F	121	9.9	0	144	1	0	0	0	1	0	0	121
4	5334	F	118	11.6	0	144	1	0	0	0	1	0	0	118
4	5334	F	106	13.8	0	144	1	0	0	0	1	0	0	106
4	5334	F	100	10.4	0	144	1	0	0	0	1	0	0	100
4	5336	F	134	12.2	0	145	1	0	0	0	1	0	0	134
4	5336	F	131	9.4	0	145	1	0	0	0	1	0	0	131
4	5336	F	128	11.6	0	145	1	0	0	0	1	0	0	128
4	5336	F	126	9.7	0	145	1	0	0	0	1	0	0	126
4	5336	M	121	11.6	1	145	1	0	0	0	1	0	1	121
4	5336	M	120	13.9	1	145	1	0	0	0	1	0	1	120
4	5336	M	120	8.6	1	145	1	0	0	0	1	0	1	120
4	5336	M	118	11.4	1	145	1	0	0	0	1	0	1	118
4	5336	F	139	9.6	0	145	1	0	0	0	1	0	0	139
4	5336	F	129	10.6	0	145	1	0	0	0	1	0	0	129
4	5336	F	127	13.9	0	145	1	0	0	0	1	0	0	127
4	5336	M	118	15.2	1	145	1	0	0	0	1	0	1	118
4	5336	F	109	11.6	0	145	1	0	0	0	1	0	0	109
5	1975	M	120	11.3	1	146	1	0	0	0	0	1	1	120
5	1975	M	118	10.1	1	146	1	0	0	0	0	1	1	118
5	1975	M	114	6.6	1	146	1	0	0	0	0	1	1	114
5	1975	F	113	6.9	0	146	1	0	0	0	0	1	0	113
5	1975	F	99	7.3	0	146	1	0	0	0	0	1	0	99
5	1975	M	97	5.3	1	146	1	0	0	0	0	1	1	97
5	1979	F	119	5.3	0	147	1	0	0	0	0	1	0	119
5	1979	M	117	10.3	1	147	1	0	0	0	0	1	1	117
5	1979	M	114	8.7	1	147	1	0	0	0	0	1	1	114
5	1979	F	111	8.1	0	147	1	0	0	0	0	1	0	111
5	1979	M	102	9.4	1	147	1	0	0	0	0	1	1	102
5	1982	F	118	11.4	0	148	1	0	0	0	0	1	0	118
5	4902	F	130	9.9	0	149	1	0	0	0	0	1	0	130
5	4902	M	138	12.3	1	149	1	0	0	0	0	1	1	138
5	4902	M	137	13	1	149	1	0	0	0	0	1	1	137
5	4902	M	134	11.7	1	149	1	0	0	0	0	1	1	134
5	4902	M	132	11.7	1	149	1	0	0	0	0	1	1	132

5	4902	F	132	11.5	0	149	1	0	0	0	0	1	0	132
5	4902	M	132	11.4	1	149	1	0	0	0	0	1	1	132
5	4902	F	131	10.2	0	149	1	0	0	0	0	1	0	131
5	4905	M	123	11.2	1	150	1	0	0	0	0	1	1	123
5	4905	F	146	10.9	0	150	1	0	0	0	0	1	0	146
5	4906	F	109	9.7	0	151	1	0	0	0	0	1	0	109
5	4906	M	166	8	1	151	1	0	0	0	0	1	1	166
5	4906	F	164	10.4	0	151	1	0	0	0	0	1	0	164
5	4906	M	137	9	1	151	1	0	0	0	0	1	1	137
5	4914	F	122	12.1	0	152	1	0	0	0	0	1	0	122
5	4914	M	165	9.5	1	152	1	0	0	0	0	1	1	165
5	4914	F	162	12	0	152	1	0	0	0	0	1	0	162
5	4914	F	127	11.2	0	152	1	0	0	0	0	1	0	127
5	4914	F	123	10.2	0	152	1	0	0	0	0	1	0	123
5	4914	F	121	7.3	0	152	1	0	0	0	0	1	0	121
5	4914	M	119	10.4	1	152	1	0	0	0	0	1	1	119
5	4914	M	116	9.6	1	152	1	0	0	0	0	1	1	116
5	4914	M	115	10.5	1	152	1	0	0	0	0	1	1	115
5	4914	F	140	12.3	0	152	1	0	0	0	0	1	0	140
5	4914	M	139	13.4	1	152	1	0	0	0	0	1	1	139
5	4914	F	139	10	0	152	1	0	0	0	0	1	0	139
5	4914	F	134	11.6	0	152	1	0	0	0	0	1	0	134
5	4914	F	131	11.1	0	152	1	0	0	0	0	1	0	131
5	4918	F	125	9	0	153	1	0	0	0	0	1	0	125
5	4918	F	119	8.6	0	153	1	0	0	0	0	1	0	119
5	4918	F	118	6.8	0	153	1	0	0	0	0	1	0	118
5	4918	F	115	9.3	0	153	1	0	0	0	0	1	0	115
5	4919	M	164	10	1	154	1	0	0	0	0	1	1	164
5	4919	F	163	10.2	0	154	1	0	0	0	0	1	0	163
5	4919	F	153	9.8	0	154	1	0	0	0	0	1	0	153
5	4919	M	113	9.4	1	154	1	0	0	0	0	1	1	113
5	4919	M	109	5.2	1	154	1	0	0	0	0	1	1	109
5	4919	F	101	8.7	0	154	1	0	0	0	0	1	0	101
5	4919	F	129	9.2	0	154	1	0	0	0	0	1	0	129
5	4919	M	126	8.2	1	154	1	0	0	0	0	1	1	126
5	4919	F	123	7.7	0	154	1	0	0	0	0	1	0	123
5	4919	M	119	8.9	1	154	1	0	0	0	0	1	1	119
5	4919	M	118	10.6	1	154	1	0	0	0	0	1	1	118
5	4919	F	117	10.1	0	154	1	0	0	0	0	1	0	117
5	4919	M	114	7.2	1	154	1	0	0	0	0	1	1	114
5	4919	M	134	12	1	154	1	0	0	0	0	1	1	134
5	4919	M	132	12.5	1	154	1	0	0	0	0	1	1	132
5	4919	F	125	9.8	0	154	1	0	0	0	0	1	0	125
5	4921	M	125	9.3	1	155	1	0	0	0	0	1	1	125
5	4921	M	164	11.8	1	155	1	0	0	0	0	1	1	164
5	4921	M	143	9.4	1	155	1	0	0	0	0	1	1	143

5	4921	F	142	9.3	0	155	1	0	0	0	0	1	0	142
5	4923	M	160	12.7	1	156	1	0	0	0	0	1	1	160
5	4923	M	136	9.8	1	156	1	0	0	0	0	1	1	136
5	5015	M	161	9	1	157	1	0	0	0	0	1	1	161
5	5015	M	155	13.4	1	157	1	0	0	0	0	1	1	155
5	5015	F	153	12.8	0	157	1	0	0	0	0	1	0	153
5	5015	F	131	11.6	0	157	1	0	0	0	0	1	0	131
5	5015	M	123	8.6	1	157	1	0	0	0	0	1	1	123
5	5015	M	121	9.1	1	157	1	0	0	0	0	1	1	121
5	5015	F	120	9.4	0	157	1	0	0	0	0	1	0	120
5	5015	F	94	8.8	0	157	1	0	0	0	0	1	0	94
5	5016	F	116	10.8	0	158	1	0	0	0	0	1	0	116
5	5016	F	115	8.5	0	158	1	0	0	0	0	1	0	115
5	5016	M	114	10.1	1	158	1	0	0	0	0	1	1	114
5	5016	F	112	9	0	158	1	0	0	0	0	1	0	112
5	5017	M	166	11.9	1	159	1	0	0	0	0	1	1	166
5	5017	M	163	11.3	1	159	1	0	0	0	0	1	1	163
5	5017	M	119	8.4	1	159	1	0	0	0	0	1	1	119
5	5017	M	112	10.7	1	159	1	0	0	0	0	1	1	112
5	5017	F	141	12.3	0	159	1	0	0	0	0	1	0	141
5	5017	M	137	9.6	1	159	1	0	0	0	0	1	1	137
5	5017	M	137	10.2	1	159	1	0	0	0	0	1	1	137
5	5017	M	118	10.6	1	159	1	0	0	0	0	1	1	118
5	5018	F	164	10.6	0	160	1	0	0	0	0	1	0	164
5	5018	M	138	11.8	1	160	1	0	0	0	0	1	1	138
5	5018	F	119	10.1	0	160	1	0	0	0	0	1	0	119
5	5018	F	118	6	0	160	1	0	0	0	0	1	0	118
5	5018	F	113	7.4	0	160	1	0	0	0	0	1	0	113
5	5018	M	112	9	1	160	1	0	0	0	0	1	1	112
5	5019	F	152	14.5	0	161	1	0	0	0	0	1	0	152
5	5019	M	119	13.5	1	161	1	0	0	0	0	1	1	119
5	5019	F	99	10.7	0	161	1	0	0	0	0	1	0	99
5	5019	F	135	7.9	0	161	1	0	0	0	0	1	0	135
5	5019	M	135	13.8	1	161	1	0	0	0	0	1	1	135
5	5019	M	133	8.6	1	161	1	0	0	0	0	1	1	133
5	5019	M	131	10.9	1	161	1	0	0	0	0	1	1	131
5	5019	F	123	8.9	0	161	1	0	0	0	0	1	0	123
5	5020	F	166	11.1	0	162	1	0	0	0	0	1	0	166
5	5020	F	165	12.2	0	162	1	0	0	0	0	1	0	165
5	5020	F	159	10	0	162	1	0	0	0	0	1	0	159
5	5020	M	118	12.8	1	162	1	0	0	0	0	1	1	118
5	5020	F	106	7.9	0	162	1	0	0	0	0	1	0	106
5	5020	M	85	8.7	1	162	1	0	0	0	0	1	1	85
5	5020	F	138	9.7	0	162	1	0	0	0	0	1	0	138
5	5020	M	135	9.3	1	162	1	0	0	0	0	1	1	135
5	5020	F	135	8.1	0	162	1	0	0	0	0	1	0	135

5	5020	M	126	10.9	1	162	1	0	0	0	0	1	1	126
5	5020	F	118	9.2	0	162	1	0	0	0	0	1	0	118
5	5020	M	116	10.8	1	162	1	0	0	0	0	1	1	116
5	5020	M	116	11.5	1	162	1	0	0	0	0	1	1	116
5	5204	M	116	12.8	1	163	1	0	0	0	0	1	1	116
5	5204	M	115	9.7	1	163	1	0	0	0	0	1	1	115
5	5204	M	114	11.5	1	163	1	0	0	0	0	1	1	114
5	5204	M	112	9.1	1	163	1	0	0	0	0	1	1	112
5	5204	F	101	6	0	163	1	0	0	0	0	1	0	101
5	5207	M	121	11.4	1	164	1	0	0	0	0	1	1	121
5	5207	M	115	12.3	1	164	1	0	0	0	0	1	1	115
5	5207	M	114	11.9	1	164	1	0	0	0	0	1	1	114
5	5207	F	98	9.8	0	164	1	0	0	0	0	1	0	98
5	5208	M	116	16.4	1	165	1	0	0	0	0	1	1	116
5	5208	M	112	12	1	165	1	0	0	0	0	1	1	112
5	5208	F	106	10.5	0	165	1	0	0	0	0	1	0	106
5	5331	F	132	7	0	166	1	0	0	0	0	1	0	132
5	5331	M	115	9.7	1	166	1	0	0	0	0	1	1	115
5	5331	M	102	6.5	1	166	1	0	0	0	0	1	1	102
5	5331	M	139	11.4	1	166	1	0	0	0	0	1	1	139
5	5331	F	127	8.8	0	166	1	0	0	0	0	1	0	127
5	5331	F	119	10.8	0	166	1	0	0	0	0	1	0	119
5	5331	M	116	9.3	1	166	1	0	0	0	0	1	1	116
5	5331	M	111	9.7	1	166	1	0	0	0	0	1	1	111
5	5332	M	118	11	1	167	1	0	0	0	0	1	1	118
5	5332	M	117	11.9	1	167	1	0	0	0	0	1	1	117
5	5332	F	116	9.4	0	167	1	0	0	0	0	1	0	116
5	5332	M	115	9.1	1	167	1	0	0	0	0	1	1	115
5	5332	F	133	6.3	0	167	1	0	0	0	0	1	0	133
5	5332	F	133	10.5	0	167	1	0	0	0	0	1	0	133
5	5332	F	130	9.9	0	167	1	0	0	0	0	1	0	130
5	5332	F	127	8.7	0	167	1	0	0	0	0	1	0	127
5	5332	M	126	7.9	1	167	1	0	0	0	0	1	1	126
5	5332	F	125	12.1	0	167	1	0	0	0	0	1	0	125
5	5332	M	116	11.3	1	167	1	0	0	0	0	1	1	116
5	5333	M	134	12.3	1	168	1	0	0	0	0	1	1	134
5	5333	M	125	12.4	1	168	1	0	0	0	0	1	1	125
5	5333	M	124	7.7	1	168	1	0	0	0	0	1	1	124
5	5333	M	119	13.3	1	168	1	0	0	0	0	1	1	119
5	5334	M	133	10.9	1	169	1	0	0	0	0	1	1	133
5	5334	F	128	9.9	0	169	1	0	0	0	0	1	0	128
5	5334	F	122	11.2	0	169	1	0	0	0	0	1	0	122
5	5334	F	114	10.6	0	169	1	0	0	0	0	1	0	114
5	5334	F	114	7.4	0	169	1	0	0	0	0	1	0	114
5	5334	M	107	11.7	1	169	1	0	0	0	0	1	1	107
5	5334	F	96	8.6	0	169	1	0	0	0	0	1	0	96

5	5336	M	135	9.6	1	170	1	0	0	0	0	1	1	135
5	5336	M	119	11.9	1	170	1	0	0	0	0	1	1	119
5	5336	M	116	11.5	1	170	1	0	0	0	0	1	1	116
5	5336	F	103	7	0	170	1	0	0	0	0	1	0	103
5	5336	F	141	12.2	0	170	1	0	0	0	0	1	0	141
5	5336	F	137	7.9	0	170	1	0	0	0	0	1	0	137
5	5336	M	134	11.3	1	170	1	0	0	0	0	1	1	134
5	5336	M	114	12.7	1	170	1	0	0	0	0	1	1	114
5	5336	F	98	10.4	0	170	1	0	0	0	0	1	0	98
6	1972	M	101	4.7	1	171	1	0	0	0	0	0	1	101
6	1991	F	121	8.9	0	172	1	0	0	0	0	0	0	121
6	4908	F	166	11.5	0	173	1	0	0	0	0	0	0	166
6	4908	F	156	11	0	173	1	0	0	0	0	0	0	156
6	4910	F	125	10.2	0	174	1	0	0	0	0	0	0	125
6	4910	M	98	10.1	1	174	1	0	0	0	0	0	1	98
6	4915	M	110	10.1	1	175	1	0	0	0	0	0	1	110
6	4915	M	104	10	1	175	1	0	0	0	0	0	1	104
6	5001	F	169	11.2	0	176	1	0	0	0	0	0	0	169
6	5002	F	170	11.5	0	177	1	0	0	0	0	0	0	170
6	5002	F	165	10.3	0	177	1	0	0	0	0	0	0	165
6	5003	M	119	7.5	1	178	1	0	0	0	0	0	1	119
6	5003	F	158	9.3	0	178	1	0	0	0	0	0	0	158
6	5004	F	102	7.2	0	179	1	0	0	0	0	0	0	102
6	5005	M	154	9.7	1	180	1	0	0	0	0	0	1	154
6	5007	F	168	12.1	0	181	1	0	0	0	0	0	0	168
6	5007	M	167	11.6	1	181	1	0	0	0	0	0	1	167
6	5007	F	161	11.3	0	181	1	0	0	0	0	0	0	161
6	5007	M	149	10.4	1	181	1	0	0	0	0	0	1	149
6	5007	M	120	13.9	1	181	1	0	0	0	0	0	1	120
6	5008	M	172	12.7	1	182	1	0	0	0	0	0	1	172
6	5008	F	142	10.8	0	182	1	0	0	0	0	0	0	142
6	5008	M	112	11.8	1	182	1	0	0	0	0	0	1	112
6	5008	M	108	8.5	1	182	1	0	0	0	0	0	1	108
6	5008	F	96	5.6	0	182	1	0	0	0	0	0	0	96
6	5009	F	174	11.8	0	183	1	0	0	0	0	0	0	174
6	5009	M	167	12	1	183	1	0	0	0	0	0	1	167
6	5009	F	158	10.8	0	183	1	0	0	0	0	0	0	158
6	5010	F	165	10.8	0	184	1	0	0	0	0	0	0	165
6	5010	M	164	14.4	1	184	1	0	0	0	0	0	1	164
6	5010	M	146	12.2	1	184	1	0	0	0	0	0	1	146
6	5011	F	164	12.6	0	185	1	0	0	0	0	0	0	164
6	5011	F	159	11.6	0	185	1	0	0	0	0	0	0	159
6	5011	M	157	11.1	1	185	1	0	0	0	0	0	1	157
6	5011	M	121	12.3	1	185	1	0	0	0	0	0	1	121
6	5011	F	107	8.1	0	185	1	0	0	0	0	0	0	107
6	5012	M	166	15.2	1	186	1	0	0	0	0	0	1	166

6	5012	M	162	16.2	1	186	1	0	0	0	0	0	1	162
6	5013	F	148	12.9	0	187	1	0	0	0	0	0	0	148
6	5071	M	104	9.6	1	188	1	0	0	0	0	0	1	104
6	5073	M	113	13.8	1	189	1	0	0	0	0	0	1	113
6	5076	M	112	11.1	1	190	1	0	0	0	0	0	1	112
6	5205	M	120	12.2	1	191	1	0	0	0	0	0	1	120
6	5205	F	110	6.2	0	191	1	0	0	0	0	0	0	110
6	5206	M	119	14.2	1	192	1	0	0	0	0	0	1	119
6	5206	F	117	11.2	0	192	1	0	0	0	0	0	0	117
6	5322	F	115	11.5	0	193	1	0	0	0	0	0	0	115
6	5322	F	98	8.7	0	193	1	0	0	0	0	0	0	98
6	5324	M	130	11.8	1	194	1	0	0	0	0	0	1	130
6	5324	F	128	7.9	0	194	1	0	0	0	0	0	0	128
6	5324	M	111	13	1	194	1	0	0	0	0	0	1	111
6	5324	F	106	10.3	0	194	1	0	0	0	0	0	0	106
6	5326	M	100	9.8	1	195	1	0	0	0	0	0	1	100
6	5326	F	139	11.5	0	195	1	0	0	0	0	0	0	139
6	5326	F	131	9.9	0	195	1	0	0	0	0	0	0	131
6	5328	M	130	14.8	1	196	1	0	0	0	0	0	1	130
6	5328	F	140	11.2	0	196	1	0	0	0	0	0	0	140
6	5328	F	130	11.3	0	196	1	0	0	0	0	0	0	130
6	5328	F	127	7.3	0	196	1	0	0	0	0	0	0	127
6	5329	M	123	9.9	1	197	1	0	0	0	0	0	1	123
6	5329	M	118	7.5	1	197	1	0	0	0	0	0	1	118
6	5329	F	146	14.4	0	197	1	0	0	0	0	0	0	146
6	5329	F	133	7	0	197	1	0	0	0	0	0	0	133
6	5329	F	130	9.7	0	197	1	0	0	0	0	0	0	130
6	5329	F	100	11.5	0	197	1	0	0	0	0	0	0	100
6	5330	M	123	10.9	1	198	1	0	0	0	0	0	1	123
6	5330	F	137	10.1	0	198	1	0	0	0	0	0	0	137
6	5330	F	134	11.8	0	198	1	0	0	0	0	0	0	134
6	5337	F	129	11.3	0	199	1	0	0	0	0	0	0	129
6	5337	M	125	13.5	1	199	1	0	0	0	0	0	1	125
6	5337	M	119	9.7	1	199	1	0	0	0	0	0	1	119
6	5337	M	116	9.6	1	199	1	0	0	0	0	0	1	116
6	5337	F	103	8.3	0	199	1	0	0	0	0	0	0	103
6	5338	F	127	11.4	0	200	1	0	0	0	0	0	0	127
6	5338	M	123	10	1	200	1	0	0	0	0	0	1	123
6	5338	F	137	12.9	0	200	1	0	0	0	0	0	0	137

APPENDIX D

Two drugs against trypanosomosis (parasitic disease transmitted by tsetse flies), Berenil and Samorin, are studied in a situation where there is widespread evidence of high levels of drug resistance. Herds of N'Dama breeds are randomly assigned to Berenil and herds of Boran breeds to Samorin treatment. The aim of the study was then to see whether there are differences in the change in PCV between the two breeds following a trypanosome infection.

PCV is a variable often measured to evaluate the severity of the diseases is packed cell volume (PCV), which is the percentage of the volume of the blood serum taken up by the red blood cells. Low PCV corresponds to anaemia and can indicate infection with the disease. That determined at the time of treatment was designated PCV0, that a month later following treatment was designated PCV35 (with 14 measurements in between). Animals belonging to a herd to which Berenil has been assigned, however, are randomly assigned to receive a high or a low dose of Berenil when detected parasitaemic with trypanosomes.

		Animal					
Breed	Days	1	2	3	4	5	6
Boran	0	36.2	35.9	29.5	28.5	30.4	33.7
	2	35.9	38.5	33.3	27.6	29.5	36.2
	4	35.3	35.9	29.2	27.9	28.8	33.3
	7	35.4	36	29.9	27.7	28.7	32.2
	9	35.4	36.3	29	29.3	28.7	30.9
	14	31.5	36.3	29.9	26.7	27.1	29.6
	17	25.5	25.2	21.3	21.3	20.7	22.6
	18	34.4	31.5	27.4	26.7	25.2	30.3
	21	34.1	30.6	25.5	25.2	22.9	28.3
	23	25.8	28.7	25.5	23.6	23.2	25.8
	25	28.7	29	24.8	23.6	22.9	24.5
	29	21.6	23.9	23.6	20	20.7	21.6
	31	21.3	21.3	22.6	19.4	19.1	17.5
	35	17.8	18.1	20.4	17.2	18.5	15.9
N'Dama	0	30.4	37.5	32.4	34.3	30.4	40.4
	2	33	37.8	30.4	33	32.1	37.5
	4	33.3	36.5	31.7	37.5	32.1	38.8
	7	31.9	35.7	31.2	36.3	30.9	37
	9	30.6	35.7	31.5	34.1	30.6	38.9
	14	31.2	33.8	27.7	30.6	29.6	31.9
	17	27.7	33.4	27.1	27.7	23.6	27.1
	18	28	31.5	27.4	29.9	29	31.5
	21	28.3	32.5	29	28	29.6	31.2
	23	27.7	34.7	28	27.1	28.7	33.1
	25	25.8	30.6	27.4	26.7	27.1	35.4
	29	26.1	31.5	28.3	28.7	25.8	30.6
	31	24.5	25.2	26.1	23.9	26.1	28.7
	35	22.6	28.7	22.9	22.6	24.2	26.1

% PCV measured at 0,2,4,..,35 days following infection

APPENDIX E

Estimated breeding values for the complete data set used in Chapter 3.

DPP = Dirichlet Process when *M* is simulated given the data

		REML		Trad. Bayes	DPP
SIRE_ID	BREED	Estimate	SE	Estimate	Sim *M*
1971	1	-0.1061	0.6654	-0.1024	-0.1097
1972	1	0.5241	0.5349	0.5689	0.5081
1973	1	0.3888	0.6399	0.4464	0.3934
1974	1	1.9339	0.5486	1.9627	1.9026
1980	1	0.9299	0.5975	0.9635	0.9193
1991	1	0.3611	0.6396	0.3741	0.3438
1999	1	0.9266	0.6654	0.9939	0.8936
4907	1	0.2289	0.5790	0.2676	0.2293
4908	1	1.6614	0.4921	1.6662	1.649
4909	1	-0.7628	0.6653	-0.7645	-0.7711
4910	1	0.4627	0.5123	0.4396	0.4511
4911	1	0.6681	0.5795	0.6867	0.653
4912	1	-0.3258	0.6655	-0.3581	-0.3125
4915	1	0.2351	0.6655	0.1955	0.224
4916	1	-0.1544	0.6658	-0.1383	-0.1681
5001	1	-0.3829	0.7309	-0.3286	-0.3592
5002	1	-0.4267	0.6398	-0.4600	-0.4385
5003	1	0.2938	0.7307	0.2753	0.2474
5004	1	0.6637	0.6401	0.6546	0.6468
5005	1	0.1155	0.6397	0.1555	0.0971
5007	1	-0.7786	0.6396	-0.7543	-0.7531
5008	1	-0.4999	0.6655	-0.5137	-0.5016
5009	1	-0.0700	0.7309	-0.0531	-0.0297
5010	1	0.1004	0.6956	0.1027	0.0818
5011	1	-0.6332	0.6956	-0.6175	-0.5763
5012	1	0.1259	0.7311	0.1063	0.1
5013	1	-0.3030	0.6961	-0.2939	-0.3329
5071	1	0.3608	0.6655	0.3945	0.3812
5073	1	-0.2421	0.7732	-0.2776	-0.2338
5076	1	-0.1386	0.7730	-0.1264	-0.1087
5205	1	-0.4303	0.7731	-0.4454	-0.3821

5324	1	-0.7496	0.7730	-0.7590	-0.7039
5326	1	-0.5001	0.7308	-0.5171	-0.4887
5328	1	-0.2392	0.6393	-0.2737	-0.232
5329	1	-0.5741	0.6167	-0.6062	-0.5698
5330	1	-1.4541	0.6393	-1.4747	-1.3737
5337	1	-0.5825	0.6654	-0.5811	-0.5837
5338	1	-0.6268	0.6954	-0.6131	-0.5882
1975	2	-0.1147	0.6173	-0.0893	-0.0912
1976	2	0.4043	0.6399	0.4477	0.3964
1979	2	0.8504	0.5495	0.8168	0.8382
1981	2	0.1681	0.5799	0.1769	0.1699
1982	2	-0.2903	0.6663	-0.2016	-0.3346
1983	2	1.4790	0.6404	1.5350	1.4576
1984	2	0.3396	0.6175	0.4279	0.3094
1986	2	0.1270	0.7310	0.1526	0.142
1988	2	0.9403	0.6658	0.9514	0.98
4901	2	0.6692	0.5497	0.6900	0.6585
4902	2	0.3625	0.5502	0.3446	0.3446
4903	2	0.1536	0.5249	0.1349	0.1255
4905	2	-0.0434	0.4942	-0.0279	-0.0769
4906	2	0.6141	0.5035	0.6523	0.5717
4913	2	0.1971	0.6664	0.2227	0.2353
4914	2	-1.0541	0.5800	-1.1073	-1.0039
4918	2	0.7037	0.6962	0.6847	0.6331
4919	2	-0.9475	0.6666	-0.9576	-0.9186
4921	2	-0.6580	0.5638	-0.6768	-0.6184
4923	2	-0.1434	0.5978	-0.1706	-0.1427
5015	2	0.0903	0.6404	0.0550	0.0552
5016	2	-0.8617	0.6177	-0.8988	-0.8042
5017	2	-1.1521	0.6176	-1.1860	-1.1127
5018	2	-0.2094	0.7308	-0.2098	-0.2311
5019	2	-0.4845	0.5975	-0.4910	-0.458
5020	2	-0.1989	0.6403	-0.2144	-0.1768
5204	2	-0.2315	0.7732	-0.2585	-0.2016
5207	2	-0.1528	0.7731	-0.1341	-0.1835
5331	2	-0.0178	0.7731	-0.0770	-0.0006
5334	2	-0.2860	0.7731	-0.2829	-0.2741
5336	2	-0.2528	0.7309	-0.2626	-0.2231
1971	3	-0.5916	0.5597	-0.6159	-0.5838
1972	3	1.0193	0.5062	1.0300	1.0054
1973	3	0.8188	0.6145	0.8021	0.8213
1974	3	-0.1343	0.6642	-0.1161	-0.1457
1980	3	0.2847	0.6149	0.2463	0.314
1991	3	0.4573	0.6380	0.4638	0.4214
1999	3	0.3507	0.6639	0.3162	0.3172
4907	3	0.1848	0.5438	0.1809	0.1652

4908	3	0.6656	0.4767	0.6633	0.6427
4909	3	0.6248	0.5598	0.6198	0.6078
4910	3	-0.1241	0.5179	-0.1408	-0.1418
4911	3	0.9102	0.6381	0.9922	0.8869
4912	3	0.7009	0.5306	0.6865	0.6699
4915	3	-0.5662	0.6152	-0.6052	-0.5554
4916	3	0.0111	0.5315	0.0232	0.007
5001	3	0.0392	0.6388	0.0851	0.0638
5002	3	0.2011	0.5439	0.2123	0.1804
5003	3	0.7758	0.5437	0.7462	0.7322
5004	3	0.0456	0.6154	0.0304	0.0351
5005	3	0.6248	0.5302	0.6826	0.5889
5006	3	-0.0257	0.7729	-0.0039	-0.0546
5007	3	-0.7342	0.5592	-0.7001	-0.7081
5008	3	-0.2444	0.5753	-0.2629	-0.2602
5009	3	-0.4822	0.5753	-0.4978	-0.4396
5010	3	0.1952	0.6951	0.2524	0.1997
5011	3	-0.5054	0.5179	-0.5245	-0.5325
5012	3	0.4238	0.6952	0.3924	0.4272
5013	3	-0.8623	0.6384	-0.8832	-0.8186
5071	3	-0.0328	0.5459	-0.0596	-0.0237
5073	3	-0.0160	0.7729	-0.0171	-0.0068
5076	3	0.1324	0.7730	0.1270	0.1257
5205	3	-0.4053	0.6643	-0.3947	-0.3858
5206	3	0.0652	0.5944	0.0957	0.0482
5321	3	0.1343	0.7302	0.1409	0.1009
5322	3	-0.8229	0.6147	-0.8742	-0.7943
5324	3	0.2162	0.7306	0.2634	0.2084
5326	3	-0.4500	0.6947	-0.5162	-0.4068
5328	3	-0.5569	0.5941	-0.5849	-0.5516
5329	3	-0.8199	0.5446	-0.8603	-0.7923
5330	3	-1.1606	0.5587	-1.2002	-1.0854
5337	3	0.1048	0.6377	0.1210	0.1019
5338	3	-0.4520	0.5177	-0.4651	-0.4289
1975	4	0.7474	0.5061	0.7256	0.7081
1976	4	0.8951	0.6149	0.9307	0.8653
1979	4	-0.3871	0.5064	-0.3926	-0.3982
1981	4	0.4741	0.6146	0.4465	0.4463
1982	4	-0.0710	0.7729	-0.0575	-0.0396
1983	4	0.1010	0.5937	0.1088	0.1157
1984	4	0.8469	0.5756	0.8823	0.805
1986	4	0.4516	0.6376	0.4461	0.4418
1988	4	-0.3328	0.5592	-0.3485	-0.3107
4901	4	0.7953	0.4953	0.8141	0.7761
4902	4	-0.1491	0.4385	-0.1395	-0.1696
4903	4	1.1514	0.5300	1.2001	1.1224

4905	4	-0.1528	0.4615	-0.1144	-0.1808
4906	4	0.2633	0.4856	0.2961	0.2459
4913	4	0.2709	0.5765	0.2676	0.2398
4914	4	0.4437	0.4323	0.4366	0.4123
4918	4	0.1952	0.5199	0.2266	0.1741
4919	4	-0.9511	0.4390	-0.9531	-0.8958
4921	4	-0.6402	0.4856	-0.6476	-0.6073
4923	4	0.1171	0.4958	0.1508	0.0728
5015	4	-0.2059	0.5308	-0.1867	-0.2039
5016	4	-0.6323	0.5587	-0.5925	-0.6073
5017	4	-0.1042	0.4955	-0.1211	-0.1358
5018	4	0.0329	0.5304	0.0047	0.035
5019	4	-0.4168	0.4857	-0.3861	-0.4184
5020	4	-0.3328	0.4600	-0.3295	-0.3613
5204	4	-0.5484	0.6380	-0.5679	-0.5169
5207	4	-0.4170	0.6644	-0.3610	-0.3791
5208	4	-0.4533	0.6645	-0.4677	-0.4507
5331	4	-0.2136	0.5312	-0.2106	-0.2016
5332	4	-0.4640	0.5302	-0.4453	-0.4668
5333	4	-0.5464	0.5941	-0.5769	-0.5264
5334	4	0.1250	0.6381	0.0967	0.0945
5336	4	0.1079	0.5063	0.1247	0.0842
1975	5	-0.7175	0.6191	-0.7509	-0.7053
1979	5	-0.4920	0.6411	-0.4991	-0.5028
1982	5	0.2473	0.7732	0.2446	0.2412
4902	5	0.4902	0.5818	0.5047	0.4392
4905	5	0.1193	0.7311	0.1769	0.1122
4906	5	-0.5996	0.6668	-0.6063	-0.5691
4914	5	0.3284	0.5074	0.3184	0.3123
4918	5	-0.3650	0.6670	-0.3851	-0.3673
4919	5	-0.6073	0.4894	-0.6402	-0.5904
4921	5	-0.4122	0.6669	-0.4361	-0.4205
4923	5	-0.0513	0.7314	-0.0619	-0.0329
5015	5	-0.0328	0.5817	-0.0313	-0.019
5016	5	0.0768	0.6667	0.0967	0.0806
5017	5	-0.1279	0.5823	-0.1145	-0.1349
5018	5	-0.4122	0.6187	-0.4317	-0.3841
5019	5	0.4651	0.5816	0.4822	0.4183
5020	5	-0.0474	0.5169	-0.0868	-0.0788
5204	5	0.0700	0.6413	0.0693	0.0713
5207	5	0.6159	0.6667	0.6394	0.5962
5208	5	1.0117	0.6963	1.0674	1.0026
5331	5	-0.4083	0.5817	-0.3608	-0.3985
5332	5	-0.0742	0.5391	-0.1131	-0.0521
5333	5	0.3543	0.6668	0.3541	0.3415
5334	5	0.2657	0.5994	0.2868	0.2627

5336	5	0.3029	0.5660	0.3118	0.2618
1972	6	-0.6139	0.7735	-0.6616	-0.6009
1991	6	-0.1264	0.7734	-0.1488	-0.119
4908	6	-0.1184	0.7325	-0.0705	-0.0998
4910	6	0.0678	0.7322	0.0918	0.0199
4915	6	0.0147	0.7325	-0.0133	-0.0129
5001	6	-0.1183	0.7736	-0.1402	-0.0864
5002	6	-0.2610	0.7326	-0.2281	-0.2262
5003	6	-0.5894	0.7321	-0.6160	-0.5468
5004	6	-0.2265	0.7735	-0.2923	-0.2258
5005	6	-0.3030	0.7735	-0.3106	-0.2561
5007	6	0.0075	0.6456	0.0763	0.0034
5008	6	-0.2902	0.6450	-0.3131	-0.3151
5009	6	-0.2190	0.6988	-0.1699	-0.234
5010	6	0.0975	0.6985	0.1176	0.0747
5011	6	-0.0125	0.6451	-0.0613	-0.0067
5012	6	0.6699	0.7326	0.7915	0.6654
5013	6	0.2102	0.7735	0.1985	0.2045
5071	6	-0.0298	0.7735	-0.0459	-0.0133
5073	6	0.4341	0.7735	0.5039	0.4147
5076	6	0.1085	0.7735	0.0970	0.1196
5205	6	-0.1755	0.7322	-0.1493	-0.1477
5206	6	0.5590	0.7322	0.6348	0.5343
5322	6	0.1846	0.7325	0.1772	0.1723
5324	6	0.2054	0.6696	0.2060	0.2179
5326	6	0.0374	0.6981	0.0459	0.0101
5328	6	0.1950	0.6695	0.2470	0.1955
5329	6	-0.1601	0.6239	-0.1136	-0.1856
5330	6	0.0850	0.6980	0.0255	0.0713
5337	6	0.1022	0.6452	0.0809	0.104
5338	6	0.2649	0.6981	0.2788	0.2485

APPENDIX F

The data set consists of 44 age-adjusted milk production record (305 days) obtained in the same year and herd from cows whose sires and dams were considered randomly representative of a large population. The records were taken from full-sib daughters.

Sires	**Dams**	**Production Records**		
1	1	4379	6560	
	2	5560	7733	7198
	3	4637	5639	8072
	4	5726	5576	
	5	4968	4574	
2	6	5355	7057	7052
	7	4605	4180	
	8	4393	4530	
3	9	5195		
	10	6137	4748	7351
	11	6253		
	12	5553	6026	6666
4	13	6268	7575	7024
	14	7112		
	15	5840	7316	6382
	16	6246	5595	
	17	5400	6440	
	18	7301	6615	
	19	5453		
	20	7374	6693	6592