NEW MODEL TO CAPTURE THE CONVERSION OF FLOW FROM CONFINED TO UNCONFINED

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DECLARATION

I, Makosha Ishmaeline Charlotte Morakaladi, hereby declare that the dissertation hereby submitted by me to the Institute for Groundwater Studies in the Faculty of Natural and Agricultural Sciences at the University of the Free State, in fulfilment of the degree of Magister Scientiae, is my own independent work and I have not previously submitted it for a qualification at another institution of higher education. In addition, I declare that all sources used or quoted have been indicated or acknowledged as complete references.

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Makosha Ishmaeline Charlotte Morakaladi
January 2020

DEDICATION

I dedicate this thesis to my parents: Maboka Lucy and Phuti Moses Morakaladi.

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ABSTRACT

The conversion of flow from confined to unconfined aquifers, while it has attracted the attention of many researchers within and outside of the field of geo-hydrology, although many results and mathematical models have been suggested to replicate such a physical problem, one will inform that, up to now the phenomena has not yet been fully understood. The available literature provides some important mathematical model that can be used to replicate the conversion; however, it is clear that such a model is highly nonlinear as the numerical simulation suggests high value decrease in water level. Such a mathematical model cannot really be used for practical purpose. In this work, a new mathematical model is suggested to minimize high nonlinearity also the mathematical model includes into the mathematical formulation the fading memory effect due to the properties of geological formations. The new model suggested here is a system that consists of partial differential equations, where the first equation presents the flow within the confined aquifer, Theis suggested such a model. The second model is an integro-differential partial differential equation, with a new fading memory term. Due to the complexity of the second equations, we adopted a numerical scheme known as Adams-Bashforth to derive the numerical solution. Using the Von Neumann stability analysis, conditions under which the used numerical scheme is efficient have been established. Numerical simulations have been performed using a mathematical software called Matlab. The mathematical model suggested in this thesis will open doors for new investigation and could be extended to 3-dimensional case, also the model could be extended to the framework of fractional differentiation and integration.

Keywords: Confined to unconfined flow, Theis model, mathematical models, integro-differential partial differential equation, Adams-Bashforth method, Von Neumann stability analysis, numerical simulations

LIST OF GREEK NOTATIONS

α	alpha
β	beta
τ	tau
∂	partial differential
δ	delta
$\sum$	sigma
Δ	delta
ω	omega
λ	lambda
$\mathcal{L}$	Laplace Transform Operator
π	pi
μ	mu
ξ	xi

ABBREVIATIONS AND NOTATIONS

B	aquifer thickness
K	hydraulic conductivity
h	hydraulic head
Q	discharge
q	darcy flux
s	drawdown
T	transmissivity
t	time
S	storativity
S_S	specific storage
S_y	specific yield
S_c	storage coefficient
r	radial distance
A	cross sectional area of flow
MP Model	Moench and Prickett Model
BDF	Backward Differentiation Formulas
AB Method	Adams-Bashforth Method
AM Method	Adams-Moulton Method

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

1.1 Background Study

Groundwater is freshwater that comes to the surface of the earth in a form of precipitation such as melting snow and rainfall. Precipitation is the principle origin of groundwater. This precipitation penetrates towards the ground surface and percolates down to deeper depths to make its way through fractures of rocks and pore spaces that are saturated with water. This results in the availability of groundwater beneath the surface of the earth (Rai, 2014). As a result, the groundwater is then stored and flows in small open spaces that are found between sand and gravel in which a saturated unit called an aquifer can form. As the groundwater moves through the aquifer system, the groundwater behaviour is mostly controlled by the medium in which it moves and probably by its own characteristics (Xiao, 2014).

An aquifer is a permeable geological formation that has the ability to store and transmit a sufficient amount of water that can be pumped from underground to the surface of the earth to be used for domestic purposes or economically for agricultural, irrigation and municipal use (Yeh and Chang, 2013; Rai, 2014). The aquifer transports water from recharge areas and provides enough supply of water to wells, springs and streams. The aquifer should have sufficient interconnected pores to do this. Two principle types of aquifer categories exist, which are unconfined and confined aquifers (see Figure 1 and Figure 2).

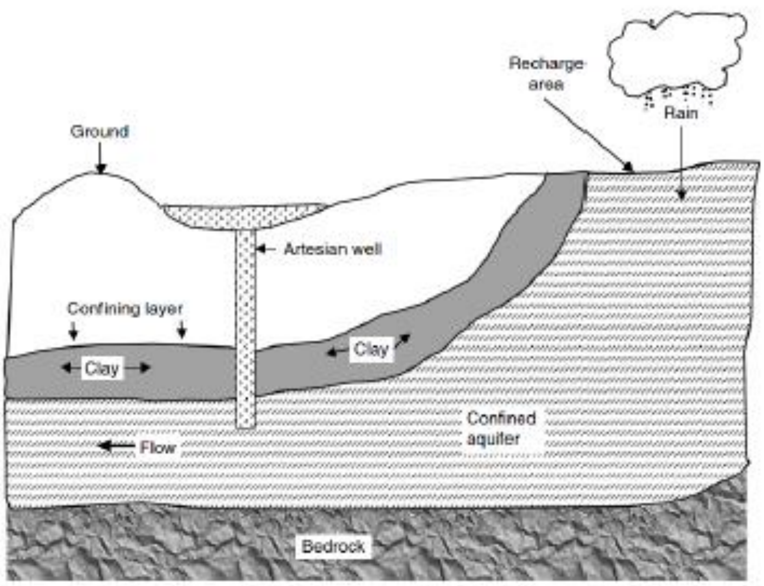

Figure 1: Confined aquifer (Spellman and Whiting, 2004)

A confined aquifer, which is also known as an artesian or pressure aquifer, is enclosed on top and bottom by a confining or impermeable geological formation (Yeh and Chang, 2013). The groundwater within the aquifer is kept away from the atmosphere and is restricted under pressure considerably higher than the atmospheric pressure by an impermeable formation (Alley, 2009; Yeh and Chang, 2013 and Rai, 2014). The confining layers protect the groundwater from being contaminated from surface pollution.

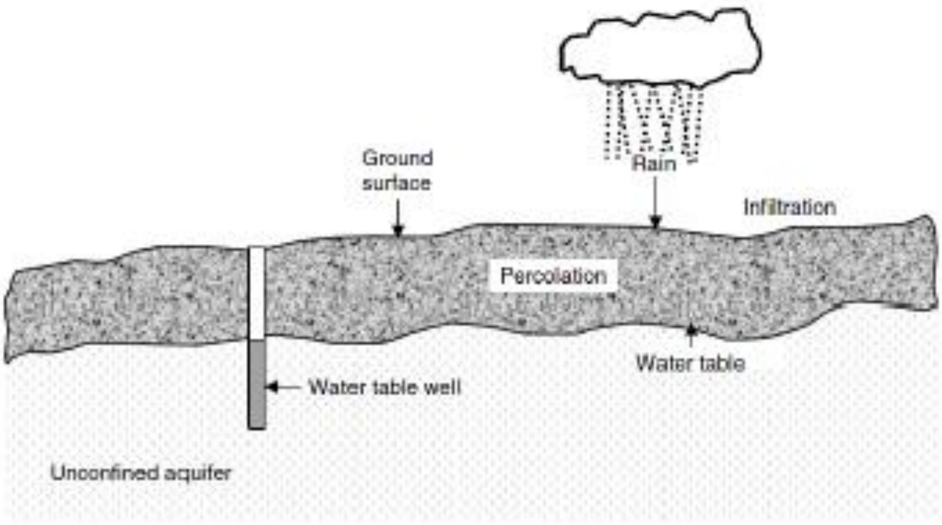

Figure 2: Unconfined aquifer (Spellman and Whiting, 2004)

An unconfined aquifer is enclosed at the top by a very permeable layer, where a lot of rainwater can easily percolate through permeable soil, and recharge the aquifer (Şen, 2015). This means that there is no overlying confining layer that prevents percolation of rainwater into the underlying aquifer. The aquifer receives recharge from the surface of the earth after rainfall, or from surface water found on the earth such as lakes, rivers or ponds. The water table is under atmospheric pressure and freely fluctuates up and down due to variations in recharge and discharge rates (Alley, 2009). These types of aquifers are susceptible to contamination which may be introduced to the aquifer material from land use if something spills on the surface (Şen, 2015). During wet seasons the water table can rise since there is significant amount of water that recharges the aquifer, the opposite of this occurs during dry seasons in cases of long periods of severe drought. If a well is penetrated into an unconfined aquifer it has to extend deep below the water table surface to be able to abstract water from the aquifer (Şen, 2015).

1.2 Groundwater Modelling

Groundwater is an important resource, so knowing and understanding the dynamics and behaviour of groundwater systems is essential, and this is achieved in many respects, including conceptual, analytical and numerical models. Groundwater modelling is a good tool that is used for the management of groundwater, remediation processes and groundwater protection and a simple way to represent reality of investigating a certain situation or predicting future behaviour (Baalousha, 2011). The models can be basic, such as spreadsheet models or simple one-dimensional analytical solutions, or ones that are developed to a high degree of complexity such as three-dimensional models.

When models are not properly designed and interpreted, they can be complicated and give wrong results. Models are used to predict the behaviour of groundwater systems before carrying out a project or a remediation scheme, making it a basic and affordable solution as compared to starting a real-life project (Baalousha, 2011). Therefore, models make life much simpler. To be able to apply a model it will depend on the objectives of the model. Even though models are not perfect, they are very beneficial when coming to hydrogeology. The challenge facing groundwater modelling is to make reality easier in a manner that does not negatively affect the precision and capacity of the model production to achieve anticipated goals. It becomes a difficult task for a modeller to depict a real-life problem to a simple way without compromising the precision or making assumptions that are invalid. Certain steps should be followed in modelling and modelling objectives should be clear and well defined in order to select a proper model. It is important to design a proper and simple conceptual model, if it is not designed well and constructed the modelling process will then be of no use because it will be a waste of time and effort (Baalousha, 2011). It is good to attempt to get the best description of the real world by gathering enough data as possible and supplying new information and data to the models.

Simple assumptions are made to understand the flow of groundwater in groundwater models using mathematical equations. The assumptions can be based on geometry of an aquifer, the course along which groundwater flow is moving, and the heterogeneous or anisotropic nature of the sediments or of the bedrock of the aquifer.

Models can be defined as conceptual representations or approximations that represent a real-world physical systems or process by the application of equations, therefore, models are not considered as exact representations physical processes or systems (Baalousha, 2011). The mathematical representation of a simplified version in the form of a model representing a hydrogeological system can help to produce reasonable scenarios which can be predicted, compared and then tested. How useful a model is, is determined by how accurately the mathematical equations can estimate the physical system that is modelled (Kumar, 2004).

By mathematically depicting a simplified version in the form of a model of a hydrogeological system, sensible summaries can be said to happen in future, compared, and tested. To understand groundwater movement and behaviour in the two types of aquifers (confined and unconfined aquifers), groundwater models including deterministic mathematical models are used to explain the process of flow in groundwater systems.

1.2.1 Deterministic Mathematical Models

Deterministic mathematical models are currently used for many groundwater models. The models are formed upon conservation of mass, energy, momentum and description of cause and the effect of relations (Delleur, 2007). Deterministic mathematical models generally make use of solution of partial differential equations in order to simulate the flow and transportation processes that are associated with the groundwater system (Bear, 1972; Anderson and Woessner, 1992). An analytical or numerical method can be used to solve deterministic mathematical models. From the assumption that groundwater flow is a time dependent problem, for a deterministic mathematical model the governing equations, boundary and initial conditions have to be properly and fully detailed (Xiao, 2014). For analytical models, parameters and boundary conditions need to be properly interpreted. Exact solutions are often analytically obtained. Various deterministic models take properties of a porous media as grouped or a combination of parameters, which prevents the explicit depiction of heterogeneous hydraulic characteristics of the model. The heterogeneous nature of aquifer properties is something that is found in nature of geological processes and is the main key influencing the flow of groundwater. Due to this, it is desirable to use distributed-parameter models, this gives more realistic representation of the properties of the system. Numerical methods are estimate solutions to the governing equation by discretizing time and space. Boundary conditions, varying parameters of the system, problem domain and stresses of the hydrogeologic system are estimated (Delleur, 2007).

1.3 Problem Statement

The increase in the human population around the world has increased the demand for fresh groundwater, thus this resulted in many confined aquifers to be pumped extensively resulting in the conversion from confined to unconfined conditions (Xiao, 2014). The conditions occur when the confined (artesian) aquifer is over pumped and the pumping period is too long (Wang et al., 2009; Wang and Zhan, 2009; Xiao et al., 2018). Literature has shown that it is impossible to model flow in a confined or unconfined aquifer using the same mathematical model. The confined flow is captured using the Theis model whereas flow in leaky aquifers is captured using the Hantush model. Both models are partial differential equations and linear equations that cannot capture the movement of groundwater in an unconfined aquifer. The Theis (1935) equation is one of the fundamental keys in working out groundwater flow problems in confined aquifers and for that reason it is used as the key methods for the flow of groundwater deterministic mathematical models. Various assumptions were focused on when the Theis (1935) equation was derived. The assumptions were that an aquifer is homogeneous, has uniform thickness, infinite in aerial extend, is isotropic and is pumped at a constant discharge rate. During investigations it has shown that in actual fact the opposite of this is true because in reality aquifers are heterogeneous, have finite aerial extend due to impermeable boundaries, are anisotropic and pumped at different discharge rates. The MP (Moench and Prickett) model which was proposed by Moench and Prickett (1972) suggested a mathematical solution for the conversion of confined to unconfined using constant transmissivity in unconfined aquifer. This model was later known as the MP model. Awodwa and Atangana (2019) recently proved that the MP model gives non-realistic results due to nonlinearity, so when solving the system this gives something that is exaggerated. This is because the theoretical model predicts higher flow than what is obtained from the field. For this study, a new numerical method is suggested that depicts the conversion of flow from confined to unconfined taking into account the delay process. The suggested method gives a clear insight and realistic approximation on what is observed in the field. Instead of taking the systems that were introduced before, a delay process is introduced. With this delay process it is not known when exactly will the pumped water reach 'B' (aquifer thickness) (figure 5), so it is assumed that the pumped water might reach 'B' at an earlier or later stage. This small delay may be due to the resistance of the soil or any factor that may make water not to reach the confined aquifer at that expected time.

1.4 Aim and Objectives

The main aim of this study is to suggest a new numerical method that depicts the conversion of flow from confined to unconfined taking into account the delay process.

1.4.1 Objectives of the Study

The research objectives of the study are as follows:

- Analyse the equation of conversion of flow from confined to unconfined groundwater flow using the existing model (Moench and Prickett model).
- Prove that the above equation which is nonlinear gives results which are exaggerated.
- Suggest a new model that will capture the conversion which is not highly linear and can account for delay in the process.
- To derive exact solution using Laplace or Fourier transform.
- Suggest a new numerical method that will capture the conversion of flow from confined to unconfined.
- To compare our results with Moench and Prickett model.
- To provide a realistic model that depicts the conversion from confined to unconfined flow.
- To introduce a delay process due to the retardation caused by the geological formation.
- To derivate exact solution using integral transform operators.
- Solve the conversion of flow from confined to unconfined using the new numerical method.

1.5 Research Outline

The dissertation consists of five chapters;

In chapter one we review the literature on flow in unconfined and confined aquifers, provide background on the significance of groundwater modelling, we derive confined flow and provide the limitations of the existing model. We also state the aim and objectives of this work.

In chapter two we suggest a new mathematical model to depict conversion of flow from confined to unconfined aquifer taking into account the delay process.

Chapter three provides the derivation of exact solution of the new model. We applied integral transform operators such as the Laplace Transform to our equation.

Chapter four provides the new numerical solution using a numerical scheme known as Adams-Bashforth method. Also, in chapter four for stability analysis we used the Von Neumann stability analysis.

Chapter five shows the numerical solutions of our model, we analyse and discuss our results and lastly give the conclusion based on the whole dissertation.

1.6 Research Framework

The following data was used to obtain the aim and objectives of the study.

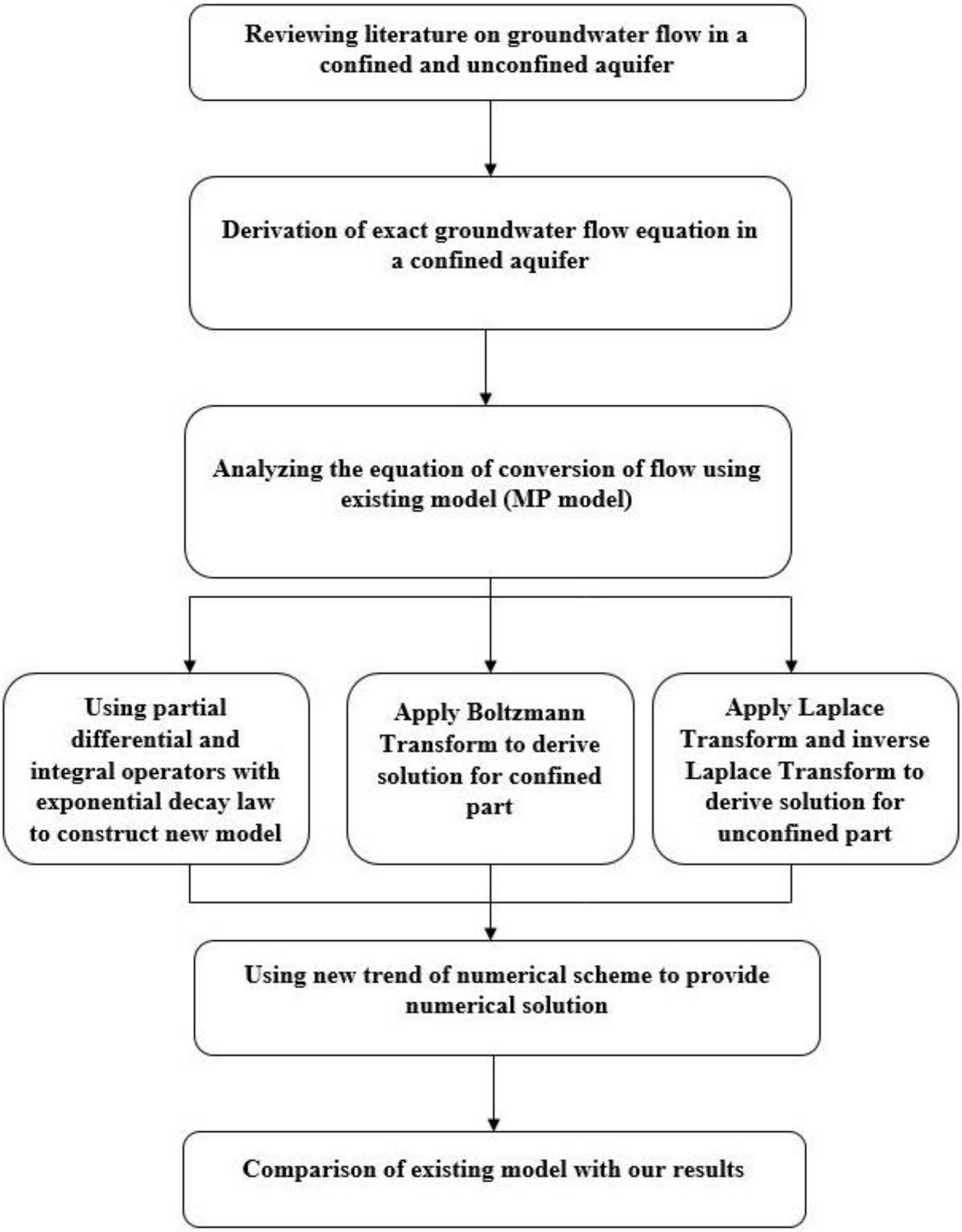

Figure 3: Research framework for the study

1.7 Confined Groundwater Flow

Confined groundwater flow is considered as the main pathway for transporting water from recharge areas to wells and springs. It is impossible to capture confined flow or unconfined flow using the same mathematical models. Therefore, the flow inside a confined aquifer is captured using the Theis model.

1.7.1 Derivation of Groundwater Flow in Confined Aquifer

In deterministic mathematical models of groundwater flow, the Theis (1935) equation is considered as the key fundamental analytical solution. The derivation of movement of groundwater within a confined aquifer begins from Darcy's Law where we have:

$$q = -K\frac{\partial h}{\partial r} \tag{1.1}$$

$$Q = -KA\frac{\partial h}{\partial r} \tag{1.2}$$

Where q represents the Darcy flux (m/s), K is the hydraulic conductivity (m/day), A is the cross-sectional area of flow (m^2), Q is the discharge (m^3/day) and $\frac{\partial h}{\partial r}$ is the hydraulic gradient. The equation has a sign that is negative which emphasizes that groundwater takes the direction of head loss.

Due to the principle of continuity equation of flow, considering an annular cylinder the difference in the rate of inflow and outflow is equal to the rate of change in volume of water in the cylinder.

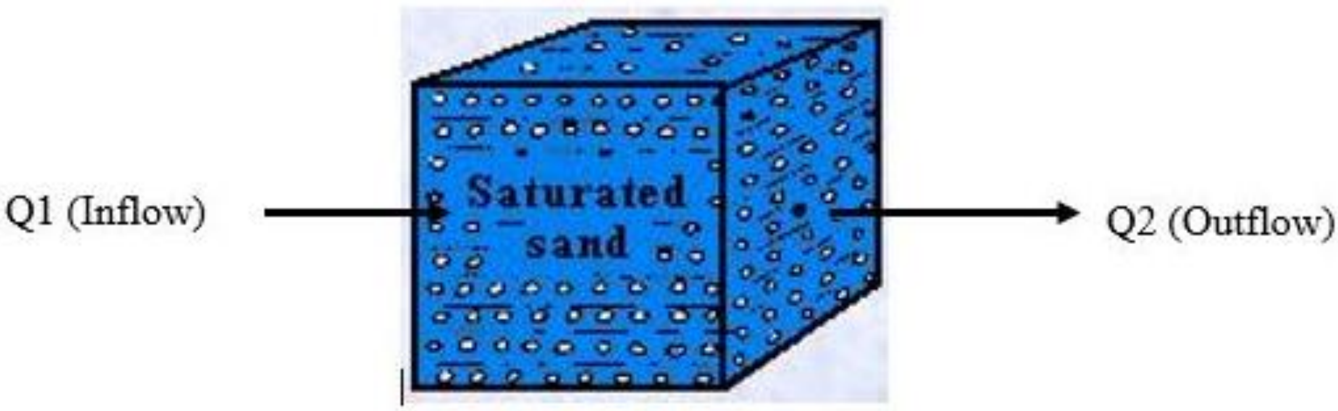

Figure 4: Diagram showing Inflow and Outflow in a porous medium (Google images, 2019)

Thus,

$$Q_{1-}\,Q_2 = \frac{\partial v}{\partial t} \tag{1.3}$$

Where the rate of inflow and outflow is given by Q_1 and Q_2 , respectively and $\frac{\partial v}{\partial t}$ is the rate of change of volume (V) of the cylinder.

The hydraulic gradient line which is the piezometric surface found at the inner surface is $\frac{\partial h}{\partial t}$, h is the height of the piezometric surface over the impermeable layer. Thus, the slope of the hydraulic gradient line at the outer surface is equal to,

$$i = \frac{\partial h}{\partial t} + \frac{\partial^2 h}{\partial r^2} dr \tag{1.4}$$

From Darcy's law,

$$Q = KIA \tag{1.5}$$

Substituting slope of hydraulic gradient Eq. (1.4) and area in Eq. (1.5), we get the inflow given by;

$$Q_1 = K \left[\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} dr\right].2\pi(r + dr)b \tag{1.6}$$

The outflow is given by;

$$Q_2 = K\frac{\partial h}{\partial r}(2\pi r)b \tag{1.7}$$

Where S is the storage coefficient, the volume of water discharged per unit surface area per unit change in head normal to the surface. Hence, the change in volume is written as;

$$\partial V = S(2\pi r)dr.\partial h \tag{1.8}$$

Therefore,

$$\frac{\partial V}{\partial t} = S(2\pi r)dr\frac{\partial h}{\partial t} \tag{1.9}$$

Where t is the time from when the pumping started.

By substituting Eq. (1.6) and Eq. (1.9) in Eq. (1.3), we get;

$$Kb\left[\frac{\partial h}{\partial r}+\frac{\partial^2 h}{\partial r^2}\partial r\right].2\pi(r+\partial r)-Kb\left[\frac{\partial h}{\partial r}\right].(2\pi r)=S(2\pi r)\partial r\frac{dh}{dt} \tag{1.10}$$

Eq. (1.10) is divided by $Kb\ (2\pi r)\ dr$ on both sides and when higher order terms are neglected we get;

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}.\frac{\partial h}{\partial r}=\frac{S}{Kb}.\frac{\partial h}{\partial t} \tag{1.11}$$

Transmissivity is given as $T = Kb$, the transmissivity as Kb is substituted in Eq. (1.11) to obtain;

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}.\frac{\partial h}{\partial r}=\frac{S}{T}.\frac{\partial h}{\partial t} \tag{1.12}$$

Eq. (1.12) is the unsteady state flow equation towards the well. Where h represents the head, r is the radial distance from the well, s is the storage coefficient, T is the transmissivity and t is the time since pumping started.

For a confined aquifer the governing form of equation that is generally used is;

$$\frac{\partial}{\partial x}\left(T_X\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)=S\frac{\partial h}{\partial t}-R+L \tag{1.13}$$

Where h is the hydraulic head, T_x and T_y are the horizontal components of transmissivity, S is the storage coefficient, R is the sink or source term that is intrinsically positive in order to represent the recharge and L is the leakage through the confined layer (Anderson and Woessner, 1992).

However, for the unconfined aquifer it is assumed that the components that represent transmissivity which are, T_x and T_y in Eq. (1.13) are replaced by $T_x = K_\chi h$ and $T_y = K_y h$ respectively. The component of L is equal to zero. The Bousinessq equation is produced which is a nonlinear governing equation (Bear, 1972; Anderson and Woessner, 1992).

1.8 Unconfined Groundwater Flow

Unconfined groundwater flow has been studied largely throughout the past years due to its use in many areas of groundwater such as agricultural drainage, catchment hydrology and groundwater hydraulics (Li et al., 2003). The flow in unconfined aquifers has over the years caused much interest due to factors such as the sudden and unexpected differences in the boundary head which is a problem that has been long considered to be unique in theoretical hydrology (Tolikas et al., 1984; Lockington, 1997).

1.8.1 Unconfined Flow Problem

There are five different approaches in which unconfined flow can be modelled. The very first approach is the application of the equation for confined flow to help model problems for unconfined flow. The second approach is the most common one which makes use of the Boussinesq equation. The third approach looks at radial confined flow that has the delayed yield term. The fourth approach considers an aquifer that is heterogeneous and anisotropic and uses the three-dimensional groundwater flow equation that is given as;

$$\frac{\partial}{\partial x}\left[Kx(x,y,z)\frac{\partial h}{\partial x}\right]+\frac{\partial}{\partial y}\left[Ky(x,y,z)\frac{\partial h}{\partial y}\right]+\frac{\partial}{\partial z}\left[Kz(x,y,z)\frac{\partial h}{\partial z}\right]=S_s\frac{\partial h}{\partial t}+R \qquad (1.14)$$

Where Kx, Ky and Kz represent the hydraulic conductivity tensor components, S_s is known as the specific storage, where R is the term for recharge that has a positive sign and is the volume of inflow to flow system per unit volume of aquifer per unit of time and t is the time from the beginning of test.

The last approach is solving the unconfined flow equation by taking into consideration the unsaturated flow that is above the water table (Yeh and Chang, 2013).

1.8.2 The Bousinessq Equation to Model Unconfined Flow

Unconfined flow is generally captured using the Bousinessq equation that is based on Dupuit assumption (Li et al., 2003). The Bousinessq equation is a nonlinear governing equation because it consists of products of h and $\partial h/\partial x$, making the analytical solution of the Bousinessq equation complicated to get (Yeh and Chang, 2013). This approach ignores the vertical flow and uses the assumption that the water table is a surface that is horizontal before pumping starts. The flow equation within unconfined aquifers is obtained by the combination of the equation of motion that was changed slightly by the Dupuit assumption with mass balance equation (Bear, 1979). The Bousinessq equation is given below;

$$\frac{\partial}{\partial \mathrm{x}}\left(K_\chi h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y h\frac{\partial h}{\partial y}\right)=S_y\frac{\partial h}{\partial t}-R \qquad (1.15)$$

For the equation, h represents the saturated thickness of an unconfined aquifer, S_y represents specific yield of an unconfined aquifer, K_χ and K_y show the horizontal components of conductivity tensor. S_y represents storage coefficient.

For the past years the equation has been applied to model and further analyse distributions of heads in different views of flow in unconfined aquifers (Bear, 1979). There are two methods proposed by Bear of linearization to make it easier to find the solution of Eq. (1.15). The very first method mentioned is for the replacement of the thickness variable, h in Eq. (1.15) with the thickness of the aquifer , b, this occurs when the change of the variable thickness, h from the result of pumping or from recharge is less comparing with the saturated thickness of the aquifer (Yeh and Chang, 2013). As a result, the Bousinessq equation changes to a linear equation that has a similar arrangement as the equation for confined flow if $K_\chi h$, $K_y h$ are replaced by T_χ and T_y respectively. Then the second method mentioned by Bear is to rearrange the right side of the equation as;

$$\left(S_y/b\right)\partial(h^2/2)/\partial t \tag{1.16}$$

This makes Eq. (1.15) to become a linear equation in h^2.

Dupuit-Forchheimer assumptions are considered as one of the most commonly used approaches that simplify the problem of flow in unconfined aquifers, which in turn makes it analytically understandable. The problem with unconfined flow is the position and head of the water table both are unknown (Mishra and Kuhlman, 2013), the Dupuit (1857) assumptions completely remove one of the unknowns. Dupuit (1857) developed the assumptions looking at unconfined aquifers that an aquifer is homogeneous, is infinite aerial extent, has uniform thickness, is isotropic and is pumped at a constant discharge rate. As mentioned before in this work for confined aquifers, in actual fact the opposite is also true for unconfined aquifers. Understanding unconfined flow is complex since the saturation thickness and transmissivity decrease within the cone of depression as the groundwater flow approaches the well (Şen, 2015).

1.9 Conversion from Confined to Unconfined

The increase in the human population for the past years caused a rapid increase in the demand of fresh groundwater all over the world (Xiao, 2014). Fresh groundwater may be required for human consumption, agricultural use, industrial use and environmental activities. Due to an increase in the human population, groundwater is over abstracted to meet requirements that are not met by surface water alone. Less knowledge and understanding of different aquifers result in confined aquifers to be pumped heavily over a long period of time and results in the conversion from confined to unconfined conditions, this changes the natural state of an aquifer (Xiao, 2014). When the extensive pumping rate and pumping period is long, the piezometric surface close to the abstraction well can drop that it becomes beneath the confining zone. The confined aquifer consequently becomes unconfined close to the pumping well (Wang et al., 2009; Wang and Zhan, 2009; Xiao et al., 2018). The heavy pumping may occur during the process of groundwater over-exploration and mine dewatering (Springer and Bair 1992; Chen 1996; Xiao et al., 2018). This conversion occurs in many large aquifers around the world (Wang and Zhan, 2009; Xiao, 2014). The conversion from confined to unconfined can result in variations in hydraulic properties such as storativity, transmissivity and diffusion between confined and unconfined zones (Xiao et al., 2018). The understanding of the conversion is an important factor as it helps with the management of groundwater resources.

1.10 Solutions of Transient Confined to Unconfined Flow

The study on the confined to unconfined conversion has been done in the past before, in the last five decades to be exact. The studies were based on numerical and analytical solutions and were carried out by several investigators. Rushton and Weddeburn (1971) for numerical solutions used a resistance-capacitance electrical analogue to investigate the behaviour of aquifers throughout the confined to unconfined conversion. In the numerical solution, the specific yield for unconfined aquifers replaced the storativity of the confined region. Elango and Swaminathan (1980) proposed a finite-element numerical solution for the confined to unconfined conversion. Using the Dupuit's assumptions, they simulated the conversion of flow from confined to unconfined by applying the finite element methods that consisted of four-sided mixed-curved isoperimetric elements that was restricted to only analyse a steady-state flow (Xiao, 2014). A semi-numerical solution for the confined to unconfined flow was proposed by Wang and Zhan (2009), which took into consideration both changes in transmissivity and storativity throughout the conversion. The solution was able to solve the nonlinearity of unconfined flow using the Runge-Kutta method.

Analytical solutions were introduced by Moench and Prickett (1972), Chen et al., (2006), Hu and Chen (2008) which have improved the understanding of the conversion of flow from confined to unconfined throughout the past years. The models were later known as the MP (Moench and Prickett) and Chen models, respectively. Moench and Prickett (1972) suggested a mathematical solution for the conversion of flow making use of a constant transmissivity for the unconfined layer. The MP model was acquired based on the similar case of heat flow in a cylindrical symmetry in which both melting or freezing can take place (Xiao, 2014). Hu and Chen (2008) explained their model deriving it from the Girinskii's potential function, a potential of steady-state flow of groundwater in a porous medium that is horizontally layered and is used to outline a variation of transmissivity in an unconfined zone of the Chen model (Xiao, 2014). The Chen model was obtained looking at the assumption that the Theis equation that is used for transient flow was correct in the calculation for Girinskii's potential (Wang and Zhan, 2009). However, the error that is made by the approximation is still unclear. To understand a natural problem such as the flow of groundwater since it is out of sight starts with a good construction of a mathematical model.

1.11 Existing Model- Moench and Prickett Model (MP Model)

The existing model is an approximate solution where the governing equation for the unconfined flow is linearized depending on the assumption of a constant transmissivity than making use of a varying transmissivity (Wang and Zhan, 2009). If the variation in water table is less than the saturated thickness of the unconfined layer, it is appropriate to make use of the idea of a constant transmissivity (Bear, 1972). This is not the same if the water table differences are similar with the saturated thickness of the unconfined layer, where the nonlinearity of unconfined flow is considered (Kompani-Zare and Zhan, 2006).

The existing model is based on the schematic sketch (figure 5). Consider a confined aquifer that is pumped and discharges with a constant rate (Q). The piezometric head (H) of the aquifer is higher than the aquifer thickness (b). When the extensive pumping rate and pumping period is long, the hydraulic head near the pumping well drops below the confining layer, making H less than the aquifer thickness. The drop in the piezometric surface will result in the drop in the water table and cause the conversion from confined to unconfined flow close to the abstraction wells. With continuous pumping the piezometric head drops and the converted zone from confined to unconfined becomes larger. The conversion can be described in Eq. (1.17) which shows the equation of flow for the confined region and Eq. (1.23) which is flow equation in the unconfined region. Both equations are symmetrical form of the Boussinesq equation, Eq. (1.15) and are given by the equations below;

Equation of flow for the confined region:

$$Sc\frac{dh}{dt} = \frac{KB}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right), h \geq B \tag{1.17}$$

Equation (1.17) can be expanded to:

$$S_y\frac{\partial h}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right) \tag{1.18}$$

$$S_y = \frac{\partial h}{\partial t}(x,t) = \frac{K}{r}\frac{\partial}{\partial r}\left[rh(r,t)\frac{\partial h}{\partial r}(r,t)\right] \tag{1.19}$$

$$S_y\frac{\partial h(r,t)}{\partial t} = \frac{K}{r}\left[\frac{\partial}{\partial r}[rh(r,t)]\frac{\partial h(r,t)}{\partial r} + \frac{\partial^2 h(r,t)}{\partial r^2}r\frac{\partial h}{\partial r}(r,t)\right] \tag{1.20}$$

$$S_y\frac{\partial h(r,t)}{\partial t} = \frac{K}{r}\left[\left(h(r,t) + \frac{\partial h(r,t)}{\partial r}r\right)\frac{\partial h(r,t)}{\partial r} + r\frac{\partial^2 h(r,t)}{\partial r^2}\frac{\partial h(r,t)}{\partial r}\right] \tag{1.21}$$

$$S_y\frac{\partial h(r,t)}{\partial t} = \frac{K}{r}\left[h(r,t)\frac{\partial h(r,t)}{\partial r} + r\left(\frac{\partial h(r,t)}{\partial r}\right)^2 + r\frac{\partial^2 h(r,t)}{r^2}\frac{\partial h(r,t)}{r^2}\right] \tag{1.22}$$

Equation of flow in the unconfined zone:

$$S_y\frac{dh}{dt} = \frac{K}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right), 0 \leq h \leq B \tag{1.23}$$

Equation (1.17) can be rewritten as;

$$\frac{Sc}{T}\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} \tag{1.24}$$

Equation (1.23) can be rewritten as;

$$Sy\frac{dh}{\partial t} = \frac{Kh}{r}\frac{\partial h}{\partial r} + K\left(\frac{\partial h}{\partial r}\right)^2 + Kh\frac{\partial^2 h}{\partial r^2} \tag{1.25}$$

Where b is the aquifer thickness, h represents the hydraulic head, r shows the radial distance from the pumping well centre, rs is the radial separation of interface that occurs between the confined and the unconfined regions, t is time since the commencement of pumping, where K is the hydraulic conductivity, Sy is the specific yield within an unconfined aquifer and Sc is storage coefficient for the confined aquifer.

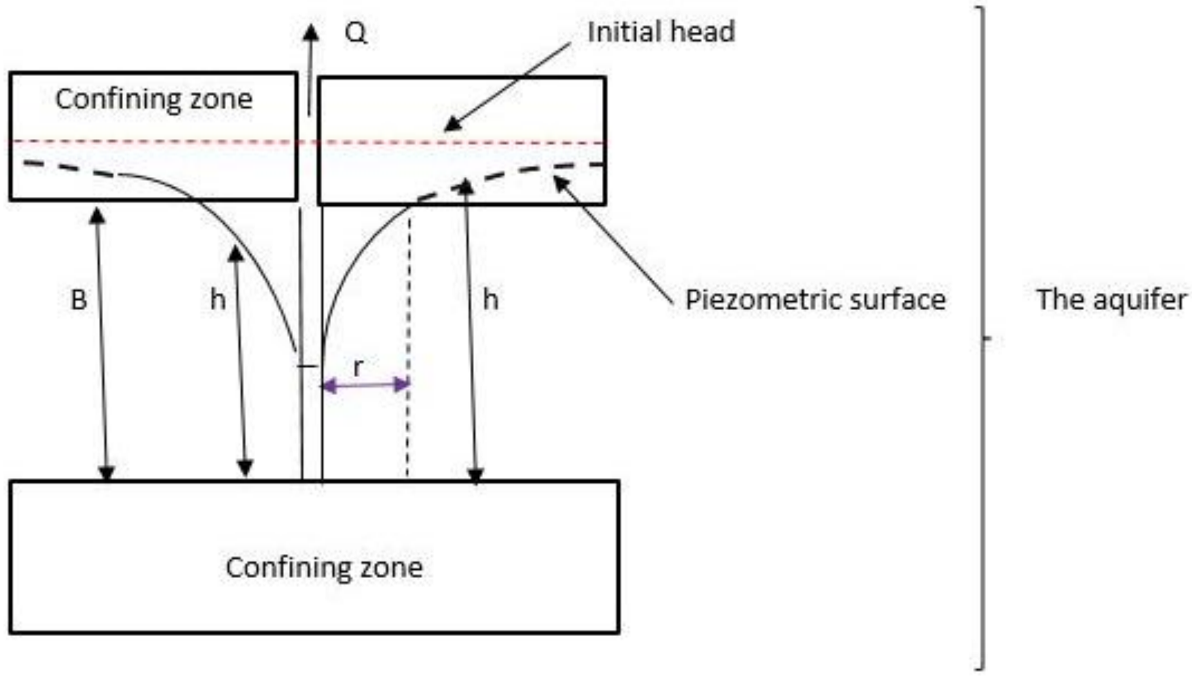

Figure 5: A sketch illustrating the confined to unconfined conversion of flow for the MP model (Wang and Zhan, 2009)

The MP model was proposed on the assumption that the height of the piezometric surface (h) of the unconfined zone is roughly the same as the thickness (B) of the confined aquifer, therefore the MP model can only be accepted when the thickness in the unsaturated zone of the unconfined layer which is near the area where the conversion takes place is notably much less than the thickness of the confined aquifer. However, if the thickness of the unsaturated zone in the area that is far from where the conversion is takes place is very large, when one uses the MP model, sufficiently great errors can occur.

1.12 Limitations of the Existing Model

The developments of analytical solutions such as the MP and Chen Models has tried to improve the understanding of the conversion of flow from confined to unconfined. However, the analytical solutions have several aspects which have not been dealt with in the past years. Limitations of the models are;

- The mathematical equation used to depict the process in the unconfined part is highly non-linear therefore one would expect to obtain an exaggerated prediction.

- The existing model does not take into account the delay process that could occur due to the geological formation that may make water not to reach the confined aquifer at that expected time.
- The heterogeneous nature of the geological formation is not considered in the existing model.
- The existing models do not touch the aspect of hydraulic properties of an aquifer during the conversion because the change of diffusivity is ignored in the Chen model and the variation of transmissivity in the MP model is neglected as well, constant transmissivity is only considered-for the unconfined aquifer (Xiao, 2014).
- The MP model makes use of a constant transmissivity in unconfined aquifer, so the model is only accepted as the unsaturated thickness of an unconfined aquifer is less than the confined aquifer thickness (Xiao, 2014).

CHAPTER 2: NEW MATHEMATICAL MODEL TO CAPTURE THE CONVERSION WITH DELAY

The progress of any field of science starts when existing theories and practises are questioned and even challenged. The nature is highly complex and requires complex models that could be able to replicate their dynamical processes. The conversion problem perhaps is one of the most difficult physical problems to be described using mathematical formulas. Although the existing mathematical model has been used with some success in many problems, it is worthwhile to recall that this model does not really capture the process taking place during the conversion. The mathematical equation used to depict the process in the unconfined part is highly non-linear therefore one would expect to obtain an exaggerated prediction. In addition to this limitation, the model does not take into account the delay process that could occur due to the geological formation. We also point out the fact that, the heterogeneity of the geological formation is not being considered in the existing model. For this section, we propose a new model that accounts for the delay, the delay of course will follow the process of fading memory.

The abstraction of water at constant rate from a well that has been drilled in an unconfined aquifer causes unconfined aquifer water levels to often drop at various quantities from those anticipated by Theis (1935) equation (Neuman, 1972). The Theis (1935) equation for confined flow only accounts for early or late portions also known as segments that are found on the unconfined aquifer time-drawdown curve, so Boulton (1954a) did not prefer the Theis solution for flow of groundwater in unconfined aquifers as it did not consider vertical flow towards the abstraction well. Boulton (1954b, 1963) suggested solutions that reproduce the unconfined aquifer time-drawdown curve for the three portions of the three individual segments of the time-drawdown curve exist, that are identified under water table conditions. The kind of behaviour that is depicted in three portions indicates that the storage coefficient of unconfined aquifers varies with time (Neuman, 1972). Boulton (1954a) and Dagan (1967a, b) proposed solutions to the problem of asymmetric movement of groundwater towards a well completely and partially penetrating respectively an unconfined aquifer and water is slowly discharged from storage instead of instantly. However, the solutions do not reproduce the early segment of the curve but reproduces the early and intermediate portions of the curve, this suggests that water levels at points beneath the unconfined zone must drop instantaneously immediately when pumping begins.

2.1 Delayed Process for Unconfined Flow

The Theis transient principles on confined theory was further analysed and improved by Boulton (1954b) to take into account the delayed yield effect. This extension was because of the changes in the rising and lowering of the water table of unconfined aquifers. Boulton (1954b; 1963) suggested solutions where three segments of the unconfined aquifer time-drawdown curve were able to be reproduced. Using the effect of the delayed yield Boulton assumed that at the same time the water table drops, water moves freely from storage through drainage, slowly instead of instantly as depicted in free-surface solutions proposed by Boulton (1954a) and Dagon (1967) (Mishra and Kuhlman, 2013).

Boulton (1954b) proposed solutions produced an integro-differential flow equation with regard to an averaged drawdown s^* as;

$$\frac{\partial^2 s^*}{\partial r^2} + \frac{1}{r}\frac{\partial s^*}{\partial r} = \left[\frac{S}{T}\frac{\partial s^*}{\partial t}\right] + \left\{\alpha S_y \int_0^t \frac{\partial s^*}{\partial \tau} e^{-\alpha(t-\tau)} d\tau\right\} \tag{2.1}$$

The model proposed by Boulton (1954b, 1963) was made on an assumption that the quantity of water allowed to move from drainage per unit horizontal area of the aquifer due to a unit drawdown that occurs at τ (time) is due to the combination of two components. The components include; S where an amount of water instantaneously escapes for storage at τ which is the time and S_y that represents the amount of water in which the discharge of the water is delayed with time due to $\alpha S_y \; exp[-\alpha(t-\tau)]$ which is the empirical formula, τ represents the time and α is an empirical constant. The solution is linearized by taking T as a constant.

Boulton (1963) proposed that there is a certain time where the effects of the delayed yield are of minor importance is equal to $1/\alpha$ (could be in minutes or hours), which is called the Boulton “delay index” (Neuman, 1972). The effect of the delay index was introduced based on the motion of the water table in unconfined units (Mishra and Kuhlman, 2013). Prickett (1965) came up with an empirical link connecting the delay index and physical properties of the aquifer by analysing field drawdown data that was obtained. Prickett suggested an implemented methodology for K, S, Sy and α for unconfined aquifers with the help of observing pumping tests and using the solution proposed by Boulton (1963).

The “delay index” suggested by Boulton (1963) did not go into detail about the delayed yields physical nature. Boulton’s model did not describe in more detail the delayed process’ physical mechanism but was further able to produce a copy of the segments of the unconfined aquifer time-drawdown curve (Mishra and Kuhlman, 2013). An approximate solution for the full penetration of the abstraction and observation wells and water table decline was developed by Streltsova (1972a). The water table was represented by Streltsova (1972a) as a material boundary that is sharp, therefore the two-dimensional equation of flow with averaged depth (∂^2) is as follows;

$$\frac{\partial^2 s^*}{\partial r^2} + \frac{1}{r}\frac{\partial s^*}{\partial r} = \frac{S}{T}\left(\frac{\partial s^*}{\partial t} - \frac{\partial \xi}{\partial t}\right) \tag{2.2}$$

For the equation above, the amount of decrease of the water table was assumed by Streltsova (1972a) to be linearly proportional with the difference between the vertical averaged head $b - s^*$ and the elevation ξ of the water table as;

$$\frac{\partial \xi}{\partial t} = \frac{K_z}{S_y b_z}(s^* - b + \xi\ ) \tag{2.3}$$

Where $bz = b/3$ is the aquifer effective thickness where the water level below the ground surface is recharged with water into the deep aquifer. $\xi(r, t = 0) = b$ which is the initial condition and identical boundary conditions were used by Streltsova at the abstraction well and the condition of the outer boundary which is $r \to \infty$ that was used both by Theis (1935) and Boulton (1963). Therefore, Eq. (2.2) gives the solution;

$$\boldsymbol{\frac{\partial \xi}{\partial t} = -\alpha_T \int_0^t e^{-\alpha_T(t-r)} \frac{\partial s^*}{\partial \tau} d_\tau} \tag{2.4}$$

Where $\alpha_T = K_z/(S_y bz)$, because both solutions are similar, by replacing them, this is by Eq. (2.4) into Eq. (2.3) to get the solution (2.1) which was proposed by Boulton (1954b). The Boulton "delay index" theory which is the same as that proposed by Streltsova both do not take into account the flow of groundwater within the unsaturated layer but takes the water table as being a dividing layer going downwards vertically under the force of gravity. Meyer (1962) collected raw data from the field that was used by Streltsova to show that the delayed process observed was not affected by the unsaturated flow of water in any way. Even though the delay index was linked to the physical properties of aquifers by Streltsova (1972a), going forward it was later a function of r (Neuman, 1975; Herrera et al.,1978).

Even though one may try to simplify assumptions, the delayed yield theory proposed by Boulton and Streltsova does not consider groundwater flow that is vertical in unconfined aquifers. Both the solutions cannot be expanded to consider for partially penetrating abstraction and observation wells. Some pumping tests were made by Prickett close to Lawrenceville in Illinois which is in the United States of America, (Prickett, 1965) then demonstrated that the specific storage for unconfined aquifers can be higher than the observed values in confined aquifers probably because of air bubbles that are trapped or particles of loosely compacted together shallow sediments. It is very much important to consider the elastic characteristics for unconfined aquifers.

2.2 The Delayed Water Table Response for Unconfined Aquifer

Some difficulties such as conceptual problems were faced by models implemented by Boulton (1954; 1963) in terms of showing the physical nature of water released from storage of unconfined aquifers. Just like Boulton and Streltsova, Neuman (1972) implemented a physically focused mathematical model which considered an unconfined aquifer as something that can be squeezed or flattened by pressure and also the water table as being a moving layer (similarly as Boulton (1954a) and Dagan (1967)). The delayed aquifer response implemented by Neuman, was caused by the release of the physical water table, then he suggested changing the term "delayed yield" with "delayed water table response". Later, the diffusion equation replaced the Laplace equation used by Boulton (1954a) and Dagan (1967). The Neuman (1972) solution is given as:

$$\frac{\partial^2 s_D}{\partial {r^2}_D} + \frac{1}{r_D}\frac{\partial s_D}{\partial r_D} + K_D\frac{\partial^2 s_D}{\partial {z^2}_D} = \frac{\partial s_D}{\partial t_D} \tag{2.5}$$

Neuman considered the water table as a layer that is constantly moving similar with the one of Boulton and Dagan, and later tried to linearize it and demonstrated the anisotropy of the aquifer as a boundary that is three-dimensional, made of the same parts that are facing each other around an axis, showing symmetry. Similarly to Dagon (1967), Neuman (1974) then considered partial penetration. The unconfined aquifer time-drawdown curve has three segments, Neuman was able to reproduce the three segments and produce approximate parameters (with the ability to estimates K_z) same with the delayed yield models. By the application of confined storage in the Neuman (1972), Eq. (2.5).

The Neuman solution was able to display similar data adjustments as compared with the delay index models. The delay index suggested by Boulton was not fully described in terms of its physical nature, Neuman (1975) was not convinced about it, he then tried to find a relationship between the two solutions by Boulton and Neuman, this resulted in a connection of the two solutions.

$$\alpha = \frac{K_z}{S_y b}\left[3.063 - 0.567 log\left(\frac{K_D r^2}{b^2}\right)\right] \tag{2.6}$$

Where α has a linear decrease with log r and is not an aquifer constant. By not taking into consideration the logarithmic phrase in Eq. (2.6), the relation $\alpha = 3K_z/S_y b$ proposed by Streltsova (1972a) is returned to normal state roughly. Following the diverse analysis of several approaches for deciding specific yields, Neuman (1987) was able to deduce that during the abstraction of water, the water table reacts faster as compared to the drainage for the unsaturated zone that is found just on top of it. Later, Malama (2011) proposed a linearization of approximately involving the impacts of unnoticed second order terms, that eventually guides to the boundary conditions of a replacement water table of;

$$S_y \frac{\partial s}{\partial t} = -K_z\left(\frac{\partial s}{\partial z} + \beta \frac{\partial^2 s}{\partial z^2}\right) \qquad z = h_0 \tag{2.7}$$

Where β is a coefficient of linearization represented by [L]. This variable β gives more change to the physical shape of the middle portion of the time-drawdown curve, which leads to the increased approximations of S_y.

2.3 New Mathematical Model to Depict Conversion from Confined to Unconfined

For this chapter we propose a new mathematical model that depicts the conversion from confined to unconfined taking into account the delay process. Eq. (2.8) and (2.9) describes the conversion of flow in confined and unconfined zone respectively. Where h, the hydraulic head (from Eq. 1.17 and 1.23) is replaced by s, due to the drawdown in Eq. (2.8).

The following equation is for the flow in confined zone;

$$S_c \frac{dh}{dt} = \frac{KB}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right), h \geq B \tag{2.8}$$

The equation for the flow in unconfined zone;

$$S_y \frac{\partial s}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(rs\frac{\partial s}{\partial r}\right),\ 0 \leq s \geq B \tag{2.9}$$

The equation below is used for the purpose of this work.

$$\frac{S}{T}\frac{\partial h(r,t)}{\partial t}+\left\{\alpha S_y\int_0^t\frac{\partial h(r,t)}{\partial \tau}e^{-\alpha(t-\tau)}d\tau\right\}=\frac{1}{r}\frac{\partial h(r,t)}{\partial r}+\frac{\partial^2 h(r,t)}{\partial r^2} \tag{2.10}$$

The above equation, Eq. (2.10), is highly non-linear and provides information that is a bit exaggerated in which the results are non-realistic, giving a higher flow than expected. For this work we include a delay process, although this has been used before by many other investigators, we realized that the second part of the equation is highly non-linear and does not take into account the delay process in terms of the water being introduced from confined to unconfined regions. From previous studies it is assumed that the soil is homogeneous in nature, but in reality the opposite is true because it can happen that the soil does not have the same properties. Meaning that the heterogeneous nature of the geological formation is not considered in the existing model. We assume a kind of heterogeneity that will delay the water to arrive at the precise time, by suggesting the delay process we will be able to reduce the nonlinearity of the equation and give something that is approximately what is seen in the field.

Thus, in this work the confined to unconfined conversion will be presented by the following mathematical formula;

$$\begin{cases} S_c\dfrac{\partial h(r,t)}{\partial t}=\dfrac{KB}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{hs}{\partial r}\right), h>B \\ \dfrac{S}{T}\dfrac{\partial h(r,t)}{\partial t}=\dfrac{1}{r}\dfrac{\partial h(r,t)}{\partial r}+\dfrac{\partial^2 h(r,t)}{\partial r^2}-\alpha Sy\displaystyle\int_0^t\dfrac{\partial h(r,t)}{\partial \tau}e^{-\alpha(t-\tau)}d\tau, h\le B \end{cases} \tag{2.11}$$

From Eq. (2.11) above we can say that the system is now well defined, where $e^{-\alpha(t-\tau)}$ is the delay process.

CHAPTER 3: DERIVATION OF EXACT AND NUMERICAL SOLUTION OF THE NEW MODEL

The process of modelling a real-world problem can be evaluated in three steps, the first step is usually the observation of the real-world problem over a period of time and space and then the observed facts can be converted into mathematical models. The second step is to provide an analytical method that could be used to solve such a model analytically in case the model is linear. For this there exist analytical methods including; Laplace transform method, Fourier transform method, variational method, Green function method, the separation of variable method, Boltzmann method used specially for parabolic equations including partial and ordinary. If solving the model analytically is a problem, the model can be solved numerically to give an approximated solution of such a model. The final part of the process is then simulations that lead to prediction. In this chapter, we present in detail the derivative of exact solutions of both equations if possible, if not we shall use some existing or a new numerical scheme to solve the model. For the case of confined aquifer, we present the derivation using exclusively the Boltzmann method to derive the exact solution for the confined part. This is done in the following subsection.

3.1 Derivation for Confined Flow

The Theis (1935) equation for confined flow is derived depending on a governing equation with related initial and boundary of a two-dimensional radial flow in a point source because the aquifer is confined, homogeneous and infinite (Xiao, 2014). Theis (1935) used a comparison between heat conduction and transient groundwater flow to propose an analytical solution that is used for confined transient flow in an abstraction well (Mishra and Kuhlman, 2013). Theis (1935) firstly suggested the unsteady-state flow equation which proposes the storativity and time factor. Boltzmann (1894) first proposed a similarity transform method which solves the differential equation (Xiao, 2014). Botlzmann's method was realized by Birkhoff (1950) that it was developed due to an algebraic symmetry of a partial differential equation, it was also found that the similarity solution of the partial differential equation could be achieved by solving a related normal differential equation (Debnath, 2004). Guessed priori, $u = Sr^2/4tT$ was used by Perina (2010) to change independent variable for a two-dimensional radial flow and proposed a similarity transformation method for the derivation of the Theis equation (Xiao, 2014). For this work, the Boltzmann transform is used to derive the solution for the confined part.

When a confined aquifer is over pumped at a constant rate by a fully penetrating well, the influence of the discharge rate extends outwards as time changes. The storativity multiplied by the rate of decline of head, totalled over the area of influence is equal to the discharge rate Q. Using the existing models, when the extensive pumping rate and pumping period is long, the hydraulic head near the pumping well drops below the confining zone, making the piezometric head less than the aquifer's thickness (figure 5). The drop in the piezometric surface will result in the drop in the water table and cause the conversion from confined to unconfined flow close to the abstraction wells. As mentioned before in this work, the derivation for flow in confined flow is given by;

$$S_c \frac{\partial h(r,t)}{\partial t} = \frac{KB}{r}\frac{\partial}{\partial r}\left(r\frac{h(r,t)}{\partial r}\right), h \geq B \tag{3.1}$$

Where h represents the hydraulic head that is measured from the bottom part of the confining layer of the aquifer (figure 5), t is the time, r is the radial distance from the centre of the well, B represents the aquifer's thickness, Sc is the storage coefficient in the confined aquifer and K is the hydraulic conductivity. We shall mention that, the derivation presented here could be found in other sources.

The equation solution;

$$\frac{\partial h(r,t)}{\partial t} = \frac{T}{S}\left\{\frac{\partial h(r,t)}{r\partial r} + \frac{\partial^2 h(r,t)}{\partial r^2}\right\} \tag{3.2}$$

Where $T = Kb$ is the constant transmissivity of the confined zone.

The solution can be derived using the Boltzmann transform $h(\lambda) = h(r,t)$ with the following transformation:

$$\lambda = \frac{r^2}{t} \tag{3.3}$$

Using the Derivative:

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial t} \tag{3.4}$$

This is because:

$$\frac{\partial\lambda}{\partial t} = -\frac{r^2}{t^2} \tag{3.5}$$

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial\lambda}\left(-\frac{r^2}{t^2}\right) \tag{3.6}$$

Deriving $\lambda = \frac{r^2}{t}$ in respect to r:

$$\frac{\partial\lambda}{\partial r} = \frac{2r}{t} \tag{3.7}$$

$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial\lambda}\cdot\frac{\partial\lambda}{\partial r} = \frac{\partial h}{\partial\lambda}\cdot\frac{2r}{t} \tag{3.8}$$

$$\frac{\partial^2 h}{\partial r^2} = \frac{\partial h}{\partial\lambda}\left(\frac{\partial h}{\partial\lambda}\cdot\frac{2r}{t}\right)\frac{\partial\lambda}{\partial r} = \frac{\partial^2 h}{\partial\lambda^2}\cdot\frac{4r^2}{t^2} \tag{3.9}$$

Substituting back into Eq. (3.2):

$$\frac{\partial h}{\partial\lambda}\cdot\frac{-r^2}{t^2} = \frac{T}{S}\left(\frac{\partial^2 h}{\partial\lambda^2}\cdot\frac{4r^2}{t^2} + \frac{1}{r}\cdot\frac{\partial h}{\partial\lambda}\cdot\frac{2r}{t}\right) \tag{3.10}$$

Dividing by $-\frac{r^2}{t^2}$:

$$\frac{\partial h}{\partial\lambda} = \frac{T}{S}\left(-\frac{4\partial^2 h}{\partial\lambda^2} - \frac{2t}{r^2}\cdot\frac{\partial h}{\partial\lambda}\right) \tag{3.11}$$

Take the inverse of $\lambda = \frac{r^2}{t}$

$$\frac{\partial h}{\partial\lambda} = \frac{T}{S}\left(\frac{-4\partial^2 h}{\partial\lambda^2} - \frac{2}{\lambda}\frac{\partial h}{\partial\lambda}\right) \tag{3.12}$$

Cross multiplication and like-terms together:

$$\frac{S}{T}\frac{\partial h}{\partial\lambda} + \frac{2\partial h}{\lambda\partial\lambda} = \left(\frac{-4\partial^2 h}{\partial\lambda^2}\right) \tag{3.13}$$

Taking out the common factor and division:

$$\frac{\partial^2 h}{\partial\lambda^2} = \frac{\partial h}{\partial\lambda}\left(\frac{-S}{4T} - \frac{1}{2\lambda}\right) \tag{3.14}$$

If $A = \frac{\partial h}{\partial\lambda}$ then $A' = \frac{\partial^2 h}{\partial\lambda^2}$

$$A' = \left(\frac{-S}{4T} - \frac{1}{2\lambda}\right) A \tag{3.15}$$

$$\frac{A'}{A} = \left(\frac{-S}{4T} - \frac{1}{2\lambda}\right) \tag{3.16}$$

Integration:

$$\ln A = -\left(\frac{S\lambda}{4T} + \frac{1}{2}\ln\lambda\right) \tag{3.17}$$

$$\int \frac{A'}{A} = \int \frac{S}{4T} + \int \frac{1}{2\lambda} \tag{3.18}$$

$$\ln A = -\left(\frac{S}{4T}\int 1 + \frac{1}{2}\int \frac{1}{\lambda}\right) \tag{3.19}$$

$$\ln A = -\left(\frac{S}{4T} + \frac{1}{2}\ln\lambda\right) \tag{3.20}$$

$$A = exp\left\{\left[-\left(\frac{S\lambda}{4T} + \frac{1}{2}\ln\lambda\right)\right] + C\right\} \tag{3.21}$$

Where C is a constant

$$\frac{\partial h}{\partial \lambda} = exp\left\{\left[-\left(\frac{S\lambda}{4T} + \frac{1}{2}\ln\lambda\right)\right] \cdot e^{C}\right\} \tag{3.22}$$

From the equation above $e^{C} = W$:

$$\frac{\partial h}{\partial \lambda} = W \cdot exp\left\{\left[-\left(\frac{S\lambda}{4T} + \frac{1}{2}\ln\lambda\right)\right]\right\} \tag{3.23}$$

$$\lim_{n\to\infty} \frac{\partial h}{\partial r} = \frac{-Q}{2\pi T} = W \tag{3.24}$$

Using the boundary addition. If r $\longrightarrow 0$ $then\ \lambda \longrightarrow 0$

$$\frac{\partial h}{\partial \lambda} = \frac{-Q}{2\pi T} \cdot exp\left[-\left(\frac{S\lambda}{4T} + \frac{1}{2}\ln\lambda\right)\right] \tag{3.25}$$

But $n \ln x = \ln x^{n}$

$$\frac{\partial h}{\partial \lambda} = \frac{-Q}{4\pi T} \cdot exp\left[-\left(\frac{S\lambda}{4T} + \ln\lambda^{\frac{1}{2}}\right)\right] \tag{3.26}$$

Separating exponential terms:

$$\frac{\partial h}{\partial \lambda} = \frac{-Q}{4\pi T} \cdot exp\frac{-S\lambda}{4T} + exp\ln\lambda^{-\frac{1}{2}} \tag{3.27}$$

$$\frac{\partial h}{\partial \lambda} = \frac{-Q}{4\pi T} \cdot exp \frac{-S\lambda}{4T} \cdot \frac{1}{\sqrt{\lambda}} \tag{3.28}$$

Let $\lambda = u$

$$\frac{\partial h}{\partial u} = \frac{-Q}{4\pi T} \int_0^{\lambda} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du + C \tag{3.29}$$

$$0 = \frac{-Q}{4\pi T} \int_0^{\infty} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du + C \tag{3.30}$$

$$C = \frac{Q}{4\pi T} \int_0^{\infty} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du \tag{3.31}$$

$$\int \frac{\partial h}{\partial u} = \frac{Q}{4\pi T} \int_0^{\alpha} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du + \frac{Q}{4\pi} \int_0^{\infty} exp\left(\frac{-Su}{4T}\right) \frac{1}{\sqrt{u}} \tag{3.32}$$

$$h(\lambda) = \frac{Q}{4\pi T} \int_0^{\alpha} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du + \frac{Q}{4\pi} \int_0^{\infty} exp\left(\frac{-Su}{4T}\right) \frac{1}{\sqrt{u}} \tag{3.33}$$

$$h(\lambda) = \frac{Q}{4\pi T} \int_u^{\infty} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du \tag{3.34}$$

$$h(\lambda, u) = \frac{Q}{4\pi T} \int_{\alpha}^{\infty} exp\left(\frac{-Su}{4T}\right) \cdot \frac{1}{\sqrt{u}} du \tag{3.35}$$

Let $r = \sqrt{u}$

$$r^2 = u \tag{3.36}$$

$\lambda = u$ but

Where $\lambda = \frac{r^2}{t}$, $\frac{r^2 S}{4Tt} = u$

If substitute u into $\lambda = \frac{r^2}{t}$

$$\frac{u}{t} = \frac{r^2 S}{4Tt} \tag{3.37}$$

For the Theis solution the drawdown will be a function of Q, r, t, T and S. Both S and Q are not included in the definition of the dimensionless variable because S is a dimensionless variable and Q is in the boundary condition only. There is no characteristic length or time for the Theis problem, so a dimensionless combination of r and T should be found. The chosen dimensionless variable is:

$$U = \frac{r^2 S}{4T} = y \tag{3.38}$$

Let $y = \frac{Sr^2}{4T}$

The Theis (1935) equation derived from the comparison between groundwater flow and heat conduction is given by the solution:

$$\boldsymbol{h(r,t) = \frac{Q}{4\pi T}\int_u^\infty exp\frac{(-y)}{y}\cdot dy} \tag{3.39}$$

3.2 Laplace Transform

Theis (1935) came to a conclusion that the Darcy's law was much more like the law of conduction of heat flow in solids and used a solution of distribution of temperature because of an instantaneous line heat source to derive his 1935 equation (Xiao, 2014). Jacob (1940) proposed an initial guess of drawdown gradient that was later used by Li (1972) to propose a derivation process of the Theis (1935) equation. Later, Verruift (1982) introduced the solution of the Theis problem using the Laplace transform. Marquis Pierre-Simon de Laplace who was an astronomer and later a mathematician (1749-1827), proposed the transform based on his work he made on the probability theory. Since then the Laplace transform has been an extensively used integral transform in a lot of physics and engineering applications.

Partially differential equation can be used to mathematically describe groundwater flow with related boundary and initial conditions (Yeh and Cheng, 2013). The Laplace transform is an integral transform that gives a simple and effective way of working out problems in hydraulics. The Laplace transform is generally good in reducing differential equation to a simpler form or to an algebraic expression, that can be solved by the use of algebraic rules and the inverse Laplace transform is used for solving the original differential equation (Yeh and Cheng, 2013). The Laplace transform can solve both linear and nonlinear equations.

It is generally used for "time-like" variables to remove the time derivative, if $f(t)$ is a function of t, then the Laplace transform $\tilde{f}(p)$ is defined by:

$$\tilde{f}(p) = \mathcal{L}\{f(t); t \to p\} = \int_0^\infty f(t).\, e^{-st}\, dt \tag{3.40}$$

Where s is complex parameter, the over-bar represents the Laplace transform and the transformed variable. The integral that describes the Laplace transform has to be convergent so that the transformation can exist, in this case the original function $f(t)$ should meet the following conditions:

1. The function $f(t)$ has to be continuous over the interval $0 < t < \infty$, if the interval is able to be divided into a restricted number of intervals that are not intersecting and the function has to have finite limits at the end of every subinterval (Lobontiu, 2018).
2. The function $f(t)$ must be an exponential order, and has to be in the form:

$$\lim_{t\to\infty} |f(t)|\,.\,e^{-\sigma.t} = 0 \tag{3.41}$$

Where the values of the constant σ are higher than the threshold value σ_c, this is known as the abscissa of convergence.

Eq. (3.40) is a product of two functions and can be solved by integration of parts of the derivative, so df/dt becomes:

$$\mathcal{L}\{f'(t)\} = e^{-st}.f(t) + \int_0^\infty se^{-st}\,.f(t)dt \tag{3.42}$$

From Eq. (3.40), we differentiated e^{-pt} and integrated $f(t)$ to get Eq. (3.42) above.

$$\mathcal{L}\{f'(t)\} = 0 - e^{-s(0)}.f(0) + s\int_0^\infty e^{-st}\,.f(t)dt \tag{3.43}$$

Where p is a constant and is outside of the intergral which is the Laplace transform of the function $f(t)$

$$\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\} \tag{3.44}$$

From Eq. (3.44) above we will be able to reduce the Laplace transform of $f(t)$ into Eq. (3.44) above to give:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{3.45}$$

Where Eq. (3.45) is the first-order derivative.

Now that we have $\mathcal{L}\{(f(t))'\} = s.\mathcal{L}\{f(t)\} - f(0)$ which is the first-order derivative.

Then $\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\} = s.\mathcal{L}\{f'(t)\} - f'(0)$ (3.46)

$$\mathcal{L}\{f''(t)\} = s(s\mathcal{L}\{f(t)\} - f(0)) - f'(p) \tag{3.47}$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - pf(0) - f'(0) \tag{3.48}$$

Where Eq. (3.48) is the second-order derivative.

The derivation becomes a multiplication. Therefore, this reduces the complexity of a differential equation and allows to achieve the answer of the differential equation where the real solution can be achieved by back transformation (Delleur, 2007). Transient groundwater flow problems are generally solved using the Laplace transform, this is when a first order time derivative of a function such as hydraulic head or drawdown is involved. The principle key properties of Laplace transforms include differentiation, linearity and time delay of a function. Think of the Laplace transform as some sort of a function machine where there is a function of time $f(t)$ as the input function and the output function is represented as $F(s)$ (Figure 6). In this case the input is taken as the time domain and the output is the s-domain. The principle of Laplace transform is to make use information of a system in the time domain and transform it to information in the s-domain.

Figure 6: The Laplace transform "machine"

3.3 Inverse Laplace Transform

Converting the Laplace transform as a function of time gives the inverse Laplace transform, this can be shown using the Fourier transform. An alternative way of solving groundwater flow problems is analytically by the use of the Fourier transformation which is also a mathematical approach. However, the Fourier transform shows a function as a series of vibrations known as frequencies.

The inverse Laplace transform is written as;

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-\infty}^{\sigma+\infty} \hat{F}\,(s) e^{st} dp \tag{3.49}$$

Where $\sigma > 0$ represents a real number, such that the contour of integration is to the right of any singularities of $\hat{F}(s)$.

There are a lot of problems with the inversion of the Laplace transform and is quite complex whereby it usually cannot be easily controlled in a few groundwater problems. An analytical solution such as the inverse Laplace transform is a time-domain solution which is usually expressed as improper integrals having limits of integration that range from zero to infinity with integrands that consist of singularity at the beginning (Yeh and Cheng, 2013). The integrands are expressed as oscillatory functions that consist of first and second order terms of the Bessel functions. It gets difficult to perform solutions to these numerical calculations precisely as they are time consuming. For this reason, the transform cannot be inverted analytically. Therefore, a number of methods are used for the Laplace transform's numerical inversion. However, these different numerical methods give accurate answers for different classes of functions so for this reason there is no best method (Hoog et al., 1982). To achieve the inverse of Laplace transform, methods or rules including partial fractions, the shift theorem and convolution theorem are used. For this work we focus on the convolution theorem to acquire the inverse Laplace transform.

3.4 The Convolution Theorem

The convolution theorem is another good approach of getting the inverse of the Laplace transform of an s-domain function that can then be given as the product of two functions (Lobontiu, 2018). The theorem focuses on the convolution of two functions $f(t)$ and $g(t)$. Convolution introduces the symbol "*" for the two functions $f(t)$ and $g(t)$ to give:

$$f(t) * g(t) = \int_0^t f(\tau).\, g(t-\tau) d\tau \tag{3.50}$$

From Eq. (3.50) it shows that one term operating on the second term is equivalent to the second term operating on the first term. This means that the convolution of two functions gives a commutative operation.

For example, if the change of variable $\tau_1 = t - \tau$ is introduced in Eq. (3.50) the equation can now change to:

$$\int_0^t f(\tau).\,g(1-\tau)d\,\tau = -\int_t^0 f(t-\tau_1).\,g(\tau_1)d\tau_1 = \int_0^t g(\tau_1).\,f(t-\tau_1)d\tau_1 \tag{3.51}$$

$$= g(t) * f(t) \tag{3.52}$$

The convolution theorem states that:

$$\mathcal{L}[f(t) * g(t)] = F(s).\,G(s) \tag{3.53}$$

To validate Eq. (3.53) apply the Laplace transform to Eq. (3.50) with respect to the definition:

$$\mathcal{L}[f(t) * g(t)] = \int_0^\infty \left[e^{-s.t}.\int_0^t f(t-\tau).\,g(\tau)d\tau\right]dt \tag{3.54}$$

Where t is a variable that ranges between 0 to ∞, τ is bounded by 0 and t. The order of integration on Eq. (3.54) is swapped because dt is outside of the bracket and $d\tau$ is inside of the bracket. The inside integral of which its variable becomes t has limits which are τ and ∞ whereas the outside intergral of which its varable is τ has limits which still remain 0 and ∞. Therefore, Eq. (3.54) becomes:

$$\mathcal{L}[f(t) * g(t)] = \int_0^\infty \left[\int_0^t e^{-s.t}.\,f(t-\tau).\,g(\tau)dt\right]d\tau \tag{3.55}$$

Where $t = \tau + t_1$ is used in the inside integral of Eq. (3.55) in which t_1 is a new variable and τ acts as a constant term with respect to the integral. Therefore, Eq. (3.55) changes to:

$$\mathcal{L}[f(t) * g(t)] = \int_0^\infty \left[e^{-s.t}.\,g(\tau).\int_0^\infty e^{-s.t_1}.\,f(t_1)dt_1\right]d\tau \tag{3.56}$$

$$= \left[\int_0^\infty e^{-s.t}.\,g(\tau)d\tau\right].\left[\int_0^\infty e^{-s.t_1}f(t_1)dt_1\right] = G(s).\,F(s)$$

$$= F(s).\,G(s) \tag{3.57}$$

This proves the convolution theorem.

If you now apply the inverse Laplace transform to Eq. (3.53) it gives:

$$\mathcal{L}^{-1}[F(s).\,G(s)] = \mathcal{L}^{-1}\big[\mathcal{L}[f(t) * g(t)]\big] = f(t) * g(t) = \int_0^t f(\tau).\,g(t-\tau)d\tau \tag{3.58}$$

Eq. (3.58) shows the inversion of the Laplace transform of the product of two functions in the s-domain (Laplace) is equal to the convolution of the first (time domain) functions.

3.5 Applying the Laplace Transform to our Equation

In the following subsection we apply the Laplace transform to reduce our differential equation to a much simpler form that can be expressed as an algebraic expression that will be able to be solved using algebraic rules.

The following equation is nonlinear, when we apply the Laplace transform, we get;

$$\frac{S}{T}\frac{\partial h(r,t)}{\partial t}+\left\{\alpha S_y\int_0^t\frac{\partial h(r,t)}{\partial \tau}e^{-\alpha(t-\tau)}d\tau\right\}=\frac{1}{r}\frac{\partial h(r,t)}{\partial r}+\frac{\partial^2 h(r,t)}{\partial r^2} \tag{3.59}$$

$$\mathcal{L}_t\left(\frac{S}{T}\frac{\partial h(r,t)}{\partial t}+\left\{\alpha S_y\int_0^t\frac{\partial h(r,t)}{\partial \tau}e^{-\alpha(t-\tau)}d\tau\right\}\right)=\mathcal{L}\left(\frac{1}{r}\frac{\partial h(r,t)}{\partial r}+\frac{\partial^2 h(r,t)}{\partial r^2}\right) \tag{3.60}$$

$$\frac{S}{T}\mathcal{L}_t\left(\frac{\partial h(r,t)}{\partial r}\right)+\alpha S_y\mathcal{L}_t\left(\int_0^t\frac{\partial h(r,\tau)}{\partial \tau}e^{-\alpha(t-\tau)}d\tau\right)=\frac{1}{r}\frac{\partial \tilde{h}(r,p)}{\partial r}+\frac{\partial^2 \tilde{h}(r,p)}{\partial r^2} \tag{3.61}$$

$$\frac{S}{T}\left[p\tilde{h}(r,p)-h(r,0)\right]+\alpha S_y\mathcal{L}\left(\frac{\partial h(r,t)}{\partial t}\right)\mathcal{L}\left(e^{(-\alpha t)}\right)=\frac{1}{r}\frac{\partial \tilde{h}(r,p)}{\partial r}+\frac{\partial^2 \tilde{h}(r,p)}{\partial r^2} \tag{3.62}$$

$$\frac{S}{T}\left[p\tilde{h}(r,p)-h(r,0)\right]+\alpha S_y\left[p\tilde{h}(r,p)-h(r,0)\right]\frac{1}{p+\alpha}=\frac{1}{r}\frac{\partial \tilde{h}(r,p)}{\partial r}+\frac{\partial^2 \tilde{h}(r,p)}{\partial r^2} \tag{3.63}$$

By grouping, put $\tilde{h}(r,p)$ together to get:

$$\tilde{h}(r,p)\left[\frac{Sp}{T}+\alpha\frac{Syp}{p+\alpha}\right]-h(r,0)\left[\frac{S}{T}+\frac{\alpha Sy}{p+\alpha}\right]=\frac{1}{r}\frac{\partial \tilde{h}(r,p)}{\partial r}+\frac{\partial^2 \tilde{h}(r,p)}{\partial r^2} \tag{3.64}$$

By factorization, let $\beta(p)=\frac{S}{T}p+\frac{\alpha Sy}{p+\alpha}p$

$$\tilde{h}(r,p)\beta(p)-\frac{1}{r}\frac{\partial \tilde{h}(r,p)}{\partial r}-\frac{\partial^2 \tilde{h}(r,p)}{\partial r^2}=h(r,0)\,\beta(p) \tag{3.65}$$

For simplicity, we put $\tilde{h}(r,p)=B(r)$ to get:

$$B(r)\beta(p)-\frac{1}{r}\frac{d}{dr}B(r)-\frac{d^2B(r)}{dr^2}=B_1(r)\beta(p) \tag{3.66}$$

Multiplying Eq. (3.66) by r we get:

$$rB(r)\beta(p) - \frac{d}{dr}B(r) - \frac{rd^2B(r)}{dr^2} = rB_1(r)\beta(p) \qquad (3.67)$$

We know that where $\mathcal{L}\{t.f(t)\} = \frac{d}{ds}F(s)$, so:

$$\mathcal{L}\left(\frac{d}{dr}B(r)\right) = -\frac{d}{ds}\left[\mathcal{L}\left(\frac{d}{dr}B(r)\right)\right] \qquad (3.68)$$

$$= -\frac{d}{ds}[s\tilde{B}(s) - B(0)] \qquad (3.69)$$

$$= -s\frac{d}{ds}\tilde{B}(s) + \tilde{B}(s) \qquad (3.70)$$

Applying the Laplace transform on the second order derivative, we get:

$$\mathcal{L}\left(r\frac{d^2}{dr^2}B(r)\right) = -\frac{d}{ds}\left[\mathcal{L}\left(\frac{d^2}{dr^2}B(r)\right)\right] \qquad (3.71)$$

$$= -\frac{d}{ds}[s^2\tilde{B}(s) - sB(0) + B'(0)] \qquad (3.72)$$

$$= -\left[2s\tilde{B}(s) + s^2\frac{d}{ds}\tilde{B}(s) - B(0)\right] \qquad (3.73)$$

$$= -2s\tilde{B}(s) + s^2\frac{d}{ds}\tilde{B}(s) - B(0) \qquad (3.74)$$

Applying Laplace transform on $B(r)\beta(p)$ to get:

$$\mathcal{L}(rB(r)\beta(p)) = -\beta(p)\frac{d}{ds}\tilde{B}(s) \qquad (3.75)$$

Applying Laplace transform on $B_1(r)\beta(p)$ to get:

$$\mathcal{L}(B_1(r)\beta(p)) = -\beta(p)\mathcal{L}\big(B_1(r)\big) \qquad (3.76)$$

$$= -\beta(p)\tilde{B}_1(s) \qquad (3.77)$$

Now the equation becomes:

$$-\beta(p)\frac{d}{ds}\tilde{B}(s) - s\frac{d}{ds}\tilde{B}(s) + \tilde{B}(s) - 2s\tilde{B}(s) + s^2\frac{d}{ds}\tilde{B}(s) - B(0) = -\beta(p)\tilde{B}_1(s) \qquad (3.78)$$

By factorizing $\frac{d}{ds}\tilde{B}(s)$ we get;

$$\frac{d}{ds}\tilde{B}(s)[s^2 - s - \beta(p)] + [1 - 2s]\tilde{B}(s) = B(0) - \beta(p)\tilde{B}_1(s) \tag{3.79}$$

Divide everything by $s^2 - s - \beta(p)$;

$$\frac{d}{ds}\tilde{B}(s) + \frac{1 - 2s}{s^2 - s - \beta(p)}\tilde{B}(s) = \frac{B(0)}{s^2 - s - \beta(p)} - \frac{\beta(p)\tilde{B}_1(s)}{s^2 - s - \beta(p)} \tag{3.80}$$

$$\frac{d}{ds}\tilde{B}(s) + q(s)\tilde{B}(s) = g(s) \tag{3.81}$$

Where $q(s) = \frac{1-2s}{s^2-s-\beta(p)}$ and $g(s) = \frac{B(0)}{s^2-s-\beta(p)} - \frac{\beta(p)\tilde{B}_1(s)}{s^2-s-\beta(p)}$

Now we can solve the above equation using linear differential equations.

3.6 Linear Differential Equations

The type of first order differential equations focused on is the linear first order differential equation, where a formula is derived for the general solution, in this case making it different from majority of the first order cases. It is important not to memorize the formula but instead memorize and have a clear understanding of the process that is used to drive the formula. In this case, it makes it easier to solve problems by using the process instead of the formula.

The main goal of solving first order linear differential equations is to get an answer that is in the form $y = y(t)$. When one solves a linear first order differential equation, the equation should be in the form that is given below, if not the process that will be used will not work in this case.

$$\frac{dy}{dt} + p(t)y = g(t) \tag{3.82}$$

Where $p(t)$ and $g(t)$ are continuous functions. A continuous function is a function that is continuous when you could draw the graph without lifting a pencil, this function has no breaks in it from left to right.

Now, assume the existence of a function $\mu(t)$ known as an integrating factor. Then multiply everything in the above equation, Eq. (3.82) with $\mu(t)$ to get;

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{3.83}$$

Now we are going to assume that $\mu(t)$ will satisfy this;

$$\mu(t)p(t) = \mu'(t) \tag{3.84}$$

At this stage it is not a concern where we will find a $\mu(t)$ that will fulfil the above equation, we will find it provided that $p(t)$ is continuous. Therefore, substituting the above equation, we have;

$$\mu(t)\frac{dy}{dt} + \mu'(t)y = \mu(t)g(t) \tag{3.85}$$

The above equation shows that on the left of the equation is the product rule.

$$\mu(t)\frac{dy}{dt} + \mu'(t)y = (\mu(t)y(t))'$$

We can replace the left side of Eq. (3.85) with this product rule, so Eq. (3.85) now becomes;

$$(\mu(t)y(t))' = \mu(t)g(t) \tag{3.86}$$

From this point we have to do something about $y(t)$, so we have to integrate both sides after using a bit of algebra and we will get the solution. So, by integrating the above equation we get;

$$\int (\mu(t)y(t))'dt = \int \mu(t)g(t)dt \tag{3.87}$$

$$\mu(t)y(t) + c = \int \mu(t)g(t)dt \tag{3.88}$$

Note that it is important to include the constant of integration, which is c. The constant is on the left of the equation above, if it is left out, a wrong answer will be obtained all the time.

For the final step of the process, basic algebra will be used to solve for $y(t)$, the solution.

$$\mu(t)y(t) = \int \mu(t)g(t)dt - c \tag{3.89}$$

$$y(t) = \frac{\int \mu(t)g(t)dt - c}{\mu(t)} \tag{3.90}$$

Since the constant of integration, c is an unknown constant, to make things easier we will replace the minus sign in front of the constant with a plus sign instead, keep in mind that this will not affect the answer for the solution, so we get;

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)} \tag{3.91}$$

Now that we have a general solution to Eq. (3.82), we need to go back to the function $\mu(t)$ to determine exactly what it is. We will start with Eq. (3.84).

$$\mu(t)p(t) = \mu'(t) \tag{3.92}$$

Dividing the equation on both sides by $\mu(t)$ to get;

$$\frac{\mu'(t)}{\mu(t)} = p(t) \tag{3.93}$$

From Calculus I class we recognize that the left of the equation is the following derivative;

$$(\ln \mu(t))' = p(t) \tag{3.94}$$

Integrate both sides to get;

$$\ln \mu(t) + k = \int p(t)dt \tag{3.95}$$

$$\ln \mu(t) = \int p(t)dt + k \tag{3.96}$$

The constant of integration which is k is moved to the right side from the left side of the equation and absorbed the minus sign to it as we did earlier on. Note that k is used because we have already used c, to avoid confusion we used different letters because they will both be in the same equation and will have different values.

Form the above equation, Eq. (3.96), we then exponentiate both sides to move $\mu(t)$ from the natural logarithm.

$$\mu(t) = e^{\int p(t)dt+k} \tag{3.97}$$

It will be a problem to have k in the exponent, so we will remove it as an exponent in the following way;

$$\mu(t) = e^{\int p(t)dt+k} \tag{3.98}$$

$$= e^k e^{\int p(t)dt} \tag{3.99}$$

Remember that $x^{a+b} = x^a x^b$.

Both k and e^k are unknown constants so we can rename e^k to k to make things much easier to give;

$$\mu(t) = ke^{\int p(t)dt} \tag{3.100}$$

Now we have both formulas for the general solution, Eq. (3.91) and for the integrating factor Eq. (3.100). The problem now is the two unknown constants, therefore it is important to eliminate at least one of them. This is done by putting Eq. (3.92) into Eq. (3.91) and rearrange the constants.

$$y(t) = \frac{\int ke^{\int p(t)dt} g(t)dt + c}{ke^{\int p(t)dt}} \tag{3.101}$$

$$= \frac{k \int e^{\int p(t)dt} g(t)dt + c}{ke^{\int p(t)dt}}$$

$$u(s,p) \; = \frac{\int e^{\int p(t)dt} g(t)dt + \frac{c}{k}}{e^{\int p(t)dt}} \tag{3.102}$$

After using initial and boundaries conditions in Laplace space, we applied the inverse Laplace transform respect to r, to obtain;

$$\bar{h}(r,s) = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n s^n r^{2n}}{2^{2n}(n!)^2 (s+\alpha)^{\frac{n}{2}}} + \frac{a}{s} \tag{3.103}$$

Using the inverse Laplace transform with respect to s we get the following general solution.

$$h(r,t)$$
$$= a\delta(t)$$
$$+ \frac{Q}{2\pi T} \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n}}{2^{2n}(n!)^2} t^{-1-\frac{n}{2}} Hypergeometric\ 1F1Regularized\ \left[\frac{n}{2}, -\frac{n}{2}, -at\right] \tag{3.104}$$

The above solution is the exact solution of the second equation, we shall note that such solution is first derived only in this work.

Therefore, the exact solution of our system is given by;

$$\begin{cases} \boldsymbol{h}(\boldsymbol{r},\boldsymbol{t}) = \dfrac{\boldsymbol{Q}}{\boldsymbol{4\pi T}}\displaystyle\int_{\boldsymbol{u}}^{\infty} \boldsymbol{exp}\frac{(-\boldsymbol{y})}{\boldsymbol{y}} \cdot \boldsymbol{dy}, h > B \\ h(r,t) = a\delta(t) + \dfrac{Q}{2\pi T}\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n r^{2n}}{2^{2n}(n!)^2} t^{-1-\frac{n}{2}} Hypergeometric\ 1F1Regularized\ \left[\frac{n}{2}, -\frac{n}{2}, -at\right], h \leq B \end{cases} \tag{3.105}$$

While we have completed with great success the derivation of the exact solution, while such a solution is novel, we must point out that the obtained exact solution is in the form of a series, which could be problematic when dealing with real world data. In particular, one will not be able to consider all the terms of the series. In this case, we shall now deviate our focus to derive the numerical solution using some accurate numerical scheme.

CHAPTER 4: NEW NUMERICAL SCHEME USING ADAMS-BASHFORTH METHOD

The Laplace transform was successfully used to derive the exact solution. Due to the complicated nature of the equation, we used an appropriate numerical scheme such as the Adams-Bashforth method to derive the numerical solution. This is done in this chapter.

4.1 Crank-Nicolson Method

British mathematicians John Crank (1916-2006) and Phyllis Nicolson (1917-1968) first introduced the Crank-Nicolson method during the early 20th century. The method was used for working out parabolic differential equations. Crank (1947) came up with a finite difference method for numerical evaluation of solving heat conduction type equations and partial differential equations. It is generally considered as the best method for solving numerical integration of diffusion problems. The method is of second order in space, implied in time, of higher order frequency and stable (Fadugba et al., 2013).

Frankel (1953) modified the simple explicit scheme and was able to show that the method is stable as compared to the simple explicit scenario which enabled higher time steps that will be of use. Hofman (1992) gave thought to the accuracy and stability of finite difference method for option pricing. Later, Britz (1988) found that the precision of a simple explicit method is hardly made better upon.

The Crank-Nicolson method is considered as a finite difference method that tries to work out partial differential equations, the approximation of the differential equation over the area of integration using a group of algebraic equations. This helps to achieve numerical solutions to partial differential equations (Fadugba et al., 2013). In the method we substitute the partial derivative found in the partial differential equations by estimations that are looked upon the expansion of the Taylor series that are close to the point or points that are of interest. Both derivatives at time steps n and $n + 1$ are applied to evaluate the spatial second derivative for the finite difference form of heat equations.

4.2 Linear Multistep Method

Linear multistep methods make use for numerically solving ordinary differential equations. When using numerical methods, we begin from a starting point and then take a step that is short to go forward in time and find the following solution point. A linear multistep method is used to find approximate solutions for initial value problems that take the form; $y' = f(t, y)$, $y(t_0) = y_0$. This results in estimations for the $y(t)$ value at separate times t_i; $y \approx y(t_i)$ where $t_i = t_0 + ih$. Where h is the time step which is written as Δt and i represents an integer.

The methods use details from last s steps to compute the next value and makes use of a linear combination of y_1 and $f(t_i, y_1)$ for the calculation of the value of y to get the anticipated present step. Therefore, a linear multistep method takes the form;

$$y_{n+1} + a_{s-1}.y_{n+1-s} + a_{s-2}.y_{n+s-2} + \cdots + a_0.y_n = h.(b_s.f(t_{n+s}, y_{n+s}) + b_{s-1}.f(t_{n+s-1}, y_{n+s-1}) + \cdots + b_0.f(t_n, y_n)) \quad (4.1)$$

$$\Leftrightarrow \sum_{j=0}^{s} a_j y_{n+j} = h \sum_{j=0}^{s} b_j f\left(t_{n+j}, y_{n+j}\right) \quad (4.2)$$

Where $a_s = 1$ and the coefficients $a_0, \dots, a_{s-1}$ and $b_0, \dots, b_s$ determine the method.

Differentiating between explicit and implicit is simple. If $b_s = 0$, the method is known to be explicit because the formula directly calculates y_{n+s}, and if $b \neq 0$, the method is known to be implicit. This is because the value of y_{n+s} is depended on the value of $f(t_{n+s}, y_{n+s})$, so the equation can be solved for y_{n+s}. Methods including Newton's method are applied to work out the implicit formula. The value of y_{n+s} can at times be predicted by using the multistep method. Thus, the value is applied in an implicit formula so as to "correct" the value, which results in the predictor-corrector method.

The families of commonly used linear multistep methods include; Adams-Bashforth (AB) and, Adams-Moulton (AM) methods and lastly the backward differentiation formulas (BDFs).

4.2.1 Adams-Bashforth Method (AB)

The method is abbreviated as AB and is an explicit method, so the method makes use of details that were obtained from the present and last time-steps to calculate the answer at t_{n+1}, where s is the order of the method.

The coefficients of the method are $a_{s-1} = 1$ and $a_{s-2} = \cdots a_0 = 0$ while the b_j are selected so that the methods have order s which determines the methods uniquely.

The forward Euler method is given by Adams-Bashforth method of first order while Adams-Bashforth method of second order is written as;

$$y_{n+1} = y_n + \frac{h}{3}\Big(3f\big(y_n, t_n - f(y_{n-1}, t_{n-1})\big)\Big) \tag{4.3}$$

The second order AB method is conditionally stable and needs the solution from the $n-1$th and the nth step.

4.2.2 Adams-Moulton Method (AM)

The method is abbreviated as AM and is an implicit method. The methods make use of information at t_{n+1} to calculate y^{n+1}. AM methods are considered to be the same as Adams-Bashforth methods because they both use $a_{s-1} = 1$ and $a_{s-2} = \cdots a_0 = 0$, the b coefficients are selected to get the highest order possible. With the AM method as an implicit method, taking out the limitation that $b_s = 0$ and an s-step Adams-Moulton method can get to order $s+1$, whereas an s-step AB method only has order s.

The first order AM methods are the backward Euler methods, where the AM methods of second order can sometimes be called the trapezoidal rule. The second order AM method has a stepping equation that is;

$$y_{n+1} = y_n + \frac{3h}{2} f\big(t_{n,} y_n\big) - \frac{h}{2} f\,\big(t_{n-1,} y_{n-1}\big) \tag{4.4}$$

4.2.3 Backward Differentiation Formulas (BDF)

The formulas are different from the Adams methods. The methods are implicit with $b_{s-1} = \cdots = b_0 = 0$ where other coefficients are selected so the method achieve order s which is the highest possible. BDF methods are commonly used for solutions of stiff differential equations.

4.3 Predictor-Corrector Method

This method uses both explicit and implicit methods to obtain a method that gives good convergence characteristics. This method integrates ordinary differential equations to get a particular unknown function that can fulfil a given differential equation. The AB methods and

AM methods are used simultaneously in the predictor-corrector method. The idea is to take an AB methods and AM methods of the same order. This is done by using the AB method to predict y^{n+1}, then call the predicted value y^{n-1} and now use y^{-n+1} in the AM method. In this way the AM method can no longer be completely implicit and no solving is required yet, so we have approximation to y^{n+1} than if the AB method was used alone.

For example, taking the forward Euler Method for the predictor equation to obtain y_{n+1}^{p} and later use the second order AM method as a corrector equation to get the finale calculated solution that is y_{n+1}.

The method is called the Euler-Trapezoidal method and is shown below;

$y_{n+1}^{p} = y_n + hf(y_n, t_n)$ Predictor

$y_{n+1} = y_n + \frac{h}{2}\left[f\left(y_{n+1}^{p}, t_{n+1}\right) + f(y_n, t_n)\right]$ Corrector (4.5)

For the corrector step the implicit term for second order AM method, $f(y_{n+1}, t_{n+1})$ is replaced with $f\left(y_{n+1}^{p}, t_{n+1}\right)$, so the value of f assessed at the predicted y_{n+1}^{p} is used. Therefore, the predictor-corrector method that is reported above is an explicit method.

Now using both the predictor and corrector we have;

$$y_{n+1}^{p} = y_n + hf(y_n, t_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f\left(y_{n+1}^{p}, t_{n+1}\right) + f(y_n, t_n)\right] \tag{4.6}$$

We know;

$$\frac{d}{dt}y(t) = f(y(t), t) \tag{4.7}$$

Using our equation, we have;

$$\frac{\partial h(r,t)}{\partial t} = \frac{1}{r}\frac{\partial h(r,t)}{\partial r} + \frac{\partial^2 h(r,t)}{\partial r^2} - \alpha S_y \int_0^t \frac{\partial h(r,t)}{\partial \tau} e^{-\alpha(t-\tau)} d\tau \tag{4.8}$$

Now;

$$\frac{\partial h(r,t)}{\partial t} = f\left(r, t, h(r,t)\right) \tag{4.9}$$

At the point t_{n+1}, we can let $y(t_{n+1}) = y^{n+1}$, $y^n = y(t_n)$

So,

$$y^p(t_{n+1}) = y(t_n) + \Delta t f(y(t_n), t_n) \tag{4.10}$$

In which the above equation is the predictor where $y(t_{n+1})$ and $y^{n+1} = y(t_n)$ are substituted into the equation in the place of y_{n+1}^p and y_n.

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}[f(y^p(t_{n+1}), t_{n+1}) + ft_n, y(t_n)] \tag{4.11}$$

The above equation represents the corrector where again where $y(t_{n+1})$ and $y^{n+1} = y(t_n)$ are substituted into the equation in the place of y_{n+1}^p and y_n.

At r_i, t_{n+1} and (r_i, t_n)

$$h^p(r_i, t_{n+1}) = h(r_i, t_n) + \Delta t f\big(r_i, t_n, h(r_i, t_n)\big) \tag{4.12}$$

$$h(r_i, t_{n+1}) = h(r_i, t_n) + \frac{\Delta t}{2}[f(r_i, t_{n+1}, h^p(r_i, t_{n+1}) + f(r_i, t_n, h(r_i, t_n)] \tag{4.13}$$

Where for both the above equations, the integer r_i is included as part of the corrector and predictor where there is t_{n+1} and t_n. The integer ordered pair (r_i, t_n) is specified.

4.4 Finite Difference Derivative Approximations

Finite difference method converts partial differential equations to a group of discrete algebraic equations and results in approximate solutions (Yeh and Chang, 2013). The methods were first proposed by Brook Taylor in 1715. Finite difference approximation method is a commonly used method that approximate differential equations and they are conceptually the simplest. The finite difference method uses a concept that approximates the differential operator (∂) by replacing the approximations which are in the equation the application of differential quotients (Δ) using the concept of the limit principle when x approaches zero. An error can occur between the exact solution and numerical solution and this error occurs by moving from ∂ to Δ. The error that occurs is known as the discretisation error or truncation error. The finite component of the Taylor series is applied in the approximation.

4.4.1 The Taylor Series

The Taylor Series is an infinite sum meaning that it is able to expand a function at a certain point. The series is ground upon finite difference approximations (Khan and Ohba, 2003). Based on the finite difference approximations the Taylor Series can be classified into three, the first is the forward difference approximations, second is difference approximations and lastly is the central difference approximations. The central difference approximations are found to be more precise than both the backward and forward difference approximations particularly for periodic and oscillating functions. All three approximations make use of values of a function from a particular group of mesh points that are uniformly spaced. This is to find the approximation of the value of a derivative from the central mesh, left and right most points respectively (Khan and Ohba, 2003). This is known as reference mesh point and the approximation order is defined by the amount of mesh points that are used for the approximation, this does not include the reference mesh point.

For the central difference approximation, we have;

$$\frac{\partial h(r_i, t_n)}{\partial r} = \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r} \tag{4.14}$$

The above approximation is obtained by subtracting the backward and forward approximation equations. Where the head at the point represented by the indices (r_i, t_n) at $h(r_i, t_n)$, along the horizontal we get line where $h(r_{i-1}, t_n)$, $h(r_i, t_n)$ and $h(r_{i+1}, t_n)$.

The central difference approximation is considered as the average of the backward and forward approximation equations, if the values of the data are equally distant. They are good for working out partial differential equations. A truncation error is found for the central difference which has order of $O(h^2)$, the step size h is chosen. The step size decides the precision of the approximate solutions and the number of all computations (Bui, 2003).

To obtain the finite difference approximation for a second order partial differential equation $\left(\frac{\partial^2 h(r_i,t_n)}{\partial r^2}\right)$ we have;

$$\frac{\partial^2 h(r_i, t_n)}{\partial r^2} = \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2} \tag{4.15}$$

The above equation is a combination of first order forward and backward approximations. Forward difference approximations are good in working out ordinary differential equations that are from single step predictor-corrector method such as Euler methods. When the data values are uniformly separated and have a step size h, this makes the truncation error for the forward difference approximation to have order of $O(h)$. The backward differences are used for approximations of derivatives where in which the data is still yet to be available in the future. The future data relies on derivatives that are approximated from previous data for instance those in control problems. Again, if the data values are uniformly separated and have a step size h, this makes the truncation error for the backward difference approximation to have an order of $O(h)$.

Now going back to our equation, where the time derivative is approximated using the forward difference approximation to give;

$$\int_0^{t_n} \frac{\partial h(r_i, t_n)}{\partial \tau} exp[-\alpha(t_n - \tau)]d\tau$$
$$= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{h\big(r_i, t_{j+1}\big) - h\big(r_i, t_j\big)}{\Delta t} exp[-\alpha(t_n - \tau)]d\tau \tag{4.16}$$

For the above equation, for the integer ordered pair (i, j), the head is represented by the indices (i, j) along the vertical line to give $h\big(r_i, t_{j+1}\big)$ and $h\big(r_i, t_j\big)$.

Now the head at one point should already be known, this is at j. In order to calculate the head for the new time step at $j + 1$, the one known head is calculated using the spatial derivatives at that point in time. This will give the following equation;

$$= \sum_{j=0}^{n-1} \frac{{h_i}^{j+1} - {h_i}^{j}}{\Delta t} \int_{t_j}^{t_{j+1}} exp[-\alpha(t_n - \tau)]d\tau \tag{4.17}$$

Now going back to (r_i, t_{n+1}) and (r_i, t_n), we know that;

$$h^p(r_i, t_{n+1}) = h(r_i, t_n) + \Delta t f(r_i, t_n, h\big(r_i, t_n, h(r_i, t_n)\big) \tag{4.18}$$

And;

$$h(r_i, t_{n+1}) = h(r_i, t_n) + \frac{\Delta t}{2}\big[f\big(r_i, t_n, h^p(r_i, t_{n+1})\big) + f(r_i, t_n, h(r_i, t_n)\big] \tag{4.19}$$

So now what is $f\big(r_i, t_n, h(r_i, t_n)\big) =?$

$$f\left(r_i, t_n, h(r_i, t_n)\right)$$
$$= \frac{1}{r_i}\frac{\partial h(r_i, t_n)}{\partial r} + \frac{\partial^2 h(r_i, t_n)}{\partial r^2}$$
$$- \alpha S_y \int_0^{t_n} \frac{\partial h(r_i, t_n)}{\partial \tau} exp[-\alpha(t_n - \tau)]d\tau \tag{4.20}$$

We let $y = t_n - \tau$ when $e = t_j$, $y = t_n - t_j$, $\tau = t_{j+1}$ and $y = t_n - t_{j+1}$

Now $dy = dt_n - d\tau$, => $dy = -d\tau$, =>$d\tau = -dy$

$$\int_{t_j}^{t_{j+1}} exp[-\alpha(t_n - \tau)]d\tau = \int_{t_n - t_j}^{t_n - t_{j+1}} exp[-\alpha y](-dy) \tag{4.21}$$

Where $t_n - t_{j+1}$, y which is $= t_n - \tau$ and $-dy$ are substituted into the equation.

The equation becomes;

$$-\int_{t_n - t_j}^{t_{n-}t_{j+1}} exp[-\alpha y]dy \tag{4.22}$$

We know that an integral changes signs when the boundaries are switched. This is taken from a rule that states that;

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \tag{4.22}$$

If we let $c = a$ from the fact that $\int_a^a f(x)dx = 0$, we will then have;

$$-\int_a^b f(x)dx = \int_b^a f(x)dx \tag{4.23}$$

Therefore, for the rule to still hold from one of the integrals being "backwards", switching the boundaries changes the sign of the integral. This is an important and useful integration property when trying to make sense of some integrals and trying to solve some of them.

Now the equation becomes;

$$= \int_{t_n - t_{j+1}}^{t_n - t_j} exp[-\alpha y]\, dy \tag{4.24}$$

Where the sign of the integral changed and the bounds of integration swapped.

Therefore;

$$\int_a^b exp(ay)dy = \frac{1}{a} exp(ay) \tag{4.25}$$

$$\int_{t_n-t_{j+1}}^{t_n-t_j} exp[-\alpha y]\, dy \quad = -\frac{1}{\alpha} exp(-\alpha y)| \begin{matrix} t_n - t_j \\ t_n - t_{j+1} \end{matrix} = \frac{1}{\alpha} exp[-\alpha(t_{n-j})] + \frac{1}{\alpha} exp[-\alpha(t_n - t_{j+1})] \tag{4.26}$$

Which can be simplified to;

$$= \frac{1}{\alpha}\left[\exp[-\alpha(t_n - t_{j+1})] - exp[-\alpha(t_{n-j})]\right] \tag{4.27}$$

$$\int_0^{t_n} \frac{\partial h(r_i, t_n)}{\partial \tau} exp[-\alpha(t_n - \tau)]d\tau = \sum_{j=0}^{n-1} \frac{h_i^{j+1} - h_i^j}{\alpha \Delta r} \left[\exp[-\alpha(t_n - t_{j+1})] - exp[-\alpha(t_{n-j})]\right] \tag{4.28}$$

We know that at $f(r_i, t_n, h(r_i, t_n))$ we have;

$$f(r_i, t_n, h(r_i, t_n)) = \frac{1}{r_i}\frac{\partial h(r_i, t_n)}{\partial r} + \frac{\partial^2 h(r_i, t_n)}{\partial r^2} - \alpha Sy \int_0^{t_n} \frac{\partial h(r_i, t_n)}{\partial r} exp[-\alpha(t_n - \tau)]d\tau \tag{4.29}$$

The following equation shows the finite difference approximation for the equation for the selection of forward differences in time and explicit temporal scheme.

$$= \frac{1}{r_i}\frac{h_{i+1}^n - h_{i-1}^n}{2\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{\Delta r^2} - \alpha Sy \sum_{j=0}^{n-1} \frac{h_i^{j+1} - h_i^j}{\alpha \Delta r}\{exp[-\alpha(t_n - t_{j+1})] - exp[-\alpha(t_n - j)]\} \tag{4.30}$$

The above equation is the forward explicit forward approximation. The approximation does not take into account the spatial derivative terms at the time level of $n + 1$ and thus only uses terms that are already defined at time of n. The head at the time of $n + 1$ is achieved at every node explicitly by the use of head values from time of n and not the form $n + 1$.

Also;

$$f\left(r_i, t_n, h^p(r_i, t_{n+1})\right) = \frac{1}{r_i}\frac{\partial h^p(r_i, t_{n+1})}{\partial r} + \frac{\partial^2 h^p(r_i, t_{n+1})}{\partial r^2} \\ -\alpha Sy \int_0^{t_n} \frac{\partial h^p(r_i, t_{n+1})}{\partial r} exp[-\alpha(t_{n+1} - \tau)] d\tau \tag{4.31}$$

4.5 Von Neumann Stability Analysis

The stability analysis is generally used to stabilize finite differential equations. The method analysis is also called the Fourier stability analysis method. It is based on numerical decomposition errors found in Fourier and was first developed and described in 1947 by John Crank and Phyllis Nicolson. The stability analysis was later investigated thoroughly in an article by John von Neumann.

The following equation shows the finite difference approximation for the equation for the selection of forward differences in time and implicit temporal scheme.

$$= \frac{1}{r_i}\frac{h_{i+1}^{p,n+1} - h_{i-1}^{p,n+1}}{2\Delta r} + \frac{h_{i+1}^{p,n+1} - 2h_i^{p,n+1} + h_{i-1}^{p,n+1}}{\Delta r^2} \\ - \alpha Sy \sum_{j=0}^{n-1} \frac{h_i^{p,j+1} - h_i^{p,j}}{\alpha \Delta r} \{exp[-\alpha(t_{n+1} - t_{j+1})] \\ - exp[-\alpha(t_{n+1} - t_j)]\} \tag{4.32}$$

For the above equation it is a fully implicit finite difference approximation where the only the head values at $n + 1$ are considered to represent the spatial derivative. The head at $n + 1$ is acquired at each node implicitly by the use of head values from only time $n + 1$ and none from time n.

Now let,

$$\{exp[-\alpha(t_{n+1} - t_{j+1})] - exp[-\alpha(t_{n+1} - t_j)]\} = \delta_n^j \tag{4.33}$$

The equation now becomes;

$$= \frac{1}{r_i}\frac{h_{i+1}^{p,n+1} - h_{i-1}^{p,n+1}}{2\Delta r} + \frac{h_{i+1}^{p,n+1} - 2h_i^{p,n+1} + h_{i-1}^{p,n+1}}{\Delta r^2} - \alpha Sy \sum_{j=0}^{n-1} \frac{h_i^{p,j+1} - h_i^{p,j}}{\alpha \Delta r} \delta_n^j \tag{4.34}$$

The above equation was simplified to make solving the equation much easier.

Now going to the predictor-corrector, at (r_i, t_{n+1}) and (r_i, t_n) we have;

$$\begin{cases} h^p(r_i, t_{n+1}) = h(r_i, t_n) + \Delta t f\big(r_i, t_n, h(r_i, t_n)\big) \\ h(r_i, t_{n+1}) = h(r_i, t_n) + \dfrac{\Delta t}{2}\Big[f\big(r_i, t_{n+1}, h^p(r_i, t_{n+1})\big) + f\big(r_i, t_n, h(r_i, t_n)\big)\Big] \end{cases} \tag{4.35}$$

By linking the equations, at (r_i, t_{n+1}) and (r_i, t_n) we now have;

$$\begin{cases} h_i^{p,n+1} = h_i^n + \Delta t f(r_i, t_n, h_i^n) \\ h_i^{n+1} = h_i^n + \dfrac{\Delta t}{2}\Big[f\big(r_i, t_{n+1}, h_i^n + \Delta t f(r_i, t_n, h_i^n)\big) + f(r_i, t_n, h_i^n)\Big] \end{cases} \tag{4.36}$$

At Eq. (4.36), the function of the finite difference approximation for the equation with the selection of forward differences in time and explicit temporal scheme is substituted into the predictor step. For the corrector step, again the function of the finite difference approximation for the equation with the selection of forward differences in time and implicit temporal scheme is substituted into the corrector step. This results in the following system;

$$\begin{cases} h_i^{p,n+1} = h_i^n + \Delta t\left[\dfrac{1}{r_i}\dfrac{h_{i+1}^n - h_{i-1}^n}{2\Delta r} + \dfrac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{\Delta r^2} - \alpha Sy\displaystyle\sum_{j=0}^{n-1}\dfrac{h_i^{j+1} - h_i^j}{\alpha\Delta r}\delta_n^\alpha\right] \\ h_i^{n+1} = h_i^n + \dfrac{\Delta t}{2}\left\{\begin{matrix} f\left[h_i^n + \Delta t\left[\begin{matrix}\dfrac{1}{r_i}\dfrac{h_{i+1}^n - h_{i-1}^n}{2\Delta r} + \dfrac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{\Delta r^2} \\ -\alpha Sy\displaystyle\sum_{j=0}^{n-1}\dfrac{h_i^{j+1} - h_i^j}{\alpha\Delta r}\delta_n^\alpha\end{matrix}\right]\right] + \\ \dfrac{1}{r_i}\dfrac{h_{i+1}^{n+1} - h_{i-1}^{n+1}}{2\Delta r} + \dfrac{h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}}{\Delta r^2} \\ -\alpha Sy\displaystyle\sum_{j=0}^{n-1}\dfrac{h_i^{j+1} - h_i^j}{\alpha\Delta r}\delta_n^j \end{matrix}\right\} \end{cases} \tag{4.37}$$

From the Von Neumann stability analysis that was performed, we can say that the equation is now stable and the conditions of the system are met.

The newly derived numerical scheme uses the Adams-Bashforth method. The above system is solved using predictor-corrector method which shows a combination of both an explicit and implicit methods and gives better convergence characteristics. The system was achieved by the integration of ordinary differential equations so as to determine an unknown function to fulfil a given differential equation.

A lot of differential equations are impossible to solve using explicitly so the Euler implicit method was introduced and was found to be very useful method to approximate the solution (Bui, 2009). By trying to manipulate explicit and implicit methods one can find ways to give good approximations than to the exact solution of partial differential parabolic equations and nonlinear differential parabolic equations (Bui, 2009).

Explicit methods compute the system at a later stage from the state of the system at the present time without working out the algebraic equations. An example of a type of numerical method include the forward Euler method that is explicit. Explicit temporal schemes are generally simple and calculate fast. Implicit methods are the opposite to explicit methods. These methods are not directly expressed in the terms of independent variables. The implicit finite difference approximations are usually paired with explicit finite difference approximations to a decided in advance to the degree that is indicated by the variable $(0 < \alpha < 1)$, where $\alpha = 0.5$ which is the Crank-Nicolson scheme.

CHAPTER 5: NUMERICAL SIMULATIONS

One of the greatest achievements by humankind is perhaps their ability to use computers in order to perform simulations of complex models that cannot be solved analytically. Indeed computer simulations can be viewed as a process of mathematical modelling, implemented using computers, with the aim of predicting or replicating the behavior of a given real world problem, or even a theoretical system. With the great increase in technology, computer simulations are considered as outstanding mathematical tools for predicting real world behavior where the analytical counterpart fail, this in all the fields of science, technology and engineering. We shall note that, this technique was developed hand-in-hand with the rapid increase of computers; this can be traced back to its first large-scale deployment during the Manhattan mission in world war two, with the main aim to model the process of nuclear detonation. There are many softwares that can be used to help perform simulations for more complicated real-world problems, this includes: Matlab, Mathematica, Maple, Python and many others that will not be listed here. In this work, we use Matlab to perform simulations for different values of theoretical parameters.

In this chapter, in order to access the efficiency of the suggested mathematical model depicting the conversion from confined to unconfined aquifer together with the used numerical scheme, we present numerical simulation for different theoretical value of aquifer parameters. The numerical simulations are depicted in figures 7 to 21.

Parameters	Values
S_c	0.001 to 0.009
S_y	0.0001 to 0.00091
B	4
T	800
α	(0.04 to 0.1) x100

Table 1: Theoretical used aquifer parameters to perform simulations

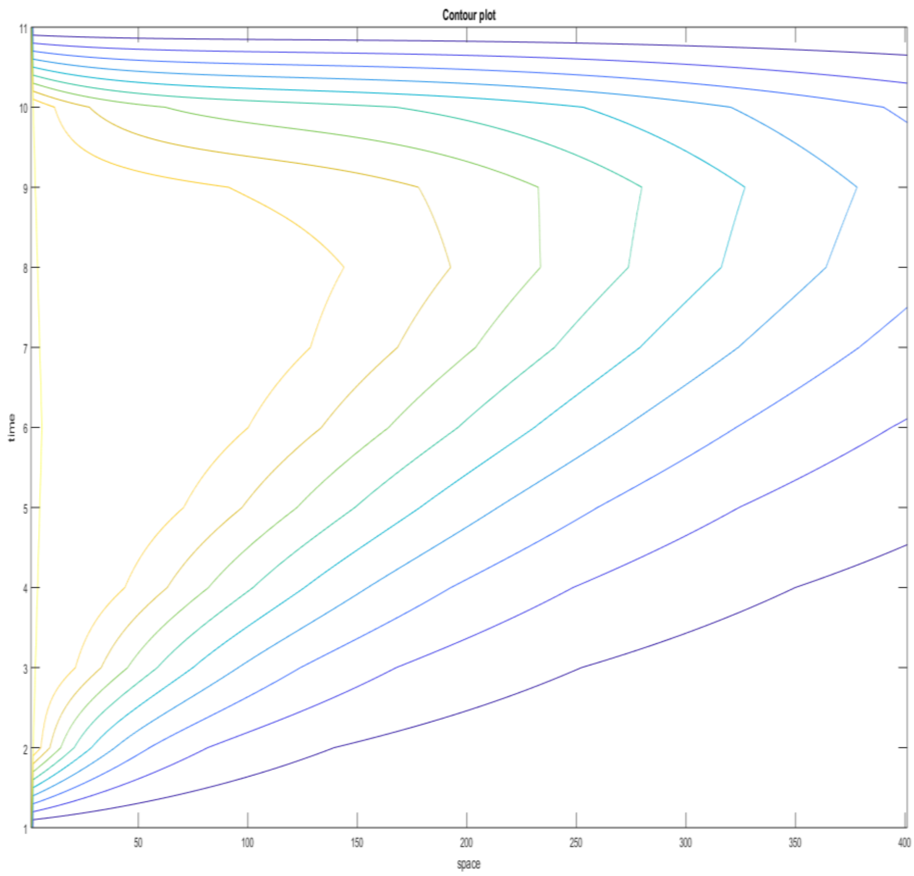

Figure 7: Contour plots showing flow in unconfined with high permeability

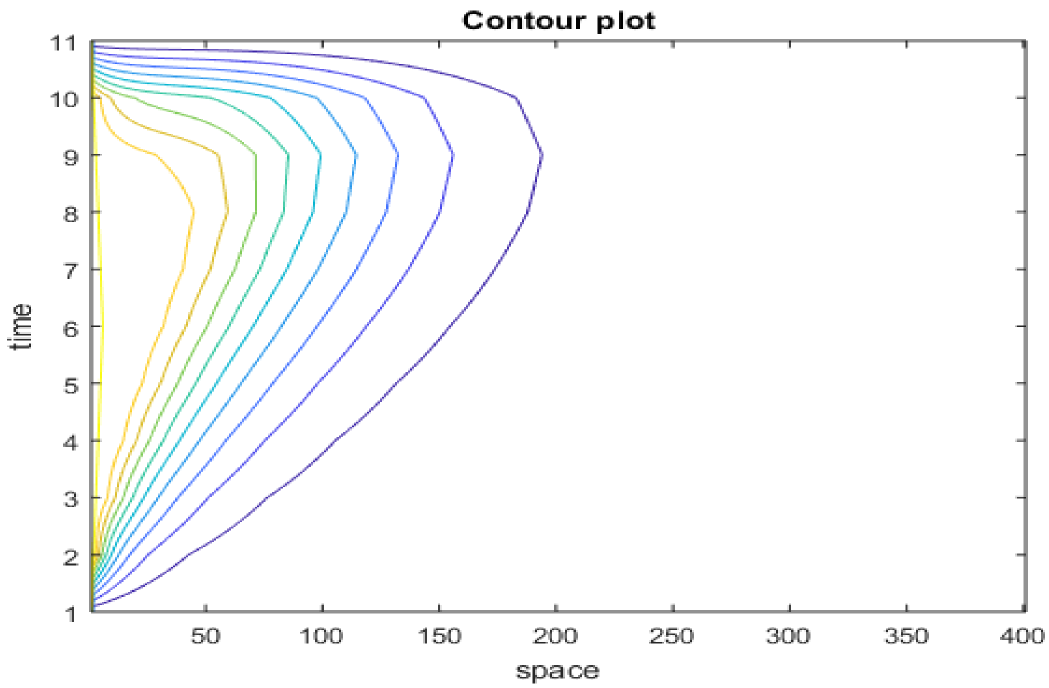

Figure 8: Contour plot showing the flow with normal permeability

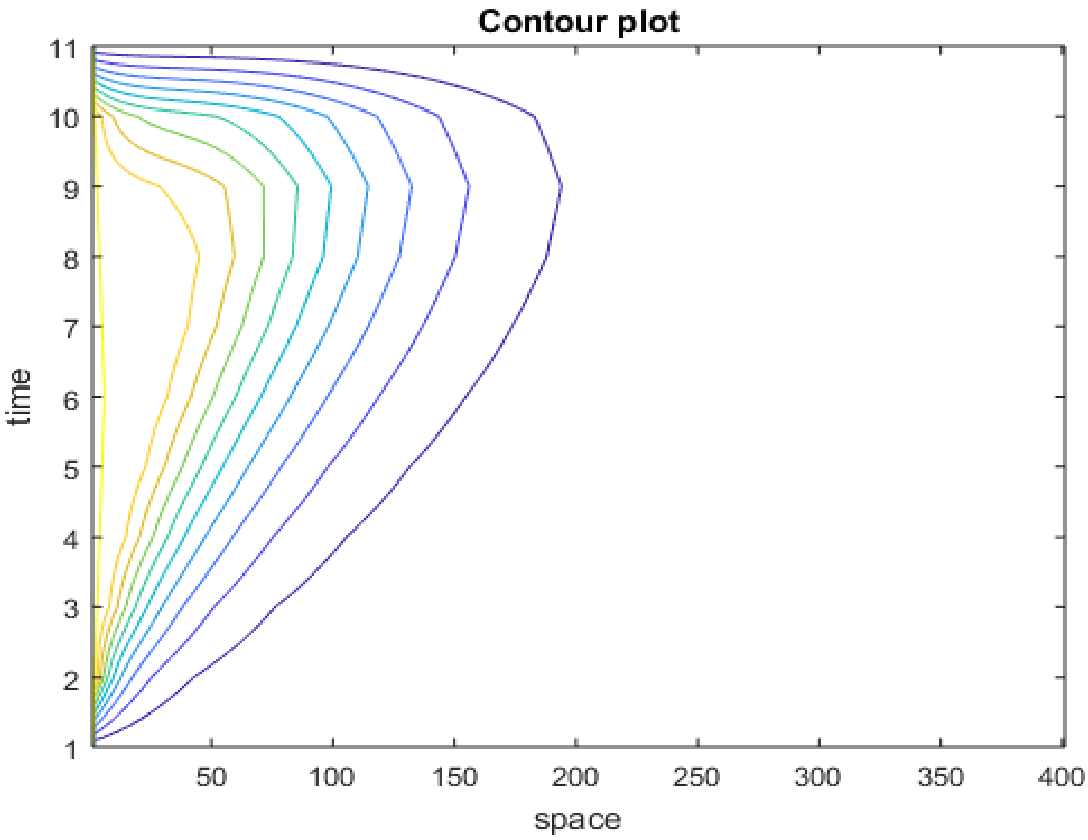

Figure 9: Contour plot showing the flow with low permeability

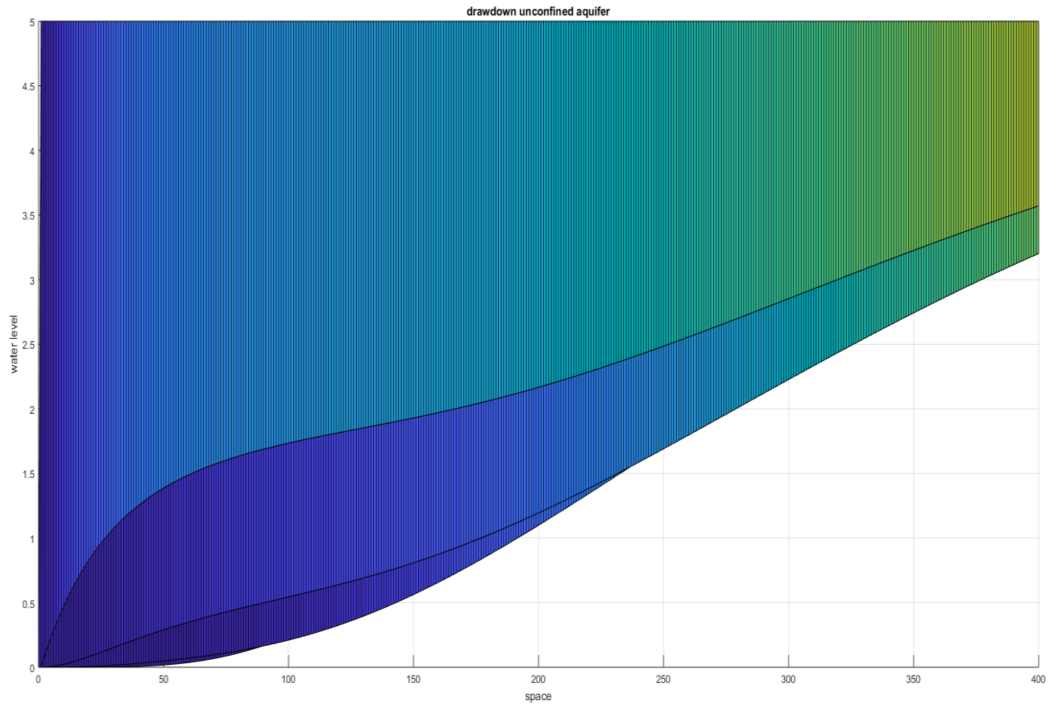

Figure 10: Drawdown with high transmissivity

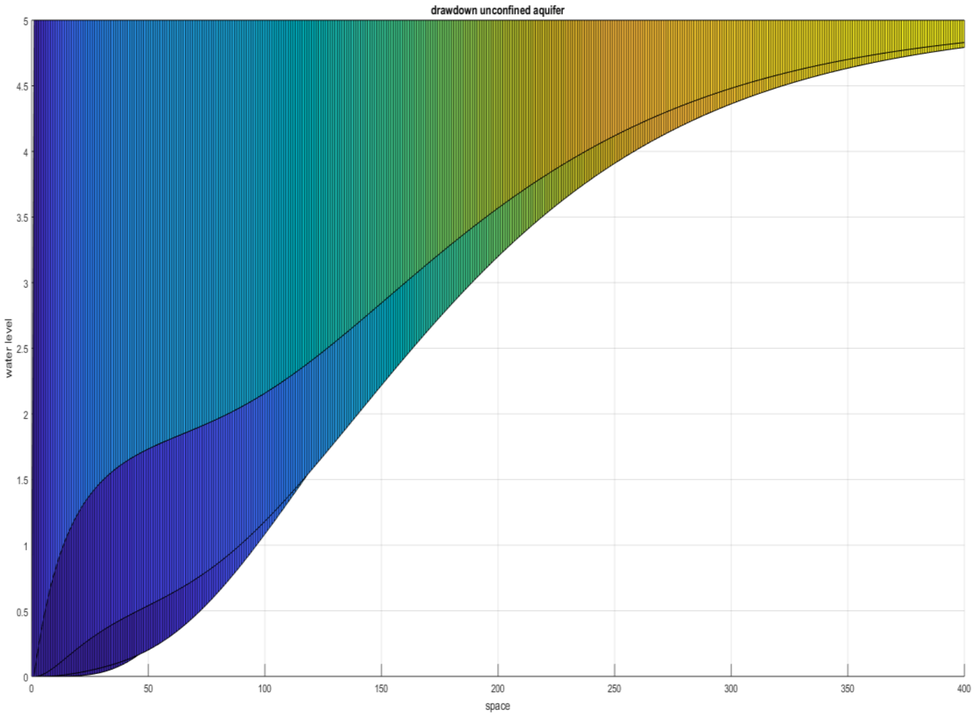

Figure 11: Drawdown with normal transmissivity

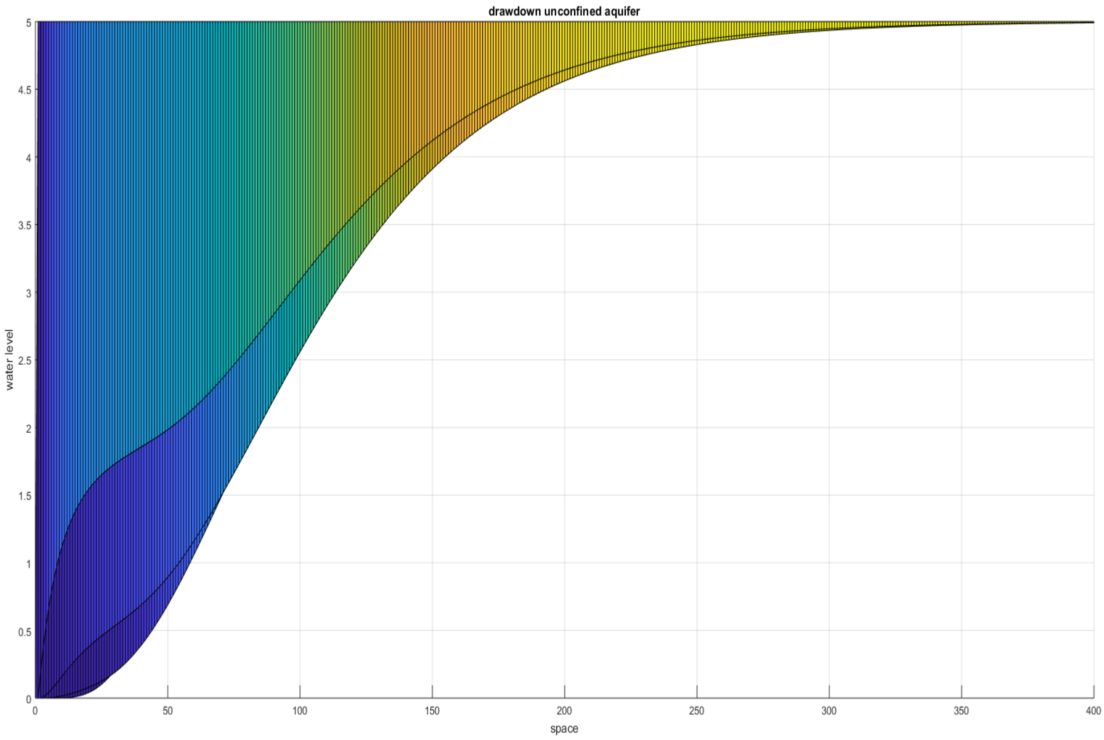

Figure 12: Drawdown with low transmissivity

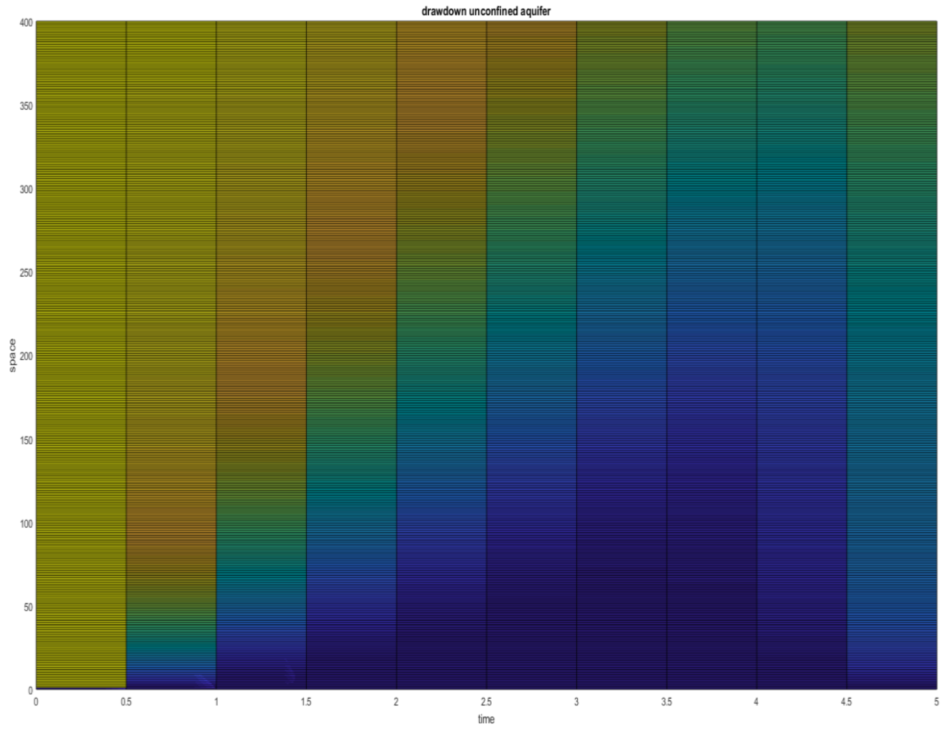

Figure 13: Histogram showing flow with high transmissivity

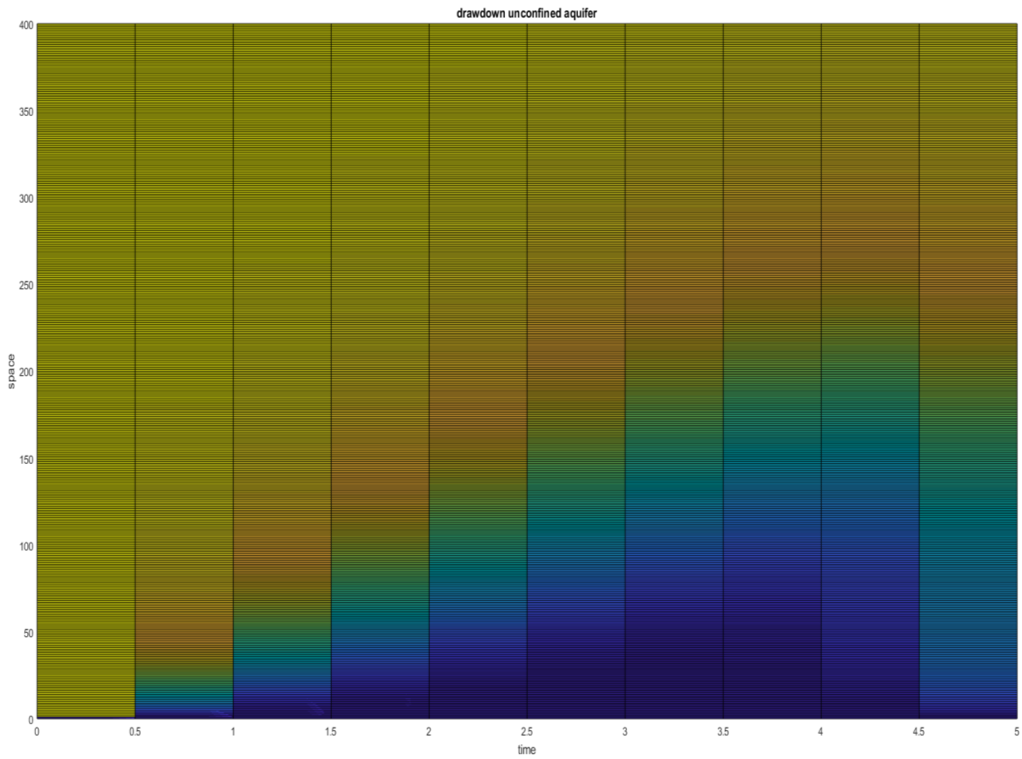

Figure 14: Histogram showing flow with normal transmissivity

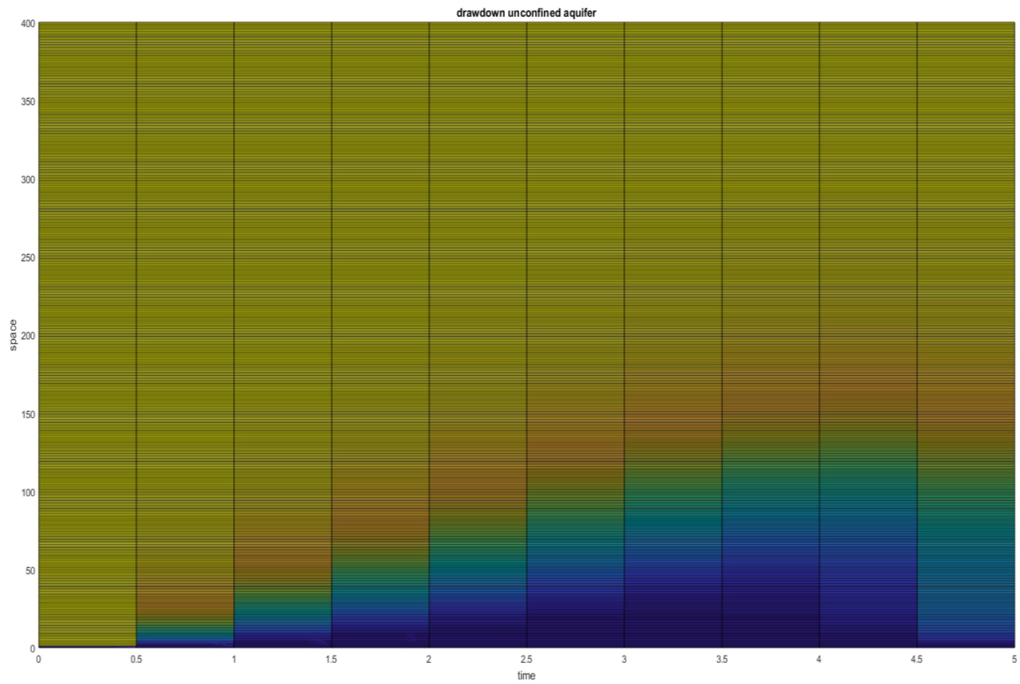

Figure 15: Histogram showing flow with low transmissivity

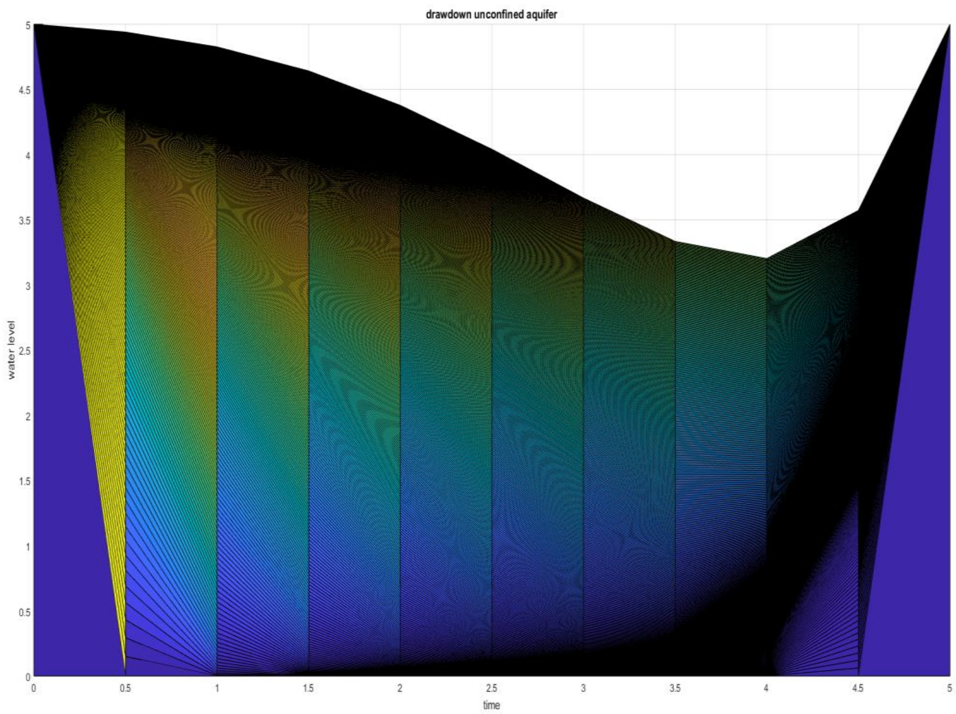

Figure 16: Flow in time with high transmissivity

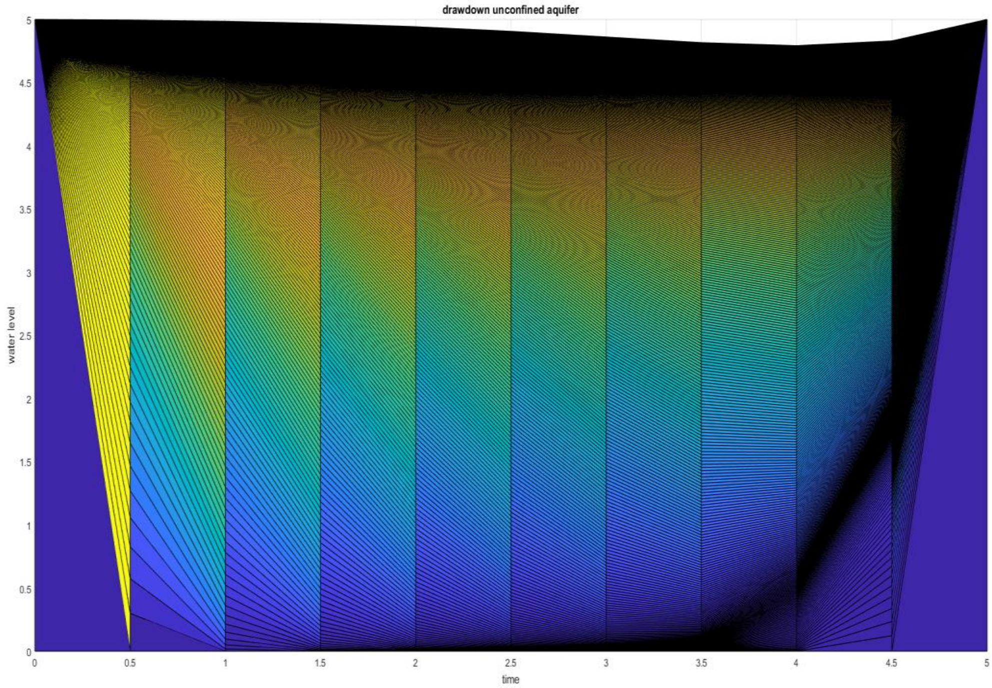

Figure 17: Flow in time with normal transmissivity

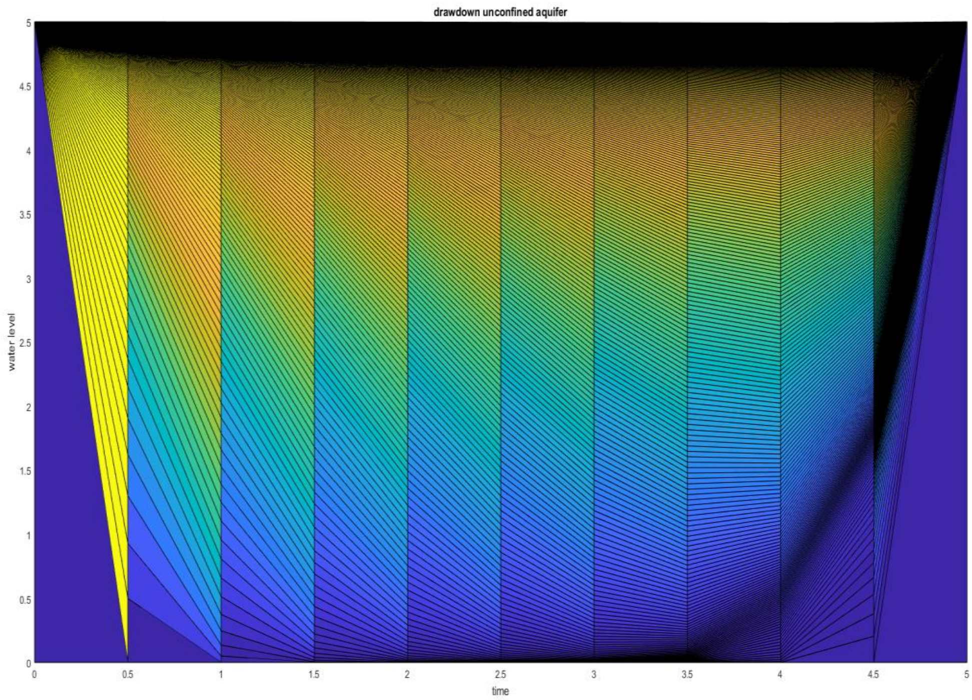

Figure 18: Flow in time with low transmissivity

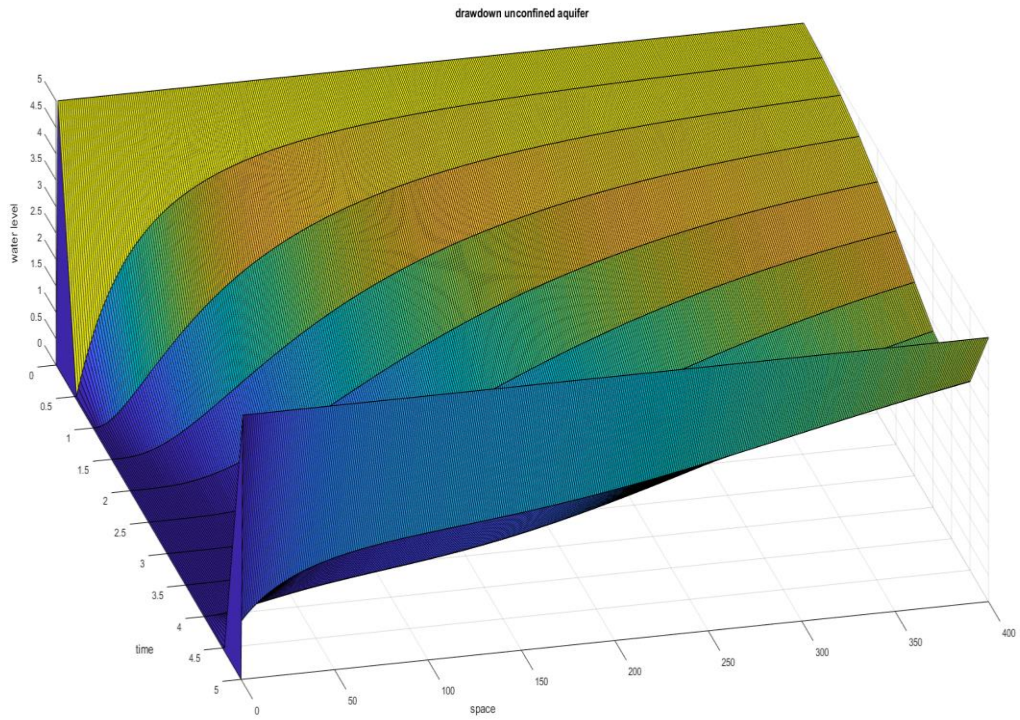

Figure 19: Space-time cone of depression with high transmissivity

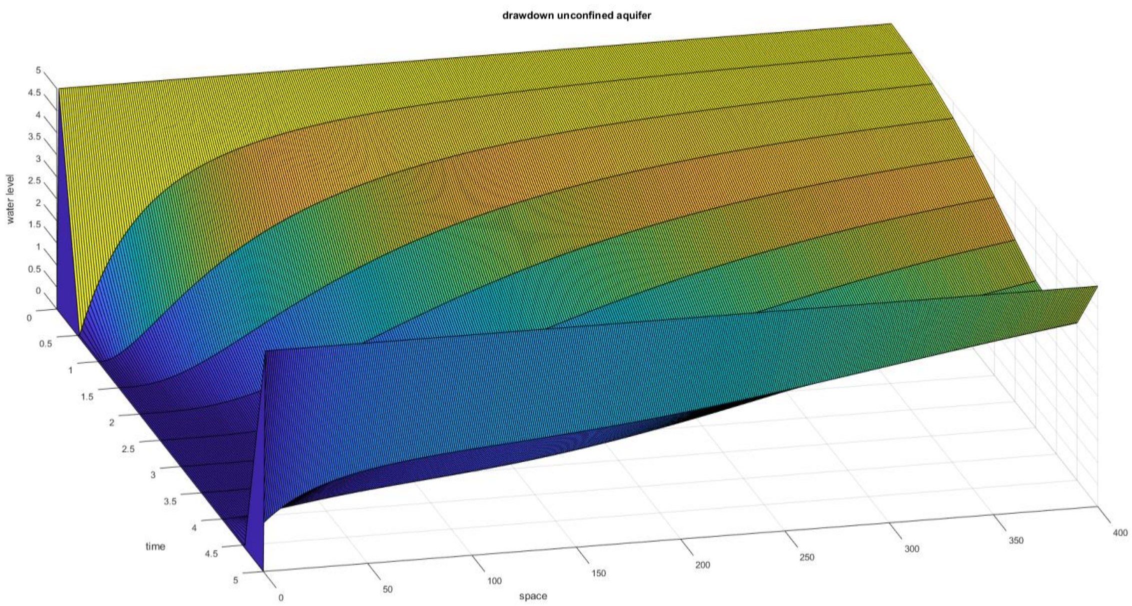

Figure 20: Space-time cone of depression for normal transmissivity

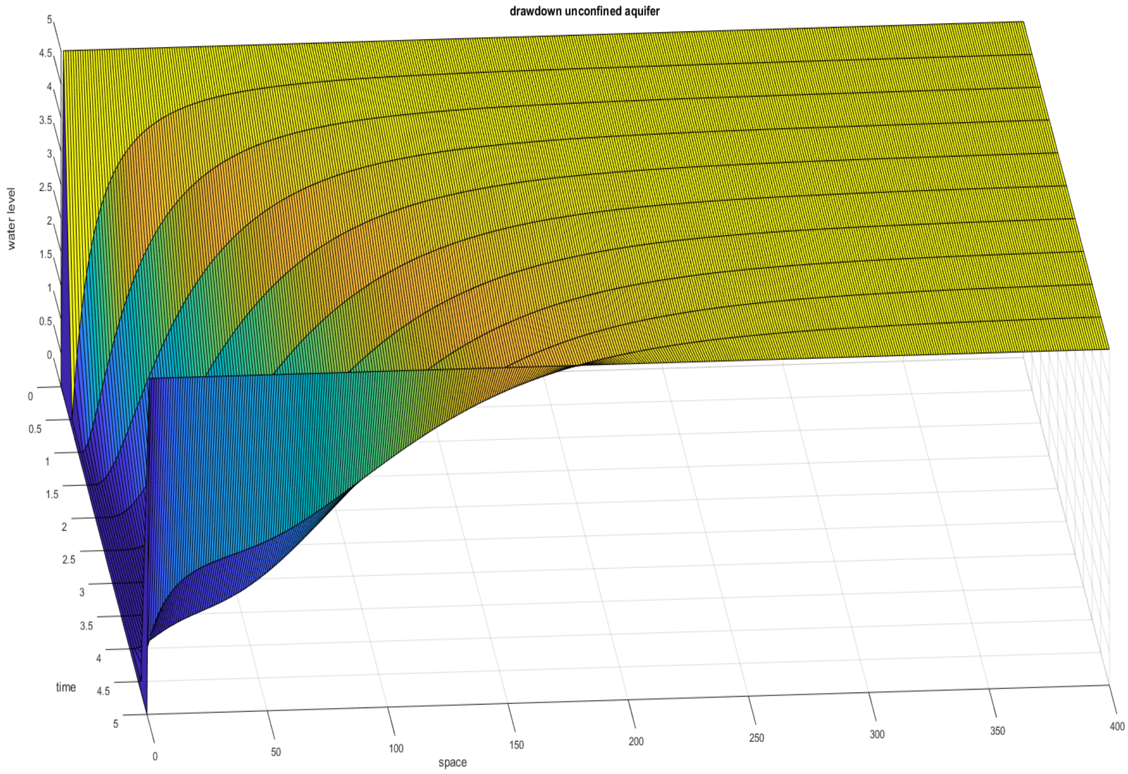

Figure 21: Space-time cone of depression for low transmissivity

RESULTS AND DISCUSSIONS

The main aim of a mathematical model is to replicate real world observed facts. Therefore, the translation from observed facts into mathematical equations should be able to capture reality. In fact, such a model should be able to be in good agreement with purely obtained experimental data if any available. The numerical simulation when agreed with experimental data can now be used for prediction, thus if such simulations are inaccurate, the prediction will be misleading and this can cause damage, or even be deadly in some cases.

The mathematical models that were suggested to depict the conversion of flow from confined to unconfined aquifer in the literature have included high nonlinearity that lead to exaggeration of determined aquifer parameters, as sometimes the numerical solution predicts high flow while the actual flow may be less. Using such a model will with no doubt lead to wrong prediction. In this work, the obtained numerical simulation predicts the flow within the unconfined aquifer with no highly non-linearity, however the model is able to account for delay and fading memory process.

CONCLUSION

While surface water has been recognized as fresh source of water for many countries, however during drought humankind relies on subsurface water. To get this water, people rely on drilling boreholes and extracting water from confined or unconfined aquifers. As the surface water is in high demand due to the increase in population, it becomes an obligation for humans to over abstract water from confined aquifers, this eventually leads to the conversion from confined to unconfined aquifers. The conversion could possibly lead latter to depletion of the aquifer. Many researchers have investigated the conversion from confined to unconfined in the last decades using mathematical models. Their suggested models could be used accurately for some problems; however, it was observed that some of those models estimate highly the aquifer parameters, which could result in misleading predictions. For example, some farmers rely on aquifer parameters to estimate how much water could be taken to have a sustainable borehole. If the used model suggests high transmissivity, while the actual transmissivity is less; the prediction given will lead to wrong results. In this work, we suggested a new model for the conversion from confined to unconfined. The model takes into account the delay in the flow, which can be linked to retardation factor of the geological formation. The new model is a system of partial differential equations, where the first equation was suggested by Theis and has been used with some success in the last decades; the second equation has a fading memory element that could be used to capture memory. The solution of the first equation is well-known and the derivation of its exact solution. The second equation was solved analytically and numerically. The Laplace transform operator was used to obtain the solution in Laplace space, while a newly introduced numerical scheme was used to solve such an equation. The conditions under which the used method is stable and converge were presented. Some numerical simulations were performed. We can conclude that our method is suitable in predicting the conversion from confined to unconfined with a delay.

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