MODELLING GROUNDWATER FLOW IN A CONFINED AQUIFER WITH DUAL LAYERS

DISEBO VENOLIAH CHAKA

STUDENT NUMBER: 2014077966

Submitted in fulfilment of the requirements for the degree

Magister Scientiae in Geohydrology

In the

Faculty of Natural and Agricultural Sciences

(Institute for Groundwater Studies)

at the

University of the Free State

Supervisor: Prof Abdon Atangana

BLOEMFONTEIN

JANUARY 2020

DECLARATION

I, **DISEBO VENOLIAH CHAKA** declare that the dissertation titled "Modelling Groundwater flow in a confined aquifer with dual layers" submitted by me under the supervision of Prof Atangana to the Institute of Groundwater Studies in the faculty of Natural and Agricultural Sciences in fulfilment of a Magister Scientiae is my own independent work that has not been previously submitted by me to any other higher learning institution.

In addition, I declare that all sources cited have been acknowledged by means of a list of references.

I furthermore, I cede the copy of the dissertation in favour of the University of the Free State.

In addition, the following article has been submitted and is under review:

Chaka D.V. and Atangana A., 2020. Modelling groundwater flow in a confined aquifer with dual layers.

DISEBO VENOLIAH CHAKA

JANUARY 2020

ACKNOWLEDGEMENTS

"Not by might, nor by power by your spirit Lord". First and foremost, I would like to thank God for giving me this opportunity. If it was not for your grace, I wouldn't have achieved all this. Thank you for your strength, your protection and above all your Wisdom.

A special note of thanks to my supervisor Prof. Abdon Atangana for sharing his knowledge and for guiding me in this journey. The quality of work produced in this article would not have been possible if his door was not always open when I ran into problems. Thank you for your constructive criticism, suggestions and the freedom to produce my own work. I will be forever grateful, **Thank you, Prof.**

I would also like to extend my appreciation to the Institute of Groundwater Studies for accepting my application and for giving an opportunity to do my MSc research with them.

I extend my appreciation and big thanks to the Old Mutual Scholarship foundation and Mr Van der Walt for making sure that all my financial needs are catered for during my period of study. Mr Van der Walt, no words can ever describe how grateful I am for all that you have done for me. **Thank you, Sir.**

To my lovely family, I send the outmost appreciation for your support, love, comfort and understanding throughout this journey. Your prayers and motivation kept me strong and focused. To my Mother Makhala Chaka, my grandparents Molaeng Chaka and Teboho Chaka your love is indeed unconditional. To my Uncles Mafanyane Chaka, Motebang Chaka and big brother Teboho Chaka your words of encouragement and your immense support kept me grounded. To my younger siblings, Tatolo Chaka and Onalenna Chaka thank you for always looking up to me as this made me work even harder to make you proud.

THANK YOU

ABSTRACT

Groundwater flow occurring in a confined aquifer has been modelled in the past using deterministic mathematical models such as the Theis (1935) equation providing fundamental solutions. However, for this model to be applicable, certain assumptions must be taken into consideration which sometimes leads to the misinterpretation of the real-life scenarios. Because of the limitations arising from this model, flow in a confined aquifer layer cannot be accurately modelled by this equation as it is too simplified. Therefore, the main aim of this study is to model flow in a confined aquifer with a dual layers using non-conventional differential operators and integral operators. Due to the complexity of the geological formation within which the flow is taking place, classical calculus operators are not suitable mathematical operators to be used. Further, research was done on appropriate mathematical operators in order to find the one to be used in this study and the application of the combination of fractal-fractional operators were adopted. Fractal-fractional derivatives are used to solve a complex physical problem and were found to be effective in modelling anomalous diffusion. In order to include into mathematical formulation some complexities of the geological formation, the concept of fractal-fractional differential and integral operators were used. These new differential operators are able to depict scenarios that combine behaviours following the power-law together with self-similarities, or fading memory with self-similarities or crossover behaviours with self-similarities that are observed when the geological formation is equipped with fractures with a self-similar feature. In this study, the Theis groundwater flow model was extended, where the classical differentiation was replaced by three different types of fractal-fractional operators. The modified models were solved numerically using the newly introduced numerical scheme. For each case, a detailed analysis of stability and convergence was presented. The obtained numerical solutions were used to depict numerical simulations showing different values of fractional order and fractal dimension. Obtained figures present a new class of flow different from the normal flow. The fractal dimension has brought new flow trends that can be observed in fractured rock aquifers. The application of these differential operators will open a new way to capture heterogeneity associated with the geological formation.

Keywords: **Groundwater flow equation; Theis; Fractal; Fractional; Uncertainty; Confined aquifer; Von Neumann stability; Classical calculus; Numerical simulation**

LIST OF GREEK NOTATIONS

α	Alpha
β	Beta
$\int$	Definite integral
Δ	Delta
∀	For all
ϵ	Epsilon
Γ	Gamma
∞	Infinity
λ	Lambda
μ	Mu
Ω	Omega
∂	Partial differential operator
∅	Phi
π	Pi
$\sum$	Summation
σ	Sigma
τ	Tau
θ	Theta
ξ	Xi

LIST OF ABBREVIATIONS AND NOTATIONS

b/B	Aquifer Thickness
e.q.	Equation
Exp	Exponent
f	Function
h	Hydraulic head
K	Hydraulic conductivity
S	Storativity
t	Time
T	Transmissivity
t^a	Fractal
V	Volume

Table of Contents

TABLE OF FIGURES

CHAPTER 1: INTRODUCTION

1.1 Background of the study

With the recent demand for water resources especially in South Africa, groundwater has become the interest as a new source for freshwater supply (de Graaf et.al, 2017). Groundwater is found below the earth's surface and is stored in pore spaces in a geological formation which is often of a porous media nature or is fractured (Xiao, 2014). The geological formation in which the groundwater is stored is referred to as an aquifer (Fetter, 1994).

An aquifer is referred to as a saturated geological unit that is permeable and occurs usually below the water table, contains a measurable amount of water and has the ability to store and transmit water to wells (Fetter, 1994). Examples of aquifers vary but not limited to unconsolidated sands and gravels, limestones, dolomites and sandstones (Freeze et.al, 1979). A geological formation with low permeability is termed an aquitard. An aquitard has the ability to transmit water, but its permeability is not of the acquired quality to support the production of a well due to restriction of movement. According to Christiansen et.al (2014), an aquiclude is a geological unit with no permeability and porosity and lack the ability to allow water through it nor the ability to store water. The common examples of aquiclude include dense un-fractured igneous or metamorphic rocks (Kruseman et.al, 2000). Because groundwater cannot be accessed easily, the method of modelling its flow was then introduced.

Groundwater is found to flow in a certain manner from the area that has the highest hydraulic head to an area which possess a hydraulic head that is lower. This path that groundwater takes is termed hydraulic gradient and it is the change in the hydraulic head from the highest to the lowest. All the factors that are found to be controlling the movement of water and the physical factors controlling water sources must be taken into consideration (Focazio et.al., 2003). These factors including the magnitude of the hydraulic conductivity and the hydraulic gradient of the aquifer and they often affect the velocity of which the groundwater will flow. On a regional scale, the patterns that the groundwater flow takes often follow the path of surface water patterns and this is why the groundwater flow direction in shallow, unconfined aquifers can be inferred by observing the surface water drainage patterns (Barackman et.al., 2002). The type of lithology in the aquifer will mostly determine the way in which the water flows (Cherry et, al., 2004).

There are different types of groundwater flow conditions, this depends on the behaviour of water in certain situations, usually how it enters a system and how it exits the system. There is steady state flow, unsteady-state flow and pseudo-state flow. The steady state flow occurs when an outside source (e.g. rainfall) recharges the well and the change occurring in the water inside both the piezometers and the well is so small that it ends up being neglected. Here, a balance occurs whereby the amount of water that enters and flows out of the system is equal. When the flow is under unsteady state, the amount of water that enters the system does not equal to the amount of water that flows out of the system. The pseudo-steady state flow is dependent on time with the drawdown and hydraulic gradient being monitored against time. There is a fairly small change in the drawdown and the hydraulic gradient (Kruseman et.al, 2000).

The modelling of groundwater was developed to conceptualize hydro-geologic processes as well as analysing information from the field by providing a quantitative framework. According to (Wang and Anderson, 1982), models are described as a “representation of the complex nature of the world”. The most common purpose of a groundwater model is to predict the behaviour of some hydrological actions and can be used to model conditions that occurred in the past (hindcasting and as interpretative tools. The term “forecasting” is used to emphasize the fact the model can in some cases contain some uncertainty (Anderson et.al, 2015).

Groundwater models are divided broadly into mathematical models and physical models. Physical models include the physical measurements of groundwater flows and heads in a model aquifer in the laboratory and this often has equipment filled with sand simulating an aquifer and wells or boreholes (Anderson et.al, 2015). In mathematical models, processes that occur in the field are represented by using equations and the data captured in the field are used to calculate different parameters (Fetter, 1998). For the purpose of this research, the focus will only be on mathematical models.

1.2. Theis model

The Theis equation was developed by Charles Vernon Theis in 1935. This is the first mathematical model of transient flow of water which can be used for modelling flow and calculating transmissivity, storativity and other hydraulic properties of a confined aquifer (Loăiciga, 2010). The hydraulic head change in a confined aquifer that has a well that is penetrating can also be solved by the Theis analytical solution (Flores et.al, 2018). It is mostly used in confined aquifers and unconfined aquifers when the impacts of the pumping well on flow fields are determined. The solution of the Theis is applied when the pumping-induced streamflow depletion is estimated (Flores et.al, 2016).

1.2.1. Derivation of Theis

The derivation of Theis was developed based on Darcy's law and comparison of the rate at which water flows in and flow out of the system. According to Darcy's law, the rate of discharge(*Q*) is given as:

$$Q = Ki \tag{1.1),}$$

where the *K* represents the hydraulic conductivity and the hydraulic gradient is i

From Darcy, the rate of inflow can then be given by:

$$Q_{1=} K\left(\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}\right)\big(2\pi(r + dr)\big)b \tag{1.2),}$$

where *b* represents the thickness of the aquifer

In addition, the rate of outflow is:

$$Q_2 = K\frac{\partial h}{\partial r}(2\pi r)b \tag{1.3).}$$

Change in volume is given by:

$$\delta V = S(2\pi r)drdh \tag{1.4).}$$

Therefore:

$$\frac{\partial V}{\partial t} = S(2\pi r)dr\frac{dr}{dt} \tag{1.5).}$$

Because the difference between the rate of flow is given by:

$$Q_1 - Q_2 = \frac{\partial V}{\partial t} \quad (1.6),$$

Therefore:

$$K\left(\frac{\partial h}{\partial r} + \frac{\partial^2 h}{dr^2}\right)\big(2\pi(r + dr)\big)b - Kb\left(\frac{\partial h}{\partial r}\right)(2\pi r)dr\frac{dh}{dt} \quad (1.7).$$

Theis equation is then derived to be:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{Kb}\frac{\partial h}{\partial t} \quad (1.8)$$

According to (Waghmare, 2016) and (Loăiciga, 2010), Theis equation solution development was based on the conduction of heat and the flow of groundwater. The following boundaries were considered:

$$\begin{aligned} &at\ t = 0 \quad , \qquad h = h_0 \\ &at\ t = \infty \quad , \qquad h = h_0 \end{aligned} \quad (1.9)$$

The solution for the equation for t≥0 is

$$s(r,t) = \frac{Q}{4\pi T}W(u) \quad (1.10)$$

From eq. 1.10, the s (r, t) in the equation represents the drawdown where r is the radial distance and t is the time at the well.

Therefore:

$$u = \frac{r^2 S_s}{4Tt}\ and\ W(u) = \int_u^\infty \frac{e^{-u}}{u}du \quad (1.11)$$

W(u) is famously called the well function (Waghmare, 2016), and is approximated as:

$$W(u) = -0.5772 - \ln(u) + u - \frac{u^2}{2.2!} + \frac{u^2}{3.3!} + \frac{u^4}{4.4!} \tag{1.12}$$

Assumptions on which the Theis is based include the following (Watson and Burnett, 1993):

- The potentiometric surface is approximated to be horizontal before any pumping has occurred on the aquifer,
- The aquifer is confined, and the extent is presumed to be infinite,
- The part of the aquifer where pumping is taking place, the aquifer is homogeneous, naturally isotropic and the thickness is of uniform nature,
- The well is being pumped at a constant rate,
- There is full penetration of the well,
- The head declines instantaneously as water is removed from storage,
- The well diameter is small so that well storage is negligible.

The use of Theis to model groundwater flow has been used extensively but there are limitations that arise in the application of this model in the field:

1. The model does not compensate for the heterogeneous nature of aquifers in the field and aquifers in the real world are rarely isotropic.
2. Aquifers in the field are imperfect and cannot however represent a uniform thickness and aquifers show hydraulic boundaries.

1.3. Problem Statement

Groundwater flow in a confined aquifer has been modelled in the past using deterministic mathematical models such as the Theis (1935) equation providing fundamental solutions. For the Theis equation to be applicable, certain assumptions have to be considered. These include the homogeneity of the aquifer, the uniform thickness, the infinite aerial extent, the fact that aquifer is isotropic and discharge rate it is pumped at which is constant. These assumptions ignore the fact that in the field, the aquifers are of heterogeneous nature, anisotropic, has finite aerial extent due to hydraulic boundaries and pumped at different discharge rate. The Theis equation is therefore simplified and does not account for the high order terms in the modelling of flow of groundwater in a confined aquifer with dual layers. This research will account for these problems by developing a model for groundwater flow equation for a confined aquifer with dual layers. The complexity and imperfect flow in the aquifers and special aquifers (confined with a dual layer) will be taken into consideration so as to not limit the equation by assumptions and simplifications.

1.4. Aims and Objectives

The main aim of this study is to develop a model for groundwater flow in a confined aquifer with dual layers.

Objectives include:

1.4.1 Deriving a new or modified equation(s) for groundwater flow in a confined aquifer with dual layers.

1.4.2 Proving the uniqueness of the above equation(s).

1.4.3 Solve the equation(s) of groundwater flow within confined aquifer with dual layers using the new numerical schemes.

1.4.4 Checking the stability of new numerical solutions.

1.4.5 Creating numerical simulations to represent groundwater flow.

1.5 Research Framework

The following framework (Figure 1) is going to be used to achieve the main aims and objectives of this study:

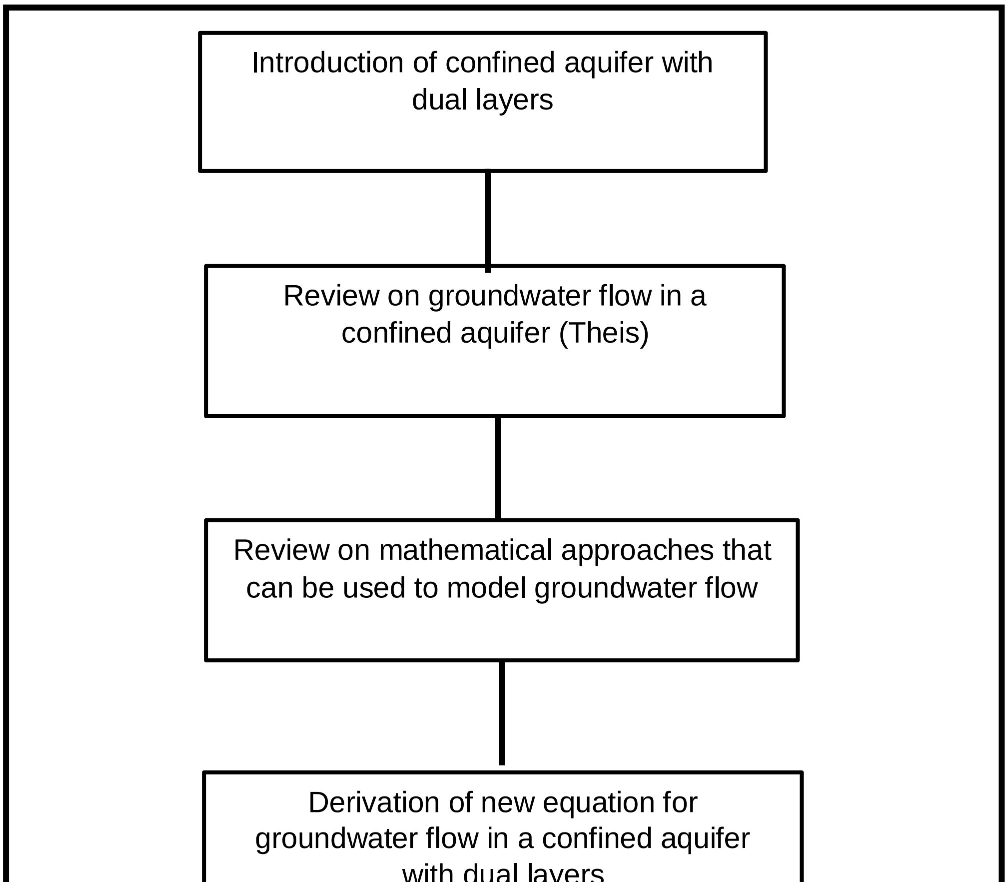

Figure 1: Research Framework of the study

1.6. Research Outline

This dissertation is divided into six chapters. Chapter one provides the introduction to the study which includes the background study on groundwater in general and the groundwater flow in a confined aquifer which is the focus. This chapter also provides the introduction to what has been used to model groundwater flow in a confined layer in the past, in addition, its derivation, the assumptions it is based on and the limitations of the model has also been included. Chapter two provides literature on aquifers and what makes the aquifer being investigated differently from the rest which is a confined aquifer with dual layers. The factors affecting the process of modelling flow is also represented in the study. Furthermore, chapter three addresses different mathematical approaches which can be used to model flow in a confined aquifer. The fractal-fractional derivatives concept is introduced here. The classical, fractional, and fractal derivatives are discussed here in detail. In chapter four the three operators of fractal-fractional are numerically approximated and numerical solutions are solved. Chapter five is the stability analysis of the numerical solutions. Finally, chapter 6 presents the numerical simulations results and discussions.

CHAPTER 2: REVIEW ON AQUIFERS

2.1. Types of Aquifers

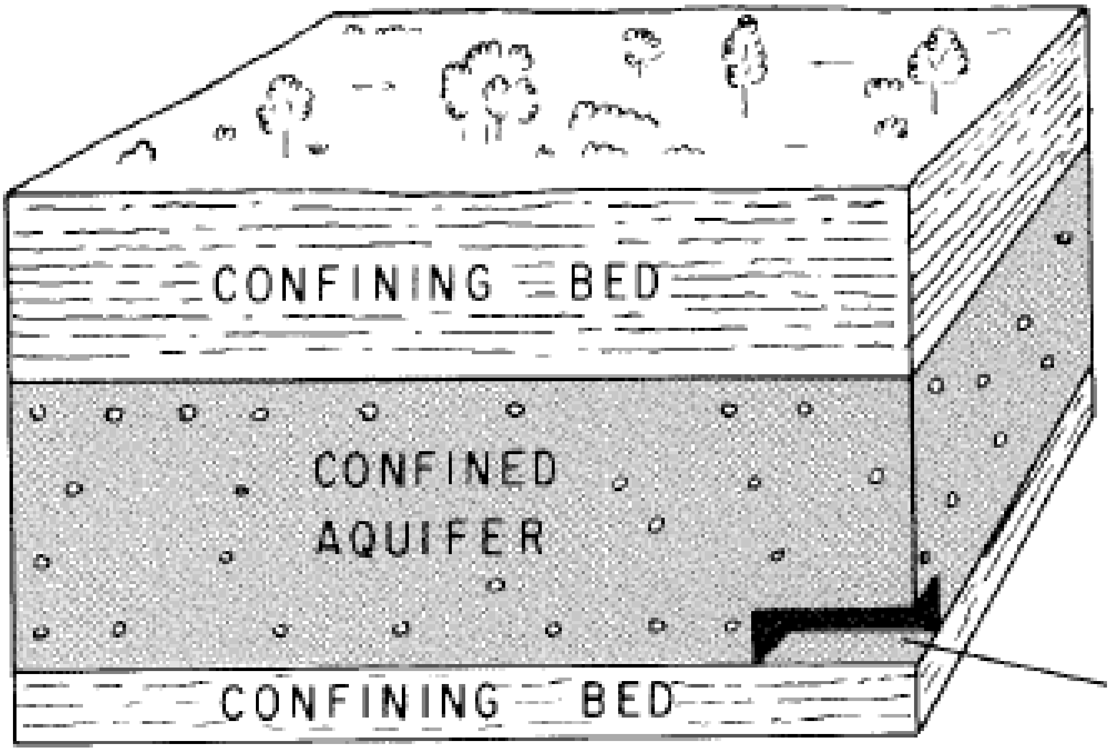

Figure 2: A confined aquifer (Heath R.C., 1987).

Aquifers have been divided into different types based on the geology and the basis of the existence or non-existence of the water table: Confined, unconfined aquifers and leaky aquifers (Fetter, 1994). A confined aquifer (Figure 2) is described as an aquifer that is bounded between two confining aquiclude layers (Christiansen et.al, 2014). Due to the confining layer on the top, recharge cannot happen directly from above, therefore it only happens in certain parts where the confining layer is non-existence. Although the top confining layer is not ideal for recharge, it does help when it comes to lessening the vulnerability to pollution (Christiansen et.al, 2014). The atmospheric pressure does not affect the aquifer due to the confining layer, but the pressure inside the confined aquifer is far greater than the atmospheric pressure (Kruseman et.al, 2000). Because of this pressure, there is a rise in the level of water occurring in the piezometer and this rise is above the top of the aquifer (Christiansen et.al, 2014).

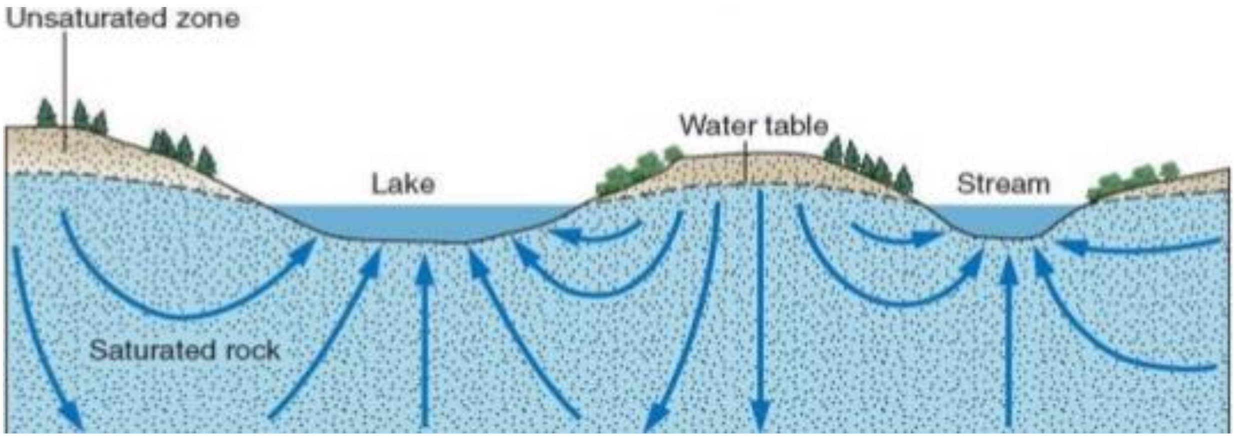

Figure 3: An unconfined aquifer. (Bishnoi B., 2016)

An unconfined aquifer (Figure 3) only consists of one confining layer at the bottom with the top part completely unbounded by a confining layer. The aquifer does not have a confining layer between the ground and the water table, so the top is open to the atmosphere. Recharge can occur directly from the top of the aquifer and because of the effect of atmospheric pressure, the water table fluctuates up and down. An unconfined aquifer acts more like a sponge and there is an increased risk of pollution because of the absence of a top confining layer (Christiansen et.al, 2014).

A semi-confined aquifer is often referred to as a leaky aquifer and is a type of aquifer confined on top and on the bottom by aquitard layers or in some cases consist of one aquitard layer and one aquiclude layer (Kruseman et.al, 2000). For this investigation, the focus will be solely on confined aquifers.

Aquifers can occur as more than one layer of an aquifer, these are termed multi-layered aquifers. Multi-layered aquifers can occur in many forms, Kruseman et, al., 2000 described three systems. They can occur as two or more aquifer layers separated by aquiclude layers. The second type of multi-layered aquifer system is an aquifer with two or more aquifer layers not separated by a layer and confining the top aquifer on the top and the bottom aquifer on its bottom by aquiclude layers. The third type of a multi-layered aquifer system consists of two or more aquifer layers which are separated by aquitard layers (Kruseman et.al, 2000).

2.2. Dual layers in a confined aquifer

Figure 4 below shows the type of aquifer investigated in this study. The aquifer is a typical confined aquifer and has two types of layers in the saturated layer. The layers are classified as separate quantities because they have different lithologies and therefore cannot be simplified or reduced to one homogeneous layer.

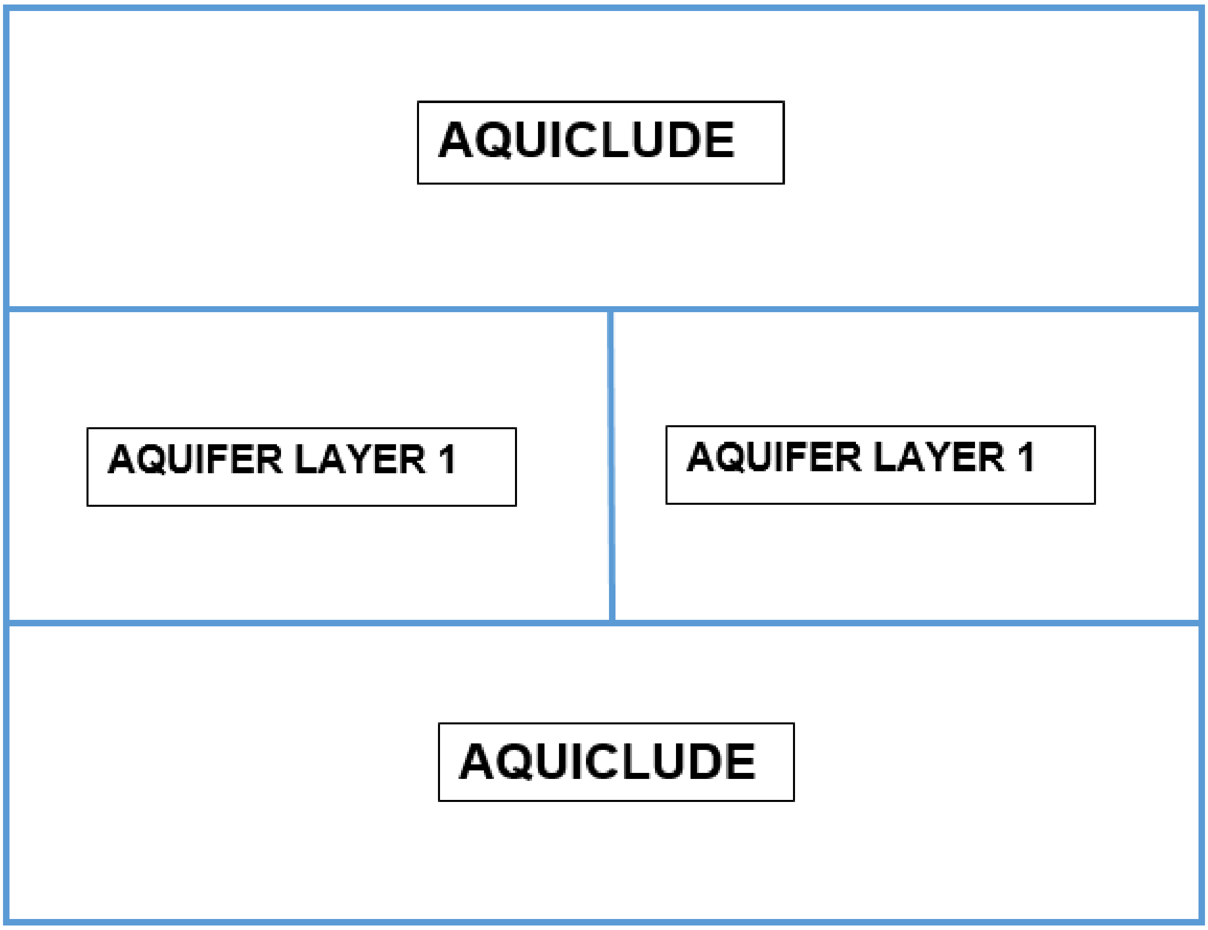

Figure 4: A confined aquifer with dual layers

2.3. Factors affecting modelling of groundwater flow

Hydrogeologists often use aquifer testing to characterize aquifer systems, aquitards and flow system boundaries. This can also be used to simulate an aquifer and results interpreted using an analytical model. The most widely used is the Theis equation. A numerical model can be proposed to solve the problem however better results cannot be apprehended by just adding complexity (Atangana and Bildik, 2013). The main effort of modelling groundwater is to come up with a mathematical relation existing between different observables which is suitable. The model should provide a better understanding of the phenomenon and should not only provide links between these different observables (Atangana and Botha, 2013).

Modelling groundwater problems using mathematical formulation has been regarded to be one of the most demanding real-world problems. The behaviour of the medium through which the water is flowing has to be precisely identified to be able to model this problem accurately. This precision is usually not achievable because the medium where water moves can change and ultimately changing the hydraulic conductivity of an aquifer's complexity (Atangana and Bildik, 2013). The mathematical model can be solved by using numerical models (Bear and Verruijt, 1987).

Researchers have proposed a prediction of how water flow in an aquifer by modelling, but limitations always arise. The fractional order derivatives were then suggested and a generalized equation by Botha and Cloot (2006). The fractional order derivative was then extended to the concept of the fractional-variation order by Atangana and Botha (2013). The numerical scheme's stability and also the convergence of the numerical scheme of groundwater flow using the fractional order derivative was presented via the Crank-Nicolson method which then presented complexity (Atangana and Bildik, 2013).

Factors such as the underlying simplifying approximations, the non-uniqueness and the uncertainty often limit the groundwater models. Forecasting simulations for groundwater uses the same concept as the forecasting that is done for the weather and they have uncertainty that should be reported and evaluated. The groundwater results that are received from the models have to be analysed for the nature and the magnitude of the uncertainties (Anderson et.al, 2015).

Different combinations of the model inputs are often producing results that correspond to the data captured in the field and this is referred to as the non-uniqueness of the model. It has been discovered that groundwater models cannot give one accurate answer, and this is why during a modelling project a professional judgment guided by a hydrogeological principle is critical in the interpretation phase. There is a number of factors that can cause uncertainty in the modelled groundwater models (Anderson et.al, 2015).

Assumptions regarding the hydrologic processes which are important modelling objectives are indirectly chosen by selecting a certain code. The unexpected hydrological features including heterogeneities in the subsurface properties, and the future stress that occurs unexpectedly and is often factors that a modeller should take into consideration when modelling groundwater and trying to limit uncertainty in forecasting models (Anderson et.al, 2015).

Some properties of groundwater system such as the hydraulic conductivity and dispersivity cannot be directly measured and the input and the output measurements need to be analysed in order to estimate them. Some of the predictions of groundwater system are from the scope of observations and therefore introduces uncertainty. The groundwater simulations have to be reliable and stable as they are the basis of the quantitative analysis of groundwater resources and that of acceptable assessment. When analyzing for the uncertainty of groundwater modelling the following factors need to be accounted for: Studying extensively uncertainties in terms of where they occur, what drives them and what processes produce them, the state and the characteristics should also be described and then evaluated (Wu et.al, 2013).

The errors that occur when the conceptual model is constructed for groundwater and the deviations which are caused by the groundwater mathematical model with an approximate solution have an effect on the results that are going to be retrieved. The modelling of uncertainty can be improved by the scarcity and also the observation error of measurement data. The separation of uncertainty sources explicitly and the independent definition is relatively difficult. The classification of these uncertainty sources is described differently by different authors (Wu et.al, 2013).

Yen et.al (1986), classified these sources into five parts:

- The natural uncertainty which is caused by the natural processes possessing inherent randomness.
- The model uncertainty emerged from the defective model which does not have the ability to represent the real-life physical processes.
- The model parameter’s uncertainty.
- The uncertainty that is caused by the error in the observations.
- The operating uncertainty derived from factors caused by humans.

Three levels were proposed by Van Asself (2000) namely: the generation location, level of managing and natural quality and these were proposed as a result of supporting management of groundwater model. The sources of uncertainty are interpreted as stochastic, fuzzy and gray according to discipline nature by Liu and Shu (2008). The two types of modelling uncertainty; aleatory and epistemic uncertainties were proposed by Merz and Thieken (2009). Generally, according to the process of groundwater modelling logically there are three sources of uncertainty stems: observation data, model parameter and the conceptual model (Wu et.al, 2013).

More researchers have been more open to using a framework to analyzing groundwater modelling uncertainty in order to establish statistics for modelling uncertainty that are simplified and reasonable and are able to satisfy the actual needs of hydrogeological workers. The input of a set of parameters are needed in order to find a numerical model, these are the storage coefficient, transmissivity, specific yield, dispersivity and hydraulic conductivity (Wu et.al, 2013).

Because of the limited hydrogeological data in the field the parameter uncertainty is mostly derived from the following factors: the temporal variability and the spatial variability, unreasonable parameter division and the scaling effect of parameters. A conceptual model is a simplification of the actual hydrogeological conditions and it sets a tone for the construction of the numerical model. Therefore, the factors influencing the model structure uncertainty include sources and sinks, unreasonable estimating of the boundaries of a groundwater model, approximation of processes of groundwater and setting the model aquifer incorrectly (location, type, number of layers, distribution) (Wu et.al, 2013).

There are two aspects for which the observation data is used for groundwater simulation. These are for the input data which is used to build a groundwater model and the conditioning data or calibration data used to model calibration. The observation uncertainty consists of a very broad range which includes the human error, error that is caused by the stochastic distribution of the observed variable, the error of the measuring device, the sampling error of the observed variable and the error caused by indirect measurement (Wu et.al, 2013).

During the assessment of the output's uncertainty, the statistical properties of the inherent structure and the input of a system are considered and deriving the statistical information of output directly is the most direct method. This direct method is regarded as not being inconvenient and has been restrained in two aspects. The first one is the troubles that occur when deriving statistical result both in mathematics and the numerical solution (Wu et.al, 2013). The second one is the actual condition where the detailed statistical properties on the system structure and generally the input is unknown. The uncertainty analysis of hydrogeological process is traditionally used to be done by probability theory and also used for other science fields. This changed when the statistical information approach came into play because of effectively analysing uncertainty from a broad perspective (Wu et.al, 2013).

Because of all these factors affecting the modelling process, every model that is created should be checked for stability in order to account for the uncertainty and the sensitivity that might arise in future.

CHAPTER 3: MATHEMATICAL APPROACHES TO MODELLING GROUNDWATER FLOW

3.1. Introduction

Due to the many simplifications and limitations of the Theis equation which are discussed in the previous chapter, other options have been investigated to accurately represent this phenomenon. This chapter explains the classical calculus, the fractional differentiation, the fractal differentiation and finally connection of the fractional and the fractal differentiation. These concepts are studied with the focus of developing a new method that will help with the modelling of groundwater flow problems.

3.2. Classic calculus

The history of differential calculus dates back to the 17th century when it was introduced by among other researchers, G. Leibniz after L. Newton introduced the concept of 'fluxional equations' (Archibald et.al, 2004). The main dynamic of the derivation is the representation of the sensitivity of a function in terms of the rate or slope of a quantity (Matlob et.al, 2017). The first and second derivatives are said to represent the velocity and the acceleration respectively. Although the use of differential calculus has been applied extensively to model the physical phenomena in many fields, its inability to focus on complex phenomena has been a concerning issue to many researchers (Matlob et.al, 2017). This throwback lead to many questions being asked, such as what does the half order of a derivative represent? (Khalil et.al, 2013). This led to a shift from the classic calculus to fractional calculus and fractal derivative to account for the more complex phenomena.

3.3. Fractional Derivation

The idea of creating a derivative with a fractional calculus can be traced back to the genesis of differential calculus and was first proposed by Leibniz and L'Hôspital 1675 (Atangana et.al, 2018). The fractional derivatives are regarded as non-local operators used to generalize the ordinary differentiation and the integration of non-integer orders (Owalabi, 2018). It took Leibniz 15 years to answer the question of what the half order of a derivative is whereby he created a fractional derivative of the function exponential by reiterating the differentiation in the following function (Atangana, 2018):

$$\frac{\mathrm{d^n} e^{\beta x}}{\mathrm{d}x^n} = \beta^{\mathrm{n}} e^{\beta x} \ \ , n = 1{,}2{,}3 \tag{3.1}$$

n was then replaced by *v* to have:

$$\frac{\mathrm{d^v} e^{\beta x}}{\mathrm{d}x^v} = \beta^{\mathrm{v}} e^{\beta x} \ , n = 1.2.3 \tag{3.2}$$

The following relationship is established after the connection of the Fourier space is applied:

$$\hat{}\, f^{\,(v)}(x) = (-ix)^{v}\hat{}\, f\,(x) \tag{3.3}$$

In 1932, Liouville then considered a function *f(x)* with the following representation:

$$f(x) = \sum_{j=1}^{\infty} a_x\, e^{\beta jx} \tag{3.4}$$

The Leibniz formulation or the v-derivative of the exponential function was then applied, and the following was achieved:

$$\frac{\mathrm{d}^v}{\mathrm{d}x^v} f(x) = \sum_{j=1}^{\infty} a_x \beta^v j e^{\beta jx} \tag{3.5}$$

After this, Liouville then suggested another formula, which is based on the integral as follows:

$$J^{\lambda} = \int_0^{\infty} e^{-xt}\, t^{\lambda-1} \mathrm{d}t\,, \quad \lambda > 0 \quad\quad x > 0 \tag{3.6}$$

By changing the variable, the following was obtained:

$$x^{-\lambda} = \frac{1}{\Gamma(\lambda)} J(x) \tag{3.7}$$

The following is then obtained if the fractional derivative in Liouville is considered with order v:

$$\frac{\mathrm{d}^v}{\mathrm{d}x^v} = \frac{1}{\Gamma(\lambda)}\int_0^\infty \frac{\mathrm{d}^v}{\mathrm{d}x^v} e^{-xt} t^{\lambda-1} dt,$$

$$= \frac{(-1)^v}{\Gamma(\lambda)}\int_0^\infty e^{-xt} t^{\lambda+v-1} \mathrm{d}t = (-1)\frac{\Gamma(\lambda+v)}{\Gamma(\lambda)} x^{-\lambda-v}, \quad \lambda, v > 0 \tag{3.8}$$

In 1876, Riemann then provided the following fractional derivative of function f(x):

$$f^{(v)}(x) = \int_a^x (x-\tau)^{-v-1} f(\tau) + \psi(x) \tag{3.9}$$

This contribution of Liouville and Riemann resulted in the combination of the two equations resulting in the well-known Riemann-Liouville definition of fractal derivation (Atangana, 2017).

3.3.1. The Riemann-Liouville definition

The Riemann-Liouville (RL) definition is a combination of contributions made by Riemann by obtaining the fractional derivative of the exponential function in 1832 and Liouville by obtaining the fractional derivative of power function in 1847 (Matlob et.al,2017). This definition is based on the power law kernel and has been applied successfully in many fields of science including in physics with the simulations of viscoelasticity flows (Baleanu and Fernandez, 2017; Liu and Li, 2015; Mainardi, 2010) and anomalous diffusion (Zhuang et.al, 2008). The RL derivative has also been found to be useful with the Laplace transform with respect to time and space components (Atangana and Kilicman,2013). This was later disapproved by (Mirza and Vieru, 2016) who argued that it's Laplace transform present physical terms that are insignificant.

Khalil et.al, 2013 defines the definition as follows:

For $\alpha \epsilon$ $[\mathrm{n}-1, n)$, the derivative of *f* is

$$D_a^{\alpha}(f)(\mathrm{t}) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{\mathrm{d}^\mathrm{n}}{\mathrm{dt}^\mathrm{n}} \int_\alpha^\mathrm{t} \frac{f(x)}{(t-x)^{\alpha-\mathrm{n}+1}} dx \tag{3.10}$$

3.3.2. The Caputo definition

Amongst many researchers, M. Caputo is also one of the contributors to fractional calculus. The Caputo definition was based on classic Riemann-Liouville definition, which was reformulated in order to use the classic initial conditions as one of the throwbacks of the Riemann-Liouville and is the requirement of what is described as a "strange" set of initial conditions (Lazarević, et.al, 2014). The Caputo definition is based on the power function with the ability to enhance the description of the memory effect but is limited when it comes to accuracy because of the singularity of its kernel (Gomez-Aguilar, et.al, 2016; Caputo and Fabrizio, 2015). The initial standard description and the boundary conditions are a few of the real-world problems appropriately addressed by the Caputo definition as compared to the Riemann-Liouville definition (Kavvas et.al, 2017; Podlubny,1998).

Khalil et.al, 2013 defines the Caputo definition as follows

For α϶ $[\mathrm{n}-1, n)$, the derivative of *f* is

$$D_a^\alpha(\mathrm{f})(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_\alpha^t \frac{\mathrm{f}^{(\mathrm{n})}(x)}{(t-x)^{\alpha-\mathrm{n}+1}} dx \tag{3.11}$$

The one property inherited from the first derivative by all definitions is that the fractional derivative is linear (Khalil et.al, 2013). These are some of the setbacks of the other definitions:

- If α is not a natural number, the Riemann-Liouville derivative does not satisfy$D_a^\alpha(1) = 0(D_a^\alpha = 0)$(for the Caputo derivation).
- All the fractional derivatives cannot be applied the product rule of derivatives

$$D_a^\alpha(fg) \neq fD_a^\alpha(g) + gD_a^\alpha(f)$$

- The formula of the derivative of quotient of two functions cannot be applied to all the fractional derivatives

$$D_a^\alpha(f/g) \neq \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}$$

- The Caputo definition assumes that the function f is differentiable.
- The chain rule cannot be used in all the fractional derivatives:

$$D_a^{\alpha}(f\text{o g})(t) \neq f^{(\alpha)}(\text{g}(t))\text{g}^{(\alpha)}(t)$$

- In general, no fractional derivatives satisfy: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$

Both the Riemann-Liouville and the Caputo definition are non-local fractional derivatives with singular kernels which restricts their applications when some of the real-world problems must be solved (Zhang et.al, 2017).

3.4. Fractal calculus

Fractal calculus is somewhat new branch of mathematics dealing with kinetics (He, 2018). Fractal derivation is a non-standard derivation used in mathematics and applied mathematics and is scaled according to t^{α} . The fractal derivation is used to model the physical problems encountered in Fick's law, Darcy's law and Fourier's law to name a few. These problems cannot be applied to the media that consists of a non-integral fractal dimension because they are based on Euclidian geometry (Atangana, 2017).

Chen et.al, 2010 defines the fractal derivative as follows:

$$\frac{\partial u}{\partial t^{\alpha}} = \lim_{t_1 \to t} \frac{\text{u}(t_1) - \text{u}(t)}{{t_1}^{\alpha} - t^{\alpha}}, \quad 0 < \alpha \tag{3.12}$$

A generalized formula:

$$\frac{\partial \text{u}^{\beta}}{\partial t^{\alpha}} = \frac{\text{u}^{\beta}(t_1) - \text{u}^{\beta}(t)}{{t_1}^{\alpha} - t_{\alpha}} \quad ,0 < \alpha \,, 0 < \beta \tag{3.13}$$

3.5. Connecting Fractional and Fractal Derivations

The fractal-fractional differentiation is a new concept combining both the fractional differentiation and the fractal derivative, used to solve physical problems which are complex requiring complex mathematical operators of differentiation (Atangana, 2017). The fractal and fractional derivatives are found to be effective in modelling anomalous diffusion (Chen et.al, 2010). The fractal-fractional differentiation can be described by several definitions, a few are described below:

Definition 1: Fractal- Fractional derivative of f of order α in Caputo sense with power law

Let f (t) be differentiable in an open interval(a, b), if f is fractal differentiable on $(a, b)$with order β, therefore:

$$ {}^{FFP}_{\ a}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma[\text{n-α}]} \int_a^t \frac{\mathrm{d}f}{\mathrm{d}y^\beta} (t-y)^{n-\alpha-1} \mathrm{d}y, \qquad (3.14) $$

$$ \mathrm{n} - 1 < \alpha \le n \qquad 0 < n - 1 < \beta \le n $$

$$ \frac{\mathrm{d}f(y)}{\mathrm{d}y^\beta} = \lim_{t-y} \frac{f(t) - f(y)}{t^\beta - y^\beta} \qquad (3.15) $$

The generalized version is given as:

$$ {}^{FFP}_{\ a}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma[\text{n-α}]} \int_a^t \frac{\mathrm{d}^\lambda f(y)}{\mathrm{d}y^\beta} (t-y)^{n-\alpha-1} \mathrm{d}y, \qquad (3.16) $$

$$ n - 1 < \alpha \le n, \quad 0 < n - 1 < \beta \le n $$

$$ \frac{\mathrm{d}^\lambda f(y)}{\mathrm{d}y^\beta} = \lim_{t-y} \frac{f^\lambda(t) - f^\lambda(y)}{t^\beta - y^\beta} \qquad (3.17) $$

Definition 2: The Fractal- Fractional derivative of *f* of order α in Caputo sense with the generalized Mittag-Leffler kernel

Let f (t) be differentiable in an open interval (a, b)if f is fractal differentiable on(a, b) with order β, therefore:

$$ {}^{FFM}_{\ a}D_t^{\alpha,\beta} f(t) = \frac{AB}{[1-\alpha]} \int_a^t \frac{\mathrm{d}f(y)}{\mathrm{d}y^\beta} E_\alpha \left[-\frac{\alpha}{1-\alpha} (t-y)^\alpha \right] \mathrm{d}y, \qquad (3.18) $$

$$ AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} $$

Generalized version is given as:

$$ {}^{FFM}_{\ a}D_t^{\alpha,\beta,\lambda} f(t) = \frac{AB}{[1-\alpha]} \int_a^t \frac{\mathrm{d}^\lambda f(y)}{\mathrm{d}y^\beta} E_\alpha \left[-\frac{\alpha}{1-\alpha} (t-y)^\alpha \right] \mathrm{d}y, \qquad (3.19) $$

$$ 0 < \alpha, \beta, \lambda \le 1 $$

Definition 3: The Fractal- Fractional derivative of f of order α in Riemann-Liouville sense with exponential decay kernel

Let f (t) be differentiable in an open interval (a, b), if *f* is fractal differentiable on (a, b) with order β therefore:

$$ {}^{FFE}_{a}D_t^{\alpha,\beta} f(t) = \frac{M(\alpha)}{[1-\alpha]} \frac{\mathrm{d}}{\mathrm{d}t^{\beta}} \int_a^t f(y) \exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right] \mathrm{d}y, \quad 0 < \alpha, \beta, \leq n \tag{3.20} $$

Generalized version is given as:

$$ {}^{FFE}_{a}D_t^{\alpha,\beta,\lambda} f(t) = \frac{M(\alpha)}{[1-\alpha]} \frac{\mathrm{d}^{\lambda}}{\mathrm{d}y^{\beta}} \int_a^t f(y) \exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right] \mathrm{d}y, \quad 0 < \alpha, \beta, \lambda \leq 1 \tag{3.21} $$

The new concept of differentiation by connecting fractional and fractal derivatives by combining e.q. 3.22 and e.q. 3.23.

The fractional derivative is defined by:

$$ {}^{c}_{o}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathrm{d}f(t)}{\mathrm{d}\tau} (t-\pi)^{-\alpha} \mathrm{d}t \tag{3.22} $$

The fractal derivative is defined by:

$$ {}^{F}_{o}D_t^{\beta} f(t) = \lim_{t-t_1} \frac{\mathrm{d}f(t) - f(t_1)}{t^{\beta} - {t_1}^{\beta}} \tag{3.23} $$

Therefore, the combination of the two derivatives is defined by:

$$ {}^{FF}_{o}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t^{\beta}} \int_o^t f(\tau)\,(t-\tau)^{-\alpha} \mathrm{d}\tau \tag{3.24} $$

The connection between the fractional and the fractional derivatives can be defined as:

$$ {}^{FF}_{o}D_t^{\alpha,\beta} f(t) = u(t) \tag{3.25} $$

The fractal derivation can be written as:

$$\lim_{t-t_1} \frac{f(t)-f(t_1)}{t-t_1} * \frac{t-t_1}{t^\beta - {t_1}^\beta} \tag{3.26}$$

Applying the first principle:

$$f'(t) = \frac{\frac{1}{t^\beta - {t_1}^\beta}}{t-t_1} \tag{3.27}$$

If $f(t) = t^\beta$, And

$$\frac{t-t_1}{t^\beta - {t_1}^\beta} * \frac{f(t)-f(t_1)}{t-t_1} = f(t) \tag{3.28}$$

Then:

$$f'(t) = \left(t^\beta\right)' = \beta t^{\beta-1} \tag{3.29}$$

Then Fractional-fractal derivation is:

$${}^{FF}_{o}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} f(t) \tag{3.30}$$

Where

$$f(t) = \int_0^t t(\tau)\,(t-\tau)^{-\alpha} d\tau \tag{3.31}$$

Therefore:

$${}^{FF}_{o}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t^\beta} f(t) * \frac{1}{\beta t^{\beta-1}} \tag{3.32}$$

Replace $f(t)$ by a value:

$$ {}_{o}^{FF}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau * \frac{1}{\beta t^{\beta-1}} = u(t) \tag{3.33} $$

Simplifying the equation

$$ {}_{o}^{FF}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau = \beta t^{\beta-1} u(t) \tag{3.34} $$

Apply the Riemann-Liouville definition of fractional derivative on both sides to obtain:

$$ f(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} (t-\tau)^{\alpha-1} u(t) d\tau \tag{3.35} $$

Therefore:

$$ {}_{o}^{FF}J_t^{\alpha,\beta} f(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{3.36} $$

Qureshi et.al, 2019 suggested six fractal-fractional differentiation definitions which they consider as basic to understanding this concept.

Definition 1: Suppose that $y(t)$ be continuous and fractal differentiable on (a, b) with order Δ then the fractal-fractional derivative of $y(t)$ with order Ω in the Riemann Liouville sense having power law type kernel is as follows:

$$ {}_{0}^{FFP}D_t^{\Omega,\Delta}\big(y(t)\big) = \frac{1}{\Gamma(m-\Omega)} \frac{d}{dt^{\Delta}} \int_0^t (t-s)^{m-\Omega-1} y(s) ds \tag{3.37} $$

Where $n-1 < \Omega, \Delta \leq n \in \mathbb{N}$

and

$$ \frac{dy(s)}{ds^{\Delta}} = \lim_{t \to s} \frac{y(t) - y(s)}{t^{\Delta} - s^{\Delta}} \tag{3.38} $$

Definition 2: Suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional derivative of $y(t)$ with order Ω in the Riemann Liouville sense having exponentially decaying type kernel is defined as follows:

$$ {}^{FFE}_{\ 0}D_t^{\Omega,\Delta}\big(y(t)\big) = \frac{M(\Omega)}{1-\Omega}\frac{d}{dt^{\Delta}}\int_0^t exp\left(-\frac{\Omega}{1-\Omega}(t-s)\right)y(s)ds\ , \qquad (3.39) $$

Where $\Omega > n \in \mathbb{N}$ and $M(1) = 1$.

Definition 3: Suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional derivative of $y(t)$ with order Ω in the Riemann Liouville sense having generalized Mittag-Leffler type kernel is defined as follows:

$$ {}^{FFE}_{\ 0}D_t^{\Omega,\Delta}\big(y(t)\big) = \frac{AB(\Omega)}{1-\Omega}\frac{d}{dt^{\Delta}}\int_0^t E_{\Omega}\left(-\frac{\Omega}{1-\Omega}(t-s)^{\Omega}\right)y(s)ds $$

Where $0 < \Omega, \Delta \leq 1$ and (3.40)

$AB(\Omega) = 1 - \Omega + \frac{\Omega}{\Gamma(\Omega)}$.

Definition 4: Suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional integral of $y(t)$ with order Ω having power law type kernel is defined as follows:

$$ {}^{FFP}_{\ 0}J_t^{\Omega,\beta}\big(y(t)\big) = \frac{\beta}{\Gamma(\Omega)}\int_0^t (t-s)^{\Omega-1}s^{\beta-1}y(s)ds \qquad (3.41) $$

Definition 5: Suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional integral of $y(t)$ with order Ω having exponentially type kernel is defined as follows:

$$ {}^{FFE}_{\ 0}J_t^{\Omega,\Delta} = \frac{\Omega\Delta}{M(\Omega)}\int_0^t s^{\Omega-1}y(s)ds + \frac{\Omega(1-\Omega)t^{\Delta-1}y(t)}{M(\Omega)} \qquad (3.42) $$

Definition 6: Suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional integral of $y(t)$ with order Ω having generalized Mittag-Leffler type kernel is defined as follows:

$$ {}^{FFM}_{\;\;0}J_t^{\Omega,\Delta}\big(y(t)\big) = \frac{\Omega\Delta}{AB(\Omega)}\int_0^t s^{(\Omega-1)} y(s)(t-s)^{\Omega-1}\, ds + \frac{\Delta(1-\Omega)t^{\Delta-1}y(t)}{AB(\Omega)} \tag{3.43} $$

This combination of fractional differentiation and the fractal derivative introduces more complex mathematical operators of differentiation allowing more complex physical problems to be solved.

CHAPTER 4: NUMERICAL APPROXIMATION

4.1. Introduction

The purpose of the numerical approximation in this study is to create acceptable operators that can be subsequently used to represent the groundwater flow by making use of computer simulations. For this to be achieved the following must be taken into consideration; the accuracy, the stability, including the computational ability of the new numerical scheme. The numerical approximation application allows the numerical analysis of the partial differential equations to achieve some extent of algorithms simplicity.

This chapter presents some fractal-fractional derivatives definitions and how they are applied to the modelling of groundwater flow problems. Each operator of the fractal-fractional is then solved.

4.1.1. Understanding of the first and the second derivative

The first derivative with one variable:

$$\frac{df(t)}{dt} = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} \tag{4.1}$$

Moving from t_n to t_{n+1} the first derivative is given as follows:

$$\frac{df(t)}{dt} \cong \frac{f(t_{n+1})-f(t_n)}{\Delta t} \tag{4.2}$$

Moving from tn to t_{n-1}

$$\frac{df(t)}{dt} \cong \frac{f(t_n)-f(t_{n-1})}{\Delta t} \tag{4.3}$$

The first derivative with two variables (respect to t):

Moving from t_n to t_{n+1}

$$\frac{\partial f}{\partial t}\left(x_{i,}t_n\right) \cong \frac{f(x_i, t_{n+1}) - f(x_i, t_n)}{\Delta t} \tag{4.4}$$

Moving from t_n to t_{n-1}

$$\frac{\partial f}{\partial t}\left(x_{i,}t_n\right) \cong \frac{f(x_i, t_n) - f(x_i, t_{n-1})}{\Delta t} \tag{4.5}$$

The first derivative with two variables (respect to x):

Moving from x_{i-1} to x_{i+1}

$$\frac{\partial f}{\partial x}\left(x_{i,}t_n\right) \cong \frac{f(x_{i+1}, t_n) - f(x_{i-1}, t_n)}{\Delta x} \tag{4.6}$$

Moving from x_i to x_{i+1}

$$\frac{\partial f}{\partial x}\left(x_{i,}t_n\right) \cong \frac{f(x_{i+1}, t_n) - f(x_{i-1}, t_n)}{\Delta x} \tag{4.7}$$

Therefore, the second derivative is given as:

$$\frac{\partial^2 f}{\partial x^2}\left(x_{i,}t_n\right) \cong \frac{f(x_{i+1}, t_n) - 2f(x_i, t_n) + f(x_{i-1}, t_n)}{(\Delta x)^2} \tag{4.8}$$

And

$$\frac{\partial^2 f}{\partial x^2}\left(x_{i,}t_n\right) \cong \frac{f(x_{i+1}, t_{n+1}) - 2f(x_i, t_n + 1) + f(x_{i-1}, t_{n-1})}{(\Delta x)^2} \tag{4.9}$$

The combination of the two is then:

$$\frac{\partial^2 f}{\partial x^2}\left(x_{i,}t_n\right) \cong \frac{1}{2}\left[\frac{f^n{}_{i+1} - 2f_i^{\,n} + f_{i-1}{}^n}{(\Delta x)^2} + \frac{f_{i+1}{}^{n+1} - 2f_i^{\,n+1} + f_{i-1}{}^{n-1}}{(\Delta x)^2}\right] \tag{4.10}$$

4.2. First operator

Definition 4 from Qureshi et.al, 2019 is the first operator of this study. For this definition suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional integral of $y(t)$ with order Ω having power law type kernel. The numerical approximation for the first operator is as follows.

$$ {}_{0}^{FFP}J_t^{\alpha,\beta}\big(y(t)\big) = \frac{\beta}{\Gamma(1-\Omega)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}y(\tau)d\tau \tag{4.11} $$

$$ {}_{0}^{FFP}J_t^{\alpha,\beta}\big(y(t)\big) = \frac{\beta}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}y(\tau)d\tau\frac{1}{\beta t^{\beta-1}} \tag{4.12} $$

For better understanding, the numerical approximation can be explained by finding the definition from moving from 0 to t_{n+1}. From equation 4.12, the equation is simplified to be equal to

$$ {}_{0}^{FFP}J_t^{\alpha,\beta}\big(y(t)\big) = \frac{d}{dt}\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\big(y(t)\big)d\tau\frac{1}{\beta t^{\beta-1}} \tag{4.13} $$

Representing the terms by $f(t)$, therefore:

$$ {}_{0}^{FFP}J_t^{\alpha,\beta}\big(y(t)\big) = \frac{d}{dt}f(t) \tag{4.14} $$

Moving from 0 to t_{n+1}

$$ \frac{df(t)}{dt} = \frac{f(t_{n+1}) - f(t_n)}{\Delta t} \tag{4.15} $$

Replacing t byt_{n+1} in equation 4.13 therefore:

$$ \frac{df(t_{n+1})}{dt} = \int_0^{t_n} \frac{(t_{n+1}-\tau)^{-\alpha}}{\Gamma(1-\alpha)}y(t)d\tau\frac{1}{\beta t_{n+1}^{\beta-1}} \tag{4.16} $$

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\beta t_{n+1}^{\beta-1}} \int_0^{t_n+1} (t_{n+1} - \tau)^{-\alpha} y(\tau) d\tau \tag{4.17}$$

To move from 0 to t_{n+1}, we first must move from 0 to t_1, to t_2, to t_3 and so on. To represent the sum factor, the equation is modified as follows:

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\mathrm{\alpha})} \frac{1}{\beta t_{n+1}^{\beta-1}} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-\alpha}\, y(t_j) d\tau \tag{4.18}$$

Further simplification

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\beta t_{n+1}^{\beta-1}} \sum_{j=0}^{n} y^j \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-\alpha}\, d\tau \tag{4.19}$$

Let $z = t_{n+1} - \tau$, therefore $z = t_{n+1} - t_j$ and $z = t_{n+1} - t_{j+1}$

Then $dz = -d\tau$, therefore

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\beta t_{n+1}^{\beta-1}} \sum_{j=0}^{n} y^j \int_{t_{n+1}-t_j}^{t_{n+1}-t_{j+1}} z^{-\alpha}(-dz) \tag{4.20}$$

Apply rule: $-\int_a^b x = \int_b^a x$

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\beta t_{n+1}^{\beta-1}} \sum_{j=0}^{n} y^j \int_{t_{n+1}-t_{j+1}}^{t_{n+1}-t_j} z^{-\alpha}(-dz) \tag{4.21}$$

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\beta t_{n+1}^{\beta-1}} \sum_{j=0}^{n} y^j \Big|_{t_{n+1} - t_{j+1}}^{t_{n+1} - t_j} \tag{4.22}$$

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\beta t_{n+1}^{\beta-1}} \sum_{j=0}^{n} y^j \left[\frac{(t_{n+1} - t_j)^{1-\alpha}}{1-\alpha} - \frac{(t_{n+1} - t_{j+1})^{1-\alpha}}{1-\alpha}\right] \tag{4.23}$$

Represent the following by

$$t_n = \Delta t_{n,} \quad t_{n+1} = \Delta t(n+1)\,, \quad t_{j+1} = \Delta t(j+1) \quad , \qquad t_j = \Delta t_j$$

Therefore

$$\frac{df(t_{n+1})}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_{n+1}^{\beta-1}}\sum_{j=0}^{n} y^j \left[\frac{((n+1)\Delta t-(j+1)\Delta t)^{1-\alpha}}{1-\alpha} - \frac{((n+1)\Delta t-(j+1)\Delta t)^{1-\alpha}}{1-\alpha}\right] \quad (4.24)$$

Taking out the common factor

$$\frac{df(t_{n+1})}{dt} = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)}\frac{1}{\beta t_{n+1}^{\beta-1}}\sum_{j=0}^{n} y^j[(n+1-j)^{1-\alpha}-(n-j)^{1-\alpha}] \quad (4.25)$$

Numerical approximation in terms of moving from 0 to t_n.

$$\frac{df(t_n)}{dt} = \int_0^{t_n}\frac{(t_n-\tau)^{-\alpha}}{\Gamma(1-\alpha)}y(t)d\tau\frac{1}{\beta t_n^{\beta-1}} \quad (4.26)$$

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\int_0^{t_n}(t_n-\tau)^{-\alpha}y(\tau)d\tau \quad (4.27)$$

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\mathrm{\alpha})}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_n-\tau)^{-\alpha}\,y(t_j)d\tau \quad (4.28)$$

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j\int_{t_j}^{t_{j+1}}(t_n-\tau)^{-\alpha}\,d\tau \quad (4.29)$$

Represent the following by $z = t_n - \tau$, therefore $z = t_n - t_j$ and $z = t_n - t_{j+1}$

And then let $dz = -d\tau$, therefore

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j\int_{t_n-t_j}^{t_n-t_{j+1}} z^{-\alpha}(-dz) \quad (4.30)$$

Apply rule: $-\int_a^b x = \int_b^a x$

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j \int_{t_n-t_j+1}^{t_n-t_j} z^{-\alpha}(-dz) \tag{4.31}$$

After Integrating

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j \Big|_{t_n-t_j}^{t_n-t_{j+1}} \tag{4.32}$$

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j \left[\frac{(t_n-t_j)^{1-\alpha}}{1-\alpha} - \frac{(t_n-t_{j+1})^{1-\alpha}}{1-\alpha}\right] \tag{4.33}$$

Represent the following by $t_n = \Delta t_{n,}$ $t_n = \Delta t(n)$, $t_{j+1} = \Delta t(j+1)$, $t_j = \Delta t_j$

Therefore

$$\frac{df(t_n)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j \left[\frac{(n\Delta t - j\Delta t)^{1-\alpha}}{1-\alpha} - \frac{(n\Delta t-(j+1)\Delta t)^{1-\alpha}}{1-\alpha}\right] \tag{4.34}$$

Taking out the common factor

$$\frac{df(t_n)}{dt} = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)}\frac{1}{\beta t_n^{\beta-1}}\sum_{j=0}^{n} y^j [(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}] \tag{4.35}$$

Substituting equation (4.25) and (4.35) into (4.15):

$$\frac{df(t)}{dt} = \frac{f(t_{n+1}) - f(t_n)}{\Delta t}$$

$$\frac{df(t)}{dt} = \frac{1}{\Delta t}\left[\left(\frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)}\frac{1}{\beta t_{n+1}^{\beta-1}}\sum_{j=0}^{n} y^j[(n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}]\right) - \left(\frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)}\frac{1}{\beta t_{n}^{\beta-1}}\sum_{j=0}^{n-1} y^j[(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}]\right)\right] \tag{4.36}$$

Taking out the common factor

$$\frac{df(t)}{dt} = \frac{1}{\Delta t}\left[\frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n} y^j\left[\left(\frac{1}{\beta t_{n+1}^{\beta-1}}[(n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}]\right) - \sum_{j=0}^{n-1} y^j\left(\frac{1}{\beta t_{n}^{\beta-1}}[(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}]\right)\right]\right] \tag{4.37}$$

Further simplification

$$\frac{df(t)}{dt} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{j=0}^{n} y^j\left(\frac{1}{\beta t_{n+1}^{\beta-1}}[(n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}]\right) - \sum_{j=0}^{n-1} y^j\left(\frac{1}{\beta t_{n}^{\beta-1}}[(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}]\right)\right] \tag{4.38}$$

When there is more than one variable in a function

$$ {}^{FFP}_{\ \ 0}D_t^{\alpha,\beta} y(r_i,t_n) = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{j=0}^{n} y_i^j\left(\frac{1}{\beta t_{n+1}^{\beta-1}}[(n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}]\right) - \sum_{j=0}^{n-1} y_i^j\left(\frac{1}{\beta t_{n}^{\beta-1}}[(n-j)^{1-\alpha} - (n-j-1)^{1-\alpha}]\right)\right] \tag{4.39}$$

Common factor

$$ {}^{FFP}_{\ \ 0}D_t^{\alpha,\beta} h(r_i,t_n) = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{j=0}^{n} \frac{1}{\beta t_{n+1}^{\beta-1}} h_i^j\, \delta_{n,j}^{\alpha,\beta} - \sum_{j=0}^{n-1} \frac{1}{\beta t_{n}^{\beta-1}} h_i^j\, \tau_{n,j}^{\alpha,\beta}\right] \tag{4.40}$$

Then:

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{j=0}^{n}\frac{1}{\beta t_{n+1}^{\beta-1}}h_i^j\,\delta_{n,j}^{\alpha,\beta}-\sum_{j=0}^{n-1}\frac{1}{\beta t_n^{\beta-1}}h_i^j\,\tau_{n,j}^{\alpha,\beta}\right] = \frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^n-h_{i-1}^n}{\Delta r}+\frac{h_{i+1}^n-2h_i^n+h_{i-1}^n}{(\Delta r)^2}\right] \tag{4.41}$$

From the first sum when j=n and the second sum when j=n-1

$$\begin{aligned}\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Bigg[&\frac{1}{\beta t_{n+1}^{\beta-1}}h_i^n\delta_{n,n}^{\alpha,\beta}+\frac{1}{\beta t_{n+1}^{\beta-1}}h_i^{n-1}\delta_{n,n-1}^{\alpha,\beta}+\sum_{j=0}^{n-2}\frac{1}{\beta t_{n+1}^{\beta-1}}h_i^j\delta_{n,j}^{\alpha,\beta}\\ &-\frac{1}{\beta t_n^{\beta-1}}h_i^{n-1}\tau_{n,n-1}^{\alpha,\beta}-\sum_{j=0}^{n-2}\frac{h_i^j}{\beta t_n^{\beta-1}}\tau_{n,j}^{\alpha,\beta}\Bigg]\\ &=\frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^n-h_{i-1}^n}{\Delta r}+\frac{h_{i+1}^n-2h_i^n+h_{i-1}^n}{(\Delta r)^2}\right]\end{aligned} \tag{4.42}$$

$$\begin{aligned}h_i^n&\left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}.\frac{\delta_{n,n}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}}+\frac{T}{S}\frac{2}{(\Delta r)^2}\right)+h_i^{n-1}\left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}}-\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_n^{\beta-1}}\right)\\ &-h_{i+1}^n\left(\frac{T}{S}\frac{1}{\Delta r(r_i)}-\frac{T}{S}\frac{1}{(\Delta r)^2}\right)-h_{i-1}^n\left(\frac{T}{S}\frac{1}{\Delta r(r_i)}-\frac{T}{S}\frac{1}{(\Delta r)^2}\right)\\ &=\sum_{j=0}^{n}\frac{1}{\beta t_{n+1}^{\beta-1}}h_i^j\delta_{n,j}^{\alpha,\beta}+\sum_{j=0}^{n-1}\frac{1}{\beta t_n^{\beta-1}}h_i^j\,\tau_{n,j}^{\alpha,\beta}\end{aligned} \tag{4.43}$$

4.3. Second Operator

Definition 2 of fractional-fractal differentiation from Qureshi et.al, 2019 is the second operator. For this operator suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional derivative of $y(t)$ with order Ω in the Riemann Liouville sense having exponentially decaying type kernel. The numerical approximation for the second operator is as follows.

$$^{FFE}{}_0D_t^{\alpha,\Delta}\big(y(t)\big)=\frac{M(\alpha)}{1-\alpha}\int_0^t f(\tau)\,exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau\,\frac{1}{\beta t^{\beta-1}} \tag{4.44}$$

$$
{}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{M(\alpha)}{\beta t^{\beta-1}(1-\alpha)}\Bigg[exp\left[-\frac{\alpha}{1-\alpha}t\right]f(0) + \int_0^t \frac{d}{d\tau}f(\tau)exp\left(-\frac{\alpha}{1-\alpha}\right)d\tau\Bigg] \tag{4.45}
$$

At t_{n+1}

$$
{}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{M(\alpha)}{\beta t_{n+1}^{\beta-1}(1-\alpha)}\Bigg[exp\left[-\frac{\alpha}{1-\alpha}t_{n+1}\right]f(0) + \int_0^{t_{n+1}} \frac{d}{d\tau}f(\tau)exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right]d\tau\Bigg] \tag{4.46}
$$

$$
{}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \int_0^{t_{n+1}} \frac{df(\tau)}{d\tau}exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right]d\tau + \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right]\frac{f^{j+1}-f^j}{\Delta t}d\tau \tag{4.47}
$$

Further simplification

$$
{}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\int_{t_j}^{t_{j+1}} exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right]d\tau \tag{4.48}
$$

Representing the following by $y = t_{n+1} - \tau$, $y = t_{n+1} - t_j$, $y = t_{n+1} - t_{j+1}$, $d\tau = -dy$

$$
{}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\int_{t_{n+1}-t_{j+1}}^{t_{n+1}-t_j} exp\left(-\frac{\alpha}{1-\alpha}y\right)dy \tag{4.49}
$$

$$
{}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{\alpha}{1-\alpha}exp\left[-\frac{\alpha}{1-\alpha}y\right]\Big|_{t_{n+1}-t_{j+1}}^{t_{n+1}-t_j} \tag{4.50}
$$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1}-f^{j}}{\Delta t}\frac{\alpha}{1-\alpha}\left[exp\left[-\frac{\alpha}{1-\alpha}\left(t_{n+1}-t_{j}\right)\right] - exp\left[-\frac{\alpha}{1-\alpha}\left(t_{n+1}-t_{j+1}\right)\right]\right] \quad (4.51) $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1}-f^{j}}{\Delta t}\frac{\alpha}{1-\alpha}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j+1)\Delta t\right] - exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]\right] \quad (4.52) $$

Therefore:

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{M(\alpha)}{\beta t_{n+1}^{\beta-1}(1-\alpha)}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]f(0) + \sum_{j=0}^{n} \frac{f^{j+1}-f^{j}}{\Delta t}\frac{\alpha}{1-\alpha}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j+1)\Delta t\right] - exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]\right]\right] \quad (4.53) $$

When we have more than one variable forward, the solution then becomes:

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{M(\alpha)}{\beta t_{n+1}^{\beta-1}(1-\alpha)}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]h(r,0) + \sum_{j=0}^{n} \frac{h_i^{j+1}-h_i^{j}}{\Delta t}\frac{\alpha}{1-\alpha}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j+1)\Delta t\right] - exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]\right]\right] = \frac{T}{S}\left[\frac{h_{i+1}^{n+1}-2h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2} + \frac{1}{r_i}\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}\right] \quad (4.54) $$

4.4. Third operator

Definition 3 from Qureshi et.al, 2019 is the third operator of this study. For this operator suppose that $y(t)$ be continuous and the fractal differentiation on (a,b) with order Δ then the fractal-fractional derivative of $y(t)$ with order Ω in the Riemann Liouville sense having generalized Mittag-Leffler type kernel. The numerical approximation for the third operator is as follows.

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{AB(\alpha)}{1-\alpha}\frac{d}{dt^{\Delta}}\int_0^t E_{\Omega}\left(-\frac{\Omega}{1-\Omega}(t-s)^{\alpha}\right)y(\tau)d\tau \tag{4.55} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{AB(\alpha)}{(1-\alpha)}\left[E_{\alpha}\left[-\frac{\alpha}{1-\alpha}t^{\alpha}\right]f(0) + \int_0^t \frac{d}{d\tau}f(\tau)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}\right)d\tau\right] \tag{4.56} $$

At t_{n+1}

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \frac{AB(\alpha)}{(1-\alpha)}\left[E_{\alpha}\left[-\frac{\alpha}{1-\alpha}t_{n+1}^{\alpha}\right]f(0) + \int_0^{t_{n+1}} \frac{d}{d\tau}f(\tau)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)\right]d\tau\right] \tag{4.57} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \int_0^{t_{n+1}} \frac{df(\tau)}{d\tau}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)^{\alpha}\right]d\tau \tag{4.58} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)^{\alpha}\right]\frac{f^{j+1}-f^{j}}{\Delta t}d\tau \tag{4.59} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\int_{t_j}^{t_{j+1}} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-\tau)^{\alpha}\right] \tag{4.60} $$

Represent the following by $y = t_{n+1} - \tau$, $y = t_{n+1} - t_j$, $y = t_{n+1} - t_{j+1}$, $d\tau = -dy$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1} - f^j}{\Delta t} \int_{t_{n+1}-t_{j+1}}^{t_{n+1}-t_j} E_\alpha\left(-\frac{\alpha}{1-\alpha} y^\alpha\right) dy \tag{4.61} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1} - f^j}{\Delta t} \int_{t_{n+1}-t_{j+1}}^{t_{n+1}-t_j} E_\alpha\left[-\frac{\alpha}{1-\alpha} y^\alpha\right] d\tau \tag{4.62} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1} - f^j}{\Delta t} E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha} y^\alpha\right] - E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha} y^\alpha\right] \Big|_{t_{n+1}-t_{j+1}}^{t_{n+1}-t_j} \tag{4.63} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1} - f^j}{\Delta t} E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha} (n-j)^\alpha \Delta t\right] - E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha} (n-j+1)^\alpha \Delta t\right] \tag{4.64} $$

$$ {}^{FFE}_{\ 0}D_t^{\alpha,\Delta}\big(y(t)\big) = \sum_{j=0}^{n} \frac{f^{j+1} - f^j}{\Delta t} \left[E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha} (n-j)^\alpha \Delta t\right] - E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha} (n-j+1)^\alpha \Delta t\right]\right] \tag{4.65} $$

Therefore:

$$
\begin{aligned}
&{}^{FFE}_{0}D_t^{\alpha,\Delta}\big(y(t)\big) = \\
&\qquad = \frac{AB(\alpha)}{(1-\alpha)}\Bigg[E_\alpha\left[-\frac{\alpha}{1-\alpha}(n-j)^\alpha\Delta t\right]f(0) \\
&\qquad + \sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\left[E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(n-j)^\alpha\Delta t\right]\right. \\
&\qquad \left.- E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(n-j+1)^\alpha\Delta t\right]\right]\Bigg]
\end{aligned}
\tag{4.66}
$$

When there is more than one variable:

$$
\begin{aligned}
{}^{FFE}_{0}D_t^{\alpha,\Delta}\big(y(t)\big) &= \frac{AB(\alpha)}{(1-\alpha)}\Bigg[E_\alpha\left[-\frac{\alpha}{1-\alpha}(n-j)^\alpha\Delta t\right]h(r,0) \\
&+ \sum_{j=0}^{n}\left(h_i^{j+1}-h_i^{j}\right)\left[(n-j)E_{\alpha,2}\left(-\frac{\alpha}{1-\alpha}(n-j)^\alpha\Delta t^\alpha\right)\right. \\
&\left.- (n-j+1)E_{\alpha,2}\left(-\frac{\alpha}{1-\alpha}(n-j+1)^\alpha\Delta t^\alpha\right)\right]\Bigg] \\
&= \frac{T}{S}\left[\frac{h_{i+1}^{n+1}-2h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{1}{r_i}\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}\right]
\end{aligned}
\tag{4.67}
$$

Due to the appearance of the generalized Mittag-Leffler function on the discretized equation, use could revert the equation to the Volterra type then suggest a comparable approximation integral.

$$
{}^{AB}_{0}D_t^{\alpha,\beta}h(r,t) = \frac{T}{S}\left[\frac{\partial^2 h(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\right] \tag{4.68}
$$

Represent $\frac{T}{S}\left[\frac{\partial^2 h(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial h(r,t)}{\partial r}\right]$ $=F(r,t,h(r,t)$

We then apply the fractal-fractional integral on both sides to have:

$$
\begin{aligned}
&h(r,t)-h(r,0)\\
&\qquad=\frac{(1-\alpha)\beta t^{\beta-1}}{AB(\alpha)}F\big(r,t,h(r,t)\big)\\
&\qquad+\frac{\alpha\beta}{AB\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1}F\big(r,\tau,h(r,\tau)\big)d\tau
\end{aligned}
\tag{4.69}
$$

At (r_i, t_{n+1}) we have

$$
\begin{aligned}
&h(r_i,t_{n+1})-h\big(r_{i,}0\big)\\
&\qquad=\frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1}F\left(r_{i,}t_{n,}h\big(r_{i,}t_n\big)\right)\\
&\qquad+\frac{\alpha\beta}{AB\Gamma(\alpha)\Gamma(\alpha)}\int_0^{t_{n+1}}\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\,F\left(r_{i,}\tau,h\big(r_{i,}\tau\big)\right)d\tau
\end{aligned}
\tag{4.70}
$$

$$
\begin{aligned}
&h(r_i,t_{n+1})-h\big(r_{i,}0\big)\\
&\qquad=\frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1}F\left(r_{i,}t_{n,}h\big(r_{i,}t_n\big)\right)\\
&\qquad+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}(t_{n+1}\\
&\qquad-\tau)^{\alpha-1}\,F\left(r_{i,}\tau,h\big(r_{i,}\tau\big)\right)d\tau
\end{aligned}
\tag{4.71}
$$

For simplicity we let $\gamma(r_{i,}\tau,\alpha,\beta)=\tau^{\beta-1}F\left(r_{i,}\tau,h\big(r_{i,}\tau\big)\right)$

$$
\begin{aligned}
&h\big(r_{i,}t_{n+1}\big)-h(r_i,0)\\
&\qquad=\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\gamma(r_{i,}\,\tau,\alpha,\beta)(t_{n+1}-\tau)^{\alpha-1}d\tau\\
&\qquad+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}(t_{n+1}\\
&\qquad-\tau)^{\alpha-1}\,F\left(r_{i,}\tau,h\big(r_{i,}\tau\big)\right)d\tau
\end{aligned}
\tag{4.72}
$$

At the integral $\big[t_{j,}t_{j+1}\big]$ we approximate $r\big(r_{i,}\tau,\alpha,\beta\big)$ using the Lagrange interpolation

The Lagrange interpolation formula is a method used to find polynomials and it was first published by Waring in 1779. The method was then re-introduced by Euler in 1783 and then was officially published in 1795 by Lagrange. The Lagrange interpolation polynomial is referred to as a polynomial that passes through all the points that are pre-defined and it is often used in the construction of Newton-Cotes formula in mathematics (Vučković, 2008).

The Lagrange interpolation has a consistent problem of a trade-off between having a better fit and having a smooth well-behaved fitting function. The higher degree of the resulting polynomial is achieved when using more data points and a greater oscillation in interpolation function existing between the data points. And this is why the number of data points must be optimal. The function between the points with greater error can then be able to be predicted by a high-degree Lagrange interpolation (Vučković, 2008).

Applying the Lagrange interpolation:

$$P_j(\tau) = \frac{\tau - t_{j-1}}{t_j - t_{j-1}} \gamma\left(r_{i,}\tau, \alpha, \beta\right) + \frac{\tau - t_j}{t_{j-1} - t_j} \gamma(r_{i,}t_{j-1,}\alpha, \beta) \approx r(r_{i,}\tau, \alpha, \beta) \tag{4.73}$$

$$\begin{aligned} h\left(r_{i,}t_{n+1}\right) - h(r_i, 0) \\ &= \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\frac{\tau - t_{j-1}}{\Delta t} \gamma\left(r_{i,}t_{j,}\alpha, \beta\right) \right. \\ &\left. - \frac{\tau - t_j}{\Delta t} \gamma(r_{i,}t_{j-1,}\alpha, \beta) \right) (t_{n+1} - \tau)^{\alpha-1}\, d\tau \\ &- \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \tau^{\beta-1}\, (t_{n+1} \\ &- \tau)^{\alpha-1} F\left(r_{i,}\tau, h(r, \tau)\right) d\tau \end{aligned} \tag{4.74}$$

$$h(r_{i,}t_{n+1}) - h(r_i,0) = \frac{\alpha\beta}{AB(\alpha)}\sum_{j=0}^{n}\Bigg[\frac{\Delta t^\alpha}{\Gamma(\alpha+2)}\gamma(r_{i.}t_{j,}\alpha,\beta)\{(n-j+1)^\alpha(n-j+2+\alpha)-(n-j)^\alpha(n-j+2+2\alpha\} - \frac{\Delta t^\alpha}{\Gamma(\alpha+2)}\gamma(r_{i,}t_{j-1,}\alpha,\beta)\{(n-j+1)^{\alpha+1}-(n-j)^\alpha(n-j+\alpha+1)\}\Bigg] \tag{4.75}$$

Replacing the $\gamma(r_{i,}\tau,\alpha,\beta)$ by $\tau^{\beta-1}F(r_{i,}\tau,h(r,\tau)$

$$h(r_{i,}t_{n+1}) - h(r_i,0) = \frac{\alpha\beta}{AB(\alpha)}\sum_{j=0}^{n}\Bigg[\frac{\Delta t^\alpha}{\Gamma(\alpha+2)}t_j^{\beta-1}F(r_{i,}t_{j,}h(r_{i,}t_j)\{(n-j+1)^\alpha(n-j+2+2\alpha)-(n-j)^\alpha(n-j+2+2\alpha)\} - \frac{\Delta t^\alpha}{AB(\alpha)}t_{j-1}^{\beta-1}F(r_{i,}t_{j-1,}\alpha,\beta)\{(n-j+1)^\alpha-(n-j)^\alpha(n-j+\alpha+1)\}\Bigg] \tag{4.76}$$

Thus, replacing the integral in the original equation, we get

$$
\begin{aligned}
&h\left(r_{i,}t_{n+1}\right)-h(r_i,0)\\
&=\frac{\alpha\beta}{AB(\alpha)}\sum_{j=0}^{n}\left[\frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)}t_j^{\beta-1}\frac{T}{S}\left(\frac{h\left(r_{i+1,}t_j\right)-2h\left(r_{i,}t_j\right)+h\left(r_{i-1,}t_j\right)}{\Delta r^2}\right.\right.\\
&\left.\left.+\frac{1}{r_i}\frac{h\left(r_{i+1,}t_j\right)-h\left(r_{i-1,}t_j\right.}{2\Delta r}\right)\right]\{(n-j+1)^{\alpha}(n-j+2+2\alpha)\\
&-(n-j)^{\alpha}(n-j+2+2\alpha)\}\\
&-\frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)}\frac{T}{S}\left[\frac{h\left(r_{i+1,}t_{j-1}\right)-2h\left(r_{i,}t_{j-1}\right)+h\left(r_{i-1,}t_{j-1}\right)}{\Delta r^2}\right.\\
&\left.+\frac{1}{r_i}\frac{h\left(r_{i+1,}t_{j-1}\right)-h\left(r_{i+1,}t_{j-1}\right)}{2\Delta r}\right]\{(n-j+1)^{\alpha+1}\\
&-(n-j)^{\alpha}(n-j+1+\alpha)\}\\
&+\frac{(1-\alpha)}{AB(\alpha)}\beta t_{n+1}^{\beta-1}\left[\frac{h\left(r_{i+1,}t_n\right)-2h\left(r_{i,}t_n\right)+h(r_{i-1,}t_n)}{\Delta r^2}\right.\\
&\left.+\frac{1}{r_i}\frac{h\left(r_{i+1,}t_n\right)-h(r_{i-1,}t_n)}{2\Delta r}\right]
\end{aligned}
\tag{4.77}
$$

CHAPTER 5: NUMERICAL STABILITY ANALYSIS

This chapter provides the numerical stability of the three operators' numerical solutions that have been generated in the previous chapter.

5.1. The Von Neumann stability analysis

The Von Neumann stability analysis was developed by Von Neumann at the Los Alamos National laboratory but was only introduced to the public when it was described briefly in an article by Crank and Nicolson in 1947. Since the beginning of the 20th, the stability analysis has been an issue. The Cauchy problem leads to the development of the Von Neumann stability analysis and now it is known as the classical method to determine stability condition including the problems with the periodic boundary conditions. This method is mostly used to determine necessary and enough stability conditions (Sousa, 2009).

The Von Neumann analysis is based on the Fourier decomposition of numerical solution if the periodic boundary conditions are assumed (Sousa, 2009):

$$U_j^n = \sum_{P=0}^{N-1} k_p^n \, e^{i\xi p(\Delta x)} \tag{5.1}$$

Where $i = \sqrt{-1}$, k_p^n is the amplification factor of the pth harmonic and $\xi p = \frac{p2\pi}{N\Delta x}$. The product of $\xi p \Delta x$ is usually called the phase angle: $\theta = \xi p \Delta x$ and covers the domain $[0,2\pi)$ in steps of $\frac{2\pi}{N}$. The regions around $\theta = 0$ and around $\theta = \pi$ are said to be associated with low frequencies and high frequencies respectively. The value $\theta = \pi$ corresponds to the highest frequency resolvable; the frequency of the wavelength $2\Delta x$. The time evolution of a single mode $k^n e^{ij\theta}$ is determined by the same numerical scheme as the complete numerical solution U_j^n. Therefore by inserting a representation of this form into a numerical scheme, a stability condition is obtained by imposing an upper bound to the amplification factor, *k* (Sousa E., 2009). The amplification factor satisfies the von Neumann condition if there is a constant *k* such that:

$$|k(\xi)| \leq 1 + k\Delta t, \qquad \forall \xi \epsilon \mathbb{R} \tag{5.2}$$

The presence of the arbitrary constant in equation (5.2) is regarded too generous for practical purposes, but also adequate for eventual convergence in the limit $\Delta t \to 0$. Therefore the inequality in (5.2) is replaced by the following stronger condition:

$$|k(\xi)| \leq 1, \ \ \forall \xi \epsilon \mathbb{R} \tag{5.3}$$

Or in terms of phase angle,

$$|k(\theta)| \leq 1, \quad \forall \theta \epsilon [0,2\pi) \tag{5.4}$$

This is referred to as practical stability or strict stability by many authors (Sousa E., 2009). In certain instances, condition (5.2) allows numerical modes to grow exponentially in time for finite values of Δt. Therefore condition (5.3) is recommended in order to prevent numerical modes from growing faster than the physical modes of the differential equation. The amplification factor satisfies the practical von Neumann condition if:

$$|k(\xi)| \leq 1, \ \ \forall \xi \epsilon \mathbb{R} \tag{5.5}$$

5.1.1. The numerical stability analysis for the numerical solution of the first operator

The first operator 3.12 is numerically approximated to find a solution for the groundwater flow and the solution is then checked for stability. The solution in 3.43 is then simplified as follows:

$$\begin{aligned} h_i^n(a_1) + h_i^{n-1}(a_2) - h_{i+1}^n(a_3) - h_{i-1}^n(a_4) \\ = \sum_{j=0}^{n} \frac{1}{\beta t_{n+1}^{\beta-1}} h_i^j \delta_{n,j}^{\alpha,\beta} + \sum_{j=0}^{n-1} \frac{1}{\beta t_n^{\beta-1}} h_i^j \tau_{n,j}^{\alpha,\beta} \end{aligned} \tag{5.6}$$

If $n = j$ then the sums can then be written as:

$$h_i^n(a_1) + h_i^{n-1}(a_2) - h_{i+1}^n(a_3) - h_{i-1}^n(a_4)$$
$$= h_i^n \frac{1}{\beta t_{n+1}^{\beta-1}} \delta_{n,n}^{\alpha,\beta} + h_i^{n-1} \frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}} + \sum_{j=0}^{n-2} h_n^j \frac{1}{\beta t_{n+1}^{\beta-1}} \delta_{n,j}^{\alpha,\beta} - h_i^{n-1} \frac{1}{\beta t_n^{\beta-1}} \tau_{n,n-1}^{\alpha,\beta} - \sum_{j=0}^{n-2} h_i^j \frac{\tau_{n,j}^{\alpha,\beta}}{\beta t_n^{\beta-1}} \tag{5.7}$$

After grouping like terms:

$$h_i^n \left(a_1 - \frac{1}{\beta t_{n+1}^{\beta-1}} \delta_{n,n}^{\alpha,\beta} \right) + h_i^{n-1} \left(a_2 - \frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}} + \frac{1}{\beta t_n^{\beta-1}} \tau_{n,n-1}^{\alpha,\beta} \right) - h_{i+1}^n(a_3) - h_{i-1}^n(a_4) = \sum_{j=0}^{n-2} h_n^j \frac{1}{\beta t_{n+1}^{\beta-1}} \delta_{n,j}^{\alpha,\beta} - \sum_{j=0}^{n-2} h_i^j \frac{\tau_{n,j}^{\alpha,\beta}}{\beta t_n^{\beta-1}} \tag{5.8}$$

If:

$$h_i^j = \delta_n[ik_m \Delta r] \tag{5.9}$$
$$h_i^n = \delta_n exp[ik_m \Delta r] \tag{5.10}$$
$$h_i^{n-1} = \delta_{n-1} exp[ik_m \Delta r] \tag{5.11}$$
$$h_{i+1}^n = \delta_n exp[ik_m(r + \Delta r] \tag{5.12}$$
$$h_{i-1}^n = \delta_n exp[ik_m(r - \Delta r] \tag{5.13}$$

Substitute the above equations:

$$\delta_n exp[ik_m \Delta r] \left(a_1 - \frac{1}{\beta t_{n+1}^{\beta-1}} \delta_{n,n}^{\alpha,\beta} \right) + \delta_{n-1} exp[ik_m \Delta r] \left(a_2 - \frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}} + \frac{1}{\beta t_n^{\beta-1}} \tau_{n,n-1}^{\alpha,\beta} \right) - \delta_n exp[ik_m(r + \Delta r](a_3) - \delta_n exp[ik_m(r - \Delta r](a_4) = \sum_{j=0}^{n-2} \delta_n exp[ik_m \Delta r] \frac{1}{\beta t_{n+1}^{\beta-1}} \delta_{n,j}^{\alpha,\beta} - \sum_{j=0}^{n-2} \delta_n exp[ik_m \Delta r] \frac{\tau_{n,j}^{\alpha,\beta}}{\beta t_n^{\beta-1}} \tag{5.14}$$

Apply the exponent rule: $e^{a+b} = e^{a}.e^{b}$

$$\begin{aligned}
&\delta_n exp[ik_m\Delta r]\left(a_1 - \frac{1}{\beta t_{n+1}^{\beta-1}}\delta_{n,n}^{\alpha,\beta}\right) \\
&\qquad + \delta_{n-1} exp[ik_m\Delta r]\left(a_2 - \frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}} + \frac{1}{\beta t_n^{\beta-1}}\tau_{n,n-1}^{\alpha,\beta}\right) \\
&\qquad - \delta_n \exp(ik_m r).(ik_m\Delta r) - \delta_n \exp(ik_m\, r).(-ik_m\,\Delta r.a_4) \\
&\qquad = \sum_{j=0}^{n-2} \delta_n exp[ik_m\Delta r]\frac{1}{\beta t_{n+1}^{\beta-1}}\delta_{n,j}^{\alpha,\beta} \\
&\qquad - \sum_{j=0}^{n-2} \delta_n exp[ik_m\Delta r]\frac{\tau_{n,j}^{\alpha,\beta}}{\beta t_n^{\beta-1}}
\end{aligned} \tag{5.15}$$

Then:

$$\begin{aligned}
&\delta_n\left(a_1 - \frac{1}{\beta t_{n+1}^{\beta-1}}\delta_{n,n}^{\alpha,\beta}\right) + \delta_{n-1}\left(a_2 - \frac{\delta_{n,n-1}^{\alpha,\beta}}{\beta t_{n+1}^{\beta-1}} + \frac{1}{\beta t_n^{\beta-1}}\tau_{n,n-1}^{\alpha,\beta}\right) \\
&\qquad - \delta_n exp[ik_m\Delta r]a_3 - \delta_n \exp -ik_m\,\Delta r.a_4 \\
&\qquad = \sum_{j=o}^{n-2}\frac{1}{\beta t_{n+1}^{\beta-1}}\delta_{n,j}^{\alpha,\beta} - \sum_{j=o}^{n-2}\frac{1}{\beta t_n^{\beta-1}}\tau_{n,j}^{\alpha,\beta}
\end{aligned} \tag{5.16}$$

If $n = 1$ then $\sum_{j=o}^{n-2}\frac{1}{\beta t_{n+1}^{\beta-1}}\delta_{n,j}^{\alpha,\beta} - \sum_{j=o}^{n-2}\frac{1}{\beta t_n^{\beta-1}}\tau_{n,j}^{\alpha,\beta} = 0$

$$\begin{aligned}
&\delta_1\left(a_1 - \frac{1}{\beta t_2^{\beta-1}}\delta_{1,1}^{\alpha,\beta}\right) + \delta_0\left(a_2 - \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{1}{\beta t_1^{\beta-1}}\tau_{1,0}^{\alpha,\beta}\right) \\
&\qquad - \delta_1 exp[ik_m\Delta r]a_3 - \delta_1 \exp(-ik_m\,\Delta r.a_4) = 0
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
&\delta_0\left(a_2 - \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{1}{\beta t_1^{\beta-1}}\tau_{1,0}^{\alpha,\beta}\right) \\
&\qquad = \delta_1\left(e^{ik_m\Delta r}a_3 + e^{-ik_m\Delta r}a_4 - a_1 - \frac{1}{\beta t_2^{\beta-1}}\delta_{1,1}^{\alpha,}\right)
\end{aligned} \tag{5.18}$$

$$\left|\frac{\delta_1}{\delta_0}\right| < 1 \Rightarrow \left|\frac{a_2 - \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{1}{\beta t_1^{\beta-1}}\tau_{1,0}^{\alpha,\beta}}{e^{ik_m\Delta r}a_3 + e^{-ik_m\Delta r}a_4 - a_1 - \frac{1}{\beta t_2^{\beta-1}}\delta_{1,1}^{\alpha,\beta}}\right| < 1 \tag{5.19}$$

Substituting $a_1, a_2, a_3\ and\ a_4$

$$\left|\frac{\left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} - \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_1^{\beta-1}}\right) - \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{1}{\beta t_1^{\beta-1}}\tau_{1,0}^{\alpha,\beta}}{e^{ik_m\Delta r}\frac{T}{S}\frac{1}{\Delta r(r_i)} + \frac{T}{S}\frac{1}{(\Delta r)^2} + e^{-ik_m\Delta r}\frac{T}{S}\frac{1}{\Delta r(r_i)} - \frac{T}{S}\frac{1}{(\Delta r)^2} - \left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}.\frac{\delta_{1,1}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{T}{S}\frac{2}{(\Delta r)^2} - \frac{1}{\beta t_2^{\beta-1}}\delta_{1,1}^{\alpha,\beta}\right)}\right| < 1 \tag{5.20}$$

Apply rule: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$

$$\left|\frac{\left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} - \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_1^{\beta-1}}\right) - \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{1}{\beta t_1^{\beta-1}}\tau_{1,0}^{\alpha,\beta}}{2i.\frac{T}{S}\frac{1}{\Delta r(r_i)} + \frac{T}{S}\frac{1}{(\Delta r)^2}\cos(k_m\Delta r) - \left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}.\frac{\delta_{1,1}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{T}{S}\frac{2}{(\Delta r)^2} - \frac{1}{\beta t_2^{\beta-1}}\delta_{1,1}^{\alpha,\beta}\right)}\right| < 1 \tag{5.21}$$

We assume for n that $|\delta_n| < |\delta_0|$, equation (5.21) as $\left|\frac{A}{B}\right|$and we want to prove that $\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1$

$$\left|\frac{\delta_{n+1}}{\delta_n}\right| < \left|\frac{B}{A}\right| + \sum_{j=0}^{n-2}\delta_i^j\,\phi_{n,j}^{\alpha,\beta} - \sum_{j=0}^{n-2}\delta_i^j\,\Phi_{n,j}^{\alpha,\beta} \tag{5.22}$$

$$\delta_{n+1} < \frac{B}{A}\delta_n + \sum_{j=0}^{n-2}\delta_i^j\,\phi_{n,j}^{\alpha,\beta} - \sum_{j=0}^{n-2}\delta_i^j\,\Phi_{n,j}^{\alpha,\beta} \tag{5.23}$$

Apply rule: $|a + b| = |a| + |b|$

$$|\delta_{n+1}| < \left|\frac{B}{A}\right||\delta_n| + \sum_{j=0}^{n-2}\left|\delta_i^j\right|\left|\phi_{n,j}^{\alpha,\beta}\right| - \sum_{j=0}^{n-2}\left|\delta_i^j\right|\left|\Phi_{n,j}^{\alpha,\beta}\right| \tag{5.24}$$

Because $|\delta_n| < |\delta_0|$, therefore:

$$|\delta_{n+1}| < \left|\frac{A}{B}\right| |\delta_0| + \sum_{j=0}^{n-2} |\delta_0| \left|\phi_{n,j}^{\alpha,\beta}\right| + \sum_{j=0}^{n-2} |\delta_0| \left|\Phi_{n,j}^{\alpha,\beta}\right| \tag{5.25}$$

$$\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1 \Rightarrow \frac{|A|}{|B|} + \sum_{j=0}^{n-2} \left|\phi_{n,j}^{\alpha,\beta}\right| + \sum_{j=0}^{n-2} \left|\Phi_{n,j}^{\alpha,\beta}\right| < 1 \tag{5.26}$$

Then the condition for stability is:

$$\begin{gathered} \min_{n \geq 0} \left[\left(\left|\frac{\left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} - \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_1^{\beta-1}}\right) - \frac{\delta_{1,0}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{1}{\beta t_1^{\beta-1}} \tau_{1,0}^{\alpha,\beta}}{2i.\frac{T}{S} \frac{1}{\Delta r(r_i)} + \frac{T}{S} \frac{1}{(\Delta r)^2} \cos(k_m \Delta r) - \left(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} . \frac{\delta_{1,1}^{\alpha,\beta}}{\beta t_2^{\beta-1}} + \frac{T}{S} \frac{2}{(\Delta r)^2} - \frac{1}{\beta t_2^{\beta-1}} \delta_{1,1}^{\alpha,\beta}\right)}\right|, \frac{|A|}{|B|}\right.\right. \\ \left.\left. + \sum_{j=0}^{n-2} \left|\phi_{n,j}^{\alpha,\beta}\right| - \sum_{j=0}^{n-2} \left|\Phi_{n,j}^{\alpha,\beta}\right| \right) < 1\right] \end{gathered} \tag{5.27}$$

5.1.2. The numerical stability analysis for the numerical solution of the second operator

This section represents the numerical stability of the numerical solution of the second operator chosen for this study.

We recall our numerical solution in eq. 3.44

$$\frac{M(\alpha)}{\beta t_{n+1}^{\beta-1}(1-\alpha)}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]h(r,0)\right.$$
$$+\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\frac{\alpha}{1-\alpha}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j+1)\Delta t\right]\right. \tag{5.28}$$
$$\left.\left.-\,exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]\right]\right]$$
$$=\frac{T}{S}\left[\frac{h_{i+1}^{n+1}-2h_i^{n+1}+h_{i-1}^{n+1}}{(\Delta r)^2}+\frac{1}{r_i}\frac{h_{i+1}^{n+1}-h_{i-1}^{n+1}}{\Delta r}\right]$$

Further simplification:

$$\frac{M(\alpha)}{\beta t_{n+1}^{\beta-1}(1-\alpha)}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]h(r,0)\right.$$
$$+\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\frac{\alpha}{1-\alpha}\left[exp\left[-\frac{\alpha}{1-\alpha}(n-j+1)\Delta t\right]\right.$$
$$\left.\left.-\,exp\left[-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right]\right]\right] \tag{5.29}$$
$$=h_{i+1}^{n+1}\left(\frac{T}{S(\Delta r)^2}+\frac{T}{Sr_i\Delta r}\right)+-h_i^{n+1}\left(\frac{2T}{S(\Delta r)^2}\right)$$
$$+\,h_{i-1}^{n+1}\left(\frac{T}{S((\Delta r)^2)}-\frac{T}{S(\Delta r)}\right)$$

To simplify the equation

$$G^{\alpha}(r_i,0)+\sum_{j=0}^{n}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\,\delta_{n,j}^{\alpha}$$
$$=h_{i+1}^{n+1}\left(\frac{T}{S(\Delta r)^2}+\frac{T}{Sr_i\Delta r}\right)+-h_i^{n+1}\left(\frac{2T}{S(\Delta r)^2}\right) \tag{5.30}$$
$$+h_{i-1}^{n+1}\left(\frac{T}{S((\Delta r)^2)}-\frac{T}{S(\Delta r)}\right)$$

$$\frac{h_i^{n+1}-h_i^{n}}{\Delta t}\,\delta_{n,n}^{\alpha}$$
$$+\sum_{j=0}^{n-1}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\,\delta_{n,j}^{\alpha} \tag{5.31}$$
$$=h_{i+1}^{n+1}\left(\frac{T}{S(\Delta r)^2}+\frac{T}{Sr_i\Delta r}\right)-h_i^{n+1}\left(\frac{2T}{S(\Delta r)^2}\right)$$
$$+h_{i-1}^{n+1}\left(\frac{T}{S((\Delta r)^2)}-\frac{T}{S(\Delta r)}\right)$$

$$h_i^{n+1}\left(\frac{\delta_{n,n}^{\alpha}}{\Delta t}\right)-h_i^{n}\left(\frac{\delta_{n,n}^{\alpha}}{\Delta t}\right)$$
$$+\sum_{j=0}^{n-2}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\,\delta_{n,j}^{\alpha} \tag{5.32}$$
$$=h_{i+1}^{n+1}\left(\frac{T}{S(\Delta r)^2}+\frac{T}{Sr_i\Delta r}\right)-h_i^{n+1}\left(\frac{2T}{S(\Delta r)^2}\right)$$
$$+h_{i-1}^{n+1}\left(\frac{T}{S((\Delta r)^2)}-\frac{T}{S(\Delta r)}\right)$$

After grouping like terms

$$h_i^{n+1}\left(\frac{\delta_{n,n}^{\alpha}}{\Delta t}+\frac{2T}{S(\Delta r)^2}\right)-h_i^{n}\left(\frac{\delta_{n,n}^{\alpha}}{\Delta t}\right)-h_{i+1}^{n+1}\left(\frac{T}{S(\Delta r)^2}+\frac{T}{Sr_i\Delta r}\right) \tag{5.33}$$
$$-h_{i-1}^{n+1}\left(\frac{T}{S((\Delta r)^2)}-\frac{T}{S(\Delta r)}\right)+\sum_{j=0}^{n-2}\frac{h_i^{j+1}-h_i^{j}}{\Delta t}\,\delta_{n,j}^{\alpha}=0$$

Using the Von Neumann analysis to check for stability

$$h_i^{n+1}(a_1) - h_i^n(a_2) - h_{i+1}^{n+1}(a_3) - h_{i-1}^{n+1}(a_4) + \sum_{j=0}^{n-2} \frac{h_i^{j+1} - h_i^j}{\Delta t} \delta_{n,j}^{\alpha} = 0 \tag{5.34}$$

Representation

$$h_i^n = \delta_n exp[ik_m \Delta r] \tag{5.35}$$

$$h_i^{n+1} = \delta_{n+1} exp[ik_m \Delta r] \tag{5.36}$$

$$h_{i+1}^{n+1} = \delta_{n+1} exp[ik_m (r + \Delta r] \tag{5.37}$$

$$h_{i-1}^{n+1} = \delta_{n+1} exp[ik_m (r - \Delta r] \tag{5.38}$$

Substitute the above equations

$$\delta_{n+1} exp[ik_m \Delta r] a_1 - \delta_n exp[ik_m \Delta r] a_2 - \delta_{n+1} exp[ik_m (r + \Delta r] a_3 - \delta_{n+1} exp[ik_m (r - \Delta r] a_4 + \sum_{j=0}^{n-2} \frac{h_i^{j+1} - h_i^j}{\Delta t} \delta_{n,j}^{\alpha} = 0 \tag{5.39}$$

Apply the exponent rule:$e^{a+b} = e^a . e^b$

$$\delta_{n+1} exp[ik_m \Delta r] a_1 - \delta_n exp[ik_m \Delta r] a_2 - \delta_{n+1} exp(ik_m r).(ik_m \Delta r) a_3 - \delta_{n+1} exp(ik_m r)(-ik_m \Delta r) a_4 + \sum_{j=0}^{n-2} \frac{h_i^{j+1} - h_i^j}{\Delta t} \delta_{n,j}^{\alpha} = 0 \tag{5.40}$$

Then simplify by cancelling out:

$$\delta_{n+1} a_1 - \delta_n a_2 - \delta_{n+1} (ik_m \Delta r) a_3 - \delta_{n+1} exp(-ik_m \Delta r) a_4 + \sum_{j=0}^{n-2} \frac{h_i^{j+1} - h_i^j}{\Delta t} \delta_{n,j}^{\alpha} = 0 \tag{5.41}$$

If $n = 1$ then

$$\sum_{j=0}^{n-2} \frac{h_i^{j+1} - h_i^j}{\Delta t} \delta_{n,j} = 0 \tag{5.42}$$

Therefore:

$$\delta_2 a_1 - \delta_1 a_2 - \delta_2 exp(ik_m\Delta r)a_3 - \delta_2 exp(-ik_m\Delta r)a_4 = 0 \tag{5.43}$$

$$\delta_2\left(a_1 - e^{ik_m\Delta r}a_3 - e^{-ik_m\Delta r}a_4\right) = \delta_1 a_2 \tag{5.44}$$

$$\left|\frac{\delta_2}{\delta_1}\right| < 1 \Rightarrow \left|\frac{a_2}{a_1 - e^{ik_m\Delta r}a_3 - e^{-ik_m\Delta r}a_4}\right| < 1 \tag{5.45}$$

Substituting $a_1, a_2, a_3\ and\ a_4$

Therefore:

$$\left|\frac{\frac{\delta_{n,n}^{\alpha}}{\Delta t}}{\frac{\delta_{n,n}^{\alpha}}{\Delta t} + \frac{2T}{S(\Delta r)^2} - e^{ik_m\Delta r}\frac{T}{S(\Delta r)^2} - \frac{T}{Sr_i\Delta r} - e^{-ik_m\Delta r}\frac{T}{S(\Delta r)^2} - \frac{T}{S(\Delta r)}}\right| < 1 \tag{5.46}$$

Apply rule: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$$\left|\frac{\frac{\delta_{n,n}^{\alpha}}{\Delta t}}{\frac{\delta_{n,n}^{\alpha}}{\Delta t} + \frac{2T}{S(\Delta r)^2} - \frac{T}{S(\Delta r)^2} + \frac{2T}{Sr_i\Delta r}\cos(k_m\Delta r)}\right| < 1 \tag{5.47}$$

$$\left|\frac{\frac{\delta_{n,n}^{\alpha}}{\Delta t}}{\frac{\delta_{n,n}^{\alpha}}{\Delta t} + \frac{T}{S(\Delta r)^2} - \frac{2T}{Sr_i\Delta r}\cos(k_m\Delta r)}\right| < 1 \tag{5.48}$$

If $\frac{\delta_{n,n}^{\alpha}}{\Delta t} + \frac{T}{S(\Delta r)^2} - \frac{2T}{Sr_i\Delta r}\cos(k_m\Delta r) > 0$ Then:

$$\frac{\delta_{n,n}^{\alpha}}{\Delta t} < \frac{\delta_{n,n}^{\alpha}}{\Delta t} + \frac{T}{S(\Delta r)^2} - \frac{2T}{Sr_i\Delta r}\cos(k_m\Delta r) \tag{5.49}$$

$$\frac{2T}{Sr_i\Delta r}\cos(k_m\Delta r) < \frac{T}{S(\Delta r)^2} \tag{5.50}$$

Since $-1 \leq \cos(k_m\Delta r) \leq 1$

Therefore

$$\frac{2T}{Sr_i\Delta r} < \frac{T}{S(\Delta r)^2} \tag{5.51}$$

After simplifying then:

$$\frac{2}{r_i} < \frac{1}{\Delta r} \tag{5.52}$$

Then:

$$\frac{\Delta r}{r_i} < \frac{1}{2} \tag{5.53}$$

5.1.3. The numerical stability analysis for the numerical solution of the Third operator

This section represents the numerical stability of the numerical solution of the third operator chosen for this study.

We recall our numerical solution in eq. 3.55

$$
\begin{aligned}
&h\left(r_{i,}t_{n+1}\right) - h(r_i, 0) \\
&= \frac{\alpha\beta}{AB(\alpha)} \sum_{j=0}^{n} \left[\frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} t_j^{\beta-1} \frac{T}{S} \left(\frac{h\left(r_{i+1,}t_j\right) - 2h\left(r_{i,}t_j\right) + h\left(r_{i-1,}t_j\right)}{\Delta r^2}\right.\right. \\
&\left.\left. + \frac{1}{r_i} \frac{h\left(r_{i+1,}t_j\right) - h\left(r_{i-1,}t_j\right)}{2\Delta r}\right)\right] \{(n-j+1)^{\alpha}(n-j+2+2\alpha) \\
&- (n-j)^{\alpha}(n-j+2+2\alpha)\} \\
&- \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \frac{T}{S} \left[\frac{h\left(r_{i+1,}t_{j-1}\right) - 2h\left(r_{i,}t_{j-1}\right) + h\left(r_{i-1,}t_{j-1}\right)}{\Delta r^2}\right. \\
&\left. + \frac{1}{r_i} \frac{h\left(r_{i+1,}t_{j-1}\right) - h\left(r_{i+1,}t_{j-1}\right)}{2\Delta r}\right] \{(n-j+1)^{\alpha+1} \\
&- (n-j)^{\alpha}(n-j+1+\alpha)\} \\
&+ \frac{(1-\alpha)}{AB(\alpha)} \beta t_{n+1}^{\beta-1} \left[\frac{h\left(r_{i+1,}t_n\right) - 2h\left(r_{i,}t_n\right) + h\left(r_{i-1,}t_n\right)}{\Delta r^2}\right. \\
&\left. + \frac{1}{r_i} \frac{h\left(r_{i+1,}t_n\right) - h\left(r_{i-1,}t_n\right)}{2\Delta r}\right]
\end{aligned} \tag{5.54}
$$

We then replace $h(r_i, t_{n+1}) = h_i^{n+1}$and $h(r_i, 0) = h_i^0$for simplicity and then we represent the following by:

$$
a_1^{j,n} = \frac{\alpha\beta}{AB(\alpha)} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} t_j^{\beta-1} \frac{T}{S\Delta r^2} \{(n-j+1)^{\alpha}(n-j+2+2\alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\} \tag{5.55}
$$

$$
a_2^{j,n} = \frac{\alpha\beta}{AB(\alpha)} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} t_j^{\beta-1} \frac{T}{2Sr_i\Delta r} \{(n-j+1)^{\alpha}(n-j+2+2\alpha) - (n-j)^{\alpha}(n-j+2+2\alpha)\} \tag{5.56}
$$

$$
a_3^{j,n} = \frac{\alpha\beta}{AB(\alpha)} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \frac{T}{S\Delta r^2} \{(n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\} \tag{5.57}
$$

$$
a_4^{j,n} = \frac{\alpha\beta}{AB(\alpha)} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \frac{T}{2Sr_i\Delta r} \{(n-j+1)^{\alpha+1} - (n-j)^{\alpha}(n-j+1+\alpha)\} \tag{5.58}
$$

$$
a_5^{j,n} = \frac{(1-\alpha)}{AB(\alpha)} \beta t_{n+1}^{\beta-1} \frac{1}{\Delta r^2} \tag{5.59}
$$

$$a_6^{j,n} = \frac{(1-\alpha)}{AB(\alpha)} \beta t_{n+1}^{\beta-1} \frac{1}{r_i \Delta r} \tag{5.60}$$

Then substitute the simplifications into the equation:

$$\begin{aligned} h_i^{n+1} - h_i^0 = \sum_{j=0}^{n} & a_1^{j,n}\left(h_{i+1}^j - 2h_i^j + h_{i-1}^j\right) + a_2^{j,n}\left(h_{i+1}^j - h_{i-1}^j\right) \\ & - a_3^{j,n}\left(h_{i+1}^{j-1} - 2h_{i-1}^{j-1} + h_{i-1}^{j-1}\right) + a_4^{j,n}\left(h_{i+1}^{j-1} - h_{i+1}^{j-1}\right) \\ & + a_5^{j,n}\left(h_{i+1}^n - 2h_i^n + h_{i-1}^n\right) + a_6^{j,n}\left(h_{i+1}^n - h_{i-1}^n\right) \end{aligned} \tag{5.61}$$

Further simplification

$$h_i^j = \delta_j exp[ik_m r] \tag{5.62}$$

$$h_{i+1}^j = \delta_j exp[ik_m(\mathrm{r} + \Delta r)] \tag{5.63}$$

$$h_{i-1}^j = \delta_j exp[ik_m(\mathrm{r} - \Delta r)] \tag{5.64}$$

$$h_{i+1}^{j+1} = \delta_{j+1} exp[ik_m(\mathrm{r} + \Delta r)] \tag{5.65}$$

$$h_{i-1}^{j-1} = \delta_{j-1} exp[ik_m(\mathrm{r} - \Delta r)] \tag{5.66}$$

$$h_{i+1}^{j-1} = \delta_{j-1} exp[ik_m(\mathrm{r} + \Delta r)] \tag{5.67}$$

$$h_i^{j-1} = \delta_j exp[ik_m(\mathrm{r} - \Delta r)] \tag{5.68}$$

$$h_i^n = \delta_n exp[ik_m r] \tag{5.69}$$

$$h_i^0 = \delta_0 exp[ik_m r] \tag{5.70}$$

$$h_i^{n+1} = \delta_{n+1} exp[ik_m r] \tag{5.71}$$

$$h_{i+1}^n = \delta_n exp[ik_m(r + \Delta r] \tag{5.72}$$

$$h_{i-1}^n = \delta_n exp[ik_m(r - \Delta r] \tag{5.73}$$

Then substituting everything into the equation:

$$
\begin{aligned}
&\delta_{n+1} exp[ik_m r] - \delta_0 exp[ik_m r] \\
&\qquad = \sum_{j=0}^{n} a_1^{j,n} \big(\delta_j exp[ik_m(\mathrm{r} + \Delta r)] - 2\big(\delta_j exp[ik_m r]\big) \\
&\qquad + \delta_j exp[ik_m(\mathrm{r} - \Delta r)]\big) \\
&\qquad + a_2^{j,n} \big(\delta_j exp[ik_m(\mathrm{r} + \Delta r)] - \delta_j exp[ik_m(\mathrm{r} - \Delta r)]\big) \\
&\qquad - a_3^{j,n} \big(\delta_{j-1} exp[ik_m(\mathrm{r} + \Delta r)] - 2\big(\delta_{j-1} exp[ik_m(\mathrm{r} - \Delta r)]\big) \qquad (5.74) \\
&\qquad + \delta_{j-1} exp[ik_m(\mathrm{r} - \Delta r)]\big) \\
&\qquad + a_4^{j,n} \big(\delta_{j-1} exp[ik_m(\mathrm{r} + \Delta r)] - \delta_{j-1} exp[ik_m(\mathrm{r} + \Delta r)]\big) \\
&\qquad + a_5^{j,n} (\delta_n exp[ik_m(r + \Delta r] - 2(\delta_n exp[ik_m r]) \\
&\qquad + \delta_n exp[ik_m(r - \Delta r]) \\
&\qquad + a_6^{j,n} (\delta_n exp[ik_m(r + \Delta r] - \delta_n exp[ik_m(r - \Delta r])
\end{aligned}
$$

Apply the exponent rule: $e^{a+b} = e^a . e^b$

$$
\begin{aligned}
&\delta_{n+1} exp[ik_m r] - \delta_0 exp[ik_m r] \\
&\quad = \sum_{j=0}^{n} a_1^{j,n} \Big(\delta_j exp(ik_m r). exp(ik_m \Delta r) - 2\big(\delta_j exp[ik_m r]\big) \\
&\quad + \delta_j exp(ik_m r). exp(-ik_m \Delta r) \Big) \\
&\quad + a_2^{j,n} \Big(\delta_j exp(ik_m r). exp(ik_m \Delta r) \\
&\quad - \delta_j exp(ik_m r). exp(-ik_m \Delta r) \Big) \\
&\quad - a_3^{j,n} \Big(\delta_{j-1} exp(ik_m r). exp(ik_m \Delta r) \\
&\quad - 2 \Big(\delta_{j-1} exp(ik_m r). exp(-ik_m \Delta r) \Big) \\
&\quad + \delta_{j-1} exp(ik_m r). exp(-ik_m \Delta r) \Big) \\
&\quad + a_4^{j,n} \Big(\delta_{j-1} exp(ik_m r). exp(ik_m \Delta r) \\
&\quad - \delta_{j-1} exp(ik_m r). exp(ik_m \Delta r) \Big) \\
&\quad + a_5^{j,n} \big(\delta_n exp(ik_m r). exp(ik_m \Delta r) - 2(\delta_n exp[ik_m r]) \\
&\quad + \delta_n exp(ik_m r). exp(-ik_m \Delta r) \big) \\
&\quad + a_6^{j,n} \big(\delta_n exp(ik_m r). exp(ik_m \Delta r) \\
&\quad - \delta_n exp(ik_m r). exp(-ik_m \Delta r) \big)
\end{aligned}
\tag{5.75}
$$

After simplifying by cancelling out the equation becomes:

$$
\begin{aligned}
\delta_{n+1} - \delta_0 = \sum_{j=0}^{n} & a_1^{j,n} \Big(\delta_j exp(ik_m \Delta r) - 2\big(\delta_j\big) + \delta_j exp(-ik_m \Delta r) \Big) \\
& + a_2^{j,n} \Big(\delta_j exp(ik_m \Delta r) - \delta_j exp(-ik_m \Delta r) \Big) \\
& - a_3^{j,n} \Big(\delta_{j-1} exp(ik_m \Delta r) - 2 \Big(\delta_{j-1} exp(-ik_m \Delta r) \Big) \\
& + \delta_{j-1} exp(-ik_m \Delta r) \Big) \\
& + a_4^{j,n} \Big(\delta_{j-1} exp(ik_m \Delta r) - \delta_{j-1} exp(ik_m \Delta r) \Big) \\
& + a_5^{j,n} \big(\delta_n exp(ik_m \Delta r) - 2(\delta_n) + \delta_n exp(-ik_m \Delta r) \big) \\
& + a_6^{j,n} \big(\delta_n exp(ik_m \Delta r) - \delta_n exp(-ik_m \Delta r) \big)
\end{aligned}
\tag{5.76}
$$

After factorizing, the equation becomes:

$$\begin{aligned}\delta_{n+1}-\delta_0 = \sum_{j=0}^{n}\Big(& a_1^{j,n}\delta_j\big(exp(ik_m\Delta r)-2+exp(-ik_m\Delta r)\big)\\ &+a_2^{j,n}\delta_j\big(exp(ik_m\Delta r)-exp(-ik_m\Delta r)\big)\\ &-a_3^{j,n}\delta_{j-1}\Big(exp(ik_m\Delta r)-2\big(exp(-ik_m\Delta r)\big)\\ &+exp(-ik_m\Delta r)\Big)\\ &+a_4^{j,n}\delta_{j-1}\big(exp(ik_m\Delta r)-exp(ik_m\Delta r)\big)\Big)\\ &+a_5^{j,n}\delta_n\big(exp(ik_m\Delta r)-2+exp(-ik_m\Delta r)\big)\\ &+a_6^{j,n}\delta_n\big(exp(ik_m\Delta r)-exp(-ik_m\Delta r)\big)\end{aligned} \tag{5.77}$$

Applying rule $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$

$$\begin{aligned}\delta_{n+1}-\delta_0 = \sum_{j=0}^{n}\Bigg(& a_1^{j,n}\delta_j\left(-4isin^2\left(\frac{k_m\Delta r}{2}\right)\right)+a_2^{j,n}\delta_j\big(2isin(k_m\Delta r)\big)\\ &-a_3^{j,n}\delta_{j-1}\left(-4isin^2\left(\frac{k_m\Delta r}{2}\right)\right)+a_4^{j,n}\delta_{j-1}\big(2isin(k_m\Delta r)\big)\Bigg)\\ &+a_5^{j,n}\delta_n\left(-4isin^2\left(\frac{k_m\Delta r}{2}\right)\right)+a_6^{j,n}\delta_n\big(2isin(k_m\Delta r)\big)\end{aligned} \tag{5.78}$$

The next step is to find in general the condition under which $\forall, j\in[0,\mathbb{N}]|\delta_n|<|\delta_0|$

When $n=1$, the following condition exists:

$$\begin{aligned}\delta_1-\delta_0 = {}& a_5^{j,0}\delta_0\left(-4sin^2\left(\frac{k_m\Delta r}{2}\right)\right)+2ia_6^{j,0}\delta_0\big(sin(k_m\Delta r)\big)\\ &-a_1^{j,0}\delta_0\left(-4sin^2\left(\frac{k_m\Delta r}{2}\right)\right)+2ia_2^{j,0}\delta_0\big(sin(k_m\Delta r)\big)\end{aligned} \tag{5.79}$$

Then taking δ_0 to the other side and factorizing:

$$\delta_1 = \delta_0\left(1 - 4a_5^{j,0} sin^2\left(\frac{k_m \Delta r}{2}\right)\right) + 2ia_6^{j,0}\left(sin(k_m \Delta r)\right)$$
$$- 4a_1^{j,0} sin^2\left(\frac{k_m \Delta r}{2}\right) + 2ia_2^{j,0}\left(sin(k_m \Delta r)\right) \tag{5.80}$$

Then dividing both sidesδ_0:

$$\left|\frac{\delta_1}{\delta_0}\right| = \left|1 - 4sin^2\left(\frac{k_m \Delta r}{2}\right)\left(a_5^{j,0} - a_1^{j,0}\right) + 2i\left(sin(k_m \Delta r)\right)\left(a_6^{j,0} + a_2^{j,0}\right)\right| \tag{5.81}$$

By representing A= $1 - 4sin^2\left(\frac{k_m \Delta r}{2}\right)\left(a_5^{j,0} - a_1^{j,0}\right)$and B=$2\left(sin(k_m \Delta r)\right)\left(a_6^{j,0} + a_2^{j,0}\right)$, then substitute in the equation:

$$\left|\frac{\delta_1}{\delta_0}\right| = |A + iB| = \sqrt{A^2 + B^2} \tag{5.82}$$

The condition is as follows:

$$\left|\frac{\delta_1}{\delta_0}\right| < 1 \quad \rightarrow \sqrt{A^2 + B^2} < 1 \tag{5.83}$$

Therefore, the numerical scheme is stable if

$$\sqrt{\left(1 - 4sin^2\left(\frac{k_m \Delta r}{2}\right)\left(a_5^{j,0} - a_1^{j,0}\right)\right)^2 + \left(2\left(sin(k_m \Delta r)\right)\left(a_6^{j,0} + a_2^{j,0}\right)\right)^2} < 1 \tag{5.84}$$

Now we assume that $\forall n \geq 1$. $\left|\frac{\delta_n}{\delta_0}\right| < 1$ then we prove that $\left|\frac{\delta_{n+1}}{\delta_0}\right|$

However:

$$\delta_{n+1} = \delta_0 + \sum_{j=0}^{n} \left\{ a_1^{j,n} \delta_j \left(-4sin^2 \left(\frac{k_m \Delta r}{2} \right) \right) + a_2^{j,n} \delta_j \big(2isin(k_m \Delta r) \big) \right.$$
$$\left. - a_3^{j,n} \delta_{j-1} \left(-4isin^2 \left(\frac{k_m \Delta r}{2} \right) \right) + a_4^{j,n} \delta_{j-1} \big(2isin(k_m \Delta r) \big) \right\} \tag{5.85}$$
$$+ a_5^{j,n} \delta_n \left(-4isin^2 \left(\frac{k_m \Delta r}{2} \right) \right) + a_5^{j,n} \delta_n \big(2isin(k_m \Delta r) \big)$$

And then

$$|\delta_{n+1}| \le |\delta_0| + \left\{ \sum_{j=0}^{n} |\delta_j| \left(4sin^2 \left(\frac{k_m \Delta r}{2} \right) a_1^{j,n} \delta_j + a_2^{j,n} \delta_j \big(2isin(k_m \Delta r) \big) \right) \right\}$$
$$+ \left\{ \sum_{j=0}^{n} |\delta_{j-1}| \left(4sin^2 \left(\frac{k_m \Delta r}{2} \right) a_3^{j,n} \delta_j \right. \right. \tag{5.86}$$
$$\left. \left. + a_4^{j,n} \delta_j \big(2isin(k_m \Delta r) \big) \right) \right\}$$
$$+ \left\{ \sum_{j=0}^{n} |\delta_n| \left(4sin^2 \left(\frac{k_m \Delta r}{2} \right) a_5^{j,n} \delta_j + a_6^{j,n} \delta_j \big(2isin(k_m \Delta r) \big) \right) \right\}$$

By indication, we know that $\forall n \ge 1$ and $|\delta_n| \le |\delta_0|$ therefore

$$|\delta_{n+1}| \le |\delta_0| \left(1 \right.$$
$$+ \sum_{j=0}^{n} \left\{ 4sin^2 \left(\frac{k_m \Delta r}{2} \right) a_1^{j,n} + a_2^{j,n} \big(2isin(k_m \Delta r) \big) \right. \tag{5.87}$$
$$\left. + 4sin^2 \left(\frac{k_m \Delta r}{2} \right) a_3^{j,n} + \big(2isin(k_m \Delta r) \big) a_4^{j,n} \right\}$$
$$\left. + 4a_5^{j,n} sin^2 \left(\frac{k_m \Delta r}{2} \right) + 2isin(k_m \Delta r) \right)$$

Dividing both sides by δ_0

$$\frac{\delta_{n+1}}{\delta_0} \leq \left(1 + \sum_{j=0}^{n} \left\{4sin^2\left(\frac{k_m \Delta r}{2}\right) a_1^{j,n} + a_2^{j,n}\left(2isin(k_m \Delta r)\right) + 4sin^2\left(\frac{k_m \Delta r}{2}\right) a_3^{j,n} + \left(2isin(k_m \Delta r)\right) a_4^{j,n}\right\} + 4a_5^{j,n} sin^2\left(\frac{k_m \Delta r}{2}\right) + 2isin(k_m \Delta r) a_6^{j,n}\right) \tag{5.88}$$

The condition when $n = 1$

$$\left|\frac{\delta_{n+1}}{\delta_0}\right| \leq \left| 1 - 4sin^2\left(\frac{k_m \Delta r}{2}\right)\left(a_1^{j,1} + a_3^{j,1} a_5^{j,1}\right) + 2isin(k_m \Delta r)\left(a_2^{j,1} + a_4^{j,1} a_6^{j,1}\right)\right| \tag{5.89}$$

For further simplification, the following representations are made:

$$C = 4sin^2\left(\frac{k_m \Delta r}{2}\right)\left(a_1^{j,1} + a_3^{j,1} a_5^{j,1}\right) \; and \quad D = 2isin(k_m \Delta r)\left(a_2^{j,1} + a_4^{j,1} a_6^{j,1}\right) \tag{5.90}$$

Therefore, the condition is

$$\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1 \quad \rightarrow \sqrt{C^2 + D^2} < 1 \tag{5.91}$$

Represented by

$$\sqrt{\left(4sin^2\left(\frac{k_m \Delta r}{2}\right)\left(a_1^{j,1} + a_3^{j,1} a_5^{j,1}\right)\right)^2 + \left(2sin(k_m \Delta r)\left(a_2^{j,1} + a_4^{j,1} a_6^{j,1}\right)\right)^2} < 1 \tag{5.92}$$

Therefore, the numerical scheme is stable if

$$min\left\{\begin{array}{l}\sqrt{\left(1-4sin^2\left(\frac{k_m\Delta r}{2}\right)\left(a_5^{j,0}-a_1^{j,0}\right)\right)^2+\left(2\left(sin(k_m\Delta r)\right)\left(a_6^{j,0}+a_2^{j,0}\right)\right)^2} \\ ,\sqrt{\left(4sin^2\left(\frac{k_m\Delta r}{2}\right)\left(a_1^{j,1}+a_3^{j,1}a_5^{j,1}\right)\right)^2+\left(2sin(k_m\Delta r)\left(a_2^{j,1}+a_4^{j,1}a_6^{j,1}\right)\right)^2}\end{array}\right\} \quad (5.93)$$

$$< 1$$

CHAPTER 6: NUMERICAL SIMULATIONS

6.1. Introduction

In this chapter, numerical simulations of the new numerical solution are represented to model the groundwater flow in a confined aquifer with dual layers. For the purpose of this task, MATLAB is used, and the simulations are portrayed from figure 5 to 36.

6.2. Results and Discussions

In the last passed decades, researchers relied on the classical differentiation and integration to depict groundwater flow within a confined aquifer. The model that was introduced by Theis was used in many situations to determine aquifer's parameters including storativity and transmissivity, however, when matching the collected data with a mathematical formula, they always observed disagreement between the mathematical formula and the collected data. A scientific question to be asked at this stage is: Does the collected data depict real world situation and is free from uncertainties? Or does the used mathematical model really replicate the situation? Now if the answer happens to be: Yes, the collected data represents the real-world problem, then one will, therefore, question the ability and the accuracy of the model.

To solve this problem researchers introduced the concept of differential operators with power law process, exponential decay law process and finally the generalized Mittag-Leffler kernel. While these new concepts have been used very successfully to capture the groundwater flow with power law, exponential decay law and the crossover processes, it was noticed that there was some flow behaviour that could not be really replicated using these three concepts. Therefore, a new concept that combines self-similarities and (power law, exponential decay law and the crossover process) were introduced, which was used in this thesis.

Mathematical software was used to produce and represent numerical simulations of groundwater flow in a confined aquifer with dual layers using modified mathematical approaches of fractal-fractional operators. Numerical simulations are depicted in figure 5 to 36 for different values of fractional orders and fractal dimensions. The three concepts were used in this simulation including the model with fractal-fractional with power law, the model with fractal-fractional with exponential decay law and finally the model with crossover effect. Three majors flow types are observed from the numerical simulation including fast flow, normal flow and slow flow. These three flows are mostly observed when the fractal dimension is reduced to zero which helps capture only natural flow with no self-similar properties.

These are of course in agreement with the properties of the new concept as the fractal-fractional differential and integral operators can be reverted to classical differential and integral operators with fractional order. On the other hand, when the fractal dimensions are considered, then the model exhibit fast flow with self-similarities, slow flow withs self-similarities and finally flow with crossover behaviour with self-similarities. In the real field observation, these results can be connected to a geological formation with dual media. The numerical solution obtained here suggest that non-conventional differential and integral operators are powerful mathematical tools able to replicate very accurately heterogeneity of the geological formation within which the sub-surface water flow.

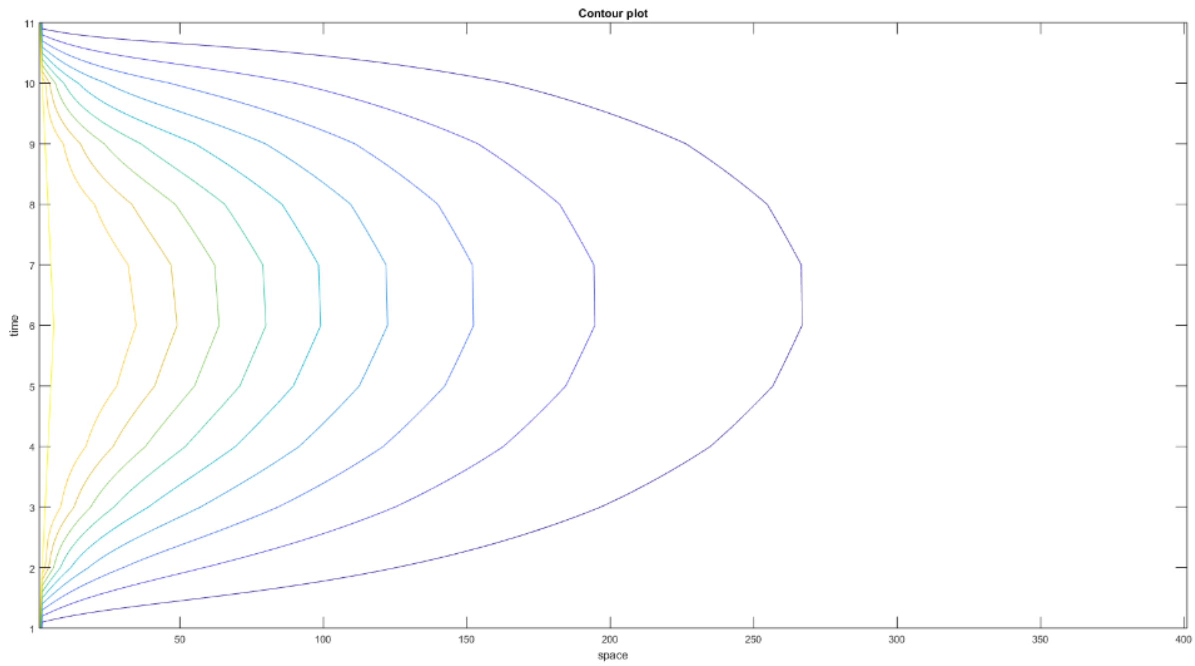

Figure 5: Contour plot numerical simulation for scale factor 1

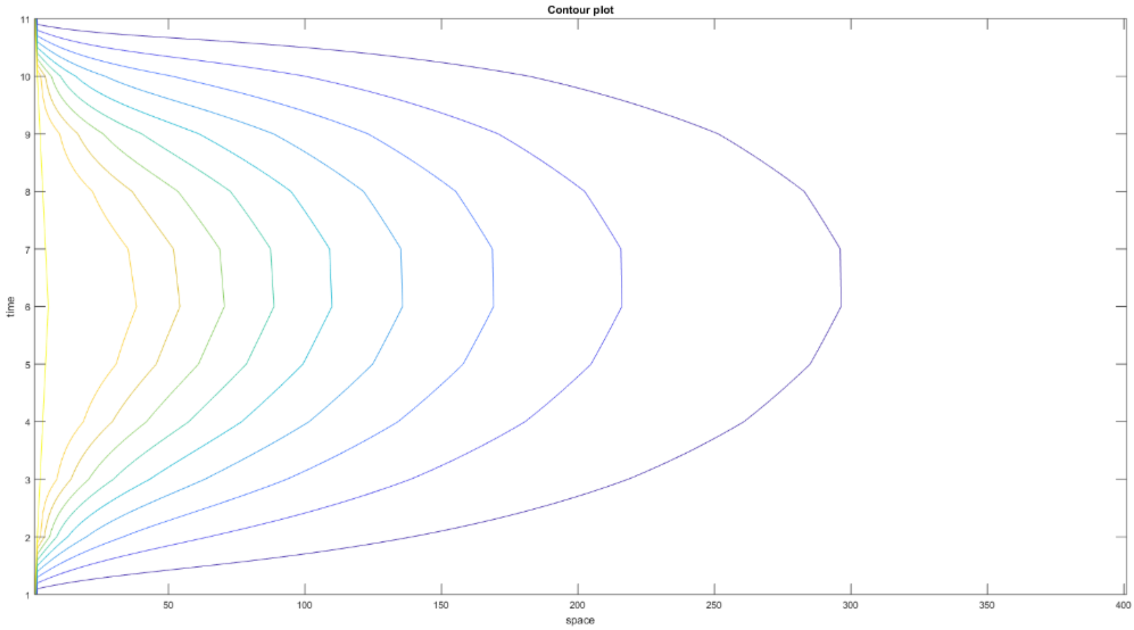

Figure 6: Contour plot numerical simulation for scale factor 0.9

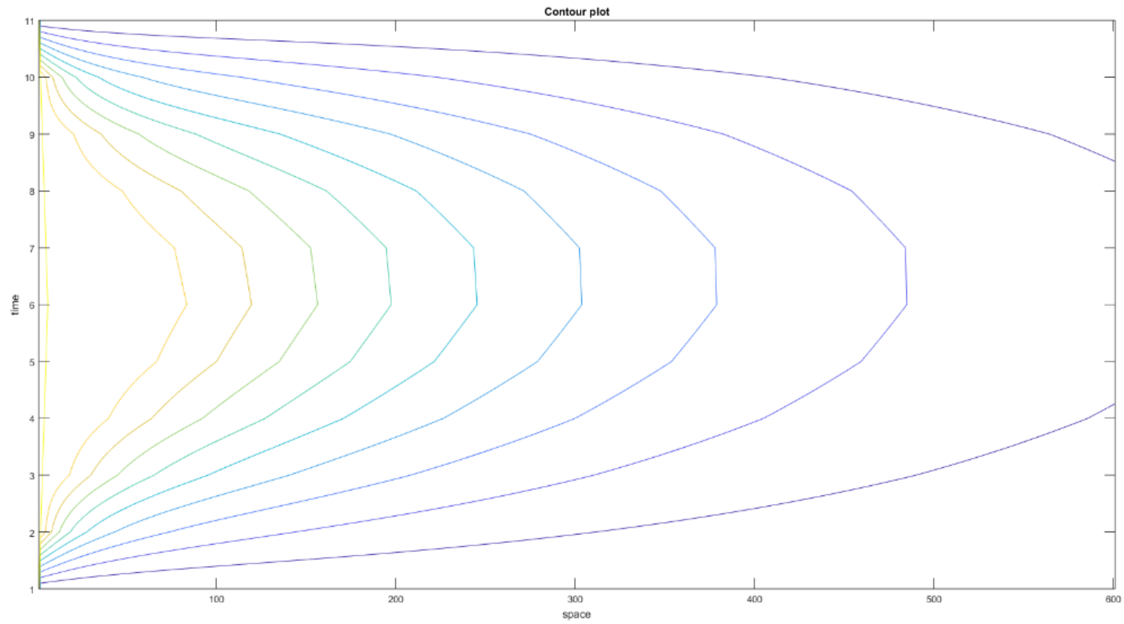

Figure 7: Contour plot numerical simulation for scale factor 0.4

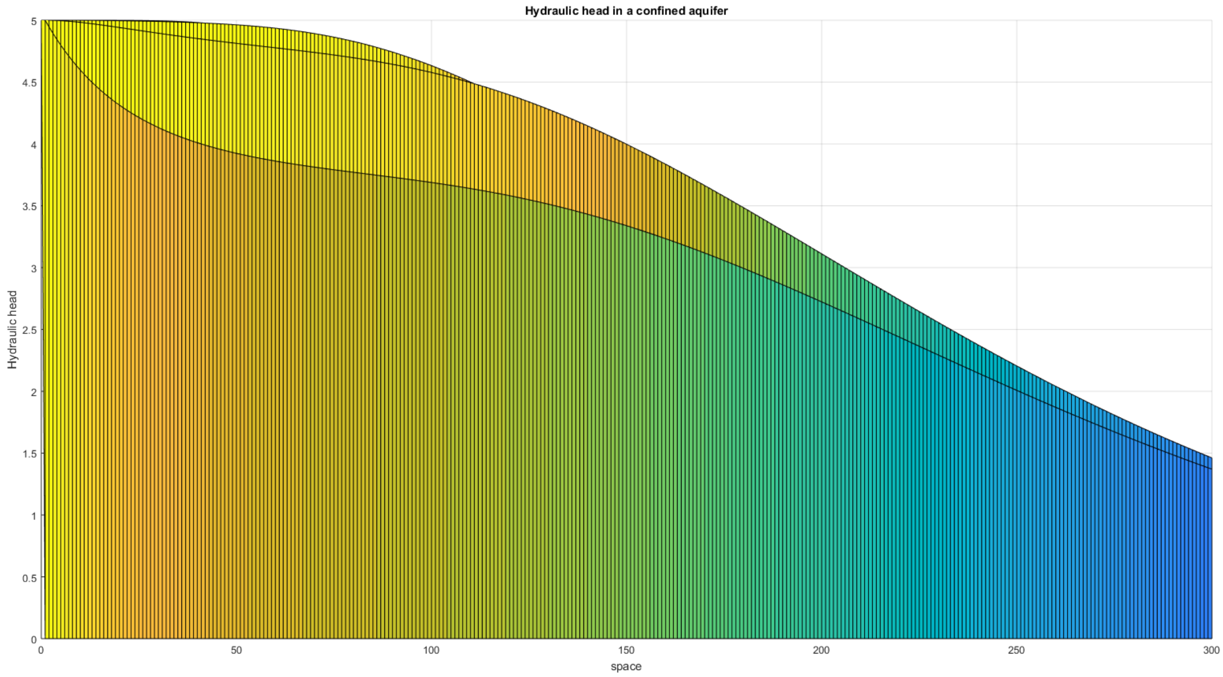

Figure 8: Numerical presentation groundwater flow in a confined aquifer with respect to hydraulic head and space with a scale factor of 0.4

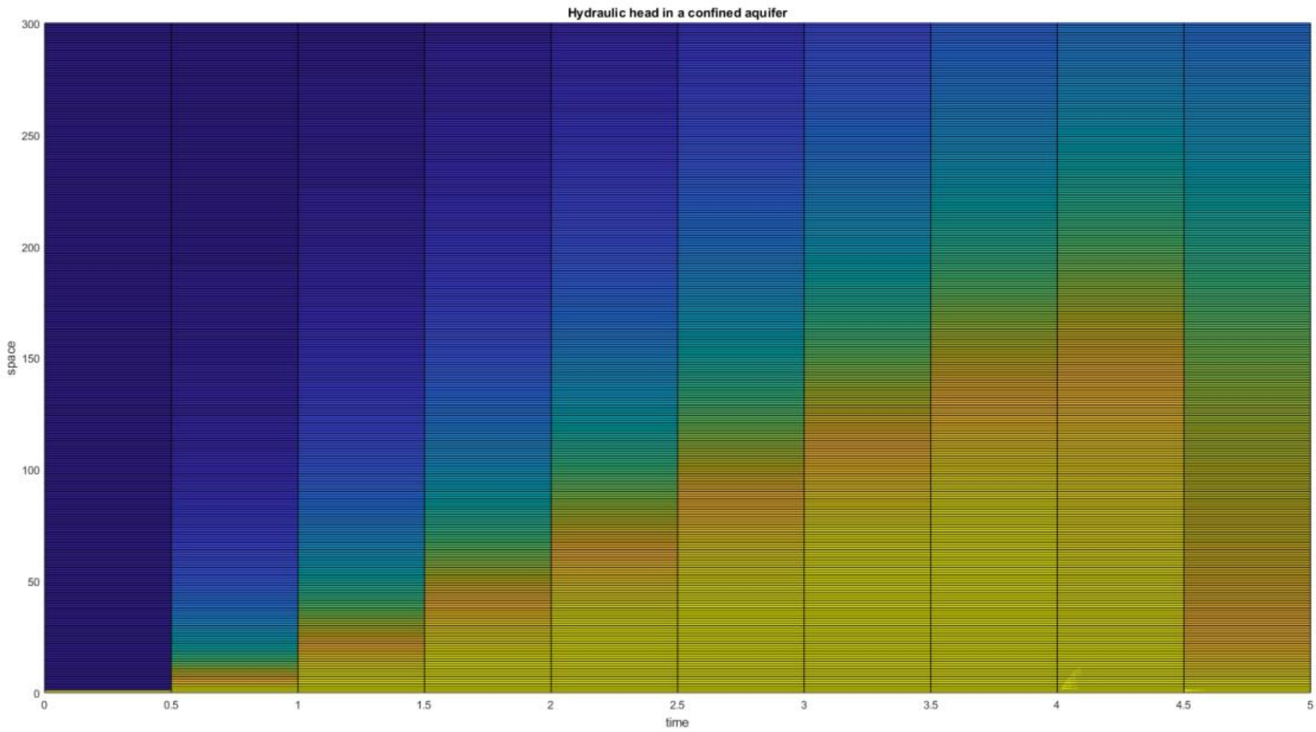

Figure 9: Numerical presentation of groundwater flow in a confined aquifer with respect to space and time with a scale factor of 0.4

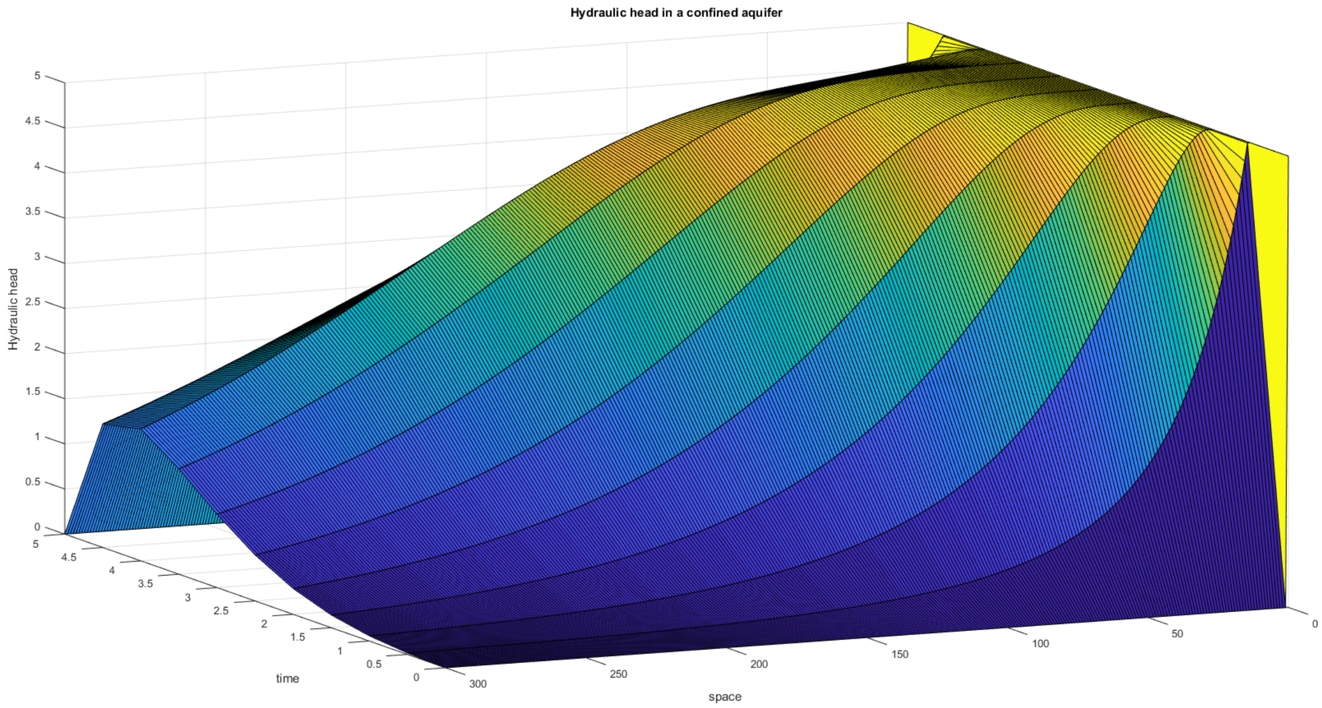

Figure 10: Numerical presentation of groundwater flow in a confined aquifer with respect to time and space, also showing the hydraulic head with a scale factor of 0.4

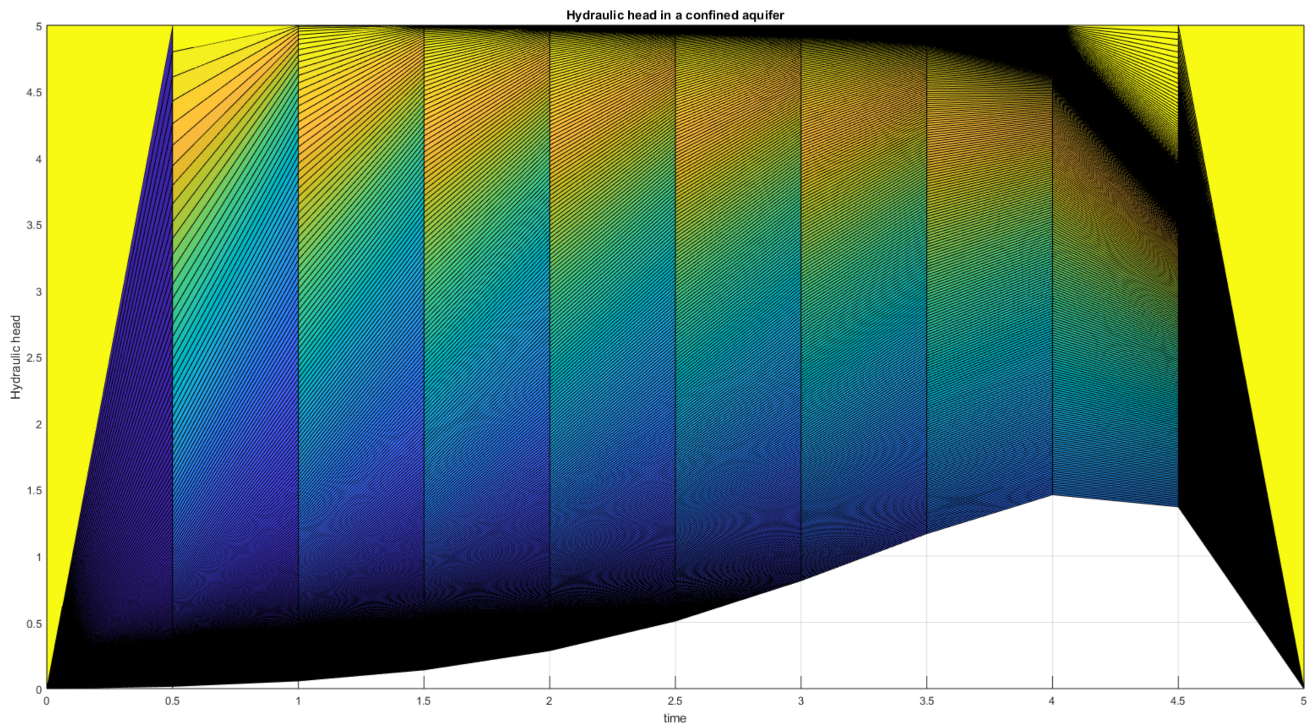

Figure 11: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time using a scale factor of 0.4

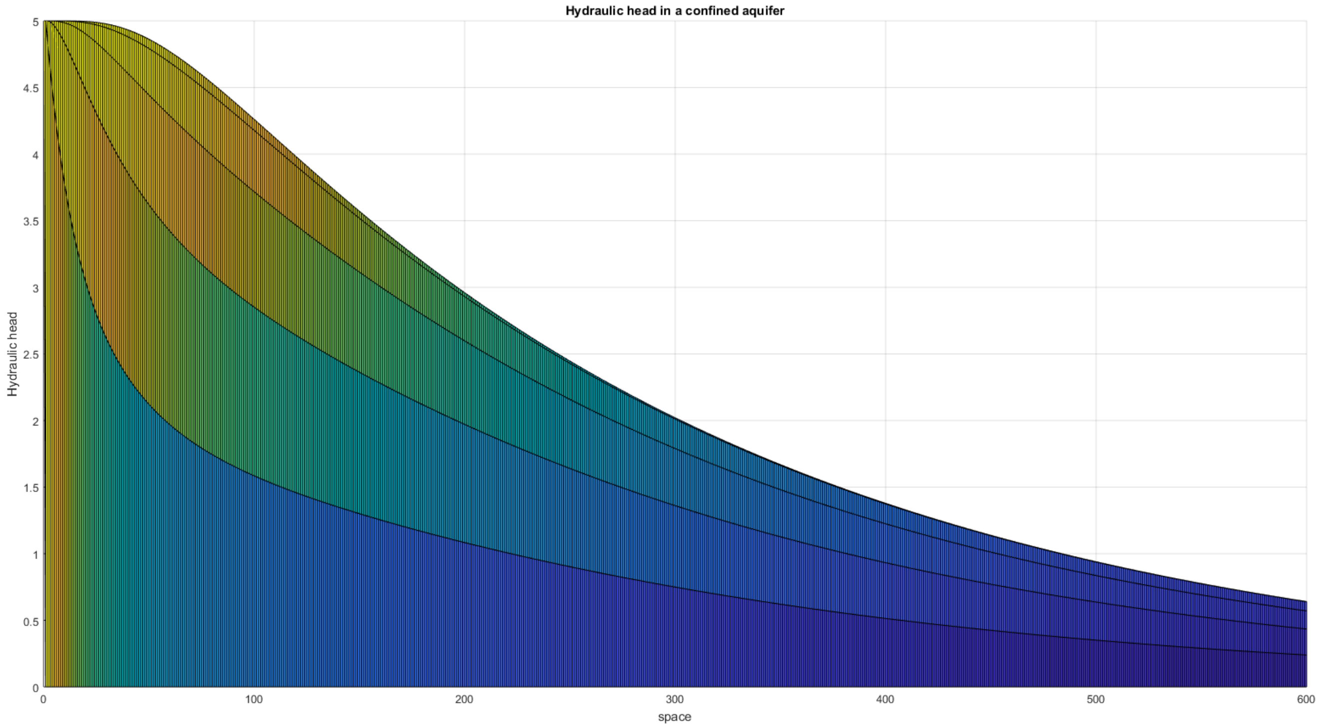

Figure 12: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space with a scale factor of 0.4

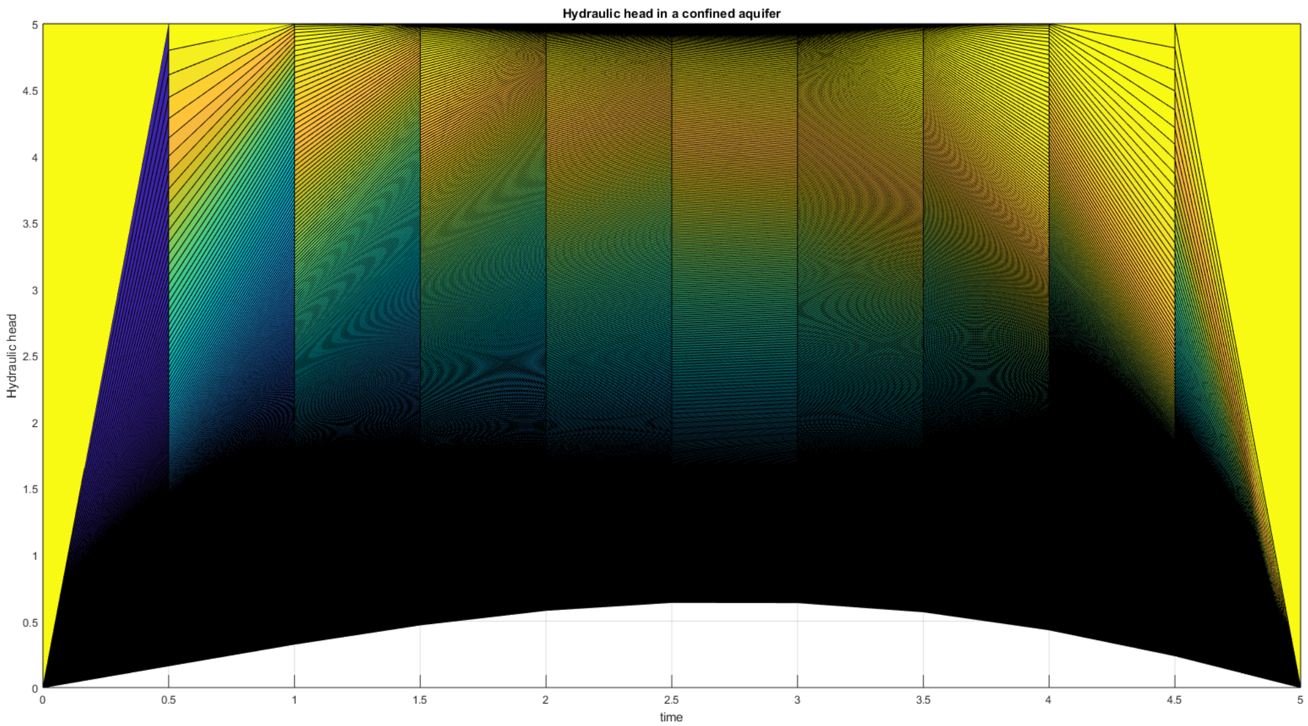

Figure 13:Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time with scale factor of 0.4

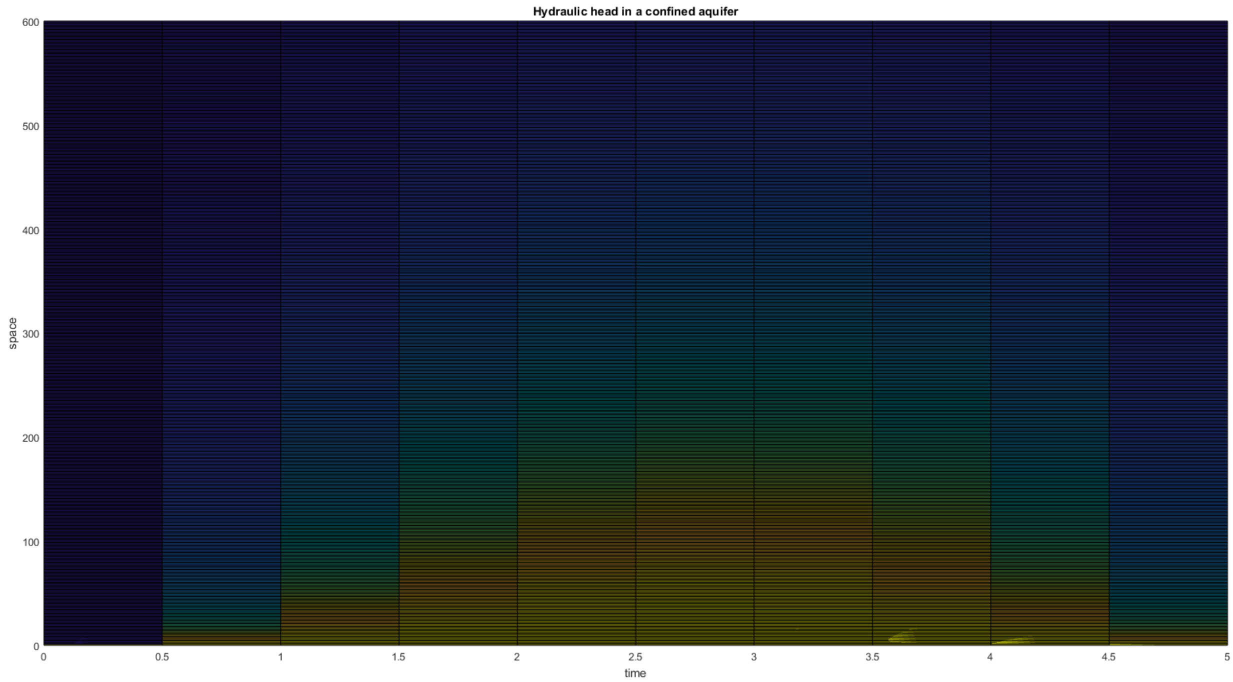

Figure 14: Numerical simulation of groundwater flow in a confined aquifer with respect to space and time with a scale factor of 0.4

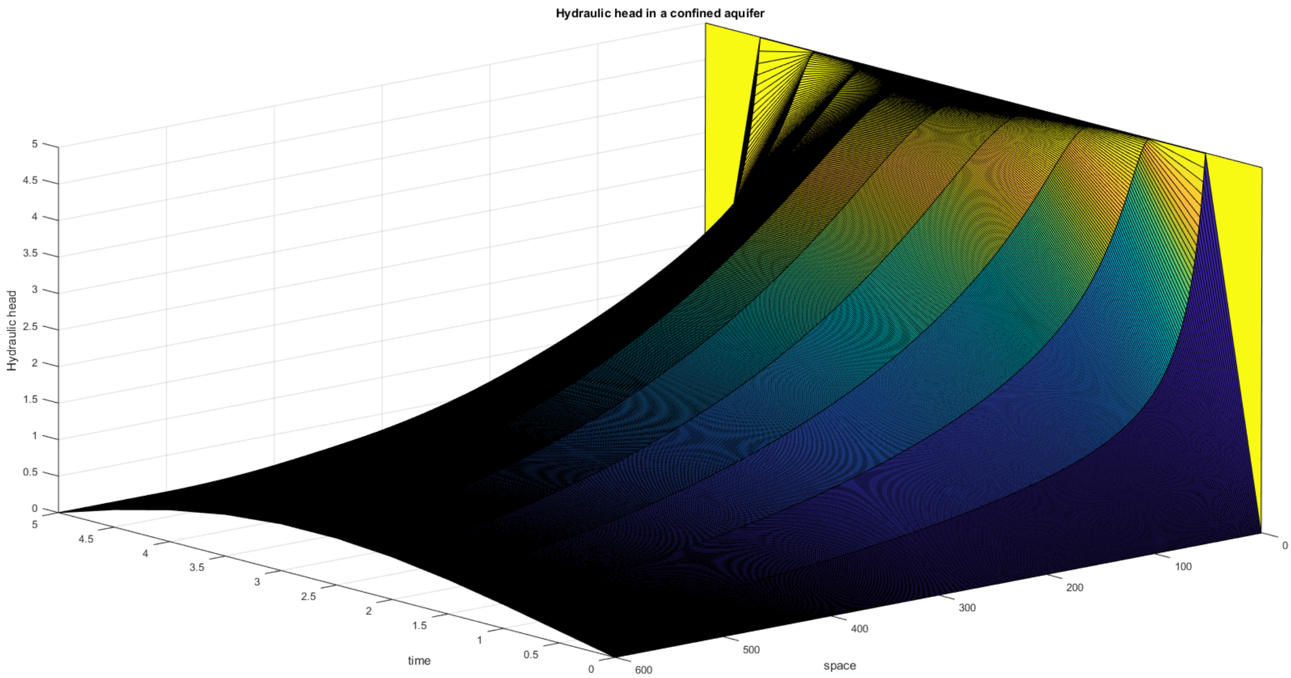

Figure 15: Numerical simulation of groundwater flow in a confined aquifer with respect space and time, showing the hydraulic head and using a scale factor of 0.4

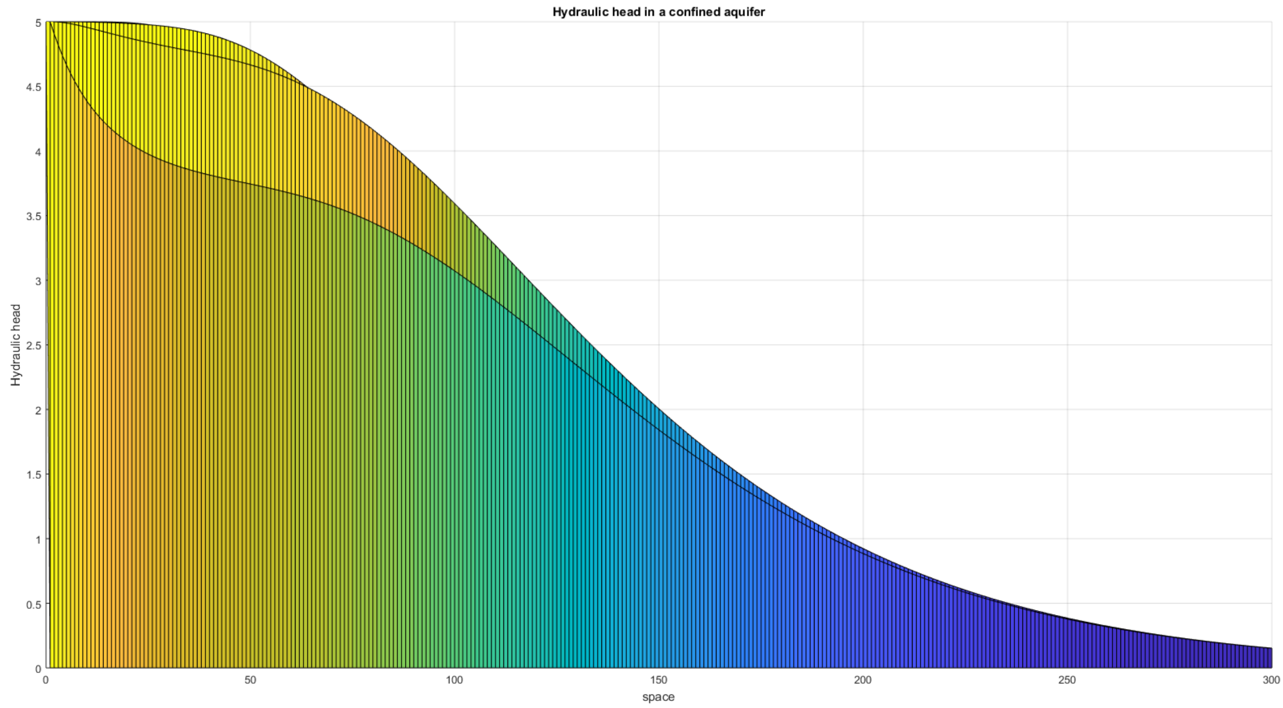

Figure 16: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space with a scale factor of 0.7

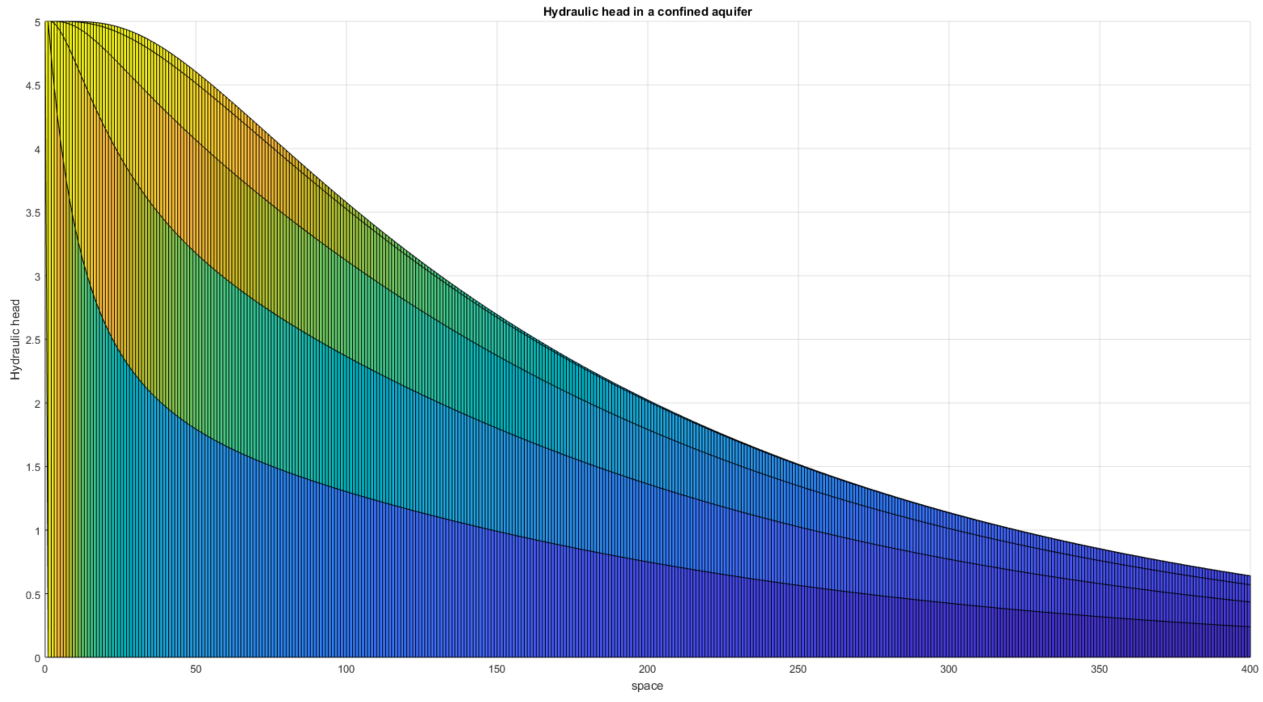

Figure 17:Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space with a scale factor of 0.7

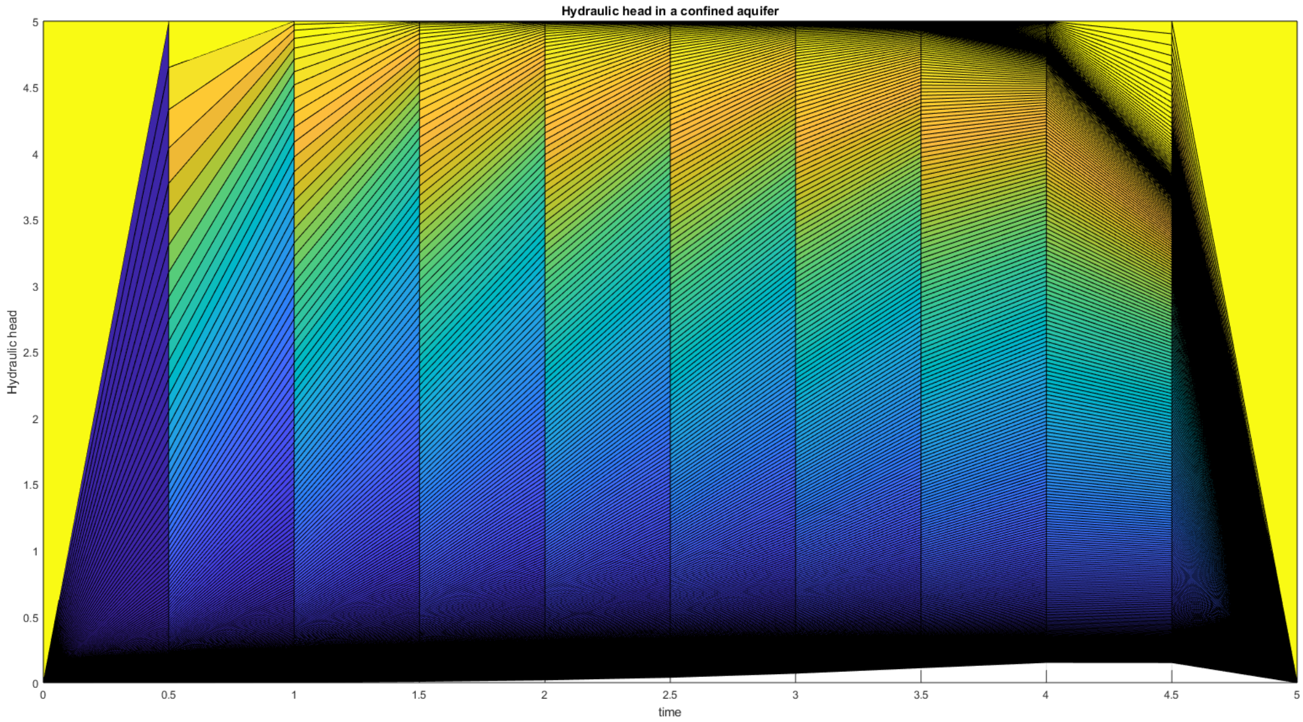

Figure 18:Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time with a scale factor of 0.7

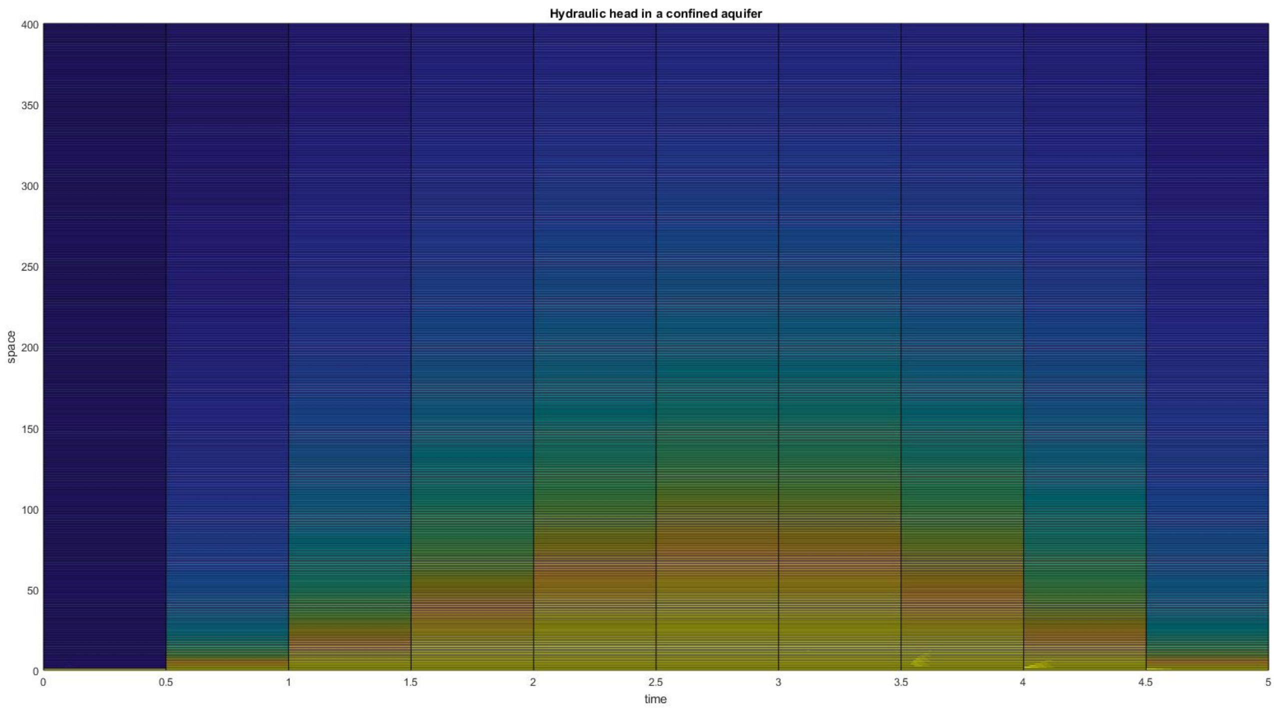

Figure 19: Numerical simulation of groundwater flow in a confined aquifer with respect to space and time with a scale factor of 0.7

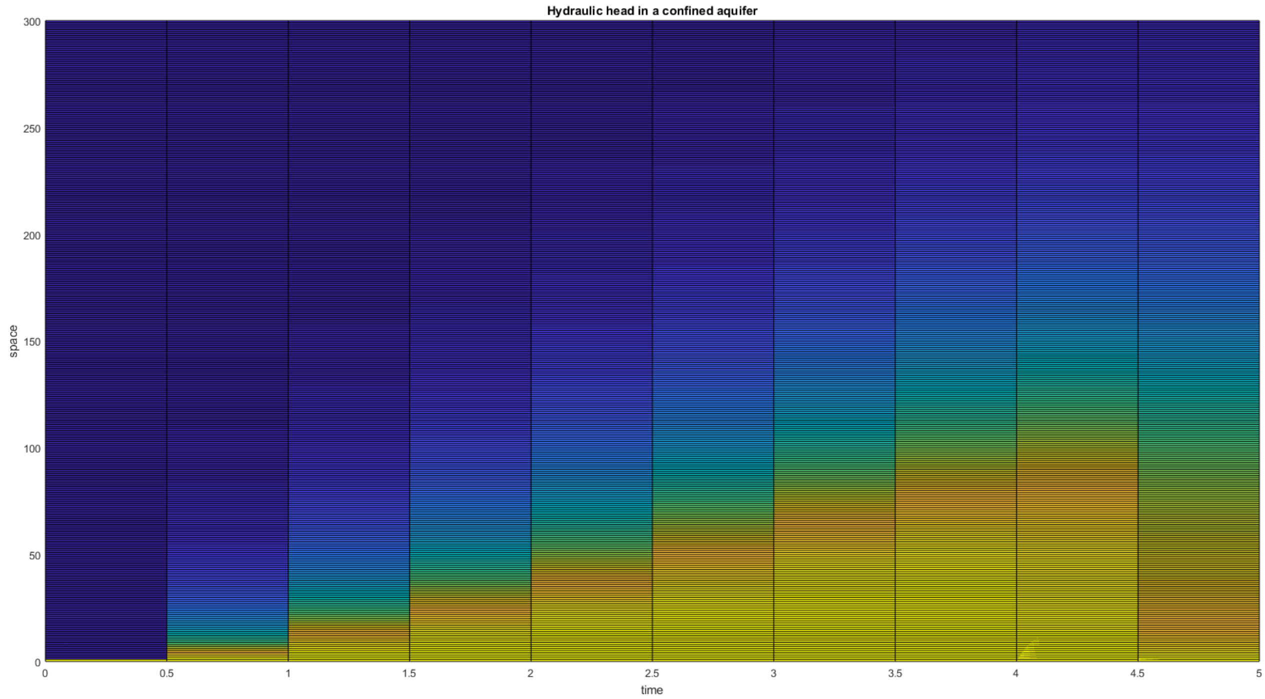

Figure 20:Numerical simulation of groundwater flow in a confined aquifer with respect to space and time with a scale factor of 0.7

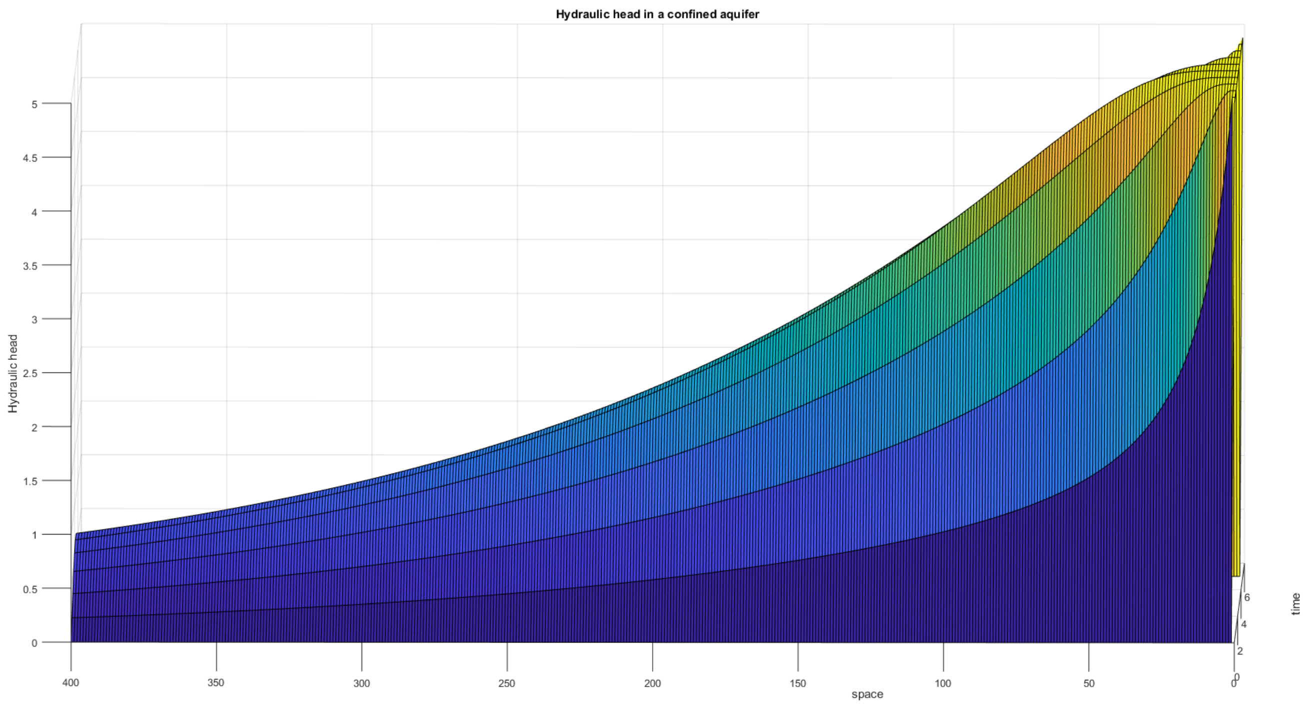

Figure 21: Numerical simulation of groundwater flow in a confined aquifer with respect to space and time, showing the hydraulic head using a scale factor of 0.7

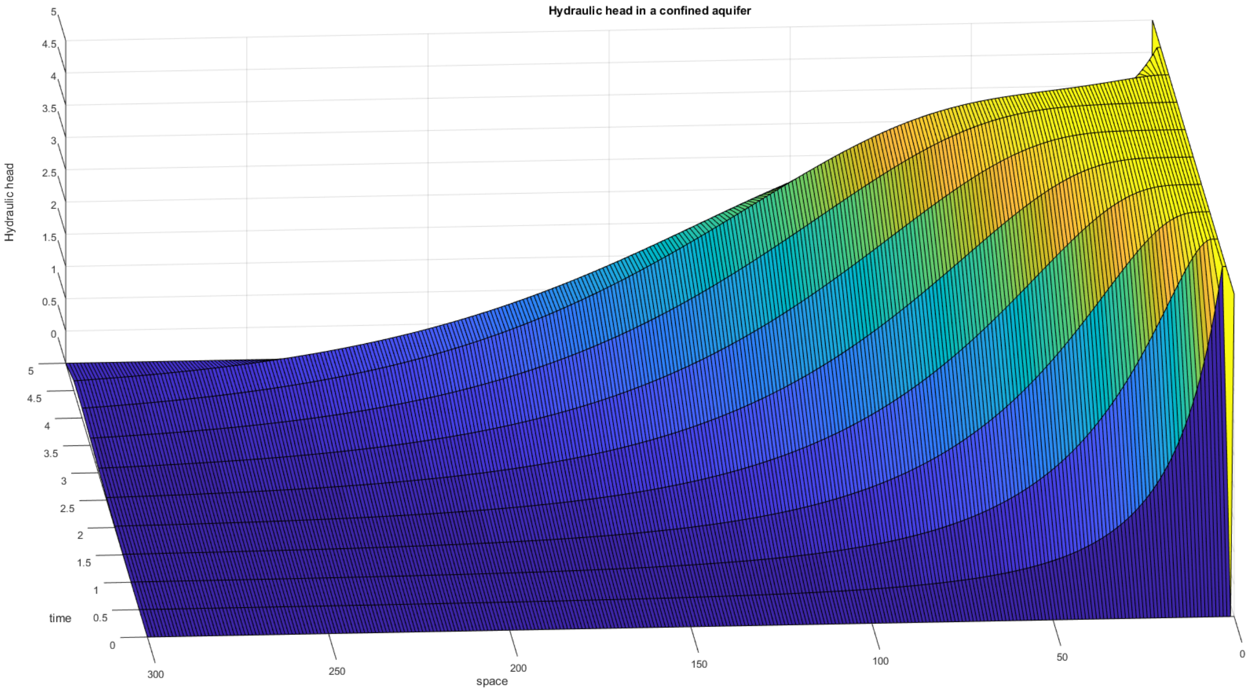

Figure 22: Numerical simulation of groundwater flow in a confined aquifer with respect to space and time, showing the hydraulic head using a scale factor of 0.7

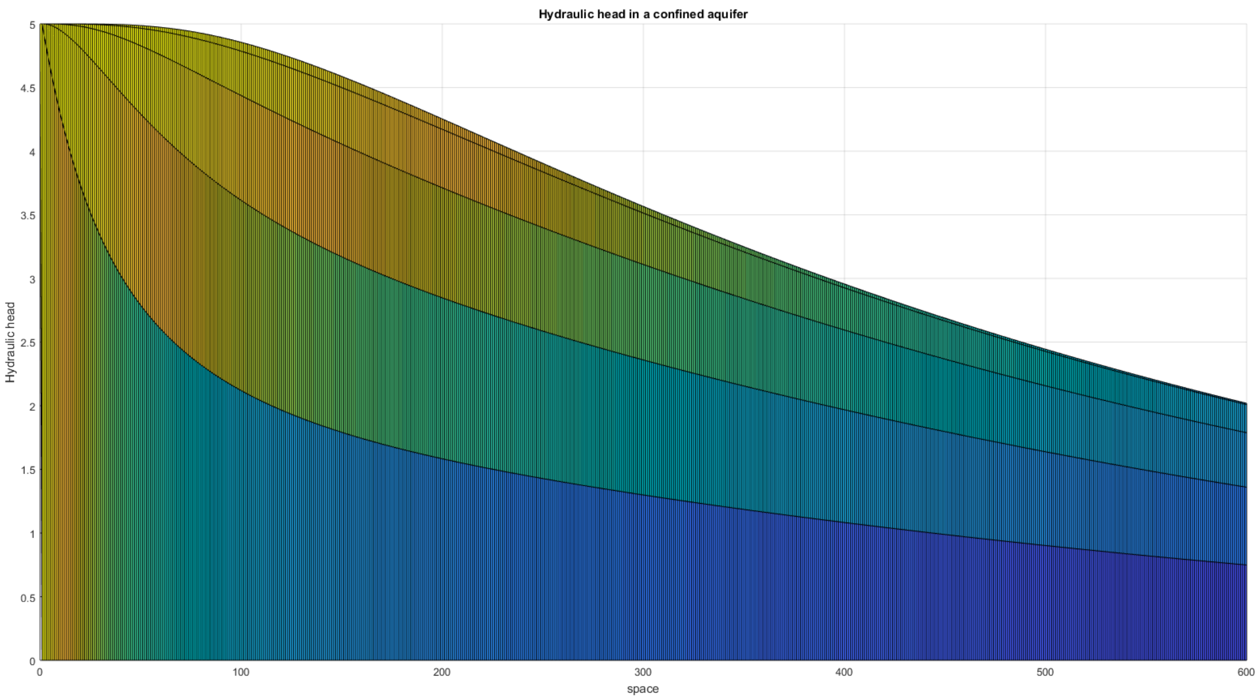

Figure 23: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space using a scale factor of 0.2

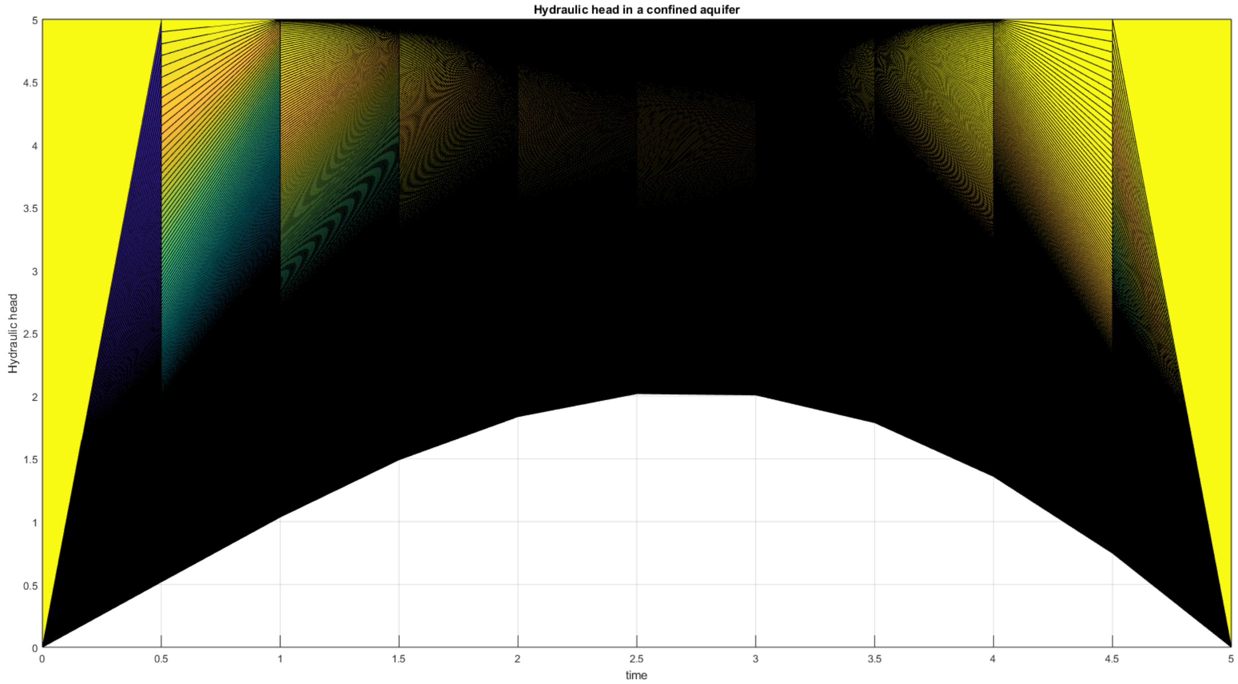

Figure 24: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time using a scale factor of 0.2

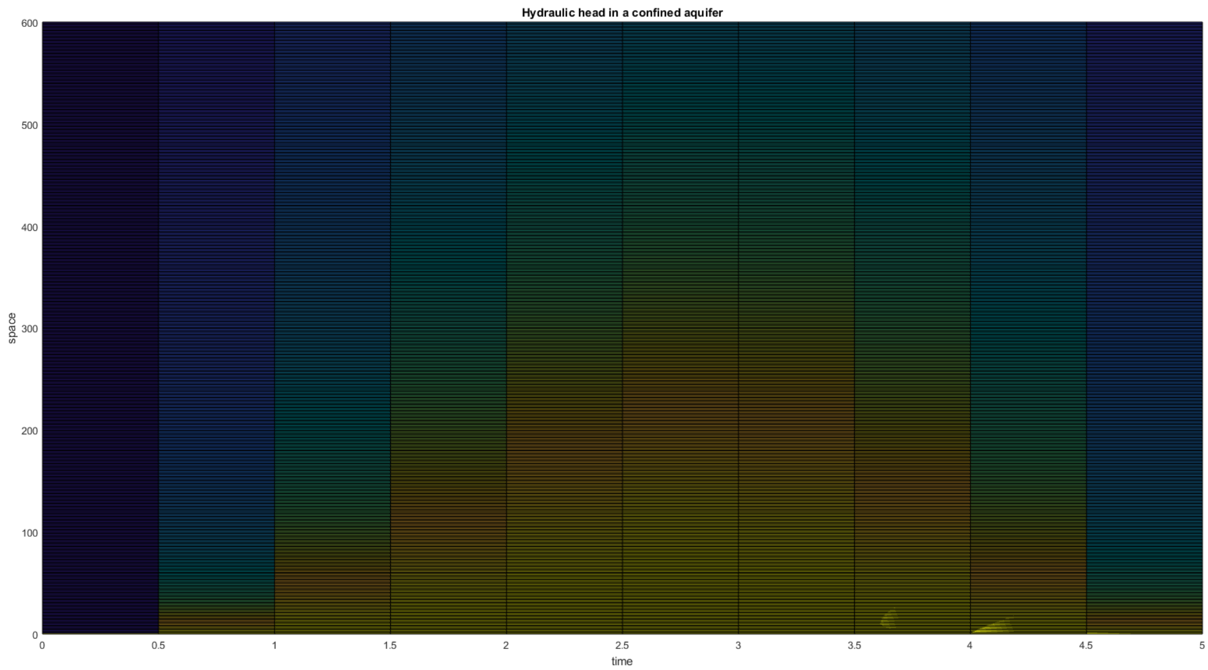

Figure 25: Numerical simulation of groundwater flow in a confined aquifer with respect to space and time using a scale factor of 0.2

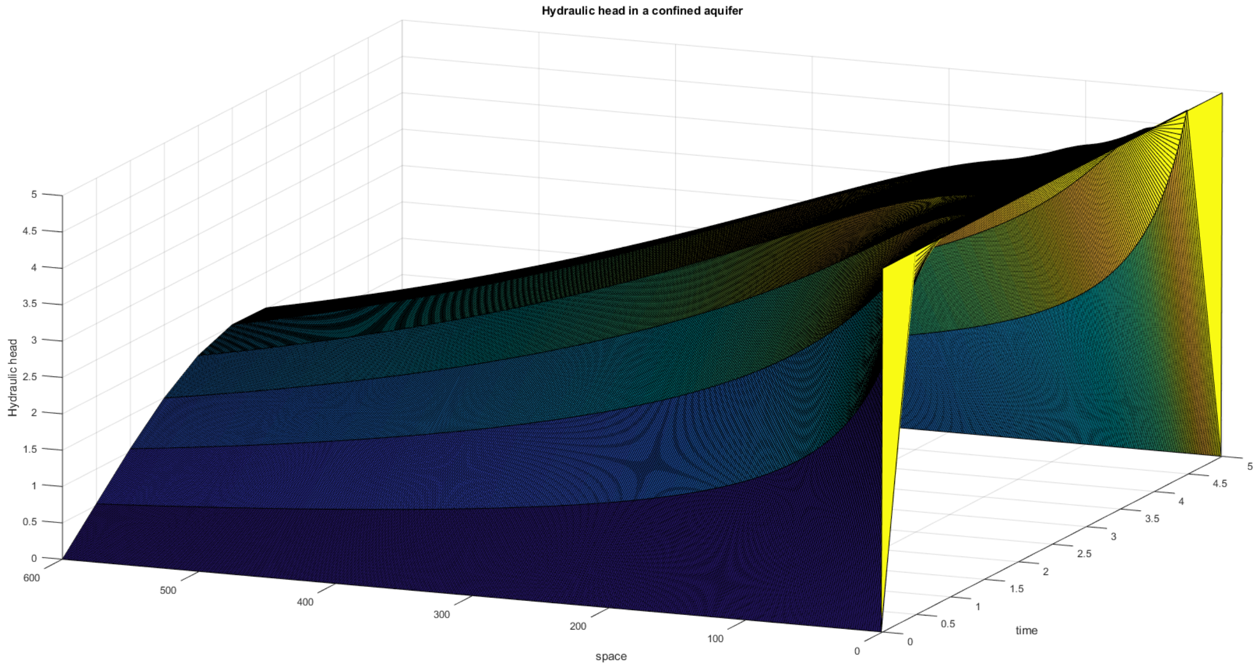

Figure 26:Numerical simulation of groundwater flow in a confined aquifer with respect to space and time, showing the hydraulic head using a scale factor of 0.2

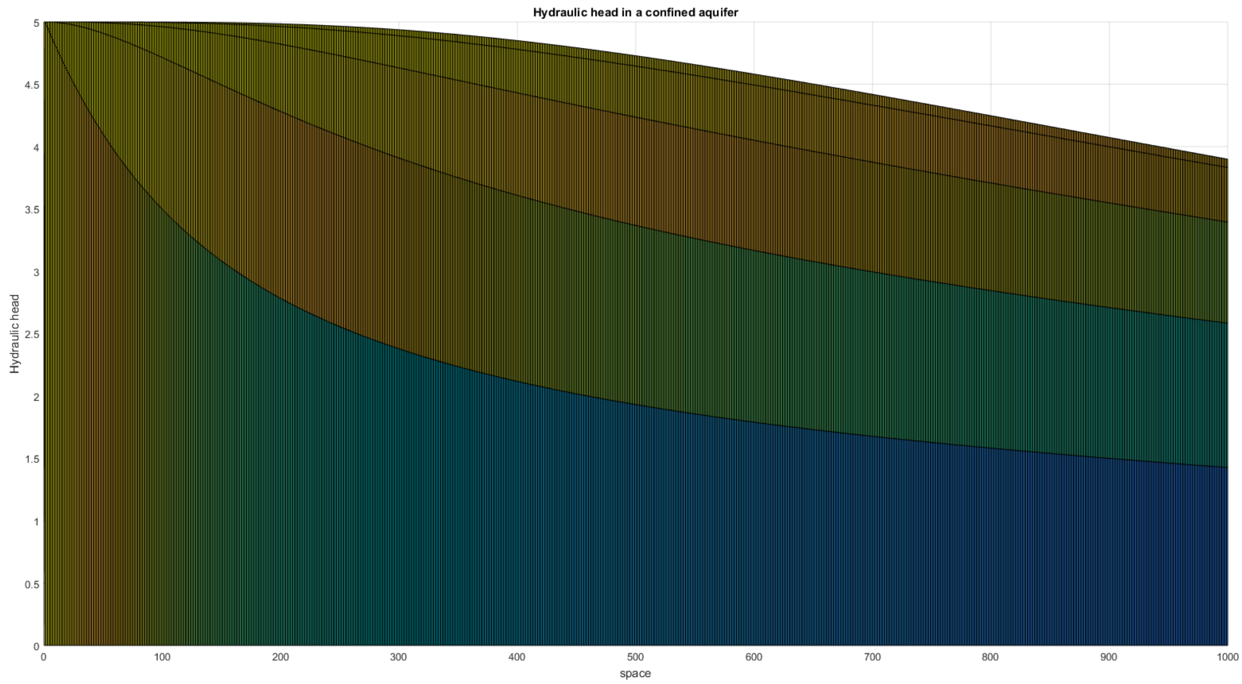

Figure 27: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time, showing the hydraulic head using a scale factor of 0.05

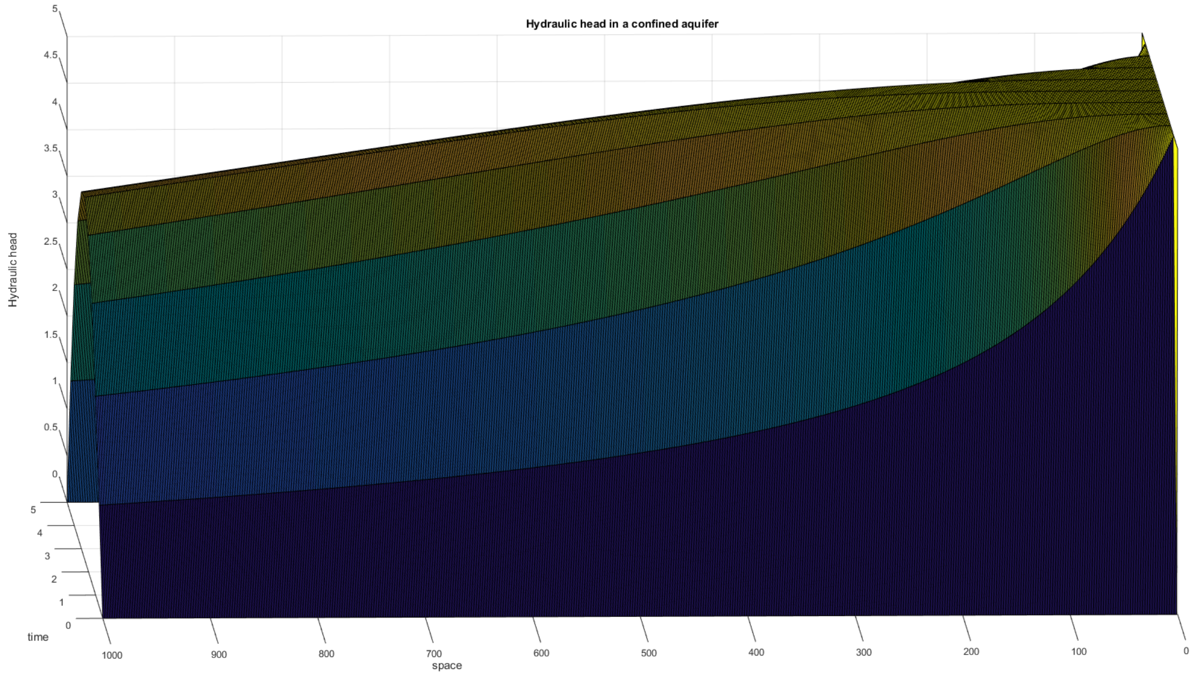

Figure 28: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time, showing the hydraulic head using a scale factor of 0.05

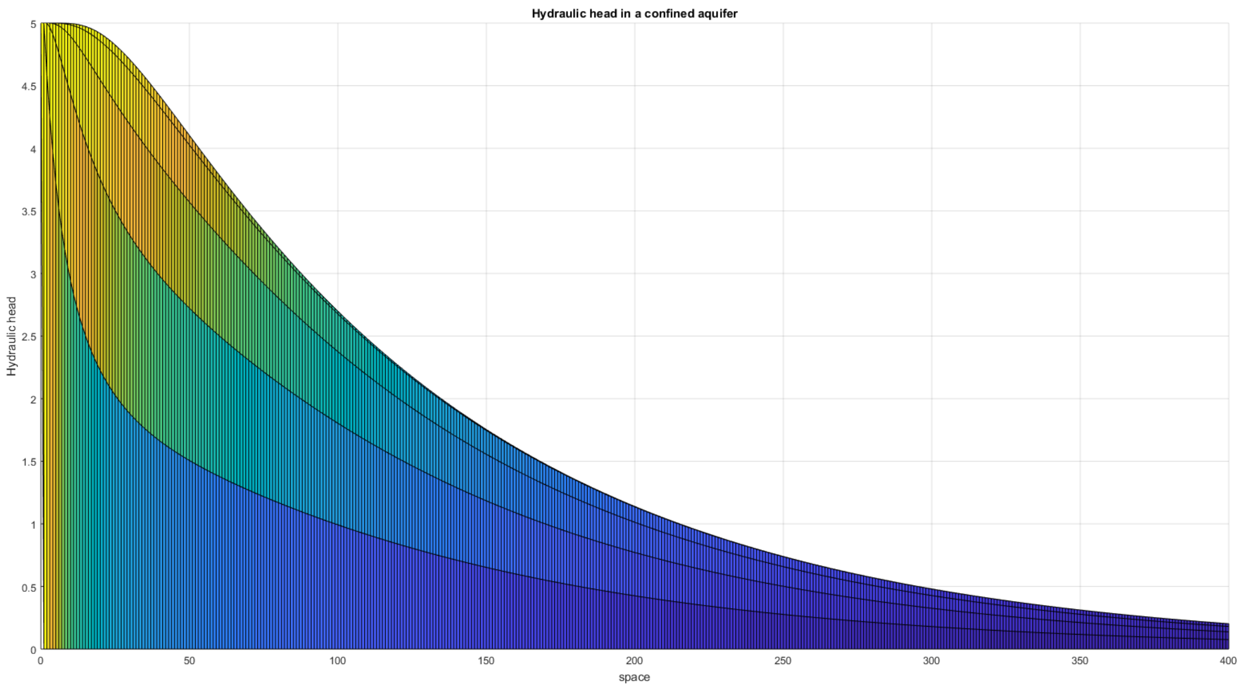

Figure 29: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space, showing the hydraulic head using a scale factor of 0.9

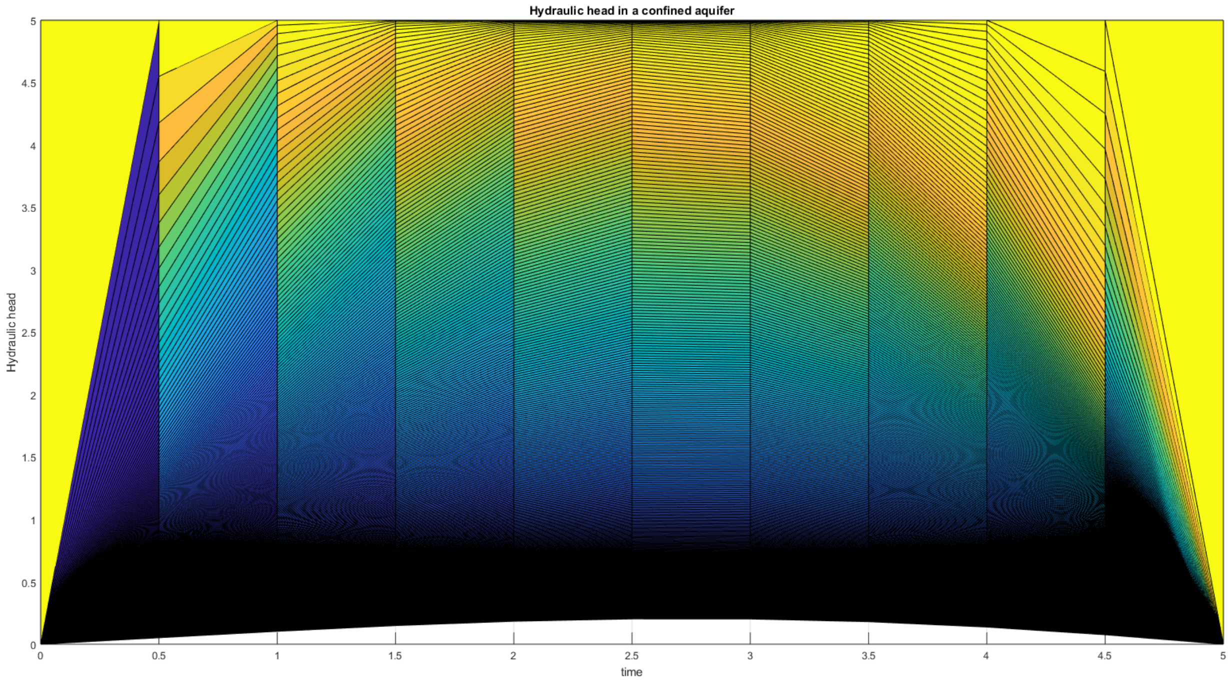

Figure 30: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time using a scale factor of 0.9

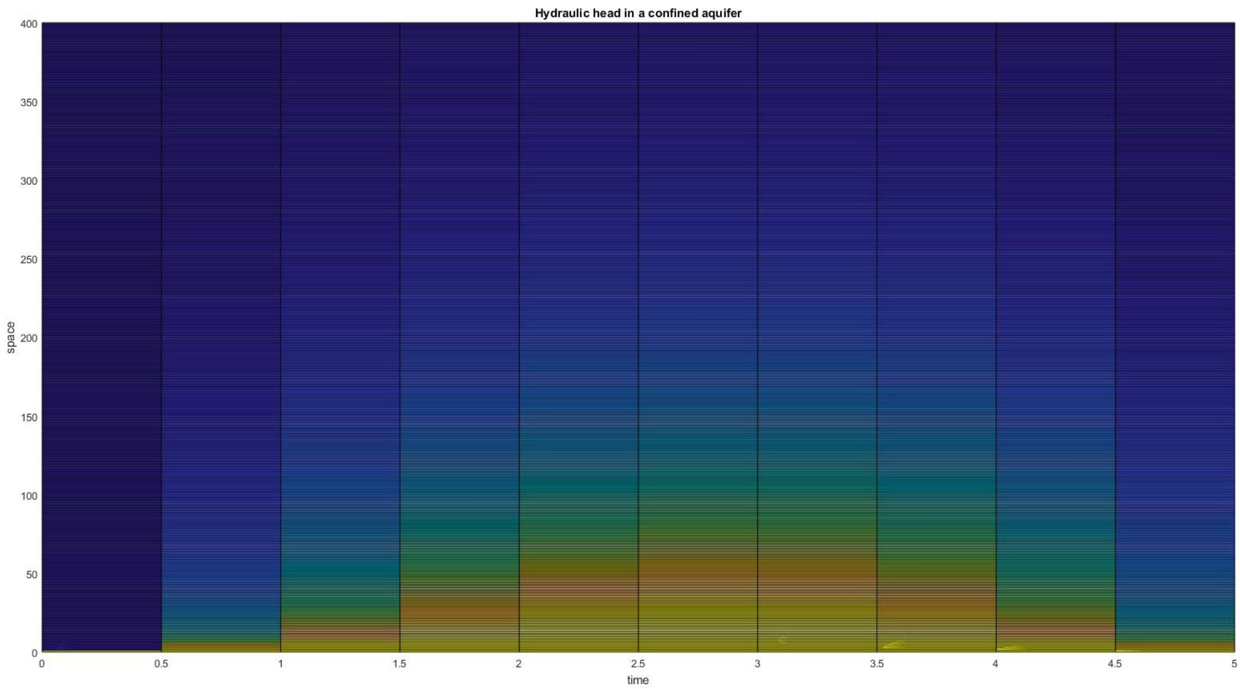

Figure 31: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time, using a scale factor of 0.9

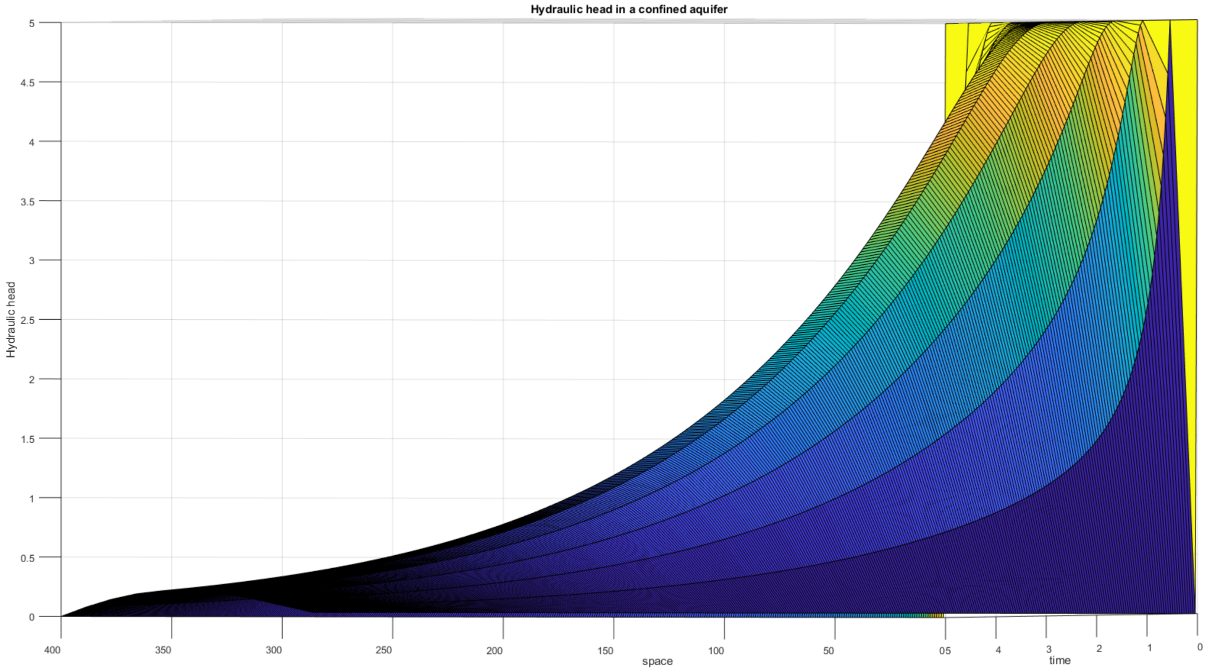

Figure 32: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space, using a scale factor of 0.9

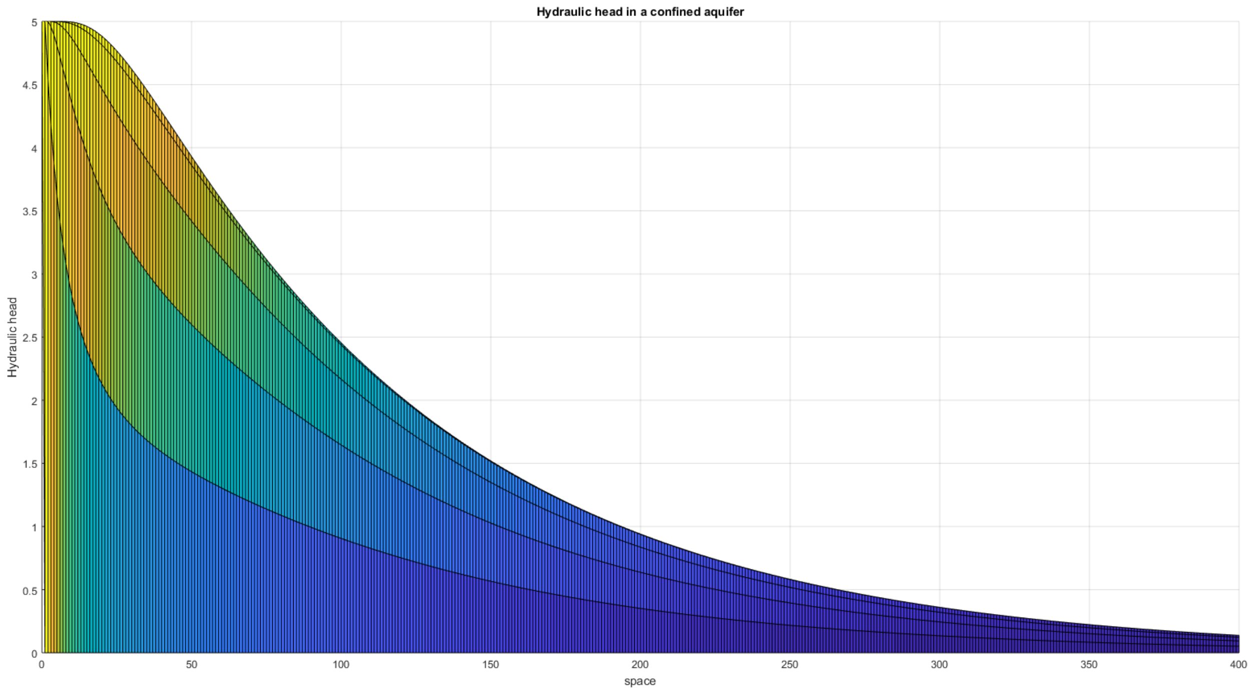

Figure 33: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space, using a scale factor of 1

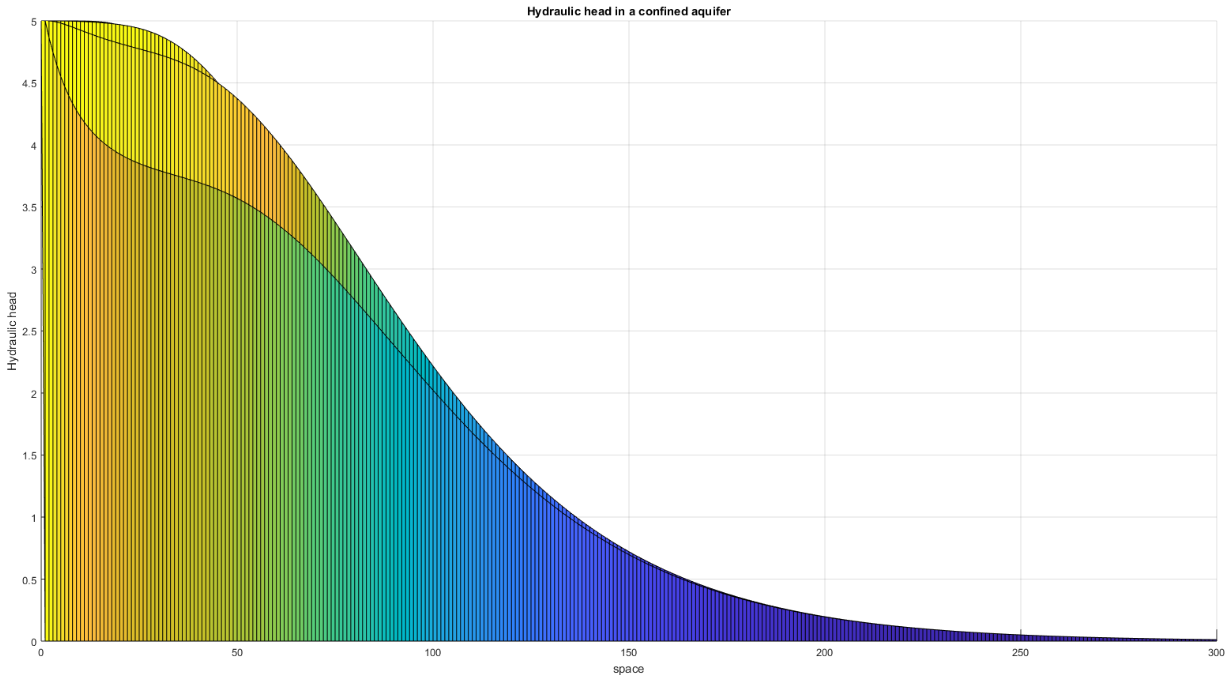

Figure 34: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and space, using a scale factor of 1

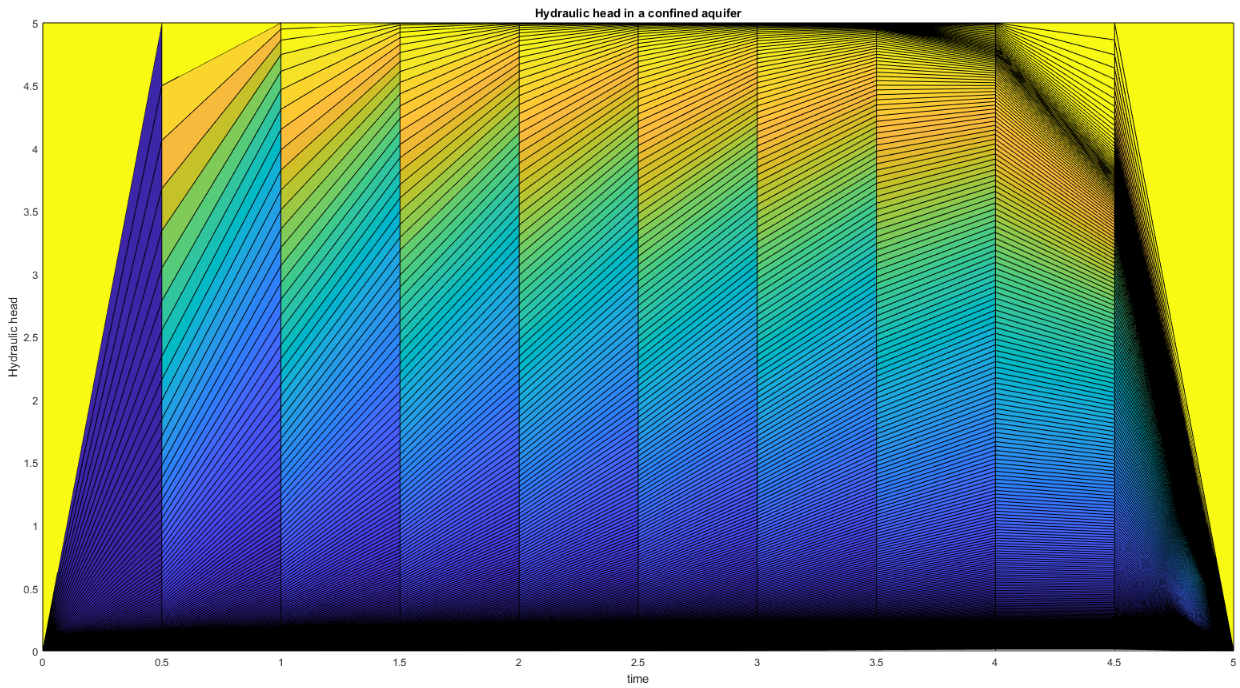

Figure 35: Numerical simulation of groundwater flow in a confined aquifer with respect to hydraulic head and time, using a scale factor of 1

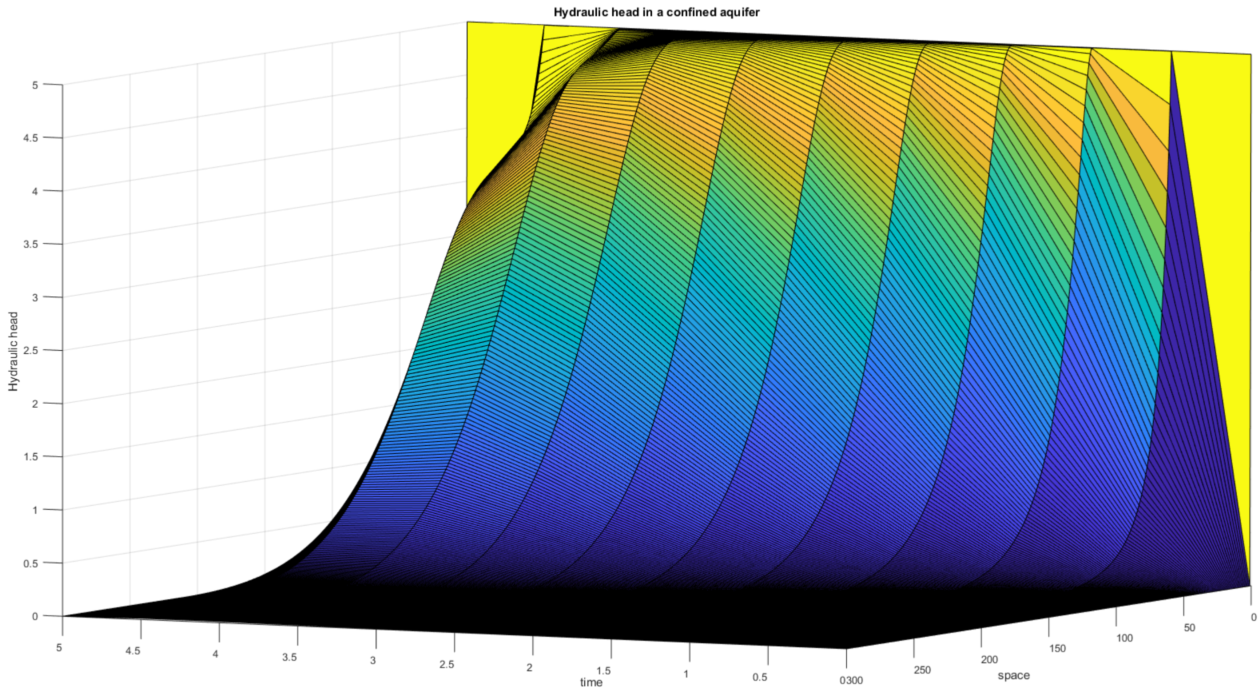

Figure 36: Numerical simulation of groundwater flow in a confined aquifer with respect to time and space showing also the hydraulic head, using a scale factor of 1

CONCLUSION

With the new challenges faced by researchers when trying to understand the flow within a given confined aquifer, researchers look for new and sophisticated mathematical operators able to include into mathematical formulation the heterogeneity of nature. The genesis of such model was initiated by Theis who used the concept of inflow, outflow and the continuity to derivative the flow equation within a confined aquifer. While this model was used successfully in many cases, the deviation of mathematical model with the observed facts let no doubt that the suggested mathematical model has some limitations. The limitations are associated with the fact that the Theis model in addition to the simplification made to obtain it, then used the concept of differential operator based on the rate of change. This operator cannot consider flow following the non-Markovian process. It cannot depict flow following the crossover process where the water flow from matrix soil to fracture or the reverse. This model cannot depict flow within the fracture, also the flow following fading velocity cannot be depicted. However, to solve this problem, fractional differential operators based on power law have been used intensively to model the flow within the geological formation as the power law has some important properties helping to capture long-range that can be associated to flow within a fracture, a great feature that the classical differentiation cannot account for. Nevertheless, while such model can also be used to replicate flow with fading velocity, however, the limitation of such model is that the fading velocity does not have a beginning and an end due to the very long tail of the power law. Additionally, the flow within a self-similar feature cannot be replicated here. In this thesis new operators called fractal-fractional derivatives were used to generalise the flow within a confined-fractured aquifer with dual media, three different physical laws are used. The new models are solved numerically using the newly suggested numerical scheme. Numerical simulation shows a connection between the fractional order, fractal dimension and the heterogeneity of the geological formation.

REFERENCES

Anderson M., Woessner W. and Hunt R. (2015). *Applied groundwater modelling: Simulation of flow and advective transport*. Academic Press. The United States of America.

Archibald T., Fraser C., and Grattan-Guinness I. (2004). *The History of Differential Equations 1670—1950*. Oberwolfach Reports, 2729–2794.

Atangana A. (2017). Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system. *Chaos, Solitons and Fractals 102*, 396-406.

Atangana A. and Bildik N. (2013). The use of fractional order derivative to predict the groundwater flow. *Mathematical problems in engineering 2013*(543026), 1-9.

Atangana A. and Botha J.F. (2013). Generalized groundwater flow equation using the concept of variable order derivation. *Boundary value problems 2013*(53).

Atangana A. and Gomez-Aguilar J.F. (2018). Fractional derivatives with no-index law property: Application to chaos and statistics. *Chaos, Solitons and Fractals 2018*(114), 516-555.

Atangana A. and Kilicman A. (2013). Analytical solutions of the space-Time fractional derivative of advection-dispersion equation (G. -C. Guo-Cheng Wu, Ed). *Mathematical problems in engineering 2013*(853127) 1-9.

Baleanu D. and Fernandez A. (2017). On some new properties of fractional derivatives with Mittag-Leffler Kernel. *Communications in Nonlinear Science and Numerical Simulation 2017*(59) pp. 444-462.

Barackman, M., & Brusseau, M. L. (2004). *Chapter 8: Groundwater sampling*, in Artiola J.F., Pepper I.L. and Brusseau M.L. (eds.), *Environmental monitoring and characterization*, Academic Press, Cambridge.

Bear J. and Verruijt A. (1987). *Modelling groundwater flow and pollution with computer programs for sample cases.* D. Reidel Publishing Company, Dordrecht, Holland.

Bishnoi B., (2016). Types of aquifers. MSc powerpoint presentation, Jai Narayan Vyas University Jodhpur, Department of Geology.

Botha J.F. and Cloot A.H. (2006). A generalized groundwater flow equation using the concept of non-integer order derivatives. *Water SA*, 1(32), 1-7.

Caputo M. and Fabrizio M. (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractal Differentiation and Application* 1(2), 73-85.

Chen W., Sun H., Zhang X., Korosvak D. (2010). Anomalous diffusion modelling by fractal and fractional derivatives. *Computers and mathematics with applications* 2010(59) 1754-1758.

Cherry, A., Parker, B. and Borchardt A. (2004). Role of aquitards in the protection of aquifers from contamination. USA: Awwa Research Foundation.

Christiansen E.H., Pommerville J.C., Krasner R.I., Hamblin K. and Issel L.M. (2014). *Dynamic Earth: Introduction to physical geology*, Jones and Bartlett Publishers, US.

De Graaf I.E.M., Van Beek R.L.P.H., Gleeson T., Moosdorf N., Schmitz O., Sutanudjaja E.H., Bierkens M.F.P. (2017). A global-scale 2-layer transient groundwater model: Development groundwater depletion. *Advances in Water Resources 2017*(102), 53-67.

Fetter C.W. (1994). *Applied hydrogeology* 4th edition. Prentice-Hall, Inc. Upper Saddle River, New Jersey.

Flores L. and Bailey R.T. (2018). Review: Revisiting the Theis solution derivation to enhance understanding and application. *Hydrogeology Journal 27*(1), 55-60.

Focazio M., Reilly T.E, Rupert M.G., Helsel D.R., (2003). Assessing groundwater vulnerability to contamination: Providing scientifically defensible information for decision-makers, *US Geological Survey Circular*.

Freeze R.A. and Cherry J.A. (1979). *Groundwater*, Prentice-Hall, Englewood Cliffs, N.J.

Gomez-Aguilar J.F., Torres L., Yepez-Martinez H., Baleanu D., Reyes J.M. and Sosa I.O. (2016). Fractional Lienard Type model of a pipeline within the fractional derivative without singular kernel. *Advances Difference Equation 2016*(173), 1-13.

He J.H. (2018). Fractal calculus and its geometrical explanation. *Results in Physics 2018*(10), 272-276.

Heath R.C., (1987). Basic Groundwater Hydrology, course-notes. U*.S. Geological survey water supply paper* 2220.

Kavvas M.L., Tu T., Ercan A. and Polsinelli J. (2017). Fractional Governing equations of transient groundwater flow in confined aquifers with multi-fractional dimensions in fractional time. *Earth System Dynamics 8*, 921-929.

Khalil R., Al Horani M., Yousef A., Sababheh M. (2014). A new definition of fractional derivative. *Journal of computational and applied mathematics* 264, 65-70.

Kruseman G.P., de Ridder N.A. (2000). *Analysis and evaluation of pumping test data*, 2nd Ed, Publication 47, ILRI, Netherlands.

Lazarević M.P., Rapaic M.R., Sekara T.B. (2014). *Introduction to fractional calculus with brief historical background*, Mladenov V. and Mastorakis N. (eds) In book: *Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modelling*, WSEAS Press.

Liu Z and Li X. (2015). Approximate controllability of fractional evolution systems with Riemann-Liouville fractional derivatives. *SIAM Journal on Control and Optimization*, *53*(4), 1920-1933.

Liu P.G. and Shu L.C. (2008). Uncertainty on numerical simulation of groundwater flow in the riverside well field, *J Jilin University (Earth science edition)* 38, 639-644.

Loăiciga H.A. (2010). Derivation approaches for the Theis 1935 equation, Groundwater, *Technical commentary*, *48*(1), 1-5.

Marinardi F. (2010). *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models*, Imperial College Press, London.

Matlob M.A. and Jamali Y. (2017). The concepts and application of fractional order differential calculus in modelling of viscoelastic systems: *A primer*.

Merz B. and Thieken A.H. (2009). Flood risk curves and uncertainty bounds, *Natural hazards 51*, 437-458.

Mirza I.A. and Viera D. (2017). Fundamental solutions to Advection-Diffusion equation with Time-fractional Caputo-Fabrizio derivative, *Computers and mathematics with applications 73*(1) 1-10.

Owolabi K.M. (2018). Rienmann-Liouville fractional derivative and application to model chaotic differential equations. *Progress in Fractional Differentiation and application 4*(2), 99-110.

Podlubny I. (1998). Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. *In fractional Differential equations* 198, Academic Press, Boston, USA.

Qureshi S., Atangana A., and Ali Shalkh A. (2019). Strange chaotic attractors under fractal-fractional operators using newly proposed numerical methods. *The European Physical Journal Plus 134* (523), 1-21.

Sousa E. (2009). On the edge of the stability analysis. *Applied Numerical Mathematics 59*(3), 1322-1336.

Theis C.V. (1935). The relation between the lowering of the piezometric surface and the rate and duration of discharge of well using groundwater storage. *American Geophysical union Trans 16*, 519-524.

Van Asselt M. (2000). *Perspectives on uncertainty and risk: The PRIMA approach to decision support*, Kluwer Academic Publishers, Dordrecht.

Vučković V. (2008). The reconstruction of the compressed digital signal using Lagrange Polynomial interpolation, 16th Telecommunications forum, TELFOR Serbia, Belgrade,1-4.

Waghmare R. V. (2016). Mathematical Modelling of Flow in Confined Aquifer. *International Journal of Novel Research in Physics Chemistry & Mathematics 3* (2) 1-16.

Wang H.F. and Anderson M.P. (1982). *Introduction to groundwater modelling: Finite difference and Finite element methods*, Academic Press, California.

Watson I. and Burnett A.D. (1995). *Hydrology: An environmental approach*. Ft. Lauderdale: Buchanan Books, Cambridge.

Wu J.C., Zen X. (2013). Review of the uncertainty analysis of groundwater numerical simulation. *Chinese Science Bulletin. Special Topic: Groundwater geology*.

Xiao L. (2014). *Evaluation of groundwater flow theories and aquifer parameters estimation*, PhD dissertation, University of Western Cape, Cape Town.

Yen B.C., Cheng S.T., Melching C.S. (1986). Stochastic and risk analysis in hydraulic engineering. Littleton: Water resources publications.

Zhang J., Ma X. and Li L. (2017). Optimally conditions for fractional variational problems with Caputo-Fabrizio derivative. *Advances in Difference Equations*, *2017*(357), 1-14.

Zhuang P., Liu F., Anh V. and Turner N. (2008). New solution and analytical techniques of the implicit numerical method for the anomalous sub-diffusion equation. *Journal on Numerical Analysis*, *46*(2), 1079-1095.