

BAYESIAN TOLERANCE INTERVALS FOR VARIANCE COMPONENT MODELS

by

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THESIS

Submitted in fulfillment of the requirements of the degree

PHILOSOPHIAE DOCTOR

in

THE FACULTY OF NATURAL AND AGRICULTURAL SCIENCES

Department of Mathematical Statistics

University of the Free State

Bloemfontein : January 2012

PROMOTER : PROFESSOR A.J. VAN DER MERWE

Vir Wie dit Mag Aangaan

INDIENING VAN PROEFSKRIF GETITELD “BAYESIAN TOLERANCE INTERVALS FOR VARIANCE COMPONENT MODELS”

Hiermee verklaar ek, Johan Hugo, dat die proefskrif getiteld “ Bayesian Tolerance Intervals for Variance Component Models” wat hierby vir die kwalifikasie Philosophiae Doctor aan die Universiteit van die Vrystaat deur my ingedien word, my selfstandige werk is en nie voorheen deur my vir ‘n graad aan ‘n ander universiteit / fakulteit ingedien is nie.

Ek verklaar ook dat ek hiermee afstand doen van outeursreg van die bogenoemde proefskrif ten gunste van die Universiteit van die Vrystaat.

Johan Hugo

Geteken te op hierdie dag van, 20..... .

Abstract

The improvement of quality has become a very important part of any manufacturing process. Since variation observed in a process is a function of the quality of the manufactured items, estimating variance components and tolerance intervals present a method for evaluating process variation. As apposed to confidence intervals that provide information concerning an unknown population parameter, tolerance intervals provide information on the entire population, and, therefore address the statistical problem of inference about quantiles and other contents of a probability distribution that is assumed to adequately describe a process. According to Wolfinger (1998), the three kinds of commonly used tolerance intervals are, the (α, δ) tolerance interval (where α is the content and δ is the confidence), the α - expectation tolerance interval (where α is the expected coverage of the interval) and the fixed - in - advance tolerance interval in which the interval is held fixed and the proportion of process measurements it contains, is estimated. Wolfinger (1998) presented a simulation based approach for determining Bayesian tolerance intervals in the case of the balanced one - way random effects model. In this thesis, the Bayesian simulation method for determining the three kinds of tolerance intervals as proposed by Wolfinger (1998) is applied for the estimation of tolerance intervals in a balanced univariate normal model, a balanced one - way random effects model with standard $N(0, \sigma_{\epsilon}^2)$ measurement errors, a balanced one - way random effects model with student t - distributed measurement errors and a balanced two - factor nested random effects model. The proposed models will be applied to data sets from a variety of fields including flatness

measurements measured on ceramic parts, measuring the amount of active ingredient found in medicinal tablets manufactured in small batches, measurements of iron concentration in parts per million determined by emission spectroscopy and a South - African data set collected at SANS Fibres (Pty.) Ltd. concerned with measuring the percentage increase in length before breaking of continuous filament polyester. In addition, methods are proposed for comparing two or more α quantiles in the case of the balanced univariate normal model. Also, the Bayesian simulation method proposed by Wolfinger (1998) for the balanced one - way random effects model will be extended to include the estimation of tolerance intervals for averages of observations from new or unknown batches. The Bayesian simulation method proposed for determining tolerance intervals for the balanced one - way random effects model with student t - distributed measurement errors will also be used for the detection of possible outlying part measurements. One of the main advantages of the proposed Bayesian approach, is that it allows explicit use of prior information. The use of prior information for a Bayesian analysis is however widely criticized, since common non - informative prior distributions such as a Jeffreys' prior can have an unexpected dramatic effect on the posterior distribution. In recognition of this problem, it will also be shown that the proposed non - informative prior distributions for the α quantiles and content of fixed - in - advance tolerance intervals in the cases of the univariate normal model, the proposed random effects model for averages of observations from new or unknown batches and the balanced two - factor nested random effects model, are reference priors (as proposed by Berger and Bernardo (1992c)) as well as probability matching priors (as proposed by Datta and Ghosh (1995)). The unique and flexible features of the Bayesian simulation method were illustrated since all mentioned models performed well for the determination of tolerance intervals.

Key Words: *Bayesian Procedure, Random Effects, Variance Components, Tolerance Intervals, Reference Priors, Probability Matching Priors, Monte Carlo Simulation, Weighted Monte Carlo Method, Student t - Distributed Measurements Errors, Gibbs Sampling.*

Opsomming

In enige vervaardigings proses, het die verbetering van gehalte essensieel geword. Aangesien die waargenome variasie in 'n proses 'n funksie van die gehalte van die vervaardigde items is, bied die beraming van variansie komponente en toleransie intervale, 'n metode waardeur die waargenome proses variasie geëvalueer kan word. In teenstelling met vertrouens intervale wat slegs inligting rakende 'n onbekende populasie parameter bied, bied toleransie intervale inligting rakende die populasie in sy geheel. Die statistiese probleem met betrekking tot die afleiding van gevolgtrekkings uit kwantiele van waarskynlikheids verdelings wat veronderstel is om 'n proses genoegsaam te beskryf, word dus deur toleransie intervale aangespreek. Luidens Wolfinger (1998), is die (α, δ) toleransie interval (waar α die inhoud en δ die vertrouens van die interval is), die α - verwagtings toleransie interval (waar α die verwagte oordekking van die interval is) en die vooraf vasgestelde toleransie interval (waar die interval reeds vasgestel is en die persentasie proses waarnemings wat hierin voorkom, beraam word), die drie toleransie intervale wat meestal gebruik word. In die geval van die gebalanseerde een rigting toevallige effekte model, het Wolfinger (1998) 'n simulasië gebaseerde beskouing vir die bepaling van Bayesiaanse toleransie intervale voorgestel. In hierdie proefskrif, word Wolfinger (1998) se voorgestelde Bayesiaanse simulasië metode vir die bepaling van die drie algemene toleransie intervale, toegepas vir die beraming van toleransie intervale in die gevalle van die gebalanseerde enkelveranderlike normaal model, die gebalanseerde een rigting toevallige effekte model met $N(0, \sigma_\epsilon^2)$ verdeelde foute, die gebalanseerde een rigting toevallige effekte model met student t - verdeelde foute en die gebalanseerde geneste toevallige effekte model.

Die voorgestelde modelle sal toegepas word op data stelle afkomstig uit verskillende terreine. Dit sluit data stelle in aangaande gelykheids mates gemeet op keramiek parte, die hoeveelheid aktiewe bestanddeel teenwoordig in klein gegroepeerde stelle medisinale tablette, die hoeveelheid yster konsentraat in deeltjies per miljoen teenwoordig, bepaal deur emissie spektroskopie, en 'n eg Suid - Afrikaanse data stel aangaande die persentasie toename in lengte van 'n aaneenlopende poliëster vesel voordat dit breek. Die Suid - Afrikaanse data stel is deur Prof. Nico Laubscher by SANS Fibres (Pty.) Ltd. versamel. Daarbenewens word metodes vir die vergelyking van twee of meer α kwantiele, in die geval van die gebalanseerde enkelveranderlike normaal model, voorgestel. Bykomend, word Wolfinger (1998) se simulasiemetode aangepas om die beraming van toleransieintervalle in die geval van die gemiddeld van waarnemings uit nuwe of onbekende gegroepeerde stelle in te sluit. Deur van die Bayesiaanse simulasiemetode gebruik te maak vir die voorgestelde toevallige effekte model met student t - verdeelde foute, word die identifisering van moontlike uitskieters ook geïllustreer. Die gebruik van spesifieke prior inligting is een van die voordele van die voorgestelde Bayesiaanse simulasiemetode. Dit is egter juis die gebruik van hierdie prior inligting wat wyd veroordeel word, aangesien algemene nie - inliggende prior verdelings, soos 'n Jeffreys' prior, 'n dramatiese onverwagte uitwerking op die posterior verdeling tot gevolg kan hê as meer as een parameter ter sprake is. Ter erkenning van die probleem, word daar gewys dat die nie - inliggende prior verdelings, voorgestel vir die α kwantiele en inhoud van die vooraf vasgestelde toleransieintervalle in die gevalle van die enkelveranderlike normaal model, die voorgestelde toevallige effekte model vir die gemiddeld van waarnemings uit onbekende of nuwe gegroepeerde stelle en die gebalanseerde geneste twee rigting toevallige effekte model, beide verwysings priors (soos voorgestel deur Berger en Bernardo (1992c)) en waarskynlikheids ooreenstemmende priors (soos voorgestel deur Datta en Ghosh (1995)), is. Aangesien al die voorgestelde modelle goed gevaar het vir die bepaling van toleransieintervalle, is die unieke en buigsame kenmerke van die Bayesiaanse simulasiemetode geïllustreer.

Research Outputs

Research results obtained for inclusion in this thesis, have thus far led to the following research outputs:

Peer Reviewed Journal Publications:

VAN DER MERWE, A.J. and HUGO, J. (2007). "Bayesian Tolerance Intervals for the Balanced Two - Factor Nested Random Effects Model". *Test*, 16, 598-612.

HUGO, J. and VAN DER MERWE, A.J. (2009). "Bayesian Tolerance Intervals for the Balanced One - Way Random Effects Model with t - Distributed Measurement Errors". *South African Statistical Journal*, 43, 219-268.

Technical Reports

VAN DER MERWE, A.J., HUGO, J. and LAUBSCHER, N.F. (2004). *Tolerance Intervals for the Balanced Two - Factor Nested Random Effects Model*. Technical Report No. 326, Department of Mathematical Statistics, University of the Free State, South - Africa.

HUGO, J. and VAN DER MERWE, A.J. (2005). *Bayesian Tolerance Intervals for the Balanced One - Way Random Effects Model with Non - Standard Measurement Errors*. Technical Report No. 356, Department of Mathematical Statistics, University of the Free State, South - Africa.

VAN DER MERWE, A.J. and HUGO, J. (2008). *Tolerance Intervals for Averages of Observations from New or Unknown Batches in the Case of the Random Effects Model*. Technical Report No. 388, Department of Mathematical Statistics, University of the Free State, South - Africa.

International Conferences: Poster

VAN DER MERWE, A.J. and HUGO, J. (2004). *A Bayesian Approach to Tolerance Interval Estimation for Statistical Process Control in a Two - Way Nested Random Effects Model*. ISBA, 2004 World Meeting, Hotel Marina Del Rey, Vina Del Mar, Chile, May 23 - 27.

Other Conferences: Papers

HUGO, J. and VAN DER MERWE, A.J. (2003). *The Bayesian Approach to Tolerance Interval Estimation for Statistical Process Control*. SASA 50th Anniversary Conference, Caesar's Palace, Johannesburg, South - Africa, November, 2003.

General Scientific Paper

HUGO, J. and VAN DER MERWE, A.J. (2004). *The Bayesian Approach to Variance Component and Tolerance Interval Estimation for Statistical Process Control*. Free State Chapter of the South - African Statistical Association (SASA) Seminar, Department of Mathematical Statistics, University of the Free State, Bloemfontein, August, 27.

Acknowledgments

The author wishes to express his sincere appreciation and gratitude to the following persons:

Prof. Abrie van der Merwe who acted as promoter for this study. Prof. Abrie, thank you for your able guidance, support, constant encouragement and patience during the course of this study. It was a real privilege being able to work with you.

My wife Amoré and two sons Louis and Albert for all their sacrifices, love, support and constant encouragement during the course of this study.

My parents, Johan and Thea Hugo for their love, support and constant encouragement for the duration of this study.

My in-laws, Stephanus and Hettie Meiring for their constant encouragement to complete the study.

The Nelson Mandela Metropolitan University and in particular Prof. Andrew Leitch (Dean of the Faculty of Science) and Proff. Booth and Straeuli (current and past director of the School of Mathematical Sciences) for their support and constant encouragement.

Colleagues from the Department of Statistics at the Nelson Mandela Metropolitan University, Prof. and Mrs. IN Litvine, David Friskin, Warren Brettenny and Lulama Kepe for constant encouragement. In particular, I would like to thank Dr. Gary Sharp for his support, advice and constant encouragement, particularly during my final year of study.

The Department of Mathematical Statistics at the University of the Free State for financial support for two years and colleagues from the same department for valuable advice.

Colleagues from the Department of Mathematics at the Nelson Mandela Metropolitan University.

Prof. Nico Laubscher for making available his spun yarn data.

My Examiners whos' constructive criticism and suggestions have led to an improvement of this thesis.

All my family and friends, not named above, for support, prayers and constant encouragement.

My Heavenly Farther, without His love, kindness and blessings, this study would not have been possible.

Contents

Contents	ix
List of Tables	xiv
List of Figures	xvii
1 General Introduction	1
1.1 Introduction	1
1.2 Outline of the Research	24
2 Simple Linear Model - Univariate Normal Model	28
2.1 Introduction	29
2.2 The Normal Linear Model with One Variance Component	30
2.3 The Prior Distribution	31
2.4 The Posterior Distribution	33
2.5 Bayesian Simulation	35
2.6 Tolerance Intervals	42
2.6.1 One - Sided (α, δ) Tolerance Interval	44
2.6.2 Two - Sided (α, δ) Tolerance Interval	52
2.6.3 α - Expectation Tolerance Interval	54

2.6.4	Fixed - in - Advance Tolerance Intervals	60
2.6.5	Tolerance Intervals for the Difference Between Quantiles	66
2.6.6	Difference Between More Than Two α Quantiles	73
2.6.6.1	Multiple Comparisons Using Simultaneous Contrasts	73
2.6.6.2	Multiple Comparisons Using Pairwise Differences	76
2.6.6.3	An Example	78
2.7	Appendix A	81
3	The One - Way Random Effects Model	146
3.1	Introduction	146
3.2	The Bayesian Approach	149
3.3	The Variance Component Model	150
3.4	The Prior Distribution	151
3.5	The Posterior Distribution	151
3.6	Bayesian Simulation	154
3.7	Tolerance Intervals	163
3.7.1	One - Sided (α, δ) Tolerance Intervals	164
3.7.2	Two - Sided (α, δ) Tolerance Interval	169
3.7.3	α - Expectation Tolerance Interval	172
3.7.4	Fixed - in - Advance Tolerance Intervals	178
4	The Balanced One - Way Random Effects Model Continued	182
4.1	Introduction	183
4.2	Non - Informative Priors in the Case of Averages of Observations from New or Unknown Batches for the Balanced One - Way Random Effects Model . .	184

4.3	Reference and Probability Matching Priors	185
4.4	Calculation of the Tabulated Values (Coverage, Interval Length and Standard Deviation)	186
4.5	Theoretical Results for the Posterior Distribution of \tilde{q}	196
4.6	An Example	197
4.6.1	One - Sided (α, δ) Tolerance Intervals	198
4.6.2	Two - Sided (α, δ) Tolerance Interval	201
4.6.3	α - Expectation Tolerance Interval	204
4.6.4	Probability Matching Prior for the Content c of the Fixed - in - Advance Tolerance Interval	206
4.6.4.1	The Weighted Monte Carlo Method	207
4.7	Appendix B	213
5	Student t - Distributed Measurement Error Model	224
5.1	Introduction	225
5.2	A Bayesian Procedure for the Student t - Distributed Measurement Error Model	229
5.3	The Prior Distribution	233
5.4	The Posterior Distribution	236
5.5	Bayesian Computation	237
5.5.1	The Gibbs Sampler	238
5.6	Conditional Posterior Distributions	241
5.7	Marginal Posterior Distributions	244
5.8	A Prior Distribution for the Degrees of Freedom ν	246
5.9	Results and Discussions	249

5.10	Tolerance Intervals	259
5.10.1	α - Expectation Tolerance Intervals	260
5.10.2	One - Sided (α, δ) Tolerance Intervals	263
5.10.3	Two - Sided (α, δ) Tolerance Intervals	268
5.10.4	Fixed - in - Advance Tolerance Intervals	271
5.11	Checking for Outliers	274
5.12	Appendix C	279
6	Two - Factor Nested Random Effects Model	292
6.1	Introduction	293
6.2	The Data	296
6.3	The Balanced Two - Factor Nested Random Effects Model	299
6.4	The Bayesian Method	302
6.4.1	The Likelihood Function	302
6.4.2	The Prior Distribution	303
6.4.3	The Joint Posterior Distribution	306
6.4.4	The Conditional Posterior Distributions	307
6.4.5	Marginal Posterior Distributions	310
6.5	Tolerance Intervals	316
6.5.1	α - Expectation Tolerance Intervals	317
6.5.2	One - Sided (α, δ) Tolerance Intervals	323
6.5.3	Two - Sided (α, δ) Tolerance Intervals	327
6.5.4	Fixed - in - Advance Tolerance Intervals	329
6.6	Appendix D	335

7	Conclusions and Future Research	366
7.1	Conclusions	366
7.2	Future Research	370
	Bibliography	372
	List of Algorithms	389

List of Tables

Table 2.1	Example 1: Summary Data of Actual Flatness Measurements Obtained from Two Industrial Processes.	35
Table 2.2	Example 2: Summary Data of Actual Flatness Measurements Obtained from Three Industrial Processes.	44
Table 2.3	Fixed - in - Advance Tolerance Intervals for the First Sample Given in Table 2.1 for an Upper Specification Limit $s = 0.009$.	65
Table 2.4	Results Obtained for Comparing $g = 3$ 0.95 Quantiles q_d ($d = 1, 2, 3$) using Simultaneous Contrasts and Pairwise Differences for the Summary Data Given in Table 2.2.	78
Table 2.5	Results from the Simulation Study Performed to Investigate the Frequentist Properties of the Two Bayesian Multiple Comparisons Procedures.	80
Table 3.1	Amount of Active Drug per Tablet Measured in Milligrams.	155
Table 3.2	Comparative Results Between (0.95, 0.95) Tolerance Intervals and 0.95 - Expectation Tolerance Intervals for the Medicinal Tablets Data Given in Table 3.1.	177
Table 4.1	Coverage Rate of the 95% Credibility Intervals for \tilde{q} using the Two Described Methods and Different Values of ρ , b and k .	189

Table 4.2	Fixed - in - Advanced Tolerance Intervals for Averages of Observations from New or Unknown Batches for a Lower Specification Limit $s = 150.30$ mg.	211
Table 5.1	Measurements of Iron in Parts per Million (ppm) Determined by Emission Spectroscopy. (A Part with Under 225 ppm of Iron is Acceptable).	227
Table 5.2	Estimated Posterior Median Values for Variance Components, Functions of Variance Components and λ_{20} .	256
Table 5.3	95% Equal Tail Credibility Intervals and Interval Widths of Quantities of Interest.	257
Table 5.4	Median Values and 95% Upper Prediction Limits of the Predictive Part Distribution Determined for the Iron Data Given in Table 5.1.	263
Table 5.5	95% Upper Credibility Limits of the 0.95 th Quantiles of the Part Distributions Determined for the Iron Data Given in Table 5.1.	265
Table 5.6	Two - Sided (0.90, 0.95) Tolerance Intervals for Different Cases for the Data Given in Table 5.1.	270
Table 5.7	Posterior Median Values and 95% Equal Tail Credibility Intervals of the Content of the Interval $[225, \infty]$ for a Fixed - in - Advanced Lower Limit $s = 225$ ppm of Iron for Different Cases for the Data Given in Table 5.1.	273
Table 5.8	90% Equal Tail Credibility Intervals for λ_{i1} and $\log_{10}(\lambda_{i1})$ for the Case where ν was Simulated for the First Measurement of the Data Given in Table 5.1.	277
Table 5.9	90% Equal Tail Credibility Intervals for λ_{i2} and $\log_{10}(\lambda_{i2})$ for the Case where ν was Simulated for the Second Measurement of the Data Given in Table 5.1.	278

Table 6.1	The Physical Property of “Extension” of a Synthetic Yarn, Measured over the First 15 Consecutive Days of January 1995.	297
Table 6.2	Estimated Mean Values and 95% Equal Tail Credibility Intervals for the Two Estimated Unconditional Predictive Distributions Depicted in Figure 6.5.1. Obtained for the Spun Yarn Data Given in Table 6.1.	321
Table D1	The Physical Property of “Extension” of a Synthetic Yarn, Measured over 15 Consecutive Days of January, 1995. Eight Packages of Yarn were Sampled Each Day and Each Measurement Represents the Average of Five Replicate Samples per Package. Data Collected by Prof. N.F. Laubscher at SANS Fibres (Pty.) Ltd.	335

List of Figures

Figure 2.5.1	Marginal Posterior Distribution of μ (using the Student t - Distribution).	37
Figure 2.5.2	Marginal Posterior Distribution of σ_{ε}^2 (using the Inverse Gamma Distribution).	37
Figure 2.5.3	Histogram of the Estimated Marginal Posterior Distribution of σ_{ε}^2 .	40
Figure 2.5.4	Histogram of the Estimated Marginal Posterior Distribution of μ .	41
Figure 2.5.5	Estimated Marginal Posterior Distribution of μ (using the Rao Blackwell Method).	41
Figure 2.6.1.1	Histogram of the Estimated Marginal Posterior Distribution of the 0.95^{th} Quantile of $N(\mu, \sigma_{\varepsilon}^2)$ for the First Sample Given in Table 2.1.	46
Figure 2.6.1.2	Estimated Marginal Posterior Distribution of the 0.95^{th} Quantile of $N(\mu, \sigma_{\varepsilon}^2)$ for the First Sample Given in Table 2.1.	48
Figure 2.6.1.3	Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile of $N(\mu, \sigma_{\varepsilon}^2)$ for the First Sample Given in Table 2.1.	49
Figure 2.6.1.4	Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile of $N(\mu, \sigma_{\varepsilon}^2)$ for the First Sample Given in Table 2.1.	50

Figure 2.6.2.1	Constructing a Two - Sided $(0.95, 0.95)$ Tolerance Interval for the First Sample Given in Table 2.1.	53
Figure 2.6.3.1	Estimated Unconditional Predictive Distribution for the First Sample Given in Table 2.1. Constructed using Method 1.	57
Figure 2.6.3.2	Estimated Unconditional Predictive Distribution for the First Sample Given in Table 2.1. Constructed using Method 2.	58
Figure 2.6.3.3	Unconditional Predictive Distribution $p(y_f \mathbf{y})$ for the First Sample Given in Table 2.1. Constructed using Method 3.	59
Figure 2.6.4.1	Histogram of the Estimated Posterior Distribution of the Content of the Interval $[0.009, \infty]$, i.e. the Fraction of Process Measurements that Lie Above the Fixed - in - Advance Upper Specification Limit $s = 0.009$ for the First Sample Given in Table 2.1.	64
Figure 2.6.5.1	Estimated Marginal Posterior Distribution of the Difference Between Two Process Population Means $(\mu_1 - \mu_2) \mathbf{y}_1, \mathbf{y}_2$ for the Summary Data Given in Table 2.1.	67
Figure 2.6.5.2	Histogram of the Estimated Marginal Posterior Distribution of the Difference Between the Two 0.95 Quantiles $\gamma = (q_1 - q_2)$ for the Data Given in Table 2.1.	69
Figure 2.6.5.3	Estimated Marginal Posterior Distribution of the Difference Between the Two 0.95 Quantiles for the Summary Data Given in Table 2.1.	70
Figure 3.6.1	Histogram of the Estimated Marginal Posterior Distribution of σ_a^2 for the Data Given in Table 3.1.	158
Figure 3.6.2	Histogram of the Estimated Marginal Posterior Distribution of the Residual Variance Component σ_ε^2 for the Data Given in Table 3.1.	159

Figure 3.6.3	Histogram of μ and Estimated Marginal Posterior Distribution of the Target Value μ for the Medicinal Tablets Data Given in Table 3.1.	160
Figure 3.6.4	Estimated Marginal Posterior Distributions $p(a_i \mathbf{y})$ ($i = 1, \dots, 15$) of the Random (Batch) Effects for the Medicinal Tablets Data Given in Table 3.1.	162
Figure 3.7.1	Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.9)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 1 as Proposed by Wolfinger (1998).	166
Figure 3.7.2	Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.9)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 2.	168
Figure 3.7.3	Estimated Marginal Posterior Distribution of the $(1 - 0.9)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using the Alternative to Step ii.) of Method 2.	169
Figure 3.7.4	Constructing a Two - Sided $(0.90, 0.95)$ Tolerance Interval for the Medicinal Tablets Data Given in Table 3.1.	171
Figure 3.7.5	Histogram and Smooth Curve of the Estimated Unconditional Predictive Distribution for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 1.	174
Figure 3.7.6	Estimated Unconditional Predictive Distribution $p(y_f \mathbf{y})$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 2.	176

Figure 3.7.7	Histogram of the Estimated Posterior Distribution of the Content of the Interval $[-\infty, 150.30]$, i.e. the Fraction of Medicinal Tablets Containing an Amount of Active Ingredient Less than the Preselected Fixed - in - Advance Lower Specification Limit $s = 150.30$ mg for the Data Given in Table 3.1.	180
Figure 4.6.1	Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile \tilde{q} for the Distribution of the Average of Observations from New or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1. Method 1 was used.	200
Figure 4.6.2	Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile \tilde{q} for the Distribution of the Average of Observations from New or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1. Method 2 was used.	201
Figure 4.6.3	Construction of a Two - Sided $(0.95, 0.95)$ Tolerance Interval for the Average of Observations from New or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1.	203
Figure 4.6.4	Estimated Unconditional Predictive Distribution of the Average Weight of the Amount of Active Ingredient Present in Newly Manufactured or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 1.	206
Figure 4.6.5	Histogram of the Posterior Distribution of the Content of the Interval $[-\infty, 150.30]$, i.e. the Proportion of Batches with an Average Active Ingredient Weight that is Below the Specified Fixed - in - Advance Lower Limit $s = 150.30$ mg for the Medicinal Tablets Data Given in Table 3.1.	211
Figure 5.9.1	Unknown Conditional Posterior Distribution of the Degrees of Freedom ν and a Normal Distribution.	251
Figure 5.9.2	Histogram of the Estimated Marginal Posterior Distribution of μ .	252

Figure 5.9.3	Histogram of the Estimated Marginal Posterior Distribution of σ_{ε}^2 .	253
Figure 5.9.4	Histogram of the Estimated Marginal Posterior Distribution of σ_a^2 .	253
Figure 5.9.5	Histogram of the Estimated Marginal Posterior Distribution of $(\sigma_{\varepsilon}^2 + \sigma_a^2)$.	254
Figure 5.9.6	Histogram of the Estimated Marginal Posterior Distribution of $\frac{\sigma_{\varepsilon}^2}{(\sigma_{\varepsilon}^2 + \sigma_a^2)}$.	254
Figure 5.9.7	Histogram of the Estimated Marginal Posterior Distribution of λ for the 20 th Day.	255
Figure 5.10.1	Estimated Unconditional Predictive Part Distribution, x , if ν is Simulated. Determined for the Iron Data Given in Table 5.1.	262
Figure 5.10.2	Histogram of the Estimated Marginal Posterior Distribution of the 0.95 th Quantile of the Part Distribution for the Data Given in Table 5.1.	266
Figure 5.10.3	Histograms of the Estimated Marginal Posterior Distributions of the $\left[\frac{(1 \pm 0.95)}{2}\right]^{th}$ Quantiles of the Part Distribution for the Data Given in Table 5.1.	268
Figure 5.10.4	Constructing a Two - Sided (0.90, 0.95) Tolerance Interval for the Part Distribution of the Data Given in Table 5.1.	270
Figure 5.10.5	Histogram of the Estimated Posterior Distribution of the Content of the Interval $[225, \infty]$ for a Fixed - in - Advance Lower Limit $s = 225$ ppm of Iron for the Part Distribution with ν Simulated. Determined for the Data Given in Table 5.1.	273
Figure 6.4.1	Histogram of the Estimated Marginal Posterior Distribution of σ_{ε}^2 - Error Variance.	313
Figure 6.4.2	Histogram of the Estimated Marginal Posterior Distribution of σ_p^2 - Variance Between Packages Within Days.	314

Figure 6.4.3	Histogram of the Estimated Marginal Posterior Distribution of σ_d^2 - Variance Between Days.	315
Figure 6.4.4	Histogram of the Estimated Marginal Posterior Distribution of μ .	316
Figure 6.5.1	Estimated Unconditional Predictive Distributions:	319
Figure 6.5.2	Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.90)^{th}$ Quantile q^* .	325
Figure 6.5.3	Histograms of the Estimated Marginal Posterior Distributions of the $\left[\frac{(1 \pm 0.90)}{2}\right]^{th}$ Quantiles q_l and q_u for the Data Given in Table 6.1.	326
Figure 6.5.4	Construction of a Two - Sided $(0.90, 0.95)$ Tolerance Interval in the Case of $\bar{y}_{f..}^*$ for the Spun Yarn Data Given in Table 6.1.	328
Figure 6.5.5	Histogram of the Estimated Posterior Content of the Interval $[19.0, \infty]$ in the case of $\bar{y}_{f..}^*$, Determined for the Spun Yarn Data Given in Table 6.1.	331

Chapter 1

General Introduction

1.1 Introduction

The practice of quality engineering in the manufacturing environment is changing rapidly, with many companies facing higher demands with the introduction of new systems and new products. There is furthermore an increased pressure for quality engineers as well as other manufacturing activities to support the economic objectives and profitability of the firm. More tools are needed by quality engineers to cope with these changes and to meet the intense international competition (*Black Nembhard and Valverde - Ventura, 2003*).

Manufacturers are thus frequently required to verify that products meet certain specifications (*Hahn, 1982*). A standard approach to the problem is to compare for example measurements from a sample of parts, to a certain specification. Inferences can then be made from results obtained about the entire population of parts (*Wilson, Hamada and Xu, 2004*). Situations however sometimes arise when for example it may seem that specifications are not being met, when in fact they are. These situations usually occur when the available data are subject to measurement error (*Hahn, 1982*). It is therefore important to account for the measurement system being used to characterize production performance (*Wilson, Hamada and Xu, 2004*).

The eventual aim of any manufacturing process should be to have a process that produces data according to the model

$$y_{ij} = \mu_0$$

where the measurement takes on a fixed preset value without any statistical variation. Also, $i = 1, \dots, b$ and $j = 1, \dots, k$ (*Laubscher, 1996*).

Variation however, exists in every aspect of our lives and can be observed everywhere (*Tsiamirtzis, 2000*). As an example, people have different heights, weights, attitudes, ideas etc., all characteristics that vary (*Tsiamirtzis, 2000*). While sociological variation is a blessing (imagine everyone looking the same or having the same attitudes or ideas), variation in industry is blamed as the major cause of bad quality (*Tsiamirtzis, 2000*). In an industrial setting, a quality characteristic is measured on a product after manufacture (*Tsiamirtzis, 2000*). This manufactured product, will have some ideal target value for the quality characteristic being measured (*Tsiamirtzis, 2000*). In a dream world, a manufacturing process could produce perfect products with no variation at all, i.e. all products are manufactured at the ideal target value for the quality characteristic being measured (*Tsiamirtzis, 2000*).

As with sociological variation that can be observed in people, statistical variation is a fact of life in any manufacturing process. Several variation generating components may lurk in any manufacturing process, for example, sampling variation or sample - to - sample variation (*Laubscher, 1996*). There may also be variation as a result of experimental error (*Laubscher, 1996*). It is therefore important that sources of variation such as these, be incorporated into a suitable model (*Laubscher, 1996*). Finding and fitting the simplest model incorporating the relevant sources of variation should thus form part of the continuous improvement program in the life of any manufacturing process (*Laubscher, 1996*).

Variance component models are appropriate in settings where variability and multiple sources of variability occur (*Wolfinger, 1998*). These suitable variance component

models are frequently used in quality control, since these models adequately handle multiple sources of variability (*Wolfinger, 1998*).

Once a suitable variance component model is selected, a key response that conveys information about the quality of a product can be measured. These measurements are then used to estimate model parameters, either by a single number (point estimate) or by a range of scores (interval estimate). Statistical intervals properly calculated from sample data, are likely to be substantially more informative to decision makers than obtaining a point estimate alone, and, are usually a great deal more meaningful than statistical significance or hypothesis tests. These statistical intervals are therefore of paramount interest to practitioners and thus management (*Van der Merwe and Hugo, 2007*).

Statistical intervals computed based on a random sample have wide applicability, since uncertainty about a scalar quantity associated with a sampled population can be quantified (*Krishnamoorthy and Mathew, 2009*). Since there are three types of commonly used intervals, the type of interval to be computed, will strongly depend on the underlying problem or application (*Krishnamoorthy and Mathew, 2009*). Bounds for an unknown scalar population parameter, such as the population mean or population standard deviation, are estimated using a confidence interval which is calculated using a random sample obtained from this population (*Krishnamoorthy and Mathew, 2009*). If for example a 95% confidence interval has to be estimated for a population mean, it can be interpreted as follows: If the 95% confidence interval is computed repeatedly from independent samples from this population, then in the long run, 95% of the computed intervals will contain the true value of μ (*Krishnamoorthy and Mathew, 2009*). If bounds for one or more future observations from a univariate sampled population are required, a prediction interval based on a random sample is used (*Krishnamoorthy and Mathew, 2009*). A prediction interval has an interpretation similar to that of a confidence interval, but is meant to provide information concerning a single value only (*Krishnamoorthy and Mathew, 2009*). Suppose now a selected sample is to

be used to conclude whether or not, for example, 95% of a population are below a specified threshold. Neither confidence - nor prediction intervals can be used to answer this question, since confidence intervals are concerned with, for example means, and prediction intervals with single values only (*Krishnamoorthy and Mathew, 2009*). In cases like this, tolerance intervals, and to be more specific to this case, an upper tolerance limit based on a random sample, is required (*Krishnamoorthy and Mathew, 2009*). These tolerance intervals, are intervals which are expected to contain a specified proportion (or more) of the sampled population (*Krishnamoorthy and Mathew, 2009*). Therefore, in contrast to confidence intervals which provide information about an unknown population parameter, a tolerance interval provides information on the entire population (*Krishnamoorthy and Mathew, 2009*). To be more specific, for a given confidence level, a tolerance interval is expected to capture a certain proportion or more (the content) of the population (*Krishnamoorthy and Mathew, 2009*). In order to obtain a tolerance interval, it is therefore required that the content and confidence level be specified (*Krishnamoorthy and Mathew, 2009*).

In any production process, designers will specify tolerances or externally determined specification limits. These tolerances or externally determined specification limits are specified for various characteristics. These characteristics are based on considerations of requirements for fit, or function, in use, or in subsequent levels of assembly. The dimensions within which a produced part should fall in order to be acceptable, is a typical example (*Easterling, Johnson, Bement and Nachtsheim, 1991*). To protect against measurement error and to keep the production facility on its toes, designers sometimes specify tolerance limits with an interval width less than the width of the true required tolerance limits. Since these ad hoc tolerances may impose undue costs due to scrap or rework, it is desirable to take a more systematic look at the determination of tolerances, taking measurement error as well as other sources of variation into account (*Easterling, Johnson, Bement and Nachtsheim, 1991*). Three important research questions should therefore be asked. These questions as proposed by Wolfinger (1998)

are as follows:

1. Assuming the manufacturing process is in control, can a limit t be found such that 90% of future measurements are greater than t with 95% probability?
2. Can an interval (t_ℓ, t_u) be constructed so that a new observation from one of the original parts will fall in (t_ℓ, t_u) with 95% probability?
3. What fraction of the process measurements lie above some preselected specification limit s , and how much uncertainty is associated with this estimated fraction?

The three research questions proposed by Wolfinger (1998), and many similar ones, can be addressed by the use of either classical Shewhart variable control charts or tolerance intervals. A classical Shewhart variable control chart harnesses information about the quality of a product by means of a pair of control charts. Both of these charts, one for the average and one for the variation, have its own 3σ control limits (Laubscher, 1996). This methodology was developed by Walter A. Shewhart (1931) and further developed by many other later researchers. These include Duncan (1974), Ryan (1989), Cryer and Ryan (1990) as well as Roes, Does and Schurink (1993).

A Shewhart variable control chart is based on the following model

$$y_{ij} = \mu + \varepsilon_{ij}$$

where

y_{ij} represents the j^{th} item sampled on the i^{th} period, μ represents a fixed target value, ε_{ij} denotes random variation about zero, $i = 1, \dots, b$ and $j = 1, \dots, k$ (Laubscher, 1996).

From this model, natural process limits (also called natural tolerance limits) can easily be obtained by determining $\mu \pm 3\sigma$ for normal populations. These natural process limits will then include a stated fraction of the individual parts in a population and are then

compared to the specification limits determined externally to see if a manufacturing process is in control (*Nazar and Shwartz, 2010*). Standard Shewhart control charts unfortunately only allow for within sample variation (*Nazar and Shwartz, 2010*). As a result, different models allowing for more sources of variation may provide more satisfactory results.

Tolerance intervals on the other hand, can also be used to address the three research questions as proposed by Wolfinger (1998). These tolerance intervals can be determined for variance component models, thus allowing for the inclusion of more sources of variation. The construction of tolerance intervals has a rich history dating back over half a century (*Wolfinger, 1998*). For reviews on this research see Wilks (1941), Wald (1942), Guttman (1970), Zacks (1971), Mee and Owen (1983), Mee (1984a and b), Miller (1989), Hahn and Meeker (1991), Bhaumik and Kulkarni (1991, 1996) and Vangel (1992). More recently, Wolfinger (1998) proposed the estimation of tolerance intervals using a one - way variance component model and Bayesian simulation. Wolfinger (1998) also pointed out that the frequentist analysis of tolerance intervals can become quite complex, even for balanced one - way random effects models. Furthermore, the frequentist analysis differs depending on the kind of tolerance interval and particular model under consideration. Also, based on work done by Weerahandi (1993) (see also Weerahandi, 1995), Krishnamoorthy and Mathew (2004) introduced one - sided tolerance limits for balanced and unbalanced one - way random models using the generalized confidence interval approach.

Three kinds of commonly used tolerance intervals address the three research questions proposed by Wolfinger (1998) respectively. These are:

1. The (α, δ) tolerance interval, where α represents the content (the proportion of the population to be contained by the interval) and δ represents the confidence (the reliability of the interval). Both α and δ lie between 0 and 1 and are typically assigned values of 0.90, 0.95 or 0.99 (*Wolfinger, 1998*).

2. The α - expectation tolerance interval, where α represents the expected coverage of the interval. Again, α is measured on a probability scale and is typically set to a value close to 1. In contrast to the (α, δ) tolerance intervals, the α - expectation tolerance interval focuses on prediction of one or a few future observations from the process and consequently tend to be narrower than the corresponding (α, δ) intervals (*Wolfinger, 1998*).
3. The fixed - in - advance tolerance interval, in which the interval is constant and one wishes to estimate the proportion of process measurements it contains. Fixed - in - advance intervals invert the prediction problem by considering the content of predetermined bounds (*Wolfinger, 1998*).

All three kinds of tolerance intervals can take the following forms: lower limit (t_ℓ, ∞) , an upper limit (∞, t_u) , or a two - sided limit (t_ℓ, t_u) . For further details about confidence intervals and tolerance limits see Hahn and Meeker (1991) and Wolfinger (1998). These tolerance intervals will address the statistical problem of inference about the quantiles of a probability distribution that is assumed to adequately describe a process (*van der Merwe and Hugo, 2007 and Wolfinger, 1998*).

As mentioned earlier, variance component models allowing for various sources of variation are needed to estimate the tolerance intervals mentioned. These variance component models are needed to assess the manufacturing process's performance when the measurement error variance depends on the true characteristics of for example the parts being measured, and are frequently used in quality control (*Wilson, Hamada and Xu, 2004*). Variances are usually estimated for balanced data using the minimum variance unbiased estimators (MVUE's). These MVUE's are based on the sums of squares appearing in the analysis of variance (ANOVA) table. MVUE's however do not exist for unbalanced data, since the sums of squares from the ANOVA table are not sufficient statistics (*Chaloner, 1987*). Searle (1979) presented various other estimators. For example, maximum likelihood estimators (MLE's), restricted maximum likelihood estimators (REML's), minimum norm quadratic unbiased estimators (MINQUE's),

minimum variance quadratic unbiased estimators (MIVQUE's), as well as several more variations of these approaches. The analysis of variance (ANOVA) estimators that are obtained by equating mean squares to their expected values is also another common approach (*Chaloner, 1987*). The Bayesian methodology can also be used to assess a manufacturing process's performance and thus provides a flexible alternative to process assessment through variance component and interval estimation.

For variance component models, most authors assume that the production or part measurements, x_i ($i = 1, \dots, b$), follow independent $N(\mu, \sigma_p^2)$ distributions, and that ε_{ij} ($i = 1, \dots, b$ and $j = 1, \dots, k$) also follow independent $N(0, \sigma_\varepsilon^2)$ distributions. This model has been considered in a variety of contexts. Hahn (1982) estimated the proportion of parts that meet specification under the assumption that the residual variance σ_ε^2 was known. Jaech (1984) also considered the case where the error variance σ_ε^2 was known. Tolerance intervals for the proportion of parts meeting specification were then estimated. Mee (1984b) also estimated tolerance intervals, but considered the cases where σ_ε^2 was known, the ratio of $\sigma_\varepsilon^2/\sigma_p^2$ was known and the case where the ratio of the variances was estimated through repeated measurements. More complicated models for the parts distribution (e.g. random effects, random coefficients and mixed effects) were considered by Wang and Iyer (1994). They also calculated tolerance intervals. Note also that these authors all used a frequentist perspective to approach the variance components problem.

In industry, prior information about the manufacturing process is usually available in abundance (*Tsiamirtzis, 2000*). The Bayesian approach therefore serves as an appealing alternative to the classical approach of variance component and tolerance interval estimation, since careful use of this prior information is available only through a Bayesian scheme (*Tsiamirtzis, 2000*).

In a letter dated 1763, Mr. Richard Price sent an essay which he found amongst the papers of the late Rev. Mr. Thomas Bayes to a Mr. John Canton. The essay was published in 1763 in the Philosophical Transactions of the Royal Society of London and

was titled “*An Essay Towards Solving a Problem in the Doctrine of Chance*” (Bellhouse, 2004). In this essay, Mr. Bayes explained how to make statistical inferences that build upon earlier understanding and information of a phenomenon, and how to properly join that understanding to update the degree of belief with the use of current data. The past understanding was called the “prior belief” and the new results were known as the “posterior belief” (Bayes, 1763). This updating process is called Bayesian Inference. In addition to Bayes’s publications, the work of Jeffreys (1939), James and Stein (1961) and the introduction of the Gibbs sampling method by Geman and Geman (1984), just to name a few, have led to an increase in the use of Bayesian statistics.

The goal of a Bayesian analysis is to derive the posterior distribution of a specific parameter (θ) given the data (\mathbf{y}), written as $p(\theta|\mathbf{y})$. Bayes’s theorem is a conditional probability statement which proves that $p(\theta|\mathbf{y})$ is proportional to the sampling distribution for the data, $p(\mathbf{y}|\theta)$, multiplied by an independent probability distribution for the parameter, $p(\theta)$ (independent, in this case, of the specific data, \mathbf{y}) (Wade, 2000).

In this relationship, Bayesians have named $p(\theta)$ the prior distribution for the parameter θ and $p(\theta|\mathbf{y})$ the posterior distribution for the parameter θ (in the sense that it summarizes what is known about θ prior and posterior to the examination of the data \mathbf{y}). In this context, the sampling distribution for the data, $p(\mathbf{y}|\theta)$, is often referred to as the likelihood function (Wade, 2000). The likelihood function of a set of observations Y_1, Y_2, \dots, Y_n , is their joint probability density function when viewed as a function of the unknown parameter, say θ , which indexes the distribution from which the Y_i ’s were generated. The likelihood function is denoted by $L(\theta|y_1, \dots, y_n) = L(\theta|\mathbf{y})$, and the probability density function is represented by $f(\mathbf{y}|\theta)$.

The Bayes rule for continuous random variables is then expressed as:

$$p(\theta|\mathbf{y}) = \frac{f(\mathbf{y}|\theta)p(\theta)}{f(\mathbf{y})}$$

where $f(\mathbf{y})$ represents the marginal distribution of Y .

If $\mathbf{y} = \{y_1, \dots, y_n\}$ represents a sample from the conditional distribution of the variable Y , then $f(\mathbf{y}|\theta) = L(\theta|\mathbf{y})$, and the Bayes Rule can be expressed as:

$$p(\theta|\mathbf{y}) = \frac{f(\mathbf{y}|\theta)p(\theta)}{f(\mathbf{y})} \propto f(\mathbf{y}|\theta)p(\theta)$$

In more formal terms,

$$p(\theta|\mathbf{y}) = cL(\theta|\mathbf{y})p(\theta)$$

where c represents the normalization constant.

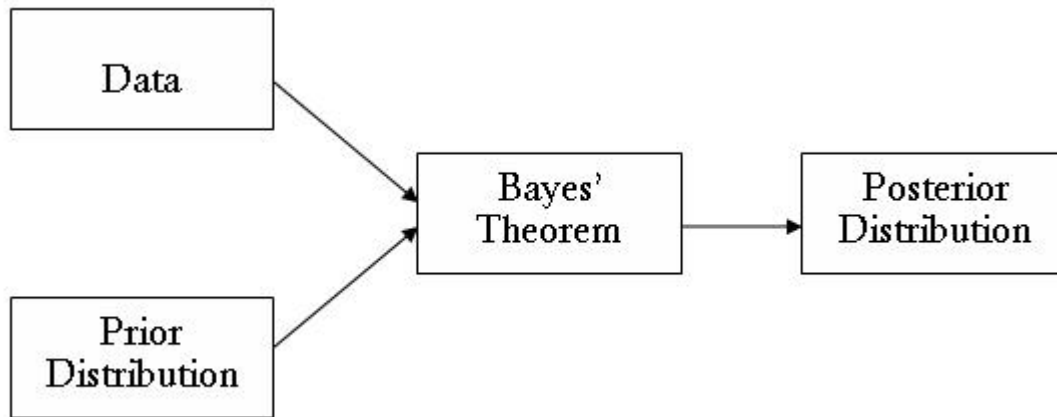
To calculate c , one needs to calculate the entire distribution of $f(\mathbf{y}|\theta) \times p(\theta)$, so one automatically calculates the entire distribution for $p(\theta|\mathbf{y})$ as well (*Wade, 2000*). Therefore, one usually speaks in terms of the posterior and prior distributions. All statistical inference is then based on the posterior distribution. The mean of the posterior distribution can serve as a point estimate for the parameter. Uncertainty in the point estimate is expressed directly in the posterior distribution and can be summarized either as percentiles of the posterior distribution or as what is termed the highest posterior density interval (*Wade, 2000*).

To calculate the posterior distribution, one has to integrate the product of the prior distribution and the likelihood function. In some simple cases, the integral can be calculated directly (in these cases it is said to have an analytical or “closed - form” solution). If no analytical solution is available, the integration can be done by numerical methods (*Wade, 2000*).

In layman’s terms, conventional statistical analyses (also called frequentist or classical statistics) calculate the probability of observing data given a specific value for a parameter, such as the value of a parameter in the case of a null hypothesis (*Wade, 2000*). Classical statistical methods therefore use sampling distributions to calculate probabilities of observing data given specific values of parameters, and as a result, use these sampling distributions directly (*Wade, 2000*). This can be illustrated using a p - value. The p - value represents the probability of observing data as extreme or more extreme than the data that were observed, given that the null hypothesis is true, on

repeated sampling of the data (*Wade, 2000*). The sampling distribution can also be used to estimate a frequentist parameter by calculating the likelihood function (*Wade, 2000*). This likelihood function is formed by calculating the probability of observing the data for every possible value of the parameter (*Wade, 2000*). In frequentist or classical statistics, this function is then interpreted to represent the relative likelihood of different parameter values and not the probability of different parameter values (*Wade, 2000*). This relative likelihood of different parameter values represents the probability of observing the data given these different parameter values (a maximum likelihood estimate is the value of the parameter that maximizes the probability of the observed data i.e. the peak of the likelihood function) (*Wade, 2000*).

In contrast to classical statistics, Bayesian methods calculate the probability of the value of a parameter given the observed data (*Wade, 2000*). In simple terms, what is known is the data, the value of the parameter is unknown. Therefore, Bayesian inference focuses on what the data reveals about this unknown parameter (*Lindley, 1986*). As with classical statistics, Bayesian methods also make use of the likelihood function, but utilize it in a different way (*Wade, 2000*). Given a prior distribution for the unknown parameter, a posterior probability distribution for this unknown parameter is calculated as the integral of the product of the likelihood function with the given prior distribution (*Wade, 2000*). The given prior distribution represents the probability distribution of the unknown parameter before consideration of the data, while the posterior distribution represents the probability distribution of this parameter after taking the data into consideration (*Wade, 2000*). This can be illustrated by the following schematic representation of this process (*Van Boekel et.al., 2004*).



All statistical inference about the unknown parameter is then made from the posterior distribution (*Wade, 2000*).

For the construction of tolerance intervals in particular, *Wolfinger (1998)* examined the differences between the Bayesian and frequentist approaches. These are summarized as follows:

The first feature involves the analysis of the intervals. The Bayesian method expresses all uncertainty about model parameters in terms of probability densities, with probability statements representing a degree of certainty (*Wolfinger, 1998*). In contrast, the frequentist approach directly regards some parameters as fixed and unknown quantities, and the confidence statements about them are typically interpreted in terms of long-run frequencies (*Wolfinger, 1998*).

- This difference is particularly apparent in the interpretation of δ in the (α, δ) tolerance interval. Taking the lower limit case as an illustration, both the Bayesian and the frequentist methods envision a true $(1 - \alpha)^{th}$ quantile and attempt to place a lower limit on it with confidence δ . The frequentist approach constructs a $100(\delta)\%$ lower confidence limit for the $(1 - \alpha)^{th}$ quantile, and surmises that confidence limits constructed in a similar manner will be greater than the true quantile $100(\delta)\%$ of the time. The Bayesian method, in contrast, constructs the posterior density of

the $(1 - \alpha)^{th}$ quantile conditional on the observed data and any prior information. Using this posterior density, the Bayesian constructs an interval containing the true $(1 - \alpha)^{th}$ quantile with subjective probability δ (Wolfinger, 1998).

- For α - expectation tolerance intervals, the Bayesian interpretation states that the interval actually obtained will contain the future observation with subjective conditional probability α . In contrast, the frequentist interpretation states that the constructed intervals will include the future observations with relative frequency α (Wolfinger, 1998).

The second distinction involves the use of prior information. The Bayesian approach formally incorporates prior information about model parameters in terms of prior density functions. Frequentist methods provide no such mechanism. Therefore, when prior information exists, the Bayesian method seems more reasonable (Wolfinger, 1998).

The third distinction is a practical one. The frequentist analysis of tolerance intervals for variance component models can become quite complex even for the balanced one-way random effects model. Frequentist analyses differ depending on the kind of tolerance interval and particular model under consideration. In contrast, the simulation-based Bayesian method can easily be applied to models with several variance components. The same analysis strategy can also be used for all three kinds of tolerance intervals (Wolfinger, 1998).

The advantages of the Bayesian approach over the frequentist approach are:

1. The Bayesian practitioner does not need to only use point estimates of variance components and other parameter values, since credibility and prediction intervals are easily obtained (Hugo, van der Merwe and Viljoen, 1997).

2. The indecision regarding the true values of the variance components is incorporated into the investigation through the use of an appropriate prior distribution (*Hugo, van der Merwe and Viljoen, 1997*).
3. The Bayesian approach provides a set of widely applicable mathematically tractable tools that are often more tailored to the requirements of users than the frequentist tools (*Jandrell and van der Merwe, 2007*).
4. Fewer mathematical problems with less proofs and theorems are associated with Bayesian methods (*Jandrell and van der Merwe, 2007*).

The problems involved with the implementation and use of the Bayesian method are:

1. The Bayesian methodology is computer intensive since integration in several dimensions is required to obtain the posterior distribution. The development of increasing computer power and numerical - integration techniques (such as Markov chain Monte Carlo methods), facilitate the use of a full analysis (*Hugo, van der Merwe and Viljoen, 1997*). However, the burden of proof rests on the monitoring of stochastic convergence and the mixing of the Markov chain (*Jandrell and van der Merwe, 2007*).
2. A prior belief about the unknown parameters needs to be set out in the form of a probability distribution. This step in any Bayesian analysis is often difficult to execute and is very controversial. This represents one of the reason for using non - informative priors in practical cases (*Hugo, van der Merwe and Viljoen, 1997*).

From the first problem mentioned with the implementation and use of the Bayesian methodology, one can see that to make appropriate inferences in a Bayesian analysis, the marginal posterior distributions and predictive densities are needed. Due to the complexity of the joint posterior distribution however, it is impossible to obtain these

marginal posterior densities analytically. It is also very difficult to obtain these marginal posterior densities numerically, due to the high number of unknowns (*van der Merwe, Pretorius and Meyer, 2003*). It is therefore recommended that a Monte Carlo simulation procedure be used to estimate these marginal posterior densities of the unknown parameters and predictive densities of future observations. A brief overview and some history of Markov chain Monte Carlo simulation will now be provided.

In recent years, statisticians have been increasingly drawn to Markov chain Monte Carlo (MCMC) simulation to examine more complex systems than would otherwise be possible (*Chib and Greenberg, 1995*). To explain Markov chain Monte Carlo simulation, suppose that we wish to generate a sample from a posterior distribution $p(\theta|\mathbf{y})$ for $\mathbf{y} \in \mathcal{R}^k$ but cannot do this directly. The key to Markov chain simulation is to create a Markov process whose stationary distribution is a specified $p(\theta|\mathbf{y})$, and run the simulation long enough so that the distribution of the current draws is close enough to the stationary distribution. Once the simulation algorithm has been implemented, it should be iterated until convergence has been approximated. Remember however that the draws are only regarded as a sample from the posterior distribution $p(\theta|\mathbf{y})$ once the effect of the fixed starting value is so small that it can be ignored (*Chib and Greenberg, 1995*).

Credit for inventing the Monte Carlo method often goes to Stanislaw Ulam, a Polish born Mathematician who worked for John von Neumann on the United States' Manhattan Project during World War II (Ulam is primarily known for inventing the hydrogen bomb in 1951 with Edward Teller). Although Ulam did not invent statistical sampling, he did however invent the Monte Carlo method in 1946 while pondering the probabilities of winning a card game of solitaire (*Eckhardt, 1987*).

As mentioned, Ulam did not invent statistical sampling. Statistical sampling had been employed before to solve quantitative problems with physical processes such as dice tosses and card draws, and W.S. Gossett, who published under the penn name "Student", also randomly sampled from height and middle finger measurements of 3000

criminals to simulate two correlated normal distributions (*obtained from the Internet website: <http://www.contingencyanalysis.com>*). Ulam did however recognize the potential for the newly invented electronic computer to automate such sampling. He developed algorithms for computer implementations and explored means of transforming non - random problems into random forms that would facilitate their solution via statistical sampling. This work was done while Ulam was working with John von Neumann and Nicholas Metropolis. Their work transformed statistical sampling from a mathematical curiosity into a formal methodology that would be applicable to a wide variety of problems. This new methodology was named after the casinos of Monte Carlo by Metropolis and the first paper on the Monte Carlo method was published by Metropolis and Ulam in 1949 (*information for this paragraph was obtained from the Internet website: <http://www.contingencyanalysis.com> and Metropolis and Ulam, 1949*).

Metropolis continued his work, and together with Rosenbluth, Rosenbluth , Teller and Teller (1953), developed the Metropolis-Hastings (M-H) algorithm which was later generalized by Hastings (1970) (*Chib and Greenberg, 1995*). Although the M-H algorithm has been used extensively in physics, it was little known to statisticians until recently, despite the paper by Hastings (1970) (*Chib and Greenberg, 1995*). The Metropolis-Hastings algorithm is extremely useful and versatile and applications are steadily appearing in literature (*Chib and Greenberg, 1995*).

The Gibbs sampling algorithm, a special case of the Metropolis-Hastings algorithm, is one of the best known Markov chain Monte Carlo methods (*Chib and Greenberg, 1995*) and will be discussed in chapter 5.

As can be seen from the second problem mentioned with the implementation and use of the Bayesian method, an integral part of traditional Bayesian analysis is the assignment of prior distributions to the unknown parameters in the model (*van der Merwe, Pretorius, Hugo and Zellner, 2001*). A Prior probability or distribution can be viewed as a description of what is in fact known about a parameter in the absence of data (*Jandrell and van der Merwe, 2007*). The choice of a prior distribution is a very difficult

and controversial step in any Bayesian analysis, since the information contained in the prior distribution, which is supposed to represent what is known about the unknown parameters before the data is available, is combined with the information supplied by the data, through the likelihood function, to form the joint posterior distribution of the parameters given the data (*Box and Tiao, 1973 and Gianola and Fernando, 1986*). It must however be stated that according to Box and Tiao (1973) some prior knowledge is employed in all inferential systems. Box and Tiao (1973) used a simple example to explain this statement. *“For example , a sampling theory analysis, using statistical methods in scientific investigation is made, as is a Bayesian analysis, as if it were believed a priori that the probability distribution of the data was exactly normal, and that each observation had exactly the same variance, and was distributed exactly independently of every other observation. But after a study of residuals had suggested model inadequacy, it might be desirable to reanalyze the data in relation to a less restrictive model into which the initial model was embedded. If non - normality was suspected, for example, it might be sensible to postulate that the sample came from a wider class of parent distributions of which the normal was a member. The consequential analysis could be difficult via sampling theory, but is readily accomplished in a Bayesian framework. Such an analysis allows evidence from the data to be taken into account about the form of the parent distribution besides making it possible to assess to what extent the prior assumption of exact normality is justified.”*

Two types of prior information are distinguished: Data based and non - data based. Data based prior information is obtained in a scientific manner from prior experimentation, while non - data based prior information is based on subjective personal opinions or beliefs and theoretical considerations. It seems to be the use of non - data based prior information to which orthodox frequentists object (*Carriquiry, 1989*).

As just mentioned, the main criticism and controversy surrounding the choice of a prior distribution, and as a result the whole Bayesian approach, is build on the principle of subjectivity, since one persons prior belief about an unknown parameter, before any

data is observed, is different from another person's (Van Boekel, et.al., 2004). Different prior beliefs about a parameter will therefore naturally lead to different posterior distributions which will be used for subsequent analyses. Subjectivity however, is actually a strength of the Bayesian methodology, since it allows for an examination of a range of posterior distributions (Van Boekel, et.al., 2004).

Even though the choice of a prior distribution might have been, and still is, a controversial and much criticized step in a Bayesian analysis, continuous research into the specification of prior distributions has assisted in reducing much of the controversy surrounding this topic (Van Boekel, et.al., 2004). The use of non - informative -, reference -, and probability matching priors have also greatly assisted in eliminating some of the controversy and criticism surrounding the choice of a prior distribution to be used for a Bayesian analysis. Non - informative prior distributions, as the name suggests, are prior distributions that play a minimal role in the posterior distribution. If prior information is vague and unsubstantial, the prior information will carry negligible weight and the posterior distribution will in effect be based entirely on information contained in the data as expressed in the likelihood function (Van Boekel, et.al., 2004). Non - informative prior distributions can be developed through the use of reference priors or probability matching priors (Jandrell and van der Merwe, 2007).

Conceptual and theoretical methods devoted to the identification of appropriate procedures for the formulation of objective prior distributions, have been studied extensively (Berger, Bernardo and Sun, 2009). One of the most utilized approaches to developing objective prior distributions, has been reference analysis introduced by Bernardo (1979) and further developed by Berger and Bernardo (1989, 1992a, 1992b, 1992c) and Sun and Berger (1998). Objective Bayesian inference is produced by reference analysis, in the sense that inferential statements depend only on the assumed model and the available data. Therefore, in a certain information - theoretic sense, the prior distribution used to make an inference is least informative (Berger, Bernardo and Sun, 2009). Informative - theoretical concepts are used in reference analysis to

make precise the idea of an objective prior which should be maximally dominated by the data, in the sense of maximizing the missing information about the parameter (Berger, Bernardo and Sun, 2009). The idea behind reference priors is therefore to formalize a function that maximizes some measure of distance between the prior and the posterior as data observations are made. By maximizing the distance, the data is allowed to have the maximum effect on the posterior estimates. More formally, the idea is to maximize the expected divergence of the posterior distribution relative to the prior. The expected posterior information about θ is therefore maximized when the prior density is $p(\theta)$. In some sense this implies therefore that $p(\theta)$ is the least informative prior about θ . The reference prior is defined in the asymptotic limit, i.e. the limit of the priors are considered as the data points approach infinity (Berger and Bernardo, 2009).

There is growing evidence, mainly through examples, suggesting that the reference prior algorithm by Berger and Bernardo (1992c) provides sensible answers from a Bayesian point of view (van der Merwe, 2000). More limited evidence also suggests that frequentist properties from reference posteriors are asymptotically "reasonable" (van der Merwe, 2000). As mentioned, the reference prior is motivated by an asymptotic argument, that of maximizing asymptotic missing information (van der Merwe, 2000). In other words, the concept behind the use of reference prior distributions is that it maximizes the expected posterior information about θ when the prior density is $p(\theta)$. In the case of scalar parameters, the Jeffreys' prior which has the feature of providing accurate frequentist inference, is used as reference prior (van der Merwe, 2000). For multiparameter settings the situation is much less clear and relatively complicated, since the reference prior algorithm depends on the ordering of the parameters and how the parameter vector is divided into sub - vectors (van der Merwe, 2000). Berger and Bernardo (1992c) however suggested that this problem can be overcome if one allows multiple groups "ordered" in terms of inferential importance (van der Merwe, 2000). The reference prior for the implied conditional problem is then determined

through a succession of analyses (*van der Merwe, 2000*). In particular, Berger and Bernardo (1992c) recommended that the reference prior be based on having each parameter in its own group (*van der Merwe, 2000*). In doing so, each conditional reference prior will be only one - dimensional (*van der Merwe, 2000*). In order to obtain a reference prior for a certain ordering of the parameters, the Fisher information matrix must first be obtained. For more information as well as a formal definition of reference priors, see Berger and Bernardo (2009).

Probability matching priors, on the other hand, are priors for which the posterior probabilities of certain aspects are exactly or approximately equal to their coverage probabilities (*Sweeting, 2005*) and was found to be appealing to both frequentists and Bayesians alike (*Ghosh et.al., 2008*). A probability matching prior is therefore a prior distribution under which the posterior probabilities of certain regions coincide either exactly or approximately with their coverage probabilities (*Datta and Sweeting, 2005*). As a simple example, consider an observation X from a $N(\theta, 1)$ distribution where the parameter θ is unknown. If an improper uniform prior π is taken over the real line of θ , then the posterior distribution of $Z = \theta - X$ is exactly the same as its sampling distribution. This implies that $pr_{\pi}\{\theta \leq \theta_{\alpha}(X)|X\} = pr_{\theta}\{\theta \leq \theta_{\alpha}(X)\} = \alpha$, where $\theta_{\alpha}(X) = X + Z_{\alpha}$ and Z_{α} represents the α quantile of a standard normal distribution. This implies that every credible interval based on the pivotal quantity Z with posterior probability α , is also a confidence interval with confidence level α . The uniform distribution therefore represents a probability matching prior. The use of probability matching priors will therefore ensure exact or approximate frequentist validity of Bayesian credible regions (*Datta and Sweeting, 2005*). For a parametric function $t(\theta)$, Datta and Ghosh (1995) derived the differential equation that a prior must satisfy in order for the posterior probability of a one - sided credibility interval and its frequentist probability to agree up to the order number $O(n^{-1})$, where n represents the sample size (*Jandrell and van der Merwe, 2007*). According to Datta and Ghosh (1995), this equation is identical to Stein's equation for a slightly different problem (see Stein, 1985). To illustrate the method for more

complex examples, suppose X_1, \dots, X_n are independently and identically distributed with density $f(x)$. Also suppose that θ represents a p - dimensional parameter vector given by $\theta = (\theta_1, \dots, \theta_p)$ (Datta and Ghosh, 1995). For θ , also consider a prior density $p(\theta)$ which has the following property of matching frequentist and posterior probability for a real - valued twice continuously differentiable parametric function $t(\theta)$:

Then

$$P_{\theta} \left[\frac{\sqrt{n}\{t(\theta) - t(\hat{\theta})\}}{\sqrt{b}} \leq Z \right] = P_{p(\theta)} \left[\frac{\sqrt{n}\{t(\theta) - t(\hat{\theta})\}}{\sqrt{b}} \leq Z \mid \mathbf{X} \right] + O_p(n^{-1}) \quad (1.1.1)$$

for all values of Z . In equation 1.1.1, $\hat{\theta}$ represents the posterior mode or maximum likelihood estimator of θ and b represents the asymptotic posterior variance of $\sqrt{n}\{t(\theta) - t(\hat{\theta})\}$ up to $O_p(n^{-\frac{1}{2}})$, $P_{\theta}(\cdot)$ represents the joint probability measure of $\mathbf{X} = [X_1, \dots, X_n]'$ under θ , and, $P_{p(\theta)}(\cdot \mid \mathbf{X})$ represents the posterior probability measure of θ under $p(\theta)$ (Datta and Ghosh, 1995). Priors such as $p(\theta)$ may be sought in an attempt to reconcile a frequentist and Bayesian approach (Peers, 1965), to find or validate a non - informative prior distribution (Berger and Barnardo, 1989) or to construct frequentist confidence sets (Stein, 1985) (Datta and Ghosh, 1995). To be more specific, Datta and Ghosh (1995) proved that the agreement between the posterior probability and the frequentist probability i.e. equation 1.1.1 holds if and only if the differential equation

$$\sum_{\alpha=1}^m \frac{\partial}{\partial \theta_{\alpha}} \{\eta_{\alpha}(\theta)p(\theta)\} = 0$$

where $p(\theta)$ represents the probability matching prior for the vector of unknown parameters, θ .

Furthermore, let

$$\nabla_t = \left[\frac{\partial}{\partial \theta_1} t(\theta), \dots, \frac{\partial}{\partial \theta_m} t(\theta) \right]'$$

and

$$\eta(\theta) = \left[\frac{F^{-1}(\theta)\nabla_t(\theta)}{\sqrt{\nabla_t'(\theta)F^{-1}(\theta)\nabla_t(\theta)}} \right] = [\eta_1(\theta), \dots, \eta_m(\theta)]'$$

From the above, it is clear that $\eta'(\boldsymbol{\theta})F(\boldsymbol{\theta})\eta(\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta}$ where $F^{-1}(\boldsymbol{\theta})$ represents the inverse of $F(\boldsymbol{\theta})$. The mentioned $F(\boldsymbol{\theta})$ represents the Fisher information matrix of $\boldsymbol{\theta}$ and $t(\boldsymbol{\theta})$ a continuously differentiable function of the parameter $\boldsymbol{\theta}$. (*Jandrell and van der Merwe, 2007, and van der Merwe, 2000*). Unlike uniform priors, a very important property of probability matching priors is that these priors always remain invariant under any one - to - one transformation of the parameters (*van der Merwe, 2000*). This probability matching criterion amounts to the requirement that the coverage probability of a Bayesian credible region is asymptotically equivalent to the coverage probability of the frequentist confidence region up to a certain order (*Ghosh, et.al., 2008*). Several probability matching criteria exist and are accomplished through either posterior quantiles, distribution functions, highest posterior density (HPD) regions or inversion of certain test statistics (*Ghosh, et.al., 2008*). As shown above, probability matching priors are obtained by solving certain differential equations (*Ghosh, et.al., 2008*). It must however be noted that probability matching priors based on posterior quantiles, distribution functions, HPD regions or inversion of certain test statistics need not always be identical (*Ghosh, et.al., 2008*). It may also happen that no prior exists which satisfies all four criteria (*Ghosh, et.al., 2008*). For the remainder of this study, reference priors as well as probability matching priors will be derived and used for some of the different models which will be discussed. For further information regarded probability matching priors, see Datta and Ghosh (1995).

The Bayesian approach to variance component estimation has been studied extensively for balanced data where $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$ (for $i = 1, \dots, b$ and $j = 1, \dots, k$). The Bayesian approach for balanced data was successfully implemented by Tiao and Tan (1965) and Box and Tiao (1973). These authors obtained some approximations for posterior densities of variance components and also derived some closed - form estimators for these components of variance. Hill (1977) developed exact and approximate Bayesian solutions for inference about variance components. The sampling properties of Bayesian and other estimators were investigated by Klotz, Milton and Zacks (1969).

Skene (1983) provided computational methods for obtaining marginal posterior densities of variance components for crossed - and nested models.

There is however little difference between the balanced and unbalanced case in a Bayesian context (*Chaloner, 1987*). Hill (1965) obtained exact and approximate posterior distributions for variance components in the balanced and unbalanced case. Hill (1965) also discussed and provided the necessary integrations. Chaloner (1987) provided a basic Bayesian approach for the unbalanced one - way variance components model using a non - informative prior distribution that is uniform on the intra-class correlation. Chaloner (1987) also illustrated that although the ANOVA estimators can easily be calculated by hand, the Bayesian estimators are generally much more efficient than these ANOVA estimators. In addition to these, Chaloner (1987) also provided a simulation study for the estimation of the ratio of the variance components and investigated the sampling properties of the highest posterior density regions for this ratio. It was also pointed out by the author that although these highest posterior density regions do not have the coverage probabilities of confidence intervals, they do lead to sensible interval estimates that are never empty. Analytical results for the estimation of a production process was provided by Hahn and Raghunathan (1988).

The Bayesian approach to variance component estimation was revisited after the development of Markov chain Monte Carlo methods by authors like Gelfand et.al. (1990), Sun et.al. (1996), Wolfinger (1998), Wolfinger and Kass (2000) and more recently Wilson et.al. (2004). Sun et.al. (1996) demonstrated that with the advent of powerful techniques such as importance sampling, Markov chain iterations and modern usage of Laplacian approximations, it became possible to provide detailed finite sample inference for many variance component models. Also, since the computations could be handled more efficiently, importance sampling, Laplacian approximations and the Gibbs sampler permitted the consideration of models and prior assumptions of high complexity. Sun et.al. (1996) also showed that the Bayesian estimates of the first stage parameters have excellent frequentist properties when uniform priors are assumed for

variance components. The authors also mentioned that if prior knowledge was available, these estimates could further be improved in a subjective sense, through the use of inverted chi - square distributions for the variance components. Interval estimates and posterior probabilities were also readily available.

Wolfinger (1998) used Markov chain Monte Carlo simulation techniques to generate a random sample from the joint posterior distribution of the mean and the variance parameters to construct a sample from other relevant posterior distributions. Wolfinger (1998) also presented a simulation based approach for determining Bayesian tolerance intervals in variance component models and illustrated that different kinds of tolerance intervals could be determined in a straightforward way. It was also pointed out that this methodology could easily be tailored to particular applications. Wolfinger and Kass (2000) indicated that although Gibbs sampling is easy to implement for balanced data using conjugate priors, simulating from full conditional posterior densities can become difficult for the analysis of unbalanced data with possibly non - conjugate priors. The authors therefore considered alternative Markov chain Monte Carlo methods and proposed and investigated a method for posterior simulation for a variance component model based on an independence chain. A default reference prior (Jeffreys' prior based on the restricted likelihood) was used (*Wolfinger and Kass, 2000*).

1.2 Outline of the Research

The purpose of this research study is to provide full Bayesian solutions to variance component and tolerance interval estimation for various variance component models. Reference priors as well as probability matching priors will be derived and then used for determining joint posterior distributions.

To make appropriate inference in Bayesian analyses however, marginal posterior distributions and predictive densities are needed. Due to the complexity of the joint

posterior distribution, it is impossible to obtain these marginal posterior densities and predictive densities analytically. Markov chain Monte Carlo (MCMC) simulation procedures will therefore be used to estimate these marginal posterior densities of the unknown parameters and predictive densities of future observations or averages of future observations. In some cases a well known Markov chain Monte Carlo simulation method known as the Gibbs sampler will be employed, while a weighted Monte Carlo simulation method will be proposed for the determination of fixed - in - advance tolerance intervals if probability matching prior distributions are used as prior for the content of these fixed - in - advance tolerance intervals.

In chapter 2, tolerance intervals will be determined for a simple linear model (univariate normal model) with one variance component. The usual univariate normal model given by

$$y_i = \mu + \varepsilon_i$$

will therefore be considered which postulates that a quantitative observation consists of a constant target value μ , plus random variation about this fixed target value (*Laubscher, 1996*). As mentioned, μ represents the fixed target value, y_i represents a single measurement of the response variable made on the i^{th} item ($i = 1, \dots, n$ where n represents the sample size) and ε_i denotes random variation about zero (*Laubscher, 1996*). It is usually assumed that the ε_i 's are independently normally distributed with unknown constant variance σ_ε^2 , i.e. $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ (*Laubscher, 1996*). If in this case the tolerance limits are defined as those limits that contain $100(\alpha)\%$ of the distribution of the quality characteristic, then the one - sided tolerance limit is simply the δ^{th} percentile of the quantile q of the $N(\mu, \sigma_\varepsilon^2)$ distribution given by

$$q = \mu + z_\alpha \sigma_\varepsilon$$

where z_α represents a standard normal ($N(0, 1)$) value for which the probability greater than z_α ($p(Z \geq z_\alpha)$) is at most equals to α .

It will also be shown that the Jeffreys' independence prior distribution given by

$$p(\mu, \sigma^2) \propto \sigma_\varepsilon^{-2}$$

is both a reference prior and a probability matching prior for this quantile.

In addition to the above, unconditional central moments for

$$q = \mu + z_\alpha \sigma_\varepsilon$$

will also be determined. This will be followed by the derivation of the posterior density and unconditional central moments for the difference between two quantiles used to determine the (α, δ) tolerance intervals. Multiple comparisons procedures for differences between more than two quantiles will also be provided and illustrated using summary data obtained from Hubele, et.al. (2005).

Chapter 3 will outline the methodology and methods used by Wolfinger (1998) for determining tolerance intervals for one - way random effects models using a non - informative prior. Since a random effects model will be discussed, two variance components will be considered. The method will be illustrated using a medicinal tablet manufacturing example.

In Chapter 4, the theory and methods proposed by Wolfinger (1998) will be extended to include tolerance intervals for averages of observations from new or unknown batches in the case of a random effects model. Reference and probability matching priors will be derived for the α^{th} quantile of the distribution of averages of observations from new or unknown batches. It will also be shown that a proposed prior distribution for the content of the fixed - in - advance tolerance interval, is a probability matching prior as well. The Bayesian simulation method will be illustrated using the same medicinal tablet manufacturing example as used in Chapter 3. A numerical experiment will be performed to investigate the frequentist properties of the Bayesian interval for the α^{th} quantile under the probability matching prior.

Chapter 5 will propose and discuss the estimation of Bayesian tolerance intervals for a balanced one - way random effects model with student t - distributed measurement

errors. A one - way random effects model with non - standard measurement errors will therefore be considered. For the estimation of the tolerance intervals, it was decided to use non - informative prior distributions for the parameters of interest, and a truncated exponential distribution as prior for the degrees of freedom ν . The procedure will be illustrated using an example obtained from Wilson, et.al. (2004). Given the nature of the problem, a well known Markov chain Monte Carlo simulation method known as the Gibbs sampler will be employed for the simulation process. In addition, since the assumption of Gaussian errors will be relaxed in the direction of the student t - family to accommodate for the possibility of outlying part measurements, it will be illustrated how the Bayesian method proposed for this specific case can be used to identify possible outlying part measurements.

In Chapter 6, tolerance intervals will be determined for a balanced two - factor nested random effects model. It will also be shown that the prior distributions proposed for the α^{th} quantile of the distribution of averages of k packages with r samples per package from any new or unknown day, and the content of the fixed - in - advance tolerance interval, are both probability matching priors. This Bayesian method will be illustrated using an example obtained from the Bellville works of SANS Fibres (Pty.) Ltd. in South Africa. The data was collected by Prof. Nico F. Laubscher, company statistician at the time of data collection at SANS Fibres.

Chapter 7 will conclude with a summary of the Bayesian methods used. Recommendations and suggestions for further research will also be provided.

The software package MATHWORKS MATLAB will be used to perform all calculations and simulations. Unless otherwise stated, and in section headings, all vectors in mathematical equations will be indicated by lower case letters typed in bold face, for example the vector \mathbf{a} compared to the non - vector a .

Chapter 2

Simple Linear Model - Univariate Normal Model

In this chapter, tolerance intervals will be determined for the univariate normal model using two Bayesian simulation methods. Exact and estimated marginal posterior distributions will be provided for the location - and variance parameters. It will also be shown that the proposed Jeffreys' independence prior is a reference prior as well as a probability matching prior for the α^{th} quantile of a $N(\mu, \sigma_\varepsilon^2)$ distribution. Similarly it will be shown that a prior distribution for the content of the fixed - in - advance tolerance interval, is also a probability matching prior. The posterior distribution of this content of the fixed - in - advance tolerance interval, can also be obtained using two illustrated Bayesian simulation methods. In addition, exact moments of the α^{th} quantile and the difference between two α quantiles will be derived. This will be followed by two proposed methods for comparing more than two α quantiles. A Bayesian simulation study will follow to investigate the frequentist properties of these two proposed methods for comparing more than two α quantiles.

2.1 Introduction

The estimation of variation or variance components serve as an integral part of the evaluation of measurement variation that is required for a variety of fields. To effectively understand these measurements, decision makers require both point and interval estimates (*van der Rijst, 2006*). Much research concerning the development of confidence intervals for variance components have been conducted for fixed effects -, random effects - and mixed models (*van der Rijst, 2006*).

The univariate normal model is by far the most developed linear model. It is interesting to note that R.A. Fisher (1924) developed a procedure for analyzing the fixed effects model (if sampling takes place from $d = 1, \dots, g$ normal populations with means μ_d and equal variances σ_d^2 and the focus is on making inferences about μ_d) and called this procedure the "analysis of variance". The procedure included optimum methods for point estimation, confidence intervals and hypothesis tests (*van der Rijst, 2006*).

During the 1930's, 1940's and 1950's a tremendous amount of work were done which generalized this model (*van der Rijst, 2006*). In a classical paper Kolodziejczyk (1935) provided general theory for linear models which also included the fixed effect model (*van der Rijst, 2006*). Most papers on variance components during the 1930's and 1940's were however concerned with point estimation (*van der Rijst, 2006*). During the 1940's and throughout the 1950's much work were done on procedures for obtaining approximate confidence intervals for variance components and also for linear combinations of variance components (*van der Rijst, 2006*). This work included contributions by Satterthwaite (1941, 1946), Crump (1951), Green (1954), Huitson (1955), Moriguti (1954), Welch (1956) and Bulmer (1957). Most of the work on variance components during this period were based on the analysis of variance as if the model was a fixed effect model (*van der Rijst, 2006*).

During the 1960's and 1970's, point estimation of variance components received a great deal of attention (*van der Rijst, 2006*). Graybill and Hultquist (1961) provided

conditions for obtaining optimum point estimates for linear combinations of variance components. Searle (1971) also provided a summary of research done concerning point estimation of variance components, while papers concerning confidence intervals for variance components started appearing in the late 1970's (*van der Rijst, 2006*). Burdick and Graybill (1988) reviewed this research and reported the present status of confidence intervals for functions of variance components (*van der Rijst, 2006*). It must be noted that up to that time, most of the results concerning confidence intervals on functions of variance components have been considered for balanced variance components models (*van der Rijst, 2006*). Searle (1988) however reviewed some of the history and results obtained for unbalanced and mixed models (*van der Rijst, 2006*).

As was mentioned, to see if a manufacturing process was in control by answering the three research questions as proposed by Wolfinger (1998) and provided in Chapter 1, three kinds of commonly used tolerance intervals can be determined.

The purpose of the remainder of this chapter is to provide a full Bayesian solution to the problem of variance component and tolerance interval estimation for a one - way fixed effect model with one variance component (the univariate normal model). It will also be shown that the proposed prior distribution to obtain the (α, δ) tolerance intervals is a reference prior as well as a probability matching prior. These proposed methods will be illustrated using summary data from flatness measurements obtained from Hubele et. al. (2005).

2.2 The Normal Linear Model with One Variance Component

To illustrate how the Bayesian approach to variance component and tolerance interval estimation for the fixed effect model can be employed, consider the usual normal linear model given by

$$y_i = \mu + \varepsilon_i \tag{2.2.1}$$

which as mentioned in Chapter 1, postulates that a quantitative observation consists of a constant target value μ , plus random variation about this fixed target value (Laubscher, 1996). Also as mentioned, μ represents the fixed target value, y_i represents a single measurement of the response variable made on the i^{th} item ($i = 1, \dots, n$ where n represents the sample size) and ε_i denotes random variation about zero (Laubscher, 1996). It is usually assumed that the ε_i 's are independently normally distributed with mean equal to zero and unknown constant variance σ_ε^2 , i.e. $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ (Laubscher, 1996).

2.3 The Prior Distribution

As was mentioned in Chapter 1, the choice of a prior distribution is a very difficult and controversial step in any Bayesian analysis, since a prior distribution can be viewed as a description of what is in fact known about a parameter before the data is observed (Box and Tiao, 1973, Gianola and Fernando, 1986, and Jandrell and van der Merwe, 2007).

For the univariate normal model given in equation 2.2.1, suppose that y_i ($i = 1, \dots, n$) follow independent and identically distributed normal distributions with mean μ and variance σ_ε^2 . For a prior distribution, we assume that the information is diffuse or vague (van der Merwe and Hugo, 2008). The following non - informative prior distribution also called the Jeffreys' independence prior is therefore proposed for the model given in equation 2.2.1.

$$p(\mu, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2} \quad (2.3.1)$$

The determination of reasonable, non - informative priors in multiparameter problems is not easy; common non - informative priors, such as a Jeffreys' prior, can have features that have an unexpected dramatic effect on the posterior. In recognition of this problem, Berger and Bernardo (1992c) proposed the *reference prior* approach to the

development of non - informative priors. This approach has the key feature of a possible dependence between the reference prior on the specification of parameters of interest and nuisance parameters. As mentioned, the solution however depends on the ordering of the parameters and how the parameter vector is divided into sub - vectors. In spite of these difficulties, there is growing evidence, that reference priors provide sensible answers from a Bayesian point of view.

As in the case of the Jeffreys' prior, the reference prior method is derived from the Fisher information matrix. To obtain the Fisher information matrix, the expected values of the second derivatives of the log likelihood must be calculated. As mentioned, since reference priors depend on the group ordering of the parameters, Berger and Bernardo (1992c) in particular recommended that the reference prior be based on having each parameter in its own group. This will have the effect that each conditional reference prior is one - dimensional.

It will now be examined to see if the Jeffreys' independence prior distribution proposed in equation 2.3.1 is in fact a reference prior distribution for the quantile given by

$$q = \mu + z_{\alpha}\sigma_{\varepsilon}. \quad (2.3.2)$$

where z_{α} denotes the α^{th} quantile of a standard normal distribution. Equation 2.3.2 is as mentioned in Chapter 1, the α^{th} quantile of $N(\mu, \sigma_{\varepsilon}^2)$.

Theorem 2.3.1

For the univariate normal model, the Jeffreys' independence prior distribution $p(\mu, \sigma_{\varepsilon}^2) \propto \sigma_{\varepsilon}^{-2}$ (i.e. equation 2.3.1) is a reference prior for the α^{th} quantile of $N(\mu, \sigma_{\varepsilon}^2)$ given by

$$q = \mu + z_{\alpha}\sigma_{\varepsilon},$$

which will be used to determine the (α, δ) tolerance intervals.

Proof

The proof of Theorem 2.3.1 is given in Appendix A.

The reference prior is but one - way to obtain useful non - informative priors. Datta and Ghosh (1995), on the other hand, derived the differential equation which a prior must satisfy if the posterior probability of a one - sided credibility interval (Bayesian confidence interval) for a parametric function and its frequentist probability agree up to $O(n^{-1})$, where n represents the sample size.

To see if the non - informative prior given in equation 2.3.1 satisfies the probability matching criteria for the quantile given in equation 2.3.2, the following theorem will now be proved.

Theorem 2.3.2

For the univariate normal model, the Jeffreys' independence prior distribution $p(\mu, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2}$ (i.e. equation 2.3.1) is a probability matching prior for the α^{th} quantile of $N(\mu, \sigma_\varepsilon^2)$ given by

$$q = \mu + z_\alpha \sigma_\varepsilon .$$

Proof

The proof of Theorem 2.3.2 is given in Appendix A.

2.4 The Posterior Distribution

A Bayesian analysis typically begins with the assignment of probability distributions to all unknown parameters associated with some parametric model of interest (*Jandrell and van der Merwe, 2007 and Wolfinger, 1998*). Once data have been observed,

inference is based on the posterior density of the parameters (*Jandrell and van der Merwe, 2007*). Using Bayes's theorem, the posterior distribution is computed in unnormalized form by multiplying the likelihood function of the data with the prior density of the unknown parameters (*Jandrell and van der Merwe, 2007 and Wolfinger, 1998*).

For the simple linear model given in equation 2.2.1, the likelihood function of the unknown parameters, μ and σ_ε^2 , is given by

$$L(\mu, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}n} \exp \left\{ -\frac{1}{2} \left[\frac{n(\mu - \bar{y})^2}{\sigma_\varepsilon^2} + \frac{(n-1)s^2}{\sigma_\varepsilon^2} \right] \right\}.$$

As was mentioned, the joint posterior distribution of μ and σ_ε^2 is then obtained by multiplying the likelihood function with the prior distribution given in equation 2.3.1. The joint posterior distribution is then given by

$$p(\mu, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(n+2)} \exp \left\{ -\frac{1}{2} \left[\frac{n(\mu - \bar{y})^2}{\sigma_\varepsilon^2} + \frac{(n-1)s^2}{\sigma_\varepsilon^2} \right] \right\}.$$

From the posterior distribution it is easy to show that the conditional posterior distribution of μ (conditional on σ_ε^2) is given by

$$\mu | \sigma_\varepsilon^2, \mathbf{y} \sim N\left(\bar{y}, \frac{\sigma_\varepsilon^2}{n}\right). \quad (2.4.1)$$

It also follows easily from the joint posterior distribution that the marginal posterior distribution of the variance component, σ_ε^2 , is given by

$$p(\sigma_\varepsilon^2 | \mathbf{y}) = \tilde{K} (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu+2)} \exp \left\{ -\frac{1}{2} \frac{(n-1)s^2}{\sigma_\varepsilon^2} \right\} \quad \forall \sigma^2 > 0 \quad (2.4.2)$$

where $\nu = n - 1$. Equation 2.4.2 is in the general form of an inverse gamma distribution where

$\mathbf{y} = [y_1, y_2, \dots, y_n]'$ represents the data vector,

$\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ represents the sample mean,

$s^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$ represents the sample variance,

\tilde{K} represents a normalization constant,

n represents the sample size, and as mentioned $\nu = n - 1$.

From equation 2.4.2 it can also easily be shown that the normalization constant \tilde{K} is given by

$$\tilde{K} = \left\{ \frac{(n-1)s^2}{2} \right\}^{\frac{1}{2}(n-1)} \cdot \frac{1}{\Gamma(\frac{n-1}{2})}.$$

2.5 Bayesian Simulation

The Bayesian method proposed for the univariate normal model given in equation 2.2.1 will now be illustrated using the following summary data of flatness measurements obtained from Example 1 in Hubele, et.al. (2005).

Summary data from Example 1 was obtained from two industrial processes used to make ceramic parts and represents summary statistics from actual flatness measurements (Hubele, et.al., 2005). The actual data used to obtain the summary statistics, were collected from stable processes and passed goodness - of - fit tests for normality (Hubele, et.al., 2005). The summary statistics from Example 1 obtained from Hubele, et.al. (2005) is provided in Table 2.1 (unfortunately the unit of measurement is not given in the article by Hubele, et.al., 2005).

Table 2.1: Example 1: Summary Data of Actual Flatness Measurements Obtained from Two Industrial Processes.

d	n_d	\bar{x}_d	s_d
1	36	0.0070	0.000986
2	27	0.0058	0.000981

The univariate normal model given in equation 2.2.1 has only two unknown parameters, i.e. μ and σ_ε^2 , and as such is not that complex. Therefore, the marginal posterior distributions of μ and σ_ε^2 can be obtained analytically for each of the two populations.

As was mentioned, the marginal posterior distribution of the variance component, σ_ε^2 , is in the general form of an inverse gamma distribution and is given by equation 2.4.2. It was also mentioned that the conditional posterior distribution of the target value μ , i.e. $p(\mu|\sigma_\varepsilon^2, \mathbf{y})$ was in the general form of a normal distribution with mean \bar{y} and variance $\frac{\sigma_\varepsilon^2}{n}$, and, is given in equation 2.4.1. The marginal posterior distribution of μ , i.e. $p(\mu|\mathbf{y})$ will now be determined.

It can easily be shown that the marginal posterior distribution of the target value μ for the univariate normal model given in equation 2.2.1 is given by

$$p(\mu|\mathbf{y}) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \cdot \frac{\sqrt{n}}{\sqrt{\pi\nu s^2}} \cdot \left[\frac{n(\mu - \bar{y})^2}{\nu s^2} + 1 \right]^{-\frac{1}{2}\nu} \quad (2.5.1)$$

which is in the general form of a student t - distribution with $\nu = n - 1$ degrees of freedom,

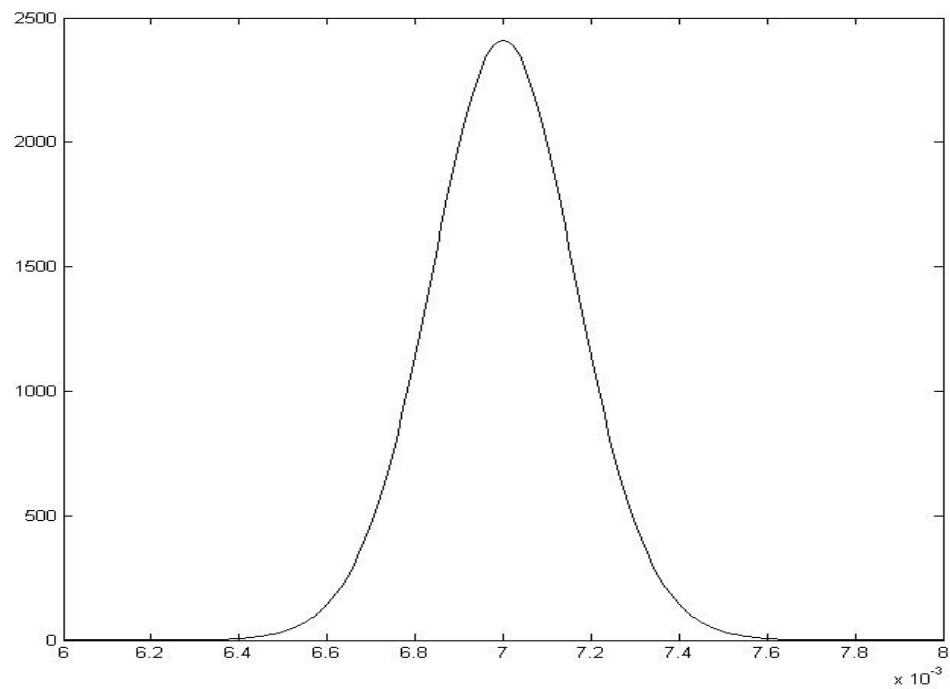
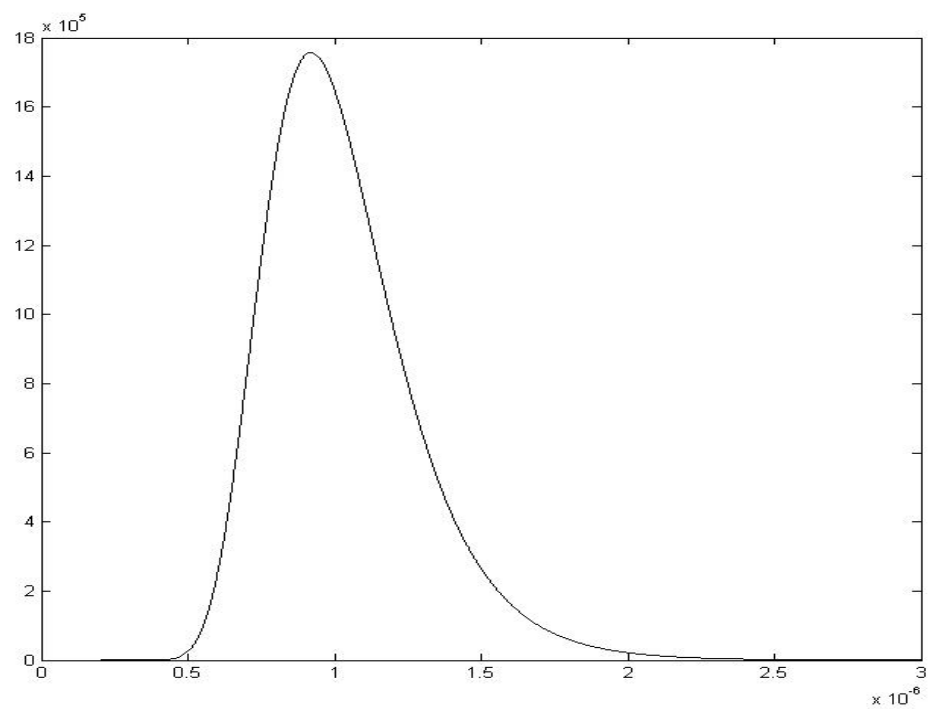
$$E(\mu|\mathbf{y}) = \bar{y} \text{ and}$$

$$Var(\mu|\mathbf{y}) = E\left(\frac{1}{n}\sigma_\varepsilon^2\right) = E\left(\frac{1}{n}\frac{\nu s^2}{\chi_\nu^2}\right) = \frac{\nu s^2}{n} E\left(\frac{1}{\chi_\nu^2}\right) = \frac{(n-1)s^2}{n} \left(\frac{1}{\nu-2}\right) = \frac{1}{n} \left(\frac{n-1}{n-3}\right) s^2.$$

The marginal posterior distribution of the target value μ for the first sample given in Table 2.1, i.e., $p(\mu|\mathbf{y})$ is given in Figure 2.5.1.

The 95% equal tail credibility interval for $p(\mu|\mathbf{y})$ is given by [0.0067, 0.0073]. Also, the marginal posterior distribution of σ_ε^2 , i.e. $p(\sigma_\varepsilon^2|\mathbf{y})$, for the first sample given in Table 2.1, is given in Figure 2.5.2.

For the marginal posterior distribution of the error variance, i.e. $p(\sigma_\varepsilon^2|\mathbf{y})$, the 95% equal tail credibility interval is given by $[6.4042 \times 10^{-7}, 1.6451 \times 10^{-6}]$.

Figure 2.5.1: Marginal Posterior Distribution of μ (using the Student t - Distribution).**Figure 2.5.2:** Marginal Posterior Distribution of σ_{ε}^2 (using the Inverse Gamma Distribution).

Markov chain Monte Carlo (MCMC) simulation can also be used to obtain random samples from the joint posterior density of the unknown model parameters by using a computer random number generator¹ (Wolfinger, 1998 and Wilson et. al., 2004).

Since the selected sample is dependent on the random number seed, different seeds will produce different samples (Wolfinger, 1998). It is however important that selected samples be large enough (Wolfinger, 1998). Differences between inferences from different samples will then be small (Wolfinger, 1998).

As was mentioned, estimated posterior distributions and predictive densities for the variance component model given in equation 2.2.1 can be obtained by using MCMC procedures (Jandrell and van der Merwe, 2007). These unconditional posterior distributions can be obtained through Monte Carlo simulations where independent samples from the joint posterior distribution of the unknown parameters are simulated. These simulated samples will represent samples from the marginal posterior distribution of the unknown parameter σ_ε^2 , i.e., $p(\sigma_\varepsilon^2|\mathbf{y})$, and conditional posterior distribution of the unknown parameter μ , i.e. $p(\mu|\sigma_\varepsilon^2, \mathbf{y})$.

Unconditional posterior distributions for the unknown parameters μ , and σ_ε^2 are simulated as follows:

a.) Simulation of σ_ε^2

Simulate $\tilde{\ell} = 10000$ independent values for σ_ε^2 . Using equation 2.4.2, if $\nu = n - 1$, it can easily be shown that

$$\frac{\nu s^2}{\sigma_\varepsilon^2} = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma_\varepsilon^2} \sim \chi_\nu^2$$

¹Computer - generated numbers are not really random, since computers are deterministic. But given a number to start with, generally called a random number seed - a number of mathematical operations can be performed on the seed so as to generate unrelated (pseudo random) numbers. If the same random number seed is used more than once identical random numbers will be generated every time. Using a different seed, would produce a different number.

which follows a chi - square distribution with ν degrees of freedom.

From this it follows that the unknown variance component σ_ε^2 can easily be simulated from this χ_ν^2 distribution by obtaining

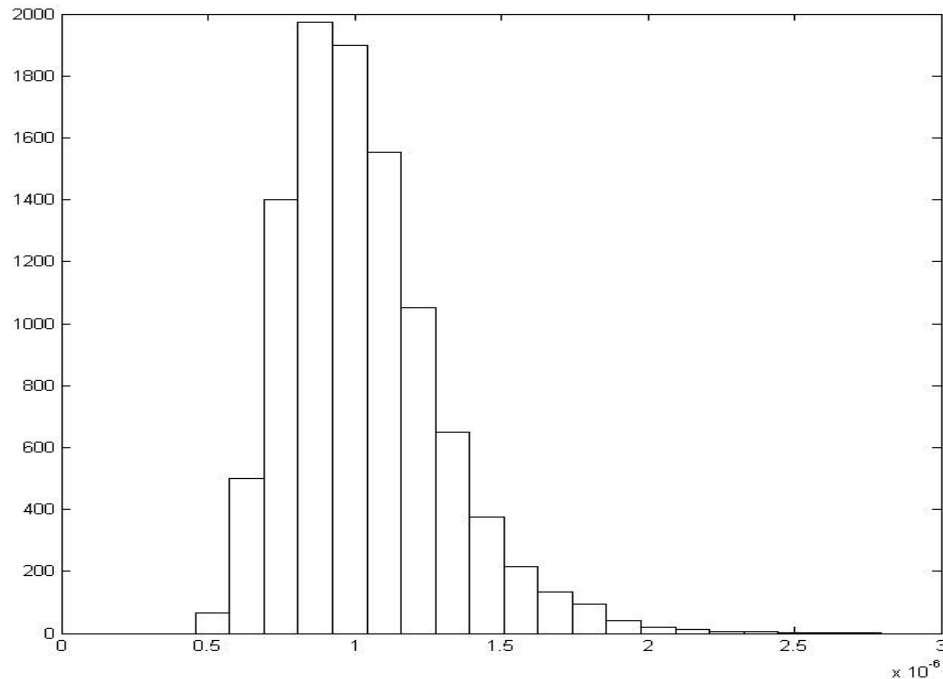
$$\sigma_\varepsilon^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\chi_\nu^2} .$$

The $\tilde{\ell} = 10000$ simulated values for σ_ε^2 can then be used to draw a histogram of the estimated marginal posterior distribution $p(\sigma_\varepsilon^2 | \mathbf{y})$. For the summary data given in Table 2.1, the histogram of the estimated marginal posterior distribution of σ_ε^2 is given in Figure 2.5.3.

b.) Simulation of μ

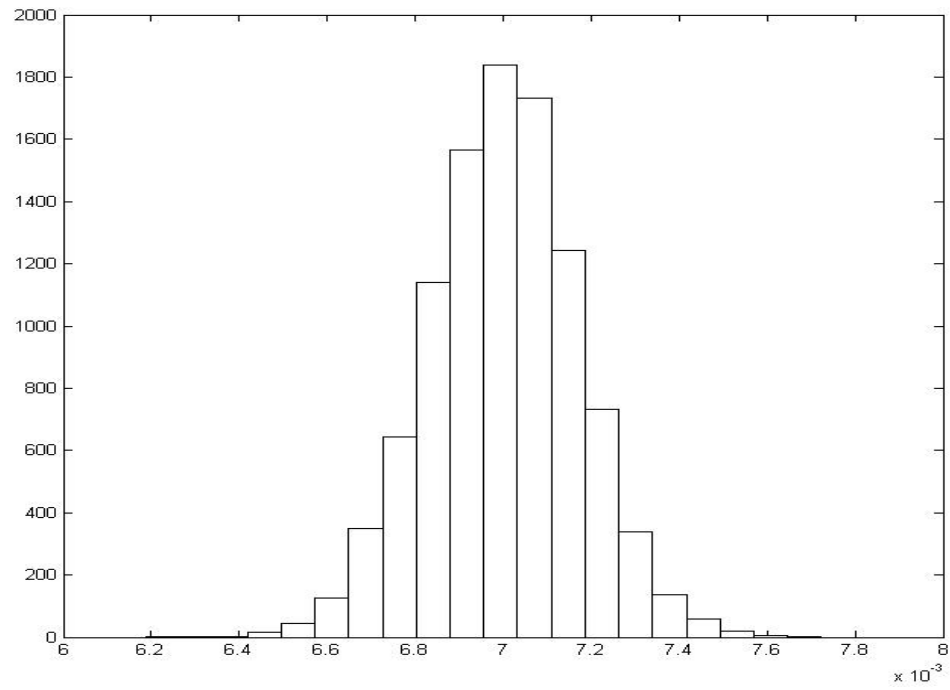
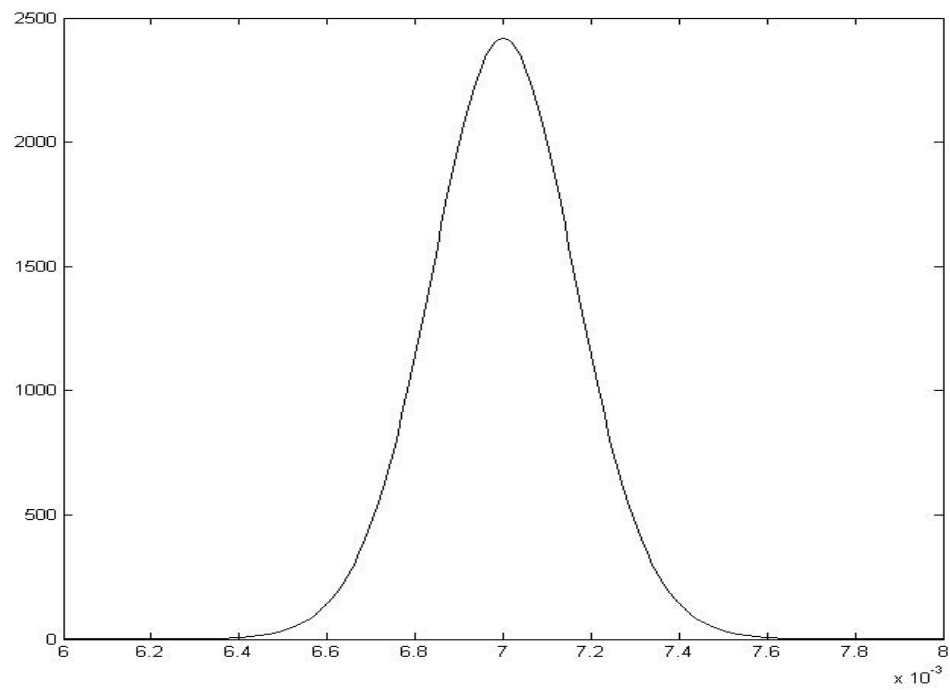
The simulation of μ involves substituting each of the $\tilde{\ell} = 10000$ simulated values for σ_ε^2 into the conditional posterior distribution of μ given by $p(\mu | \sigma_\varepsilon^2, \mathbf{y})$. This conditional posterior distribution of μ is given in equation 2.4.1. For each of the $\tilde{\ell} = 10000$ simulated values for σ_ε^2 , a value for μ will therefore be drawn from this normal distribution given in equation 2.4.1. The resulting set of $\tilde{\ell} = 10000$ simulated values for μ can then be plotted in a histogram representing the estimated marginal posterior distribution of μ . This histogram is provided in Figure 2.5.4 for the summary data given in Table 2.1.

The marginal posterior distribution of μ can also be obtained using the Rao Blackwell method (*Gelfand and Smith, 1990*). Substitute each of the simulated values for σ_ε^2 into the normal distribution given in equation 2.4.1. For each of the $\tilde{\ell} = 10000$ simulated σ_ε^2 values, plot the posterior distribution of μ conditional on $\sigma_{\varepsilon_i}^2 \forall i = 1, \dots, 10000$. The 10000 conditional posterior distribution ($p(\mu | \sigma_{\varepsilon_i}^2, \mathbf{y})$) curves are then overlaid and the average distribution of these 10000 conditional posterior distributions are obtained. This average distribution is given in Figure 2.5.5 and represents the estimated unconditional or marginal posterior distribution of μ , i.e. $p(\mu | \mathbf{y})$ which was determined for the first sample given in Table 2.1. This distribution will be similar to a student t - distribution with ν degrees of freedom.

Figure 2.5.3: Histogram of the Estimated Marginal Posterior Distribution of σ_ε^2 .

From Figure 2.5.3 we can see that the histogram is not extremely skew. This is due to the relatively high number of degrees of freedom. Remember that the sample size $n = 36$ for sample 1 given in Table 2.1, thus making the degrees of freedom necessary for simulating the error variance σ_ε^2 equal to $\nu = n - 1 = 35$. By comparing Figures 2.5.2 and 2.5.3, it is also clear that the shape of the histogram is almost identical to the shape of the marginal posterior distribution of σ_ε^2 plotted using the inverse gamma distribution. This indicates that the two methods used for obtaining $p(\sigma_\varepsilon^2|\mathbf{y})$ are equivalent. The 95% credibility intervals are also for all practical purpose the same.

As mentioned, Figures 2.5.4 and 2.5.5 represent respectively the histogram of the estimated marginal posterior distribution $p(\mu|\mathbf{y})$, and, the estimated marginal posterior distribution $p(\mu|\mathbf{y})$, obtained using the Rao Blackwell method. From Figures 2.5.4 and 2.5.5 it can be seen that the shapes of both the histogram and the plot of $p(\mu|\sigma_\varepsilon^2)$ are for all practical purposes identical to Figure 2.5.1 which was obtained by plotting the student t - distribution. Although not given here, the same can also be said for the 95%

Figure 2.5.4: Histogram of the Estimated Marginal Posterior Distribution of μ .**Figure 2.5.5:** Estimated Marginal Posterior Distribution of μ (using the Rao Blackwell Method).

credibility intervals.

Since both unknown parameters (μ and σ_ε^2) have been simulated, we can now proceed and answer the three research questions proposed by Wolfinger (1998) and given in Chapter 1. This will be done using the three tolerance intervals also proposed by Wolfinger (1998).

2.6 Tolerance Intervals

According to Krishnamoorthy and Mathew (2009), a tolerance interval can be defined if the content and confidence levels are specified. The content will be denoted by α while the confidence level will be denoted by δ . An (α, δ) tolerance interval will therefore simply be defined as a $100(\delta)\%$ confidence interval constructed using a random sample, and, is required to contain a proportion α (content) or more of the sampled population (*Krishnamoorthy and Mathew, 2009*). Simply put, a tolerance interval therefore represents a confidence interval of the content of some population.

As mentioned, statistical tolerance intervals (or limits) are determined using sample data obtained from some process (*Jandrell and van der Merwe, 2007*). The variation visible in the process are quantified by these tolerance intervals (*Jandrell and van der Merwe, 2007*). Based on sample data, the potential of the process is identified by specifying minimum and maximum values (*Jandrell and van der Merwe, 2007*). These minimum and maximum values bound a region that will probably (with probability δ) contain more than a certain proportion, α , of the total population (*Jandrell and van der Merwe, 2007*). It is however accepted that with the complimentary probability, the bound region will contain less than the proportion α (*Jandrell and van der Merwe, 2007*).

The problem of determining tolerance intervals for a distribution based in observed sample data have been investigated for a variety of applications by many authors

(Wang and Iyer, 1994). Some of the earliest work in the subject was performed by Wilks (1941, 1942), Wald (1942, 1943) and Wald and Wolfowitz (1946). The problem of estimating tolerance limits for a univariate distribution F , consists of finding two sample statistics, g_L and g_U , such that with a certain level of confidence, δ , it can be stated that at least a proportion of the population, α , is contained in the interval $[g_L, g_U]$ (Wang and Iyer, 1994). This estimated two - sided α - content, δ - confidence ((α, δ) for short) interval, is referred to as a tolerance interval for the distribution F (Wang and Iyer, 1994). Problems like the one mentioned, have been studied extensively in the case of a normal distribution with unknown mean (μ) and unknown variance (σ^2) (Wang and Iyer, 1994). To put it more generally, the two - sided (α, δ) tolerance interval problem has been studied extensively for the case of the univariate normal model (Wang and Iyer, 1994). For further discussions on the topic, see for example Odeh and Owen (1980) and Hahn and Meeker (1991).

As was mentioned in Chapter 1, Wolfinger (1998) described three commonly used tolerance intervals. These are the (α, δ) one - and two - sided tolerance intervals, the α - expectation tolerance interval and the fixed - in - advance tolerance interval.

The remainder of this chapter will be dedicated to the Bayesian approach for estimating the three commonly used tolerance intervals as proposed by Wolfinger (1998) for the univariate normal model given in equation 2.2.1. The Bayesian method will be illustrated for the first sample given in Table 2.1 and obtained from Hubele et.al. (2005). In addition, the central moments for the quantile $q = \mu + z_\alpha \sigma_\varepsilon$ and the difference between two quantiles will also be provided. The proposed Bayesian method for the differences between two quantiles and the subsequent tolerance interval will be illustrated using both samples 1 and 2 given in Table 2.1 and obtained from Hubele et.al. (2005). This will be followed by two proposed Bayesian methods for comparing more than two quantiles. These two proposed methods will also be demonstrated using summary data from flatness measurements obtained from Hubele et.al. (2005) and given in Table 2.2. For the data given in Table 2.2, a simulation study will follow to check the frequen-

tist performance of the two proposed Bayesian multiple comparisons procedures, for differences between more than two quantiles.

Table 2.2: Example 2: Summary Data of Actual Flatness Measurements Obtained from Three Industrial Processes.

d	n_d	\bar{x}_d	S_d
1	20	0.00045	0.00012
2	20	0.00045	0.00009
3	20	0.00073	0.00010

As was mentioned for Table 2.1, the summary statistics provided in Table 2.2 were obtained from three stable industrial processes used to make aluminium parts (*Hubele et.al., 2005*). The summary statistics were obtained from actual flatness measurements which passed goodness - of - fit tests for normality (*Hubele, et.al., 2005*). Similar to Table 2.1, the unit of measurement is unfortunately not given in the article by Hubele, et. al. (2005).

The Bayesian simulation procedure for obtaining the three commonly used tolerance intervals proposed by Wolfinger (1998) will now be illustrated using the first sample of Example 1 given in Table 2.1.

2.6.1 One - Sided (α, δ) Tolerance Interval

As mentioned in Chapter 1, a (α, δ) upper (lower) tolerance limit is a statistic for which at least $100(\alpha)\%$ of a population of an underlying random variable is less than (greater than) the tolerance limit with $100(\delta)\%$ confidence (*Jandrell and van der Merwe, 2007*).

These mentioned (α, δ) tolerance intervals are typically applied in cases requiring long - run forecasts about several observations from a process assumed to be in a state of

statistical control (*Jandrell and van der Merwe, 2007*). In cases such as this, inference is required about the actual quantiles of the assumed underlying probability distribution (*Jandrell and van der Merwe, 2007*). Based on an available sample of measurements, manufactures therefore use (α, δ) tolerance intervals to predict the future performance of a manufactured product (*Jandrell and van der Merwe, 2007*).

According to Wolfinger (1998), the upper (α, δ) one - sided tolerance limit for the univariate normal model given in equation 2.2.1 represents the δ^{th} sample quantile obtained from the marginal posterior distribution of the α^{th} quantile q of a $N(\mu, \sigma_\varepsilon^2)$ distribution (i.e. a quantile of a quantile), where q is given by

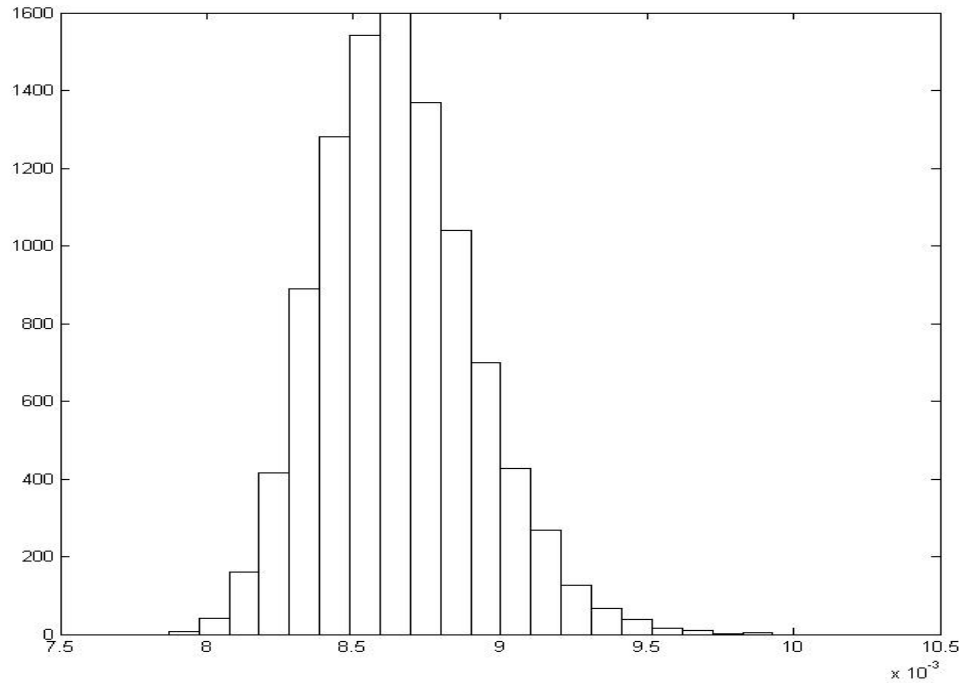
$$q = \mu + z_\alpha \sigma_\varepsilon \quad (2.6.1)$$

and z_α denotes the α^{th} quantile of a standard normal distribution. Therefore, to construct an upper one - sided (α, δ) tolerance interval, the estimated marginal posterior distribution of q must be obtained, which in this case represents the α^{th} quantile of the $N(\mu, \sigma_\varepsilon^2)$ distribution. The estimated marginal posterior distribution of q can easily be obtained using two methods, both utilizing Bayesian simulation.

Method 1

- i.) Simulate the variance component σ_ε^2 using the Bayesian simulation method described in section 2.5.
- ii.) Given the simulated variance component σ_ε^2 , simulate a value for μ using equation 2.4.1.
- iii.) Substitute these simulated values for σ_ε^2 and μ into equation 2.6.1 and calculate q .
- iv.) Repeat step i.) to iii.) for example $\tilde{\ell} = 10000$ times and plot a histogram for q .

Figure 2.6.1.1: Histogram of the Estimated Marginal Posterior Distribution of the 0.95th Quantile of $N(\mu, \sigma_\epsilon^2)$ for the First Sample Given in Table 2.1.



95% Equal Tail Credibility Interval: [0.0082, 0.0092]
 (0.95, 0.95) Upper One - Sided Tolerance Limit: 0.0091

The consequent plotted histogram represents the estimated marginal or unconditional posterior distribution of $q|\mathbf{y}$.

Using method 1, the resulting histogram representing the estimated marginal posterior distribution of $q|\mathbf{y}$ for the first sample given in Table 2.1, is given in Figure 2.6.1.1. The one - sided ($\alpha = 0.95, \delta = 0.95$) upper tolerance limit can also easily be determined by ranking the simulated q values in order of magnitude (from small to large) and finding the $100(0.95)^{th}$ percentile of the ranked simulated values.

A second method can also be used to simulate the marginal posterior distribution of q , i.e. $p(q|\mathbf{y})$. This method can be performed as follows:

Method 2

- i.) Given equation 2.6.1 and the marginal posterior distribution of the variance component σ_ε^2 given in equation 2.4.2, it follows that the conditional posterior distribution of $q|\sigma_\varepsilon^2, \mathbf{y}$ is given by

$$p(q|\sigma_\varepsilon^2, \mathbf{y}) \sim N(\bar{y} + z_\alpha \sigma_\varepsilon, \frac{\sigma_\varepsilon^2}{n}). \quad (2.6.2)$$

- ii.) Simulate a variance component σ_ε^2 using the Bayesian simulation method described in section 2.5.
- iii.) Substitute σ_ε^2 and σ_ε into equation 2.6.2 and draw the normal distribution.
- iv.) Repeat steps ii.) and iii.) for example $\tilde{\ell} = 10000$ times.
- v.) Using the Rao Blackwell method described in section 2.5, determine the estimated marginal posterior distribution of $q|\mathbf{y}$.

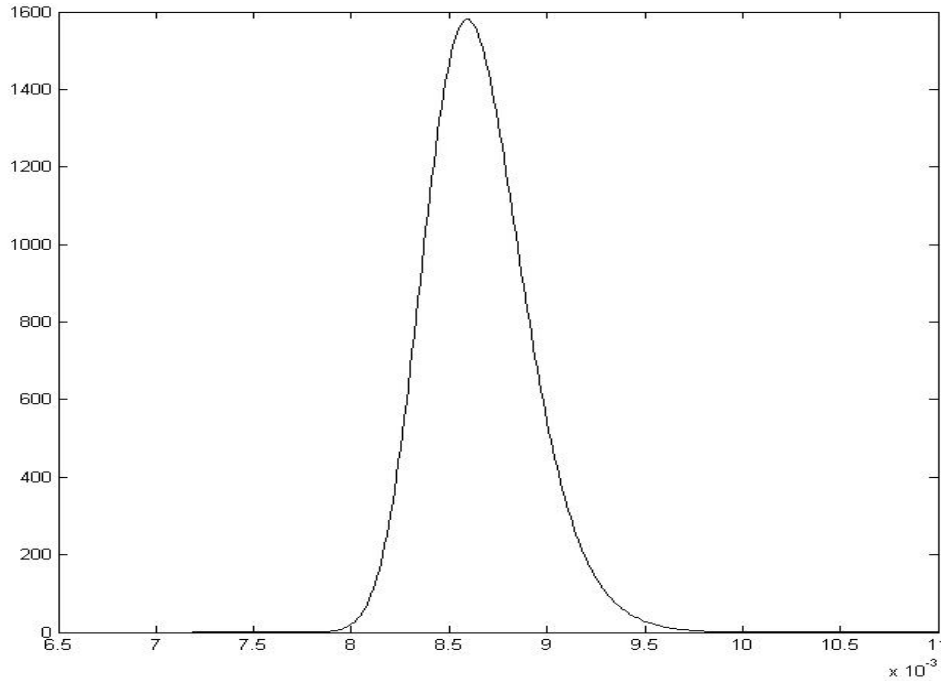
Using method 2, the estimated marginal posterior distribution of the quantile q given by

$$q = \mu + z_{0.95} \sigma_\varepsilon = \mu + 1.645 \sigma_\varepsilon \quad (2.6.3)$$

was determined for the first sample given in Table 2.1. The estimated marginal posterior distribution of q is represented in Figure 2.6.1.2.

The 95% equal tail credibility interval for the estimated marginal posterior distribution of $q|\mathbf{y}$ represented in Figure 2.6.1.2 is also equal to $[0.0082, 0.0092]$. For the first sample given in Table 2.1, the 95th percentile of the estimated marginal posterior distribution of the quantile q given in equation 2.6.3 is equal to 0.0091, thus indicating the value of which 95% of future unknown flatness measurements will be less than with probability 0.95. This therefore represents the Bayesian (0.95, 0.95) upper tolerance limit.

Figure 2.6.1.2: Estimated Marginal Posterior Distribution of the 0.95th Quantile of $N(\mu, \sigma_\varepsilon^2)$ for the First Sample Given in Table 2.1.



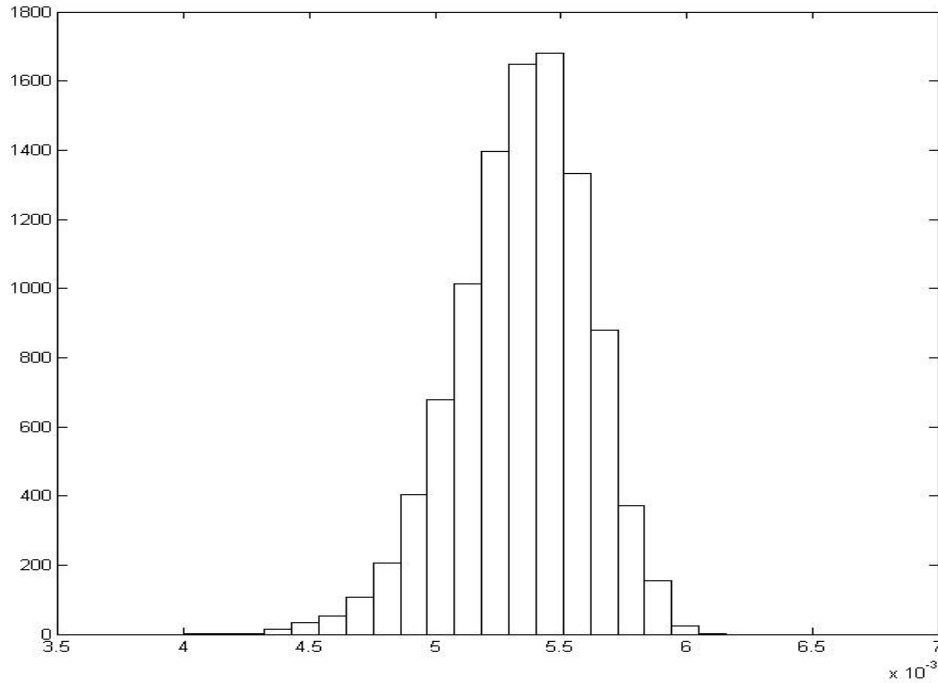
95% Equal Tail Credibility Interval: [0.0082, 0.0092]
 (0.95, 0.95) Upper One - Sided Tolerance Limit: 0.0091

Similarly (using both methods 1 and 2), to construct a lower one - sided ($\alpha = 0.95, \delta = 0.95$) tolerance interval, the estimated marginal posterior distribution of q_l must be obtained, which in this case represents the $(1 - 0.95)^{th}$ quantile of the $N(\mu, \sigma_\varepsilon^2)$ distribution with q_l given by

$$q_l = \mu - z_{0.95}\sigma_\varepsilon = \mu - 1.645\sigma_\varepsilon \quad (2.6.4)$$

where $z_{0.95} = 1.645$ represents the 0.95th quantile of a standard normal distribution. Using method 1, the histogram of the estimated marginal posterior distribution of q_l was obtained for the first sample given in Table 2.1. This histogram is given in Figure 2.6.1.3, while the estimated marginal posterior distribution of $q_l|\mathbf{y}$ determined using method 2, is provided in Figure 2.6.1.4.

Figure 2.6.1.3: Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2)$ for the First Sample Given in Table 2.1.



95% Equal Tail Credibility Interval: [0.0048, 0.0058]
 (0.95, 0.95) Lower One - Sided Tolerance Limit: 0.0049

The 95% equal tail credibility interval for the estimated marginal posterior distribution of $q_l|\mathbf{y}$, represented in Figure 2.6.1.4, is equal to [0.0048, 0.0058]. The lower one - sided (0.95, 0.95) tolerance limit is equal to 0.0049. This represents the $100(1 - 0.95)^{th}$ percentile of the estimated marginal posterior distribution of equation 2.6.4, thus indicating the value of which 95% of future unknown flatness measurements will be greater than with probability 0.95. This therefore represents the Bayesian "B - basis", ($\alpha = 0.95, \delta = 0.95$) lower tolerance limit (Wolfinger, 1998).

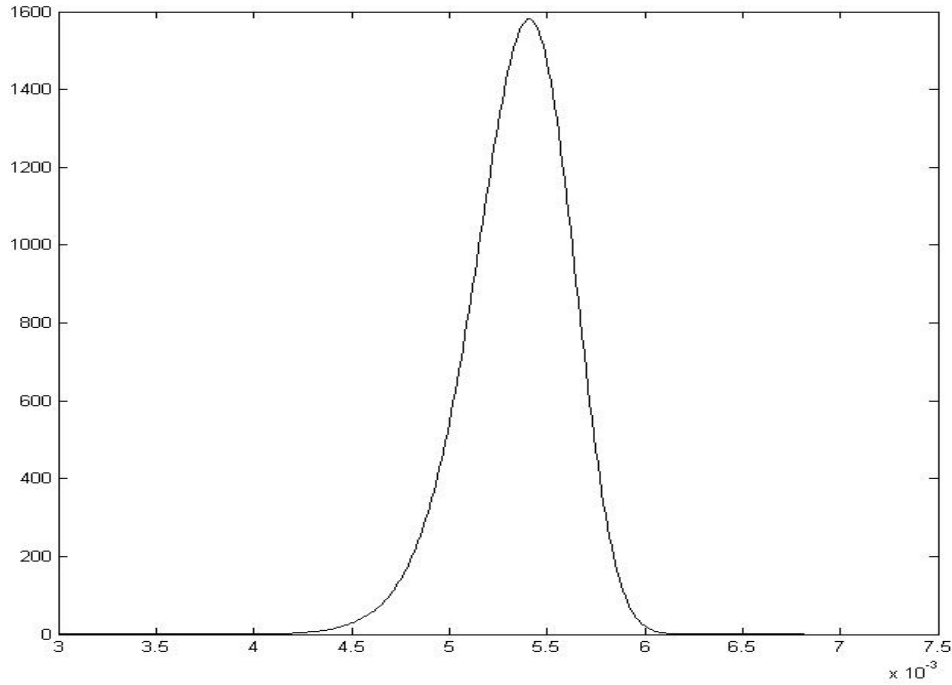
The Exact Moments of $q|\mathbf{y}$

The exact moments of $q|\mathbf{y}$ will now be determined.

For the quantile q given in equation 2.6.1, i.e. $q = \mu + z_\alpha \sigma_\varepsilon$, it is known that

$$\mu \sim N(\bar{y}, \frac{\sigma^2}{n}) \text{ and } \frac{\nu s^2}{\sigma_\varepsilon^2} \sim \chi_\nu^2 \text{ where } \nu = n - 1.$$

Figure 2.6.1.4: Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2)$ for the First Sample Given in Table 2.1.



95% Equal Tail Credibility Interval: [0.0048 , 0.0058]
 (0.95, 0.95) Lower One - Sided Tolerance Limit: 0.0049

It is therefore also known that

$$\begin{aligned} q &= \bar{y} + z \frac{\sigma_\varepsilon}{\sqrt{n}} + z_\alpha \sigma_\varepsilon \text{ where } z \sim N(0, 1) \\ &= \bar{y} + \sigma_\varepsilon \left\{ \frac{z}{\sqrt{n}} + z_\alpha \right\} \\ &= \bar{y} + \left(\frac{\nu s^2}{\chi_\nu^2} \right)^{\frac{1}{2}} \left\{ \frac{z}{\sqrt{n}} + z_\alpha \right\}. \end{aligned}$$

The moments of the χ_ν^2 distribution will now be determined.

In general, the r^{th} moment about the origin of $\left(\frac{1}{\chi_\nu^2}\right)^{\frac{1}{2}}$ is given by

$$E\left(\frac{1}{x}\right)^{\frac{r}{2}} = \frac{1}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty x^{\frac{1}{2}(\nu-r)-1} e^{-\frac{1}{2}x} dx .$$

Since

$$\int_0^\infty x^{\frac{1}{2}(\nu-r)-1} e^{-\frac{1}{2}x} dx = 2^{\frac{1}{2}(\nu-r)} \Gamma\left(\frac{\nu-r}{2}\right) ,$$

it follows that

$$E\left(\frac{1}{x}\right)^{\frac{r}{2}} = \frac{2^{\frac{1}{2}(\nu-r)}\Gamma(\frac{\nu-r}{2})}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} = \frac{2^{\frac{1}{2}\nu}2^{-\frac{r}{2}}\Gamma(\frac{\nu-r}{2})}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} = \frac{\Gamma(\frac{\nu-r}{2})}{2^{\frac{r}{2}}\Gamma(\frac{\nu}{2})}.$$

Note, for notational purposes, the first four moments about the origin of the conditional distribution of q (conditional on σ_ε^2) is given by μ'_1 , μ'_2 , μ'_3 and μ'_4 . Also, the central moments of q given σ_ε^2 , is given by μ_2 , μ_3 and μ_4 . Note also that the unconditional moments about the origin of q is given by m'_1 , m'_2 , m'_3 and m'_4 , while, the unconditional central moments of q is given by m_2 , m_3 and m_4 . We therefore have

$$\mu'_1 = \bar{y} + z_\alpha \sigma_\varepsilon,$$

$$\mu_2 = \frac{\sigma_\varepsilon^2}{n},$$

$$\mu_3 = 0,$$

$$\mu_4 = 3 \left(\frac{\sigma_\varepsilon^2}{n} \right)^2.$$

Theorem 2.6.1.1

- a.) For the univariate normal model given in equation 2.2.1, the mean (first moment about zero) of the marginal posterior distribution of q , i.e. $p(q|\mathbf{y})$ is given by

$$E(q|\mathbf{y}) = \bar{y} + z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}.$$

- b.) For the univariate normal model given in equation 2.2.1, the second central moment of the marginal posterior distribution of q , i.e. $p(q|\mathbf{y})$ is given by

$$\text{Var}(q|\mathbf{y}) = (\nu s^2) \left\{ \frac{1}{\nu-2} \left[z_\alpha^2 + \frac{1}{n} \right] - z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right\}.$$

- c.) For the univariate normal model given in equation 2.2.1, the third central moment of the marginal posterior distribution of q , i.e. $p(q|\mathbf{y})$ is given by

$$m_3 = (\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})z_\alpha}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left\{ \left[\frac{1}{(\nu-2)} \left(\frac{3}{n} - z_\alpha^2(2\nu-7) \right) \right] + z_\alpha^2(\nu-3) \frac{\Gamma^2(\frac{\nu-1}{2})}{\Gamma^2(\frac{\nu}{2})} \right\}.$$

d.) For the univariate normal model given in equation 2.2.1, the fourth central moment of the marginal posterior distribution of q , i.e. $p(q|\mathbf{y})$ is given by

$$m_4 = \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} - \frac{3z_\alpha^4 (\nu s^2)^2 \Gamma^4\left(\frac{\nu-1}{2}\right)}{2^2 \Gamma^4\left(\frac{\nu}{2}\right)} - \frac{6(\nu-1)z_\alpha^2 (\nu s^2)^2 \Gamma\left(\frac{\nu-3}{2}\right) \Gamma\left(\frac{\nu-1}{2}\right)}{2^2 n (\nu-2) \Gamma^2\left(\frac{\nu}{2}\right)} + \frac{2(\nu-5)z_\alpha^4 (\nu s^2)^2 \Gamma\left(\frac{\nu-3}{2}\right) \Gamma\left(\frac{\nu-1}{2}\right)}{2^2 (\nu-2) \Gamma^2\left(\frac{\nu}{2}\right)} .$$

Proof

The proof of Theorem 2.6.1.1 is given in Appendix A.

2.6.2 Two - Sided (α, δ) Tolerance Interval

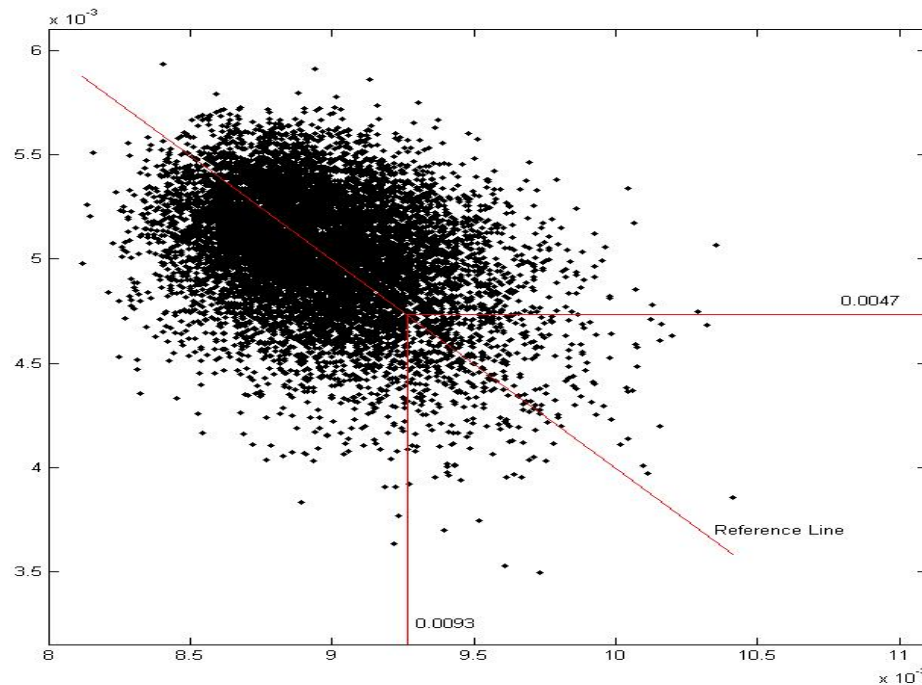
Wolfinger (1998) suggested that a two - sided (α, δ) tolerance interval can also be constructed. This two - sided interval needs to be one - dimensional and symmetric about the posterior mean or some other form of central tendency (Wolfinger, 1998). Wolfinger (1998) also mentioned that the construction of these two - sided (α, δ) tolerance intervals are slightly more complex, since the simple procedure of computing upper and lower limits separately and then combining them is not precisely valid. The reason for this is that the two quantiles, i.e. q_ℓ and q_u , do not have a posterior correlation equal to 1 (Wolfinger, 1998).

Wolfinger (1998) therefore suggested that one way of constructing a valid two - sided (α, δ) tolerance interval, is to begin by computing the two quantities, q_ℓ and q_u given by

1. $q_\ell = \mu - z_{\frac{\alpha}{2}} \sigma_\varepsilon$, and
2. $q_u = \mu + z_{\frac{\alpha}{2}} \sigma_\varepsilon$.

These (q_ℓ, q_u) pairs then form a sample from the bivariate posterior distribution of the $\left[\frac{(1-\alpha)}{2}\right]^{th}$ and $\left[\frac{(1+\alpha)}{2}\right]^{th}$ quantiles (Wolfinger, 1998). Bayesian confidence regions for these

Figure 2.6.2.1: Constructing a Two - Sided (0.95, 0.95) Tolerance Interval for the First Sample Given in Table 2.1.



bivariate samples can be obtained, but are difficult to use in practice, since they are two - dimensional ellipsoids. Wolfinger (1998) however succeeded in constructing a two - sided (α, δ) tolerance interval that is one - dimensional and symmetric about the mean. To obtain such an interval, Wolfinger (1998) suggested to first form a scatter plot of q_ℓ versus q_u , with q_ℓ on the vertical axis. A reference line given by

$$q_\ell = -q_u + 2\bar{y}_..$$

then needs to be constructed. Two additional lines then have to be drawn, one parallel to each axis and intersecting on the reference line. This intersection point is then slid along the reference line until $100(1 - \delta)\%$ of the (q_ℓ, q_u) pairs are contained in the half rectangle opening towards the lower right portion of the graph (Wolfinger, 1998). The coordinates of the resulting intersection point form a two - sided (α, δ) tolerance interval of the desired form (Wolfinger, 1998). This procedure is graphically illustrated for the first sample given in Table 2.1, in Figure 2.6.2.1.

The scatterplot and subsequent two - sided (0.95, 0.95) tolerance interval depicted in Figure 2.6.2.1 was determined in the following way:

- i.) Simulate the variance component σ_ε^2 using the Bayesian simulation method described in section 2.5.
- ii.) Given the simulated variance component σ_ε^2 , simulate a value for μ using equation 2.4.1.
- iii.) For a two - sided (0.95, 0.95) tolerance interval, calculate the simulated values for $q_\ell = \mu - 1.96\sigma_\varepsilon$ and $q_u = \mu + 1.96\sigma_\varepsilon$ by using the simulated values for σ_ε^2 and μ .
- iv.) Repeat steps i) - iii) for example $\tilde{\ell} = 10000$ times, draw the scatterplot and use the method proposed by Wolfinger (1998) which was discussed earlier in this section to obtain the two - sided ($\alpha = 0.95, \delta = 0.95$) tolerance interval.

For the first sample given in Table 2.1, the two - sided (0.95, 0.95) tolerance interval given by [0.0047, 0.0093] can be interpreted as follows: If ceramic parts are manufactured using industrial process 1, 95% of the actual flatness measurements will fall in the interval [0.0047, 0.0093] with probability 0.95.

2.6.3 α - Expectation Tolerance Interval

The tolerance intervals discussed thus far, were α - content δ - confidence tolerance intervals. Another type of tolerance interval investigated in literature is referred to as an α - expectation tolerance interval. An α - expectation tolerance interval is an interval such that the average content of the interval is α (Krishnamoorthy and Mathew, 2009).

According to Wolfinger (1998), the α - expectation tolerance interval addresses research question 2 (mentioned in Chapter 1) and focus on prediction of one or a

few future observations from a process. These α - expectation tolerance intervals are therefore also prediction intervals for future observations (*Krishnamoorthy and Mathew, 2009*). Wolfinger (1998) also mentioned that since these α - expectation tolerance intervals focus on prediction of one or a few future observations from a process, these intervals tend to be narrower than corresponding (α, δ) tolerance intervals.

To construct α - expectation tolerance intervals, Wolfinger (1998) suggested that simulations be conducted from an appropriate predictive density $p(y_f|\mathbf{y})$ where y_f represents a future observation.

For the univariate normal model given in equation 2.2.1, the unconditional predictive density of a future measurement from a process, i.e. $p(y_f|\mathbf{y})$ can analytically be obtained.

Theorem 2.6.3.1

For the univariate normal model given in equation 2.2.1 the unconditional predictive density of a future measurement from a process follows a student t - distribution with $\nu = n - 1$ degrees of freedom with,

$$E(y_f|\mathbf{y}) = \bar{y}, \text{ and}$$

$$Var(y_f|\mathbf{y}) = \frac{(n+1)}{n} \left(\frac{n-1}{n-3} \right) s^2 .$$

Proof

The proof of Theorem 2.6.3.1 is given in Appendix A.

The unconditional predictive distribution of y_f , i.e. $p(y_f|\mathbf{y})$ can also be estimated using Monte Carlo simulation. The three methods used to estimate the predictive distribution of a future observation will now be described.

Method 1

As was mentioned in the prove of Theorem 2.6.3.1, it is known that the conditional predictive density of y_f given μ and σ_ε^2 , is given by

$$y_f | \mu, \sigma_\varepsilon^2 \sim N(\mu, \sigma_\varepsilon^2). \quad (2.6.5)$$

- i.) Simulate the variance component σ_ε^2 using the Bayesian simulation method described in section 2.5.
- ii.) Given the simulated variance component σ_ε^2 , simulate a value for μ using equation 2.4.1.
- iii.) Substitute these simulated values for σ_ε^2 and μ into equation 2.6.5 and draw the normal distribution.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times.
- v.) Using the Rao Blackwell method described in section 2.5, determine the estimated unconditional predictive distribution of $y_f | \mathbf{y}$.

The estimated unconditional predictive distribution of $y_f | \mathbf{y}$, for the first sample given in Table 2.1, constructed using method 1, is depicted in Figure 2.6.3.1.

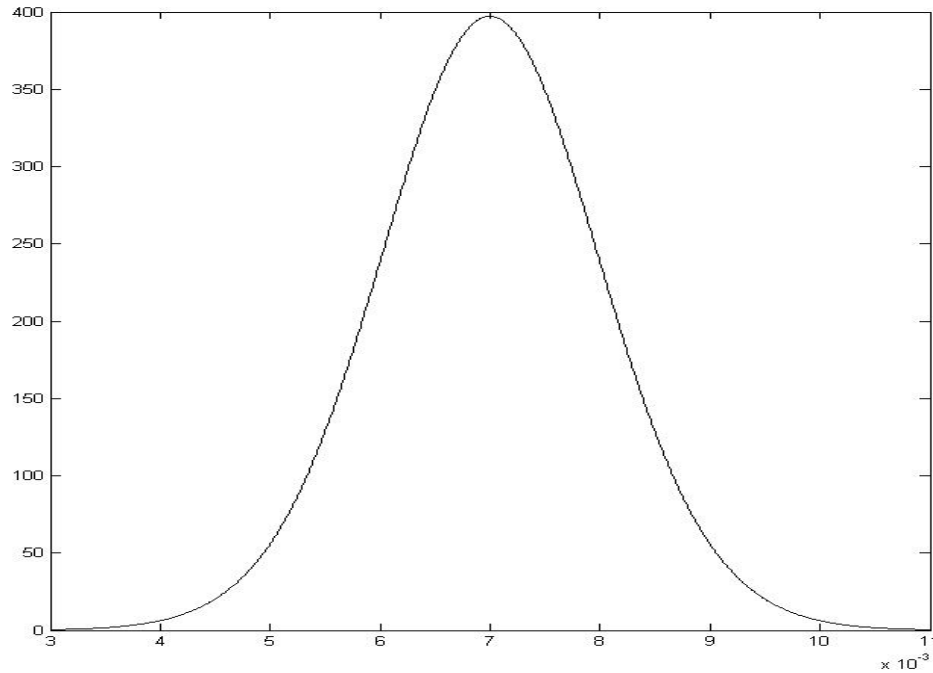
Method 2

It was also mentioned in the proof of Theorem 2.6.3.1, that the conditional predictive distribution of y_f given σ_ε^2 , is given by

$$y_f | \sigma_\varepsilon^2, \mathbf{y} \sim N\left(\bar{y}, \frac{(n+1)\sigma_\varepsilon^2}{n}\right). \quad (2.6.6)$$

- i.) Simulate the variance component σ_ε^2 using the Bayesian simulation method described in section 2.5.

Figure 2.6.3.1: Estimated Unconditional Predictive Distribution for the First Sample Given in Table 2.1. Constructed using Method 1.

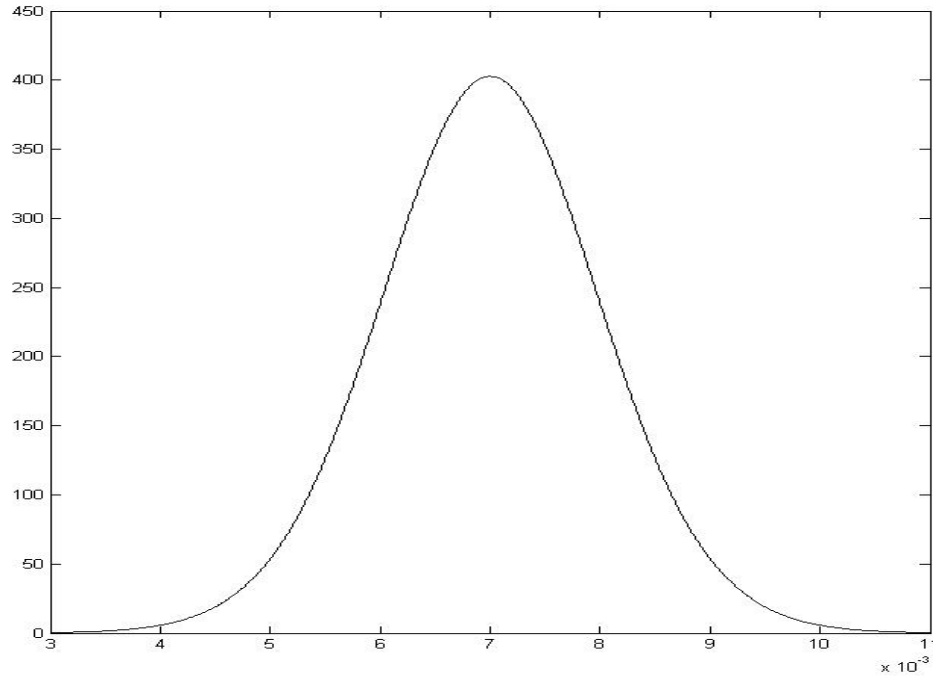


95% Equal Tail Credibility Interval or 0.95 - Expectation Tolerance Interval:
 $[0.0050, 0.0090]$

- ii.) Substitute the value for the sample mean, \bar{y} , obtained from the data, and the simulated value for σ_ε^2 into equation 2.6.6 and draw the normal distribution.
- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times.
- iv.) Using the Rao Blackwell method described in section 2.5, determine the estimated unconditional predictive distribution of $y_f | \mathbf{y}$.

For illustrative purposes, the estimated unconditional predictive distribution of $y_f | \mathbf{y}$, for the first sample given in Table 2.1, constructed using method 2, is provided in Figure 2.6.3.2.

Figure 2.6.3.2: Estimated Unconditional Predictive Distribution for the First Sample Given in Table 2.1. Constructed using Method 2.



95% Equal Tail Credibility Interval or 0.95 - Expectation Tolerance Interval:
[0.0050 , 0.0090]

Method 3

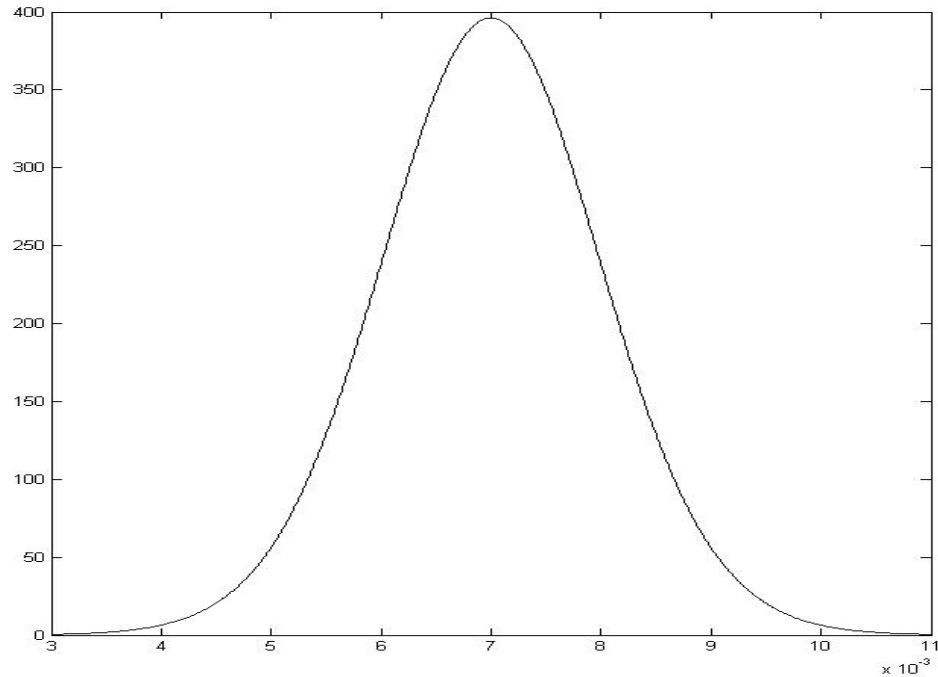
It was mentioned earlier that for the simple linear model given in equation 2.2.1, the unconditional predictive density of future measurements from a process can analytically be obtained.

It was therefore proved in Theorem 2.6.3.1 that the unconditional predictive density, i.e. $p(y_f|\mathbf{y})$ follows a student t - distribution with $\nu = n - 1$ degrees of freedom given by

$$p(y_f|\mathbf{y}) = \frac{\left\{\frac{\nu s^2}{2}\right\}^{\frac{1}{2}(n-1)}}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{n^{\frac{1}{2}}}{\sqrt{2\pi}(n+1)^{\frac{1}{2}}} \cdot \Gamma\left(\frac{n}{2}\right) \cdot \left[\frac{2}{\frac{n}{(n+1)}(y_f - \bar{y})^2 + \nu s^2}\right]^{\frac{1}{2}n}. \quad (2.6.7)$$

- i.) Calculate both the sample mean \bar{y} , and variance s^2 , using the sample data.
- ii.) Substitute both the sample mean and variance calculated from the sample data into equation 2.6.7 (derived in the prove of Theorem 2.6.3.1) and draw the student t - distribution.

Figure 2.6.3.3: Unconditional Predictive Distribution $p(y_f|\mathbf{y})$ for the First Sample Given in Table 2.1. Constructed using Method 3.



Using method 3, the unconditional predictive distribution, $p(y_f|\mathbf{y})$, for the first sample given in Table 2.1, is depicted in Figure 2.6.3.3.

From Figures 2.6.1.1 to 2.6.3.3 it is clear that all three methods for constructing the predictive distribution $p(y_f|\mathbf{y})$ are equivalent, since all three figures are for all practical purposes the same.

To determine a two - sided α - expectation tolerance interval using methods 1 to 3, obtain the $(\frac{1-\alpha}{2})^{th}$ and $(\frac{1+\alpha}{2})^{th}$ quantiles of the estimated unconditional - or unconditional predictive densities.

For the first sample given in Table 2.1, this two - sided 0.95 - expectation tolerance interval is given by $[0.0050, 0.0090]$ with posterior mean equals to 0.007. From this 0.95 - expectation tolerance interval it follows that the process will be in control if 95% or more future flatness measurements obtained from manufactured ceramic parts (manufactured using industrial process 1) will fall in the interval $[0.0050, 0.0090]$.

2.6.4 Fixed - in - Advance Tolerance Intervals

According to Wolfinger (1998), fixed - in - advance tolerance intervals answer research question 3 mentioned in Chapter 1. These fixed - in - advance tolerance intervals invert the prediction problem by considering the content of predetermined bounds (Wolfinger, 1998).

To determine the content of a fixed - in - advance tolerance interval using the Bayesian approach, the posterior density of the content has to be determined (Wolfinger, 1998). If an upper fixed - in - advance limit, s , is specified for a sample with data assumed to arise from the univariate normal model given in equation 2.2.1, then the content c of the interval $[s, \infty]$ for each observation in the simulated sample is determined by

$$c = 1 - \Phi \left[\frac{s - \mu}{\sigma_\varepsilon} \right] \quad (2.6.8)$$

where $\Phi [\cdot]$ represents a standard normal cumulative distribution function (Wolfinger, 1998). The content c of the interval $[s, \infty]$ represents the fraction of process measurements that lie above the fixed - in - advance preselected specification limit s . If the content c is therefore found for each observation in the simulated sample, these calculated c values form a sample from the posterior density of the content above a preselected specification limit s (Wolfinger, 1998).

To determine a fixed - in - advance tolerance interval for the content of the interval $[s, \infty]$, the following steps can be followed:

- i.) Simulate the variance component σ_ε^2 using the Bayesian simulation method explained in section 2.5.
- ii.) Given the simulated variance component σ_ε^2 , simulate a value for μ using equation 2.4.1.
- iii.) Substitute the simulated values for σ_ε^2 and μ into equation 2.6.8 and determine the content c of the interval $[s, \infty]$.

- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times to form a sample from the posterior density of the content above the fixed - in - advance upper specification limit s .

This sample of c values from the posterior density of the content can then be used to draw a histogram. Wolfinger (1998) also mentioned that this sample of content values can be used to determine estimates of the posterior mean, variance, quantiles or the entire density of the content. A $100(\alpha)\%$ equal tail credibility interval can also easily be obtained for the content of the interval $[s, \infty]$ by ranking the sample of c values in order of magnitude and then finding the $100(\frac{1-\alpha}{2})^{th}$ and $100(\frac{1+\alpha}{2})^{th}$ percentiles of the ranked simulated c values.

Using the method of Datta and Ghosh (1995), a probability matching prior for the content c given in equation 2.6.8 can also be derived. This is given in the following theorem.

Theorem 2.6.4.1

For the balanced univariate normal model given in equation 2.2.1, the prior distribution

$$\pi_m(\boldsymbol{\theta}) \propto \sigma_\varepsilon^{-1} \left[1 + \frac{(s - \mu)^2}{2\sigma_\varepsilon^2} \right]^{\frac{1}{2}} \quad (2.6.9)$$

is a probability matching prior for the content of the interval $[s, \infty]$ given by

$$c = 1 - \Phi \left[\frac{s - \mu}{\sigma_\varepsilon} \right]$$

where $\Phi \left[\frac{s - \mu}{\sigma_\varepsilon} \right]$ represents a standard normal cumulative distribution function. The prior distribution given by

$$\pi^m(\boldsymbol{\theta}) \propto \sigma^{-3} \left\{ 1 + \frac{(s - \mu)^2}{2\sigma_\varepsilon^2} \right\}^{-\frac{1}{2}} \quad (2.6.10)$$

is also a probability matching prior for the content c .

Proof

The proof of Theorem 2.6.4.1 is given in Appendix A.

Equations 2.6.9 and 2.6.10 are also probability matching priors for the content c^* given by

$$c^* = \Phi \left[\frac{s - \mu}{\sigma_\varepsilon} \right]$$

if s is a lower specification limit.

For the probability matching prior given in equation 2.6.10, the weighted Monte Carlo method will be used to simulate from the posterior distribution. The method proposed by Chen and Shao (1999) (See also Kim (2006)) does not require knowing the closed form of the marginal posterior distribution of c , only the Kernel of the posterior distribution of $(\mu, \sigma_\varepsilon^2)$ is needed. This weighted Monte Carlo method (sampling - importance resampling (SIR)) is especially suitable for computing Bayesian confidence intervals.

The weighted Monte Carlo algorithm aims at drawing a random sample from a target distribution π by first drawing a sample from a proposal distribution q . From this, a smaller sample is drawn with sample probabilities proportional to the importance ratios $\frac{\pi}{q}$. In the case of the credibility intervals it is not even necessary to draw the smaller sample. The weights (sample probabilities) are however important.

For the Jeffreys' independence prior distribution given in equation 2.3.1,

$$p(\mu, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2},$$

the joint posterior of the parameters μ and σ_ε^2 is

$$P_R(\mu, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu+2)} \exp \left\{ -\frac{1}{2} \left[\frac{n(\mu - \bar{y})^2}{\sigma_\varepsilon^2} + \frac{(n-1)s^2}{\sigma_\varepsilon^2} \right] \right\}. \quad (2.6.11)$$

Equation 2.6.11 represents the proposal distribution q .

In the case of the probability matching prior given in equation 2.6.10, the joint posterior distribution of the parameters is

$$P_M(\mu, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu+3)} \left[1 + \frac{(s - \mu)^2}{2\sigma_\varepsilon^2} \right]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \left[\frac{n(\mu - \bar{y})^2}{\sigma_\varepsilon^2} + \frac{(n-1)s^2}{\sigma_\varepsilon^2} \right] \right\}. \quad (2.6.12)$$

Equation 2.6.12 represents the target distribution π . It is important that q is a good approximation of π , i.e. that it does not have tails that are too thin. The sample probabilities are therefore proportional to

$$\frac{\pi}{q} = \frac{P_M(\mu, \sigma_\varepsilon^2)}{P_R(\mu, \sigma_\varepsilon^2)} = \sigma_\varepsilon^{-1} \left[1 + \frac{(s - \mu)^2}{2\sigma_\varepsilon^2} \right]^{-\frac{1}{2}}$$

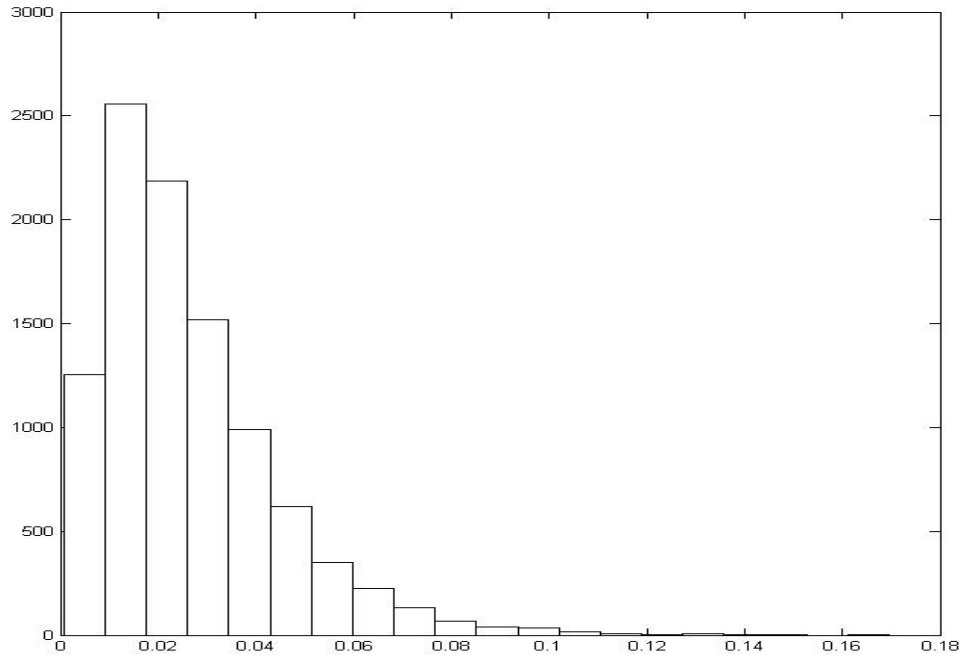
and the normalized weights for $l = 1, 2, \dots, \tilde{\ell}$ can be determined by

$$w_{(l)} = \frac{\sigma_\varepsilon^{-1(l)} \left[1 + \frac{(s - \mu^{(l)})^2}{2\sigma_\varepsilon^{2(l)}} \right]^{-\frac{1}{2}}}{\sum_{l=1}^{\tilde{\ell}} \sigma_\varepsilon^{-1(l)} \left[1 + \frac{(s - \mu^{(l)})^2}{2\sigma_\varepsilon^{2(l)}} \right]^{-\frac{1}{2}}}. \quad (2.6.13)$$

Using the weighted Monte Carlo method (sampling - importance resampling method), the fixed - in - advance tolerance interval for the probability matching prior given in equation 2.6.10 can be obtained as follows:

- i.) Simulate variance components σ_ε^2 using the Bayesian simulation method described in section 2.5, and, for the simulated variance components, simulate values for the mean μ using equation 2.4.1. This is done to obtain a Monte Carlo sample $\{\mu^{(l)}, \sigma_\varepsilon^{2(l)} \text{ for } l = 1 \dots \tilde{\ell}\}$ from the proposal distribution q and to calculate $c_l = 1 - \Phi \left[\frac{s - \mu^{(l)}}{\sqrt{\sigma_\varepsilon^{2(l)}}} \right]$ for $(l = 1, 2, \dots, \tilde{\ell})$.
- ii.) Sort $\{c_l, l = 1, 2, \dots, \tilde{\ell}\}$ to obtain the ordered values $c_{(1)} \leq c_{(2)} \leq c_{(3)} \leq \dots \leq c_{(\tilde{\ell})}$.
- iii.) Compute the weighted function $w_{(l)}$ given in equation 2.6.13 associated with the l^{th} ordered $c_{(l)}$ value, since each simulated c_l value has an associated weight.

Figure 2.6.4.1: Histogram of the Estimated Posterior Distribution of the Content of the Interval $[0.009, \infty]$, i.e. the Fraction of Process Measurements that Lie Above the Fixed - in - Advance Upper Specification Limit $s = 0.009$ for the First Sample Given in Table 2.1.



95% Fixed - in - Advance Tolerance Interval: $[0.004559, 0.071053]$

- iv.) Sum the weights associated with each $c_{(l)}$ value from left to right (small to large) until $\sum_{l=1}^{k_1} w_{(l)} = \frac{1-\alpha}{2}$. Write down the corresponding $c_{(k_1)}$ value and denote it as $c^{\frac{1-\alpha}{2}}$. Also, obtain the sum of the weights associated with each $c_{(l)}$ value from left to right until you get $\sum_{l=1}^{k_2} w_{(l)} = \frac{1+\alpha}{2}$. Write down the corresponding ordered value $c_{(k_2)}$ and denote it as $c^{\frac{1+\alpha}{2}}$.
- v.) The $100(\alpha)\%$ fixed - in - advance tolerance interval is then given by $\left[c^{\frac{1-\alpha}{2}}, c^{\frac{1+\alpha}{2}} \right]$.

For illustrative purposes, a fixed - in - advance upper specification limit $s = 0.009$ was selected for the first sample given in Table 2.1. The histogram of $\tilde{\ell} = 10000$ simulated values for the posterior content c given in equation 2.6.8 of the interval $[0.009, \infty]$, obtained using ordinary Monte Carlo simulation, is depicted in Figure 2.6.4.1.

From Figure 2.6.4.1 it can be seen that the estimated posterior distribution of the content c for the interval $[0.009, \infty]$ is positively skewed with a posterior median equals to

0.0219.

The results in Table 2.3 also represent the fixed - in - advance tolerance intervals for the first sample given in Table 2.1 with an upper specification limit of $s = 0.009$. The two fixed - in - advance tolerance intervals were obtained using both the classical Bayesian simulation method (ordinary Monte Carlo simulation) and the weighted Monte Carlo (sampling - importance resampling) method.

Table 2.3: Fixed - in - Advance Tolerance Intervals for the First Sample Given in Table 2.1 for an Upper Specification Limit $s = 0.009$.

	Monte Carlo Method		Weighted Monte Carlo Method	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Observation Number	250	9750	250	9747
Sum of Weights	—	—	0.024891	0.974936
95% Credibility Interval	0.004559	0.071053	0.004559	0.070902

From Table 2.3 it can be seen that the ordinary Monte Carlo method and the weighted Monte Carlo method (used for the probability matching prior given in equation 2.6.10) provide fixed - in - advance tolerance intervals which are for all practical purposes the same for an upper specification limit $s = 0.009$. Using the results obtained from the weighted Monte Carlo method for illustrative purposes, the 95% ($\alpha = 0.95$) equal tail credibility interval for the posterior content of the interval $[0.009, \infty]$ is given by $[0.0046, 0.0709]$. This means that between 0.46% and 7.09% of future ceramic parts manufactured by process 1, will have flatness measurements above $s = 0.009$ with probability 0.95. As was mentioned, the above fixed - in - advance upper specification limit $s = 0.009$ was solely selected for illustrative purposes. In practice, these fixed - in - advance upper or lower specification limits are often determined from engineering or regulatory considerations.

2.6.5 Tolerance Intervals for the Difference Between Quantiles

When determining tolerance intervals for processes used in the manufacturing of products, the situation sometimes arise when the quantiles need to be compared. This difference between the two α quantiles is then given by

$$\gamma = q_1 - q_2 = (\mu_1 + z_\alpha \sigma_{\varepsilon_1}) - (\mu_2 + z_\alpha \sigma_{\varepsilon_2}) = (\mu_1 - \mu_2) + z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) \quad (2.6.14)$$

where

$$\mu_1 | \sigma_{\varepsilon_1}^2, \mathbf{y}_1 \sim N(\bar{y}_1, \frac{\sigma_{\varepsilon_1}^2}{n_1}),$$

$$\mu_2 | \sigma_{\varepsilon_2}^2, \mathbf{y}_2 \sim N(\bar{y}_2, \frac{\sigma_{\varepsilon_2}^2}{n_2}),$$

$$\sigma_1^2 = \frac{\nu_1 s_1^2}{\chi_{\nu_1}^2},$$

$$\sigma_2^2 = \frac{\nu_2 s_2^2}{\chi_{\nu_2}^2},$$

$$\nu_1 = n_1 - 1,$$

$$\nu_2 = n_2 - 1,$$

z_α represents the $100(\alpha)^{th}$ percentile of a standard normal distribution, and n_1 and n_2 represent the samples sizes.

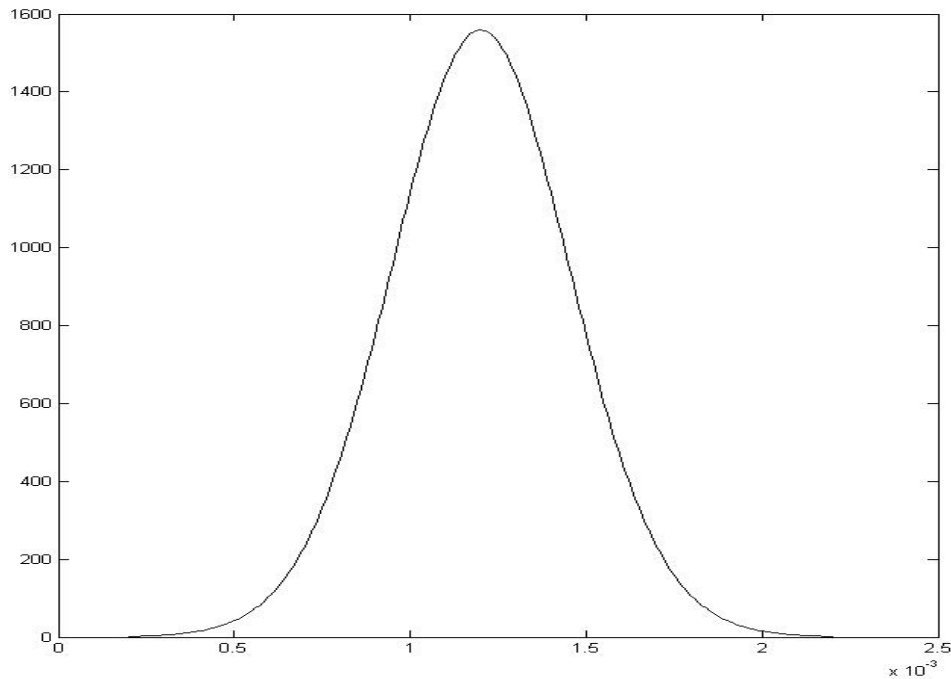
It is also clear that the conditional posterior distribution for $(\mu_1 - \mu_2)$ will be given by

$$(\mu_1 - \mu_2) | \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2 \sim N\left(\bar{y}_1 - \bar{y}_2, \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right). \quad (2.6.15)$$

The marginal posterior distribution for the difference between the two means $(\mu_1 - \mu_2) | \mathbf{y}$ can then be obtained as follows:

- i.) Simulate the variance components for the two processes $\sigma_{\varepsilon_1}^2 = \frac{\nu_1 s_1^2}{\chi_{\nu_1}^2}$, and $\sigma_{\varepsilon_2}^2 = \frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}$ using the Bayesian simulation method described in section 2.5.

Figure 2.6.5.1: Estimated Marginal Posterior Distribution of the Difference Between Two Process Population Means $(\mu_1 - \mu_2)|\mathbf{y}_1, \mathbf{y}_2$ for the Summary Data Given in Table 2.1.



95% Equal Tail Credibility Interval: [0.000704, 0.0017]

- ii.) Substitute the simulated values $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$ as well as the two sample means \bar{y}_1 and \bar{y}_2 into the conditional posterior distribution given in equation 2.6.15 and draw the normal distribution.
- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times.
- iv.) Using the Rao Blackwell method described in section 2.5, determine the estimated unconditional posterior distribution for the difference between the two population process means $(\mu_1 - \mu_2)$.

The estimated marginal posterior distribution of the difference between the average flatness measurements of ceramic parts produced by the two processes given in Table 2.1, is depicted in Figure 2.6.5.1.

For the estimated marginal posterior distribution given in Figure 2.6.5.1, the posterior mean of the difference $(\mu_1 - \mu_2)$ is equal to 0,0012 with 95% equal tail credibility interval

given by $[0.000704, 0.0017]$. Since zero does not fall in the 95% credibility interval, it can be concluded that the average flatness measurements of ceramic parts produced by the two processes given in Table 2.1, differ significantly with probability 0.95.

Since the conditional posterior distribution of the quantile $q = \mu + z_\alpha \sigma_\varepsilon$ is given by equation 2.6.2, i.e.

$$p(q|\sigma_\varepsilon^2, \mathbf{y}) \sim N(\bar{y} + z_\alpha \sigma_\varepsilon, \frac{\sigma_\varepsilon^2}{n}),$$

it is clear that the conditional posterior distribution of the difference between two α quantiles is given by

$$\gamma|\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2 \sim N\left\{(\bar{y}_1 - \bar{y}_2) + z_\alpha(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}), \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right\} \quad (2.6.16)$$

where $\gamma = (q_1 - q_2)$.

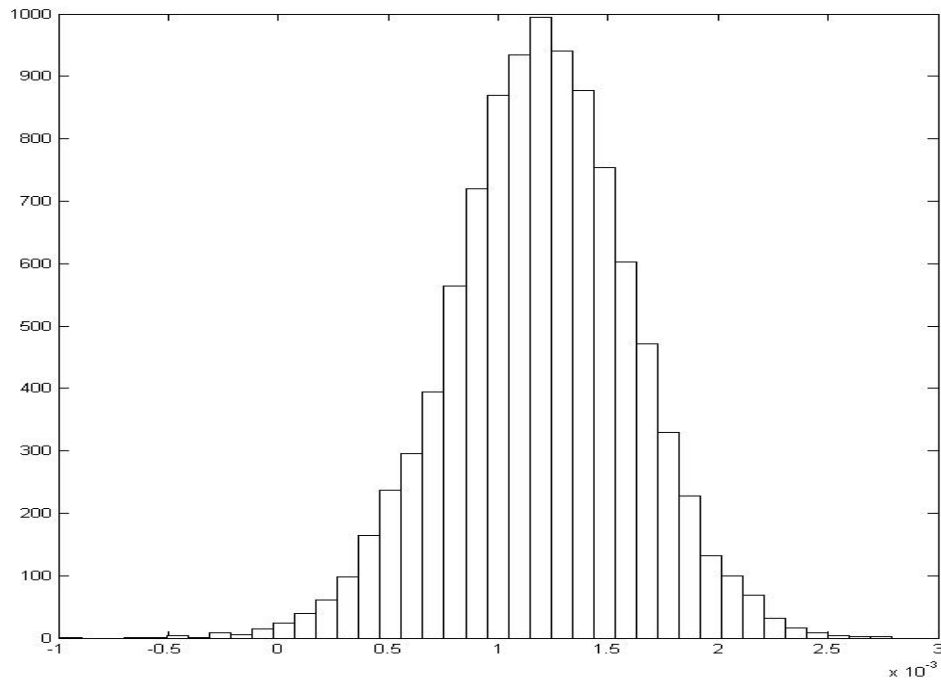
The estimated marginal posterior distribution of the difference between the quantiles $\gamma = (q_1 - q_2)$ can then be obtained in two ways using Bayesian simulation.

Method 1

- i.) Simulate the variance components for the two processes, i.e. $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$ using the Bayesian simulation method described in section 2.5.
- ii.) For the simulated variance components, simulate a value for each mean using $\mu_1|\sigma_{\varepsilon_1}^2, \mathbf{y}_1 \sim N(\bar{y}_1, \frac{\sigma_{\varepsilon_1}^2}{n_1})$, and $\mu_2|\sigma_{\varepsilon_2}^2, \mathbf{y}_2 \sim N(\bar{y}_2, \frac{\sigma_{\varepsilon_2}^2}{n_2})$.
- iii.) Substitute the simulated values for $\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mu_1$ and μ_2 into $q_1 = \mu_1 + z_\alpha \sigma_{\varepsilon_1}$, and $q_2 = \mu_2 + z_\alpha \sigma_{\varepsilon_2}$ and calculate $(q_1 - q_2)$, or substitute the simulated values for $\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mu_1$ and μ_2 into equation 2.6.14 to obtain $(q_1 - q_2)$.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and draw a histogram.

This histogram then represents the estimated marginal posterior distribution of the difference between the two α quantiles given by $\gamma = (q_1 - q_2)$. The upper one - sided (α, δ)

Figure 2.6.5.2: Histogram of the Estimated Marginal Posterior Distribution of the Difference Between the Two 0.95 Quantiles $\gamma = (q_1 - q_2)$ for the Data Given in Table 2.1.



95% Equal Tail Credibility Interval: [0.00036 , 0.0020]

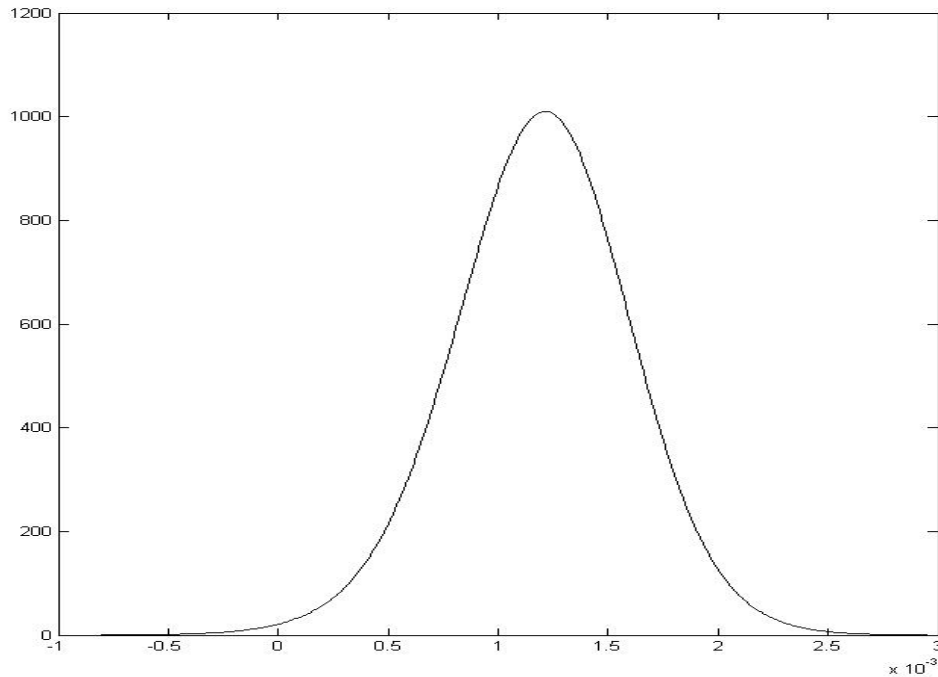
tolerance limit can then be obtained as the $100(\delta)^{th}$ percentile of the ranked simulated γ values.

The histogram of the estimated marginal posterior distribution of the difference between the two 0.95 quantiles $\gamma = (q_1 - q_2)$ is illustrated in Figure 2.6.5.2 for the summary data of the two processes given in Table 2.1.

Method 2

- i.) Simulate the variance components, $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$ using the same method as described in method 1.
- ii.) Substitute the simulated values for $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$, as well as the two sample means \bar{y}_1 and \bar{y}_2 into equation 2.6.16 and draw the normal distribution.
- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times.

Figure 2.6.5.3: Estimated Marginal Posterior Distribution of the Difference Between the Two 0.95 Quantiles for the Summary Data Given in Table 2.1.



95% Equal Tail Credibility Interval: [0.00038 , 0.0020]

- iv.) Using the Rao Blackwell method described in section 2.5, determine the estimated marginal posterior distribution for the difference between the two α quantiles.

Using method 2, the estimated marginal posterior distribution for the difference between the two 0.95 quantiles obtained for the summary data given in Table 2.1, is depicted in Figure 2.6.5.3.

For the summary data given in Table 2.1, the estimated marginal posterior distribution of the difference between the two 0.95 quantiles i.e. $\gamma = (q_1 - q_2)$ given by equation 2.6.16, has a posterior mean equals to 0.0012 and a 95% equal tail credibility interval equals to [0.00038 , 0.0020]. From both Figures 2.6.5.2 and 2.6.5.3 and the respective 95% credibility intervals, it can be seen that the two 0.95 quantiles of the flatness measurements of ceramic parts produced by the two processes given in Table 2.1, differ significantly, since zero does not fall in the 95% equal tail credibility intervals.

Note also that both methods 1 and 2 produced equivalent estimated marginal posterior distributions with 95% equal tail credibility intervals that were for all practical purposes the same.

The exact moments of the difference between two α quantiles i.e. $\gamma = (q_1 - q_2)$ can also be determined. By applying these central moments to Pearson curves or Cornish - Fisher expansions, approximations of the exact marginal posterior distribution of $(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2$ can be obtained.

The Exact Moments of $\gamma = q_1 - q_2$

The following theorem will now be proved (for notational purposes note that the first four moments about the origin of the conditional distribution of $(q_1 - q_2)$ (conditional on $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$) is given by μ'_1, μ'_2, μ'_3 and μ'_4 . Also, the central moments of $(q_1 - q_2)$ given $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$, is given by μ_2, μ_3 and μ_4 . Note also that the unconditional moments about the origin of $(q_1 - q_2)$ is given by m'_1, m'_2, m'_3 and m'_4 , while the unconditional central moments of $(q_1 - q_2)$ is given by m_2, m_3 and m_4).

Theorem 2.6.5.1

- a.) For the univariate normal model given in equation 2.2.1, the mean (first moment about zero) of the marginal posterior distribution of the difference between two α quantiles, i.e. $(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2$ is given by

$$E[\gamma|\mathbf{y}_1, \mathbf{y}_2] = (\bar{y}_1 - \bar{y}_2) + z_\alpha \left\{ (\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - (\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right\}$$

where $\gamma = (q_1 - q_2)$.

- b.) For the univariate normal model given in equation 2.2.1, the second central moment of the marginal posterior distribution of the difference between two α quantiles, i.e. $(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2$ is given by

$$\begin{aligned} \text{Var}(\gamma|\mathbf{y}_1, \mathbf{y}_2) &= (\nu_1 s_1^2) \left\{ \left(\frac{1}{\nu_1 - 2} \right) \left[z_\alpha^2 + \frac{1}{n_1} \right] - \frac{z_\alpha^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right\} \\ &\quad - (\nu_2 s_2^2) \left\{ \left(\frac{1}{\nu_2 - 2} \right) \left[z_\alpha^2 + \frac{1}{n_2} \right] - \frac{z_\alpha^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right\} \end{aligned}$$

where $\gamma = (q_1 - q_2)$.

- c.) For the univariate normal model given in equation 2.2.1, the third central moment of the marginal posterior distributions of the difference between two α quantiles, i.e. $(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2$ is given by

$$\begin{aligned} m_3 &= \frac{z_\alpha (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left\{ \left(\frac{1}{\nu_1 - 2} \right) \left[\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right] + \frac{(\nu_1 - 3) z_\alpha^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \right\} \\ &\quad - \frac{z_\alpha (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left\{ \frac{1}{(\nu_2 - 2)} \left[\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right] + \frac{(\nu_2 - 3) z_\alpha^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{\Gamma^2(\frac{\nu_2}{2})} \right\}. \end{aligned}$$

- d.) For the univariate normal model given in equation 2.2.1, the fourth central moment of the marginal posterior distribution of the difference between two α quantiles, i.e. $(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2$ is given by

$$\begin{aligned} m_4 &= \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)}{(\nu_d - 2)(\nu_d - 4)} \left(\frac{3}{n_d} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right) - \frac{6z_\alpha^2 (\nu_d - 1)(\nu_d s_d^2)^2 \Gamma(\frac{\nu_d - 1}{2}) \Gamma(\frac{\nu_d - 3}{2})}{2^2 n_d \Gamma^2(\frac{\nu_d}{2})(\nu_d - 2)} \right. \\ &\quad \left. + \frac{2(\nu_d - 5) z_\alpha^4 (\nu_d s_d^2)^2 \Gamma(\frac{\nu_d - 1}{2}) \Gamma(\frac{\nu_d - 3}{2})}{2^2 \Gamma^2(\frac{\nu_d}{2})(\nu_d - 2)} - \frac{3z_\alpha^4 (\nu_d s_d^2)^2 \Gamma^4(\frac{\nu_d - 1}{2})}{2^2 \Gamma^4(\frac{\nu_d}{2})} \right\} \\ &\quad + 6(\nu_1 s_1^2)(\nu_2 s_2^2) \left\{ \frac{1}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_1} + \frac{z_\alpha^2}{n_2} + z_\alpha^4 \right] - z_\alpha^4 \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})(\nu_2 - 2)} - z_\alpha^4 \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})(\nu_1 - 2)} \right. \\ &\quad \left. - z_\alpha^2 \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2}) n_2 (\nu_2 - 2)} - z_\alpha^2 \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2}) n_1 (\nu_1 - 2)} + z_\alpha^4 \frac{\Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} \right\}. \end{aligned}$$

Proof

The proof of Theorem 2.6.5.1 is given in Appendix A.

2.6.6 Differences Between More Than Two α Quantiles

As was mentioned for the difference between two α quantiles, the situation may also arise when more than two α quantiles need to be compared.

Bayesian significance testing and multiple comparisons for Markov chain Monte Carlo outputs have been proposed by Hoshino (2008), while Ganesh (2009) proposed simultaneous credible intervals for small area estimation problems. Two methods as proposed by Ganesh (2009) have been adapted for multiple comparisons of more than two α quantiles and will be discussed for the univariate normal model given in equation 2.2.1. The first method is based on simultaneous contrasts, while the second method is based on pairwise comparisons of more than two α quantiles. The rationale behind using simultaneous contrasts and the proposed pairwise comparisons method can be explained as follows: Suppose the probability of making a Type I error is α . If a series of confidence intervals are constructed, each with a probability α of indicating a difference between a pair of α quantiles if no difference exists, then the risk of making at least one Type I error in the series of inferences will be larger than the value of α specified for a single interval (Mendenhall and Sincich, 2003). Consequently, the selected value α is referred to as an experiment-wise error rate, rather than a comparison-wise error rate (Mendenhall and Sincich, 2003).

2.6.6.1 Multiple Comparisons Using Simultaneous Contrasts

For Bayesian multiple comparisons using simultaneous contrasts, the main focus will be on the construction of simultaneous $100(\alpha)\%$ credible intervals for all pairwise comparisons of α quantiles (Ganesh, 2009), where these pairwise comparisons represent special cases of general contrasts (Mendenhall and Sincich, 2003).

For the univariate normal model given in equation 2.2.1, it is known that the α^{th} quantile of the normal distribution $N(\mu, \sigma_\epsilon^2)$ is given by

$$q = \mu + z_\alpha \sigma_\varepsilon$$

with conditional posterior distribution given by equation 2.6.2. The α quantiles q_d is therefore given by

$$q_d = \mu_d + z_\alpha \sigma_{\varepsilon_d} \quad (d = 1, \dots, g)$$

with conditional posterior distributions given by

$$q_d | \sigma_{\varepsilon_d}^2, \mathbf{y}_d \sim N(\bar{y}_d + z_\alpha \sigma_{\varepsilon_d}, \frac{\sigma_{\varepsilon_d}^2}{n_d}) \quad (d = 1, \dots, g) \quad (2.6.17)$$

It was also proved in Theorem 2.6.1.1 a.) that

$$E(q_d | \mathbf{y}_d) = \bar{y}_d + z_\alpha (\nu_d s_d^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_d-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_d}{2})} \quad (d = 1, \dots, g) \quad (2.6.18)$$

and in Theorem 2.6.1.1 b.) that

$$Var(q_d | \mathbf{y}_d) = z_\alpha^2 \left\{ (\nu_d s_d^2) \left[\frac{1}{(\nu_d - 2)} - \frac{\Gamma^2(\frac{\nu_d-1}{2})}{2\Gamma^2(\frac{\nu_d}{2})} \right] \right\} + \frac{1}{n_d} (\nu_d s_d^2) \frac{1}{(\nu_d - 2)} \quad (d = 1, \dots, g) \quad (2.6.19)$$

For each of the g processes, simulate a q_d value by first simulating a variance $\sigma_{\varepsilon_d}^2$ using the Bayesian simulation method described in section 2.5. Substitute the simulated $\sigma_{\varepsilon_d}^2$ value into equation 2.6.17, determine q_d and call it q_d^* . Then use the g simulated q_d^* ($d = 1, \dots, g$) values to form a vector

$$\mathbf{q}^* = \begin{bmatrix} q_1^* \\ q_2^* \\ \vdots \\ q_g^* \end{bmatrix}$$

and for the g processes also determine the $E(q_d | \mathbf{y}_d)$ and $Var(q_d | \mathbf{y}_d)$ ($d = 1, \dots, g$) using equations 2.6.18 and 2.6.19 respectively.

The g expected values and variances of $q_d | \mathbf{y}_d$ ($d = 1, \dots, g$) are then used to set up a vector of expected values $E(\mathbf{q} | \mathbf{y})$ given by

$$E(\mathbf{q}|\mathbf{y}) = \begin{bmatrix} E(q_1|\mathbf{y}_1) \\ E(q_2|\mathbf{y}_2) \\ \vdots \\ E(q_g|\mathbf{y}_g) \end{bmatrix},$$

and a covariance matrix $Var(\mathbf{q}|\mathbf{y})$ given by

$$Var(\mathbf{q}|\mathbf{y}) = \begin{bmatrix} Var(q_1|\mathbf{y}_1) & 0 & 0 & \cdots & 0 \\ 0 & Var(q_2|\mathbf{y}_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & Var(q_g|\mathbf{y}_g) \end{bmatrix}$$

Proceed by setting up $g - 1$ pairwise contrasts ℓ' each with dimension $(1 \times g)$ such that $\sum_{d=1}^g \ell_d = 0$. These contrasts are for example given by

$$\ell'_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\ell'_2 = \begin{bmatrix} 0 & 1 & -1 & 0 & \cdots & 0 \end{bmatrix}$$

\vdots

$$\ell'_{g-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

which is then used to determine

$$T^{(3)} = \max_{\ell_1, \ell_2, \dots, \ell_{g-1}} \left\{ \frac{[\ell'_1(\mathbf{q}^* - E(\mathbf{q}|\mathbf{y}))]^2}{\ell'_1 var(\mathbf{q}|\mathbf{y}) \ell_1}, \frac{[\ell'_2(\mathbf{q}^* - E(\mathbf{q}|\mathbf{y}))]^2}{\ell'_2 var(\mathbf{q}|\mathbf{y}) \ell_2}, \dots, \frac{[\ell'_{g-1}(\mathbf{q}^* - E(\mathbf{q}|\mathbf{y}))]^2}{\ell'_{g-1} var(\mathbf{q}|\mathbf{y}) \ell_{g-1}} \right\}.$$

Repeat the process for example $\tilde{\ell} = 10000$ times to obtain 10000 $T^{(3)}$ values and sort the obtained $T^{(3)}$ values in order of magnitude. From the sorted $T^{(3)}$ values, obtain $T_\alpha^{(3)}$, the $100(\alpha)^{th}$ percentile of the sorted $T^{(3)}$ values.

For each of the $g - 1$ contrasts, then determine $100(\alpha)\%$ simultaneous Bayesian credibility intervals using $T_\alpha^{(3)}$. This can be done as follows using the contrasts mentioned above:

$$\begin{aligned}
1.) \quad & \ell'_1 E(\mathbf{q}|\mathbf{y}) T^{(2)} \pm \left\{ \ell'_1 \text{var}(\mathbf{q}|\mathbf{y}) \ell_1 T_\alpha^{(3)} \right\}^{\frac{1}{2}} \\
2.) \quad & \ell'_2 E(\mathbf{q}|\mathbf{y}) \pm \left\{ \ell'_2 \text{var}(\mathbf{q}|\mathbf{y}) \ell_2 T_\alpha^{(3)} \right\}^{\frac{1}{2}} \\
& \vdots \\
g-1.) \quad & \ell'_{g-1} E(\mathbf{q}|\mathbf{y}) \pm \left\{ \ell'_{g-1} \text{var}(\mathbf{q}|\mathbf{y}) \ell_{g-1} T_\alpha^{(3)} \right\}^{\frac{1}{2}}
\end{aligned}$$

If for example the first $100(\alpha)\%$ Bayesian credibility interval does contain zero, it can be interpreted that the two α quantiles of the first two processes, i.e. q_1 and q_2 do not differ significantly. Similarly, if for example the second $100(\alpha)\%$ Bayesian credibility interval does not contain zero, it can be interpreted that the two α quantiles of the second and third processes, i.e. q_2 and q_3 do differ significantly. All remaining $100(\alpha)\%$ Bayesian credibility intervals (from 3 to $g-1$) are interpreted in the same way.

2.6.6.2 Multiple Comparisons Using Pairwise Differences

For Bayesian multiple comparisons using pairwise differences, the focus is essentially on determining a critical value $T_\alpha^{(2)}$, i.e. the $100(\alpha)^{th}$ percentile of the estimated posterior distribution of $T^{(2)}$, where

$$T^{(2)} = \max_d \left\{ \left[q_d - E(q_d|\mathbf{y}_d) \right] | \mathbf{y}_d \right\} - \min_d \left\{ \left[q_d - E(q_d|\mathbf{y}_d) \right] | \mathbf{y}_d \right\} \quad \text{for } d = 1 \dots g.$$

As was mentioned for the proposed method of multiple comparisons using simultaneous contrasts for the univariate normal model given in equation 2.2.1, it is known that the $d = 1, \dots, g$ conditional posterior distributions of the α quantiles q_d are given by equation 2.6.17. It was also proved in Theorem 2.6.1.1 a.) that the unconditional expected values $E(q_d|\mathbf{y}_d)$ for the $d = 1, \dots, g$ processes are given by equation 2.6.18.

Simulate a q_d ($d = 1, \dots, g$) value for each of the g processes by first simulating a variance $\sigma_{\varepsilon_d}^2$ using the Bayesian simulation method described in section 2.5. Substitute the

simulated $\sigma_{\varepsilon_d}^2$ value into equation 2.6.17 and determine q_d ($d = 1, \dots, g$). For each of the g processes, determine $[q_d - E(q_d|\mathbf{y}_d)]|\mathbf{y}_d$ ($d = 1, \dots, g$).

Sort the g $[q_d - E(q_d|\mathbf{y}_d)]|\mathbf{y}_d$ ($d = 1, \dots, g$) values determined in order of magnitude, and, by using a Bayesian version of Tukey's method for constructing simultaneous confidence intervals, determine

$$T^{(2)} = \max_d \{[q_d - E(q_d|\mathbf{y}_d)]|\mathbf{y}_d\} - \min_d \{[q_d - E(q_d|\mathbf{y}_d)]|\mathbf{y}_d\} \quad (d = 1, \dots, g) \quad (\text{Ganesh, 2009}).$$

The logic behind using $T^{(2)}$ for this multiple comparisons procedure is that if a critical value is determined for the difference between two α quantiles as $\max_d \left\{ [q_d - E(q_d|\mathbf{y}_d)]|\mathbf{y}_d \right\} - \min_d \left\{ [q_d - E(q_d|\mathbf{y}_d)]|\mathbf{y}_d \right\}$ (this critical value implies a difference in the respective α quantiles), then any other pair of α quantiles that differ by as much as or more than this critical value would also imply a difference in the corresponding α quantiles (Mendenhall and Sincich, 2003).

Repeat the process for example $\tilde{\ell} = 10000$ times to obtain 10000 $T^{(2)}$ values. Sort the $T^{(2)}$ values in order of magnitude and determine $T_\alpha^{(2)}$, the $100(\alpha)^{th}$ percentile of the sorted $T^{(2)}$ values.

Determine $|E(q_d|\mathbf{y}_d) - E(q_h|\mathbf{y}_h)|$ where $d = 1, \dots, g$, $h = 1, \dots, g$ and $d < h$ for each pairwise comparison of the α quantiles q_d ($d = 1, \dots, g$) for the g processes. Remember these $E(q_d|\mathbf{y}_d)$'s ($d = 1, \dots, g$) are determined using equation 2.6.18.

Two quantiles q_d and q_h ($d = 1, \dots, g$, $h = 1, \dots, g$, $d < h$) differ significantly if

$$|E(q_d|\mathbf{y}_d) - E(q_h|\mathbf{y}_h)| \geq T_\alpha^{(2)},$$

and do not differ significantly if

$$|E(q_d|\mathbf{y}_d) - E(q_h|\mathbf{y}_h)| < T_\alpha^{(2)}.$$

2.6.6.3 An Example

The two multiple comparisons procedures for comparing more than two α quantiles q_d ($d = 1, \dots, g$) will be illustrated using the summary data of flatness measurements obtained from aluminium parts manufactured by three processes. As mentioned, this summary data given in Table 2.2 was obtained from Hubele, et.al. (2005).

For this example, $g = 3$. For all three manufacturing processes given in Table 2.2, $\tilde{\ell} = 10000$ variances $\sigma_{\varepsilon_d}^2$ ($d = 1, 2, 3$) were simulated using the Bayesian simulation procedure described in section 2.5. For each of the 10000 simulated $\sigma_{\varepsilon_d}^2$ ($d = 1, 2, 3$) values, 10000 q_d ($d = 1, 2, 3$) values were simulated using equation 2.6.17. The expected values $E(q_d|\mathbf{y}_d)$ ($d = 1, 2, 3$) and variances $Var(q_d|\mathbf{y}_d)$ ($d = 1, 2, 3$) were also determined using equations 2.6.18 and 2.6.19 respectively.

Results obtained from the two multiple comparisons procedure are given in Table 2.4.

Table 2.4: Results Obtained for Comparing $g = 3$ 0.95 Quantiles q_d ($d = 1, 2, 3$) using Simultaneous Contrasts and Pairwise Differences for the Summary Data Given in Table 2.2.

Simultaneous Contrasts			Pairwise Differences	
$T_{0.95}^{(3)} = 5.3108$			$T_{0.95}^{(2)} = 1.38 \times 10^{-4}$	
95% Credibility Intervals obtained by $\ell'_d E(\mathbf{q} \mathbf{y}) \pm \left\{ \ell'_d Var(\mathbf{q} \mathbf{y}) \ell_d T_{\alpha}^{(3)} \right\}^{\frac{1}{2}}$ for $d = 1, 2$			Absolute Differences $ E(q_d \mathbf{y}_d) - E(q_h \mathbf{y}_h) $ obtained for $d = 1, 2, 3, h = 1, 2, 3$ and $d < h$.	
For	Lower Limit	Upper Limit	For	$ E(q_d \mathbf{y}_d) - E(q_h \mathbf{y}_h) $
$q_1 - q_2$	-7.95×10^{-5}	1.82×10^{-4}	$q_1 - q_2$	$5.14 \times 10^{-5} < 1.38 \times 10^{-4}$
$q_2 - q_3$	-4.15×10^{-4}	-1.80×10^{-4}	$q_1 - q_3$	$0.0002457 > 1.38 \times 10^{-4}$ *
			$q_2 - q_3$	$0.0002971 > 1.38 \times 10^{-4}$ *

Using the method of simultaneous contrasts, it can be seen from Table 2.4 that the 0.95th quantile q_1 determined for flatness measurements obtained from aluminium parts

produced by process one does not differ significantly from the 0.95th quantile q_2 determined for manufacturing process two, since zero is contained in the 95% credibility interval. There is however a significant difference between the 0.95th quantile q_2 and the 0.95th quantile q_3 determined for manufacturing processes two and three, since zero is not contained in the 95% credibility interval. Using the method of pairwise differences, it can be seen from Table 2.4 that the 0.95th quantile q_1 and the 0.95th quantile q_2 determined for processes one and two respectively do not differ significantly since $|E(q_1|\mathbf{y}_1) - E(q_2|\mathbf{y}_2)| < T_{0.95}^{(2)}$. Similarly, it can be seen that the 0.95th quantile q_3 determined for manufacturing process three differ significantly from both the 0.95th quantile q_1 and the 0.95th quantile q_2 determined for manufacturing processes one and two, since in both cases

$$|E(q_1|\mathbf{y}_1) - E(q_3|\mathbf{y}_3)| > T_{0.95}^{(2)} \text{ and}$$

$$|E(q_2|\mathbf{y}_2) - E(q_3|\mathbf{y}_3)| > T_{0.95}^{(2)} .$$

A simulation study was also performed to check the frequentist properties of the two Bayesian multiple comparisons procedures for differences between more than two α quantiles q_d ($d = 1, \dots, g$). The investigation was started by simulating $\tilde{\ell} = 10000$ data sets for each of the three manufacturing processes given in Table 2.2. For the simulation of the data, the summary statistics given for population 1 were taken as the population parameters given by $\mu_1 = \mu_2 = \mu_3 = 0.00045$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0.00012$, with the sample sizes for all three processes considered equal, i.e. $n_1 = n_2 = n_3 = 20$. For each of the 10000 data sets simulated per process, 1000 Bayesian simulations were performed using equation 2.6.17 to obtain 1000 simulated q_d values. For each of the 10000 data sets, these 1000 simulated q_d values were then used to obtain the $T_{0.95}^{(3)}$ and $T_{0.95}^{(2)}$ values, and in the case of the multiple comparisons procedure using simultaneous contrasts, also the two 95% Bayesian credibility intervals. For the 10000 data sets, all significant differences indicated between the 0.95 quantiles q_d ($d = 1, 2, 3$) were then counted for each of the two proposed methods. Since the population parameters for all three manufacturing processes were considered the same, frequentist properties were met

if approximately $100(1 - 0.95)\%$ of the 95% credibility intervals for the two contrasts (in the case of the simultaneous contrasts method), and approximately $100(1 - 0.95)\%$ of the absolute differences (for the pairwise differences method), indicated differences between the 0.95 quantiles q_d ($d = 1, 2, 3$). The process was also repeated for sample sizes equal to 50 and 100. Results from the simulation study are reported in Table 2.5.

Table 2.5: Results from the Simulation Study Performed to Investigate the Frequentist Properties of the Two Bayesian Multiple Comparisons Procedures.

Sample Sizes	Simultaneous Contrasts Percentage Differences Indicated	Pairwise Differences Percentage Differences Indicated
20	4.0	3.4
50	4.7	4.1
100	5.1	5.0

From Table 2.5 it is clear that the frequentist properties of the two proposed Bayesian multiple comparisons procedures for comparing more than two 0.95 quantiles q_d ($d = 1, 2, 3$) are for all practical purposes met across the range of selected sample sizes, since the percentage differences, indicated for both method are approximately 5%. This is visible especially for larger sample sizes, although also acceptable for smaller sample sizes.

2.7 Appendix A

Proof of Theorem 2.3.1

Let $t(\boldsymbol{\theta}) = \mu + z_\alpha \sigma_\varepsilon$.

Transform μ to $t(\boldsymbol{\theta})$ and obtain the derivatives with respect to $t(\boldsymbol{\theta})$ and σ_ε^2 .

Therefore

$$\mu = t(\boldsymbol{\theta}) - z_\alpha \sigma_\varepsilon$$

with

$$\frac{\partial \mu}{\partial t(\boldsymbol{\theta})} = 1 \qquad \frac{\partial \mu}{\partial \sigma_\varepsilon^2} = -\frac{1}{2} \frac{z_\alpha}{\sigma_\varepsilon}$$

$$\frac{\partial \sigma_\varepsilon^2}{\partial t(\boldsymbol{\theta})} = 0 \qquad \frac{\partial \sigma_\varepsilon^2}{\partial \sigma_\varepsilon^2} = 1$$

Therefore

$$A = \begin{bmatrix} \frac{\partial \mu}{\partial t(\boldsymbol{\theta})} & \frac{\partial \mu}{\partial \sigma_\varepsilon^2} \\ \frac{\partial \sigma_\varepsilon^2}{\partial t(\boldsymbol{\theta})} & \frac{\partial \sigma_\varepsilon^2}{\partial \sigma_\varepsilon^2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \frac{z_\alpha}{\sigma_\varepsilon} \\ 0 & 1 \end{bmatrix}.$$

Now

$$\begin{aligned} F(t(\boldsymbol{\theta}), \sigma_\varepsilon^2) &= A' F(\mu, \sigma_\varepsilon^2) A \\ &= \begin{bmatrix} 1 & -\frac{1}{2} \frac{z_\alpha}{\sigma_\varepsilon} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{n}{\sigma_\varepsilon^2} & 0 \\ 0 & \frac{n}{2} \left(\frac{1}{\sigma_\varepsilon^2}\right)^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \frac{z_\alpha}{\sigma_\varepsilon} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{n}{\sigma_\varepsilon^2} & 0 \\ -\frac{z_\alpha n}{2\sigma_\varepsilon^3} & \frac{n}{2} \left(\frac{1}{\sigma_\varepsilon^2}\right)^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{z_\alpha}{2\sigma_\varepsilon} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{n}{\sigma_\varepsilon^2} & -\frac{n z_\alpha}{2\sigma_\varepsilon^3} \\ -\frac{z_\alpha n}{2\sigma_\varepsilon^3} & \frac{z_\alpha^2 n}{4\sigma_\varepsilon^4} + \frac{n}{2\sigma_\varepsilon^4} \end{bmatrix} \end{aligned}$$

$$\therefore F(t(\boldsymbol{\theta}), \sigma_\varepsilon^2) = \begin{bmatrix} \frac{n}{\sigma_\varepsilon^2} & -\frac{nz_\alpha}{2\sigma_\varepsilon^3} \\ -\frac{z_\alpha n}{2\sigma_\varepsilon^3} & \frac{z_\alpha^2 n}{4\sigma_\varepsilon^4} + \frac{n}{2\sigma_\varepsilon^4} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

The determinant of $F(t(\boldsymbol{\theta}), \sigma_\varepsilon^2)$ is given by

$$\frac{n^2 z_\alpha^2}{4\sigma_\varepsilon^6} + \frac{n^2}{2\sigma_\varepsilon^6} - \frac{n^2 z_\alpha^2}{4\sigma_\varepsilon^6} = \frac{n^2}{2\sigma_\varepsilon^6}.$$

Therefore

$$F^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{bmatrix} = \frac{2\sigma_\varepsilon^6}{n^2} \begin{bmatrix} \frac{n}{2\sigma_\varepsilon^4} \left(\frac{z_\alpha^2}{2} + 1 \right) & \frac{nz_\alpha}{2\sigma_\varepsilon^3} \\ \frac{nz_\alpha}{2\sigma_\varepsilon^3} & \frac{n}{\sigma_\varepsilon^2} \end{bmatrix}.$$

Therefore

$$F^{11} = \frac{\sigma_\varepsilon^2}{n} \left(\frac{z_\alpha^2}{2} + 1 \right) \text{ and } (F^{11})^{-1} = \frac{n}{\sigma_\varepsilon^2} \left(\frac{z_\alpha^2}{2} + 1 \right)^{-1}$$

and

$$p(t(\boldsymbol{\theta})) \propto h_{11}^{\frac{1}{2}} \text{ which is a constant since it does not contain } t(\boldsymbol{\theta}).$$

$$\text{Also, } h_{22} = \frac{n}{2\sigma_\varepsilon^4} \left(\frac{z_\alpha^2}{2} + 1 \right) = F_{22}.$$

and

$$p(\sigma_\varepsilon^2 | t(\boldsymbol{\theta})) \propto h_{22}^{\frac{1}{2}} \propto \sigma_\varepsilon^{-2}.$$

Therefore, the reference prior for the ordering

$$\{t(\boldsymbol{\theta}), \sigma_\varepsilon^2\} = p(t(\boldsymbol{\theta}), \sigma_\varepsilon^2) = p(t(\boldsymbol{\theta})) \cdot p(\sigma_\varepsilon^2 | t(\boldsymbol{\theta})) \propto \sigma_\varepsilon^{-2}.$$

In the $(\mu, \sigma_\varepsilon^2)$ parameterization this corresponds to

$$p(\mu, \sigma_\varepsilon^2) = p(t(\boldsymbol{\theta}), \sigma_\varepsilon^2) \left| \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} \right|.$$

where $\left\| \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} \right\|$ represents the jacobian of the transformation from $t(\boldsymbol{\theta})$ to μ .

Since $\left| \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} \right| = 1$, it follows that $p(\mu, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2}$.

Proof of Theorem 2.3.2

Let $t(\boldsymbol{\theta}) = \mu + z_\alpha \sigma_\varepsilon$.

Therefore $\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} = 1$, and

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} = \frac{z_\alpha}{2\sigma_\varepsilon}.$$

Therefore $\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})$

$$\begin{aligned} &= \begin{bmatrix} 1 & \frac{z_\alpha}{2\sigma_\varepsilon} \end{bmatrix} \begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & 0 \\ 0 & \frac{2(\sigma_\varepsilon^2)^2}{n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & \frac{z_\alpha}{2\sigma_\varepsilon} \cdot \frac{2}{n} (\sigma_\varepsilon^2)^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & \frac{z_\alpha \sigma_\varepsilon^3}{n} \end{bmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta}) &= \begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & \frac{z_\alpha \sigma_\varepsilon^3}{n} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{z_\alpha}{2\sigma_\varepsilon} \end{bmatrix} \\ &= \frac{\sigma_\varepsilon^2}{n} + \frac{z_\alpha^2 \sigma_\varepsilon^2}{2n} \\ &= \frac{\sigma_\varepsilon^2}{n} \left[1 + \frac{z_\alpha^2}{2} \right]. \end{aligned}$$

Also

$$\begin{aligned} \boldsymbol{\gamma}(\boldsymbol{\theta}) &= \frac{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})}{\sqrt{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}} \\ &= \begin{bmatrix} \gamma_1(\boldsymbol{\theta}) & \gamma_2(\boldsymbol{\theta}) \end{bmatrix} \\ &= \frac{\begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & \frac{z_\alpha \sigma_\varepsilon^3}{n} \end{bmatrix}}{\sqrt{\frac{\sigma_\varepsilon^2}{n} \left[1 + \frac{z_\alpha^2}{2} \right]}} \\ &= \begin{bmatrix} \frac{\frac{\sigma_\varepsilon^2}{n}}{\left[\frac{\sigma_\varepsilon^2}{n} \left[1 + \frac{z_\alpha^2}{2} \right] \right]^{\frac{1}{2}}} & \frac{\frac{z_\alpha \sigma_\varepsilon^3}{n}}{\left[\frac{\sigma_\varepsilon^2}{n} \left[1 + \frac{z_\alpha^2}{2} \right] \right]^{\frac{1}{2}}} \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{\frac{\sigma_\varepsilon^2}{n}}{\frac{\sigma_\varepsilon}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{\frac{1}{2}}} \quad \frac{\frac{z_\alpha \sigma_\varepsilon^2}{n}}{\frac{\sigma_\varepsilon}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{\frac{1}{2}}} \right] \\
&= \left[\frac{\frac{\sigma_\varepsilon^2 \sqrt{n}}{n \sigma_\varepsilon} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}}}{\frac{z_\alpha \sigma_\varepsilon^2 \sqrt{n}}{n \sigma_\varepsilon} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}}} \right] \\
&= \left[\frac{\frac{\sigma_\varepsilon}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}}}{\frac{z_\alpha \sigma_\varepsilon^2}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}}} \right].
\end{aligned}$$

If $p(\mu, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2}$ is considered, then

$$\frac{\partial}{\partial \mu} \{\gamma_1(\boldsymbol{\theta})p(\mu, \sigma_\varepsilon^2)\} + \frac{\partial}{\partial \sigma_\varepsilon^2} \{\gamma_2(\boldsymbol{\theta})p(\mu, \sigma_\varepsilon^2)\} = 0,$$

since

$$\begin{aligned}
\gamma_1(\boldsymbol{\theta}) \cdot p(\mu, \sigma_\varepsilon^2) &= \frac{\sigma_\varepsilon}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}} \cdot \sigma_\varepsilon^{-2} \gamma_1(\boldsymbol{\theta}) \cdot p(\mu, \sigma_\varepsilon^2) \\
&= \frac{\sigma_\varepsilon}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}} \cdot \sigma_\varepsilon^{-2} \\
&= \frac{\sigma_\varepsilon^{-1}}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}} \\
&= \frac{\sigma_\varepsilon^{-1}}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}},
\end{aligned}$$

$$\text{and, therefore } \frac{\partial}{\partial \mu} \left\{ \frac{\sigma_\varepsilon^{-1}}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}} \right\} = 0.$$

Also

$$\begin{aligned}
\gamma_2(\boldsymbol{\theta}) \cdot p(\mu, \sigma_\varepsilon^2) &= \frac{z_\alpha \sigma_\varepsilon^2}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}} \cdot \sigma_\varepsilon^{-2} \\
&= \frac{z_\alpha}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}},
\end{aligned}$$

$$\text{and, therefore } \frac{\partial}{\partial \sigma_\varepsilon^2} \left\{ \frac{z_\alpha}{\sqrt{n}} \left[1 + \frac{z_\alpha^2}{2}\right]^{-\frac{1}{2}} \right\} = 0.$$

Therefore

$$p(\mu, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2} \text{ is a probability matching prior for the } \alpha^{\text{th}} \text{ quantile } q = \mu + z_\alpha \sigma_\varepsilon.$$

Proof of Theorem 2.6.1.1

a.) The first moment about the origin of $(\frac{1}{\chi^2_\nu})^{\frac{1}{2}}$ is obtained by considering $r = 1$, therefore

$$E\left(\frac{1}{x}\right)^{\frac{1}{2}} = \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}$$

and therefore

$$E\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}} = \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}.$$

Since it is known that the first moment about the origin of $(\frac{1}{\chi^2_\nu})^{\frac{1}{2}}$ is given by

$$E\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}} = \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}, \text{ and,}$$

it is known that

$$q = \bar{y} + \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}} \left\{ \frac{z}{\sqrt{n}} + z_\alpha \right\},$$

it can be shown that the mean (first moment about zero) of the marginal posterior distribution of q is given by

$$\begin{aligned} E(q|\mathbf{y}) &= E \left[\bar{y} + \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}} \left\{ \frac{z}{\sqrt{n}} + z_\alpha \right\} \right] \\ &= E \left[\bar{y} + z \frac{\sigma_\varepsilon}{\sqrt{n}} + z_\alpha \sigma_\varepsilon \right] \\ &= E[\bar{y}] + E[z] E \left[\frac{\sigma_\varepsilon}{\sqrt{n}} \right] + E[z_\alpha] E[\sigma_\varepsilon] \\ &= \bar{y} + 0 + z_\alpha E[\sigma_\varepsilon] \quad \text{since } E[z] = 0. \\ &= \bar{y} + z_\alpha (\nu s^2)^{\frac{1}{2}} E \left[\left(\frac{1}{\chi^2_\nu} \right)^{\frac{1}{2}} \right] \\ &= \bar{y} + z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}. \end{aligned}$$

b.) The second moment about the origin of $\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}}$ is obtained by considering $r = 2$, therefore

$$E\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}} = \frac{\Gamma\left(\frac{\nu-2}{2}\right)}{2\Gamma\left(\frac{\nu}{2}\right)} = \frac{\Gamma\left(\frac{\nu-2}{2}\right)}{2\left(\frac{\nu-2}{2}\right)\Gamma\left(\frac{\nu-2}{2}\right)} = \frac{1}{\nu-2}.$$

By considering the conditional posterior distribution of q (conditional on σ_ε^2), the

$$E(q|\sigma_\varepsilon^2, \mathbf{y}) = \bar{y} + z_\alpha \sigma_\varepsilon$$

and

$$Var(q|\sigma_\varepsilon^2, \mathbf{y}) = \frac{\sigma_\varepsilon^2}{n}.$$

It is therefore known that the conditional posterior distribution of q is given by

$$p(q|\sigma_\varepsilon^2, \mathbf{y}) \sim N\left(\bar{y} + z_\alpha \sigma_\varepsilon, \frac{\sigma_\varepsilon^2}{n}\right).$$

Therefore

$$q|\chi^2_\nu, \mathbf{y} \sim N\left(\bar{y} + z_\alpha \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}}, \frac{1}{n} \frac{\nu s^2}{\chi^2_\nu}\right).$$

Therefore

$$Var(q|\mathbf{y}) = Var\left\{\bar{y} + z_\alpha \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}}\right\} + E_{\chi^2_\nu}\left\{\frac{1}{n} \frac{\nu s^2}{\chi^2_\nu}\right\}.$$

Now,

$$\begin{aligned} Var_{\chi^2_\nu}\left\{\bar{y} + z_\alpha \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}}\right\} &= Var_{\chi^2_\nu}\{\bar{y}\} + Var_{\chi^2_\nu}\left\{z_\alpha \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}}\right\} \\ &= Var_{\chi^2_\nu}\left\{z_\alpha \left(\frac{\nu s^2}{\chi^2_\nu}\right)^{\frac{1}{2}}\right\} \\ &= z_\alpha^2 (\nu s^2) Var_{\chi^2_\nu}\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}}. \end{aligned}$$

It can also be shown that

$$Var_{\chi^2_\nu}\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}} = E\left[\frac{1}{\chi^2_\nu}\right] - \left\{E\left[\frac{1}{\chi^2_\nu}\right]^{\frac{1}{2}}\right\}^2.$$

Since

$$E\left(\frac{1}{\chi_\nu^2}\right) = \frac{1}{(\nu-2)}$$

and

$$E\left(\frac{1}{\chi_\nu^2}\right)^{\frac{1}{2}} = \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}, \text{ the}$$

$$\begin{aligned} \text{Var}\left(\frac{1}{\chi_\nu^2}\right)^{\frac{1}{2}} &= \frac{1}{\nu-2} - \left\{ \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} \right\}^2 \\ &= \frac{1}{\nu-2} - \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})}. \end{aligned}$$

It can therefore be shown that

$$\text{Var}_{\chi_\nu^2} \left[\bar{y} + z_\alpha \left(\frac{vs^2}{\chi_\nu^2} \right)^{\frac{1}{2}} \right] = z_\alpha^2 (vs^2) \left\{ \frac{1}{\nu-2} - \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right\}.$$

Also,

$$E_{\chi_\nu^2} \left[\frac{1}{n} \frac{vs^2}{\chi_\nu^2} \right] = \frac{1}{n} vs^2 E_{\chi_\nu^2} \left[\frac{1}{\chi_\nu^2} \right] = \frac{vs^2}{n} \left(\frac{1}{\nu-2} \right).$$

It is therefore clear that the variance of the unconditional posterior distribution of q (the second central moment of the unconditional posterior distribution of q) is given by

$$\begin{aligned} \text{Var}(q|\mathbf{y}) &= z_\alpha^2 (vs^2) \left\{ \frac{1}{\nu-2} - \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right\} + \frac{vs^2}{n} \left(\frac{1}{\nu-2} \right) \\ &= (vs^2) \left\{ z_\alpha^2 \left[\frac{1}{\nu-2} - \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right] + \frac{1}{n} \left(\frac{1}{\nu-2} \right) \right\} \\ &= (vs^2) \left\{ \frac{z_\alpha^2}{\nu-2} - \frac{z_\alpha^2 \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} + \frac{1}{n} \left(\frac{1}{\nu-2} \right) \right\} \\ &= (vs^2) \left[\frac{1}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) - \frac{z_\alpha^2 \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right]. \end{aligned}$$

c.) The third moment about the origin of $(\frac{1}{\chi^2})^{\frac{1}{2}}$ is obtained by considering $r = 3$,

$$\text{therefore } E\left(\frac{1}{\chi^2}\right)^{\frac{3}{2}} = \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})}.$$

It is therefore also known that

$$E\left(\frac{1}{\chi^2}\right)^{\frac{3}{2}} = \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})}.$$

The third moment about the origin of the conditional posterior distribution of q (conditional on σ_ε^2) can in general be written as

$$\mu'_3 = \mu_3 + 3\mu_2\mu'_1 + (\mu'_1)^3.$$

By substituting μ'_1 , μ_2 and μ_3 into the equation for μ'_3 , the third moment about the origin of $q|\sigma_\varepsilon^2, \mathbf{y}$, is given by

$$\begin{aligned} \mu'_3 &= 0 + 3\left(\frac{\sigma_\varepsilon^2}{n}\right)(\bar{y} + z_\alpha\sigma_\varepsilon) + (\bar{y} + z_\alpha\sigma_\varepsilon)^3 \\ &= \frac{3\bar{y}\sigma_\varepsilon^2}{n} + \frac{3z_\alpha\sigma_\varepsilon^3}{n} + \bar{y}^3 + 3\bar{y}^2z_\alpha\sigma_\varepsilon + 3\bar{y}z_\alpha^2\sigma_\varepsilon^2 + z_\alpha^3\sigma_\varepsilon^3 \\ &= \bar{y}^3 + 3\bar{y}^2z_\alpha\sigma_\varepsilon + 3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)\sigma_\varepsilon^2 + \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)\sigma_\varepsilon^3 \\ &= \bar{y}^3 + 3\bar{y}^2z_\alpha\left(\frac{\nu s^2}{\chi^2}\right)^{\frac{1}{2}} + 3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)\left(\frac{\nu s^2}{\chi^2}\right) + \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)\left(\frac{\nu s^2}{\chi^2}\right)^{\frac{3}{2}}. \end{aligned}$$

It was already shown that

$$m'_1 = E(q|\mathbf{y}) = \bar{y} + (\nu s^2)^{\frac{1}{2}}z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} \text{ and } m_2 = \text{Var}(q|\mathbf{y}) = (\nu s^2) \left[\frac{1}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) - z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right].$$

Also,

$$\begin{aligned} m'_3 &= E(\mu'_3) = E \left[\bar{y}^3 + 3\bar{y}^2z_\alpha\left(\frac{\nu s^2}{\chi^2}\right)^{\frac{1}{2}} + 3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)\left(\frac{\nu s^2}{\chi^2}\right) + \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)\left(\frac{\nu s^2}{\chi^2}\right)^{\frac{3}{2}} \right] \\ &= E(\bar{y}^3) + 3\bar{y}^2z_\alpha(\nu s^2)^{\frac{1}{2}} E \left[\left(\frac{1}{\chi^2}\right)^{\frac{1}{2}} \right] + 3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)(\nu s^2) E \left[\frac{1}{\chi^2} \right] + \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)(\nu s^2)^{\frac{3}{2}} E \left[\left(\frac{1}{\chi^2}\right)^{\frac{3}{2}} \right] \\ &= \bar{y}^3 + 3\bar{y}^2z_\alpha(\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} + 3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)(\nu s^2) \left[\frac{1}{\nu-2} \right] + \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)(\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})}. \end{aligned}$$

It is also known that the third unconditional central moment of q is given by

$$\begin{aligned}
m_3 &= m'_3 - 3m_2m'_1 - (m'_1)^3 \\
&= \left\{ \bar{y}^3 + 3\bar{y}^2 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 3\bar{y} \left(\frac{1}{n} + z_\alpha^2 \right) \left(\frac{\nu s^2}{\nu-2} \right) + \left(\frac{3z_\alpha}{n} + z_\alpha^3 \right) (\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \right\} \\
&\quad - 3 \left\{ (\nu s^2) \left[\frac{1}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) - z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right] \right\} \left\{ \bar{y} + (\nu s^2)^{\frac{1}{2}} z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right\} - \left\{ \bar{y} + (\nu s^2)^{\frac{1}{2}} z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right\}^3 \\
&= \bar{y}^3 + 3\bar{y}^2 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 3\bar{y} \left(\frac{1}{n} + z_\alpha^2 \right) \left(\frac{\nu s^2}{\nu-2} \right) + \left(\frac{3z_\alpha}{n} + z_\alpha^3 \right) (\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \\
&\quad - \left\{ \frac{3(\nu s^2)}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) - 3(\nu s^2) z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \right\} \left\{ \bar{y} + (\nu s^2)^{\frac{1}{2}} z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right\} - \left\{ \bar{y} + (\nu s^2)^{\frac{1}{2}} z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right\}^3 \\
&= \bar{y}^3 + 3\bar{y}^2 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 3\bar{y} \left(\frac{1}{n} + z_\alpha^2 \right) \left(\frac{\nu s^2}{\nu-2} \right) + \left(\frac{3z_\alpha}{n} + z_\alpha^3 \right) (\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \\
&\quad - \left\{ \frac{3(\nu s^2)}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) \bar{y} + \frac{3z_\alpha (\nu s^2)^{\frac{3}{2}}}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - 3z_\alpha^2 (\nu s^2) \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \bar{y} - 3z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} \right\} \\
&\quad - \left\{ \bar{y}^3 + 3\bar{y} (\nu s^2)^{\frac{1}{2}} z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 3\bar{y} (\nu s^2) z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} + (\nu s^2)^{\frac{3}{2}} z_\alpha^3 \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} \right\} \\
&= \bar{y}^3 + 3\bar{y}^2 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 3\bar{y} \left(\frac{1}{n} + z_\alpha^2 \right) \left(\frac{\nu s^2}{\nu-2} \right) + \left(\frac{3z_\alpha}{n} + z_\alpha^3 \right) (\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \\
&\quad - \frac{3(\nu s^2)}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) \bar{y} - \frac{3z_\alpha (\nu s^2)^{\frac{3}{2}}}{\nu-2} \left(z_\alpha^2 + \frac{1}{n} \right) \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 3z_\alpha^2 (\nu s^2) \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} \bar{y} + 3z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} \\
&\quad - \bar{y}^3 - 3\bar{y} (\nu s^2)^{\frac{1}{2}} z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - 3\bar{y} (\nu s^2) z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} - (\nu s^2)^{\frac{3}{2}} z_\alpha^3 \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})}.
\end{aligned}$$

Consider the terms that contain \bar{y}^3 :

$$\bar{y}^3 - \bar{y}^3 = 0.$$

Consider the terms that contain \bar{y}^2 :

$$3\bar{y}^2 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - 3\bar{y}^2 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} = 0.$$

Consider the terms that contain \bar{y} :

$$3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)(\nu s^2)\left(\frac{1}{\nu-2}\right) - 3\bar{y}\left(\frac{1}{n} + z_\alpha^2\right)(\nu s^2)\left(\frac{1}{\nu-2}\right) + 3(\nu s^2)z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma(\frac{\nu}{2})}\bar{y} - 3\bar{y}(\nu s^2)z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2\Gamma(\frac{\nu}{2})} = 0.$$

Therefore

$$\begin{aligned} m_3 &= \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)(\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3(\nu s^2)^{\frac{3}{2}}}{\nu-2} \left(\frac{1}{n} + z_\alpha^2\right) z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} + 3(\nu s^2)^{\frac{3}{2}} z_\alpha^3 \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} - (\nu s^2)^{\frac{3}{2}} z_\alpha^3 \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ &= \left(\frac{3z_\alpha}{n} + z_\alpha^3\right)(\nu s^2)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3(\nu s^2)^{\frac{3}{2}}}{\nu-2} \left(\frac{1}{n} + z_\alpha^2\right) z_\alpha \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} + 2(\nu s^2)^{\frac{3}{2}} z_\alpha^3 \frac{\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ &= \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{n(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} - \frac{3z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} + \frac{2(\nu s^2)^{\frac{3}{2}}z_\alpha^3\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ &= \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3z_\alpha^3(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \\ &\quad + \frac{(\nu s^2)^{\frac{3}{2}}z_\alpha^3(\nu-3)\Gamma(\frac{\nu-3}{2})\Gamma^2(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \quad \text{from } \Gamma(\frac{\nu-1}{2}) = (\frac{\nu-3}{2})\Gamma(\frac{\nu-3}{2}) \\ &= \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}(\nu-2)\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{z_\alpha^3(\nu s^2)^{\frac{3}{2}}(\nu-2)\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{3z_\alpha^3(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \\ &\quad + \frac{(\nu s^2)^{\frac{3}{2}}z_\alpha^3(\nu-3)\Gamma(\frac{\nu-3}{2})\Gamma^2(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ &= \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left[(\nu-2) - (\nu-3) \right] + \frac{z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left[(\nu-2) - 3(\nu-3) \right] + \frac{(\nu s^2)^{\frac{3}{2}}z_\alpha^3(\nu-3)\Gamma(\frac{\nu-3}{2})\Gamma^2(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ &= \frac{3z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{(2\nu-7)z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{(\nu s^2)^{\frac{3}{2}}z_\alpha^3(\nu-3)\Gamma(\frac{\nu-3}{2})\Gamma^2(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ &= \frac{z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left[\frac{3}{n(\nu-2)} - \frac{z_\alpha^2(2\nu-7)}{(\nu-2)} + \frac{z_\alpha^2(\nu-3)\Gamma^2(\frac{\nu-1}{2})}{\Gamma^2(\frac{\nu}{2})} \right] \\ &= \frac{z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left\{ \left[\frac{1}{(\nu-2)} \left(\frac{3}{n} - z_\alpha^2(2\nu-7) \right) \right] + \frac{z_\alpha^2(\nu-3)\Gamma^2(\frac{\nu-1}{2})}{\Gamma^2(\frac{\nu}{2})} \right\}. \end{aligned}$$

d.) The fourth moment about the origin of $\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}}$ is obtained by considering $r = 4$, therefore

$$\begin{aligned} E\left(\frac{1}{\chi^2_\nu}\right)^{\frac{4}{2}} &= \frac{\Gamma(\frac{\nu-4}{2})}{2^{\frac{4}{2}}\Gamma(\frac{\nu}{2})} \\ &= \frac{\Gamma(\frac{\nu-4}{2})}{2^2\Gamma(\frac{\nu-2}{2})\Gamma(\frac{\nu-2}{2})} \\ &= \frac{\Gamma(\frac{\nu-4}{2})}{2 \cdot 2(\frac{\nu-2}{2})(\frac{\nu-4}{2})\Gamma(\frac{\nu-4}{2})} \\ &= \frac{1}{(\nu-2)(\nu-4)}. \end{aligned}$$

It is therefore also known that

$$E\left(\frac{1}{\chi^2_\nu}\right)^{\frac{4}{2}} = E\left(\frac{1}{\chi^2_\nu}\right)^2 = \frac{1}{(\nu-2)(\nu-4)}.$$

The fourth moment about the origin of the conditional posterior distribution of q (conditional on σ_ε^2) can in general be written as

$$\begin{aligned} \mu'_4 &= \mu_4 + 4\mu'_1\mu_3 + 6(\mu'_1)^2\mu_2 + (\mu'_1)^4 \\ &= 3\left(\frac{\sigma_\varepsilon^2}{n}\right)^2 + 0 + 6(\bar{y} + z_\alpha\sigma_\varepsilon)^2\left(\frac{\sigma_\varepsilon^2}{n}\right) + (\bar{y} + z_\alpha\sigma_\varepsilon)^4 \\ &= 3\frac{\sigma_\varepsilon^4}{n^2} + 6(\bar{y}^2 + 2\bar{y}z_\alpha\sigma_\varepsilon + z_\alpha^2\sigma_\varepsilon^2)\left(\frac{\sigma_\varepsilon^2}{n}\right) + \bar{y}^4 + 4\bar{y}^3z_\alpha\sigma_\varepsilon + 6\bar{y}^2z_\alpha^2\sigma_\varepsilon^2 + 4\bar{y}z_\alpha^3\sigma_\varepsilon^3 + z_\alpha^4\sigma_\varepsilon^4 \\ &= 3\frac{\sigma_\varepsilon^4}{n^2} + 6\bar{y}^2\frac{\sigma_\varepsilon^2}{n} + 12\bar{y}z_\alpha\frac{\sigma_\varepsilon^3}{n} + 6z_\alpha^2\frac{\sigma_\varepsilon^4}{n} + \bar{y}^4 + 4\bar{y}^3z_\alpha\sigma_\varepsilon + 6\bar{y}^2z_\alpha^2\sigma_\varepsilon^2 + 4\bar{y}z_\alpha^3\sigma_\varepsilon^3 + z_\alpha^4\sigma_\varepsilon^4 \\ &= \bar{y}^4 + 4\bar{y}^3z_\alpha\sigma_\varepsilon + 6\bar{y}^2\sigma_\varepsilon^2\left(\frac{1}{n} + z_\alpha^2\right) + 4\bar{y}z_\alpha\sigma_\varepsilon^3\left(\frac{3}{n} + z_\alpha^2\right) + \sigma_\varepsilon^4\left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4\right) \\ &= \bar{y}^4 + 4\bar{y}^3z_\alpha(\nu s^2)^{\frac{1}{2}}\left(\frac{1}{\chi^2_\nu}\right)^{\frac{1}{2}} + 6\bar{y}^2(\nu s^2)\left(\frac{1}{\chi^2_\nu}\right)\left(\frac{1}{n} + z_\alpha^2\right) + 4\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\left(\frac{1}{\chi^2_\nu}\right)^{\frac{3}{2}}\left(\frac{3}{n} + z_\alpha^2\right) \\ &\quad + (\nu s^2)^2\left(\frac{1}{\chi^2_\nu}\right)^2\left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4\right). \end{aligned}$$

Now

$$m'_4 = E(\mu'_4).$$

Therefore

$$\begin{aligned} m'_4 &= E(\bar{y}^4) + 4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} E \left[\left(\frac{1}{\chi^2_\nu} \right)^{\frac{1}{2}} \right] + 6\bar{y}^2 (\nu s^2) \left(\frac{1}{n} + z_\alpha^2 \right) E \left[\left(\frac{1}{\chi^2_\nu} \right) \right] \\ &\quad + 4\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \left(\frac{3}{n} + z_\alpha^2 \right) E \left[\left(\frac{1}{\chi^2_\nu} \right)^{\frac{3}{2}} \right] + (\nu s^2)^2 \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) E \left[\left(\frac{1}{\chi^2_\nu} \right)^2 \right] \\ &= \bar{y}^4 + 4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 6\bar{y}^2 (\nu s^2) \left(\frac{1}{n} + z_\alpha^2 \right) \left(\frac{1}{\nu-2} \right) \\ &\quad + 4\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \left(\frac{3}{n} + z_\alpha^2 \right) \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + (\nu s^2)^2 \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \left[\frac{1}{(\nu-2)(\nu-4)} \right]. \end{aligned}$$

It is known that the fourth unconditional central moment of q is given by

$$m_4 = m'_4 - 4m'_1 m_3 - 6(m'_1)^2 m_2 - (m'_1)^4.$$

Now in the equation $m_4 = m'_4 - 4m'_1 m_3 - 6(m'_1)^2 m_2 - (m'_1)^4$, consider the following terms:

$$\begin{aligned} m'_4 &= \bar{y}^4 + 4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + 6\bar{y}^2 (\nu s^2) \left(\frac{1}{n} + z_\alpha^2 \right) \left(\frac{1}{\nu-2} \right) + 4\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \left(\frac{3}{n} + z_\alpha^2 \right) \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \\ &\quad + (\nu s^2)^2 \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \left[\frac{1}{(\nu-2)(\nu-4)} \right] \\ &= \bar{y}^4 + \frac{4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + \frac{6\bar{y}^2 (\nu s^2)}{n(\nu-2)} + \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2)}{\nu-2} + \frac{12\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{n 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + \frac{4\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \\ &\quad + \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} \end{aligned}$$

$$-4m'_1 m_3$$

$$= -4 \left[\bar{y} + z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right] (\nu s^2)^{\frac{3}{2}} z_\alpha \frac{\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \left\{ \left[\frac{1}{(\nu-2)} \left(\frac{3}{n} - z_\alpha^2 (2\nu-7) \right) \right] + z_\alpha^2 (\nu-3) \frac{\Gamma^2(\frac{\nu-1}{2})}{\Gamma^2(\frac{\nu}{2})} \right\}$$

$$= -4 \left[\bar{y} + \frac{z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right] \left\{ \frac{3z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{n(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} - \frac{z_\alpha^3 (\nu s^2)^{\frac{3}{2}} (2\nu-7) \Gamma(\frac{\nu-3}{2})}{(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + \frac{z_\alpha^3 (\nu s^2)^{\frac{3}{2}} (\nu-3) \Gamma^2(\frac{\nu-1}{2}) \Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} \right\}$$

$$= -4 \left[\bar{y} + \frac{z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right] \left\{ \frac{3z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{n(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} - \frac{z_\alpha^3 (\nu s^2)^{\frac{3}{2}} (2\nu-7) \Gamma(\frac{\nu-3}{2})}{(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + \frac{2z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2 \Gamma^3(\frac{\nu}{2})} \right\}$$

$$\text{from } (\nu-3) \Gamma(\frac{\nu-3}{2}) = 2 \Gamma(\frac{\nu-1}{2})$$

$$= -\frac{12\bar{y}z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{n(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3 (2\nu-7) (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} - \frac{8\bar{y}z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})}$$

$$- \frac{12z_\alpha^2 (\nu s^2)^2 \Gamma(\frac{\nu-1}{2}) \Gamma(\frac{\nu-3}{2})}{n(\nu-2) 2^2 \Gamma^2(\frac{\nu}{2})} + \frac{4z_\alpha^4 (2\nu-7) (\nu s^2)^2 \Gamma(\frac{\nu-1}{2}) \Gamma(\frac{\nu-3}{2})}{(\nu-2) 2^2 \Gamma^2(\frac{\nu}{2})} - \frac{8z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})}$$

$$-6(m'_1)^2 m_2$$

$$= -6(\nu s^2) \left[\bar{y}^2 + 2\bar{y}z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + z_\alpha^2 (\nu s^2) \frac{\Gamma^2(\frac{\nu-1}{2})}{2 \Gamma^2(\frac{\nu}{2})} \right] \left[\frac{1}{(\nu-2)} \left(\frac{1}{n} + z_\alpha^2 \right) - z_\alpha^2 \frac{\Gamma^2(\frac{\nu-1}{2})}{2 \Gamma^2(\frac{\nu}{2})} \right]$$

$$= -6(\nu s^2) \left[\bar{y}^2 + \frac{2\bar{y}z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + \frac{z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2 \Gamma^2(\frac{\nu}{2})} \right] \left[\frac{1}{n(\nu-2)} + \frac{z_\alpha^2}{(\nu-2)} - \frac{z_\alpha^2 \Gamma^2(\frac{\nu-1}{2})}{2 \Gamma^2(\frac{\nu}{2})} \right]$$

$$= -6(\nu s^2) \left\{ \frac{\bar{y}^2}{n(\nu-2)} + \frac{\bar{y}^2 z_\alpha^2}{(\nu-2)} - \frac{\bar{y}^2 z_\alpha^2 \Gamma^2(\frac{\nu-1}{2})}{2 \Gamma^2(\frac{\nu}{2})} + \frac{2\bar{y}z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{n(\nu-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + \frac{2\bar{y}z_\alpha^3 (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{(\nu-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - \frac{2\bar{y}z_\alpha^3 (\nu s^2)^{\frac{1}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} \right.$$

$$\left. + \frac{z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{n(\nu-2) 2 \Gamma^2(\frac{\nu}{2})} + \frac{z_\alpha^4 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{(\nu-2) 2 \Gamma^2(\frac{\nu}{2})} - \frac{z_\alpha^4 (\nu s^2) \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})} \right\}$$

$$= -\frac{6\bar{y}^2 (\nu s^2)}{n(\nu-2)} - \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2)}{(\nu-2)} + \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2 \Gamma^2(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-1}{2})}{n(\nu-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-1}{2})}{(\nu-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + \frac{12\bar{y}z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})}$$

$$- \frac{6z_\alpha^2 (\nu s^2)^2 \Gamma^2(\frac{\nu-1}{2})}{n(\nu-2) 2 \Gamma^2(\frac{\nu}{2})} - \frac{6z_\alpha^4 (\nu s^2)^2 \Gamma^2(\frac{\nu-1}{2})}{(\nu-2) 2 \Gamma^2(\frac{\nu}{2})} + \frac{6z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})}$$

$$\begin{aligned}
& (m_1')^4 \\
&= - \left[\bar{y} + z_\alpha (\nu s^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \right]^4 \\
&= -\bar{y}^4 - \frac{4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} - \frac{4\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} - \frac{z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
m_4 &= \bar{y}^4 + \frac{4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} + \frac{6\bar{y}^2 (\nu s^2)}{n(\nu-2)} + \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2)}{\nu-2} + \frac{12\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{n 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + \frac{4\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} \\
&+ \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} - \frac{12\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{n(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} + \frac{4\bar{y} z_\alpha^3 (2\nu-7) (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-3}{2})}{(\nu-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu}{2})} - \frac{8\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} \\
&- \frac{12z_\alpha^2 (\nu s^2)^2 \Gamma(\frac{\nu-1}{2}) \Gamma(\frac{\nu-3}{2})}{n(\nu-2) 2^2 \Gamma^2(\frac{\nu}{2})} + \frac{4z_\alpha^4 (2\nu-7) (\nu s^2)^2 \Gamma(\frac{\nu-1}{2}) \Gamma(\frac{\nu-3}{2})}{(\nu-2) 2^2 \Gamma^2(\frac{\nu}{2})} - \frac{8z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})} \\
&- \frac{6\bar{y}^2 (\nu s^2)}{n(\nu-2)} - \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2)}{(\nu-2)} + \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} - \frac{12\bar{y} z_\alpha (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-1}{2})}{n(\nu-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - \frac{12\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma(\frac{\nu-1}{2})}{(\nu-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \\
&+ \frac{12\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} - \frac{6z_\alpha^2 (\nu s^2)^2 \Gamma^2(\frac{\nu-1}{2})}{n(\nu-2) 2\Gamma^2(\frac{\nu}{2})} - \frac{6z_\alpha^4 (\nu s^2)^2 \Gamma^2(\frac{\nu-1}{2})}{(\nu-2) 2\Gamma^2(\frac{\nu}{2})} + \frac{6z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})} \\
&- \bar{y}^4 - \frac{4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} - \frac{4\bar{y} z_\alpha^3 (\nu s^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu}{2})} - \frac{z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})}.
\end{aligned}$$

Consider the terms that contain \bar{y}^4 :

$$\bar{y}^4 - \bar{y}^4 = 0$$

Consider the terms that contain \bar{y}^3 :

$$\frac{4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} - \frac{4\bar{y}^3 z_\alpha (\nu s^2)^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} = 0$$

Consider the terms that contain \bar{y}^2 :

$$\frac{6\bar{y}^2 (\nu s^2)}{n(\nu-2)} + \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2)}{\nu-2} - \frac{6\bar{y}^2 (\nu s^2)}{n(\nu-2)} - \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2)}{(\nu-2)} + \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} - \frac{6\bar{y}^2 z_\alpha^2 (\nu s^2) \Gamma^2(\frac{\nu-1}{2})}{2\Gamma^2(\frac{\nu}{2})} = 0$$

Consider the terms that contain \bar{y} :

$$\begin{aligned} & \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3(2\nu-7)(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{8\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ & - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{n(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} + \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} - \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \end{aligned}$$

From the terms containing \bar{y} , consider the terms that contain both \bar{y} and z_α^3 :

Therefore

$$\begin{aligned} & \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3(2\nu-7)(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{8\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} \\ & + \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} - \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu}{2})} \\ & = \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3(2\nu-7)(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} \\ & = \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3(2\nu-7)(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \quad \text{from } \Gamma(\frac{\nu-1}{2}) = (\frac{\nu-3}{2})\Gamma(\frac{\nu-3}{2}) \\ & = \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}(\nu-2)\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} + \frac{4\bar{y}z_\alpha^3(2\nu-7)(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \\ & = \frac{4\bar{y}z_\alpha^3(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left[(\nu-2) + (2\nu-7) - 3(\nu-3) \right] \\ & = 0 \end{aligned}$$

From the terms containing \bar{y} , consider those terms that contain both \bar{y} and z_α :

Therefore

$$\begin{aligned}
& \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}{n(\nu-2)2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} \\
&= \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \quad \text{from } \Gamma(\frac{\nu-1}{2}) = (\frac{\nu-3}{2})\Gamma(\frac{\nu-3}{2}) \\
&= \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}(\nu-2)\Gamma(\frac{\nu-3}{2})}{(\nu-2)n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} - \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}(\nu-3)\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \\
&= \frac{12\bar{y}z_\alpha(\nu s^2)^{\frac{3}{2}}\Gamma(\frac{\nu-3}{2})}{(\nu-2)n2^{\frac{3}{2}}\Gamma(\frac{\nu}{2})} \left[(\nu-2) - 1 - (\nu-3) \right] \\
&= 0
\end{aligned}$$

The terms containing \bar{y} all therefore become equal to zero.

The fourth central moment of the unconditional posterior distribution of q i.e. $p(q|\mathbf{y})$ is therefore given by

$$\begin{aligned}
m_4 &= \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} - \frac{12z_\alpha^2(\nu s^2)^2\Gamma(\frac{\nu-1}{2})\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} + \frac{4z_\alpha^4(2\nu-7)(\nu s^2)^2\Gamma(\frac{\nu-1}{2})\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} \\
&\quad - \frac{8z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} - \frac{6z_\alpha^2(\nu s^2)^2\Gamma^2(\frac{\nu-1}{2})}{n(\nu-2)2\Gamma^2(\frac{\nu}{2})} - \frac{6z_\alpha^4(\nu s^2)^2\Gamma^2(\frac{\nu-1}{2})}{(\nu-2)2\Gamma^2(\frac{\nu}{2})} \\
&\quad + \frac{6z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} - \frac{z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} \\
&= \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} - \frac{12z_\alpha^2(\nu s^2)^2\Gamma(\frac{\nu-1}{2})\Gamma(\frac{\nu-3}{2})}{n(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} + \frac{4z_\alpha^4(2\nu-7)(\nu s^2)^2\Gamma(\frac{\nu-1}{2})\Gamma(\frac{\nu-3}{2})}{(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} \\
&\quad - \frac{8z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} - \frac{6z_\alpha^2(\nu s^2)^2(\nu-3)\Gamma(\frac{\nu-3}{2})\Gamma(\frac{\nu-1}{2})}{n(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} - \frac{6z_\alpha^4(\nu s^2)^2(\nu-3)\Gamma(\frac{\nu-3}{2})\Gamma(\frac{\nu-1}{2})}{(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} \\
&\quad + \frac{6z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} - \frac{z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} \quad \text{from } \Gamma(\frac{\nu-1}{2}) = (\frac{\nu-3}{2})\Gamma(\frac{\nu-3}{2}) \\
&= \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} + \frac{z_\alpha^4(\nu s^2)^2\Gamma^4(\frac{\nu-1}{2})}{2^2\Gamma^4(\frac{\nu}{2})} \left[-8 + 6 - 1 \right] \\
&\quad - \frac{6z_\alpha^2(\nu s^2)^2\Gamma(\frac{\nu-3}{2})\Gamma(\frac{\nu-1}{2})}{n(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} \left[2 + (\nu-3) \right] + \frac{2z_\alpha^4(\nu s^2)^2\Gamma(\frac{\nu-3}{2})\Gamma(\frac{\nu-1}{2})}{(\nu-2)2^2\Gamma^2(\frac{\nu}{2})} \left[2(2\nu-7) - 3(\nu-3) \right].
\end{aligned}$$

Therefore

$$m_4 = \left(\frac{3}{n^2} + \frac{6z_\alpha^2}{n} + z_\alpha^4 \right) \frac{(\nu s^2)^2}{(\nu-2)(\nu-4)} - \frac{3z_\alpha^4 (\nu s^2)^2 \Gamma^4(\frac{\nu-1}{2})}{2^2 \Gamma^4(\frac{\nu}{2})} - \frac{6(\nu-1)z_\alpha^2 (\nu s^2)^2 \Gamma(\frac{\nu-3}{2}) \Gamma(\frac{\nu-1}{2})}{n(\nu-2)2^2 \Gamma^2(\frac{\nu}{2})} + \frac{2(\nu-5)z_\alpha^4 (\nu s^2)^2 \Gamma(\frac{\nu-3}{2}) \Gamma(\frac{\nu-1}{2})}{(\nu-2)2^2 \Gamma^2(\frac{\nu}{2})}.$$

Proof of Theorem 2.6.3.1

Let a future observation from a process be equal to y_f which follows a normal distribution with mean equal to μ and variance equal to σ_ε^2 , i.e.

$$y_f \sim N(\mu, \sigma_\varepsilon^2). \quad (2.7.1)$$

Equation 2.7.1 can therefore be written as

$$f(y_f | \mu, \sigma_\varepsilon^2) = \frac{1}{\sigma_\varepsilon \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(y_f - \mu)^2}{\sigma_\varepsilon^2}}.$$

Now

$$f(y_f | \sigma_\varepsilon^2, \mathbf{y}) = \int_{-\infty}^{\infty} f(y_f | \mu, \sigma_\varepsilon^2) \cdot p(\mu | \mathbf{y}) du.$$

Since

$\mu | \mathbf{y} \sim N(\mu, \frac{\sigma_\varepsilon^2}{n})$, it follows that

$$\begin{aligned} f(y_f | \sigma_\varepsilon^2, \mathbf{y}) &= \int_{-\infty}^{\infty} \frac{1}{\sigma_\varepsilon \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(y_f - \mu)^2}{\sigma_\varepsilon^2}} \cdot \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon \sqrt{2\pi}} e^{-\frac{n}{2\sigma_\varepsilon^2} (\mu - \bar{y})^2} du \\ &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2 (2\pi)} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma_\varepsilon^2} (y_f^2 - 2y_f \mu + \mu^2) - \frac{1}{2\sigma_\varepsilon^2} (n\mu^2 - 2n\mu\bar{y} + n\bar{y}^2)} du \\ &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2 (2\pi)} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma_\varepsilon^2} \{ (y_f^2 - 2y_f \mu + \mu^2) + (n\mu^2 - 2n\mu\bar{y} + n\bar{y}^2) \}} du \right] \\ &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2 (2\pi)} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma_\varepsilon^2} \{ (n\mu^2 + \mu^2 - 2\mu y_f - 2n\mu\bar{y}) + (y_f^2 + n\bar{y}^2) \}} du \right]. \end{aligned}$$

Now only consider

$$\begin{aligned}
 & (n\mu^2 + \mu^2 - 2\mu y_f - 2n\mu\bar{y}) + (y_f^2 + n\bar{y}^2) \\
 &= (n+1) \left[\mu^2 - \frac{2\mu(y_f+n\bar{y})}{n+1} \right] + (y_f^2 + n\bar{y}^2) \\
 &= (n+1) \left[\mu - \frac{(y_f+n\bar{y})}{n+1} \right]^2 + \left[y_f^2 + n\bar{y}^2 - \frac{(y_f+n\bar{y})^2}{n+1} \right].
 \end{aligned}$$

Therefore

$$\begin{aligned}
 f(y_f | \sigma_\varepsilon^2, \mathbf{y}) &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2(2\pi)} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma_\varepsilon^2} \left\{ (n+1) \left[\mu - \frac{(y_f+n\bar{y})}{n+1} \right]^2 + \left[y_f^2 + n\bar{y}^2 - \frac{(y_f+n\bar{y})^2}{n+1} \right] \right\}} du \right] \\
 &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2(2\pi)} \left[\int_{-\infty}^{\infty} e^{-\frac{(n+1)}{2\sigma_\varepsilon^2} \left[\mu - \frac{(y_f+n\bar{y})}{n+1} \right]^2} \cdot e^{-\frac{1}{2\sigma_\varepsilon^2} \left[y_f^2 + n\bar{y}^2 - \frac{(y_f+n\bar{y})^2}{n+1} \right]} du \right] \\
 &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2(2\pi)} e^{-\frac{1}{2\sigma_\varepsilon^2} \left[y_f^2 + n\bar{y}^2 - \frac{(y_f+n\bar{y})^2}{n+1} \right]} \left\{ \int_{-\infty}^{\infty} e^{-\frac{(n+1)}{2\sigma_\varepsilon^2} \left[\mu - \frac{(y_f+n\bar{y})}{n+1} \right]^2} du \right\}.
 \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} e^{-\frac{(n+1)}{2\sigma_\varepsilon^2} \left[\mu - \frac{(y_f+n\bar{y})}{n+1} \right]^2} du = \frac{\sigma_\varepsilon \sqrt{2\pi}}{(n+1)^{\frac{1}{2}}},$$

it follows that

$$\begin{aligned}
 f(y_f | \sigma_\varepsilon^2, \mathbf{y}) &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon \sqrt{2\pi} (n+1)^{\frac{1}{2}}} e^{-\frac{1}{2\sigma_\varepsilon^2} \left[y_f^2 + n\bar{y}^2 - \frac{(y_f+n\bar{y})^2}{n+1} \right]} \\
 &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2 \sqrt{2\pi} (n+1)^{\frac{1}{2}}} e^{-\frac{1}{2\sigma_\varepsilon^2} \left[\frac{(n+1)y_f^2 + n(n+1)\bar{y}^2 - (y_f+n\bar{y})^2}{n+1} \right]} \\
 &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2 \sqrt{2\pi} (n+1)^{\frac{1}{2}}} e^{-\frac{1}{2\sigma_\varepsilon^2} \left[\frac{ny_f^2 - 2ny_f\bar{y} + n\bar{y}^2}{n+1} \right]} \\
 &= \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon^2 \sqrt{2\pi} (n+1)^{\frac{1}{2}}} e^{-\frac{n}{2(n+1)\sigma_\varepsilon^2} [(y_f - \bar{y})^2]}.
 \end{aligned}$$

Therefore

$$y_f | \sigma_\varepsilon^2, \mathbf{y} \sim N \left(\bar{y}, \frac{(n+1)\sigma_\varepsilon^2}{n} \right). \quad (2.7.2)$$

As already mentioned, since it is known from equation 2.7.1 that

$y_f | \mu, \sigma_\varepsilon^2 \sim N(\mu, \sigma_\varepsilon^2)$ and therefore

$E[y_f | \mu, \sigma_\varepsilon^2] = \mu$, it can also be shown that

$$E[y_f | \sigma_\varepsilon^2] = E_{p(\mu | \sigma_\varepsilon^2, \mathbf{y})} \{E[y_f | \mu, \sigma_\varepsilon^2]\} = E_{p(\mu | \sigma_\varepsilon^2, \mathbf{y})} [\mu] = \bar{y}, \text{ and,}$$

Also

$$\begin{aligned} \text{Var}(y_f | \sigma_\varepsilon^2) &= \text{Var}_{p(\mu | \sigma_\varepsilon^2, \mathbf{y})} \{E(y_f | \mu, \sigma_\varepsilon^2)\} + E_{p(\mu | \sigma_\varepsilon^2, \mathbf{y})} [\text{Var}(y_f | \mu, \sigma_\varepsilon^2)] \\ &= \text{Var}(\mu) + E(\sigma_\varepsilon^2) \\ &= \frac{\sigma_\varepsilon^2}{n} + \sigma_\varepsilon^2 \\ &= \left(\frac{n+1}{n}\right)\sigma_\varepsilon^2. \end{aligned}$$

It therefore follows that

$$y_f | \sigma_\varepsilon^2, \mathbf{y} \sim N \left(\bar{y}, \left(\frac{n+1}{n}\right)\sigma_\varepsilon^2 \right).$$

To continue,

$$\begin{aligned}
f(y_f|\mathbf{y}) &= \int_0^\infty f(y_f|\sigma_\varepsilon^2, \mathbf{y})p(\sigma_\varepsilon^2 | \mathbf{y})d\sigma_\varepsilon^2 \\
&= \int_0^\infty \frac{n^{\frac{1}{2}}}{\sigma_\varepsilon \sqrt{2\pi(n+1)}^{\frac{1}{2}}} e^{-\frac{n}{2(n+1)\sigma_\varepsilon^2}(y_f - \bar{y})^2} \cdot k(\sigma_\varepsilon^2)^{-\frac{1}{2}(n+1)} e^{-\frac{1}{2}\frac{(n-1)s^2}{\sigma_\varepsilon^2}} d\sigma_\varepsilon^2 \\
&= \frac{kn^{\frac{1}{2}}}{\sqrt{2\pi(n+1)}^{\frac{1}{2}}} \int_0^\infty (\sigma_\varepsilon^2)^{-\frac{1}{2}} \exp\left\{-\frac{n}{2(n+1)\sigma_\varepsilon^2}(y_f - \bar{y})^2\right\} \cdot (\sigma_\varepsilon^2)^{-\frac{1}{2}(n+1)} \exp\left\{-\frac{1}{2}\frac{(n-1)s^2}{\sigma_\varepsilon^2}\right\} d\sigma_\varepsilon^2 \\
&= \frac{kn^{\frac{1}{2}}}{\sqrt{2\pi(n+1)}^{\frac{1}{2}}} \int_0^\infty (\sigma_\varepsilon^2)^{-\frac{1}{2}(n+1)-\frac{1}{2}} \exp\left\{-\frac{1}{2}\frac{n}{(n+1)\sigma_\varepsilon^2}(y_f - \bar{y})^2 - \frac{1}{2}\frac{(n-1)s^2}{\sigma_\varepsilon^2}\right\} d\sigma_\varepsilon^2 \\
&= c \int_0^\infty (\sigma_\varepsilon^2)^{-\frac{1}{2}[(n+1)+1]} \exp\left\{-\frac{1}{2(n+1)\sigma_\varepsilon^2} [n(y_f - \bar{y})^2 + (n+1)(n-1)s^2]\right\} d\sigma_\varepsilon^2 \\
&= c \int_0^\infty (\sigma_\varepsilon^2)^{-\frac{1}{2}(n+2)} \exp\left\{-\frac{1}{2(n+1)\sigma_\varepsilon^2} [n(y_f - \bar{y})^2 + (n+1)(n-1)s^2]\right\} d\sigma_\varepsilon^2 \\
&= c \int_0^\infty (\sigma_\varepsilon^2)^{-\frac{1}{2}n-1} \exp\left\{-\frac{1}{2(n+1)\sigma_\varepsilon^2} [n(y_f - \bar{y})^2 + (n+1)(n-1)s^2]\right\} d\sigma_\varepsilon^2.
\end{aligned}$$

It is also known that since

$$\int_0^\infty a^{\frac{1}{2}n-1} \exp\left\{-\frac{a}{2\sigma^2}\right\} da = \Gamma\left(\frac{1}{2}n\right)(2\sigma^2)^{\frac{1}{2}n}$$

it follows that

$$\begin{aligned}
f(y_f|\mathbf{y}) &= \frac{kn^{\frac{1}{2}}}{\sqrt{2\pi(n+1)}^{\frac{1}{2}}} \Gamma\left(\frac{1}{2}n\right) \left[\frac{2(n+1)}{n(y_f - \bar{y})^2 + (n+1)(n-1)s^2}\right]^{\frac{1}{2}n} \\
&= \frac{kn^{\frac{1}{2}}}{\sqrt{2\pi(n+1)}^{\frac{1}{2}}} \Gamma\left(\frac{n}{2}\right) \left[\frac{2(n+1)}{(n+1)\left[\frac{n}{(n+1)}(y_f - \bar{y})^2 + (n+1)s^2\right]}\right]^{\frac{1}{2}n} \quad \forall -\infty \leq y_f \leq \infty \\
&= \frac{kn^{\frac{1}{2}}}{\sqrt{2\pi(n+1)}^{\frac{1}{2}}} \Gamma\left(\frac{n}{2}\right) \left[\frac{2}{\frac{n}{(n+1)}(y_f - \bar{y})^2 + \nu s^2}\right]^{\frac{1}{2}n} \quad \forall -\infty \leq y_f \leq \infty
\end{aligned}$$

where $\nu = n - 1$ and

$$k = \frac{\left\{\frac{(n-1)s^2}{2}\right\}^{\frac{1}{2}(n-1)}}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Therefore, the unconditional predictive distribution $p(y_f | \mathbf{y})$ is given by

$$\begin{aligned}
 p(y_f | \mathbf{y}) &= \frac{\left\{ \frac{\nu s^2}{2} \right\}^{\frac{1}{2}(n-1)}}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{n^{\frac{1}{2}}}{\sqrt{2\pi}(n+1)^{\frac{1}{2}}} \Gamma\left(\frac{n}{2}\right) \cdot \left[\frac{2}{\frac{n}{(n+1)}(y_f - \bar{y})^2 + \nu s^2} \right]^{\frac{1}{2}n} \quad (2.7.3) \\
 &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{(n+1)}} \left[\frac{\nu s^2}{2} \right]^{\frac{1}{2}(n-1)} \left[\frac{2}{\frac{n}{(n+1)}(y_f - \bar{y})^2 + \nu s^2} \right]^{\frac{1}{2}n} \\
 &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{(n+1)}} \left[\frac{\nu s^2}{2} \right]^{\frac{1}{2}n} \left[\frac{\nu s^2}{2} \right]^{-\frac{1}{2}} \left[\frac{\frac{n}{(n+1)}(y_f - \bar{y})^2}{2} + \frac{\nu s^2}{2} \right]^{-\frac{1}{2}n} \\
 &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{(n+1)}} \left[\frac{2}{\nu s^2} \right]^{-\frac{1}{2}n} \left[\frac{2}{\nu s^2} \right]^{\frac{1}{2}} \left[\frac{\frac{n}{(n+1)}(y_f - \bar{y})^2}{2} + \frac{\nu s^2}{2} \right]^{-\frac{1}{2}n} \\
 &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \left[\frac{n}{2\pi(n+1)} \cdot \frac{2}{\nu s^2} \right]^{\frac{1}{2}} \left[\frac{\frac{2n}{(n+1)}(y_f - \bar{y})^2}{2\nu s^2} + \frac{\nu s^2}{2} \cdot \frac{2}{\nu s^2} \right]^{-\frac{1}{2}n} \\
 &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \left[\frac{2n}{2\pi(n+1)\nu s^2} \right]^{\frac{1}{2}} \left[\frac{\frac{2n}{(n+1)}(y_f - \bar{y})^2}{2\nu s^2} + 1 \right]^{-\frac{1}{2}n} \\
 &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \left[\frac{n}{\pi(n+1)\nu s^2} \right]^{\frac{1}{2}} \left[\frac{n(y_f - \bar{y})^2}{(n+1)\nu s^2} + 1 \right]^{-\frac{1}{2}n}.
 \end{aligned}$$

This unconditional predictive distribution $p(y_f | \mathbf{y})$ is in the general form of a student t -distribution with $\nu = n - 1$ degrees of freedom,

$$E(y_f | \mathbf{y}) = \bar{y} \text{ and}$$

$$\begin{aligned}
 Var(y_f | \mathbf{y}) &= E \left[\frac{(n+1)}{n} \sigma^2 \right] \\
 &= E \left[\frac{(n+1)}{n} \frac{\nu s^2}{\chi_\nu^2} \right] \\
 &= \frac{(n+1)\nu s^2}{n} E \left[\frac{1}{\chi_\nu^2} \right] \\
 &= \frac{(n+1)\nu s^2}{n} \frac{1}{(\nu-2)} \\
 &= \frac{(n+1)(n-1)s^2}{n} \frac{1}{(n-3)} \\
 &= \frac{(n+1)}{n} \left[\frac{(n-1)}{(n-3)} \right] s^2.
 \end{aligned}$$

Proof of Theorem 2.6.4.1

The fixed - in - advance tolerance interval is defined as

$$c = 1 - \Phi \left[\frac{s - \mu}{\sigma_\varepsilon} \right] = 1 - \Phi(\theta)$$

where $\Phi(\theta)$ is the standard normal cumulative distribution function and $\theta = \frac{s - \mu}{\sigma_\varepsilon}$.

It is therefore known that

$$\Phi\left(\frac{s - \mu}{\sigma_\varepsilon}\right) = \int_{-\infty}^{\frac{s - \mu}{\sigma_\varepsilon}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

and

$$\Phi(\theta) = \int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz .$$

$$\begin{aligned} \text{Now } \frac{\partial c}{\partial \mu} &= -\frac{\partial \Phi(\theta)}{\partial \theta} \cdot \frac{\partial \theta}{\partial \mu} \\ &= -\frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} \left(\frac{1}{\sigma_\varepsilon} \right) \\ &= \frac{1}{\sigma_\varepsilon} \cdot \frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} \\ &= \frac{1}{\sigma_\varepsilon} \cdot \frac{e^{-\frac{1}{2}\left(\frac{s - \mu}{\sigma_\varepsilon}\right)^2}}{\sqrt{2\pi}} . \end{aligned}$$

$$\begin{aligned} \text{Also } \frac{\partial c}{\partial \sigma_\varepsilon^2} &= -\frac{\partial \Phi(\theta)}{\partial \theta} \cdot \frac{\partial \theta}{\partial \sigma_\varepsilon^2} \\ &= -\frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} \left(-\frac{1}{2}\right) (\sigma_\varepsilon^2)^{-\frac{3}{2}} (s - \mu) \\ &= \frac{s - \mu}{2\sigma_\varepsilon^3} \cdot \frac{e^{-\frac{1}{2}\left(\frac{s - \mu}{\sigma_\varepsilon}\right)^2}}{\sqrt{2\pi}} . \end{aligned}$$

Therefore

$$\begin{aligned} \nabla'_t(\theta) &= \begin{bmatrix} \frac{\partial c}{\partial \mu} & \frac{\partial c}{\partial \sigma_\varepsilon^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sigma_\varepsilon} \cdot \frac{e^{-\frac{1}{2}\left(\frac{s - \mu}{\sigma_\varepsilon}\right)^2}}{\sqrt{2\pi}} & \frac{(s - \mu)}{2\sigma_\varepsilon^3} \cdot \frac{e^{-\frac{1}{2}\left(\frac{s - \mu}{\sigma_\varepsilon}\right)^2}}{\sqrt{2\pi}} \end{bmatrix} \\ &= F \begin{bmatrix} \frac{1}{\sigma_\varepsilon} & \frac{s - \mu}{2\sigma_\varepsilon^3} \end{bmatrix} \quad \text{where} \quad F = \frac{e^{-\frac{1}{2}\left(\frac{s - \mu}{\sigma_\varepsilon}\right)^2}}{\sqrt{2\pi}} . \end{aligned}$$

Using the method of Datta and Ghosh (1995), it follows that since

$$F^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & 0 \\ 0 & \frac{2(\sigma_\varepsilon^2)^2}{n} \end{bmatrix}, \nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta}) \text{ is given by}$$

$$\begin{aligned} \nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta}) &= F \begin{bmatrix} \frac{1}{\sigma_\varepsilon} & \frac{s-\mu}{2\sigma_\varepsilon^3} \end{bmatrix} \begin{bmatrix} \frac{\sigma_\varepsilon^2}{n} & 0 \\ 0 & \frac{2(\sigma_\varepsilon^2)^2}{n} \end{bmatrix} \\ &= F \begin{bmatrix} \frac{\sigma_\varepsilon}{n} & \frac{(s-\mu)\sigma_\varepsilon}{n} \end{bmatrix} \\ &= \frac{\sigma_\varepsilon}{n} F \begin{bmatrix} 1 & (s-\mu) \end{bmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta}) &= F \frac{\sigma_\varepsilon}{n} \begin{bmatrix} 1 & (s-\mu) \end{bmatrix} \frac{1}{\sigma_\varepsilon} F \begin{bmatrix} 1 & \frac{(s-\mu)}{2\sigma_\varepsilon^2} \end{bmatrix} \\ &= F^2 \frac{1}{n} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2} \right]. \end{aligned}$$

Therefore

$$\sqrt{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})} = F \frac{1}{\sqrt{n}} \left(1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2} \right)^{\frac{1}{2}}.$$

Now

$$\begin{aligned} \eta(\boldsymbol{\theta}) &= \frac{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})}{\sqrt{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}} \\ &= \frac{F \frac{\sigma_\varepsilon}{n} \begin{bmatrix} 1 & (s-\mu) \end{bmatrix}}{F \frac{1}{\sqrt{n}} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2} \right]^{\frac{1}{2}}} \text{ and therefore} \end{aligned}$$

$$\eta_1(\boldsymbol{\theta}) = \frac{\frac{\sigma_\varepsilon}{\sqrt{n}}}{\left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2} \right]^{\frac{1}{2}}} \text{ and,}$$

$$\eta_2(\boldsymbol{\theta}) = \frac{\frac{1}{\sqrt{n}}\sigma_\varepsilon(s-\mu)}{\left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2} \right]^{\frac{1}{2}}}.$$

Also

$$\begin{aligned}\eta_1(\boldsymbol{\theta})\pi_m(\boldsymbol{\theta}) &= \frac{\frac{\sigma_\varepsilon}{\sqrt{n}}}{\left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{\frac{1}{2}}} \cdot \sigma_\varepsilon^{-1} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{n}}, \text{ and}\end{aligned}$$

$$\begin{aligned}\eta_2(\boldsymbol{\theta})\pi_m(\boldsymbol{\theta}) &= \frac{\frac{\sigma_\varepsilon}{\sqrt{n}}(s-\mu)}{\left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{\frac{1}{2}}} \cdot \sigma_\varepsilon^{-1} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{n}}(s-\mu) \text{ and it therefore follows that}\end{aligned}$$

$$\frac{\partial}{\partial \mu} [\eta_1(\boldsymbol{\theta})\pi_m(\boldsymbol{\theta})] + \frac{\partial}{\partial \sigma_\varepsilon^2} [\eta_2(\boldsymbol{\theta})\pi_m(\boldsymbol{\theta})] = 0$$

since

$$\frac{\partial}{\partial \mu} [\eta_1(\boldsymbol{\theta})\pi_m(\boldsymbol{\theta})] = 0, \text{ and } \frac{\partial}{\partial \sigma_\varepsilon^2} [\eta_2(\boldsymbol{\theta})\pi_m(\boldsymbol{\theta})] = 0.$$

It therefore follows that equation 2.6.9 is a probability matching prior for the fixed - in - advance tolerance interval given in equation 2.6.8.

It can also be shown that

$\pi^m(\boldsymbol{\theta}) \propto \sigma_\varepsilon^{-3} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{-\frac{1}{2}}$ is also a probability matching prior for the fixed - in - advance tolerance interval given in equation 2.6.8, since

$$\begin{aligned}\eta_1(\boldsymbol{\theta})\pi^m(\boldsymbol{\theta}) &= \frac{\frac{\sigma_\varepsilon}{\sqrt{n}}}{\left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{\frac{1}{2}}} \cdot \sigma_\varepsilon^{-3} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{-\frac{1}{2}} \\ &= \frac{\frac{1}{\sqrt{n}}}{\sigma_\varepsilon^2 \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]}, \text{ and}\end{aligned}$$

$$\begin{aligned}\eta_2(\boldsymbol{\theta})\pi^m(\boldsymbol{\theta}) &= \frac{\frac{\sigma_\varepsilon}{\sqrt{n}}(s-\mu)}{\left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{\frac{1}{2}}} \cdot \sigma_\varepsilon^{-3} \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^{-\frac{1}{2}} \\ &= \frac{\frac{1}{\sqrt{n}}(s-\mu)}{\sigma_\varepsilon^2 \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]}.\end{aligned}$$

It therefore follows that

$$\begin{aligned} \frac{\partial}{\partial \mu} [\eta_1(\boldsymbol{\theta})\pi^m(\boldsymbol{\theta})] &= \frac{\frac{1}{\sqrt{n}}(s-\mu)}{\sigma_\varepsilon^2 \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^2} \\ &= \frac{\frac{1}{\sqrt{n}}(s-\mu)}{\sigma_\varepsilon^4 \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^2}, \text{ and} \end{aligned}$$

$$\frac{\partial}{\partial \sigma_\varepsilon^2} [\eta_2(\boldsymbol{\theta})\pi^m(\boldsymbol{\theta})] = \frac{-\frac{1}{\sqrt{n}}(s-\mu)}{\sigma_\varepsilon^4 \left[1 + \frac{(s-\mu)^2}{2\sigma_\varepsilon^2}\right]^2} \text{ with the result that}$$

$$\frac{\partial}{\partial \mu} [\eta_1(\boldsymbol{\theta})\pi^m(\boldsymbol{\theta})] + \frac{\partial}{\partial \sigma_\varepsilon^2} [\eta_2(\boldsymbol{\theta})\pi^m(\boldsymbol{\theta})] = 0.$$

Proof of Theorem 2.6.5.1

a.) For two α quantiles q_d ($d = 1, 2$) given by

$$q_1 = \mu_1 + z_\alpha \sigma_{\varepsilon_1} \text{ and}$$

$$q_2 = \mu_2 + z_\alpha \sigma_{\varepsilon_2},$$

with

$$\mu_1 \sim N(\bar{y}_1, \frac{\sigma_{\varepsilon_1}^2}{n_1}), \text{ and } \mu_2 \sim N(\bar{y}_2, \frac{\sigma_{\varepsilon_2}^2}{n_2})$$

where

$$\frac{\nu_1 s_1^2}{\sigma_{\varepsilon_1}^2} \sim \chi_{\nu_1}^2, \text{ and } \frac{\nu_2 s_2^2}{\sigma_{\varepsilon_2}^2} \sim \chi_{\nu_2}^2,$$

it is known that

$$\begin{aligned} (q_1 - q_2) &= \left[\bar{y}_1 + z \frac{\sigma_{\varepsilon_1}}{\sqrt{n_1}} + z_\alpha \sigma_{\varepsilon_1} \right] - \left[\bar{y}_2 + z \frac{\sigma_{\varepsilon_2}}{\sqrt{n_2}} + z_\alpha \sigma_{\varepsilon_2} \right] \quad \text{where } z \sim N(0, 1) \\ &= \bar{y}_1 - \bar{y}_2 + z \frac{\sigma_{\varepsilon_1}}{\sqrt{n_1}} - z \frac{\sigma_{\varepsilon_2}}{\sqrt{n_2}} + z_\alpha \sigma_{\varepsilon_1} - z_\alpha \sigma_{\varepsilon_2} \\ &= (\bar{y}_1 - \bar{y}_2) + \sigma_{\varepsilon_1} \left[\frac{z}{\sqrt{n_1}} + z_\alpha \right] - \sigma_{\varepsilon_2} \left[\frac{z}{\sqrt{n_2}} + z_\alpha \right] \\ &= (\bar{y}_1 - \bar{y}_2) + \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left[\frac{z}{\sqrt{n_1}} + z_\alpha \right] - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \left[\frac{z}{\sqrt{n_2}} + z_\alpha \right]. \end{aligned}$$

Since it is known from the proof of Theorem 2.6.1.1 a.) that

$$E\left(\frac{1}{\chi^2_d}\right)^{\frac{1}{2}} = \frac{\Gamma(\frac{\nu_d-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_d}{2})} \quad (\text{for } d = 1, 2)$$

and it is also known that

$$(q_1 - q_2) = (\bar{y}_1 - \bar{y}_2) + \left(\frac{\nu_1 s_1^2}{\chi^2_{\nu_1}}\right)^{\frac{1}{2}} \left[\frac{z}{\sqrt{n_1}} + z_\alpha\right] - \left(\frac{\nu_2 s_2^2}{\chi^2_{\nu_2}}\right)^{\frac{1}{2}} \left[\frac{z}{\sqrt{n_2}} + z_\alpha\right],$$

it can be shown that the mean (the first moment about zero) of the marginal posterior distribution of $(q_1 - q_2)$ is given by

$$\begin{aligned} E\left[(q_1 - q_2) | \mathbf{y}_1, \mathbf{y}_2\right] &= E\left[(\bar{y}_1 - \bar{y}_2) + \left(\frac{\nu_1 s_1^2}{\chi^2_{\nu_1}}\right)^{\frac{1}{2}} \left[\frac{z}{\sqrt{n_1}} + z_\alpha\right] - \left(\frac{\nu_2 s_2^2}{\chi^2_{\nu_2}}\right)^{\frac{1}{2}} \left[\frac{z}{\sqrt{n_2}} + z_\alpha\right]\right] \\ &= E\left[(\bar{y}_1 - \bar{y}_2) + z \frac{\sigma_{\varepsilon_1}}{\sqrt{n_1}} + z_\alpha \sigma_{\varepsilon_1} - z \frac{\sigma_{\varepsilon_2}}{\sqrt{n_2}} - z_\alpha \sigma_{\varepsilon_2}\right] \\ &= E\left[(\bar{y}_1 - \bar{y}_2)\right] + E\left[z\right] E\left[\frac{\sigma_{\varepsilon_1}}{\sqrt{n_1}}\right] + E\left[z_\alpha\right] E\left[\sigma_{\varepsilon_1}\right] \\ &\quad - E\left[z\right] E\left[\frac{\sigma_{\varepsilon_2}}{\sqrt{n_2}}\right] - E\left[z_\alpha\right] E\left[\sigma_{\varepsilon_2}\right] \\ &= (\bar{y}_1 - \bar{y}_2) + 0 + z_\alpha E\left[\sigma_{\varepsilon_1}\right] - 0 - z_\alpha E\left[\sigma_{\varepsilon_2}\right] \\ &= (\bar{y}_1 - \bar{y}_2) + z_\alpha \left\{ E\left[\sigma_{\varepsilon_1}\right] - E\left[\sigma_{\varepsilon_2}\right] \right\} \\ &= (\bar{y}_1 - \bar{y}_2) + z_\alpha \left\{ \left(\nu_1 s_1^2\right)^{\frac{1}{2}} E\left(\frac{1}{\chi^2_{\nu_1}}\right)^{\frac{1}{2}} - \left(\nu_2 s_2^2\right)^{\frac{1}{2}} E\left(\frac{1}{\chi^2_{\nu_2}}\right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Therefore

$$E\left[(q_1 - q_2) | \mathbf{y}_1, \mathbf{y}_2\right] = (\bar{y}_1 - \bar{y}_2) + z_\alpha \left\{ \left(\nu_1 s_1^2\right)^{\frac{1}{2}} E\left(\frac{1}{\chi^2_{\nu_1}}\right)^{\frac{1}{2}} - \left(\nu_2 s_2^2\right)^{\frac{1}{2}} E\left(\frac{1}{\chi^2_{\nu_2}}\right)^{\frac{1}{2}} \right\}.$$

Now, since $E\left(\frac{1}{\chi^2_\nu}\right) = \frac{\Gamma(\frac{\nu-2}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}$, it can easily be shown that

$$E\left[(q_1 - q_2) | \mathbf{y}_1, \mathbf{y}_2\right] = (\bar{y}_1 - \bar{y}_2) + z_\alpha \left\{ \left(\nu_1 s_1^2\right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-2}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2\right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-2}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \right\}.$$

b.) Following from the proof of Theorem 2.6.1.1 b.), it is known that

$$\begin{aligned} E\left(\frac{1}{\chi_{\nu_d}^2}\right) &= \frac{\Gamma(\frac{\nu_d-2}{2})}{2\Gamma(\frac{\nu_d}{2})} \quad (\text{for } d = 1, 2) \\ &= \frac{\Gamma(\frac{\nu_d-2}{2})}{2(\frac{\nu_d-2}{2})\Gamma(\frac{\nu_d-2}{2})} \\ &= \frac{1}{\nu_d-2}. \end{aligned}$$

By considering the conditional posterior distribution of $(q_1 - q_2)$ (conditional on both $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$), it follows that

$$E[(q_1 - q_2) | \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2] = (\bar{y}_1 - \bar{y}_2) + z_{\alpha}(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2})$$

and

$$\text{var}[(q_1 - q_2) | \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2] = \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}.$$

It is therefore known that the conditional posterior distribution of $(q_1 - q_2)$ is given by

$$(q_1 - q_2) | \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2 \sim N\left[(\bar{y}_1 - \bar{y}_2) + z_{\alpha}(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}), \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right].$$

Therefore

$$(q_1 - q_2) | \chi_{\nu_1}^2, \chi_{\nu_2}^2, \mathbf{y}_1, \mathbf{y}_2 \sim N\left\{(\bar{y}_1 - \bar{y}_2) + z_{\alpha}\left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}}\right], \frac{1}{n_1} \frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} + \frac{1}{n_2} \frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right\}$$

and

$$\text{var}[(q_1 - q_2) | \mathbf{y}_1, \mathbf{y}_2] = \text{var}\left\{(\bar{y}_1 - \bar{y}_2) + z_{\alpha}\left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}}\right]\right\} + E\left\{\frac{1}{n_1} \frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} + \frac{1}{n_2} \frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right\}.$$

$$\text{Now } \text{var}\left\{(\bar{y}_1 - \bar{y}_2) + z_{\alpha}\left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}}\right]\right\}$$

$$= \text{var}\left\{(\bar{y}_1 - \bar{y}_2)\right\} + \text{var}\left\{z_{\alpha}\left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}}\right]\right\}$$

$$= \text{var}\left\{z_{\alpha}\left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}}\right]\right\}$$

$$\begin{aligned}
&= z_\alpha^2 \left\{ \text{var} \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} + \text{var} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \right\} \\
&= z_\alpha^2 \left\{ (\nu_1 s_1^2) \text{var} \left(\frac{1}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} + (\nu_2 s_2^2) \text{var} \left(\frac{1}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \right\}.
\end{aligned}$$

It was also shown in Theorem 2.6.1.1 b.) that

$$\text{var}_{\chi_{\nu_d}^2} \left(\frac{1}{\chi_{\nu_d}^2} \right)^{\frac{1}{2}} = E \left[\frac{1}{\chi_{\nu_d}^2} \right] - \left\{ E \left[\frac{1}{\chi_{\nu_d}^2} \right] \right\}^2 \quad (\text{for } d = 1, 2).$$

$$\text{Since } E \left(\frac{1}{\chi_{\nu_d}^2} \right) = \frac{1}{(\nu_d - 2)} \quad (d = 1, 2)$$

and

$$E \left(\frac{1}{\chi_{\nu_d}^2} \right)^{\frac{1}{2}} = \frac{\Gamma(\frac{\nu_d - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_d}{2})} \quad (d = 1, 2),$$

it therefore follows that

$$\begin{aligned}
\text{var} \left(\frac{1}{\chi_{\nu_d}^2} \right)^{\frac{1}{2}} &= \frac{1}{(\nu_d - 2)} - \left\{ \frac{\Gamma(\frac{\nu_d - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_d}{2})} \right\}^2 \\
&= \frac{1}{(\nu_d - 2)} - \frac{\Gamma^2(\frac{\nu_d - 1}{2})}{2\Gamma^2(\frac{\nu_d}{2})} \quad (d = 1, 2).
\end{aligned}$$

It can therefore be shown that

$$\begin{aligned}
&\text{var}_{\chi_{\nu_1}^2, \chi_{\nu_2}^2} \left\{ (\bar{y}_1 - \bar{y}_2) + z_\alpha \left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \right] \right\} \\
&= z_\alpha^2 \left\{ (\nu_1 s_1^2) \left[\frac{1}{(\nu_1 - 2)} - \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right] - (\nu_2 s_2^2) \left[\frac{1}{(\nu_2 - 2)} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right] \right\}.
\end{aligned}$$

Also

$$\begin{aligned}
&E_{\chi_{\nu_1}^2, \chi_{\nu_2}^2} \left[\frac{1}{n_1} \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right) + \frac{1}{n_2} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right) \right] \\
&= \frac{1}{n_1} (\nu_1 s_1^2) E_{\chi_{\nu_1}^2} \left(\frac{1}{\chi_{\nu_1}^2} \right) + \frac{1}{n_2} (\nu_2 s_2^2) E_{\chi_{\nu_2}^2} \left(\frac{1}{\chi_{\nu_2}^2} \right) \\
&= \frac{\nu_1 s_1^2}{n_1} \left(\frac{1}{\nu_1 - 2} \right) + \frac{\nu_2 s_2^2}{n_2} \left(\frac{1}{\nu_2 - 2} \right).
\end{aligned}$$

It is therefore clear that the variance of the marginal posterior distribution of $(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2$ (the second central moment of the marginal posterior distribution of $(q_1 - q_2)$), is given by

$$\begin{aligned}
& \text{var} [(q_1 - q_2)|\mathbf{y}_1, \mathbf{y}_2] \\
&= z_\alpha^2 \left\{ (\nu_1 s_1^2) \left[\frac{1}{\nu_1 - 2} - \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right] \right\} - z_\alpha^2 \left\{ (\nu_2 s_2^2) \left[\frac{1}{\nu_2 - 2} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right] \right\} + \frac{\nu_1 s_1^2}{n_1} \left(\frac{1}{\nu_1 - 2} \right) + \frac{\nu_2 s_2^2}{n_2} \left(\frac{1}{\nu_2 - 2} \right) \\
&= (\nu_1 s_1^2) \left\{ z_\alpha^2 \left[\frac{1}{\nu_1 - 2} - \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right] + \frac{1}{n_1} \left(\frac{1}{\nu_1 - 2} \right) \right\} - (\nu_2 s_2^2) \left\{ z_\alpha^2 \left[\frac{1}{\nu_2 - 2} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right] + \frac{1}{n_2} \left(\frac{1}{\nu_2 - 2} \right) \right\} \\
&= (\nu_1 s_1^2) \left\{ \left(\frac{1}{\nu_1 - 2} \right) \left[z_\alpha^2 + \frac{1}{n_1} \right] - \frac{z_\alpha^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right\} - (\nu_2 s_2^2) \left\{ \left(\frac{1}{\nu_2 - 2} \right) \left[z_\alpha^2 + \frac{1}{n_2} \right] - \frac{z_\alpha^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right\}.
\end{aligned}$$

c.) It was shown in Theorem 2.6.1.1 c.) that

$$E\left(\frac{1}{\chi_{\nu_d}^2}\right)^{\frac{3}{2}} = \frac{\Gamma(\frac{\nu_d - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_d}{2})} \quad (\text{for } d = 1, 2).$$

The third moment about the origin of the conditional posterior distribution of $(q_1 - q_2)|\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2$ can in general be written as

$$\mu'_3 = \mu_3 + 3\mu_2\mu'_1 + (\mu'_1)^3.$$

By substituting

$$\mu'_1 = (\bar{y}_1 - \bar{y}_2) + z_\alpha(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}),$$

$$\mu_2 = \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}, \text{ and}$$

$$\mu_3 = 0$$

into the equation for μ'_3 , it follows that the third moment about the origin of $(q_1 - q_2)|\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2$ is given by

$$\begin{aligned}
\mu_3' &= 0 + 3\left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right) \left\{ (\bar{y}_1 - \bar{y}_2) + z_\alpha(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) \right\} + \left\{ (\bar{y}_1 - \bar{y}_2) + z_\alpha(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) \right\}^3 \\
&= \frac{3\sigma_{\varepsilon_1}^2}{n_1} (\bar{y}_1 - \bar{y}_2) + \frac{3\sigma_{\varepsilon_2}^2}{n_2} (\bar{y}_1 - \bar{y}_2) + 3z_\alpha \left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2} \right) (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + (\bar{y}_1 - \bar{y}_2)^3 \\
&\quad + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2})^2 + z_\alpha^3 (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2})^3 \\
&= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + 3(\bar{y}_1 - \bar{y}_2) \left\{ \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2} + z_\alpha^2 (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2})^2 \right\} \\
&\quad + 3z_\alpha \left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2} \right) (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + z_\alpha^3 (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2})^3 \\
&= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + 3(\bar{y}_1 - \bar{y}_2) \frac{\sigma_{\varepsilon_1}^2}{n_1} + 3(\bar{y}_1 - \bar{y}_2) \frac{\sigma_{\varepsilon_2}^2}{n_2} \\
&\quad + 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \sigma_{\varepsilon_1}^2 - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2} + 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \sigma_{\varepsilon_2}^2 + 3z_\alpha \frac{\sigma_{\varepsilon_1}^3}{n_1} - 3z_\alpha \frac{\sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2}}{n_1} \\
&\quad + 3z_\alpha \frac{\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2}{n_2} - 3z_\alpha \frac{\sigma_{\varepsilon_2}^3}{n_2} + z_\alpha^3 \sigma_{\varepsilon_1}^3 - 3z_\alpha^3 \sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2} + 3z_\alpha^3 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2 - z_\alpha^3 \sigma_{\varepsilon_2}^3 \\
&= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + 3(\bar{y}_1 - \bar{y}_2) \left\{ \frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2} + z_\alpha^2 \sigma_{\varepsilon_1}^2 + z_\alpha^2 \sigma_{\varepsilon_2}^2 \right\} \\
&\quad + 3z_\alpha \frac{\sigma_{\varepsilon_1}^3}{n_1} + z_\alpha^3 \sigma_{\varepsilon_1}^3 - 3z_\alpha \frac{\sigma_{\varepsilon_2}^3}{n_2} - z_\alpha^3 \sigma_{\varepsilon_2}^3 - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2} + 3z_\alpha \frac{\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2}{n_2} - 3z_\alpha \frac{\sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2}}{n_1} \\
&\quad - 3z_\alpha^3 \sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2} + 3z_\alpha^3 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2 \\
&= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) + 3(\bar{y}_1 - \bar{y}_2) \left\{ \sigma_{\varepsilon_1}^2 \left(z_\alpha^2 + \frac{1}{n_1} \right) + \sigma_{\varepsilon_2}^2 \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} \\
&\quad + \sigma_{\varepsilon_1}^3 \left(\frac{3z_\alpha}{n_1} + z_\alpha^3 \right) - \sigma_{\varepsilon_2}^3 \left(\frac{3z_\alpha}{n_2} + z_\alpha^3 \right) - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \sigma_{\varepsilon_1} \sigma_{\varepsilon_2} - 3z_\alpha \sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2} \left(\frac{1}{n_1} + z_\alpha^2 \right) \\
&\quad + 3z_\alpha \sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2 \left(\frac{1}{n_2} + z_\alpha^2 \right)
\end{aligned}$$

$$\begin{aligned}
&= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \right] \\
&\quad + 3(\bar{y}_1 - \bar{y}_2) \left\{ \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right) \left(z_\alpha^2 + \frac{1}{n_1} \right) + \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right) \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} + \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{3}{2}} \left(\frac{3z_\alpha}{n_1} + z_\alpha^3 \right) \\
&\quad - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{3}{2}} \left(\frac{3z_\alpha}{n_2} + z_\alpha^3 \right) - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} - 3z_\alpha \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right) \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \left(\frac{1}{n_1} + z_\alpha^2 \right) \\
&\quad + 3z_\alpha \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right) \left(\frac{1}{n_2} + z_\alpha^2 \right)
\end{aligned}$$

Now $m'_3 = E(\mu'_3)$, therefore

$$\begin{aligned}
m'_3 &= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \right] \\
&\quad + 3(\bar{y}_1 - \bar{y}_2) \left\{ \nu_1 s_1^2 E \left(\frac{1}{\chi_{\nu_1}^2} \right) \left(z_\alpha^2 + \frac{1}{n_1} \right) + \nu_2 s_2^2 E \left(\frac{1}{\chi_{\nu_2}^2} \right) \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} \\
&\quad + \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} E \left(\frac{1}{\chi_{\nu_1}^2} \right)^{\frac{3}{2}} \left(\frac{3z_\alpha}{n_1} + z_\alpha^3 \right) - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} E \left(\frac{1}{\chi_{\nu_2}^2} \right)^{\frac{3}{2}} \left(\frac{3z_\alpha}{n_2} + z_\alpha^3 \right) \\
&\quad - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \\
&\quad - 3z_\alpha \left(\nu_1 s_1^2 \right) E \left(\frac{1}{\chi_{\nu_1}^2} \right) \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \left(\frac{1}{n_1} + z_\alpha^2 \right) \\
&\quad + 3z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right) E \left(\frac{1}{\chi_{\nu_2}^2} \right) \left(\frac{1}{n_2} + z_\alpha^2 \right) \\
&= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \\
&\quad + 3(\bar{y}_1 - \bar{y}_2) \left\{ \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(z_\alpha^2 + \frac{1}{n_1} \right) + \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} + \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3z_\alpha}{n_1} + z_\alpha^3 \right] \\
&\quad - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3z_\alpha}{n_2} + z_\alpha^3 \right] - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
&\quad - 3z_\alpha \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] + 3z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right].
\end{aligned}$$

It is also known that $m_3 = m'_3 - 3m_2m'_1 - (m'_1)^3$.

Therefore

$$\begin{aligned}
m_3 &= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \\
&+ 3(\bar{y}_1 - \bar{y}_2) \left\{ \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(z_\alpha^2 + \frac{1}{n_1} \right) + \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} + \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3z_\alpha}{n_1} + z_\alpha^3 \right] \\
&- \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3z_\alpha}{n_2} + z_\alpha^3 \right] - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
&- 3z_\alpha \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] + 3z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] \\
&- 3 \left\{ z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \left(\frac{1}{\nu_1-2} - \frac{\Gamma^2(\frac{\nu_1-1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) + \left(\nu_2 s_2^2 \right) \left(\frac{1}{\nu_2-2} - \frac{\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) \right] + \frac{(\nu_1 s_1^2)}{n_1(\nu_1-2)} + \frac{(\nu_2 s_2^2)}{n_2(\nu_2-2)} \right\} \\
&\times \left\{ (\bar{y}_1 - \bar{y}_2) + z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\} \\
&- \left\{ (\bar{y}_1 - \bar{y}_2) + z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\}^3.
\end{aligned}$$

Therefore

$$\begin{aligned}
m_3 &= (\bar{y}_1 - \bar{y}_2)^3 + 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \\
&+ 3(\bar{y}_1 - \bar{y}_2) \left\{ \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(z_\alpha^2 + \frac{1}{n_1} \right) + \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} + \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3z_\alpha}{n_1} + z_\alpha^3 \right] \\
&- \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3z_\alpha}{n_2} + z_\alpha^3 \right] - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
&- 3z_\alpha \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] + 3z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] \\
&- 3 \left\{ z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \left(\frac{1}{\nu_1-2} - \frac{\Gamma^2(\frac{\nu_1-1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) + \left(\nu_2 s_2^2 \right) \left(\frac{1}{\nu_2-2} - \frac{\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) \right] + \frac{(\nu_1 s_1^2)}{n_1(\nu_1-2)} + \frac{(\nu_2 s_2^2)}{n_2(\nu_2-2)} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times (\bar{y}_1 - \bar{y}_2) - 3 \left\{ z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \left(\frac{1}{\nu_1 - 2} - \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) + \left(\nu_2 s_2^2 \right) \left(\frac{1}{\nu_2 - 2} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) \right] + \frac{(\nu_1 s_1^2)}{n_1(\nu_1 - 2)} \right. \\
& \quad \left. + \frac{(\nu_2 s_2^2)}{n_2(\nu_2 - 2)} \right\} \left\{ z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\} \\
& - (\bar{y}_1 - \bar{y}_2)^3 - 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \\
& - 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right]^2 \\
& - z_\alpha^3 \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right]^3.
\end{aligned}$$

Now consider the terms that contain $(\bar{y}_1 - \bar{y}_2)^3$

$$(\bar{y}_1 - \bar{y}_2)^3 - (\bar{y}_1 - \bar{y}_2)^3 = 0.$$

Consider the terms that contain $(\bar{y}_1 - \bar{y}_2)^2$

$$\begin{aligned}
& 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \\
& - 3(\bar{y}_1 - \bar{y}_2)^2 z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] = 0.
\end{aligned}$$

Consider the terms that contain $(\bar{y}_1 - \bar{y}_2)$

$$\begin{aligned}
& 3(\bar{y}_1 - \bar{y}_2) \left\{ \frac{(\nu_1 s_1^2)}{(\nu_1 - 2)} \left(z_\alpha^2 + \frac{1}{n_1} \right) + \frac{(\nu_2 s_2^2)}{(\nu_2 - 2)} \left(z_\alpha^2 + \frac{1}{n_2} \right) \right\} - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
& - 3 \left\{ z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \left(\frac{1}{\nu_1 - 2} - \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) \right] (\bar{y}_1 - \bar{y}_2) \right\} - 3 \left\{ z_\alpha^2 \left[\left(\nu_2 s_2^2 \right) \left(\frac{1}{\nu_2 - 2} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) \right] (\bar{y}_1 - \bar{y}_2) \right\} \\
& - 3 \left[\frac{1}{n_1} \frac{(\nu_1 s_1^2)}{(\nu_1 - 2)} \right] (\bar{y}_1 - \bar{y}_2) - 3 \left[\frac{1}{n_2} \frac{(\nu_2 s_2^2)}{(\nu_2 - 2)} \right] (\bar{y}_1 - \bar{y}_2) - 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right] \\
& + 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right] \left[\left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] - 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_2 s_2^2 \right) \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right].
\end{aligned}$$

Therefore the terms containing $(\bar{y}_1 - \bar{y}_2)$ become

$$\begin{aligned}
& 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \frac{(\nu_1 s_1^2)}{(\nu_1 - 2)} + 3(\bar{y}_1 - \bar{y}_2) \frac{(\nu_1 s_1^2)}{n_1(\nu_1 - 2)} + 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \frac{(\nu_2 s_2^2)}{(\nu_2 - 2)} + 3(\bar{y}_1 - \bar{y}_2) \frac{(\nu_2 s_2^2)}{n_2(\nu_2 - 2)} \\
& - 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - 3z_\alpha^2 (\bar{y}_1 - \bar{y}_2) \frac{(\nu_1 s_1^2)}{(\nu_1 - 2)} + 3z_\alpha^2 (\bar{y}_1 - \bar{y}_2) \frac{(\nu_1 s_1^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \\
& - 3z_\alpha^2 (\bar{y}_1 - \bar{y}_2) \frac{(\nu_2 s_2^2)}{(\nu_2 - 2)} + 3z_\alpha^2 (\bar{y}_1 - \bar{y}_2) \frac{(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} - 3(\bar{y}_1 - \bar{y}_2) \frac{(\nu_1 s_1^2)}{n_1(\nu_1 - 2)} - 3(\bar{y}_1 - \bar{y}_2) \frac{(\nu_2 s_2^2)}{n_2(\nu_2 - 2)} \\
& - 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right] - 3(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_2 s_2^2 \right) \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right] \\
& + 6(\bar{y}_1 - \bar{y}_2) z_\alpha^2 \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right] \left[\left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \\
& = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
m_3 &= \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3z_\alpha}{n_1} + z_\alpha^3 \right] - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3z_\alpha}{n_2} + z_\alpha^3 \right] \\
& - \frac{3z_\alpha (\nu_1 s_1^2) (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{(\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] + \frac{3z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{1}{n_2} + z_\alpha^2 \right] \\
& - 3 \left\{ z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \left(\frac{1}{\nu_1 - 2} - \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) + \left(\nu_2 s_2^2 \right) \left(\frac{1}{\nu_2 - 2} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) \right] + \frac{(\nu_1 s_1^2)}{n_1(\nu_1 - 2)} + \frac{(\nu_2 s_2^2)}{n_2(\nu_2 - 2)} \right\} \\
& \times \left\{ z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\} - z_\alpha^3 \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_1 - 1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
& - 3z_\alpha^2 \left(\nu_1 s_1^2 \right) \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2}) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + 3z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right) \frac{\Gamma(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2}) 2\Gamma^2(\frac{\nu_2}{2})} - z_\alpha^2 \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_2 - 1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& = \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3z_\alpha}{n_1} + z_\alpha^3 \right] - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3z_\alpha}{n_2} + z_\alpha^3 \right] - \frac{3z_\alpha (\nu_1 s_1^2) (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{(\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] \\
& + \frac{3z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{1}{n_2} + z_\alpha^2 \right] - \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 1}{2})}{(\nu_1 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} + 3z_\alpha^3 \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_1 - 1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& + 3z_\alpha^3 \frac{(\nu_1 s_1^2) (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - 3z_\alpha^3 \left(\nu_1 s_1^2 \right) \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2}) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - 3z_\alpha^3 \frac{(\nu_2 s_2^2) (\nu_1 s_1^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})}
\end{aligned}$$

$$\begin{aligned}
& + 3z_\alpha^3 \frac{(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + 3z_\alpha^3 \frac{(\nu_2 s_2^2)(\nu_1 s_1^2)^{\frac{1}{2}} \Gamma^2(\frac{\nu_2-1}{2}) \Gamma(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} - 3z_\alpha^3 \frac{(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} - \frac{3(\nu_1 s_1^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_1-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \\
& + \frac{3(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2}) z_\alpha}{n_1(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{3(\nu_2 s_2^2)(\nu_1 s_1^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1-1}{2}) z_\alpha}{n_2(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{3(\nu_2 s_2^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \\
& + \frac{3z_\alpha^3 (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma^2(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^2(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} - \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} + z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})}.
\end{aligned}$$

Now consider the terms containing $(\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{1}{2}}$

$$\begin{aligned}
& - \frac{3z_\alpha (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{3z_\alpha^3 (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha^3 (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
& - \frac{3z_\alpha^2 (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma^2(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^2(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha^3 (\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}} \Gamma^2(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^2(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})}.
\end{aligned}$$

= 0 .

Also, consider the terms containing $(\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)$

$$\begin{aligned}
& \frac{3z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2})}{(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2})}{(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \\
& + \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma^2(\frac{\nu_2-1}{2}) \Gamma(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} - \frac{3z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})}
\end{aligned}$$

= 0 .

Therefore

$$\begin{aligned}
m_3 & = \frac{3z_\alpha (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-1}{2})}{(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \\
& + \frac{3z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} + \frac{3z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{3z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} - \frac{3(\nu_1 s_1^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_1-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \\
& + \frac{3(\nu_2 s_2^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} + \frac{z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})}.
\end{aligned}$$

Consider terms from the first sample only

$$\begin{aligned}
& \frac{3z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-1}{2})}{(\nu_1-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{3z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
& - \frac{3(\nu_1 s_1^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_1-1}{2})}{n_1(\nu_1-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
& = \frac{3z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-1}{2})}{(\nu_1-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{2z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} - \frac{3(\nu_1 s_1^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_1-1}{2})}{n_1(\nu_1-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \\
& = \frac{3z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} (\nu_1-3) \Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} (\nu_1-3) \Gamma(\frac{\nu_1-3}{2}) \Gamma^2(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
& \quad - \frac{3(\nu_1 s_1^2)^{\frac{3}{2}} z_\alpha (\nu_1-3) \Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \quad \text{from } \Gamma(\frac{\nu_1-1}{2}) = (\frac{\nu_1-3}{2}) \Gamma(\frac{\nu_1-3}{2}) \\
& = \frac{3z_\alpha(\nu_1-2)(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3(\nu_1-2)(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{3z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} (\nu_1-3) \Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \\
& \quad + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} (\nu_1-3) \Gamma(\frac{\nu_1-3}{2}) \Gamma^2(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} - \frac{3(\nu_1 s_1^2)^{\frac{3}{2}} z_\alpha (\nu_1-3) \Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \\
& = \left[\frac{3z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right] \left[(\nu_1-2) - (\nu_1-3) \right] + \left[\frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right] \left[(\nu_1-2) - 3(\nu_1-3) \right] \\
& \quad + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} (\nu_1-3) \Gamma(\frac{\nu_1-3}{2}) \Gamma^2(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
& = \frac{3z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{(2\nu_1-7)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}} (\nu_1-3) \Gamma(\frac{\nu_1-3}{2}) \Gamma^2(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
& = \frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left\{ \frac{3}{n_1(\nu_1-2)} - \frac{(2\nu_1-7)z_\alpha^2}{(\nu_1-2)} + \frac{z_\alpha^2(\nu_1-3) \Gamma^2(\frac{\nu_1-1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \right\} \\
& = \frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left\{ \frac{1}{(\nu_1-2)} \left[\frac{3}{n_1} - z_\alpha^2(2\nu_1-7) \right] + \frac{z_\alpha^2(\nu_1-3) \Gamma^2(\frac{\nu_1-1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \right\}.
\end{aligned}$$

Also, consider the terms for the second sample only

$$\begin{aligned}
& -\frac{3z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{3z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& + \frac{3(\nu_2 s_2^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& = -\frac{3z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{2z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} + \frac{3(\nu_2 s_2^2)^{\frac{3}{2}} z_\alpha \Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
& = -\frac{3z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} (\nu_2-3) \Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} (\nu_2-3) \Gamma(\frac{\nu_2-3}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& \quad + \frac{3(\nu_2 s_2^2)^{\frac{3}{2}} z_\alpha (\nu_2-3) \Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \quad \text{from } \Gamma(\frac{\nu_2-1}{2}) = (\frac{\nu_2-3}{2}) \Gamma(\frac{\nu_2-3}{2}) \\
& = -\frac{3z_\alpha(\nu_2-2)(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3(\nu_2-2)(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{3z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} (\nu_2-3) \Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \\
& \quad - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} (\nu_2-3) \Gamma(\frac{\nu_2-3}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} + \frac{3(\nu_2 s_2^2)^{\frac{3}{2}} z_\alpha (\nu_2-3) \Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \\
& = \left[\frac{3z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \right] \left[-(\nu_2-2) + (\nu_2-3) \right] + \left[\frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \right] \left[-(\nu_2-2) + 3(\nu_2-3) \right] \\
& \quad - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} (\nu_2-3) \Gamma(\frac{\nu_2-3}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& = \frac{-3z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{(2\nu_2-7)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}} (\nu_2-3) \Gamma(\frac{\nu_2-3}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \\
& = \frac{-z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left\{ \frac{3}{n_2(\nu_2-2)} - \frac{(2\nu_2-7)z_\alpha^2}{(\nu_2-2)} + \frac{z_\alpha^2(\nu_2-3) \Gamma^2(\frac{\nu_2-1}{2})}{\Gamma^2(\frac{\nu_2}{2})} \right\} \\
& = -\frac{z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left\{ \frac{1}{(\nu_2-2)} \left[\frac{3}{n_2} - z_\alpha^2(2\nu_2-7) \right] + \frac{z_\alpha^2(\nu_2-3) \Gamma^2(\frac{\nu_2-1}{2})}{\Gamma^2(\frac{\nu_2}{2})} \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
m_3 & = \frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left\{ \frac{1}{(\nu_1-2)} \left[\frac{3}{n_1} - z_\alpha^2(2\nu_1-7) \right] + \frac{z_\alpha^2(\nu_1-3) \Gamma^2(\frac{\nu_1-1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \right\} \\
& \quad - \frac{z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left\{ \frac{1}{(\nu_2-2)} \left[\frac{3}{n_2} - z_\alpha^2(2\nu_2-7) \right] + \frac{z_\alpha^2(\nu_2-3) \Gamma^2(\frac{\nu_2-1}{2})}{\Gamma^2(\frac{\nu_2}{2})} \right\}
\end{aligned}$$

d.) It was shown in Theorem 2.6.1.1 d.) that the fourth moment about the origin of $\left(\frac{1}{\chi^2_{\nu_d}}\right)^{\frac{1}{2}}$ ($d = 1, 2$) is given by

$$E\left(\frac{1}{\chi^2_{\nu_d}}\right)^{\frac{1}{2}} = \frac{1}{(\nu_d-2)(\nu_d-4)} \quad (d = 1, 2).$$

The fourth moment of the conditional posterior distribution of $(q_1 - q_2) | \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, \mathbf{y}_1, \mathbf{y}_2$ can then in general be written as

$$\begin{aligned} \mu'_4 &= \mu_4 + 4\mu'_1\mu_3 + 6(\mu'_1)^2\mu_2 + (\mu'_1)^4 \\ &= 3\left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right)^2 + 4\left\{\left(\bar{y}_1 - \bar{y}_2\right) + z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)\right\}(0) + 6\left\{\left(\bar{y}_1 - \bar{y}_2\right) + z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)\right\}^2 \\ &\quad \times \left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right) + \left\{\left(\bar{y}_1 - \bar{y}_2\right) + z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)\right\}^4 \\ &= 3\left(\frac{\sigma_{\varepsilon_1}^4}{n_1^2} + 2\frac{\sigma_{\varepsilon_1}^2}{n_1} \cdot \frac{\sigma_{\varepsilon_2}^2}{n_2} + \frac{\sigma_{\varepsilon_2}^4}{n_2^2}\right) + 6\left\{\left(\bar{y}_1 - \bar{y}_2\right)^2 + 2\left(\bar{y}_1 - \bar{y}_2\right)z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right) + z_{\alpha}^2\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)^2\right\} \\ &\quad \times \left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right) + \left(\bar{y}_1 - \bar{y}_2\right)^4 + 4\left(\bar{y}_1 - \bar{y}_2\right)^3 z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right) + 6\left(\bar{y}_1 - \bar{y}_2\right)^2 z_{\alpha}^2\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)^2 \\ &\quad + 4\left(\bar{y}_1 - \bar{y}_2\right)z_{\alpha}^3\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)^3 + z_{\alpha}^4\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)^4 \\ &= \frac{3\sigma_{\varepsilon_1}^4}{n_1^2} + \frac{6\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2}{n_1n_2} + \frac{3\sigma_{\varepsilon_2}^4}{n_2^2} + 6\left(\bar{y}_1 - \bar{y}_2\right)^2\left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right) + 12\left(\bar{y}_1 - \bar{y}_2\right)z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right)\left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2}\right) \\ &\quad + 6z_{\alpha}^2\left(\frac{\sigma_{\varepsilon_1}^4}{n_1} + \frac{\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2}{n_2} - \frac{2\sigma_{\varepsilon_1}^3\sigma_{\varepsilon_2}}{n_1} - \frac{2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}^3}{n_2} + \frac{\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^4}{n_2}\right) + 4\left(\bar{y}_1 - \bar{y}_2\right)^3 z_{\alpha}\left(\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}\right) \\ &\quad + \left(\bar{y}_1 - \bar{y}_2\right)^4 + 6\left(\bar{y}_1 - \bar{y}_2\right)^2 z_{\alpha}^2\left[\sigma_{\varepsilon_1}^2 - 2\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} + \sigma_{\varepsilon_2}^2\right] \\ &\quad + 4\left(\bar{y}_1 - \bar{y}_2\right)z_{\alpha}^3\left[\sigma_{\varepsilon_1}^3 - 3\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2} + 3\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}^2 - \sigma_{\varepsilon_2}^3\right] \\ &\quad + z_{\alpha}^4\left\{\sigma_{\varepsilon_1}^4 - 4\sigma_{\varepsilon_1}^3\sigma_{\varepsilon_2} + 6\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 - 4\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}^3 + \sigma_{\varepsilon_2}^4\right\} \end{aligned}$$

$$\begin{aligned}
&= \sigma_{\varepsilon_1}^4 \left\{ \frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right\} + \sigma_{\varepsilon_2}^4 \left\{ \frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right\} + \sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2}^2 \left\{ \frac{6}{n_1 n_2} + \frac{6z_\alpha^2}{n_2} + \frac{6z_\alpha^2}{n_1} + 6z_\alpha^4 \right\} \\
&\quad - 4\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^3 \left\{ \frac{3z_\alpha^2}{n_2} + z_\alpha^4 \right\} - 4\sigma_{\varepsilon_1}^3 \sigma_{\varepsilon_2} \left\{ \frac{3z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
&\quad + (\bar{y}_1 - \bar{y}_2) \left\{ 12z_\alpha \left[\frac{\sigma_{\varepsilon_1}^3}{n_1} + \frac{\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2}{n_2} - \frac{\sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2}}{n_1} - \frac{\sigma_{\varepsilon_2}^3}{n_2} \right] + 4z_\alpha^3 \left[\sigma_{\varepsilon_1}^3 - 3\sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2} + 3\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2 - \sigma_{\varepsilon_2}^3 \right] \right\} \\
&\quad + 6(\bar{y}_1 - \bar{y}_2)^2 \left\{ z_\alpha^2 \left[\sigma_{\varepsilon_1}^2 - 2\sigma_{\varepsilon_1} \sigma_{\varepsilon_2} + \sigma_{\varepsilon_2}^2 \right] + \left(\frac{\sigma_{\varepsilon_1}^2}{n_1} + \frac{\sigma_{\varepsilon_2}^2}{n_2} \right) \right\} + (\bar{y}_1 - \bar{y}_2)^3 \left\{ 4z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) \right\} \\
&\quad + (\bar{y}_1 - \bar{y}_2)^4 \\
&= \sigma_{\varepsilon_1}^4 \left\{ \frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right\} + \sigma_{\varepsilon_2}^4 \left\{ \frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right\} + \sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2}^2 \left\{ \frac{6}{n_1 n_2} + \frac{6z_\alpha^2}{n_2} + \frac{6z_\alpha^2}{n_1} + 6z_\alpha^4 \right\} \\
&\quad - 4\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^3 \left\{ \frac{3z_\alpha^2}{n_2} + z_\alpha^4 \right\} - 4\sigma_{\varepsilon_1}^3 \sigma_{\varepsilon_2} \left\{ \frac{3z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
&\quad + (\bar{y}_1 - \bar{y}_2) \left\{ 4\sigma_{\varepsilon_1}^3 \left[\frac{3z_\alpha}{n_1} + z_\alpha^3 \right] - 4\sigma_{\varepsilon_2}^3 \left[\frac{3z_\alpha}{n_2} + z_\alpha^3 \right] - 12\sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2} \left[\frac{z_\alpha}{n_1} + z_\alpha^3 \right] + 12\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}^2 \left[\frac{z_\alpha}{n_2} + z_\alpha^3 \right] \right\} \\
&\quad + 6(\bar{y}_1 - \bar{y}_2)^2 \left\{ \sigma_{\varepsilon_1}^2 \left[\frac{1}{n_1} + z_\alpha^2 \right] + \sigma_{\varepsilon_2}^2 \left[\frac{1}{n_2} + z_\alpha^2 \right] - 2\sigma_{\varepsilon_1} \sigma_{\varepsilon_2} z_\alpha^2 \right\} + (\bar{y}_1 - \bar{y}_2)^3 \left\{ 4z_\alpha (\sigma_{\varepsilon_1} - \sigma_{\varepsilon_2}) \right\} \\
&\quad + (\bar{y}_1 - \bar{y}_2)^4.
\end{aligned}$$

Therefore μ'_4 will be equal to

$$\begin{aligned}
\mu'_4 &= \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^2 \left\{ \frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right\} + \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^4 \left\{ \frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right\} + 6 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right) \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right) \left\{ \frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
&\quad - 4 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{3}{2}} \left\{ \frac{3z_\alpha^2}{n_2} + z_\alpha^4 \right\} - 4 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{3}{2}} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \left\{ \frac{3z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
&\quad + 4(\bar{y}_1 - \bar{y}_2) z_\alpha \left\{ \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{3}{2}} \left[\frac{3}{n_1} + z_\alpha^2 \right] - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{3}{2}} \left[\frac{3}{n_2} + z_\alpha^2 \right] - 3 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right) \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \left[\frac{1}{n_1} + z_\alpha^2 \right] \right. \\
&\quad \left. + 3 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right) \left[\frac{1}{n_2} + z_\alpha^2 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& +6(\bar{y}_1 - \bar{y}_2)^2 \left\{ \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right) \left[\frac{1}{n_1} + z_\alpha^2 \right] + \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right) \left[\frac{1}{n_2} + z_\alpha^2 \right] - 2 \left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} z_\alpha^2 \right\} \\
& + (\bar{y}_1 - \bar{y}_2)^3 \left\{ 4z_\alpha \left[\left(\frac{\nu_1 s_1^2}{\chi_{\nu_1}^2} \right)^{\frac{1}{2}} - \left(\frac{\nu_2 s_2^2}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \right] \right\} + (\bar{y}_1 - \bar{y}_2)^4.
\end{aligned}$$

Now $m'_4 = E(\mu'_4)$, therefore

$$\begin{aligned}
m'_4 &= (\bar{y}_1 - \bar{y}_2)^4 + (\bar{y}_1 - \bar{y}_2)^3 \left\{ 4z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}} \right] \right\} \\
& +6(\bar{y}_1 - \bar{y}_2)^2 \left\{ \left(\nu_1 s_1^2 \right) E\left(\frac{1}{\chi_{\nu_1}^2}\right) \left[\frac{1}{n_1} + z_\alpha^2 \right] + \left(\nu_2 s_2^2 \right) E\left(\frac{1}{\chi_{\nu_2}^2}\right) \left[\frac{1}{n_2} + z_\alpha^2 \right] \right. \\
& \quad \left. - 2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} z_\alpha^2 E\left(\frac{1}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}} \right\} \\
& +4(\bar{y}_1 - \bar{y}_2) z_\alpha \left\{ \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} E\left(\frac{1}{\chi_{\nu_1}^2}\right)^{\frac{3}{2}} \left[\frac{3}{n_1} + z_\alpha^2 \right] - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} E\left(\frac{1}{\chi_{\nu_2}^2}\right)^{\frac{3}{2}} \left[\frac{3}{n_2} + z_\alpha^2 \right] \right. \\
& \quad - 3 \left(\nu_1 s_1^2 \right) \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_1}^2}\right) E\left(\frac{1}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}} \left[\frac{1}{n_1} + z_\alpha^2 \right] \\
& \quad \left. + 3 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right) E\left(\frac{1}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_2}^2}\right) \left[\frac{1}{n_2} + z_\alpha^2 \right] \right\} \\
& -4 \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_1}^2}\right)^{\frac{3}{2}} E\left(\frac{1}{\chi_{\nu_2}^2}\right)^{\frac{1}{2}} \left\{ \frac{3z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
& -4 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} E\left(\frac{1}{\chi_{\nu_1}^2}\right)^{\frac{1}{2}} E\left(\frac{1}{\chi_{\nu_2}^2}\right)^{\frac{3}{2}} \left\{ \frac{3z_\alpha^2}{n_2} + z_\alpha^4 \right\} \\
& +6 \left(\nu_1 s_1^2 \right) \left(\nu_2 s_2^2 \right) E\left(\frac{1}{\chi_{\nu_1}^2}\right) E\left(\frac{1}{\chi_{\nu_2}^2}\right) \left\{ \frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
& + \left(\nu_1 s_1^2 \right)^2 E\left(\frac{1}{\chi_{\nu_1}^2}\right)^2 \left\{ \frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right\} \\
& + \left(\nu_2 s_2^2 \right)^2 E\left(\frac{1}{\chi_{\nu_2}^2}\right)^2 \left\{ \frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right\}.
\end{aligned}$$

$$\begin{aligned}
m'_4 = & (\bar{y}_1 - \bar{y}_2)^4 + (\bar{y}_1 - \bar{y}_2)^3 \left\{ 4z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\} \\
& + 6(\bar{y}_1 - \bar{y}_2)^2 \left\{ \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left[\frac{1}{n_1} + z_\alpha^2 \right] + \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] - 2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} z_\alpha^2 \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right\} \\
& + 4(\bar{y}_1 - \bar{y}_2) z_\alpha \left\{ \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3}{n_1} + z_\alpha^2 \right] - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_2} + z_\alpha^2 \right] \right. \\
& \quad \left. - 3 \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] + 3 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] \right\} \\
& - 4z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_1} + z_\alpha^2 \right] \\
& - 4z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_2} + z_\alpha^2 \right] \\
& + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1-2)(\nu_2-2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& + \frac{(\nu_1 s_1^2)^2}{(\nu_1-2)(\nu_1-4)} \left[\frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& + \frac{(\nu_2 s_2^2)^2}{(\nu_2-2)(\nu_2-4)} \left[\frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right].
\end{aligned}$$

It is also known that m_4 is given by

$$m_4 = m'_4 - 4m'_1 m_3 - 6(m'_1)^2 m_2 - (m'_1)^4.$$

Consider m'_4 only, therefore

$$\begin{aligned}
& \left(\bar{y}_1 - \bar{y}_2 \right)^4 + \left(\bar{y}_1 - \bar{y}_2 \right)^3 \left\{ 4z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\} \\
& + 6 \left(\bar{y}_1 - \bar{y}_2 \right)^2 \left\{ \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left[\frac{1}{n_1} + z_\alpha^2 \right] + \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] - 2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} z_\alpha^2 \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right\} \\
& + 4 \left(\bar{y}_1 - \bar{y}_2 \right) z_\alpha \left\{ \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3}{n_1} + z_\alpha^2 \right] - \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_2} + z_\alpha^2 \right] \right. \\
& \quad \left. - 3 \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] + 3 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] \right\} \\
& - 4 z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-3}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_1} + z_\alpha^2 \right] \\
& - 4 z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2}) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_2} + z_\alpha^2 \right] + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1-2)(\nu_2-2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& + \frac{(\nu_1 s_1^2)^2}{(\nu_1-2)(\nu_1-4)} \left[\frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_2 s_2^2)^2}{(\nu_2-2)(\nu_2-4)} \left[\frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right] \\
& = \left(\bar{y}_1 - \bar{y}_2 \right)^4 + 4 \left(\bar{y}_1 - \bar{y}_2 \right)^3 z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - 4 \left(\bar{y}_1 - \bar{y}_2 \right)^3 z_\alpha \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \\
& + 6 \left(\bar{y}_1 - \bar{y}_2 \right)^2 \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left[\frac{1}{n_1} + z_\alpha^2 \right] + 6 \left(\bar{y}_1 - \bar{y}_2 \right)^2 \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] \\
& - 12 \left(\bar{y}_1 - \bar{y}_2 \right)^2 z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2}) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + 4 \left(\bar{y}_1 - \bar{y}_2 \right) z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{3}{n_1} + z_\alpha^2 \right] \\
& - 4 \left(\bar{y}_1 - \bar{y}_2 \right) z_\alpha \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_2} + z_\alpha^2 \right] - 12 \left(\bar{y}_1 - \bar{y}_2 \right) z_\alpha \frac{(\nu_1 s_1^2)}{(\nu_1-2)} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{n_1} + z_\alpha^2 \right] \\
& + 12 \left(\bar{y}_1 - \bar{y}_2 \right) z_\alpha \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \frac{(\nu_2 s_2^2)}{(\nu_2-2)} \left[\frac{1}{n_2} + z_\alpha^2 \right] \\
& - 4 z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{3}{2}} \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-3}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_1} + z_\alpha^2 \right] \\
& - 4 z_\alpha^2 \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \left(\nu_2 s_2^2 \right)^{\frac{3}{2}} \frac{\Gamma(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2}) 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{3}{n_2} + z_\alpha^2 \right] \\
& + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1-2)(\nu_2-2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_1 s_1^2)^2}{(\nu_1-2)(\nu_1-4)} \left[\frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_2 s_2^2)^2}{(\nu_2-2)(\nu_2-4)} \left[\frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right]
\end{aligned}$$

$$\begin{aligned}
&= (\bar{y}_1 - \bar{y}_2)^4 + \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 (\nu_1 s_1^2)}{n_1 (\nu_1 - 2)} \\
&\quad + \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_1 s_1^2)}{(\nu_1 - 2)} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 (\nu_2 s_2^2)}{n_2 (\nu_2 - 2)} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_2 s_2^2)}{(\nu_2 - 2)} - \frac{12(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_2 - 1}{2})}{2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 3}{2})}{n_1 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2 - 3}{2})}{n_2 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \\
&\quad - \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha (\nu_1 s_1^2) (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{n_1 (\nu_1 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2) (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1 - 1}{2})}{n_2 (\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \\
&\quad + \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{12 z_\alpha^2 (\nu_1 s_1^2)^{\frac{3}{2}} (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1 - 3}{2}) \Gamma(\frac{\nu_2 - 1}{2})}{n_1 2^2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} - \frac{4 z_\alpha^4 (\nu_1 s_1^2)^{\frac{3}{2}} (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1 - 3}{2}) \Gamma(\frac{\nu_2 - 1}{2})}{2^2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \\
&\quad - \frac{12 z_\alpha^2 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_2 2^2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} - \frac{4 z_\alpha^4 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{2^2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{6(\nu_1 s_1^2) (\nu_2 s_2^2)}{(\nu_1 - 2) (\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_1 s_1^2)^2}{(\nu_1 - 2) (\nu_1 - 4)} \left[\frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_2 s_2^2)^2}{(\nu_2 - 2) (\nu_2 - 4)} \left[\frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right].
\end{aligned}$$

Consider $-4m'_1 m_3$ only, therefore

$$\begin{aligned}
&-4 \left\{ (\bar{y}_1 - \bar{y}_2) + z_\alpha \left[(\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - (\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\} \\
&\quad \times \left\{ \frac{z_\alpha (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \left[\frac{1}{(\nu_1 - 2)} \left(\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right) + \frac{z_\alpha^2 (\nu_1 - 3) \Gamma^2(\frac{\nu_1 - 1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \right] \right. \\
&\quad \left. - \frac{z_\alpha (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \left[\frac{1}{(\nu_2 - 2)} \left(\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right) + \frac{z_\alpha^2 (\nu_2 - 3) \Gamma^2(\frac{\nu_2 - 1}{2})}{\Gamma^2(\frac{\nu_2}{2})} \right] \right\}.
\end{aligned}$$

Suppose

$$\begin{aligned}
a &= (\bar{y}_1 - \bar{y}_2) \quad ; \quad b_1 = (\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \quad ; \quad b_2 = (\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \quad ; \quad c_1 = \frac{z_\alpha (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \\
c_2 &= \frac{z_\alpha (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2 - 3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \quad ; \quad d_1 = \frac{z_\alpha^2 (\nu_1 - 3) \Gamma^2(\frac{\nu_1 - 1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \quad ; \quad d_2 = \frac{z_\alpha^2 (\nu_2 - 3) \Gamma^2(\frac{\nu_2 - 1}{2})}{\Gamma^2(\frac{\nu_2}{2})},
\end{aligned}$$

it therefore follows that $-4m'_1m_3$ is given by

$$\begin{aligned}
& -4 \left\{ a + z_\alpha \left[b_1 - b_2 \right] \right\} \left\{ c_1 \left[\frac{1}{(\nu_1-2)} \left(\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right) + d_1 \right] - c_2 \left[\frac{1}{(\nu_2-2)} \left(\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right) + d_2 \right] \right\} \\
& = \left(-4a - 4z_\alpha b_1 + 4z_\alpha b_2 \right) \left\{ c_1 \left[\frac{1}{(\nu_1-2)} \left(\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right) + d_1 \right] \right. \\
& \quad \left. - c_2 \left[\frac{1}{(\nu_2-2)} \left(\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right) + d_2 \right] \right\} \\
& = -4ac_1 \left[\frac{1}{(\nu_1-2)} \left(\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right) + d_1 \right] + 4ac_2 \left[\frac{1}{(\nu_2-2)} \left(\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right) + d_2 \right] \\
& \quad - 4z_\alpha b_1 c_1 \left[\frac{1}{(\nu_1-2)} \left(\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right) + d_1 \right] + 4z_\alpha b_1 c_2 \left[\frac{1}{(\nu_2-2)} \left(\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right) + d_2 \right] \\
& \quad + 4z_\alpha b_2 c_1 \left[\frac{1}{(\nu_1-2)} \left(\frac{3}{n_1} - z_\alpha^2 (2\nu_1 - 7) \right) + d_1 \right] - 4z_\alpha b_2 c_2 \left[\frac{1}{(\nu_2-2)} \left(\frac{3}{n_2} - z_\alpha^2 (2\nu_2 - 7) \right) + d_2 \right] \\
& = -\frac{12a}{n_1(\nu_1-2)} c_1 + \frac{4az_\alpha^2(2\nu_1-7)}{(\nu_1-2)} c_1 - 4ac_1 d_1 + \frac{12a}{n_2(\nu_2-2)} c_2 - \frac{4az_\alpha^2(2\nu_2-7)}{(\nu_2-2)} c_2 + 4ac_2 d_2 \\
& \quad - \frac{12z_\alpha}{n_1(\nu_1-2)} b_1 c_1 + \frac{4z_\alpha^3(2\nu_1-7)}{(\nu_1-2)} b_1 c_1 - 4z_\alpha b_1 c_1 d_1 + \frac{12z_\alpha}{n_2(\nu_2-2)} b_1 c_2 - \frac{4z_\alpha^3(2\nu_2-7)}{(\nu_2-2)} b_1 c_2 + 4z_\alpha b_1 c_2 d_2 \\
& \quad + \frac{12z_\alpha}{n_1(\nu_1-2)} b_2 c_1 - \frac{4z_\alpha^3(2\nu_1-7)}{(\nu_1-2)} b_2 c_1 + 4z_\alpha b_2 c_1 d_1 - \frac{12z_\alpha}{n_2(\nu_2-2)} b_2 c_2 + \frac{4z_\alpha^3(2\nu_2-7)}{(\nu_2-2)} b_2 c_2 - 4z_\alpha b_2 c_2 d_2.
\end{aligned}$$

By substituting $a, b_1, b_2, c_1, c_2, d_1$ and d_2 back, it follows that $-4m'_1m_3$ is given by

$$\begin{aligned}
& = -\frac{12(\bar{y}_1 - \bar{y}_2)}{n_1(\nu_1-2)} \left(\frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right) + \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^2 (2\nu_1 - 7)}{(\nu_1-2)} \left(\frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right) \\
& \quad - 4(\bar{y}_1 - \bar{y}_2) \left(\frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left(\frac{z_\alpha^2 (\nu_1 - 3) \Gamma^2(\frac{\nu_1-1}{2})}{\Gamma^2(\frac{\nu_1}{2})} \right) + \frac{12(\bar{y}_1 - \bar{y}_2)}{n_2(\nu_2-2)} \left(\frac{z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \right) \\
& \quad - \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^2 (2\nu_2 - 7)}{(\nu_2-2)} \left(\frac{z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \right) + 4(\bar{y}_1 - \bar{y}_2) \left(\frac{z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} \right) \left(\frac{z_\alpha^2 (\nu_2 - 3) \Gamma^2(\frac{\nu_2-1}{2})}{\Gamma^2(\frac{\nu_2}{2})} \right) \\
& \quad - \frac{12z_\alpha}{n_1(\nu_1-2)} \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left(\frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right) + \frac{4z_\alpha^3(2\nu_1-7)}{(\nu_1-2)} \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left(\frac{z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} \right)
\end{aligned}$$

Also consider $-6(m'_1)^2 m_2$ only, therefore

$$-6 \left\{ \left(\bar{y}_1 - \bar{y}_2 \right) + z_\alpha \left[\left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right] \right\}^2 \\ \times \left\{ z_\alpha^2 \left[\left(\nu_1 s_1^2 \right) \left[\frac{1}{\nu_1-2} - \frac{\Gamma^2(\frac{\nu_1-1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right] + \left(\nu_2 s_2^2 \right) \left[\frac{1}{\nu_2-2} - \frac{\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right] \right] + \frac{(\nu_1 s_1^2)}{n_1(\nu_1-2)} + \frac{(\nu_2 s_2^2)}{n_2(\nu_2-2)} \right\}.$$

Suppose

$$a = \left(\bar{y}_1 - \bar{y}_2 \right) ; \quad b_1 = \left(\nu_1 s_1^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} ; \quad b_2 = \left(\nu_2 s_2^2 \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})}$$

$$e_1 = \frac{(\nu_1 s_1^2)}{n_1(\nu_1-2)} ; \quad e_2 = \frac{(\nu_2 s_2^2)}{n_2(\nu_2-2)},$$

it therefore follows that $-6(m'_1)^2 m_2$ is given by

$$-6 \left\{ a + z_\alpha [b_1 - b_2] \right\}^2 \left\{ z_\alpha^2 \left[\left[\frac{(\nu_1 s_1^2)}{\nu_1-2} - b_1^2 \right] + \left[\frac{(\nu_2 s_2^2)}{\nu_2-2} - b_2^2 \right] \right] + e_1 + e_2 \right\} \\ = -6 \left\{ a^2 + 2a z_\alpha [b_1 - b_2] + z_\alpha^2 [b_1 - b_2]^2 \right\} \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1-2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2-2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\ = -6 \left\{ a^2 + 2a z_\alpha [b_1 - b_2] + z_\alpha^2 [b_1^2 - 2b_1 b_2 + b_2^2] \right\} \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1-2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2-2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\ = -6 \left\{ a^2 + 2a z_\alpha b_1 - 2a z_\alpha b_2 + z_\alpha^2 b_1^2 - 2z_\alpha^2 b_1 b_2 + z_\alpha^2 b_2^2 \right\} \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1-2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2-2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\ = -6a^2 \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1-2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2-2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\ - 12a z_\alpha b_1 \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1-2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2-2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\ + 12a z_\alpha b_2 \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1-2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2-2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\}$$

$$\begin{aligned}
& -6z_\alpha^2 b_1^2 \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\
& + 12z_\alpha^2 b_1 b_2 \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\
& - 6z_\alpha^2 b_2^2 \left\{ z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} - z_\alpha^2 b_1^2 + z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} - z_\alpha^2 b_2^2 + e_1 + e_2 \right\} \\
= & -6a^2 z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} + 6a^2 z_\alpha^2 b_1^2 - 6a^2 z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} + 6a^2 z_\alpha^2 b_2^2 - 6a^2 e_1 - 6a^2 e_2 \\
& - 12a z_\alpha^3 b_1 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} + 12a z_\alpha^3 b_1^3 - 12a z_\alpha^3 b_1 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} + 12a z_\alpha^3 b_1 b_2^2 - 12a z_\alpha b_1 e_1 - 12a z_\alpha b_1 e_2 \\
& + 12a z_\alpha^3 b_2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} - 12a z_\alpha^3 b_1^2 b_2 + 12a z_\alpha^3 b_2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} - 12a z_\alpha^3 b_2^3 + 12a z_\alpha b_2 e_1 + 12a z_\alpha b_2 e_2 \\
& - 6z_\alpha^4 b_1^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} + 6z_\alpha^4 b_1^4 - 6z_\alpha^4 b_1^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} + 6z_\alpha^4 b_1^2 b_2^2 - 6z_\alpha^2 b_1^2 e_1 - 6z_\alpha^2 b_1^2 e_2 \\
& + 12z_\alpha^4 b_1 b_2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} - 12z_\alpha^4 b_1^3 b_2 + 12z_\alpha^4 b_1 b_2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} - 12z_\alpha^4 b_1 b_2^3 + 12z_\alpha^2 b_1 b_2 e_1 + 12z_\alpha^2 b_1 b_2 e_2 \\
& - 6z_\alpha^4 b_2^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} + 6z_\alpha^4 b_1^2 b_2^2 - 6z_\alpha^4 b_2^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} + 6z_\alpha^4 b_2^4 - 6z_\alpha^2 b_2^2 e_1 - 6z_\alpha^2 b_2^2 e_2.
\end{aligned}$$

By substituting a , b_1 , b_2 , e_1 and e_2 back, it follows that $-6(m'_1)^2 m_2$ is given by

$$\begin{aligned}
= & -6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \frac{(\nu_1 s_1^2)}{\nu_1 - 2} + 6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \left((\nu_1 s_1^2) \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) - 6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \frac{(\nu_2 s_2^2)}{\nu_2 - 2} \\
& + 6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \left((\nu_2 s_2^2) \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) - 6(\bar{y}_1 - \bar{y}_2)^2 \left(\frac{(\nu_1 s_1^2)}{n_1(\nu_1 - 2)} \right) - 6(\bar{y}_1 - \bar{y}_2)^2 \left(\frac{(\nu_2 s_2^2)}{n_2(\nu_2 - 2)} \right) \\
& - 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \frac{(\nu_1 s_1^2)}{\nu_1 - 2} + 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_1 - 1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \right) \\
& - 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \frac{(\nu_2 s_2^2)}{\nu_2 - 2} + 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2) \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right) \\
& - 12(\bar{y}_1 - \bar{y}_2) z_\alpha \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left(\frac{(\nu_1 s_1^2)}{n_1(\nu_1 - 2)} \right) - 12(\bar{y}_1 - \bar{y}_2) z_\alpha \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left(\frac{(\nu_2 s_2^2)}{n_2(\nu_2 - 2)} \right) \\
& + 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) \frac{(\nu_1 s_1^2)}{\nu_1 - 2} - 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2) \frac{\Gamma^2(\frac{\nu_1 - 1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) \\
& + 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) \frac{(\nu_2 s_2^2)}{\nu_2 - 2} - 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_2 s_2^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_2 - 1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& -\frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_2-1}{2})}{(\nu_1-2)2\Gamma^2(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})}{2^2\Gamma^2(\frac{\nu_1}{2})\Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_2 s_2^2)^2\Gamma^2(\frac{\nu_2-1}{2})}{(\nu_2-2)2\Gamma^2(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_2 s_2^2)^2\Gamma^4(\frac{\nu_2-1}{2})}{2^2\Gamma^4(\frac{\nu_2}{2})} \\
& -\frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2\Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_2 s_2^2)^2\Gamma^2(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma^2(\frac{\nu_2}{2})}.
\end{aligned}$$

Consider $-(m'_1)^4$ only, therefore

$$-\left\{(\bar{y}_1 - \bar{y}_2) + z_\alpha \left[(\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - (\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \right]\right\}^4.$$

Now, suppose that

$$a = (\bar{y}_1 - \bar{y}_2) \quad ; \quad b_1 = (\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \quad ; \quad b_2 = (\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})},$$

it therefore follows that $-(m'_1)^4$ will be

$$\begin{aligned}
& -\left\{a + z_\alpha [b_1 - b_2]\right\}^4 \\
& = -\left\{a^4 + 4a^3 z_\alpha [b_1 - b_2] + 6a^2 z_\alpha^2 [b_1 - b_2]^2 + 4a z_\alpha^3 [b_1 - b_2]^3 + z_\alpha^4 [b_1 - b_2]^4\right\} \\
& = -a^4 - 4a^3 z_\alpha [b_1 - b_2] - 6a^2 z_\alpha^2 [b_1 - b_2]^2 - 4a z_\alpha^3 [b_1 - b_2]^3 - z_\alpha^4 [b_1 - b_2]^4 \\
& = -a^4 - 4a^3 z_\alpha [b_1 - b_2] - 6a^2 z_\alpha^2 [b_1^2 - 2b_1 b_2 + b_2^2] - 4a z_\alpha^3 [b_1^3 - 3b_1^2 b_2 + 3b_1 b_2^2 - b_2^3] \\
& \quad - z_\alpha^4 [b_1^4 - 4b_1^3 b_2 + 6b_1^2 b_2^2 - 4b_1 b_2^3 + b_2^4] \\
& = -a^4 - 4a^3 z_\alpha b_1 + 4a^3 z_\alpha b_2 - 6a^2 z_\alpha^2 b_1^2 + 12a^2 z_\alpha^2 b_1 b_2 - 6a^2 z_\alpha^2 b_2^2 - 4a z_\alpha^3 b_1^3 + 12a z_\alpha^3 b_1^2 b_2 \\
& \quad - 12a z_\alpha^3 b_1 b_2^2 + 4a z_\alpha^3 b_2^3 - z_\alpha^4 b_1^4 + 4z_\alpha^4 b_1^3 b_2 - 6z_\alpha^4 b_1^2 b_2^2 + 4z_\alpha^4 b_1 b_2^3 - z_\alpha^4 b_2^4.
\end{aligned}$$

Substitute a , b_1 and b_2 back, therefore $-(m'_1)^4$ is given by

$$\begin{aligned}
&= -(\bar{y}_1 - \bar{y}_2)^4 - 4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) + 4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) \\
&\quad - 6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2 \Gamma^2(\frac{\nu_1}{2})} \right) + 12(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) \\
&\quad - 6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})} \right) - 4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \right) \\
&\quad + 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2 \Gamma^2(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) - 12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})} \right) \\
&\quad + 4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 \left((\nu_2 s_2^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \right) - z_\alpha^4 \left((\nu_1 s_1^2)^2 \frac{\Gamma^4(\frac{\nu_1-1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} \right) + 4z_\alpha^4 \left((\nu_1 s_1^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} \right) \\
&\quad - 6z_\alpha^4 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2 \Gamma^2(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})} \right) + 4z_\alpha^4 \left((\nu_1 s_1^2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} \right) \left((\nu_2 s_2^2)^{\frac{3}{2}} \frac{\Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} \right) \\
&\quad - z_\alpha^4 \left((\nu_2 s_2^2)^2 \frac{\Gamma^4(\frac{\nu_2-1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} \right) \\
&= -(\bar{y}_1 - \bar{y}_2)^4 - \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_1 s_1^2) \Gamma^2(\frac{\nu_1-1}{2})}{2 \Gamma^2(\frac{\nu_1}{2})} \\
&\quad + \frac{12(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} - \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_2 s_2^2) \Gamma^2(\frac{\nu_2-1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_1}{2})} \\
&\quad + \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2) (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma^2(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^2(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2) \Gamma(\frac{\nu_1-1}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} \\
&\quad + \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}} \Gamma^3(\frac{\nu_2}{2})} - \frac{z_\alpha^4 (\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1-1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} + \frac{4z_\alpha^4 (\nu_1 s_1^2)^{\frac{3}{2}} (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma^3(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2^2 \Gamma^3(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \\
&\quad - \frac{6z_\alpha^4 (\nu_1 s_1^2) (\nu_2 s_2^2) \Gamma^2(\frac{\nu_1-1}{2}) \Gamma^2(\frac{\nu_2-1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} + \frac{4z_\alpha^4 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-1}{2}) \Gamma^3(\frac{\nu_2-1}{2})}{2^2 \Gamma(\frac{\nu_1}{2}) \Gamma^3(\frac{\nu_2}{2})} - \frac{z_\alpha^4 (\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2-1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
m_4 &= (\bar{y}_1 - \bar{y}_2)^4 + \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha (\nu_1 s_1^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 (\nu_1 s_1^2)}{n_1 (\nu_1 - 2)} \\
&\quad + \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_1 s_1^2)}{(\nu_1 - 2)} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 (\nu_2 s_2^2)}{n_2 (\nu_2 - 2)} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_2 s_2^2)}{(\nu_2 - 2)} - \frac{12(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2 (\nu_1 s_1^2)^{\frac{1}{2}} (\nu_2 s_2^2)^{\frac{1}{2}} \Gamma(\frac{\nu_1-1}{2}) \Gamma(\frac{\nu_2-1}{2})}{2 \Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_1 s_1^2)^{\frac{3}{2}} \Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3 (\nu_2 s_2^2)^{\frac{3}{2}} \Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\nu_2}{2})}
\end{aligned}$$

$$\begin{aligned}
& - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \\
& + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1-1}{2})}{(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1 2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2 2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1-2)(\nu_2-2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_1 s_1^2)^2}{(\nu_1-2)(\nu_1-4)} \left[\frac{3}{n_1^2} + \frac{6z_\alpha^2}{n_1} + z_\alpha^4 \right] + \frac{(\nu_2 s_2^2)^2}{(\nu_2-2)(\nu_2-4)} \left[\frac{3}{n_2^2} + \frac{6z_\alpha^2}{n_2} + z_\alpha^4 \right] \\
& - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}(2\nu_1-7)\Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_1-3)\Gamma^2(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu_1}{2})} \\
& + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma^2(\frac{\nu_2-1}{2})\Gamma(\frac{\nu_2-3}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^2(\nu_1 s_1^2)^2\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^2\Gamma^2(\frac{\nu_1}{2})} + \frac{4z_\alpha^4(\nu_1 s_1^2)^2(2\nu_1-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{(\nu_1-2)2^2\Gamma^2(\frac{\nu_1}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^2(\nu_1-3)\Gamma^3(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{2^2\Gamma^4(\frac{\nu_1}{2})} \\
& + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})\Gamma(\frac{\nu_2-3}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(2\nu_1-7)\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(\nu_1-3)\Gamma^2(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{2^2\Gamma^3(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^2(\nu_2 s_2^2)^2\Gamma(\frac{\nu_2-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2\Gamma^2(\frac{\nu_2}{2})} + \frac{4z_\alpha^4(\nu_2 s_2^2)^2(2\nu_2-7)\Gamma(\frac{\nu_2-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2\Gamma^2(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_2 s_2^2)^2(\nu_2-3)\Gamma^3(\frac{\nu_2-1}{2})\Gamma(\frac{\nu_2-3}{2})}{2^2\Gamma^4(\frac{\nu_2}{2})} \\
& - \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_1 s_1^2)}{\nu_1-2} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_1 s_1^2)\Gamma^2(\frac{\nu_1-1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} - \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_2 s_2^2)}{\nu_2-2} + \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_2 s_2^2)\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{6(\bar{y}_1 - \bar{y}_2)^2(\nu_1 s_1^2)}{n_1(\nu_1-2)} - \frac{6(\bar{y}_1 - \bar{y}_2)^2(\nu_2 s_2^2)}{n_2(\nu_2-2)} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})}{(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu_1}{2})} \\
& - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1-1}{2})}{(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})\Gamma^2(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \\
& - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma^2(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}}\Gamma^2(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
& + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)^2\Gamma^2(\frac{\nu_1-1}{2})}{(\nu_1-2)2\Gamma^2(\frac{\nu_1}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)^2\Gamma^4(\frac{\nu_1-1}{2})}{2^2\Gamma^4(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_1-1}{2})}{(\nu_2-2)2\Gamma^2(\frac{\nu_1}{2})}
\end{aligned}$$

$$\begin{aligned}
& + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})}{2^2\Gamma^2(\frac{\nu_1}{2})\Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)^2\Gamma^2(\frac{\nu_1-1}{2})}{n_1(\nu_1-2)2\Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_1-1}{2})}{n_2(\nu_2-2)2\Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma^3(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{2^2\Gamma^3(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_2-1}{2})}{(\nu_1-2)2\Gamma^2(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})}{2^2\Gamma^2(\frac{\nu_1}{2})\Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_2 s_2^2)^2\Gamma^2(\frac{\nu_2-1}{2})}{(\nu_2-2)2\Gamma^2(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_2 s_2^2)^2\Gamma^4(\frac{\nu_2-1}{2})}{2^2\Gamma^4(\frac{\nu_2}{2})} \\
& - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2\Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_2 s_2^2)^2\Gamma^2(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma^2(\frac{\nu_2}{2})} - (\bar{y}_1 - \bar{y}_2)^4 - \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \\
& + \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_1 s_1^2)\Gamma^2(\frac{\nu_1-1}{2})}{2\Gamma^2(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{6(\bar{y}_1 - \bar{y}_2)^2 z_\alpha^2(\nu_2 s_2^2)\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu_1-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha^3(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma^2(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}}\Gamma^2(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
& - \frac{12(\bar{y}_1 - \bar{y}_2) z_\alpha^3(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})\Gamma^2(\frac{\nu_2}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2) z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma^3(\frac{\nu_2-1}{2})}{2^{\frac{3}{2}}\Gamma^3(\frac{\nu_2}{2})} - \frac{z_\alpha^4(\nu_1 s_1^2)^2\Gamma^4(\frac{\nu_1-1}{2})}{2^2\Gamma^4(\frac{\nu_1}{2})} \\
& + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma^3(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{2^2\Gamma^3(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2)\Gamma^2(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})}{2^2\Gamma^2(\frac{\nu_1}{2})\Gamma^2(\frac{\nu_2}{2})} + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} \\
& - \frac{z_\alpha^4(\nu_2 s_2^2)^2\Gamma^4(\frac{\nu_2-1}{2})}{2^2\Gamma^4(\frac{\nu_2}{2})}.
\end{aligned}$$

Consider the terms that contain $(\bar{y}_1 - \bar{y}_2)^4$

$$(\bar{y}_1 - \bar{y}_2)^4 - (\bar{y}_1 - \bar{y}_2)^4 = 0.$$

Consider the terms that contain $(\bar{y}_1 - \bar{y}_2)^3$

$$\begin{aligned}
& \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2)^3 z_\alpha(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
& = 0.
\end{aligned}$$

$$\begin{aligned}
&= \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_1 - 2)\Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}(2\nu_1 - 7)\Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_1 - 3)\Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} \\
&\quad - \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2 - 2)\Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2 - 7)\Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2 - 3)\Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \\
&= \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} \left[(\nu_1 - 2) + (2\nu_1 - 7) - 3(\nu_1 - 3) \right] \\
&\quad - \frac{4(\bar{y}_1 - \bar{y}_2)z_\alpha^3(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \left[(\nu_2 - 2) + (2\nu_2 - 7) - 3(\nu_2 - 3) \right] \\
&= 0.
\end{aligned}$$

From the terms containing $(\bar{y}_1 - \bar{y}_2)$, consider the terms that contain both $(\bar{y}_1 - \bar{y}_2)$ and z_α , therefore

$$\begin{aligned}
&\frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 3}{2})}{n_1 2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 3}{2})}{n_2 2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
&+ \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \\
&- \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 1}{2})}{n_1(\nu_1 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
&+ \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 1}{2})}{n_2(\nu_2 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
&= \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 3}{2})}{n_1 2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1 - 1}{2})}{n_1(\nu_1 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \\
&+ \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)\Gamma(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \\
&- \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 3}{2})}{n_2 2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
&+ \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2 - 1}{2})}{n_2(\nu_2 - 2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})}
\end{aligned}$$

$$\begin{aligned}
&= \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})}{n_1(\nu_1-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_1}{2})} \\
&\quad - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2^{\frac{1}{2}}\Gamma(\frac{\nu_2}{2})} \\
&= \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1 2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} \\
&\quad - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_1-3)\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} \quad \text{from } \Gamma(\frac{\nu_1-1}{2}) = (\frac{\nu_1-3}{2})\Gamma(\frac{\nu_1-3}{2}) \\
&\quad - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2 2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \quad \text{from } \Gamma(\frac{\nu_2-1}{2}) = (\frac{\nu_2-3}{2})\Gamma(\frac{\nu_2-3}{2}) \\
&= \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_1-2)\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_1-3)\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} \\
&\quad - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-2)\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} + \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \\
&= \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_1 s_1^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-3}{2})}{n_1(\nu_1-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_1}{2})} \left[(\nu_1 - 2) - 1 - (\nu_1 - 3) \right] \\
&\quad - \frac{12(\bar{y}_1 - \bar{y}_2)z_\alpha(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^{\frac{3}{2}}\Gamma(\frac{\nu_2}{2})} \left[(\nu_2 - 2) - 1 - (\nu_2 - 3) \right] \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
&= -\frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(\nu_1-2)\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(\nu_1-2)\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(2\nu_1-7)\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(\nu_1-3)\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}(\nu_1-3)\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&= \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{n_1(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left[-(\nu_1-2) + 1 + (\nu_1-3) \right] \\
&\quad - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}\Gamma(\frac{\nu_1-3}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_1-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left[(\nu_1-2) + (2\nu_1-7) - 3(\nu_1-3) \right] \\
&= 0.
\end{aligned}$$

Consider the terms that contain $(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}$, therefore

$$\begin{aligned}
&= -\frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_2 2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_1-1}{2})\Gamma^2(\frac{\nu_2-1}{2})\Gamma(\frac{\nu_2-3}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} \\
&= -\frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_2 2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{8z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \quad \text{from } (\nu_2-3)\Gamma(\frac{\nu_2-3}{2}) = 2\Gamma(\frac{\nu_2-1}{2}) \\
&\quad - \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma^3(\frac{\nu_2-1}{2})}{2^2\Gamma(\frac{\nu_1}{2})\Gamma^3(\frac{\nu_2}{2})}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_2 2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-1}{2})}{n_2(\nu_2-2)2\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&= -\frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_2 2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \quad \text{from } \Gamma(\frac{\nu_2-1}{2}) = (\frac{\nu_2-3}{2})\Gamma(\frac{\nu_2-3}{2}) \\
&\quad + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \quad \text{from } \Gamma(\frac{\nu_2-1}{2}) = (\frac{\nu_2-3}{2})\Gamma(\frac{\nu_2-3}{2}) \\
&= -\frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-2)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-2)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(2\nu_2-7)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&\quad + \frac{12z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} + \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}(\nu_2-3)\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \\
&= \frac{12z_\alpha^2(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_1-3}{2})}{n_2(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left[-(\nu_2-2) + 1 + (\nu_2-3) \right] \\
&\quad - \frac{4z_\alpha^4(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}\Gamma(\frac{\nu_1-1}{2})\Gamma(\frac{\nu_2-3}{2})}{(\nu_2-2)2^2 \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left[(\nu_2-2) + (2\nu_2-7) - 3(\nu_2-3) \right] \\
&= 0.
\end{aligned}$$

The terms containing $(\bar{y}_1 - \bar{y}_2)^4$, $(\bar{y}_1 - \bar{y}_2)^3$, $(\bar{y}_1 - \bar{y}_2)^2$, $(\bar{y}_1 - \bar{y}_2)$, $(\nu_1 s_1^2)^{\frac{3}{2}}(\nu_2 s_2^2)^{\frac{1}{2}}$, $(\nu_1 s_1^2)^{\frac{1}{2}}(\nu_2 s_2^2)^{\frac{3}{2}}$ all therefore become equal to zero.

The fourth central moment of the marginal posterior distribution of $q_1 - q_2$ i.e. $p\left((q_1 - q_2) | \mathbf{y}_1, \mathbf{y}_2\right)$ is therefore given by

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] \right\} + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& - \frac{12z_\alpha^2(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{n_1(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^2(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_2 s_2^2)^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{n_2(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{4z_\alpha^4(\nu_1 s_1^2)^2 (2\nu_1 - 7) \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& + \frac{4z_\alpha^4(\nu_2 s_2^2)^2 (2\nu_2 - 7) \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& - \frac{8z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} - \frac{z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} \quad \text{from } (\nu_1 - 3) \Gamma(\frac{\nu_1 - 3}{2}) = 2 \Gamma(\frac{\nu_1 - 1}{2}) \\
& - \frac{8z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} - \frac{z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} \quad \text{from } (\nu_2 - 3) \Gamma(\frac{\nu_2 - 3}{2}) = 2 \Gamma(\frac{\nu_2 - 1}{2}) \\
& + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})}
\end{aligned}$$

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] \right\} + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& - \frac{12z_\alpha^2(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{n_1(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^2(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_2 s_2^2)^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{n_2(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{4z_\alpha^4(\nu_1 s_1^2)^2 (2\nu_1 - 7) \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& + \frac{4z_\alpha^4(\nu_2 s_2^2)^2 (2\nu_2 - 7) \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& - \frac{3z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} - \frac{3z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} .
\end{aligned}$$

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] \right\} + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& - \frac{12z_\alpha^2(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2)2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)^2(\nu_1 - 3) \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2)2^2 \Gamma^2(\frac{\nu_1}{2})} && \text{from } \Gamma(\frac{\nu_1 - 1}{2}) = (\frac{\nu_1 - 3}{2}) \Gamma(\frac{\nu_1 - 3}{2}) \\
& - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2)2 \Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{12z_\alpha^2(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2)2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_2 s_2^2)^2(\nu_2 - 3) \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2)2^2 \Gamma^2(\frac{\nu_2}{2})} && \text{from } \Gamma(\frac{\nu_2 - 1}{2}) = (\frac{\nu_2 - 3}{2}) \Gamma(\frac{\nu_2 - 3}{2}) \\
& - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2)2 \Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{4z_\alpha^4(\nu_1 s_1^2)^2(2\nu_1 - 7) \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)^2(\nu_1 - 3) \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^2 \Gamma^2(\frac{\nu_1}{2})} && \text{from } \Gamma(\frac{\nu_1 - 1}{2}) = (\frac{\nu_1 - 3}{2}) \Gamma(\frac{\nu_1 - 3}{2}) \\
& - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2)2 \Gamma^2(\frac{\nu_2}{2})} \\
& + \frac{4z_\alpha^4(\nu_2 s_2^2)^2(2\nu_2 - 7) \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_2 s_2^2)^2(\nu_2 - 3) \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^2 \Gamma^2(\frac{\nu_2}{2})} && \text{from } \Gamma(\frac{\nu_2 - 1}{2}) = (\frac{\nu_2 - 3}{2}) \Gamma(\frac{\nu_2 - 3}{2}) \\
& - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2)2 \Gamma^2(\frac{\nu_1}{2})} \\
& - \frac{3z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} - \frac{3z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} .
\end{aligned}$$

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] \right\} + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& - \frac{6z_\alpha^2(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2)2^2 \Gamma^2(\frac{\nu_1}{2})} \left[2 + (\nu_1 - 3) \right] - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2)2 \Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{6z_\alpha^2(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2)2^2 \Gamma^2(\frac{\nu_2}{2})} \left[2 + (\nu_2 - 3) \right] - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2)2 \Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{z_\alpha^4(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2)2^2 \Gamma^2(\frac{\nu_1}{2})} \left[4(2\nu_1 - 7) - 6(\nu_1 - 3) \right] - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2)2 \Gamma^2(\frac{\nu_2}{2})} \\
& + \frac{z_\alpha^4(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2)2^2 \Gamma^2(\frac{\nu_2}{2})} \left[4(2\nu_2 - 7) - 6(\nu_2 - 3) \right] - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2)2 \Gamma^2(\frac{\nu_1}{2})} \\
& - \frac{3z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} - \frac{3z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} .
\end{aligned}$$

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] \right\} + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& - \frac{6z_\alpha^2(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} \left[\nu_1 - 1 \right] - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{6z_\alpha^2(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} \left[\nu_2 - 1 \right] - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{z_\alpha^4(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} \left[2\nu_1 - 10 \right] - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& + \frac{z_\alpha^4(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} \left[2\nu_2 - 10 \right] - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& - \frac{3z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} - \frac{3z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} .
\end{aligned}$$

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] \right\} + \frac{6(\nu_1 s_1^2)(\nu_2 s_2^2)}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] \\
& - \frac{6z_\alpha^2(\nu_1 - 1)(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{n_1(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma(\frac{\nu_2 - 1}{2})}{n_1(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& - \frac{6z_\alpha^2(\nu_2 - 1)(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{n_2(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^2(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{n_2(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& + \frac{2(\nu_1 - 5)z_\alpha^4(\nu_1 s_1^2)^2 \Gamma(\frac{\nu_1 - 1}{2}) \Gamma(\frac{\nu_1 - 3}{2})}{(\nu_1 - 2) 2^2 \Gamma^2(\frac{\nu_1}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_2 - 1}{2})}{(\nu_1 - 2) 2 \Gamma^2(\frac{\nu_2}{2})} \\
& + \frac{2(\nu_2 - 5)z_\alpha^4(\nu_2 s_2^2)^2 \Gamma(\frac{\nu_2 - 1}{2}) \Gamma(\frac{\nu_2 - 3}{2})}{(\nu_2 - 2) 2^2 \Gamma^2(\frac{\nu_2}{2})} - \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2})}{(\nu_2 - 2) 2 \Gamma^2(\frac{\nu_1}{2})} \\
& - \frac{3z_\alpha^4(\nu_1 s_1^2)^2 \Gamma^4(\frac{\nu_1 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_1}{2})} - \frac{3z_\alpha^4(\nu_2 s_2^2)^2 \Gamma^4(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^4(\frac{\nu_2}{2})} \\
& + \frac{6z_\alpha^4(\nu_1 s_1^2)(\nu_2 s_2^2) \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} .
\end{aligned}$$

Therefore, the fourth central moment of the marginal posterior distribution of $(q_1 - q_2) | \mathbf{y}_1, \mathbf{y}_2$ is equal to

$$\begin{aligned}
m_4 = & \sum_{d=1}^2 \left\{ \frac{(\nu_d s_d^2)^2}{(\nu_d - 2)(\nu_d - 4)} \left[\frac{3}{n_d^2} + \frac{6z_\alpha^2}{n_d} + z_\alpha^4 \right] - \frac{6z_\alpha^2(\nu_d - 1)(\nu_d s_d^2)^2 \Gamma(\frac{\nu_d - 1}{2}) \Gamma(\frac{\nu_d - 3}{2})}{2^2 n_d (\nu_d - 2) \Gamma^2(\frac{\nu_d}{2})} \right. \\
& \left. + \frac{2(\nu_d - 5)z_\alpha^4 (\nu_d s_d^2)^2 \Gamma(\frac{\nu_d - 1}{2}) \Gamma(\frac{\nu_d - 3}{2})}{2^2 (\nu_d - 2) \Gamma^2(\frac{\nu_d}{2})} - \frac{3z_\alpha^4 (\nu_d s_d^2)^2 \Gamma^4(\frac{\nu_d - 1}{2})}{2^2 \Gamma^4(\frac{\nu_d}{2})} \right\} \\
& + 6 \left(\nu_1 s_1^2 \right) \left(\nu_2 s_2^2 \right) \left\{ \frac{1}{(\nu_1 - 2)(\nu_2 - 2)} \left[\frac{1}{n_1 n_2} + \frac{z_\alpha^2}{n_2} + \frac{z_\alpha^2}{n_1} + z_\alpha^4 \right] - \frac{z_\alpha^4 \Gamma^2(\frac{\nu_1 - 1}{2})}{2(\nu_2 - 2) \Gamma^2(\frac{\nu_1}{2})} - \frac{z_\alpha^4 \Gamma^2(\frac{\nu_2 - 1}{2})}{2(\nu_1 - 2) \Gamma^2(\frac{\nu_2}{2})} \right. \\
& \left. - \frac{z_\alpha^2 \Gamma^2(\frac{\nu_1 - 1}{2})}{2n_2 (\nu_2 - 2) \Gamma^2(\frac{\nu_1}{2})} - \frac{z_\alpha^2 \Gamma^2(\frac{\nu_2 - 1}{2})}{2n_1 (\nu_1 - 2) \Gamma^2(\frac{\nu_2}{2})} + \frac{z_\alpha^4 \Gamma^2(\frac{\nu_1 - 1}{2}) \Gamma^2(\frac{\nu_2 - 1}{2})}{2^2 \Gamma^2(\frac{\nu_1}{2}) \Gamma^2(\frac{\nu_2}{2})} \right\}.
\end{aligned}$$

Chapter 3

The One - Way Random Effects Model

In this chapter, the Bayesian simulation method for determining variance components and tolerance intervals for a one - way random effects model using a non - informative Jeffreys' prior distribution will be reviewed. The method was originally proposed by Wolfinger (1998). In addition to the simulation method proposed by Wolfinger (1998) for obtaining Bayesian tolerance intervals, an alternative Bayesian simulation method for obtaining the mentioned Bayesian tolerance intervals will also be discussed. The method proposed by Wolfinger (1998) and the alternative method will be illustrated using a process for the manufacturing of medicinal tablets in small batches.

3.1 Introduction

When discussing point estimation for the percentiles of y_{ij} when sampling takes place from various batches of some material, Fertig and Mahn (1974) provided a motivation for deriving one - sided tolerance limits for the one - way random effects model given by

$$y_{ij} = \mu + a_i + \varepsilon_{ij} \quad (3.1.1)$$

where

$$i = 1, \dots, b,$$

$$j = 1, \dots, k,$$

μ is a fixed target value,

$$a_i \sim N(0, \sigma_a^2) \text{ and}$$

$$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2).$$

It however appears that Lemon (1977) was the first to attempt to formally derive a lower tolerance limit for the distribution of $y_{ij} \sim N(\mu, \sigma_a^2 + \sigma_\varepsilon^2)$ and thus the random effects model given in equation 3.1.1 (*Krishnamoorthy and Mathew, 2009*). Mee and Owen (1983) pointed out that the tolerance limit determined by Lemon (1977) was quite conservative and proceeded to derive a less conservative tolerance limit using the Satterthwaite approximation. Avoiding the Satterthwaite approximation, Vangel (1992) succeeded in determining a less conservative tolerance limit compared to the tolerance limit proposed by Mee and Owen (1983). Later on, Krishnamoorthy and Mathew (2004) derived a lower tolerance limit for $y_{ij} \sim N(\mu, \sigma_a^2 + \sigma_\varepsilon^2)$. Krishnamoorthy and Mathew (2004) first defined a generalized pivotal quantity which is a function of the underlying random variables and the corresponding observed values, and then determined the lower tolerance limit using the generalized confidence interval idea (*Krishnamoorthy and Mathew, 2009*). Chen and Harris (2006) more recently computed tolerance intervals using a proposed numerical approach.

Mee (1984a) extended the Mee and Owen (1983) approach for determining a one - sided tolerance limit for the random effects model given in equation 3.1.1 to also arrive at a two - sided tolerance interval for $y_{ij} \sim N(\mu, \sigma_a^2 + \sigma_\varepsilon^2)$, also using the Satterthwaite approximation. Bechman and Tietjen (1989) also derived a two - sided tolerance interval after replacing some unknown parameter by some upper bound (*Krishnamoorthy and Mathew, 2009*). Using the generalized confidence interval idea, Liao and Iyer

(2004) and Liao, Lin and Iyer (2005) derived approximate two - sided tolerance intervals after deriving two slightly different margin of error statistics. The approximation used in Liao, Lin and Iyer (2005) seems to be an improvement over the approximation proposed by Liao and Iyer (2004) (*Krishnamoorthy and Mathew, 2009*). Krishnamoorthy and Mathew (2009) also mentioned that even though Liao, Lin and Iyer (2005) succeeded in eventually using generalized confidence intervals for a suitable linear combination of the variance components σ_a^2 and σ_ε^2 , the proposed approach is not a straightforward application of the generalized confidence interval idea, since the two - sided tolerance interval problem does not reduce to a confidence interval problem concerning percentiles.

As was mentioned in Chapter 1, an α - expectation tolerance interval is an interval where α represents the expected coverage of the interval and is also a prediction interval for a future observation. Wilks (1941), Paulson (1943) and Guttman (1970) derived such intervals for univariate normal distributions (*Krishnamoorthy and Mathew, 2009*). Mee (1984a) derived an α - expectation tolerance interval for the balanced random effects model in a similar manner as the derivation of the (α, δ) one - sided tolerance interval proposed by Mee and Owen (1983). Like Mee and Owen (1983), Mee (1984a) also used the Satterthwaite approximation for deriving this α - expectation tolerance interval (*Krishnamoorthy and Mathew, 2009*). Lin and Liao (2006) also derived α - expectation tolerance intervals for a general mixed model with balanced data which can also be adopted for the random effects model given in equation 3.1.1. For a general random effects model with balanced data, Lin and Liao (2008) also derived simultaneous prediction intervals (*Krishnamoorthy and Mathew, 2009*).

The above authors have considered the derivation of tolerance intervals for the random effects model from a frequentist perspective. For the one - way random effects model given in equation 3.1.1, tolerance intervals can also be determined using the Bayesian approach.

3.2 The Bayesian Approach

When prior information is used on a parameter space, the term Bayesian is used to describe the approach. In the investigation of a statistical quandary, Bayes's theorem can consequently improve the situation through the use of this prior information. If, however, the use of prior information is rejected (as in the "objectivist" school), then the term frequentist describes the approach. In this case, probability is interpreted exclusively related to comparative frequencies in major reproduction (*Aitchison, 1964*).

As argued by Aitchison (1964), the frequentist assertion is an intricate one, with numerous mathematical complexities. According to Aitchison (1964), the associated frequentist interpretation concerning tolerance regions is commonly misunderstood. Bayesian formulations are straightforward and seem more appropriate for use in tolerance regions. See Aitchison (1964) for an in - depth comparative discussion on the two approaches, with an accompanying illustrative example.

As mentioned, contrary to the frequentist method, Bayesian formulation requires the additional notion of a prior probability density $p(\theta)$ on the parameter space. The prior density signifies the user's prior beliefs (based on experience) or a suitable, rational form of weighting over the feasible parameters. Aitchison (1964) states that the Bayesian then bases his subsequent actions exclusively on the posterior distribution (obtained using the prior density and the likelihood) and the resulting consequences.

The methodology for determining Bayesian tolerance intervals for the one - way random effects model has originally been proposed by Wolfinger (1998) using both informative and non - informative prior distributions. Wolfinger (1998) also provided relationships with frequentist methodologies.

For the remainder of this chapter, the Bayesian method as proposed by Wolfinger (1998) for determining tolerance intervals for the one - way random effects model using a non - informative prior distribution will be discussed. The methodology pro-

posed by Wolfinger (1998) will be illustrated using an example for the manufacturing of medical tablets.

3.3 The Variance Component Model

Minimum variance unbiased estimators (MVUE's) are used to estimate variances when balanced data is considered. For unbalanced data, however, such estimators do not exist. Unbalanced data could be a result of spoiled samples, missing data, different batches having different sampling costs or poorly designed experiments (Chaloner, 1987). A variety of estimators, such as maximum likelihood estimators (MLE's), minimum norm quadratic unbiased estimators (MINQUE's) and numerous variations to these methods (see Searle (1979)), have been suggested. These estimators were studied by Swallow and Monahan (1984), but the Bayesian estimators were not considered as an alternative until the comparative study by Chaloner (1987).

Chaloner (1987) also reasoned that the Bayesian approach had several advantages over the classical methods. These advantages include always finding non - negative estimates for variances, non - empty highest posterior density regions and the entire posterior probability distribution can be reported at any one time.

The Bayesian method proposed by Wolfinger (1998) for estimating variance components and tolerance intervals for a one - way random effects model can be illustrated using the random effects model given in equation 3.1.1 and represented by

$$y_{ij} = \mu + a_i + \varepsilon_{ij}$$

where y_{ij} refers to j^{th} measurement for the i^{th} batch ($i = 1, \dots, b, j = 1, \dots, k$), the overall mean is modeled by μ , a_i denotes the random effects factor and finally ε_{ij} is the experimental error involved in the process. The random effects parameter and the error component both follow normal distributions with means equal to 0 and variances σ_a^2 and σ_ε^2 respectively.

3.4 The Prior Distribution

In Chapter 1 it was mentioned that the choice of a prior distribution is a controversial and much criticized step in any Bayesian analysis, since the prior distribution $p(\theta)$ is specified by the analyst. For the one - way random effects model given in equation 3.1.1, it was therefore decided to follow Wolfinger (1998) and also use the non - informative Jeffreys' reference prior given by

$$p(\mu, \sigma_a^2, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} \quad (3.4.1)$$

This non - informative prior distribution is constructed to be invariant to reparameterization of the variance components (Wolfinger, 1998). More general discussions on non - informative prior distributions for the one - way random effects model can be found in Box and Tiao (1973) and Chaloner (1987).

3.5 The Posterior Distribution

Using Bayes's theorem, the posterior distribution of the unknown parameters is obtained by multiplying the likelihood function with the prior distribution given in equation 3.4.1. For the balanced one - way random effects model given in equation 3.1.1, the likelihood function of the unknown parameters, μ , σ_a^2 , σ_ε^2 and a_i ($i = 1, \dots, b$) is given by

$$\begin{aligned} L(\mu, \sigma_a^2, \sigma_\varepsilon^2, a_i | \mathbf{y}) &\propto \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{1}{2}bk} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2} \sum_{j=1}^k \sum_{i=1}^b (y_{ij} - \mu - a_i)^2\right\} \\ &\times \left(\frac{1}{\sigma_a^2}\right)^{\frac{1}{2}b} \exp\left\{-\frac{1}{2\sigma_a^2} \sum_{i=1}^b a_i^2\right\}. \end{aligned} \quad (3.5.1)$$

After multiplying the likelihood function with the prior distribution, the joint posterior distribution of the unknown parameters μ , σ_a^2 , σ_ε^2 and a_i is given by

$$\begin{aligned}
p(\mu, \sigma_a^2, \sigma_\varepsilon^2, \mathbf{a}_i | \mathbf{y}) &\propto \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{1}{2}bk} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2} \sum_{j=1}^k \sum_{i=1}^b (y_{ij} - \mu - a_i)^2\right\} \\
&\times \left(\frac{1}{\sigma_a^2}\right)^{\frac{1}{2}b} \exp\left\{-\frac{1}{2\sigma_a^2} \sum_{i=1}^b a_i^2\right\} \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} \quad (3.5.2)
\end{aligned}$$

To obtain the posterior distribution of μ , σ_a^2 and σ_ε^2 , equation 3.5.2 is integrated with respect to a_i . In other words,

$$p(\mu, \sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) = \int_{-\infty}^{\infty} p(\mu, \sigma_a^2, \sigma_\varepsilon^2, \mathbf{a}_i | \mathbf{y}) d\mathbf{a}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_b \end{bmatrix}'.$$

After integrating equation 3.5.2 with respect to \mathbf{a} and completing the square, the joint posterior distribution of μ , σ_a^2 and σ_ε^2 is given by

$$\begin{aligned}
p(\mu, \sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) &\propto \left(\sigma_\varepsilon^2\right)^{-\frac{1}{2}(\nu_1+2)} \left(\sigma_\varepsilon^2 + k\sigma_a^2\right)^{-\frac{1}{2}(\nu_2+3)} \\
&\times \exp\left\{-\frac{1}{2} \left[\frac{bk(\bar{y}_{..} - \mu)^2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right]\right\} \quad (3.5.3)
\end{aligned}$$

where

$$\sigma_a^2 > 0,$$

$$\sigma_\varepsilon^2 > 0,$$

$$\nu_1 = b(k - 1),$$

$$\nu_2 = b - 1,$$

$$\nu_1 m_1 = \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - \bar{y}_{i.})^2, \text{ and}$$

$$\nu_2 m_2 = k \sum_{i=1}^b (\bar{y}_{i.} - \bar{y}_{..})^2$$

where $\bar{y}_j = \sum_{i=1}^k \frac{y_{ij}}{k}$ and $\bar{y}_{..} = \sum_{i=1}^b \sum_{j=1}^k \frac{y_{ij}}{bk}$. Also, $\nu_1 m_1$ represents the observed sum of squares within batches (SSE), while $\nu_2 m_2$ represents the observed sum of squares between batches (SSA).

From the joint posterior distribution given in equation 3.5.2 and the joint posterior distribution given in equation 3.5.3, the following conditional posterior distributions of the unknown parameters can also be obtained.

Theorem 3.5.1

For the balanced random effects model given in equation 3.1.1, the conditional posterior distribution of μ , given the variance components, is normal with mean

$$E(\mu | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) = \bar{y}_{..}$$

and variance

$$Var(\mu | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) = \frac{1}{bk} (\sigma_\varepsilon^2 + k\sigma_a^2) .$$

Therefore

$$p(\mu | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) \sim N\left(\bar{y}_{..}, \frac{1}{bk} (\sigma_\varepsilon^2 + k\sigma_a^2)\right) . \quad (3.5.4)$$

Theorem 3.5.2

The joint posterior distribution of the variance components for the balanced random effects model given in equation 3.1.1, is given by

$$p(\sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) \sim \left(\sigma_\varepsilon^2\right)^{-\frac{1}{2}(\nu_1+2)} \left(\sigma_\varepsilon^2 + k\sigma_a^2\right)^{-\frac{1}{2}(\nu_2+2)} \times \exp\left\{-\frac{1}{2}\left[\frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2}\right]\right\} \quad (3.5.5)$$

where $\sigma_\varepsilon^2 > 0, \sigma_a^2 > 0$.

Theorem 3.5.3

Given the variance components, the posterior distribution of the random effects a_i ($i = 1, \dots, b$) for the balanced random effects model given in equation 3.1.1, is given by

$$p(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) \sim N \left(\left[\frac{k\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right] (\bar{y}_i - \bar{y}_{..}), \frac{\sigma_\varepsilon^2 \sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \left[\sigma_\varepsilon^2 + \frac{k\sigma_a^2}{b} \right] \right) \text{ for } i = 1, \dots, b. \quad (3.5.6)$$

The proves of Theorems 3.5.1 - 3.5.3 have been derived previously. For more details pertaining these mathematical expressions, see Box and Tiao (1973) and Searle, Casella and McCullough (1992).

3.6 Bayesian Simulation

It was mentioned in Chapter 2, that for the balanced univariate normal model, the unconditional posterior distributions of the unknown parameters can be obtained analytically, since the derivations were not that complex . Wolfinger (1998) mentioned however that for the balanced one - way random effects model, the analytical derivations of unconditional posterior densities for the unknown parameters μ , σ_a^2 , σ_ε^2 , a_i ($i = 1, \dots, b$) and posterior densities of quantiles in order to construct tolerance limits, appear to be formidable.

It was therefore decided to follow Wolfinger (1998) and also use a straightforward, flexible and economical Bayesian simulation method to obtain these unconditional posterior densities and to construct approximate tolerance intervals of varying types. The Bayesian method proposed for the balanced one - way random effects model given in equation 3.1.1 will now be illustrated using the following simulated data set pertaining to the manufacturing of pills (medicinal tablets).

The simulated data in Table 3.1 represents the amount of milligrams of active drug per manufactured tablet from a factory manufacturing tablets in very small batches. A small batch in this instance is likely to represent a weekly or monthly intake of tablets for an individual patient. The data are assumed to arise from a normal distribution with unknown parameters, but it has more structure than a simple random sample, because it is clustered in fifteen batches and each batch contains ten tablets. A lower specification limit $s = 150.30$ mg is specified for the medicinal tablets data given in Table 3.1.

Table 3.1: Amount of Active Drug per Tablet Measured in Milligrams.

Batch	Measurements									
1	150.52	150.39	150.31	150.49	150.47	150.67	150.17	150.45	150.42	150.37
2	150.35	150.47	150.72	150.56	150.53	150.62	150.60	150.52	150.51	150.63
3	150.48	150.79	150.63	150.46	150.71	150.67	150.70	150.48	150.48	150.58
4	150.41	150.45	150.40	150.33	150.24	150.39	150.28	150.36	150.27	150.33
5	150.58	150.54	150.30	150.54	150.50	150.32	150.58	150.46	150.41	150.49
6	150.49	150.83	150.66	150.63	150.72	150.79	150.64	150.62	150.71	150.73
7	150.33	150.44	150.48	150.34	150.50	150.42	150.37	150.54	150.39	150.52
8	150.39	150.52	150.35	150.52	150.47	150.54	150.51	150.37	150.54	150.53
9	150.64	150.78	150.51	150.69	150.51	150.47	150.60	150.50	150.69	150.72
10	150.61	150.49	150.60	150.50	150.68	150.56	150.59	150.73	150.62	150.62
11	150.48	150.25	150.49	150.43	150.40	150.44	150.31	150.36	150.30	150.40
12	150.35	150.41	150.36	150.39	150.34	150.37	150.51	150.32	150.25	150.32
13	150.54	150.67	150.57	150.45	150.57	150.48	150.39	150.38	150.67	150.42
14	150.41	150.54	150.57	150.73	150.47	150.72	150.72	150.49	150.66	150.58
15	150.60	150.45	150.66	150.72	150.45	150.51	150.69	150.62	150.55	150.45

The selected lower specification limit $s = 150.30$ mg can for example indicate the minimum amount of active ingredient that has to be taken per dose to render the medication effective. The data and above limit are selected solely for illustrative purposes.

In practice, fixed - in - advance limits are often determined from medical or regulatory considerations. See for example Wolfinger (1998). Based on the data, the quantities needed for the simulation procedure are $b = 15$, $k = 10$, $\nu_1 = b(k - 1) = 135$, $\nu_2 = b - 1 = 14$, $\bar{y}_{..} = 150.5076$, $\nu_1 m_1 = \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - \bar{y}_{..})^2 = 1.26552$ and $\nu_2 m_2 = k \sum_{i=1}^b (y_{i.} - \bar{y}_{..})^2 = 1.469816$.

Similar to what was illustrated in Chapter 2, Markov chain Monte Carlo (MCMC) simulation will also be used to obtain random samples from the joint posterior distribution of the unknown model parameters using a computer random number generator. These simulated samples will represent samples from the conditional posterior distribution of the unknown parameter μ , i.e. $p(\mu | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y})$, the joint posterior distribution of the unknown variance components σ_a^2 and σ_ε^2 , i.e. $p(\sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y})$ and the conditional posterior distribution of the random effects parameter a_i ($i = 1, \dots, b$), i.e. $p(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y})$.

Estimated marginal posterior distributions for the unknown parameters σ_ε^2 , σ_a^2 , μ and a_i are simulated as follows:

a.) Simulation of σ_ε^2 and σ_a^2

For the balanced one - way random effects model under the Jeffreys' non - informative prior given in equation 3.4.1, Wolfinger (1998) mentioned that $p(\sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y})$ can be written directly as the product of two inverted gamma distributions given by

$$p(\sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) \propto IG \left[\sigma_\varepsilon^2 + k\sigma_a^2 \left| \frac{b-1}{2}, \frac{k}{2} \sum_{j=1}^b (\bar{y}_{i.} - \bar{y}_{..})^2 \right. \right] \times IG \left[\sigma_\varepsilon^2 \left| \frac{b(k-1)}{2}, \frac{1}{2} \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - \bar{y}_{i.})^2 \right. \right]$$

where b refers to the number of batches, k denotes the number of observations contained within each batch, $\bar{y}_{i.}$ represents the average of the i^{th} batch and $\bar{y}_{..}$ refers to the overall mean of the entire sample. Wolfinger (1998) therefore mentioned that one independently draws $(\sigma_\varepsilon^2 + k\sigma_a^2)$ and σ_ε^2 from the inverted gamma densities to sample from $p(\sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y})$. Therefore, $\frac{\nu_1 m_1}{\sigma_\varepsilon^2}$ follows a chi - square distribution with $\nu_1 = b(k - 1)$ degrees of freedom, i.e. $\frac{\nu_1 m_1}{\sigma_\varepsilon^2} \sim \chi_{\nu_1}^2$ where $\nu_1 m_1 = \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - \bar{y}_{i.})^2$.

From this it follows that the unknown variance component σ_ε^2 can easily be simulated using the $\chi_{\nu_1}^2$ ¹ distribution by obtaining

$$\sigma_\varepsilon^2 = \frac{\nu_1 m_1}{\chi_{\nu_1}^2}.$$

To simulate $(\sigma_\varepsilon^2 + k\sigma_a^2)$ we also know that $\frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2}$ follows a chi - square distribution with $\nu_2 = b - 1$ degrees of freedom. It is therefore known that

$$\frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} \sim \chi_{\nu_2}^2$$

where $\nu_2 m_2 = k \sum_{i=1}^b (\bar{y}_i - \bar{y}_{..})^2$.

A simulated value of $(\sigma_\varepsilon^2 + k\sigma_a^2)$ can therefore be obtained from the relationship

$$(\sigma_\varepsilon^2 + k\sigma_a^2) \sim \frac{\nu_2 m_2}{\chi_{\nu_2}^2}.$$

To obtain σ_a^2 , one therefore has to calculate

$$\sigma_a^2 = \frac{(\sigma_\varepsilon^2 + k\sigma_a^2) - \sigma_\varepsilon^2}{k}.$$

To ensure that only positive values for the variance components are obtained, the simulated variance components are kept only if

$$\sigma_a^2 = \frac{(\sigma_\varepsilon^2 + k\sigma_a^2) - \sigma_\varepsilon^2}{k} > 0, \text{ or,}$$

in other words, if

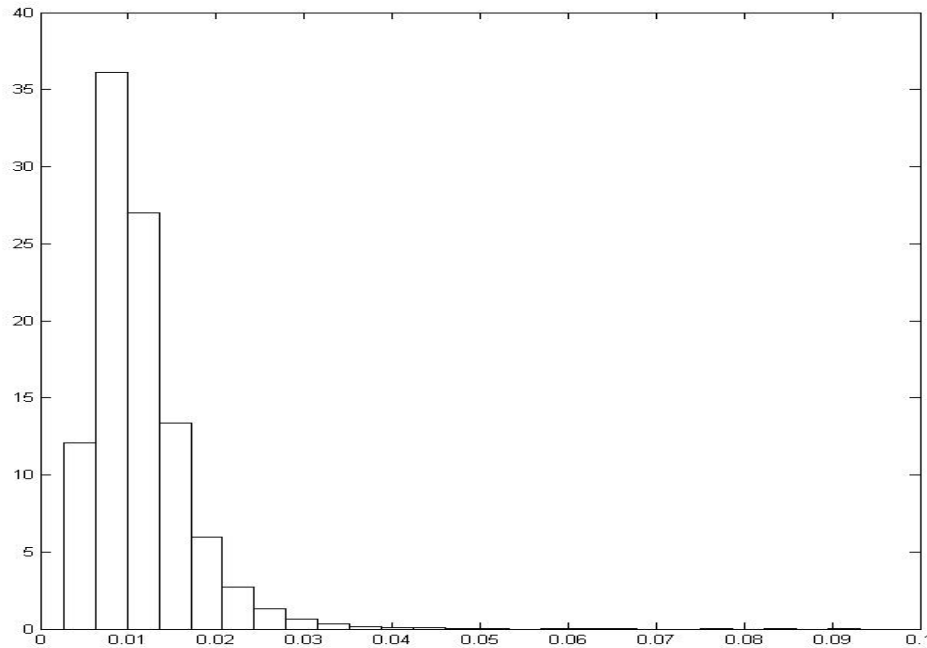
$$\sigma_{12}^2 = (\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2.$$

Repeat the process for example $\tilde{\ell} = 10000$ times, retain only the positive pairs of variance components, and draw histograms of σ_a^2 and σ_ε^2 . The histograms will represent the estimated marginal posterior distributions $p(\sigma_a^2|\mathbf{y})$ and $p(\sigma_\varepsilon^2|\mathbf{y})$.

For the data given in Table 3.1, the histograms of the estimated marginal posterior distributions of the unknown variance components i.e. $p(\sigma_a^2|\mathbf{y})$ and $p(\sigma_\varepsilon^2|\mathbf{y})$ are depicted in Figures 3.6.1 and 3.6.2 respectively.

¹ $\chi_{\nu_1}^2 = \sum_{\ell=1}^{\nu_1} z_\ell^2$ where $z_\ell \sim N(0, 1)$

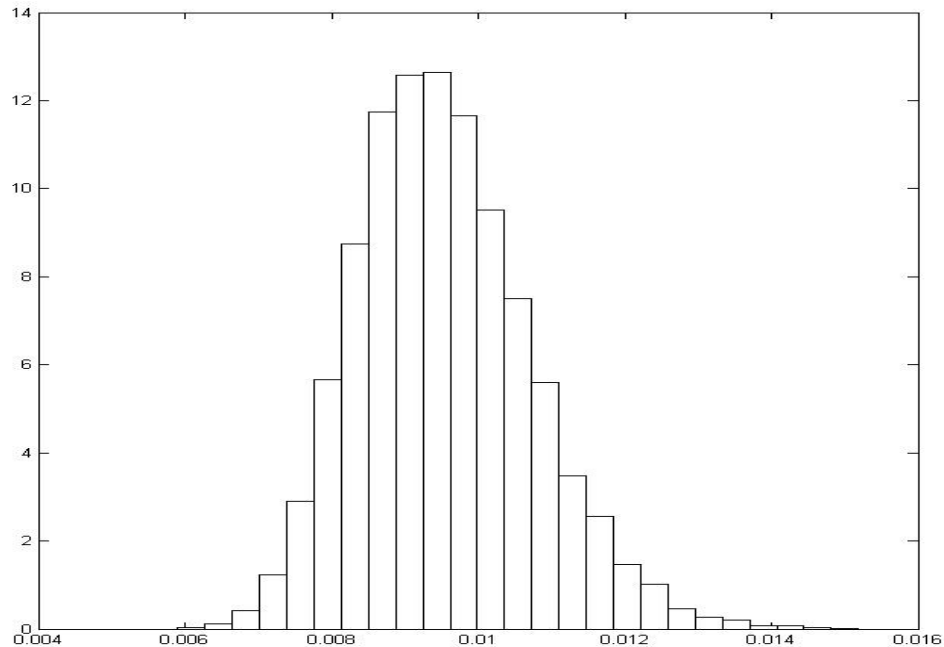
Figure 3.6.1: Histogram of the Estimated Marginal Posterior Distribution of σ_a^2 for the Data Given in Table 3.1.



From Figure 3.6.1 it can be seen that the histogram of the estimated marginal posterior distribution of the batch variance component σ_a^2 is relatively skew due to the low number of degrees of freedom $\nu_2 = 15 - 1 = 14$ associated with σ_a^2 . The 95% equal tail credibility interval of σ_a^2 can also be determined by ranking the retained simulated values for σ_a^2 and obtaining the 2.5th and 97.5th percentiles of the ranked simulated values. For the data given in Table 3.1, the 95% equal tail credibility interval of the batch variance component σ_a^2 is given by [0.0046 , 0.0249].

It can be seen from Figure 3.6.2 that the histogram of the estimated marginal posterior distribution of the residual variance component for the medicinal tablets data given in Table 3.1 is fairly symmetrical. This is due to the high number of degrees of freedom associated with σ_ε^2 . Remember that for the medicinal tablets data given in Table 3.1, there were 15 batches with observations from 10 tablets per batch. The number of degrees of freedom ν_1 is therefore equal to $\nu_1 = 15(10 - 1) = 135$. The 95% equal tail credibility interval is determined in the same way as for the batch variance compo-

Figure 3.6.2: Histogram of the Estimated Marginal Posterior Distribution of the Residual Variance Component σ_ε^2 for the Data Given in Table 3.1.



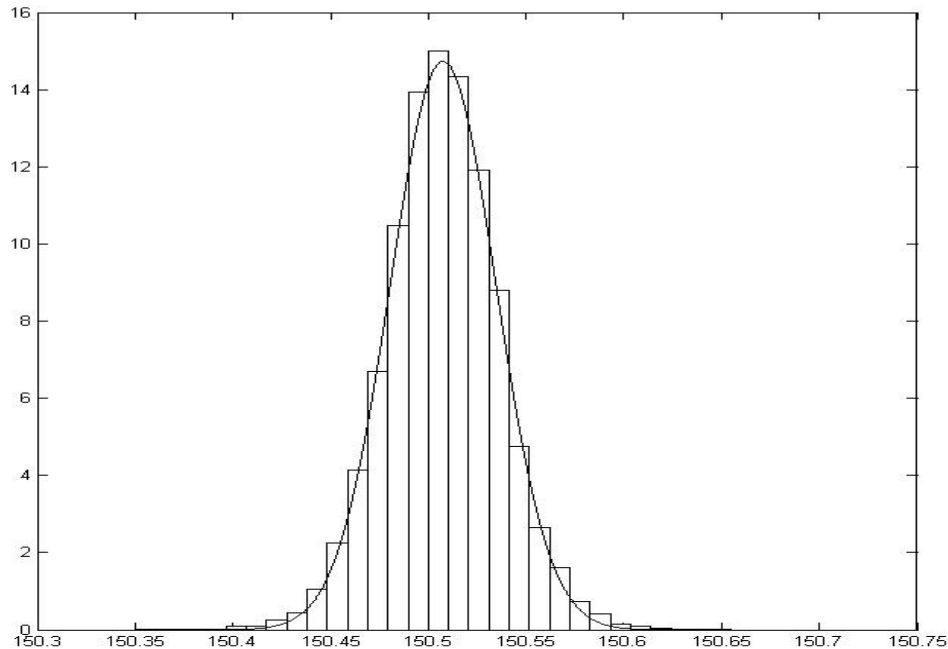
ment and is equal to $[0.0075, 0.0121]$.

b.) Simulation of the target value μ

By substituting each of the simulated and retained pairs of variance components, i.e. σ_a^2 and σ_ε^2 into equation 3.5.4 and then drawing a value μ from the normal distribution given in equation 3.5.4, values of the target value μ can be simulated. There will therefore be one simulated value μ for each pair of retained simulated variance components. The resulting set of simulated μ values can then be displayed in a histogram. This histogram will represent the estimated marginal posterior distribution of the target value μ .

The estimated marginal posterior distribution of μ can also be determined using the Rao Blackwell method described in section 2.5. For each pair of simulated variance components, the normal distribution given in equation 3.5.4 is therefore drawn. As mentioned, this process is repeated for example $\tilde{\ell}$ times, i.e. once for each pair of

Figure 3.6.3: Histogram of - and Estimated Marginal Posterior Distribution of the Target Value μ for the Medicinal Tablets Data Given in Table 3.1.



simulated retained variance components. The average distribution of these $\tilde{\ell}$ normal distributions will then represent the estimated unconditional posterior distribution of μ , i.e. $p(\mu|\mathbf{y})$.

For the medicinal tablets data given in Table 3.1, the histogram of the estimated marginal posterior distribution of the target value μ is illustrated in Figure 3.6.3. Also depicted in Figure 3.6.3 is the estimated marginal posterior distribution of μ obtained using the Rao Blackwell method described in section 2.5. This estimated marginal posterior distribution $p(\mu|\mathbf{y})$ is represented by the solid line.

The 95% equal tail credibility interval for the estimated marginal posterior distribution of the fixed target value μ is $[150.4516, 150.5640]$ for the medicinal tablets data given in Table 3.1. This 95% credibility interval is also obtained by finding the 2.5th and 97.5th percentiles of the ranked simulated μ values.

c.) Simulation of the random (batch) effects a_i ($i = 1, \dots, b$)

To obtain the posterior distribution of a_i ($i = 1, \dots, b$) given the variance components σ_a^2 and σ_ε^2 , a_i ($i = 1, \dots, b$) values can be simulated from the normal distribution given in equation 3.5.6. Equation 3.5.6 can also easily be rewritten as follows: Wolfinger (1998) and others have shown that the conditional posterior density of a_i (conditional on μ , σ_a^2 , σ_ε^2 and the data) follows a normal distribution with

$$E(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mu, \mathbf{y}) = \left[\frac{k\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right] (\bar{y}_i - \mu) \quad (i = 1, \dots, b)$$

and

$$Var(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mu, \mathbf{y}) = \frac{\sigma_\varepsilon^2 \sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \quad (i = 1, \dots, b).$$

It is also known from equation 3.5.4 that

$$p(\mu | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) \sim N\left(\bar{y}_{..}, \frac{1}{bk} (\sigma_\varepsilon^2 + k\sigma_a^2)\right).$$

It therefore follows that

$$E(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) = \left[\frac{k\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right] (\bar{y}_i - \bar{y}_{..}) \quad (i = 1, \dots, b)$$

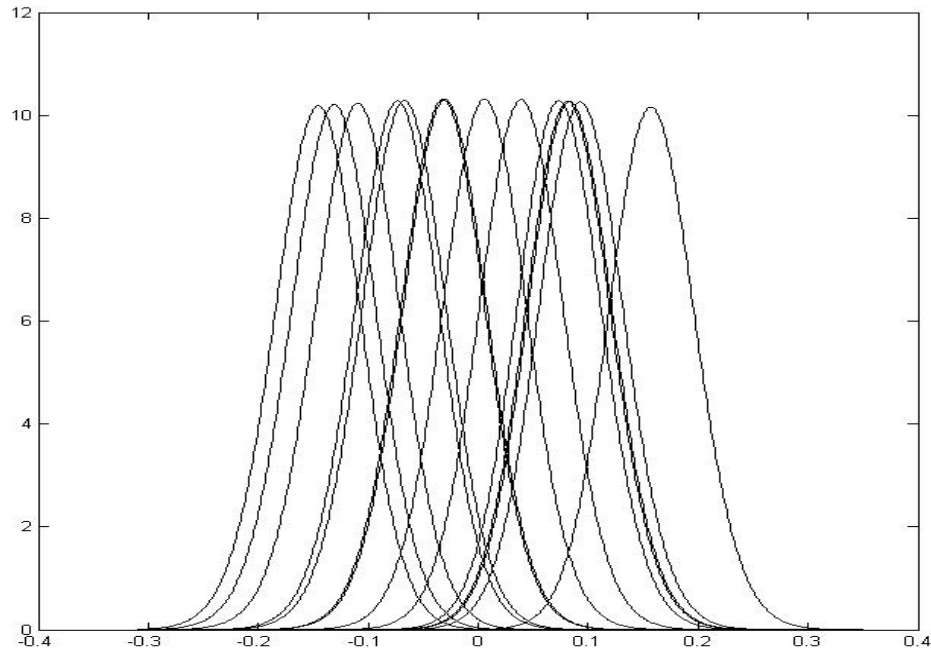
and for $i = 1, \dots, b$

$$\begin{aligned} Var(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) &= E_\mu \left[Var(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mu, \mathbf{y}) \right] + Var_\mu \left[E(a_i | \sigma_a^2, \sigma_\varepsilon^2, \mu, \mathbf{y}) \right] \\ &= \frac{\sigma_\varepsilon^2 \sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \left[\frac{k\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right]^2 \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)}{bk} \\ &= \frac{\sigma_\varepsilon^2 \sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{k(\sigma_a^2)^2}{b} \left(\frac{1}{\sigma_\varepsilon^2 + k\sigma_a^2} \right) \\ &= \frac{\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \left[\sigma_\varepsilon^2 + \frac{k\sigma_a^2}{b} \right]. \end{aligned}$$

For $i = 1, \dots, b$, the posterior distribution of a_i is therefore given by

$$p(a_i | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y}) \sim N\left(\left[\frac{k\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right] (\bar{y}_i - \bar{y}_{..}), \frac{\sigma_a^2}{\sigma_\varepsilon^2 + k\sigma_a^2} \left[\sigma_\varepsilon^2 + \frac{k\sigma_a^2}{b} \right] \right). \quad (3.6.1)$$

Figure 3.6.4: Estimated Marginal Posterior Distributions $p(a_i|\mathbf{y})$ ($i = 1, \dots, 15$) of the Random (Batch) Effects for the Medicinal Tablets Data Given in Table 3.1.



The estimated marginal posterior distributions of the random (batch) effects $p(a_i|\mathbf{y})$ ($i = 1, \dots, b$) can then be simulated using equation 3.6.1 as follows:

- i.) For each of the $\tilde{\ell}$ pairs of retained, simulated variance components σ_ε^2 and σ_a^2 , draw the conditional posterior normal distributions given in equation 3.6.1.
- ii.) Using the Rao Blackwell argument described in section 2.5, the estimated marginal posterior distributions of the random effects a_i ($i = 1, \dots, b$), i.e. $p(a_i|\mathbf{y})$ are then obtained as the average distributions of the $\tilde{\ell}$ conditional posterior distributions.

For the medicinal tablets data given in Table 3.1, the estimated marginal posterior distributions of the random effects parameters a_i ($i = 1, \dots, 15$) are depicted in Figure 3.6.4.

3.7 Tolerance Intervals

As was mentioned in Chapters 1 and 2, tolerance intervals address the statistical problem of inference about the quantiles of a probability distribution assumed to adequately describe a process (Wolfinger, 1998).

There is a profound record of more than 70 years in constructing tolerance intervals (see for example Wilks (1941) and Wald (1942)). In - depth reviews are available by Guttman (1970), Zacks (1971), Miller (1989) and Hahn and Meeker (1991). Even though significant progress has been made on studies concerning tolerance intervals, most of the previous work cannot be applied directly to the medicinal tablets data given in Table 3.1, since the medicinal tablets data display variability between and within batches of manufactured tablets (Wolfinger, 1998). The random effects model given in equation 3.1.1 will therefore also be used to estimate the three commonly used tolerance intervals proposed by Wolfinger (1998) and described in detail in Chapter 1.

The three initial questions posed by Wolfinger (1998) and given in Chapter 1 are answered by these three types of tolerance intervals respectively. Cases that entail long - run prediction classically make use of (α, δ) tolerance intervals, where inference is made on the genuine quantiles of the assumed principal distribution. Given a sample of measurements, manufacturers are able to apply the (α, δ) tolerance interval method to forecast future performance of produced products. Taking $\alpha = 0.90$ and $\delta = 0.95$ for the medicinal tablets data, a lower (α, δ) tolerance limit called the Bayesian "B - basis" interval, is obtained (Wolfinger, 1998). Wolfinger (1998) also stated that this interval of interest can classically be interpreted as a lower 95% confidence limit on the tenth percentile of the population of medicinal tablet measurements.

In comparison, α - expectation tolerance intervals have a tendency to be more constricted, since these intervals focus on prediction of future observations. As an example, Wolfinger (1998) explains that an aircraft manufacturer would be able to construct

a 0.95 - expectation interval for the next component used by analyzing measurements made on previous components.

Setting predetermined bounds and regarding the content therein, inverts the prediction problem. These intervals refer to the fixed - in - advance tolerance intervals mentioned and can be applied in the evaluation of the amount of active ingredient in manufactured medicinal tablets.

3.7.1 One - Sided (α, δ) Tolerance Intervals

A lower (α, δ) one - sided tolerance limit is a limit such that $100(\alpha)\%$ of a population of an underlying random variable is greater than the lower (α, δ) one - sided tolerance limit with $100(\delta)\%$ confidence (*Jandrell and van der Merwe, 2007*). As mentioned, α therefore represents the content (the proportion to be contained by the interval) and δ represents the confidence (reliability of the interval).

According to Wolfinger (1998), the lower (α, δ) one - sided tolerance limit for the random effects model given in equation 3.1.1 represents the $(1 - \delta)^{th}$ sample quantile obtained from the marginal posterior distribution of the $(1 - \alpha)^{th}$ quantile q of a $N(\mu, \sigma_\epsilon^2 + \sigma_a^2)$ distribution (i.e. a quantile of a quantile), where q is given by

$$q = \mu - z_\alpha(\sigma_\epsilon^2 + \sigma_a^2)^{\frac{1}{2}} \quad (3.7.1)$$

and z_α denotes the α^{th} quantile of the standard normal distribution.

Therefore, in order to construct the lower one - sided (α, δ) tolerance limit for the balanced random effects model given in equation 3.1.1, the marginal posterior distribution of q , which represents the $(1 - \alpha)^{th}$ quantile of the $N(\mu, \sigma_\epsilon^2 + \sigma_a^2)$ distribution, must be estimated. Two methods, both utilizing Bayesian simulation, can be used to obtain the mentioned marginal posterior density of q . Method 1 was proposed by Wolfinger (1998).

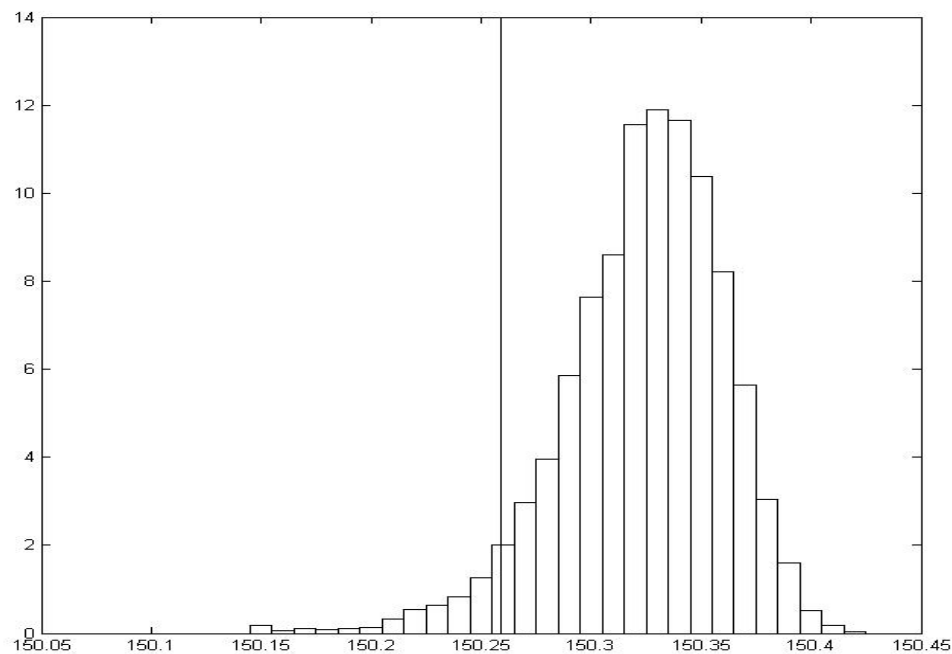
Method 1

- i.) Simulate a pair of variance components $(\sigma_a^2, \sigma_\varepsilon^2)$ subject to the condition that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ using the Bayesian simulation method described in section 3.6.
- ii.) If the condition stated in i.) is met, substitute the simulated pair of variance components into equation 3.5.4 and simulate μ using equation 3.5.4.
- iii.) Use the simulated variance components σ_a^2 and σ_ε^2 , as well as the simulated target value μ , and calculate $q = \mu - z_\alpha(\sigma_a^2 + \sigma_\varepsilon^2)^{\frac{1}{2}}$ where z_α represents the α^{th} quantile of a standard normal distribution.
- iv.) Repeat steps i.) - iii.) for example $\tilde{\ell} = 10000$ times and draw the histogram of the simulated q values. This histogram will represent the estimated marginal posterior density of q , i.e. $p(q|\mathbf{y})$.

To illustrate the use of method 1, the lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit was determined for the medicinal tablets data given in Table 3.1, with $z_{0.9} = 1.282$. This resulting histogram representing the estimated marginal posterior distribution of the $(1 - 0.9)^{th}$ quantile of the $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ distribution is depicted in Figure 3.7.1.

According to Wolfinger (1998) the histogram illustrated in Figure 3.7.1 portrays data on the amount of active ingredient present in medicinal tablets manufactured in future batches. The Bayesian, "B - basis", lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit equal to 150.2588 mg is indicated by the vertical reference line and marks the 5th percentile of the estimated marginal posterior distribution of q . Using method 1, this Bayesian, "B - basis" lower tolerance limit which is equal to 150.2588 mg represents the value of which 90% of unknown future amounts of active ingredient will be greater than with probability 0.95.

Figure 3.7.1: Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.9)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 1 as Proposed by Wolfinger (1998).



Lower (0.90, 0.95) One - Sided Tolerance Limit: 150.2588

Method 2

Since it is known that

$$\mu | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y} \sim N \left(\bar{y}_{..}, \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} \right)$$

it follows that

$$q | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y} \sim N \left(\bar{y}_{..} - z_\alpha (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}, \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} \right) \quad (3.7.2)$$

where z_α represents the α^{th} quantile of a standard normal distribution.

The estimated marginal posterior density of q , i.e. $p(q|\mathbf{y})$ can therefore be simulated as follows:

- i.) Simulate a pair of variance components $(\sigma_a^2, \sigma_\varepsilon^2)$ using the Bayesian simulation method described in section 3.6 and check that the condition $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ is being met.

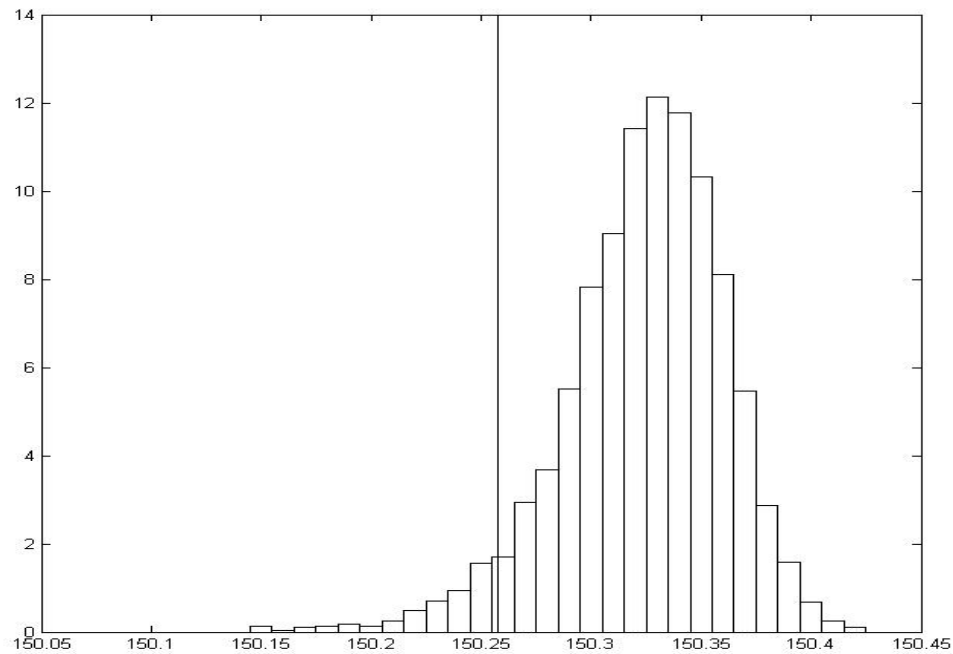
- ii.) If the condition stated in i.) is met, substitute the simulated pair of variance components into equation 3.7.2 and simulate a value q from this normal distribution.
- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times and draw the histogram representing the estimated marginal posterior distribution of q .

As an alternative to simulating q from equation 3.7.2 mentioned in step ii.), the normal distribution given in equation 3.7.2 can also be drawn after substitution of the simulated variance components. Step i.) and the alternative to step ii.) can also be repeated for example $\tilde{\ell} = 10000$ times. Using the Rao Blackwell argument described in section 2.5, the estimated marginal posterior distribution of q can then be determined by averaging the $\tilde{\ell}$ conditional posterior distributions of q given in equation 3.7.2.

Using method 2, the lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit was determined for the medicinal tablets data given in Table 3.1. Figure 3.7.2 represents the histogram of the estimated marginal posterior distribution of q , the $(1 - 0.9)^{th}$ quantile of the $N(\mu, \sigma_{\epsilon}^2 + \sigma_a^2)$ distribution, while the estimated marginal posterior distribution of q , i.e. $p(q|\mathbf{y})$ obtained using the alternative to step ii.), is depicted in Figure 3.7.3.

Similar to Figure 3.7.1, the histogram and estimated marginal posterior distribution displayed in Figures 3.7.2 and 3.7.3 respectively also represent data on the amount of active ingredient present in medicinal tablets manufactured in future batches. Using method 2, the Bayesian "B - basis" lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit is equal to 150.2583 mg. This lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit also represents the 5^{th} percentile of the estimated marginal posterior distribution of q , and, is indicated by the vertical reference line depicted in Figure 3.7.2. The Bayesian "B - basis" lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit obtained using method 2, is for all practical purposes the same as the lower ($\alpha = 0.90, \delta = 0.95$) one - sided tolerance limit obtained using method 1 as proposed by Wolfinger (1998). The interpretation is also the same.

Figure 3.7.2: Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.9)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 2.



Lower (0.90, 0.95) One - Sided Tolerance Limit: 150.2583

Although not given here, the upper (α, δ) one - sided tolerance interval can also be determined easily for the balanced one - way random effects model given in equation 3.1.1. This is done by replacing equation 3.7.1 with

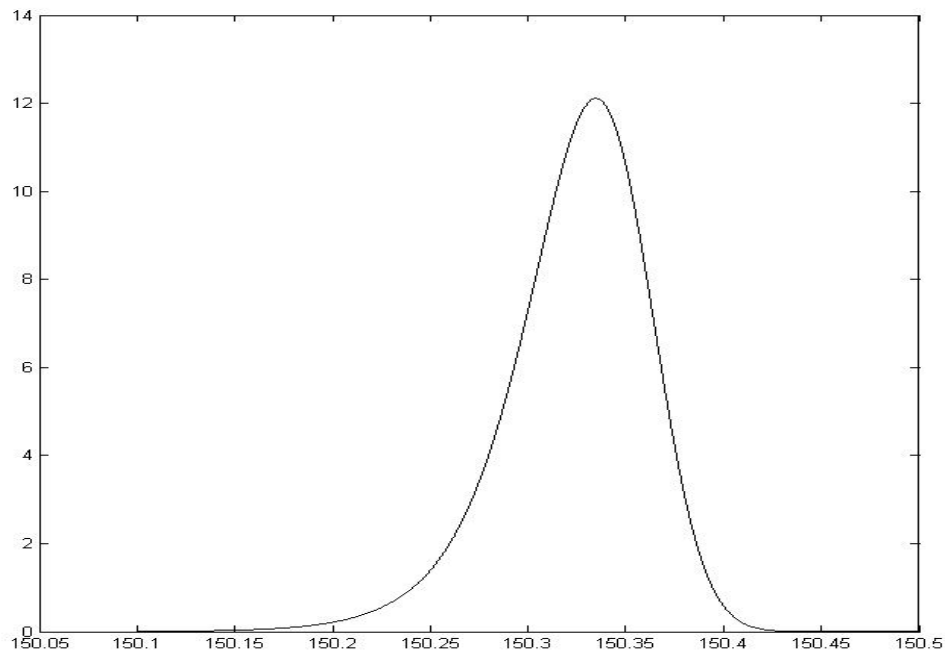
$$q = \mu + z_\alpha (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}$$

for method 1, and replacing equation 3.7.2 with

$$q | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y} \sim N\left(\bar{y}_{..} + z_\alpha (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}, \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk}\right)$$

for method 2. Both methods 1 and 2 are then applied in the same way as already discussed. It must be noted also that the interpretation for the upper (α, δ) one - sided tolerance limit is slightly different. In this case, the upper (α, δ) one - sided tolerance limit will represent the value of which $100(\alpha)\%$ of unknown future measurements will be less than with probability δ .

Figure 3.7.3: Estimated Marginal Posterior Distribution of the $(1 - 0.9)^{th}$ Quantile of $N(\mu, \sigma_\varepsilon^2 + \sigma_a^2)$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using the Alternative to Step ii.) of Method 2.



Lower (0.90, 0.95) One - Sided Tolerance Limit: 150.2583

3.7.2 Two - Sided (α, δ) Tolerance Interval

Similar to the univariate normal model discussed in Chapter 2, two - sided (α, δ) tolerance intervals can also be constructed for the balanced one - way random effects model given in equation 3.1.1. Wolfinger (1998), as well as Krishnamoorthy and Mathew (2009), mentioned that the Bayesian approach for the computation of these two - sided (α, δ) tolerance intervals is not that straightforward, but can numerically be obtained by performing Bayesian simulation.

For the construction of these two - sided (α, δ) tolerance intervals, Wolfinger (1998) suggested to still begin by computing the two quantiles q_ℓ and q_u given by

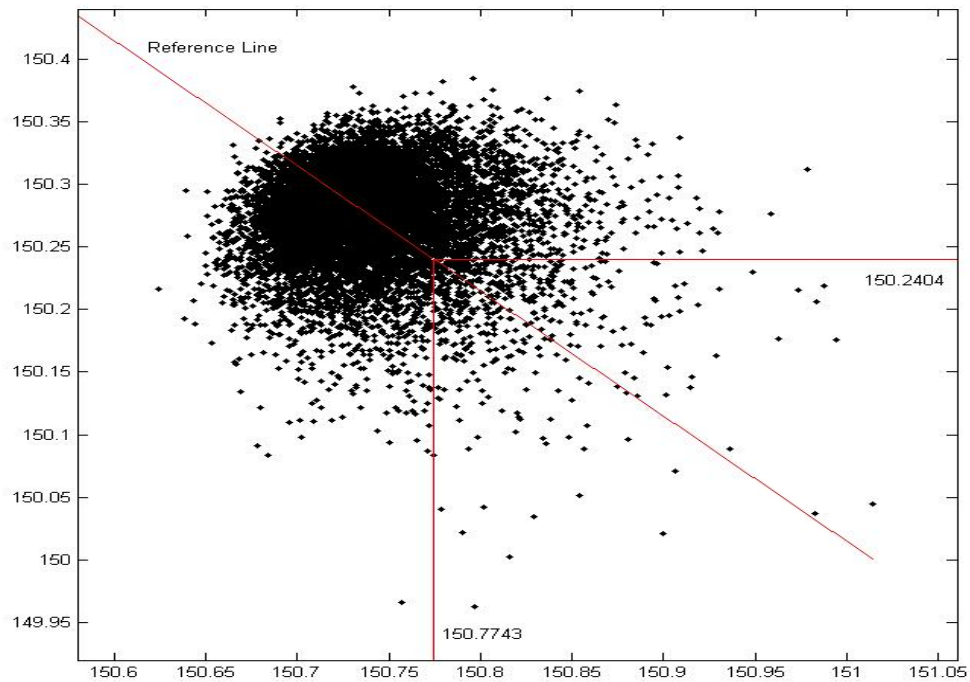
1. $q_\ell = \mu - z_{\frac{1+\alpha}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}$, and
2. $q_u = \mu + z_{\frac{1+\alpha}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}$

where $z_{\frac{1+\alpha}{2}}$ represents the $(\frac{1+\alpha}{2})^{th}$ quantile of a standard normal distribution. These (q_ℓ, q_u) pairs form a sample from the bivariate posterior distribution of the $\left[\frac{(1-\alpha)}{2}\right]^{th}$ and $\left[\frac{(1+\alpha)}{2}\right]^{th}$ quantiles (Wolfinger, 1998). According to Lee (1997), the Bayesian confidence regions for these bivariate samples usually take the form of highest posterior density regions, which according to Wolfinger (1998), are difficult to use in practice, since these regions are typically two - dimensional ellipsoids. Wolfinger (1998) also mentioned that the simple procedure of computing upper and lower limits separately on then just combining them is not valid, since the two quantiles do not have a posterior correlation equal to 1.

Wolfinger (1998) therefore suggested that a valid two - sided (α, δ) tolerance interval be constructed as follows:

- i.) Simulate the variance components σ_ε^2 and σ_a^2 using the Bayesian simulation method explained in section 3.6, subject to the condition that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$.
- ii.) If the condition stated in i.) is met, substitute the simulated retained pair of variance components into equation 3.5.4 to simulate a value for the target value μ .
- iii.) Using the retained simulated pair of variance components and the target value μ , simulate values for $q_\ell = \mu - z_{\frac{1+\alpha}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}$ and $q_u = \mu + z_{\frac{1+\alpha}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{\frac{1}{2}}$.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and plot a scatterplot of the q_ℓ and q_u simulated values with q_ℓ plotted on the vertical axis.
- v.) For the simulated q_ℓ and q_u values, construct a reference line given by $q_\ell = -q_u + 2\bar{y}_..$ and draw the reference line on the scatterplot. Two additional

Figure 3.7.4: Constructing a Two - Sided $(0.90, 0.95)$ Tolerance Interval for the Medicinal Tablets Data Given in Table 3.1.



lines also have to be drawn, one parallel to each axis and intersecting on the reference line.

- vi.) Slide the intersection point along the reference line until $100(1 - \delta)\%$ of the (q_ℓ, q_u) pairs are contained in the half rectangle opening towards the lower right portion of the graph. The coordinates of the resulting intersection point then form a two - sided (α, δ) tolerance interval of the desired form.

In Figure 3.7.4, this procedure as proposed by Wolfinger (1998) is graphically illustrated for a two - sided $(\alpha = 0.90, \delta = 0.95)$ tolerance interval, determined for the medicinal tablets data given in Table 3.1, where $z_{\frac{1+0.90}{2}}$ used for determining the q_ℓ and q_u values, was equal to 1.645.

For the medicinal tablets data given in Table 3.1, the two - sided $(\alpha = 0.90, \delta = 0.95)$ tolerance interval can be interpreted as follows: If medicinal tablets are manufactured,

90% of the amount of active ingredient present in the manufactured medicinal tablets will have a weight between 150.2404 mg and 150.7743 mg with probability 0.95.

The one - and two - sided (α, δ) tolerance intervals covered thus far, apply to observations from new batches only. Although not covered here, Wolfinger (1998) also indicated that if inference is required from the i^{th} ($i = 1, \dots, b$) batch, μ can be replaced with $\mu + \alpha_i$ as the normal mean when computing quantiles. Similarly, Wolfinger (1998) suggested that $\sigma_\varepsilon^2 + \sigma_a^2$ should be replaced with either σ_ε^2 or σ_a^2 if inference is required for hypothetical measurements involving just either the between - or within batch variance.

3.7.3 α - Expectation Tolerance Interval

It was mentioned in Chapter 2, that according to Wolfinger (1998), α - expectation tolerance intervals focus on the prediction of one or a few future observations from a process. Krishnamoorthy and Mathew (2009) therefore also call these α - expectation tolerance intervals prediction intervals of future observations, and, mentioned that these intervals are intervals such that their average content is α . Since these α - expectation tolerance intervals focus on prediction of one or a few future observations from a process, Wolfinger (1998) indicated that these intervals will be narrower than corresponding (α, δ) tolerance intervals.

To construct an α - expectation tolerance interval for the balanced one - way random effects model given in equation 3.1.1, Wolfinger (1998) suggested that simulations be conducted from an appropriate predictive distribution $p(y_f | \mathbf{y})$ where y_f represents a future observation from a new or unknown batch.

Two methods, both utilizing Bayesian simulation, can be used to construct these α - expectation tolerance intervals. Method 1 was proposed by Wolfinger (1998).

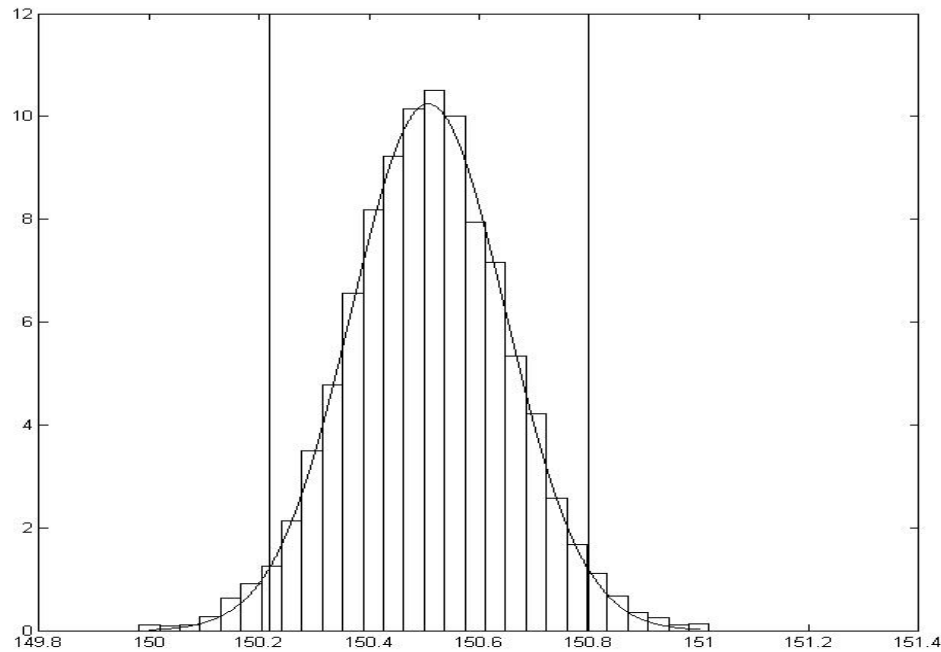
Method 1

- i.) Simulate a pair of variance components $(\sigma_a^2, \sigma_\varepsilon^2)$ subject to the condition that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ using the Bayesian simulation method described in section 3.6.
- ii.) If the condition stated in i.) is met, substitute the simulated retained pair of variance components into the normal distribution given in equation 3.5.4 and simulate a value μ from this normal distribution.
- iii.) Substitute the simulated variance components σ_a^2 and σ_ε^2 , as well as the simulated value for μ into the conditional posterior of $y_f | \mu, \sigma_a^2, \sigma_\varepsilon^2$, which according to Wolfinger (1998), is given by $y_f | \mu, \sigma_a^2, \sigma_\varepsilon^2 \sim N(\mu, \sigma_a^2 + \sigma_\varepsilon^2)$. Simulate a future observation from a new or unknown batch y_f from this normal distribution.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and draw a histogram of the simulated y_f values. This histogram represents an estimate of the unconditional predictive distribution $p(y_f | \mathbf{y})$.

Using method 1 as proposed by Wolfinger (1998), the histogram of the estimated unconditional predictive distribution was determined for the medicinal tablets data given in Table 3.1 and is provided in Figure 3.7.5. Wolfinger (1998) used a smoothing procedure to obtain the smooth curve also depicted in Figure 3.7.5. The two vertical reference lines indicate the 2.5th and 97.5th percentiles which also represents the estimated Bayesian $\alpha = 0.95$ - expectation tolerance interval.

This 95% equal tail credibility interval representing the $\alpha = 0.95$ - expectation tolerance interval for the medicinal tablets data given in Table 3.1, is equal to [150.2179, 150.7993] and can easily be determined by ranking the simulated y_f values in order of magnitude and then finding the 2.5th and 97.5th percentiles of the ranked simulated y_f values.

Figure 3.7.5: Histogram and Smooth Curve of the Estimated Unconditional Predictive Distribution for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 1.



0.95 - Expectation Tolerance Interval: [150.2179 , 150.7993]

Method 2

It was mentioned earlier that in order to obtain the α - expectation tolerance interval, the predictive density of a future observation from a new or unknown batch needs to be determined.

According to Wolfinger (1998), the conditional predictive density is given by

$$y_f | \mu, \sigma_\varepsilon^2, \sigma_a^2 \sim N(\mu, \sigma_\varepsilon^2 + \sigma_a^2).$$

It was also given in equation 3.5.4 that the conditional posterior distribution of the target value μ is given by

$$\mu | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y} \sim N(\bar{y}_{..}, \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk}).$$

Now

$$E(y_{f_i} | \mu, a_i, \sigma_a^2, \sigma_\varepsilon^2) = \mu + a_i$$

and

$$Var(y_{f_i} | \mu, a_i, \sigma_a^2, \sigma_\varepsilon^2) = \sigma_\varepsilon^2 .$$

Since

$$E(a_i) = 0 \quad (a_i = N(0, \sigma_a^2))$$

it follows that

$$E(y_f | \mu, \sigma_a^2, \sigma_\varepsilon^2) = \mu + E(a_i) = \mu$$

and

$$Var(y_f | \mu, \sigma_a^2, \sigma_\varepsilon^2) = E_{a_i} \left[Var(y_{f_i} | \mu, a_i, \sigma_\varepsilon^2, \sigma_a^2) \right] + Var_{a_i} \left[E(y_{f_i} | \mu, a_i, \sigma_\varepsilon^2, \sigma_a^2) \right] = \sigma_\varepsilon^2 + \sigma_a^2 .$$

Also,

$$E(y_f | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) = E(\mu | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y}) = \bar{y}_{..}$$

and

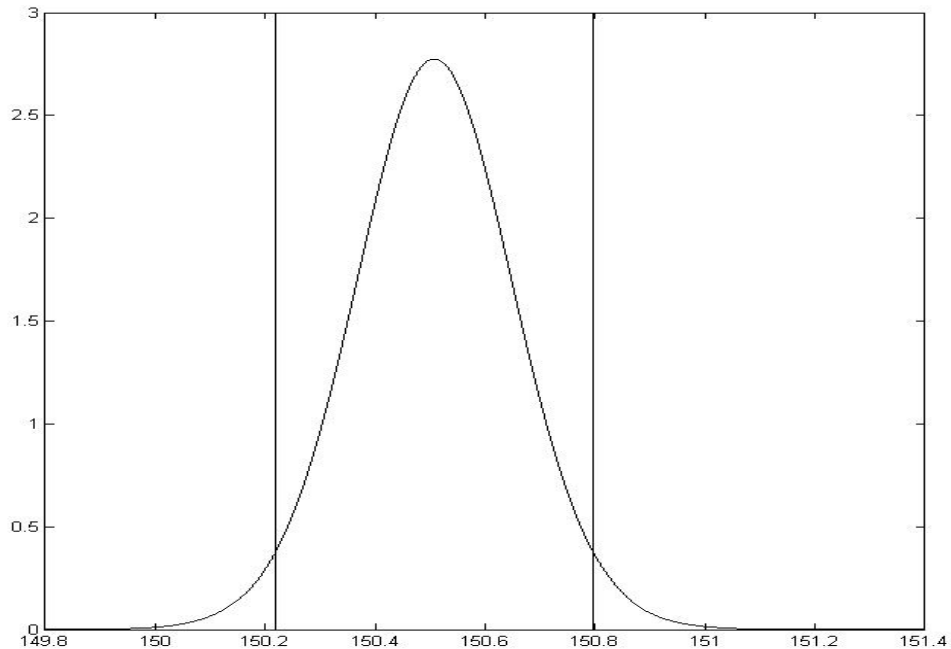
$$\begin{aligned} Var(y_f | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) &= E_\mu \left[Var(y_{f_i} | \mu, \sigma_\varepsilon^2, \sigma_a^2) \right] + Var_\mu \left[E(y_{f_i} | \mu, \sigma_\varepsilon^2, \sigma_a^2) \right] \\ &= \sigma_\varepsilon^2 + \sigma_a^2 + \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)}{bk} . \end{aligned}$$

Therefore, the conditional predictive density of y_f is given by

$$y_f | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y} \sim N \left(\bar{y}_{..}, \sigma_a^2 + \sigma_\varepsilon^2 + \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)}{bk} \right). \quad (3.7.3)$$

To simulate the estimated unconditional predictive distribution $p(y_f | \mathbf{y})$, and, hence determine the α - expectation tolerance interval, the following steps should be followed.

Figure 3.7.6: Estimated Unconditional Predictive Distribution $p(y_f|\mathbf{y})$ for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 2.



0.95 - Expectation Tolerance Interval: [150.2174 , 150.7962]

- i.) Simulate a pair of variance components $(\sigma_a^2, \sigma_\varepsilon^2)$ subject to the condition that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ using the Bayesian simulation procedure discussed in section 3.6.
- ii.) If the condition stated in i.) is met, substitute the simulated retained pair of variance components σ_a^2 and σ_ε^2 , as well as the sample mean \bar{y}_\cdot into equation 3.7.3, simulate a value y_f , and draw the normal distribution.
- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times and obtain the average density curve using the Rao Blackwell argument described in section 2.5. This average density curve represents an estimate of the unconditional predictive distribution $p(y_f|\mathbf{y})$.

The estimated unconditional predictive distribution was determined for the medicinal tablets data given in Table 3.1 using method 2, and is depicted in Figure 3.7.6.

From both Figures 3.7.5 and 3.7.6 it can be seen that methods 1 and 2 are equivalent methods for estimating the unconditional predictive distribution $p(y_f | \mathbf{y})$, since the two figures are for all practical purposes the same. The 95% equal tail credibility interval was also obtained for the estimated unconditional predictive density given in Figure 3.7.6. The two vertical reference lines depicted in Figure 3.7.6 represents the 2.5th and 97.5th percentiles or mentioned 95% equal tail credibility interval. This 95% equal tail credibility interval is in fact the $\alpha = 0.95$ - expectation tolerance interval and is obtained in a similar way as the $\alpha = 0.95$ - expectation tolerance interval obtained using method 1. Using method 2, the $\alpha = 0.95$ - expectation tolerance interval was determined for the medicinal tablets data given in Table 3.1, and is equal to [150.2174, 150.7962]. This estimated $\alpha = 0.95$ - expectation tolerance interval is for all practical purposes the same as the interval estimated using method 1, and, can be interpreted as follows: The process manufacturing the small batches of medicinal tablets will be in control if 95% or more future medicinal tablets manufactured have an amount of active ingredient weighing between 150.2174 mg and 150.7962 mg.

Since Wolfinger (1998) mentioned that the $\alpha = 0.95$ - expectation tolerance interval focus on prediction of one or a few future observations from a process, and as a result, tend to be narrower than the corresponding (α, δ) tolerance intervals, the $(\alpha = 0.95, \delta = 0.95)$ two - sided tolerance interval, $(\alpha = 0.95, \delta = 0.95)$ one - sided lower tolerance limit and the lower one - sided $\alpha = 0.95$ - expectation tolerance limit were determined for the medicinal tablets data given in Table 3.1. These results are for comparative purposes given in Table 3.2.

Table 3.2: Comparative Results Between $(0.95, 0.95)$ Tolerance Intervals and 0.95 - Expectation Tolerance Intervals for the Medicinal Tablets Data Given in Table 3.1.

	$(\alpha = 0.95, \delta = 0.95)$ Tolerance Intervals	$\alpha = 0.95$ Expectation Tolerance Intervals
Two Sided	150.1873 – 150.8291	150.2174 – 150.7962
One - Sided (Lower limit)	150.2585	150.2670

From Table 3.2 it is clear that the one - sided lower $\alpha = 0.95$ - expectation tolerance limit is larger than the corresponding one - sided ($\alpha = 0.95, \delta = 0.95$) lower tolerance limit. It can also be seen that the two - sided $\alpha = 0.95$ - expectation tolerance interval is narrower than the corresponding ($\alpha = 0.95, \delta = 0.95$) two - sided tolerance interval. It is therefore evident that α - expectation tolerance intervals are narrower than the corresponding (α, δ) tolerance intervals.

Although not given here, Wolfinger (1998) also indicated that similar to the (α, δ) tolerance intervals, different predictive densities can also be analyzed in the same way by simply adjusting the expressions used for the mean and variance parameters during the calculations.

3.7.4 Fixed - in - Advance Tolerance Intervals

According to Wolfinger (1998), fixed - in - advance tolerance intervals invert the prediction problem by considering the content of predetermined bounds. These fixed - in - advance tolerance intervals therefore answer research question 3 as proposed by Wolfinger (1998) and given in Chapter 1.

To determine the content of a fixed - in - advance tolerance interval using the Bayesian approach, the posterior density of the content has to be determined (Wolfinger, 1998). If a lower fixed - in - advance limit, s , is specified for a sample with data assumed to arise from the balanced one - way random effects model given in equation 3.1.1, the content c of the interval $[s, \infty]$ for each observation in the sample of simulated parameters $(\mu, \sigma_\varepsilon^2, \sigma_a^2)$ is determined by

$$c = 1 - \Phi \left[\frac{s - \mu}{(\sigma_a^2 + \sigma_\varepsilon^2)^{\frac{1}{2}}} \right].$$

Since a lower fixed - in - advance limit s is selected, the main focus will be on the content of the interval $[-\infty, s]$, and thus, the content less than the specified lower

specification limit s . The content of the interval $[-\infty, s]$ can therefore be determined by calculating

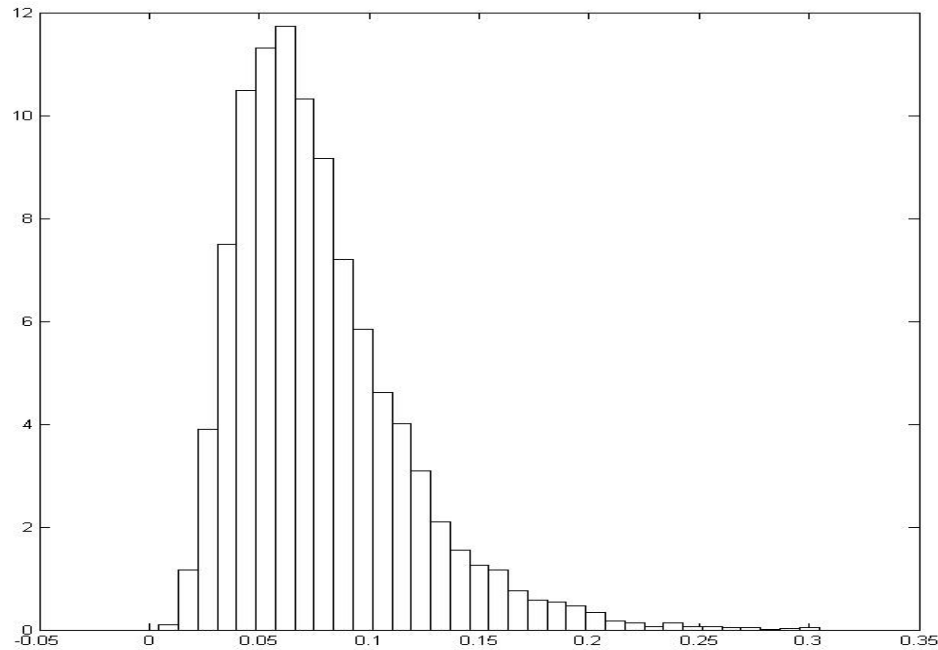
$$c^* = \Phi \left[\frac{s - \mu}{(\sigma_a^2 + \sigma_\varepsilon^2)^{\frac{1}{2}}} \right]$$

where $\Phi[\cdot]$ represents a standard normal cumulative distribution function (Wolfinger, 1998). As just mentioned, remember, the content c^* of the interval $[-\infty, s]$ represents the fraction of process measurements that lie below a preselected fixed - in - advance lower specification limit s . If the content c^* is therefore found for each observation in the sample of simulated parameters, these calculated c^* values form a sample from the posterior density of the content below the preselected specification limit s (Wolfinger, 1998).

To determine a fixed - in - advance tolerance interval for the content of the interval $[-\infty, s]$, the following steps can be followed:

- i.) Simulate a pair of variance components (σ_a^2 and σ_ε^2) using the Bayesian simulation method discussed in section 3.6. Retain only those pairs of variance components that meet the condition stating that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$.
- ii.) If the condition stated in i.) is met, substitute the simulated retained pair of variance components into the normal distribution given in equation 3.5.4 and simulate a value μ from this normal distribution.
- iii.) Substitute the simulated variance components σ_a^2 and σ_ε^2 , as well as the simulated value for μ , into the formula for the content of the interval $[-\infty, s]$ given by $c^* = \Phi \left[\frac{s - \mu}{(\sigma_a^2 + \sigma_\varepsilon^2)^{\frac{1}{2}}} \right] = \int_{-\infty}^{\frac{s - \mu}{(\sigma_a^2 + \sigma_\varepsilon^2)^{\frac{1}{2}}} } \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$.
- iv.) Repeat the simulation process explained in steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and draw a histogram representing the estimated posterior distribution of the content c^* below the preselected specification limit s .

Figure 3.7.7: Histogram of the Estimated Posterior Distribution of the Content of the Interval $[-\infty, 150.30 \text{ mg}]$, i.e. the Fraction of Medicinal Tablets Containing an Amount of Active Ingredient Less than the Preselected Fixed - in - Advance Lower Specification Limit $s = 150.30 \text{ mg}$ for the Data Given in Table 3.1.



95% Fixed - in - Advance Tolerance Interval: $[0.0262, 0.1754]$

A $100(\alpha)\%$ equal tail credibility interval can also easily be determined for the content of the interval $[-\infty, s]$ by ranking the sample of c^* values in order of magnitude and then finding the $100(\frac{1-\alpha}{2})^{th}$ and $100(\frac{1+\alpha}{2})^{th}$ percentiles of the ranked simulated c^* values. This $100(\alpha)\%$ equal tail credibility interval represents the fixed - in - advance tolerance interval of the content below a preselected specification limit s .

For illustrative purposes, a fixed - in - advance lower specification limit $s = 150.30 \text{ mg}$ was selected for the medicinal tablets data given in Table 3.1. The selected lower specification limit $s = 150.30 \text{ mg}$ can for example indicate the minimum amount of active ingredient that has to be taken per dose to render the medication effective. The histogram of the sample of $\tilde{\ell} = 10000$ simulated c^* values from the posterior distribution of the content of the interval $[-\infty, s]$ obtained using ordinary Monte Carlo simulation, is depicted in Figure 3.7.7.

From Figure 3.7.7 it can be seen that the posterior content of the interval $[-\infty, s]$ is positively skewed. Also, the 95% equal tail credibility interval or fixed - in - advance tolerance interval of the posterior content of the interval $[-\infty, s = 150.30]$ is equal to $[0.0262, 0.1754]$. This means that between 2.62% and 17.54% of medicinal tablets manufactured in future batches, will contain an amount of active ingredient less than the specified preselected limit $s = 150.30$ mg. In other words, of future medicinal tablets manufactured in small batches, between 2.62% and 17.54% of the tablets will be ineffective as treatment.

Remember, this preselected lower specification limit $s = 150.30$ mg was selected for illustrative purposes only.

Chapter 4

The Balanced One - Way Random Effects

Model Continued

In this chapter, the Bayesian simulation method for determining tolerance intervals originally proposed by Wolfinger (1998) and discussed in Chapter 3, will be extended to include Bayesian tolerance intervals for averages of observations from new or unknown batches in the case of the balanced one - way random effects model. It will also be shown, that although the non - informative Jeffreys' prior used in Chapter 3 is not a reference - nor probability matching prior for the α^{th} quantile q of the $N(\mu, \sigma_{\epsilon}^2 + \sigma_{\alpha}^2)$ distribution, it is however both a reference - and probability matching prior for the α^{th} quantile \tilde{q} of the distribution of averages of observations from new or unknown batches. An extensive simulation study using two methods will follow to investigate what kind of frequentist properties the Bayesian interval for \tilde{q} have under the mentioned probability matching prior. For both these simulation methods, the mean and variance of the posterior distribution of \tilde{q} will also be derived theoretically. In addition, it will also be shown that a proposed prior distribution for the content of the fixed - in - advance tolerance interval, is a probability matching prior. For this proposed probability matching prior, the posterior distribution of the content and subsequent fixed - in - advance tolerance interval will be determined using the weighted Monte Carlo method. The Bayesian simulation method for obtaining tolerance intervals for

averages of observation from new or unknown batches will be illustrated using the simulated medicinal tablets data presented in Chapter 3.

Parts of this chapter have been published as a technical report. For more details, see van der Merwe and Hugo (2008).

4.1 Introduction

In the previous chapter, a simulation based approach originally proposed by Wolfinger (1998), was presented for determining Bayesian tolerance intervals in the case of a balanced one - way random effects model. In this chapter, the Bayesian simulation method will be extended to include tolerance intervals for averages of observations from new or unknown batches or groups. A reference prior and a probability matching prior will also be derived for the α^{th} quantile, while a probability matching prior will be derived for the content of the interval $[s, \infty]$.

To illustrate how and when these tolerance intervals will be used, consider a factory which manufactures medicinal tablets in very small batches. As mentioned in Chapter 3, a small batch in this instance is likely to be a weekly or monthly intake of tablets for an individual patient. The interest is in whether the patient gets on average the required dosage of the active ingredient from the batch in the specified time, given that each patient must get an average dosage of at least s . The question therefore is whether the process is capable of producing to this specification.

4.2 Non - Informative Priors in the Case of Averages of Observations from New or Unknown Batches for the Balanced One - Way Random Effects Model

As before, the one - way random effects model can be described as follows:

$$y_{ij} = \mu + a_i + \varepsilon_{ij}$$

where ($i = 1, \dots, b$ and $j = 1, \dots, k$). Also, y_{ij} , μ , a_i ($i = 1, \dots, b$) and ε_{ij} are defined as given in section 3.3, and the a_i 's and ε_{ij} 's are mutually independent normally distributed random variables with $E(a_i) = E(\varepsilon_{ij}) = 0$ and variances σ_a^2 and σ_ε^2 respectively.

It can easily be shown that the predictive density of the average of k future observations $y_{f1}, y_{f2}, \dots, y_{fk}$ from a new or unknown batch, given the variance components, is normally distributed with mean

$$E(\bar{y}_f | \mu, \sigma_\varepsilon^2, \sigma_a^2) = \mu$$

and variance

$$Var(\bar{y}_f | \mu, \sigma_\varepsilon^2, \sigma_a^2) = \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}$$

$$\text{where } \bar{y}_f = \frac{1}{k} \sum_{j=1}^k y_{fj} .$$

In what follows, non - informative priors for

$$\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}} \quad (4.2.1)$$

will be considered where z_α denotes the $100(\alpha)^{th}$ percentile of a standard normal distribution. Equation 4.2.1 denotes the α^{th} quantile of the distribution for the average of future data from new or unknown batches.

To obtain the posterior distribution of \tilde{q} , the non - informative prior distribution given by

$$p(\mu, \sigma_\varepsilon^2, \sigma_a^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} \quad (4.2.2)$$

will be used. In the next section (Theorem 4.3.1) it will be proved that equation 4.2.2 is a probability matching prior as well as a reference prior for the parameter \tilde{q} .

Wolfinger (1998) also used this non - informative prior to obtain posterior distributions of variance components and tolerance intervals for individual observations. Equation 4.2.2 is however not a probability matching prior or a reference prior for the parameter

$$q = \mu + z_\alpha \sqrt{\sigma_\varepsilon^2 + \sigma_a^2}$$

which represents the α^{th} quantile of the distribution for a single (individual) observation from a new or unknown batch.

4.3 Reference and Probability Matching Priors

It will now be proved that the non - informative prior distribution given in equation 4.2.2 is a probability matching prior as well as a reference prior for the α^{th} quantile \tilde{q} given in equation 4.2.1.

Theorem 4.3.1

For the balanced one - way random effects model given in equation 3.1.1, the prior distribution $p(\mu, \sigma_\varepsilon^2, \sigma_a^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1}$ is a probability matching prior as well as a reference prior for the α^{th} quantile given by

$$\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$$

of the distribution of the average of future data from new or unknown batches.

Proof

The proof of Theorem 4.3.1 is given in Appendix B.

4.4 Calculation of the Tabulated Values (Coverage, Interval Length and Standard Deviation)

The question also arises as to what kind of frequentist coverage probabilities the Bayesian Interval for \tilde{q} under the non - informative prior given in equation 4.2.2 will have. Extensive numerical computations will be conducted in order to evaluate the procedure. The layout used will be similar to that of Krishnamoorthy and Mathew (2004) and Chen and Harris (2006). Data will be generated from the balanced one - way normal random effects model. Each design will be specified by the number of groups (batches) (b), the sample size per group (k), as well as the intraclass correlation coefficient

$$\rho = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_\varepsilon^2}.$$

To simplify the designs, it was decided to use $\mu = 0$ and $\sigma_a^2 + \sigma_\varepsilon^2 = 1$.

Recall from Theorem 3.5.2 that, under the non - informative prior distribution given in equation 4.2.2, the joint posterior distribution of the variance components is given by

$$p(\sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}(\nu_1+2)} \exp \left\{ -\frac{1}{2} \left[\frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right] \right\} \quad (4.4.1)$$

where ν_1 , ν_2 , $\nu_1 m_1$, $\nu_2 m_2$, \bar{y}_i , and $\bar{y}_{..}$ are defined as given in equation 3.5.3 and

$$\mathbf{y} = [y_{11}, y_{12}, \dots, y_{1k}, y_{21}, \dots, y_{bk}]'.$$

Also recall from Theorem 3.5.1 that the target value μ , given the variance components, follows a normal distribution given by

$$\mu | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y} \sim N \left(\bar{y}_{..}, \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{kb} \right). \quad (4.4.2)$$

Two simulation methods can now be used to obtain the 95% equal tail credibility interval of the parameter \tilde{q} .

Method 1

- i.) Since the variance component parameter in \tilde{q} given in equation 4.2.1 is of the form $\sigma_\varepsilon^2 + k\sigma_a^2$, it follows from section 3.6, that given the data, $\sigma_\varepsilon^2 + k\sigma_a^2 = \frac{\nu_2 m_2}{\delta}$ where $\delta \sim \chi_{\nu_2}^2$. This therefore implies that $\sigma_\varepsilon^2 + k\sigma_a^2$ can easily be simulated using the mentioned $\chi_{\nu_2}^2$ distribution.¹
- ii.) Given the simulated function of variance components, the target value μ can be simulated easily using equation 4.4.2.
- iii.) Substitute the simulated function of variance components ($\sigma_\varepsilon^2 + k\sigma_a^2$) as well as the simulated target value μ into equation 4.2.1 and obtain \tilde{q} .

Method 2

From equation 4.4.1 it is clear that

- i.) σ_ε^2 can be simulated using a $\chi_{\nu_1}^2$ distribution since $\sigma_\varepsilon^2 = \frac{\nu_1 m_1}{\tau}$ where $\tau \sim \chi_{\nu_1}^2$. This was described in more detail in section 3.6.
- ii.) Simulate σ_a^2 by first simulating $(\sigma_\varepsilon^2 + k\sigma_a^2)$ using step i.) explained in method 1.
- iii.) Calculate $\sigma_a^2 = \frac{(\sigma_\varepsilon^2 + k\sigma_a^2) - \sigma_\varepsilon^2}{k}$. The two simulated variance components σ_ε^2 and σ_a^2 are only retained if $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$.
- iv.) Substitute the retained simulated variance components into equation 4.4.2 and simulate a value μ .
- v.) Substitute the simulated retained variance components σ_ε^2 and σ_a^2 , as well as the simulated target value μ , into equation 4.2.1 and obtain \tilde{q} .

¹Since we are simulating a function of the two variance components ($\sigma_\varepsilon^2 + k\sigma_a^2$), it is not necessary to check the condition stating that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ mentioned in Chapter 3, section 3.6.

For each design, a data set consisting of 1000 entries will be generated. For each data set with each different intraclass correlation ρ , 1000 Bayesian simulations will be conducted to obtain the posterior distribution of \tilde{q} and the estimated 95% credibility interval. The estimated coverage rate will be the proportion of the 1000 credibility intervals that contain the true parameter \tilde{q} obtained using $\mu = 0$ and ρ given for each design. For the simulation study, since it is considered that $\sigma_a^2 + \sigma_\varepsilon^2 = 1$, it follows that the intraclass correlation coefficient $\rho = \sigma_a^2$, and, $\sigma_\varepsilon^2 = 1 - \sigma_a^2$. It therefore follows that the true population parameter \tilde{q} which will be used, be equal to $\tilde{q} = z_\alpha \sqrt{\frac{(1-\rho)+k\rho}{k}}$ since $\mu = 0$. It is expected however that method 2 will produce conservative results for low values of ρ . This particularly means that it is expected that method 2 will produce wider 95% equal tail credibility intervals for \tilde{q} than that of method 1, resulting in coverage probabilities above the nominal confidence of 0.95. One possible reason for this stems from the fact that for method 2, the condition stating that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ has to be met. If this condition is not met, the particular pair of variance components σ_a^2 and σ_ε^2 is disregarded, resulting in a possible artificial inflation of the posterior variance of \tilde{q} .

The simulation results for the two methods are presented in Table 4.1 for the 90th percentile (i.e. $z_\alpha = 1.282$) and for a nominal confidence level of 0.95. The upper tabulated value represents the estimated coverage rate, while the two lower values represent the average length and standard deviation of the credibility intervals.

Table 4.1: Coverage Rate of the 95% Credibility Intervals for \tilde{q} using the Two Described Methods and Different Values of ρ , b and k .

(b, k)	ρ	Method1	Method 2
(3, 2)	0	0.9560	1.000
		4.9512	7.5516
		2.9817	4.1246
	0.1	0.9460	0.9980
		5.2808	7.2942
		4.9360	4.0865
		0.25	0.9560
		5.3993	7.2430
		3.3787	3.5971
		0.5	0.9500
		6.2127	7.1352
		4.0415	3.5635
		0.75	0.9510
		6.3714	7.2404
		3.9205	3.8395
		0.85	0.9510
		6.7922	7.0404
		3.9560	3.9252
		0.95	0.9600
		6.8116	7.0063
			3.9474

(b, k)	ρ	Method 1	Method 2
(7, 2)	0	0.9330	0.9920
		1.8345	2.1936
		0.5743	0.4654
	0.1	0.9520	0.9960
		1.9241	2.2070
		0.5576	0.4708
		0.25	0.9470
		2.0170	2.2512
		0.5970	0.5294
		0.5	0.9430
		2.2446	2.3554
		0.6671	0.5942
		0.75	0.9570
		2.4854	2.4568
		0.7422	0.7079
		0.85	0.9610
		2.5253	2.5611
		0.7491	0.7300
		0.95	0.9500
		2.5655	2.5425
			0.7699

Table 4.1: Continued

(b, k)	ρ	Method 1	Method 2
(10, 2)	0	0.9390	0.9950
		1.4270	1.6515
		0.3578	0.3012
	0.1	0.9490	0.9910
		1.5268	1.6911
			0.3675
	0.25		0.9430
		1.6051	1.7286
		0.3949	0.3530
	0.5	0.9500	0.9570
			1.8022
	0.75	0.9430	
		1.9088	1.9021
		0.4823	0.4631
	0.85	0.9440	0.9410
			1.9708
	0.95	0.9500	
		2.0415	2.0017
		0.5110	0.5043

(b, k)	ρ	Method 1	Method 2
(3, 4)	0	0.9600	0.9990
		3.5400	5.0689
		2.1935	1.8342
	0.1	0.9480	0.9990
			4.0333
	0.25	0.9510	
		4.4768	5.3855
0.5	0.9480		
		5.4321	5.8633
0.75	0.9650		
		6.4419	7.2404
0.85	0.9420		
		6.5257	7.0404
0.95	0.9600		
		6.8116	6.9577
		3.9474	5.13

Table 4.1: Continued

(b, k)	ρ	Method 1	Method 2	(b, k)	ρ	Method 1	Method 2
(6, 4)	0	0.9440	0.9990	(4, 5)	0	0.9460	0.9990
		1.4615	1.7739			1.9690	2.5668
		0.5096	0.3559			0.8921	0.6087
	0.1	0.9470	0.9980		0.1	0.9430	0.9980
		1.6422	1.8684			2.2805	2.7247
		0.5358	0.4035			1.0199	0.7596
	0.25	0.9410	0.9890		0.25	0.9570	0.9950
		1.8996	2.0388			2.7712	3.0131
		0.6377	0.5720			1.2115	1.0652
	0.5	0.9390	0.9690		0.5	0.9510	0.9850
		2.2494	2.3492			3.3704	3.5433
		0.7513	0.7241			1.4375	1.3886
	0.75	0.9530	0.9620		0.75	0.9530	0.9580
		2.6595	2.6400			3.9224	3.9334
		0.9053	0.8514			1.6332	1.7647
	0.85	0.9440	0.9460		0.85	0.9540	0.9530
		2.7523	2.7207			3.9953	4.0710
		0.9193	0.9247			1.8138	1.7110
	0.95	0.9450	0.9650		0.95	0.9600	0.9430
		2.8461	2.9055			4.2436	4.2687
		0.9785	0.9357			1.7574	1.8578

Table 4.1: Continued

(b, k)	ρ	Method 1	Method 2
(5, 6)	0	0.9480	1.0000
		1.4116	1.7451
		0.5566	0.3435
	0.1	0.9430	0.9960
		1.6980	1.8980
		0.6650	0.4983
		0.25	0.9420
		2.1139	2.1739
		0.7846	0.6887
		0.5	0.9340
		2.6072	2.6372
		1.0219	0.8954
		0.75	0.9510
		3.0011	3.0279
		1.0851	1.0989
		0.85	0.9470
		3.2044	3.2437
		1.1928	1.1577
		0.95	0.9470
		3.3422	3.3378
		1.2384	1.1952

(b, k)	ρ	Method 1	Method 2
(3, 10)	0	0.9520	1.0000
		2.2078	3.1410
		1.3741	0.9813
	0.1	0.9490	1.0000
		3.0747	3.5913
		1.7360	1.5233
		0.25	0.9300
		4.0571	4.4777
		5.3558	7.6003
		0.5	0.9490
		5.0036	5.2867
		2.9980	3.1010
		0.75	0.9480
		3.0180	6.1292
		3.5878	6.4635
		0.85	0.9440
		6.3749	6.5916
		4.1076	3.7769
		0.95	0.9510
		6.9328	6.7823
		3.9474	4.0250

Table 4.1: Continued

(b, k)	ρ	Method 1	Method 2
(10, 10)	0	0.9480	0.9950
		0.6583	0.7268
		0.1590	0.1069
	0.1	0.9550	0.9840
		0.8846	0.9120
		0.2094	0.1969
	0.25	0.9400	0.9550
		1.1709	1.1747
		0.2871	0.2853
	0.5	0.9420	0.9510
		1.5250	1.5259
		0.3813	0.3716
	0.75	0.9500	0.9460
		1.7995	1.8091
		0.4335	0.4652
	0.85	0.9550	0.9550
		1.9214	1.8994
		0.4788	0.4712
	0.95	0.9500	0.9510
		1.9886	2.0223
		0.5099	0.4900

(b, k)	ρ	Method 1	Method 2
(15, 10)	0	0.9450	0.9990
		0.5105	0.5571
		0.1001	0.0765
	0.1	0.9460	0.9670
		0.7025	0.7140
		0.1428	0.1308
	0.25	0.9510	0.9630
		0.9250	0.9232
		0.1781	0.1811
	0.5	0.9410	0.9460
		1.2045	1.1262
		0.2365	0.2460
	0.75	0.9510	0.9530
		1.4288	1.4445
		0.2952	0.2782
	0.85	0.9470	0.9500
		1.5071	1.5216
		0.2964	0.3100
	0.95	0.9520	0.9390
		1.5707	1.5834
		0.3038	0.3150

Table 4.1: Continued

(b, k)	ρ	Method 1	Method 2
(35, 25)	0	0.9440	0.9910
		0.2100	0.2100
		0.0288	0.0235
	0.1	0.9400	0.9370
		0.3877	0.3839
		0.0527	0.0518
	0.25	0.9630	0.9450
		0.5539	0.5554
		0.0741	0.0775
	0.5	0.9450	0.9450
		0.7551	0.7546
		0.1049	0.0998
	0.75	0.9450	0.9490
		0.9139	0.9146
		0.1210	0.1208
	0.85	0.9460	0.9500
		0.9721	0.9683
		0.1289	0.1291
	0.95	0.9450	0.9500
		1.0191	1.0171
		0.1348	0.1360

(b, k)	ρ	Method 1	Method 2
(3, 200)	0	0.9530	1.0000
		0.4901	0.7007
		0.3095	0.2214
	0.1	0.9370	0.9390
		2.3148	2.2731
		1.4280	1.3512
	0.25	0.9370	0.9490
		3.4424	3.5190
		2.0432	2.5949
	0.5	0.9420	0.9430
		4.9074	4.9722
		3.1825	3.2106
	0.75	0.9490	0.9530
		5.8988	6.1695
		3.4861	3.9086
	0.85	0.9440	0.9450
		6.3450	6.5919
		3.8942	4.8099
	0.95	0.9430	0.9510
		6.7827	6.7776
		4.6001	4.8793

Table 4.1: Continued

(b, k)	ρ	Method 1	Method 2
(10, 200)	0	0.9580	0.9970
		0.1446	0.1639
		0.0356	0.0230
	0.1	0.9500	0.9610
		0.6676	0.6625
		0.1624	0.1573
	0.25	0.9440	0.9520
		1.0295	1.0477
		0.2610	0.2520
	0.5	0.9610	0.9470
		1.4495	1.4437
		0.3465	0.3479
	0.75	0.9500	0.9590
		1.7727	1.7639
		0.4157	0.4461
	0.85	0.9390	0.9420
		1.8588	1.8786
		0.4785	0.4817
	0.95	0.9540	0.9460
		1.9746	2.0076
		0.4989	0.4917

The simulation results substantiate what was expected. The second method is conservative for low values of ρ , with large average interval lengths, particularly for small values of b and k . It is clear that the coverage of the first method is near the nominal confidence of 0.95 uniformly across the range of selected values for ρ , b and k . Note that the coverage probabilities, interval widths and standard deviations for the two methods are close to each other for large values of ρ . In the next section more theoretical results of the distribution of \tilde{q} will be derived.

4.5 Theoretical Results for the Posterior Distribution of \tilde{q}

It was stated in Chapter 2 that Wolfinger (1998) mentioned that the analytical derivations of exact unconditional posterior densities for the unknown parameters μ , σ_a^2 , σ_ε^2 , a_i ($i = 1, \dots, b$) and exact posterior densities of quantiles to construct tolerance intervals for the balanced one - way random effects model appeared to be formidable. The same can be said about the analytical derivations of the quantiles in the case of averages of observations from new or unknown batches for the random effects model given in equation 3.1.1.

It is however possible to obtain exact moments for the quantiles in the case of averages of observations from new or unknown batches. In what follows, the exact first moment about zero and second central moment, i.e. the mean and variance, of the posterior distribution of $\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ will be determined analytically for both methods 1 and 2.

Theorem 4.5.1

- i.) The mean and variance of the posterior distribution of $\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ in the case of method 1, is

$$E(\tilde{q}|\mathbf{y}) = \bar{y}_{..} + z_\alpha \frac{1}{k^{\frac{1}{2}}} (\nu_2 m_2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{\sqrt{2}\Gamma(\frac{\nu_2}{2})} \text{ and}$$

$$Var(\tilde{q}|\mathbf{y}) = \frac{\nu_2 m_2}{bk(\nu_2-2)} + \frac{z_\alpha^2 \nu_2 m_2}{k} \left\{ \frac{1}{\nu_2-2} - \frac{\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \right\}.$$

- ii.) The mean and the variance for the posterior distribution of $\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ in the case of method 2 is

$$E(\tilde{q}|\mathbf{y}) = \bar{y}_{..} + z_\alpha \left(\frac{\nu_2 m_2}{2k} \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{\sqrt{2}\Gamma(\frac{\nu_2}{2})} \frac{Pr\{F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1}\}}{Pr\{F_{\nu_2, \nu_1} < \frac{m_2}{m_1}\}} \text{ and}$$

$$Var(\tilde{q}|\mathbf{y}) = \frac{\nu_2 m_2}{bk(\nu_2-2)} \frac{Pr\{F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1}\}}{Pr\{F_{\nu_2, \nu_1} < \frac{m_2}{m_1}\}} + \frac{z_\alpha^2 \nu_2 m_2}{k} \left\{ \frac{1}{\nu_2-2} - \frac{Pr\{F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1}\}}{2\Gamma^2(\frac{\nu_2}{2})} - \frac{[\Gamma(\frac{\nu_2-1}{2})]^2}{[\Gamma(\frac{\nu_2}{2})]^2} \frac{Pr\{F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1}\}}{Pr\{F_{\nu_2, \nu_1} < \frac{m_2}{m_1}\}} \right\}$$

where F_{ν_2, ν_1} represents an F - distribution with ν_2 and ν_1 degrees of freedom.

Pr stands for probability.

Proof

The proof of Theorem 4.5.1 is given in Appendix B.

The third and fourth central moments of \tilde{q} can also be derived although it is not given here. By applying these moments to Pearson curves or Cornish - Fisher expansions, approximations of the exact posterior distribution of \tilde{q} can be obtained. In section 4.6, the two Bayesian simulation methods for obtaining posterior distributions of $\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ will be applied to a real problem.

4.6 An Example

The data used for the example is the same data set used to illustrate the methods proposed by Wolfinger (1998) for determining Bayesian tolerance intervals for the balanced one - way random effects model discussed in Chapter 3. The data was originally presented in Table 3.1. Recall that the data are assumed to arise from a normal distribution with unknown parameters, but it has more structure than a simple random sample because it is clustered in fifteen batches and each batch contains ten tablets. As mentioned, the random effects model given in equation 3.1.1 will also be used and is given by

$$y_{ij} = \mu + a_i + \varepsilon_{ij}$$

$$i = 1, \dots, b \text{ and } j = 1, \dots, k$$

where y_{ij} , μ , a_i and ε_{ij} are defined as in Chapter 3.

4.6.1 One - Sided (α, δ) Tolerance Intervals

It was already mentioned in section 4.2 that the α^{th} quantile of the average of observations from new or unknown batches in the case of the balanced one - way random effects model, is given by

$$\tilde{q} = \mu + z_{\alpha} \sqrt{\frac{(\sigma_{\varepsilon}^2 + k\sigma_a^2)}{k}}$$

where z_{α} represents the $100(\alpha)^{th}$ percentile of a standard normal distribution (see equation 4.2.1).

It was also mentioned in section 4.4 that the \tilde{q} values can be simulated using two Bayesian simulation procedures. Method 1, directly simulated $(\sigma_{\varepsilon}^2 + k\sigma_a^2)$, while method 2, simulated first σ_{ε}^2 and then σ_a^2 . For the second method, the condition that $(\sigma_{\varepsilon}^2 + k\sigma_a^2) > \sigma_{\varepsilon}^2$ had to be met in order to retain the two simulated variance components. If this condition was not met, the two simulated variance components were disregarded and a new pair simulated under the same condition. For both methods the simulation procedure was repeated for example $\tilde{\ell} = 10000$ times. Also, for both methods, histograms depicting the estimated marginal posterior distributions $p(\tilde{q}|\mathbf{y})$, can be produced. Similarly, a one - sided lower (α, δ) tolerance interval can also be constructed using both methods.

In order to construct a lower one - sided (α, δ) tolerance limit, Wolfinger (1998) suggested that the marginal posterior distribution of \tilde{q} , which represents the $(1 - \alpha)^{th}$ quantile of the distribution of averages of observations from new or unknown batches, be estimated. For the construction of the lower one - sided (α, δ) tolerance limit, \tilde{q} is given by

$$\tilde{q} = \mu - z_{\alpha} \sqrt{\frac{(\sigma_{\varepsilon}^2 + k\sigma_a^2)}{k}}$$

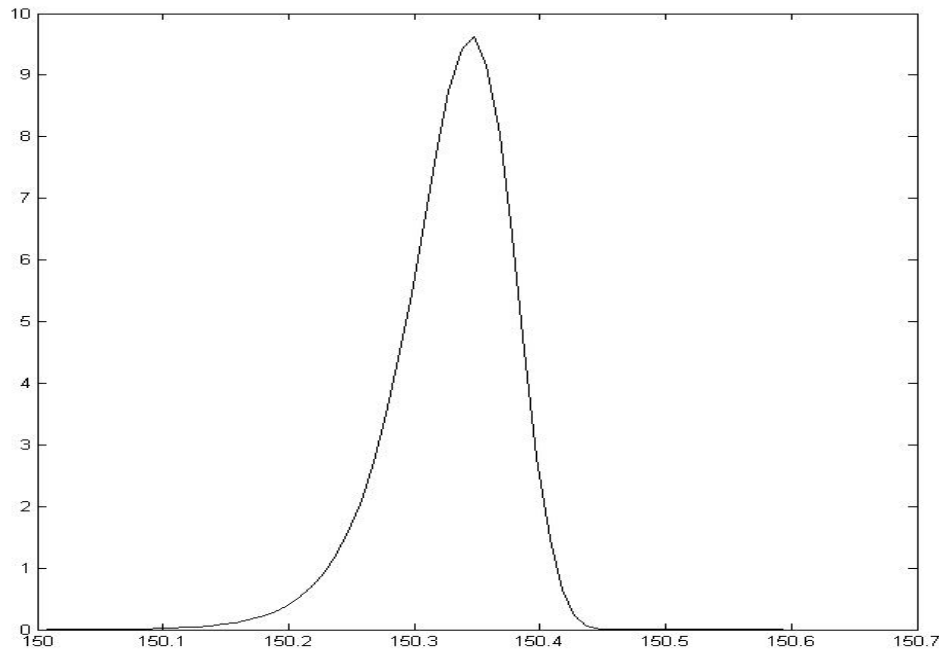
where z_{α} represents the $100(\alpha)^{th}$ percentile of a standard normal distribution. Methods 1 and 2 can be applied in the following way:

- i.) For method 1, simulate $(\sigma_\varepsilon^2 + k\sigma_a^2)$ using the Bayesian simulation procedure described in method 1. For method 2, simulate a pair of variance components σ_ε^2 and σ_a^2 using the Bayesian simulation procedure discussed in section 3.6 of Chapter 3, ensuring that the condition stating that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$, is met.
- ii.) Given either $(\sigma_\varepsilon^2 + k\sigma_a^2)$ as simulated using method 1 or the pair of variance components σ_ε^2 and σ_a^2 simulated using method 2, the conditional posterior distribution of \tilde{q} can be simulated from a normal distribution with mean equals to $E(\tilde{q}|\sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) = \bar{y}_{..} - z_\alpha \sqrt{\frac{(\sigma_\varepsilon^2 + k\sigma_a^2)}{k}}$ and variance given by $Var(\tilde{q}|\sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) = \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)}{bk}$.
- iii.) Substitute the variance components and sample mean of the data, $\bar{y}_{..}$, into the normal distribution given in step ii.) above, and, draw the normal distribution.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times, each time drawing the normal distribution.
- v.) Using the Rao Blackwell argument explained in section 2.5, the estimated marginal posterior distribution of \tilde{q} , i.e. $p(\tilde{q}|\mathbf{y})$ can be obtained by averaging the $p(\tilde{q}|\sigma_a^2, \sigma_\varepsilon^2, \mathbf{y})$ distributions over the $\tilde{\ell}$ repetitions.

The two estimated marginal posterior distributions of $\tilde{q} = \mu - z_{0.95} \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ obtained using methods 1 and 2, are displayed in Figures 4.6.1 and 4.6.2 respectively. For both figures $z_{0.95} = 1.645$ and $\tilde{\ell} = 10000$ simulations were run.

Also, for both methods, the one - sided ($\alpha = 0.95, \delta = 0.95$) lower tolerance limits can be determined easily by ranking the simulated \tilde{q} values in order of magnitude, and finding the $100(1 - 0.95)^{th}$ percentile of the ranked simulated \tilde{q} values. For both methods 1 and 2, the lower one - sided ($\alpha = 0.95, \delta = 0.95$) tolerance limits were equal to 150.3931 mg.

Figure 4.6.1: Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile \tilde{q} for the Distribution of the Average of Observations from New or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1. Method 1 was used.



Lower One - Sided $(0.95, 0.95)$ Tolerance Limit: 150.3931

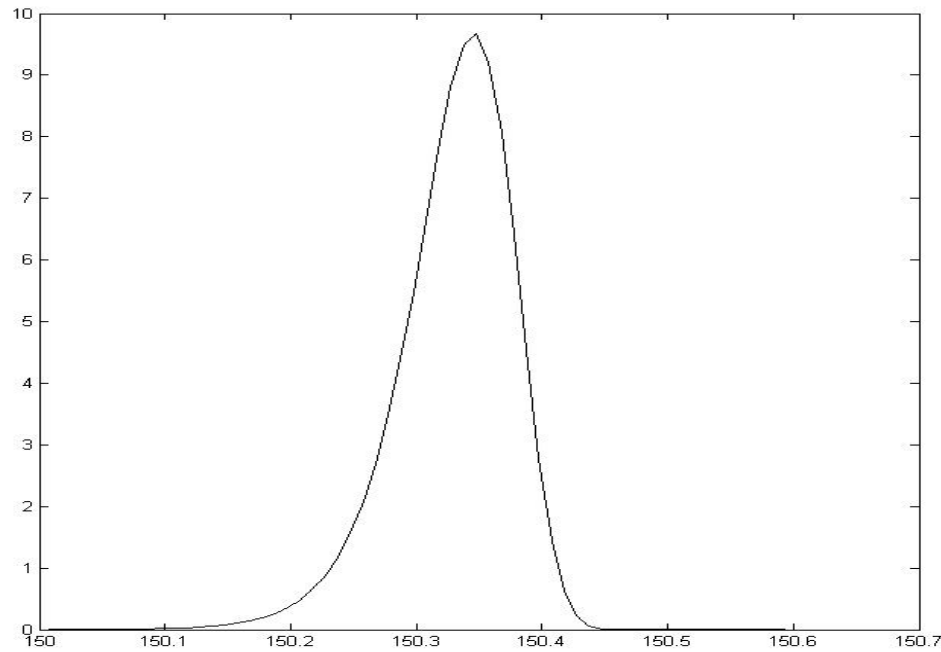
The one - sided $(\alpha = 0.95, \delta = 0.95)$ lower tolerance limits for both methods equal to 150.3931 mg represents the milligrams of active ingredient of which 95% of unknown future averages of manufactured batches of tablets will be greater than with probability 0.95.

From both Figures 4.6.1 and 4.6.2, as well as the two one - sided $(\alpha = 0.95, \delta = 0.95)$ lower tolerance limits obtained, it is clear that there is not much to choose between the two methods for obtaining the posterior distribution of

$$\tilde{q} = \mu - z_{0.95} \sqrt{\frac{\sigma_{\varepsilon}^2 + k\sigma_a^2}{k}}.$$

This can be explained using the results obtained from the simulation study conducted in section 4.4 and given in Table 4.1. For the medicinal tablets data given in Table 3.1,

Figure 4.6.2: Estimated Marginal Posterior Distribution of the $(1 - 0.95)^{th}$ Quantile \tilde{q} for the Distribution of the Average of Observations from New or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1. Method 2 was used.



Lower One - Sided $(0.95, 0.95)$ Tolerance Limit: 150.3931

the estimated intraclass correlation coefficient is equal to

$$\hat{\rho} = \frac{\hat{\sigma}_a^2}{\hat{\sigma}_a^2 + \hat{\sigma}_\varepsilon^2} = 0.5049$$

which is quite large. The above results were thus expected.

4.6.2 Two - sided (α, δ) Tolerance Interval

In order to construct a two - sided (α, δ) tolerance interval for the average of observations from new or unknown batches in the case of the balanced one - way random effects model, the two quantiles \tilde{q}_ℓ and \tilde{q}_u need to be determined. The two quantiles \tilde{q}_ℓ and \tilde{q}_u are determined by calculating

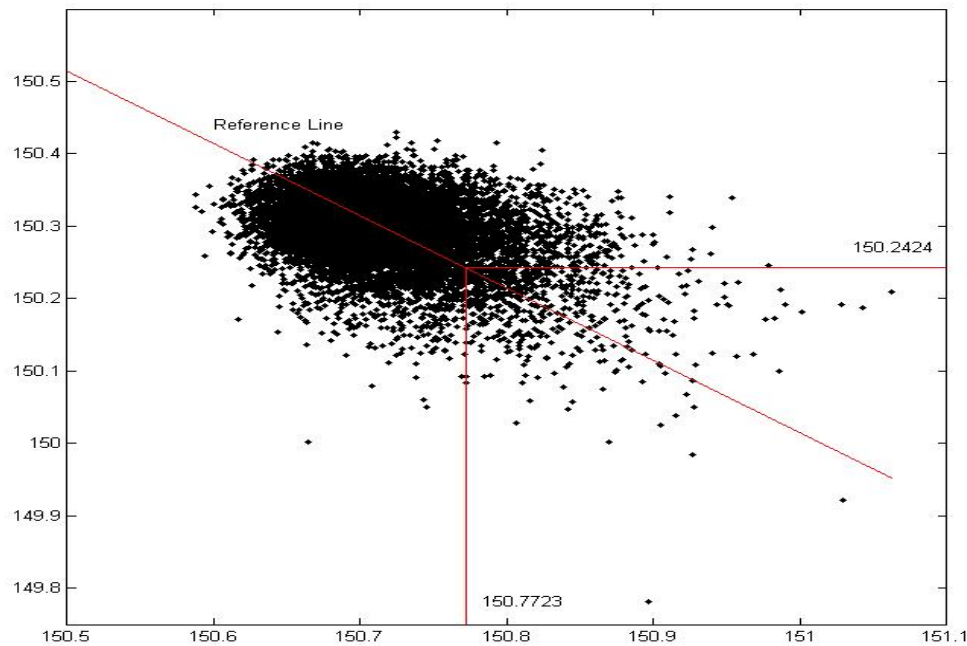
1. $\tilde{q}_u = \mu + z_{\frac{1+\alpha}{2}} \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$, and
2. $\tilde{q}_\ell = \mu - z_{\frac{1+\alpha}{2}} \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$

where $z_{\frac{1+\alpha}{2}}$ represents the $100(\frac{1+\alpha}{2})^{th}$ percentile of a standard normal distribution. Remember, as Wolfinger (1998) indicated, these $(\tilde{q}_\ell, \tilde{q}_u)$ pairs form a sample from the bivariate posterior distribution of the $[\frac{(1-\alpha)}{2}]^{th}$ and $[\frac{(1+\alpha)}{2}]^{th}$ quantiles, with the highest posterior density region typically forming a two - dimensional ellipsoid, since the two quantiles \tilde{q}_ℓ and \tilde{q}_u do not have a posterior correlation equal to 1.

Following Wolfinger (1998), it is therefore suggested that a valid two - sided (α, δ) tolerance interval for the average of observations from new or unknown batches in the case of the balanced one - way random effects model be constructed as follows:

- i.) Simulate $(\sigma_\varepsilon^2 + k\sigma_a^2)$ using the Bayesian simulation method described in method 1, or using method 2, simulate a pair of variance components σ_a^2 and σ_ε^2 using the Bayesian simulation technique explained in section 3.6, ensuring that the condition stating that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ is met. It should be remembered that if the above condition is not met, the simulated pair of variance components be disregarded and a new pair be simulated.
- ii.) Substitute the retained simulated pair of variance components or $(\sigma_\varepsilon^2 + k\sigma_a^2)$ into equation 4.4.2 to simulate a value for the target value μ .
- iii.) Using the simulated retained pair of variance components σ_a^2 and σ_ε^2 or $(\sigma_\varepsilon^2 + k\sigma_a^2)$ and the target value μ , simulate the values for $\tilde{q}_\ell = \mu - z_{\frac{1+\alpha}{2}} \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ and $\tilde{q}_u = \mu + z_{\frac{1+\alpha}{2}} \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and plot a scatterplot of the \tilde{q}_ℓ and \tilde{q}_u simulated values with \tilde{q}_ℓ plotted on the vertical axis.
- v.) For the simulated \tilde{q}_ℓ and \tilde{q}_u values, construct a reference line given by $\tilde{q}_\ell = -\tilde{q}_u + 2\bar{y}_..$ and draw the reference line on the scatterplot.

Figure 4.6.3: Construction of a Two - Sided (0.95, 0.95) Tolerance Interval for the Average of Observations from New or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1.



- vi.) Construct two additional lines, one parallel to each axis and intersecting on the reference line. This intersection point is then slid along the reference line until $100(1 - \delta)\%$ of the $(\tilde{q}_\ell, \tilde{q}_u)$ pairs are contained in the half rectangle opening towards the lower right portion of the graph. The resulting coordinates of this intersection point will then form a two - sided (α, δ) tolerance interval of the desired form.

This procedure as proposed by Wolfinger (1998) is graphically illustrated in Figure 4.6.3 for a two - sided $(\alpha = 0.95, \delta = 0.95)$ tolerance interval constructed for the medicinal tablets data given in Table 3.1, where $z_{\frac{1+0.95}{2}}$ used for determining the \tilde{q}_ℓ and \tilde{q}_u values, is equal to 1.96.

The two - sided $(\alpha = 0.95, \delta = 0.95)$ tolerance interval for the average of observations from new or unknown batches is equal to $[150.2424, 150.7723]$ and, can be interpreted as follows: If medicinal tablets are manufactured, 95% of the average weights of the

active ingredient present in new or unknown batches will fall between 150.2424 mg and 150.7723 mg with probability 0.95.

4.6.3 α - Expectation Tolerance Interval

It was already mentioned in section 4.2 that the predictive distribution of the average of k future observations $y_{f_1} \dots y_{f_k}$ from a new or unknown batch, given the variance components, follows a normal distribution given by

$$p(\bar{y}_f | \sigma_a^2, \sigma_\varepsilon^2, \mathbf{y}) \sim N\left(\mu, \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}\right)$$

where $\bar{y}_f = \frac{1}{k} \sum_{j=1}^k y_{f_j}$.

In order to construct an α - expectation tolerance interval for the average of future observations from new or unknown batches in the case of the random effects model, simulations need to be conducted from an appropriate predictive distribution $p(\bar{y}_f | \mathbf{y})$, where \bar{y}_f represents the average of future observations from new or unknown batches. Following Wolfinger (1998), the Bayesian simulation procedure proposed is as follows:

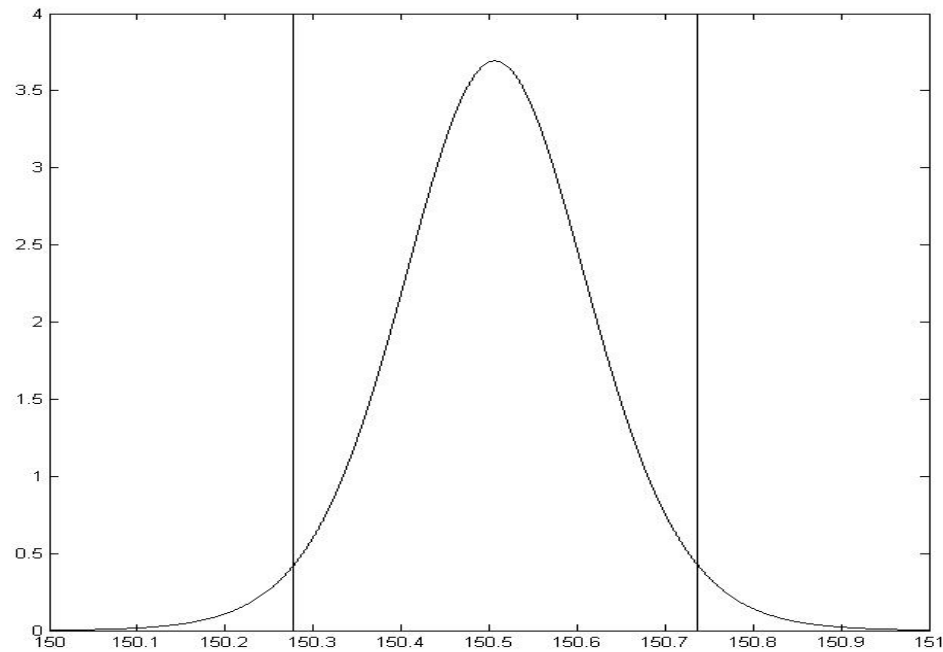
- i.) Simulate $(\sigma_\varepsilon^2 + k\sigma_a^2)$ using the Bayesian simulation method described in method 1, or simulate a pair of variance components σ_a^2 and σ_ε^2 subject to the condition stating that $(\sigma_\varepsilon^2 + k\sigma_a^2) > \sigma_\varepsilon^2$ as explained in section 3.6. This condition will ensure that σ_a^2 is positive. If this condition is not met, disregard the simulated pair of variance components and simulate a new pair.
- ii.) Substitute either the retained pair of variance components σ_a^2 and σ_ε^2 obtained using method 2 discussed in section 4.4, or $(\sigma_\varepsilon^2 + k\sigma_a^2)$ simulated using method 1 also discussed in section 4.4, into equation 4.4.2 to simulate a target value μ .

- iii.) Substitute the simulated variance components σ_a^2 and σ_ε^2 or $(\sigma_\varepsilon^2 + k\sigma_a^2)$, as well as the target value μ , into the predictive distribution of the average of future observations from new or unknown batches given above and in section 4.2. Simulate a \bar{y}_f value from this normal distribution, and draw the normal distribution given above.
- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and obtain the average conditional predictive distribution using the Rao Blackwell argument discussed in section 2.5. This average distribution represents an estimate of the unconditional predictive distribution of the average of future observations from new or unknown batches $p(\bar{y}_f | \mathbf{y})$.

Using method 1 discussed in section 4.4, the estimated unconditional predictive distribution of the average of future measurements from new or unknown batches $p(\bar{y}_f | \mathbf{y})$ was determined for the medicinal tablets data given in Table 3.1, and, is depicted in Figure 4.6.4. The two vertical lines also depicted in Figure 4.6.4 represents the 2.5th and 97.5th percentiles of the estimated unconditional predictive distribution of the average of future observations from new or unknown batches, and in fact, represent the estimated $\alpha = 0.95$ - expectation tolerance interval for this average of future observations from new or unknown batches.

The estimated $\alpha = 0.95$ - expectation tolerance interval is equal to [150.2775, 150.7361] for the medicinal tablets data given in Table 3.1 and can be interpreted as follows: The process used for manufacturing the small batches of medicinal tablets will be in control if 95% of the average weights of the active ingredient present in tablets manufactured in new or unknown batches, is between 150.2775 mg and 150.7361 mg.

Figure 4.6.4: Estimated Unconditional Predictive Distribution of the Average Weights of the Amount of Active Ingredient Present in Newly Manufactured or Unknown Batches for the Medicinal Tablets Data Given in Table 3.1. Obtained using Method 1.



0.95 - Expectation Tolerance Interval: [150.2775 , 150.7361]

4.6.4 Probability Matching Prior for the Content c of the Fixed - in - Advance Tolerance Interval

Wolfinger (1998) mentioned that the content of a Bayesian fixed - in - advance tolerance interval is determined by constructing the posterior distribution of this content. As an example, suppose that an upper fixed - in - advance limit s is specified for averages of data assuming to arise from new batches. Then

$$c = 1 - \Phi \left[\frac{s - \mu}{\sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}} \right]$$

must be computed, where as mentioned, $\Phi[\cdot]$ represents a standard normal cumulative distribution function.

The following theorem can now be proved.

Theorem 4.6.1

For the balanced one - way random effect model, the prior distribution

$$\pi(\mu, \sigma_\varepsilon^2, \sigma_a^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{3}{2}} \left\{ 1 + \frac{k(s - \mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\}^{-\frac{1}{2}} \quad (4.6.1)$$

is a probability matching prior for the content c given by

$$c = 1 - \Phi \left[\frac{s - \mu}{\sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}} \right]$$

of a fixed - in - advance tolerance interval, where $\Phi \left[\frac{s - \mu}{\sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}} \right]$ represents the standard normal cumulative distribution function.

Proof

The proof of Theorem 4.6.1 is given in Appendix B.

Equation 4.6.1 is also a probability matching prior for $1 - \Phi \left[\frac{\mu - s}{\sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}} \right]$ if s is a lower specification limit.

4.6.4.1 The Weighted Monte Carlo Method

In this section a weighted Monte Carlo method is described which will be used to simulate from the posterior distribution in the case of the probability matching prior given in equation 4.6.1. As mentioned, this method is especially suitable for computing Bayesian confidence intervals. Chen and Shao's (1999) method (See also Kim (2006)) does not require knowing the closed form of the marginal posterior distribution of c , only the kernel of the posterior distribution of $(\mu, \sigma_a^2, \sigma_\varepsilon^2)$ is needed.

The weighted Monte Carlo (sampling - importance resampling (SIR)) algorithm aims at drawing a random sample from a target distribution π by first drawing a sample from a proposal distribution γ , and from this a smaller sample is drawn with sample probabilities proportional to the importance ratios $\frac{\pi}{\gamma}$. As also mentioned before, in the case of the credibility intervals it is not even necessary to draw the smaller sample. The weights (sample probabilities) are however important.

For the non - informative prior distribution given in equation 4.2.2, i.e.

$$p_R(\mu, \sigma_a^2, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1},$$

the joint posterior distribution of the unknown parameters μ , σ_a^2 and σ_ε^2 is given by

$$p_R(\mu, \sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}(\nu_2+3)} \\ \times \exp \left\{ -\frac{1}{2} \left[\frac{kb(\mu - \bar{y})^2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} + \frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right] \right\}. \quad (4.6.2)$$

Equation 4.6.2 represents the proposal distribution γ .

In the case of the probability matching prior given by

$$p_M(\mu, \sigma_a^2, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{3}{2}} \left\{ 1 + \frac{k(s - \mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\}$$

and provided in equation 4.6.1, the joint posterior distribution of the unknown parameters μ , σ_a^2 and σ_ε^2 is given by

$$p_M(\mu, \sigma_a^2, \sigma_\varepsilon^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}(\nu_2+4)} \times \left\{ 1 + \frac{k(s-\mu)^2}{2(\sigma_\varepsilon^2+k\sigma_a^2)} \right\}^{-\frac{1}{2}} \\ \times \exp \left\{ -\frac{1}{2} \left[\frac{kb(\mu - \bar{y})^2}{\sigma_\varepsilon^2 + k\sigma_a^2} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} + \frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} \right] \right\}. \quad (4.6.3)$$

Equation 4.6.3 represents the target distribution π . It is important that γ is a good approximation of π , i.e. that it does not have tails that are too thin. The sample probabilities are therefore proportional to

$$\frac{\pi}{\gamma} = \frac{p_M(\mu, \sigma_a^2, \sigma_\varepsilon^2)}{p_R(\mu, \sigma_a^2, \sigma_\varepsilon^2)} = (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}} \left\{ 1 + \frac{k(s - \mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\}^{-\frac{1}{2}}$$

with the resulting normalized weights for $l = 1, 2, \dots, \tilde{\ell}$ given by

$$w_l = \frac{\left(\sigma_\varepsilon^{2(l)} + k\sigma_a^{2(l)}\right)^{-\frac{1}{2}} \left\{1 + \frac{k(s-\mu^{(l)})^2}{2(\sigma_\varepsilon^{2(l)} + k\sigma_a^{2(l)})}\right\}^{-\frac{1}{2}}}{\sum_{l=1}^{\tilde{\ell}} \left(\sigma_\varepsilon^{2(l)} + k\sigma_a^{2(l)}\right)^{-\frac{1}{2}} \left\{1 + \frac{k(s-\mu^{(l)})^2}{2(\sigma_\varepsilon^{2(l)} + k\sigma_a^{2(l)})}\right\}^{-\frac{1}{2}}}. \quad (4.6.4)$$

Using the weighted Monte Carlo method (sampling - importance resampling method), the fixed - in - advance tolerance interval for the probability matching prior given in equation 4.6.1 can be obtained as follows:

Step 1

Using the Bayesian simulation method described in method 1 given in section 4.4, obtain a Monte Carlo sample $\{(\mu^{(l)}, (\sigma_\varepsilon^2 + \sigma_a^2)^{(l)}, l = 1, \dots, \tilde{\ell})\}$ from the proposal distribution γ and calculate the content c^* of the interval $[-\infty, s]$ given by

$$c^* = \Phi \left[\frac{s - \mu^{(l)}}{\sqrt{\frac{(\sigma_\varepsilon^2 + k\sigma_a^2)^{(l)}}{k}}} \right] \quad (\text{for } l = 1, 2, \dots, \tilde{\ell}),$$

since the main focus will be on the content less than the specified lower specification limit s .

Step 2

Sort the $\tilde{\ell} \{c_i^*, l = 1, 2, \dots, \tilde{\ell}\}$ values to obtain the ordered values

$$c_{(1)}^* \leq c_{(2)}^* \leq \dots \leq c_{(\tilde{\ell})}^*.$$

Step 3

Each simulated c_i^* ($l = 1, 2, \dots, \tilde{\ell}$) value has an associated weight. Therefore compute the weighted function $w_{(l)}$ associated with each of the l^{th} ordered $c_{(l)}^*$ values using equation 4.6.4.

Step 4

Sum the weights associated with each $c_{(l)}^*$ value from left to right (small to large) until $\sum_{l=1}^{k_1} w_{(l)} = \frac{1-\alpha}{2}$. Write down the corresponding ordered $c_{(k_1)}^*$ value and denote it as $c^{*\frac{1-\alpha}{2}}$. Also, obtain the sum of the weights associated with each $c_{(l)}^*$ value from left to right until you get $\sum_{l=1}^{k_2} w_{(l)} = \frac{1+\alpha}{2}$. Write down the corresponding ordered value $c_{(k_2)}^*$ and denote it as $c^{*\frac{1+\alpha}{2}}$.

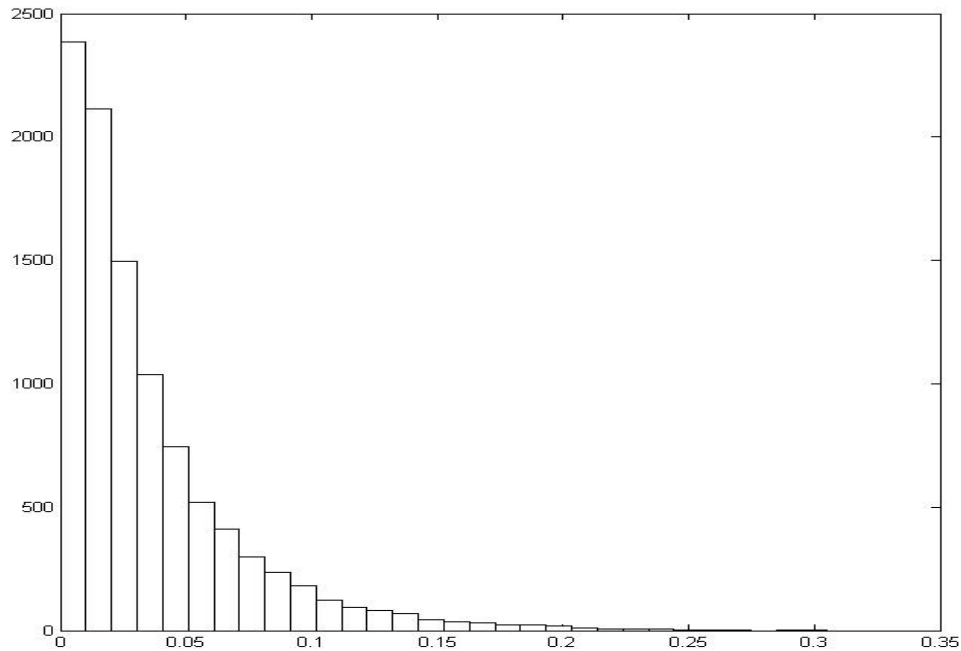
Step 5

The $100(\alpha)\%$ Bayesian confidence interval or $100(\alpha)\%$ Bayesian fixed - in - advance tolerance interval is then given by $\left[c^{*\frac{1-\alpha}{2}}, c^{*\frac{1+\alpha}{2}} \right]$.

As mentioned, for the medicinal tablets data given in Table 3.1, the lower specification limit selected was $s = 150.30$ mg. Since we consider a lower specification limit, the fixed - in - advance tolerance interval in this case estimates the proportion of batches not meeting the minimum average dose specification. This therefore represents the proportion of batches with an average weight of the active ingredient that is below the minimum specified limit of 150.30 mg. Since the average content of these batches is below the minimum, this proportion of batches will not be effective as treatment. The posterior distribution of the content of the interval $[-\infty, s = 150.30]$ was determined for the medicinal tablets data given in Table 3.1 and is depicted in Figure 4.6.5.

From Figure 4.6.5 it can be seen that the posterior distribution of the content of the interval $[-\infty, s = 150.30$ mg] is positively skewed. Also, for illustrative purposes, using the results obtained from the weighted Monte Carlo method, the 95% equal tail credibility interval of the posterior distribution of the content of the interval $[-\infty, s = 150.30$ mg] obtained for the average active ingredient weight of manufactured tablets from new or unknown batches, is given by $[0.001736, 0.13337]$. This 95% equal tail credibility interval, also represents the 95% fixed - in - advance tolerance interval.

Figure 4.6.5: Histogram of the Posterior Distribution of the Content of the Interval $[-\infty, 150.30]$, i.e. the Proportion of Batches with an Average Active Ingredient Weight that is Below the Specified Fixed - in - Advance Lower Limit $s = 150.30$ mg for the Medicinal Tablets Data Given in Table 3.1.



95% Fixed - in - Advance Tolerance Interval: $[0.001768, 0.135048]$

Comparative results obtained using the ordinary Monte Carlo method and weighted Monte Carlo method for obtaining the fixed - in - advance tolerance interval for averages of observations from new or unknown batches using a lower specification limit $s = 150.30$ mg, is provided in Table 4.2.

Table 4.2: Fixed - in - Advanced Tolerance Intervals for Averages of Observations from New or Unknown Batches for a Lower Specification Limit $s = 150.30$ mg.

	Monte Carlo Method		Weighted Monte Carlo Method	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Observation Number	250	9750	246	9737
Sum of Weights	—	—	0.02495	0.97498
95% Credibility Interval	0.001768	0.135048	0.001736	0.13337

From Table 4.2 it can be seen that the ordinary Monte Carlo method and the weighted Monte Carlo method provide similar fixed - in - advanced tolerance intervals for a lower specification limit $s = 150.30$ mg. Using results obtained for the weighted Monte Carlo method, it can therefore be interpreted that with 95% confidence, between 0.1736% and 13.337% of the batches manufactured will on average not contain sufficient active ingredient to render the drug effective. As mentioned, this lower specification limit $s = 150.30$ mg was selected for illustrative purposes only.

4.7 Appendix B

Proof of Theorem 4.3.1

Probability Matching Prior

It is well known that for the balanced random effects model, the Fisher information matrix for the parameters μ , σ_a^2 and σ_ε^2 is given by

$$F(\mu, \sigma_a^2, \sigma_\varepsilon^2) = F(\tilde{\boldsymbol{\mu}}) = \begin{bmatrix} \frac{bk}{\sigma_\varepsilon^2 + k\sigma_a^2} & 0 & 0 \\ 0 & \frac{bk^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} & \frac{bk}{2(\sigma_\varepsilon^2 + k\sigma_a^2)^2} \\ 0 & \frac{bk}{2(\sigma_\varepsilon^2 + k\sigma_a^2)^2} & \frac{b(k-1)}{2(\sigma_\varepsilon^2)^2} + \frac{b}{2(\sigma_\varepsilon^2 + k\sigma_a^2)^2} \end{bmatrix} \quad (4.7.1)$$

with its inverse given by

$$F^{-1}(\mu, \sigma_a^2, \sigma_\varepsilon^2) = F^{-1}(\tilde{\boldsymbol{\mu}}) = \begin{bmatrix} \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} & 0 & 0 \\ 0 & \frac{2\{(k-1)(\sigma_\varepsilon^2 + k\sigma_a^2)^2 + (\sigma_\varepsilon^2)^2\}}{bk^2(k-1)} & \frac{-2(\sigma_\varepsilon^2)^2}{bk(k-1)} \\ 0 & \frac{-2(\sigma_\varepsilon^2)^2}{bk(k-1)} & \frac{-2(\sigma_\varepsilon^2)^2}{b(k-1)} \end{bmatrix}. \quad (4.7.2)$$

Define $t(\tilde{\boldsymbol{\mu}}) = \tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$.

Now,

$$\frac{\partial t(\tilde{\boldsymbol{\mu}})}{\partial \mu} = 1,$$

$$\frac{\partial t(\tilde{\boldsymbol{\mu}})}{\partial \sigma_a^2} = z_\alpha (k)^{\frac{1}{2}} \frac{1}{2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}},$$

$$\frac{\partial t(\tilde{\boldsymbol{\mu}})}{\partial \sigma_\varepsilon^2} = z_\alpha \left(\frac{1}{k}\right)^{\frac{1}{2}} \frac{1}{2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}},$$

and

$$\nabla'_t(\tilde{\boldsymbol{\mu}}) = \left[1 \quad z_\alpha (k)^{\frac{1}{2}} \frac{1}{2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}} \quad z_\alpha \left(\frac{1}{k}\right)^{\frac{1}{2}} \frac{1}{2} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}} \right].$$

Further

$$\nabla'_t(\tilde{\boldsymbol{\mu}}) F^{-1}(\tilde{\boldsymbol{\mu}}) = \left[\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} \quad \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)^{\frac{3}{2}}}{bk^{\frac{3}{2}}} \quad 0 \right]$$

and

$$\nabla'_t(\tilde{\boldsymbol{\mu}})F^{-1}(\tilde{\boldsymbol{\mu}})\nabla_t(\tilde{\boldsymbol{\mu}}) = \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} \left(1 + \frac{1}{2}z_\alpha^2\right).$$

Also

$$\boldsymbol{\eta}(\tilde{\boldsymbol{\mu}}) = \frac{\nabla'_t(\tilde{\boldsymbol{\mu}})F^{-1}(\tilde{\boldsymbol{\mu}})}{\sqrt{\nabla'_t(\tilde{\boldsymbol{\mu}})F^{-1}(\tilde{\boldsymbol{\mu}})\nabla_t(\tilde{\boldsymbol{\mu}})}} = \begin{bmatrix} \eta_1(\tilde{\boldsymbol{\mu}}) & \eta_2(\tilde{\boldsymbol{\mu}}) & \eta_3(\tilde{\boldsymbol{\mu}}) \end{bmatrix}$$

where

$$\eta_1(\tilde{\boldsymbol{\mu}}) = \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)^{\frac{1}{2}}}{(bk)^{\frac{1}{2}}} \left(1 + \frac{1}{2}z_\alpha^2\right)^{-\frac{1}{2}},$$

$$\eta_2(\tilde{\boldsymbol{\mu}}) = \frac{(\sigma_\varepsilon^2 + k\sigma_a^2)}{b^{\frac{1}{2}}k} \left(1 + \frac{1}{2}z_\alpha^2\right)^{-\frac{1}{2}}, \text{ and}$$

$$\eta_3(\tilde{\boldsymbol{\mu}}) = 0.$$

The prior distribution $p(\tilde{\boldsymbol{\mu}}) = p(\mu, \sigma_a^2, \sigma_\varepsilon^2)$ is a probability matching prior if the following differential equation is satisfied:

$$\frac{\partial\{\eta_1(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\mu} + \frac{\partial\{\eta_2(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\sigma_a^2} + \frac{\partial\{\eta_3(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\sigma_\varepsilon^2} = 0.$$

Now, take $p(\tilde{\boldsymbol{\mu}}) \propto \sigma_\varepsilon^{-2x} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1}$, then

$$\text{a) } \frac{\partial\{\eta_1(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\mu} = 0,$$

$$\text{b) } \frac{\partial\{\eta_2(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\sigma_a^2} = 0,$$

and

$$\text{c) } \frac{\partial\{\eta_3(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\sigma_\varepsilon^2} = 0.$$

It therefore follows that

$$\frac{\partial\{\eta_1(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\mu} + \frac{\partial\{\eta_2(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\sigma_a^2} + \frac{\partial\{\eta_3(\tilde{\boldsymbol{\mu}})p(\tilde{\boldsymbol{\mu}})\}}{\partial\sigma_\varepsilon^2} = 0.$$

If we take $x = 1$, then $p(\mu, \sigma_a^2, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + k\sigma_a^2)^{-1}$ is a probability - matching prior for the parameter $\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$.

Reference Prior

In the case of the reference prior, the Fisher information matrix for the parameters $t(\tilde{\mu})$, ν and σ_ε^2 , where $\nu = \frac{\sigma_a^2}{\sigma_\varepsilon^2}$ and $t(\tilde{\mu}) = \tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ must first be obtained. This will be done in two stages. In the first stage the Fisher information matrix for μ , ν and σ_ε^2 will be derived and in the second stage, it will be derived for $t(\tilde{\mu})$, ν and σ_ε^2 .

Let

$$A = \frac{\partial(\mu, \sigma_a^2, \sigma_\varepsilon^2)}{\partial(\mu, \nu, \sigma_\varepsilon^2)} \text{ where } \nu = \frac{\sigma_a^2}{\sigma_\varepsilon^2} \text{ and } \sigma_a^2 = \nu \sigma_\varepsilon^2,$$

then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & \nu \\ 0 & 0 & 1 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & 0 \\ 0 & \nu & 1 \end{bmatrix}$$

and

$$F(\mu, \nu, \sigma_\varepsilon^2) = A' F(\mu, \sigma_a^2, \sigma_\varepsilon^2) A.$$

For the Fisher information matrix, $F(\mu, \nu, \sigma_\varepsilon^2)$, as defined in equation 4.7.2, σ_a^2 must be substituted by $\nu \sigma_\varepsilon^2$.

Therefore

$$\begin{aligned} F(\mu, \nu, \sigma_\varepsilon^2) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & \nu \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{bk}{\sigma_\varepsilon^2(1+k\nu)} & 0 & 0 \\ 0 & \frac{bk^2}{2(\sigma_\varepsilon^2)^2(1+k\nu)^2} & \frac{bk}{2(\sigma_\varepsilon^2)^2(1+k\nu)^2} \\ 0 & \frac{bk}{2(\sigma_\varepsilon^2)^2(1+k\nu)^2} & \frac{b(k-1)}{2(\sigma_\varepsilon^2)^2} + \frac{b}{2(\sigma_\varepsilon^2)^2(1+k\nu)^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & 0 \\ 0 & \nu & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{bk}{\sigma_\varepsilon^2(1+k\nu)} & 0 & 0 \\ 0 & \frac{bk^2}{2(1+k\nu)^2} & \frac{bk}{2(\sigma_\varepsilon^2)(1+k\nu)} \\ 0 & \frac{bk}{2(\sigma_\varepsilon^2)(1+k\nu)} & \frac{b}{2(\sigma_\varepsilon^2)^2} \end{bmatrix}. \end{aligned}$$

Now

$$t(\tilde{\boldsymbol{\mu}}) = \mu + z_{\alpha} \sqrt{\frac{\sigma_{\varepsilon}^2(1+k\nu)}{k}}$$

and

$$\mu = t(\tilde{\boldsymbol{\mu}}) - z_{\alpha} \sqrt{\frac{\sigma_{\varepsilon}^2(1+k\nu)}{k}} .$$

Also

$$\frac{\partial \mu}{\partial t(\tilde{\boldsymbol{\mu}})} = 1,$$

$$\frac{\partial \mu}{\partial \nu} = -\frac{1}{2} z_{\alpha} (\sigma_{\varepsilon}^2)^{\frac{1}{2}} k^{\frac{1}{2}} (1+k\nu)^{-\frac{1}{2}},$$

$$\frac{\partial \mu}{\partial \sigma_{\varepsilon}^2} = -\frac{1}{2} z_{\alpha} (\sigma_{\varepsilon}^2)^{-\frac{1}{2}} k^{-\frac{1}{2}} (1+k\nu)^{\frac{1}{2}},$$

and

$$\tilde{A} = \begin{bmatrix} \frac{\partial \mu}{\partial t(\tilde{\boldsymbol{\mu}})} & \frac{\partial \mu}{\partial \nu} & \frac{\partial \mu}{\partial \sigma_{\varepsilon}^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The Fisher information matrix for the parameters $t(\tilde{\boldsymbol{\mu}}), \nu, \sigma_{\varepsilon}^2$ is therefore given by

$$F(t(\tilde{\boldsymbol{\mu}}, \nu, \sigma_{\varepsilon}^2) = \tilde{A}' F(\mu, \nu, \sigma_{\varepsilon}^2) \tilde{A} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$$

where

$$F_{11} = \frac{bk}{\sigma_{\varepsilon}^2(1+k\nu)} ,$$

$$F_{12} = F_{22} = -\frac{1}{2} z_{\alpha} (\sigma_{\varepsilon}^2)^{-\frac{1}{2}} bk^{\frac{3}{2}} (1+k\nu)^{-\frac{3}{2}} ,$$

$$F_{13} = F_{31} = -\frac{1}{2} z_{\alpha} (\sigma_{\varepsilon}^2)^{-\frac{3}{2}} bk^{\frac{1}{2}} (1+k\nu)^{-\frac{1}{2}} ,$$

$$F_{23} = F_{32} = \frac{1}{2} bk^2 (\sigma_{\varepsilon}^2)^{-1} (1+k\nu)^{-1} \left(\frac{1}{2} z_{\alpha}^2 + 1 \right) .$$

$$F_{22} = \frac{1}{2}bk^2(1+k\nu)^{-2} \left(\frac{1}{2}z_\alpha^2 + 1 \right)$$

and

$$F_{33} = \frac{1}{2}b(\sigma_\varepsilon^2)^{-2} \left(\frac{1}{2}z_\alpha^2 + k \right).$$

Consider the submatrix

$$\tilde{F} = \begin{bmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{bmatrix}.$$

The inverse of this submatrix is

$$\tilde{F}^{-1} = \begin{bmatrix} \frac{2(k+\frac{1}{2}z_\alpha^2)(1+k\nu)^2}{bk^2(k-1)(\frac{1}{2}z_\alpha^2+1)} & \frac{-2(\sigma_\varepsilon^2)(1+k\nu)}{bk(k-1)} \\ \frac{-2(\sigma_\varepsilon^2)(1+k\nu)}{bk(k-1)} & \frac{2(\sigma_\varepsilon^2)^2}{b(k-1)} \end{bmatrix}.$$

Now

$$\begin{aligned} h_1 = F_{11.2} &= F_{11} - \begin{bmatrix} F_{12} & F_{13} \end{bmatrix} \tilde{F}^{-1} \begin{bmatrix} F_{21} \\ F_{31} \end{bmatrix} \\ &= \frac{bk}{\sigma_\varepsilon^2(1+k\nu)} - \begin{bmatrix} \frac{(1+k\nu)^{\frac{1}{2}}}{(\sigma_\varepsilon^2)^{\frac{1}{2}}k^{\frac{1}{2}}(k-1)} \left\{ z_\alpha^2 - \frac{(k+\frac{1}{2}z_\alpha^2)}{(\frac{1}{2}z_\alpha^2+1)} \right\} & 0 \end{bmatrix} \begin{bmatrix} F_{21} \\ F_{31} \end{bmatrix} \\ &= \frac{bk}{\sigma_\varepsilon^2(1+k\nu)} - \left\{ 1 + \frac{1}{2(k-1)} \left[z_\alpha^2 - \frac{(k+\frac{1}{2}z_\alpha^2)}{(\frac{1}{2}z_\alpha^2+1)} \right] \right\}. \end{aligned}$$

Note, h_1 does not contain $t(\tilde{\mu})$.

Also

$$\begin{aligned} H &= \begin{bmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{bmatrix}^{-1} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \\ &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} - \frac{1}{F_{33}} \begin{bmatrix} F_{13} \\ F_{23} \end{bmatrix} \begin{bmatrix} F_{31} & F_{32} \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} H_{22} &= \frac{bk^2(\frac{1}{2}z_\alpha^2+1)}{2(1+k\nu)^2} - \frac{2(\sigma_\varepsilon^2)^2}{b(k+\frac{1}{2}z_\alpha^2)} \frac{b^2k^2(\frac{1}{2}z_\alpha^2+1)^2}{4(\sigma_\varepsilon^2)^2(1+k\nu)^2} \\ &= \frac{bk^2(\frac{1}{2}z_\alpha^2+1)}{2(1+k\nu)^2} \left\{ 1 - \frac{(\frac{1}{2}z_\alpha^2+1)^2}{(k+\frac{1}{2}z_\alpha^2)} \right\} = h_2, \end{aligned}$$

and

$$h_3 = \frac{1}{2}b(\sigma_\varepsilon^2)^{-2}(k + \frac{1}{2}z_\alpha^2).$$

From this it follows that

$$p(t(\tilde{\boldsymbol{\mu}})) \propto h_1^{\frac{1}{2}} = 1 \text{ (because } h_1 \text{ does not contain } t(\tilde{\boldsymbol{\mu}})),$$

$$p(\nu|t(\tilde{\boldsymbol{\mu}})) \propto h_2^{\frac{1}{2}} = (1+k\nu)^{-1},$$

and

$$p(\sigma_\varepsilon^2|t(\tilde{\boldsymbol{\mu}}), \nu) \propto h_3^{\frac{1}{2}} = (\sigma_\varepsilon^2)^{-1}.$$

The reference prior relative to the ordered parameterization $(t(\tilde{\boldsymbol{\mu}}), \nu, \sigma_\varepsilon^2)$ is therefore given by

$$\begin{aligned} p(t(\tilde{\boldsymbol{\mu}}), \nu, \sigma_\varepsilon^2) &= p(t(\tilde{\boldsymbol{\mu}}))p(\nu|t(\tilde{\boldsymbol{\mu}}))p(\sigma_\varepsilon^2|t(\tilde{\boldsymbol{\mu}}), \nu) \\ &\propto (1+k\nu)^{-1}\sigma_\varepsilon^{-2}. \end{aligned}$$

Now,

$$t(\tilde{\boldsymbol{\mu}}) = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2(1+k\nu)}{k}},$$

$$\frac{\partial t(\tilde{\boldsymbol{\mu}})}{\partial \mu} = 1, \nu = \frac{\sigma_a^2}{\sigma_\varepsilon^2} \text{ and } \frac{\partial \nu}{\partial \sigma_a^2} = \frac{1}{\sigma_\varepsilon^2}.$$

From this it follows that the reference prior of the $(t(\tilde{\boldsymbol{\mu}}), \sigma_a^2, \sigma_\varepsilon^2)$ parameterization is given by

$$\begin{aligned} p(t(\tilde{\boldsymbol{\mu}}), \sigma_a^2, \sigma_\varepsilon^2) &\propto (1 + k\nu)^{-1} \sigma_\varepsilon^{-2} (\sigma_\varepsilon^{-2}) & (4.7.3) \\ &= \left(\sigma_\varepsilon^2 + k \frac{\sigma_a^2}{\sigma_\varepsilon^2} \cdot \sigma_\varepsilon^2 \right)^{-1} (\sigma_\varepsilon^{-2}) . \\ &= (\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} \sigma_\varepsilon^{-2} . \end{aligned}$$

In a similar manner it can also be shown that equation 4.7.3 is also a reference prior for the ordered parameterization $(\mu, \sigma_a^2, \sigma_\varepsilon^2)$.

Proof of Theorem 4.5.1

For Method 1

Since $\mu | \sigma_\varepsilon^2, \sigma_a^2, \mathbf{y} \sim N(\bar{y}_{..}; \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{kb})$, the posterior distribution of $\tilde{q} = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$

conditional on σ_ε^2 and σ_a^2 , is normal with mean $\bar{y}_{..} + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}$ and variance $\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{kb}$.

Since

$$\frac{\nu_2 m_2}{\sigma_\varepsilon^2 + k\sigma_a^2} \sim \chi_{\nu_2}^2 \text{ and}$$

$$E \left\{ \left(\chi_{\nu_2}^2 \right) \right\}^r = \frac{2^r \Gamma(\frac{\nu_2}{2} + r)}{\Gamma(\frac{\nu_2}{2})}, \text{ it follows that}$$

$$\begin{aligned} E(\tilde{q} | \mathbf{y}) &= \bar{y}_{..} + z_\alpha \frac{1}{k^{\frac{1}{2}}} (\nu_2 m_2)^{\frac{1}{2}} E \left(\frac{1}{\chi_{\nu_2}^2} \right)^{\frac{1}{2}} \\ &= \bar{y}_{..} + z_\alpha \frac{1}{k^{\frac{1}{2}}} (\nu_2 m_2)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2 - 1}{2})}{2^{\frac{1}{2}} \Gamma(\frac{\nu_2}{2})} . \end{aligned}$$

Also

$$Var(\tilde{q} | \mathbf{y}, \sigma_\varepsilon^2 + k\sigma_a^2) = \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{kb}$$

and

$$Var(\tilde{q}|\mathbf{y}) = E \{Var(\tilde{q}|\mathbf{y}, \sigma_\varepsilon^2 + k\sigma_a^2) + Var \{E(\tilde{q}|\mathbf{y}, \sigma_\varepsilon^2 + k\sigma_a^2)\} \}.$$

Now

$$\begin{aligned} E \{Var(\tilde{q}|\mathbf{y}, \sigma_\varepsilon^2 + k\sigma_a^2)\} &= \frac{1}{bk} E(\sigma_\varepsilon^2 + k\sigma_a^2) \\ &= \frac{1}{bk} E \left(\frac{\nu_2 m_2}{\chi_{\nu_2}^2} \right) = \frac{\nu_2 m_2}{bk(\nu_2 - 2)} \end{aligned}$$

and

$$\begin{aligned} Var \{E(\tilde{q}|\mathbf{y}, \sigma_\varepsilon^2 + k\sigma_a^2)\} &= Var \left\{ \bar{y}_{..} + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}} \right\} \\ &= \frac{z_\alpha^2}{k} Var \left\{ \sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right\}. \end{aligned}$$

Further

$$Var \left\{ \sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right\} = E \left\{ \left(\sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right)^2 \right\} - \left\{ E \left(\sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right) \right\}^2.$$

Now

$$\begin{aligned} E \left\{ \left(\sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right)^2 \right\} &= E \{(\sigma_\varepsilon^2 + k\sigma_a^2)\} \\ &= E \left(\frac{\nu_2 m_2}{\chi_{\nu_2}^2} \right) = \frac{\nu_2 m_2}{\nu_2 - 2} \end{aligned}$$

and

$$\begin{aligned} E \left\{ \left(\sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right) \right\}^2 &= \left\{ \frac{(\nu_2 m_2)^{\frac{1}{2}} \Gamma(\frac{\nu_2 - 1}{2})}{\sqrt{2} \Gamma(\frac{\nu_2}{2})} \right\}^2 \\ &= \frac{(\nu_2 m_2) \Gamma^2(\frac{\nu_2 - 1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})}. \end{aligned}$$

Therefore, it follows that

$$Var \{E(\tilde{q}|\mathbf{y}, \sigma_\varepsilon^2 + k\sigma_a^2)\} = \frac{z_\alpha^2}{k} \left\{ \frac{\nu_2 m_2}{\nu_2 - 2} - \frac{(\nu_2 m_2) \Gamma^2(\frac{\nu_2 - 1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})} \right\}$$

and

$$Var(\tilde{q}|\mathbf{y}) = \frac{\nu_2 m_2}{bk(\nu_2 - 2)} + \frac{z_\alpha^2 \nu_2 m_2}{k} \left\{ \frac{1}{\nu_2 - 2} - \frac{\Gamma^2(\frac{\nu_2 - 1}{2})}{2 \Gamma^2(\frac{\nu_2}{2})} \right\}.$$

For Method 2

In Box and Tiao (1973) it is proved that for method 2

$$E \left\{ \left(\frac{1}{\sigma_\varepsilon^2 + k\sigma_a^2} \right)^r \right\} = \left(\frac{2}{\nu_2 m_2} \right)^r \frac{\Gamma(\frac{\nu_2}{2} + r)}{\Gamma(\frac{\nu_2}{2})} \frac{Pr \left\{ F_{\nu_2+2r, \nu_1} < \frac{\nu_2}{\nu_2+2r} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}}$$

where F_{ν_2, ν_1} represents an F - distribution with ν_2 and ν_1 degrees of freedom.

From this it follows that

$$E \left(\sqrt{\sigma_\varepsilon^2 + k\sigma_a^2} \right) = \left(\frac{\nu_2 m_2}{2} \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{\Gamma(\frac{\nu_2}{2})} \frac{Pr \left\{ F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}}$$

and

$$E(\sigma_\varepsilon^2 + k\sigma_a^2) = \left(\frac{\nu_2 m_2}{\nu_2 - 2} \right)^{\frac{1}{2}} \frac{Pr \left\{ F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}}.$$

Also

$$E(\tilde{q}|\mathbf{y}) = \bar{y}_{..} + z_\alpha \left(\frac{\nu_2 m_2}{2k} \right)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu_2-1}{2})}{\Gamma(\frac{\nu_2}{2})} \frac{Pr \left\{ F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}}$$

and

$$\begin{aligned} Var(\tilde{q}|\mathbf{y}) &= \frac{\nu_2 m_2}{bk(\nu_2-2)} \frac{Pr \left\{ F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}} \\ &+ \frac{z_\alpha^2 \nu_2 m_2}{k} \left\{ \frac{1}{(\nu_2-2)} \frac{Pr \left\{ F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}} - \frac{\Gamma^2(\frac{\nu_2-1}{2})}{2\Gamma^2(\frac{\nu_2}{2})} \left[\frac{Pr \left\{ F_{\nu_2-1, \nu_1} < \frac{\nu_2}{\nu_2-1} \frac{m_2}{m_1} \right\}}{Pr \left\{ F_{\nu_2, \nu_1} < \frac{m_2}{m_1} \right\}} \right]^2 \right\}. \end{aligned}$$

Proof of Theorem 4.6.1

The content c of the interval $[s, \infty]$ is defined as

$$c = 1 - \Phi \left[\frac{s - \mu}{\sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}} \right] = 1 - \Phi(\boldsymbol{\theta})$$

where $\Phi \left[\frac{s - \mu}{\sqrt{\frac{\sigma_\varepsilon^2 + k\sigma_a^2}{k}}} \right]$ represents the standard normal cumulative distribution function.

Now

$$\frac{\partial c}{\partial \mu} = \frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} k^{\frac{1}{2}} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}},$$

$$\frac{\partial c}{\partial \sigma_\varepsilon^2} = \frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} \frac{1}{2} (s - \mu) k^{\frac{3}{2}} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{3}{2}}$$

and

$$\frac{\partial c}{\partial \sigma_a^2} = \frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} \frac{1}{2} (s - \mu) k^{\frac{1}{2}} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{3}{2}}.$$

Therefore

$$\begin{aligned} \nabla'_t(\boldsymbol{\theta}) &= \begin{bmatrix} \frac{\partial c}{\partial \mu} & \frac{\partial c}{\partial \sigma_a^2} & \frac{\partial c}{\partial \sigma_\varepsilon^2} \end{bmatrix} \\ &= \frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} k^{\frac{1}{2}} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}} \begin{bmatrix} 1 & \frac{1}{2}(s - \mu)k(\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} & \frac{1}{2}(s - \mu)(\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} \end{bmatrix} \\ &= f \begin{bmatrix} 1 & \frac{1}{2}(s - \mu)k(\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} & \frac{1}{2}(s - \mu)(\sigma_\varepsilon^2 + k\sigma_a^2)^{-1} \end{bmatrix} \end{aligned}$$

where

$$f = \frac{e^{-\frac{1}{2}\theta^2}}{\sqrt{2\pi}} k^{\frac{1}{2}} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{1}{2}}.$$

Using the method of Datta and Ghosh (1995), it follows that

$$\nabla'_t(\boldsymbol{\theta})F^{-1}(\mu, \sigma_a^2, \sigma_\varepsilon^2) = f \begin{bmatrix} \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} & \frac{(s - \mu)(\sigma_\varepsilon^2 + k\sigma_a^2)}{bk} & 0 \end{bmatrix}$$

and

$$\{\nabla'_t(\boldsymbol{\theta})F^{-1}(\mu, \sigma_a^2, \sigma_\varepsilon^2)\nabla_t(\boldsymbol{\theta})\}^{\frac{1}{2}} = f \left\{ \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} + \frac{(s - \mu)^2}{2b} \right\}^{\frac{1}{2}}$$

where $F^{-1}(\mu, \sigma_a^2, \sigma_\varepsilon^2)$ is given in equation 4.7.2.

Therefore

$$\eta(\boldsymbol{\theta}) = \frac{\nabla'_t(\boldsymbol{\theta})F^{-1}(\mu, \sigma_a^2, \sigma_\varepsilon^2)\nabla_t(\boldsymbol{\theta})}{\{\nabla'_t(\boldsymbol{\theta})F^{-1}(\mu, \sigma_a^2, \sigma_\varepsilon^2)\nabla_t(\boldsymbol{\theta})\}^{\frac{1}{2}}} = \begin{bmatrix} \eta_1(\boldsymbol{\theta}) & \eta_2(\boldsymbol{\theta}) & \eta_3(\boldsymbol{\theta}) \end{bmatrix}$$

where

$$\eta_1(\boldsymbol{\theta}) = \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} \left\{ \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} + \frac{(s - \mu)^2}{2b} \right\}^{-\frac{1}{2}},$$

$$\eta_2(\boldsymbol{\theta}) = \frac{(s-\mu)(\sigma_\varepsilon^2 + k\sigma_a^2)}{bk} \left\{ \frac{\sigma_\varepsilon^2 + k\sigma_a^2}{bk} + \frac{(s-\mu)^2}{2b} \right\}^{-\frac{1}{2}},$$

and

$$\eta_3(\boldsymbol{\theta}) = 0.$$

$$\text{The prior } \pi_M(\mu, \sigma_a^2, \sigma_\varepsilon^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + k\sigma_a^2)^{-\frac{3}{2}} \left\{ 1 + \frac{k(s-\mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\}^{-\frac{1}{2}}$$

is a probability matching prior since

$$\begin{aligned} & \frac{\partial}{\partial \mu} \left\{ \eta_1(\boldsymbol{\theta}) \pi(\mu, \sigma_a^2, \sigma_\varepsilon^2) \right\} \\ &= \frac{\partial}{\partial \mu} \left\{ \sigma_\varepsilon^{-2} \left(\frac{1}{bk} \right)^{\frac{1}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{-1} \left[1 + \frac{k(s-\mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right]^{-1} \right\} \\ &= \sigma_\varepsilon^{-2} \left(\frac{1}{bk} \right)^{\frac{1}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{-1} \left\{ 1 + \frac{k(s-\mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\}^{-2} \frac{k(s-\mu)}{(\sigma_\varepsilon^2 + k\sigma_a^2)} \\ &= \sigma_\varepsilon^{-2} \left(\frac{1}{bk} \right)^{\frac{1}{2}} (\sigma_\varepsilon^2 + \sigma_a^2)^{-2} \left\{ 1 + \frac{k(s-\mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\}^{-2} k(s-\mu) \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial}{\partial \sigma_a^2} \left\{ \eta_2(\boldsymbol{\theta}) \pi(\mu, \sigma_a^2, \sigma_\varepsilon^2) \right\} \\ &= \frac{\partial}{\partial \sigma_a^2} \left\{ (s-\mu) \sigma_\varepsilon^{-2} (bk)^{-\frac{1}{2}} \left[(\sigma_\varepsilon^2 + k\sigma_a^2) + \frac{k(s-\mu)^2}{2} \right]^{-1} \right\} \\ &= -\sigma_\varepsilon^{-2} (bk)^{\frac{1}{2}} (\sigma_\varepsilon^2 + k\sigma_a^2)^{-2} \left\{ 1 + \frac{k(s-\mu)^2}{2(\sigma_\varepsilon^2 + k\sigma_a^2)} \right\} k(s-\mu). \end{aligned}$$

Also

$$\frac{\partial}{\partial \sigma_\varepsilon^2} \left\{ \eta_3(\boldsymbol{\theta}) \pi(\mu, \sigma_a^2, \sigma_\varepsilon^2) \right\} = 0.$$

Therefore

$$\begin{aligned} & \frac{\partial}{\partial \mu} \left\{ \eta_1(\boldsymbol{\theta}) \pi(\mu, \sigma_a^2, \sigma_\varepsilon^2) \right\} + \frac{\partial}{\partial \sigma_a^2} \left\{ \eta_2(\boldsymbol{\theta}) \pi(\mu, \sigma_a^2, \sigma_\varepsilon^2) \right\} + \frac{\partial}{\partial \sigma_\varepsilon^2} \left\{ \eta_3(\boldsymbol{\theta}) \pi(\mu, \sigma_a^2, \sigma_\varepsilon^2) \right\} \\ &= 0. \end{aligned}$$

Chapter 5

Student t - Distributed Measurement Error

Model

In Chapter 5, the simulation of variance components and quantiles used for the estimation of tolerance intervals will be considered for the balanced one - way random effects model with errors not of the usual $N(0, \sigma_\varepsilon^2)$ form. Rather, the assumption of Gaussian errors will be relaxed in the direction of the student t - distribution family. A non - informative prior distribution is proposed for the location and variance parameters while a normal prior distribution is proposed for the random effects parameter. It was also decided to use a truncated exponential prior distribution for the degrees of freedom. A prior distribution proportional to a gamma distribution was also proposed for λ_{ij} . For iron data originally presented by Wilson Hamada and Xu (2004), the Gibbs sampler will be used to obtain marginal posterior distributions for the unknown parameters using different degrees of freedom. This is followed by the determination of the three kinds of tolerance intervals originally proposed by Wolfinger (1998). In addition, the student t - distributed measurement error model will be used for the detection of possible outlying part measurements, using the scale parameter $\log_{10}(\lambda_{ij})$.

Parts of this chapter have been published in the South - African Statistical Journal. For more details see Hugo and van der Merwe (2009).

5.1 Introduction

It was mentioned in Chapter 1 that manufacturers are frequently required to verify that products meet certain specifications (*Hahn, 1982*). A standard approach to the problem would then be to compare for example measurements from a sample of parts, to a certain specification. Inferences can then be made from these results about the entire population of parts (*Wilson, Hamada and Xu, 2004*). The situation may however sometimes arise when for example it may seem that specifications are not being met, when in fact they are. This usually occurs when the available data is subject to measurement error (*Hahn, 1982*). It is therefore important to account for the measurement system being used to characterize production performance based on, for example, this available sample of parts (*Wilson, Hamada and Xu, 2004*).

Wilson, Hamada and Xu (2004) recently investigated such a case by considering the assessment of a manufacturing process's performance when the sample of parts produced by the process was measured with error, and as a result, would generate measurement errors not of the standard $N(0, \sigma_\varepsilon^2)$ form. Wilson et. al. (2004) firstly provided the standard Bayesian formulation of the one - way variance component model using a normal prior on μ , an inverse gamma (IG) prior on σ_ε^2 , and the uniform shrinkage prior proposed by Daniels (1999), on σ_a^2 given σ_ε^2 . The authors also illustrated a model where the error variance was assumed to be proportional to the true part value. They modeled the measurements as $N(x_i, (\rho x_i)^2)$, with ρ unknown. The priors proposed for μ and σ_a^2 were normal and inverse gamma (IG) respectively, while the prior distribution assumed for ρ , was a gamma distribution. The authors then illustrated a model where the measurement errors followed a log - normal distribution. For this model the authors used the same prior distributions as for the standard Bayesian model mentioned above. In addition to these, the authors showed how to handle censored data using a truncated normal distribution, calculated tolerance intervals for the parts distribution to assess the manufacturing process's performance, and, determined release specifications based on the producer's risk (probability of rejecting a good part) and

consumer's risk (probability of accepting a bad part).

To illustrate how a process's performance can be assessed when a sample of parts produced by a process is measured with error, consider the following example from a new manufacturing process. The data in Table 5.1 was obtained from a new manufacturing process and represents measurements of iron concentration in parts per million (ppm) (*Wilson, Hamada and Xu, 2004*). Each row of Table 5.1 represents two measurements of the same part determined by emission spectroscopy. According to *Wilson, Hamada and Xu (2004)*, a part is considered to be acceptable if it has under 225 ppm (parts per million) of iron. The engineers involved in the process are interested in understanding production characteristics while the chemists, who measure the parts, are interested in understanding the measurements system (*Wilson, Hamada and Xu, 2004*).

Table 5.1: Measurements of Iron in Parts per Million (ppm) Determined by Emission Spectroscopy. (A Part with Under 225 ppm of Iron is Acceptable).

Part	Measurement 1	Measurement 2
1	206	258
2	181	197
3	185	162
4	195	195
5	170.5	143.8
6	193.8	224.8
7	244.8	217
8	191.5	196.8
9	209.3	189.5
10	134.5	143.8
11	223.8	198.5
12	103	129.3
13	99.7	201.8
14	137.5	119.8
15	144.5	130
16	159	166.5
17	140.5	138
18	207	230
19	195.5	190.5
20	142.3	163.8
21	74.3	86.5
22	439.5	211.5
23	130.3	114
24	99.7	201.8

Wilson, Hamada and Xu (2004) suggested that a one - way model be used which is mathematically given by

$$y_{ij} = x_i + \varepsilon_{ij} \quad (5.1.1)$$

where y_{ij} represents the j^{th} measurement on the i^{th} part, x_i is the true value for the i^{th} part and ε_{ij} is the measurement error associated with the j^{th} measurement of the i^{th} part. The real interest is in x_i and ε_{ij} separately, but what is observed, are the y'_{ij} s. A variety of issues can be addressed if it is possible to estimate the distributions of x_i and ε_{ij} . These include many of the common ways of characterizing production performance. Specifically, Wilson, Hamada and Xu (2004) mentioned some of these common ways, if for example parts with an upper specification limit U are considered. These include the proportion of parts meeting the specification, as one measure of production performance. Another measure of production performance uses particular quantiles which summarizes the whole production distribution (i.e. the distribution of x) (Wilson, Hamada and Xu, 2004). According to Wilson et. al. (2004) a related issue is the setting of release or test specifications. For example, in deciding to accept a part or not, one needs to account for the measurement error. According to Wilson et. al. (2004), a typical approach is to tighten the specification U , to a release specification U_r , where the selection of U_r depends on the trade off that needs to be made between two types of errors, a good part can be rejected or a bad part can be accepted. The probabilities of these events are known as the producer's - and consumer's risks respectively, and are given by

$$P(x \leq U | y > U_r) \text{ and } P(x > U | y \leq U_r)$$

(Wilson, Hamada and Xu, 2004).

The novel features of the above example arise because the measurement system cannot be characterized in the traditional way, i.e., as following a $N(0, \sigma_\varepsilon^2)$ distribution for ε_{ij} . Rather, the measurement system may not be normally distributed and may have for example a multiplicative (or different) structure for the variance (Wilson, Hamada and Xu, 2004). The remainder of this chapter will be dedicated to the characterization of the measurement system in Table 5.1 using a student t - distribution. To be more specific the principle purpose of the rest of this chapter is to specify a Bayesian model

for the data given in Table 5.1, and, to study the effect of departure from the usual model which assumes that $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$. This is followed by the computational methods for implementing the Bayesian approach. The Bayesian method is then applied to the data given in Table 5.1, and to conclude, Bayesian tolerance intervals in the case where the residuals follow a student t - distribution will then be provided.

5.2 A Bayesian Procedure for the Student t - Distributed Measurement Error Model

Casual inspection of the residuals obtained for the iron data given in Table 5.1, has revealed that the underlying distribution is symmetrical, but with heavy tails. It is therefore feasible to replace the normal distribution with some other distribution which is also symmetric, behaves like the normal distribution in the central area, but has heavier tails (*Bernardo and Smith, 1994 and Tsiamyrtzis, 2000*). Therefore, to accommodate for the possibility of outlying measurements, the assumption of Gaussian errors will be relaxed in the direction of the student t - distributed family.

Consider the series of independent errors $\varepsilon_{ij} | \sigma_\varepsilon^2, \lambda_{ij} \sim N(0, \frac{\sigma_\varepsilon^2}{\lambda_{ij}})$ for $i = 1, \dots, b$ and $j = 1, \dots, k$. By placing a prior distribution on λ_{ij} , enables a wide variety of model error densities $f(\varepsilon_{ij} | \sigma_\varepsilon^2)$ to emerge as scale mixtures of normal distributions, i.e. $f(\varepsilon_{ij} | \sigma_\varepsilon^2) = \int_0^\infty p(\varepsilon_{ij} | \sigma_\varepsilon^2, \lambda_{ij}) p(\lambda_{ij}) d\lambda_{ij}$ (*Andrews and Mallows (1974), Carlin and Polson (1991) and Wakefield et. al. (1994)*). The following theorems can now be proved.

Theorem 5.2.1

If the errors for the one - way variance component model given by equation 5.1.1 are assumed to be independently student t - distributed, and if

$$(\varepsilon_{ij} | \lambda_{ij}, \sigma_\varepsilon^2) \sim N(0, \frac{\sigma_\varepsilon^2}{\lambda_{ij}}) \text{ for } i = 1, \dots, b, j = 1, \dots, k \text{ and where } \nu \lambda_{ij} \sim \chi_\nu^2.$$

then

$$f(\varepsilon_{ij}|\lambda_{ij}, \sigma_\varepsilon^2) = \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \lambda_{ij}^{\frac{1}{2}} \exp\left[\frac{1}{\sigma_\varepsilon^2} \lambda_{ij} \varepsilon_{ij}^2\right]. \quad (5.2.1)$$

Also, if we define

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1k} & \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{bk} \end{bmatrix}' \text{ and}$$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} & \lambda_{21} & \lambda_{22} & \cdots & \lambda_{bk} \end{bmatrix}'.$$

then

$$f(\boldsymbol{\varepsilon}|\boldsymbol{\lambda}, \sigma_\varepsilon^2) = \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{bk}{2}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}} \exp\left[\frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} \varepsilon_{ij}^2\right]. \quad (5.2.2)$$

Proof

The proof of Theorem 5.2.1 is given in Appendix C.

Theorem 5.2.2

If it is supposed that $\nu\lambda_{ij} \sim \chi_\nu^2$ for $i = 1, \dots, b$ and $j = 1, \dots, k$ then

$$p(\lambda_{ij}) = \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{1}{2}\nu} \Gamma(\frac{\nu}{2})} \lambda_{ij}^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu\lambda_{ij}} \text{ for } \lambda_{ij} > 0. \quad (5.2.3)$$

Proof

The proof of Theorem 5.2.2 is given in Appendix C.

Theorem 5.2.3

If the conditional density of $\varepsilon_{ij}|\lambda_{ij}, \sigma_\varepsilon^2$ is given as in equation 5.2.1, the conditional density of $\boldsymbol{\varepsilon}|\boldsymbol{\lambda}, \sigma_\varepsilon^2$ is given as in equation 5.2.2 and the density of λ_{ij} is given as in equation 5.2.3, then the conditional density of $\varepsilon_{ij}|\sigma_\varepsilon^2$ is represented by a univariate t - distribution with ν degrees of freedom given by

$$p(\varepsilon_{ij}|\sigma_\varepsilon^2) = \frac{\nu^{\frac{1}{2}\nu} \Gamma\left[\frac{1}{2}(\nu+1)\right] \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{1}{2}}}{\pi^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \left\{ \nu + \frac{1}{\sigma_\varepsilon^2} \varepsilon_{ij}^2 \right\}^{-\frac{1}{2}(\nu+1)} \quad (5.2.4)$$

and the joint density of $\boldsymbol{\varepsilon}|\sigma_\varepsilon^2$ follows a multivariate t - distribution given by

$$p(\boldsymbol{\varepsilon}|\sigma_\varepsilon^2) = \frac{\nu^{\frac{1}{2}\nu bk} \left\{ \Gamma\left[\frac{1}{2}(\nu + 1)\right] \right\}^{bk}}{(\pi\sigma_\varepsilon^2)^{\frac{1}{2}bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \left\{ \nu + \frac{1}{\sigma_\varepsilon^2} \boldsymbol{\varepsilon}_{ij}^2 \right\}^{-\frac{1}{2}(\nu+1)}. \quad (5.2.5)$$

Proof

The proof of Theorem 5.2.3 is given in Appendix C.

Now, reconsider equation 5.2.2 and define

$$H = \begin{bmatrix} \lambda_{11} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{12} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{13} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{bk} \end{bmatrix}.$$

We can therefore write equation 5.2.2 in matrix notation as follows:

$$p(\boldsymbol{\varepsilon}|\boldsymbol{\lambda}, \sigma_\varepsilon^2) = \left(\frac{1}{2\pi\sigma_\varepsilon^2} \right)^{\frac{bk}{2}} |H|^{\frac{1}{2}} \exp \left[\frac{1}{\sigma_\varepsilon^2} \boldsymbol{\varepsilon}' H \boldsymbol{\varepsilon} \right]. \quad (5.2.6)$$

For this example, it is assumed that $\nu\lambda_{ij} \sim \chi_\nu^2$ so that $\boldsymbol{\varepsilon}_{ij}|\sigma_\varepsilon^2 \sim t_\nu(0, \sigma_\varepsilon^2)$, i.e. representing a student t - distribution with mean 0, variance σ_ε^2 and degrees of freedom ν .

Now, the random effects model defined in equation 5.1.1 can be written as

$$y_{ij} = \mu + a_i + \boldsymbol{\varepsilon}_{ij} \quad \text{for } i = 1, \dots, b, j = 1, \dots, k \quad (5.2.7)$$

where y_{ij} represents the j^{th} measured value on the i^{th} part, μ is the overall mean, a_i represents the row or part effect (random effect) where $a_i|\sigma_a^2 \sim N(0, \sigma_a^2)$, and, $\boldsymbol{\varepsilon}_{ij}$ represents the measurement error for the j^{th} measurements on the i^{th} part, where as mentioned,

$$\boldsymbol{\varepsilon}_{ij}|\lambda_{ij}, \sigma_\varepsilon^2 \sim N\left(0, \frac{\sigma_\varepsilon^2}{\lambda_{ij}}\right) \text{ with } \nu\lambda_{ij} \sim \chi_\nu^2.$$

In matrix notation the random effects model given in equation 5.2.7 can be written as

$$\mathbf{y} = \mu \mathbf{j} + Z \mathbf{a} + \boldsymbol{\varepsilon} \quad (5.2.8)$$

where

$$\mathbf{j} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}' \text{ with dimension } (bk \times 1),$$

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_b \end{bmatrix}' \text{ and } \mathbf{a} \sim N(\mathbf{0}, \sigma_a^2 I_b),$$

$$\mathbf{y} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1k} & y_{21} & y_{22} & \cdots & y_{bk} \end{bmatrix}'$$

and for the example given in Table 5.1, $b = 24$ and $k = 2$.

Also,

$$\nu_1 = b(k - 1), \nu_2 = b - 1,$$

$$\nu_1 m_1 = \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - \bar{y}_{i.})^2, \text{ and,}$$

$$\nu_2 m_2 = k \sum_{i=1}^b (\bar{y}_{i.} - \bar{y}_{..})^2$$

where

$$\bar{y}_{i.} = \frac{1}{k} \sum_{j=1}^k y_{ij}, \text{ and}$$

$$\bar{y}_{..} = \frac{1}{bk} \sum_{i=1}^b \sum_{j=1}^k y_{ij}.$$

Also

$$Z = I_b \otimes \mathbf{j}$$

where $I_b \otimes \mathbf{j}$ denotes the Kronecker product of the matrices I_b and \mathbf{j} with I_b representing a $(b \times b)$ identity matrix and \mathbf{j} a $(k \times 1)$ vector of ones.

Z is therefore equal to

$$Z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & & \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} .$$

Now, since $\varepsilon = \mathbf{y} - \mu\mathbf{j} - Z\mathbf{a}$ and the jacobian of the transformation from ε to \mathbf{y} is $\left| \frac{\partial \varepsilon}{\partial \mathbf{y}} \right| = 1$, it follows from equation 5.2.6 that the distribution of \mathbf{y} is given by

$$L(\mathbf{y}|\mu, \mathbf{a}, \boldsymbol{\lambda}, \sigma_\varepsilon^2) = \left(\frac{1}{2\pi\sigma_\varepsilon^2} \right)^{\frac{1}{2}bk} |H|^{\frac{1}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} (\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H (\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a}) \right\}. \quad (5.2.9)$$

Equation 5.2.9 is known as the likelihood function, and, can be regarded as the function through which the data \mathbf{y} modifies prior knowledge of the unknown parameters (Box and Tiao, 1973).

5.3 The Prior Distribution

It was mentioned in Chapter 1, that an integral part of traditional Bayesian analysis is the assignment of prior distributions to the unknown parameters in the model (van der Merwe, Pretorius, Hugo and Zellner, 2001). The choice of a prior distribution is a very difficult and controversial step in any Bayesian analysis, since the information contained in the prior distribution, which is supposed to represent what is known about the unknown parameters before the data is available, is combined with the information supplied by the data, through the likelihood function, to form the joint posterior distribution of the parameters given the data (Box and Tiao, 1973 and Gianola and Fernando, 1986).

It must also be reiterated that two types of prior information are distinguished: Data based and non - data based. Data based prior information is obtained in a scientific manner from prior experimentation, while non - data based prior information is based on subjective personal opinions or beliefs and theoretical considerations. It seems to be the use of non - data based prior information to which orthodox frequentists object (*Carrquiry, 1989*).

The choice of a prior distribution can either express prior ignorance or prior belief. One can therefore either choose a non - informative prior distribution, having a “flat” probability density function, or a prior distribution assigning probability one to a single value (*Carrquiry, 1989*). In problems of scientific inference it is usually better, if at all possible, to let the data “speak for itself”. Box and Tiao (1973) therefore pointed out that it is usually better to conduct the analysis as if a state of relative ignorance existed, and thus recommended the use of non - informative prior distributions.

For the prior distribution of the parameters $\mu, \sigma_a^2, \sigma_\varepsilon^2$, it is assumed that little is known about these parameters initially. It was decided to use non - informative prior distribution to represent these parameters i.e.

$$p(\mu, \sigma_a^2, \sigma_\varepsilon^2) \propto \sigma_a^{-1} \sigma_\varepsilon^{-2}.$$

Since $p(\sigma_a^2) \propto \sigma_a^{-2}$ will result in improper posterior distributions, it was suggested by Gelman (2006) to use a uniform density on σ_a which is equivalent to $p(\sigma_a^2) \propto \sigma_a^{-1}$. This density can be interpreted as a limit of the half t - family on σ_a , when the scale approaches infinity (and any value of the degrees of freedom).

Another non - informative prior distribution sometimes proposed in Bayesian literature, is uniform on σ_a^2 . According to Gelman (2006), this is not recommended as it seems to have a miscalibration towards higher values, and, also requires $b \geq 4$ groups for a proper posterior distribution.

For the random effects \mathbf{a} , a normal prior distribution given by

$$p(\mathbf{a}) = \left(\frac{1}{2\pi\sigma_a^2} \right)^{\frac{1}{2}b} \exp\left(-\frac{1}{2\sigma_a^2} \mathbf{a}' \mathbf{a}\right)$$

will be used.

It was also decided that the prior distribution of the parameters λ_{ij} for $i = 1, \dots, b$ and $j = 1, \dots, k$ be taken as

$$p(\lambda_{ij}) = \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp(-\frac{1}{2}\lambda_{ij}\nu), \text{ where } \nu\lambda_{ij} \text{ follows a gamma distribution.}$$

Since $p(\boldsymbol{\lambda}) = \prod_{i=1}^b \prod_{j=1}^k p(\lambda_{ij})$, the prior distribution of $\boldsymbol{\lambda}$ is therefore given as

$$\begin{aligned} p(\boldsymbol{\lambda}) &= \prod_{i=1}^b \prod_{j=1}^k \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp(-\frac{1}{2}\lambda_{ij}\nu) \\ &= \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{\nu}{2}bk} \left[\Gamma(\frac{\nu}{2}) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu-1} \exp(-\frac{1}{2}\nu \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij}). \end{aligned}$$

The joint prior distribution of the parameters $(\mu, \sigma_a^2, \sigma_\varepsilon^2, \mathbf{a}$ and $\boldsymbol{\lambda})$ is then given by

$$\begin{aligned} p(\mu, \sigma_a^2, \sigma_\varepsilon^2, \mathbf{a}, \boldsymbol{\lambda}) &\propto \sigma_\varepsilon^{-2} \left(\frac{1}{2\pi\sigma_a^2} \right)^{\frac{1}{2}b} \exp\left(-\frac{1}{2\pi\sigma_a^2} \mathbf{a}' \mathbf{a}\right) \\ &\quad \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{\nu}{2}bk} \left[\Gamma(\frac{\nu}{2}) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu-1} \exp(-\frac{1}{2}\nu \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij}). \end{aligned} \quad (5.3.1)$$

This completes the prior specification for the random effects model given in equation 5.2.7, if the degrees of freedom ν is held fixed. A prior distribution for ν will later be selected and ν will also be simulated.

5.4 The Posterior Distribution

Given the likelihood function and the prior distributions, the joint posterior density of the unknown parameters can always be determined.

Bayes's theorem states that for data \mathbf{y} and unknowns γ ,

$$p(\gamma|\mathbf{y}) \propto L(\mathbf{y}|\gamma)p(\gamma).$$

This implies that the prior information about γ , as described by the prior density $p(\gamma)$, is updated by the information contained in the data, as described by the likelihood or the joint density of the data $L(\mathbf{y}|\gamma)$, to yield the joint posterior of γ given by $p(\gamma|\mathbf{y})$ (Wilson *et. al.*, 2004).

For this problem, $\gamma = (\mu, \sigma_a^2, \sigma_\varepsilon^2, \mathbf{a}, \boldsymbol{\lambda})$ and the joint posterior density of γ is therefore given by

$$\begin{aligned} p(\mu, \sigma_a^2, \sigma_\varepsilon^2, \mathbf{a}, \boldsymbol{\lambda}) &\propto \text{Likelihood function} \times \text{Prior distributions} \\ &\propto \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}bk} \left|H\right|^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})\right\} \sigma_\varepsilon^{-2} \\ &\quad \left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}b} \exp\left(-\frac{1}{2\pi\sigma_a^2}\mathbf{a}'\mathbf{a}\right) \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{\nu}{2}bk} \left[\Gamma\left(\frac{\nu}{2}\right)\right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left(-\frac{1}{2}\nu \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij}\right) \end{aligned} \quad (5.4.1)$$

where H is defined as in equation 5.2.6 and \mathbf{y} , \mathbf{j} , \mathbf{a} and Z defined as in equation 5.2.8.

From the joint posterior distribution given in equation 5.4.1 the conditional densities can be determined.

5.5 Bayesian Computation

To make appropriate inferences in a Bayesian analysis, the marginal posterior distributions and predictive densities are needed. Due to the complexity of the joint posterior distribution however, it is impossible to obtain these marginal posterior densities analytically. It is also very difficult to obtain these marginal posterior densities numerically, due to the high number of unknowns (*van der Merwe, Pretorius and Meyer, 2003*). It is therefore recommended that a Monte Carlo simulation procedure be used to estimate these marginal posterior densities of the unknown parameters and predictive densities of future observations.

In recent years, statisticians have been increasingly drawn to Markov chain Monte Carlo (MCMC) simulation to examine more complex systems than would otherwise be possible (*Chib and Greenberg, 1995*).

It was mentioned in Chapter 1, that to explain Markov chain Monte Carlo simulation, suppose a sample needs to be generated from a posterior distribution $p(\mu|\mathbf{y})$ for $\mu \in \tau \subset \mathbb{R}^k$, but it cannot be done directly. The key to Markov chain simulation is then to create a Markov process whose stationary distribution is a specified $p(\mu|\mathbf{y})$ and run the simulation long enough so that the distribution of the current draws is close enough to the stationary distribution. Once the simulation algorithm has been implemented, sufficient iterations should be performed until convergence has been approximated.

It was also mentioned in Chapter 1, that Metropolis together with Rosenbluth, Rosenbluth, Teller and Teller (1953), developed the Metropolis-Hastings (M-H) algorithm which was later generalized by Hastings (1970). Although the M-H algorithm has been used extensively in physics, it was little known to statisticians until recently, despite the paper by Hastings (*Chib and Greenberg, 1995*). The Metropolis-Hastings algorithm is extremely useful and versatile and applications are steadily appearing in literature (*Chib and Greenberg, 1995*).

The Gibbs sampling algorithm, a special case of the Metropolis-Hastings algorithm, is one of the best known Markov chain Monte Carlo methods (*Chib and Greenberg, 1995*) and will be discussed in the following section.

5.5.1 The Gibbs Sampler

Gibbs sampling is a numerical integration method for generating random variables from a (marginal) distribution indirectly, without having to calculate the density. This technique is based only on elementary properties of Markov chains (*Casella and George, 1990*).

Consider the following example: Suppose we are given a joint density function $f(w, x, y, z)$ and we want to obtain the characteristics, such as the mean and variance, of the marginal density

$$f(w) = \int \int \int f(w, x, y, z) dx dy dz.$$

The most natural approach would be to calculate $f(w)$ analytically or even numerically, and then use this result to obtain the desired characteristics such as the mean and variance. However, there are some cases where the marginal distribution $f(w)$ of the density function $f(w, x, y, z)$ cannot be readily calculated. In these cases an alternative approach is provided by Gibbs sampling.

The Gibbs sampler allows us to effectively simulate a random sample

$$w_1, \dots, w_n \sim f(w)$$

without requiring $f(w)$. By simulating a large enough sample, the mean, variance or any other characteristics of $f(w)$ can be calculated to any desired degree of accuracy.

This characteristic can easily be proved by the fact that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n w_i = \int_{-\infty}^{\infty} w f(w) dw = E(w).$$

Thus, by taking the sample size (n) large enough, any degree of accuracy can be obtained.

The Gibbs sampler is therefore a method for generating a sample from $f(w)$ in an indirect way, by sampling from the conditional distributions $f(w|x, y, z)$, $f(x|w, y, z)$, $f(y|w, x, z)$ and $f(z|w, x, y)$ which are often known in linear statistical models. It is therefore possible to generate samples from these conditional distributions given the specified values of the conditioning variables (*Casella and George, 1990*).

The Gibbs sampler generates a random sample indirectly as follows:

The initial values $X^{(0)} = x^{(0)}$, $Y^{(0)} = y^{(0)}$, $Z^{(0)} = z^{(0)}$ are specified and the rest of the Gibbs sequence of random variables are obtained iteratively by alternately generating values in the following way:

Draw

$$w^{(1)} \text{ from } f(w|x^{(0)}, y^{(0)}, z^{(0)})$$

then

$$x^{(1)} \text{ from } f(x|w^{(1)}, y^{(0)}, z^{(0)}),$$

also draw

$$y^{(1)} \text{ from } f(y|w^{(1)}, x^{(1)}, z^{(0)})$$

and lastly

$$z^{(1)} \text{ from } f(z|w^{(1)}, x^{(1)}, y^{(1)})$$

to complete one iteration of the scheme. At the p^{th} iteration we draw

$w^{(p)}$ from $f(w|x^{(p-1)}, y^{(p-1)}, z^{(p-1)})$

then

$x^{(p)}$ from $f(x|w^{(p)}, y^{(p-1)}, z^{(p-1)})$,

then

$y^{(p)}$ from $f(y|w^{(p)}, x^{(p)}, z^{(p-1)})$

and lastly

$z^{(p)}$ from $f(z|w^{(p)}, x^{(p)}, y^{(p)})$.

Geman and Geman (1984) have shown that under fairly general conditions, the distribution of $w^{(p)}$ converges to $f(w)$ (the true marginal distribution of w) as p approaches infinity. Thus, the value $w^{(p)}$ can be regarded as a simulated observation from $f(w)$ if p is large enough. By repeating the Gibbs sequence np times, the Gibbs sampler generates n observations

$w_1^{(p)}, \dots, w_n^{(p)}$.

Therefore, the Gibbs sampler can generate n realized values of the random variable $w^{(p)}$ if the Gibbs sequence is repeated n times. If the repetitions are independent, using predetermined initial values $x^{(0)}$, $y^{(0)}$ and $z^{(0)}$ for each sequence, the final values will be independent. Thus, by simulating a large enough sample, characteristics such as the mean and variance of $f(w)$ can be determined to any desired degree of accuracy (*van der Merwe and Botha, 1993*). Characteristics of $f(x)$, $f(y)$ and $f(z)$ can be obtained in a similar way.

By looking at the above statements, one can see that the Gibbs sampler is relatively easy to implement and subsequently has had an immense impact on Bayesian statistics.

For the random effects model given in equation 5.2.7 and the data given in Table 5.1, the Gibbs sampler will succeed, because the problem of dealing simultaneously with a large number of intricately related unknown parameters is reduced into a much simpler problem of dealing with one unknown quantity at a time, sampling each from its full conditional posterior distribution (*Gilks, Thomas and Spiegelhalter, 1994*).

As was mentioned above, one issue however with Gibbs sampling, referred to as the convergence of the Gibbs sampler, is whether the draws are approximately a random sample from the posterior distribution (*Wilson, Hamada and Xu, 2004*). By running the Gibbs sampler however a number of times and then discarding the draws obtained, also known as the burn - in period, the impact of initially chosen values are decreased. Also, by retaining every p^{th} draw, dependence between draws can be reduced (*Wilson, Hamada and Xu, 2004*). The Gibbs sampler on the other hand also has an advantage when calculating quality characteristics, since an approximate randomly selected sample from the posterior densities of the unknown parameters is obtained after running the Gibbs sampler np times. This sample can then be used in a straightforward way to calculate a distribution for a particular characteristic (*Wilson, Hamada and Xu, 2004*).

5.6 Conditional Posterior Distributions

As mentioned, to implement the Gibbs sampler, the full conditional posterior distributions of the unknown parameters are needed.

The following theorems can now be proved.

Theorem 5.6.1

For the random effects model given in equation 5.2.7 with joint posterior distribution given by equation 5.4.1, the conditional posterior distribution of μ is given by

$$p(\mu|\mathbf{y}, \sigma_\varepsilon^2, H, \mathbf{a}) \sim N\left(\left(\frac{1}{\sigma_\varepsilon^2}\mathbf{j}'H\mathbf{j}\right)^{-1}\frac{1}{\sigma_\varepsilon^2}\mathbf{j}'H(\mathbf{y} - Z\mathbf{a}), \sigma_\varepsilon^2(\mathbf{j}'H\mathbf{j})^{-1}\right) \quad (5.6.1)$$

where H is defined as in equation 5.2.6 and \mathbf{y} , \mathbf{a} and Z defined as in equation 5.2.8.

Proof

The proof of Theorem 5.6.1 is given in Appendix C.

Theorem 5.6.2

For the random effects model given in equation 5.2.7 with joint posterior distribution given by equation 5.4.1, the conditional posterior distribution of the vector of random effects \mathbf{a} is given by

$$p(\mathbf{a}|\mu, \sigma_\varepsilon^2, \sigma_a^2, H, \mathbf{y}) \sim N\left(\left(\frac{1}{\sigma_\varepsilon^2}Z'HZ + \frac{1}{\sigma_a^2}I_b\right)^{-1}\left(\frac{1}{\sigma_\varepsilon^2}Z'H\mathbf{y}^*\right), \left(\frac{1}{\sigma_\varepsilon^2}Z'HZ + \frac{1}{\sigma_a^2}I_b\right)^{-1}\right) \quad (5.6.2)$$

where $\mathbf{y}^* = \mathbf{y} - \mu\mathbf{j}$, H is defined as in equation 5.2.6 and \mathbf{a} , \mathbf{y} and Z defined as in equation 5.2.8.

Proof

The proof of Theorem 5.6.2 is given in Appendix C.

Theorem 5.6.3

The conditional posterior distribution of σ_ε^2 for the random effects model given by equation 5.2.7 with joint posterior distribution given by equation 5.4.1, is given by

$$p(\sigma_\varepsilon^2 | \mu, \mathbf{a}, \sigma_a^2, H, \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(bk+2)} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})\right\} \quad (5.6.3)$$

which is in the general form of an inverse gamma distribution where H is defined as in equation 5.2.6 and \mathbf{a} , \mathbf{y} and Z defined as in equation 5.2.8.

Proof

The proof of Theorem 5.6.3 is given in Appendix C.

Theorem 5.6.4

For the random effects model given in equation 5.2.7 with joint posterior distribution given by equation 5.4.1, the conditional posterior distribution of σ_a^2 is given by

$$p(\sigma_a^2 | \mathbf{a}, \mathbf{y}) \propto (\sigma_a^2)^{-\frac{1}{2}b} \exp\left\{-\frac{1}{2\sigma_a^2} \mathbf{a}' \mathbf{a}\right\} \quad (5.6.4)$$

which is in the general form of an inverse gamma distribution where \mathbf{a} is defined as in equation 5.2.8.

Proof

The proof of Theorem 5.6.4 is given in Appendix C.

Theorem 5.6.5

For the random effects model given in equation 5.2.7 with joint posterior distribution given by equation 5.4.1, the conditional posterior distribution of λ_{ij} is given by

$$p(\lambda_{ij} | \mathbf{y}, \mathbf{a}, \mu, \sigma_\varepsilon^2, \sigma_a^2) \propto \lambda_{ij}^{\frac{1}{2}(\nu+1)-1} \exp\left\{-\frac{1}{2}\lambda_{ij}\left[\nu + \frac{1}{\sigma_\varepsilon^2}(y_{ij} - \mu - a_i)^2\right]\right\} \quad (5.6.5)$$

which is in the general form of a gamma distribution where \mathbf{y} and \mathbf{a} are defined as in equation 5.2.8, $i = 1, \dots, b$ and $j = 1, \dots, k$.

Proof

The proof of Theorem 5.6.5 is given in Appendix C.

5.7 Marginal Posterior Distributions

As mentioned, Markov chain Monte Carlo simulation methods can be used to obtain the random samples from the joint posterior distribution. It was also mentioned that one of the best known MCMC methods, is the Gibbs sampler, which consists of repeated cycles of draws from the full conditional posterior distribution. The Gibbs sampler however has a principle requirement that all conditional posterior densities must be available in the sense that random variates can be generated from them.

To implement the Gibbs sampling procedure, random numbers have to be generated from the required posterior distributions. The Gibbs sampling procedure for the random effects model given in equation 5.2.7 can be implemented as follows:

- i.) Start the iterative process for the burn - in period by specifying initial values for ν (which is held fixed), $\sigma_\varepsilon^{2(0)}$, $\sigma_p^{2(0)}$, $\mathbf{a}^{(0)}$ and $\boldsymbol{\lambda}^{(0)}$. These preselected values and vectors are then used to draw $\mu^{(1)}$, the first value of μ , from a normal distribution given by $\mu | \mathbf{y}, \sigma_\varepsilon^2, \mathbf{a}, H \sim N\left((\mathbf{j}' H \mathbf{j})^{-1} \mathbf{j}' H (\mathbf{y} - Z \mathbf{a}), \sigma_\varepsilon^2 (\mathbf{j}' H \mathbf{j})^{-1}\right)$

where H is defined as in equation 5.2.6.¹

ii.) The preselected values $\sigma_a^{2(0)}$, $\mathbf{a}^{(0)}$ and $\boldsymbol{\lambda}^{(0)}$ are then used together with the first simulated value for μ i.e. $\mu^{(1)}$, to draw $\sigma_\varepsilon^{2(1)}$ from the conditional posterior density of $\sigma_\varepsilon^2 | \mu, \mathbf{a}, H, \mathbf{y}$ by simulating $\tau = \frac{(\mathbf{y} - \mu \mathbf{j} - Z\mathbf{a})' H (\mathbf{y} - \mu \mathbf{j} - Z\mathbf{a})}{\sigma_\varepsilon^2}$ from a chi - square distribution with bk degrees of freedom, and, then by calculating $\sigma_\varepsilon^2 = \frac{(\mathbf{y} - \mu \mathbf{j} - Z\mathbf{a})' H (\mathbf{y} - \mu \mathbf{j} - Z\mathbf{a})}{\tau}$ where $\tau \sim \chi_{bk}^2$ as mentioned.

iii.) By now, using the preselected values for \mathbf{a} and $\boldsymbol{\lambda}$, together with the first simulated values $\mu^{(1)}$ and $\sigma_\varepsilon^{2(1)}$, $\sigma_a^{2(1)}$ can be drawn from the conditional posterior density of $\sigma_a^2 | \mu, \sigma_\varepsilon^2, \mathbf{a}, H, \mathbf{y}$ by simulating $\theta = \frac{\mathbf{a}' \mathbf{a}}{\sigma_a^2}$ from a chi - square distribution with $b - 1$ degrees of freedom, and, then by obtaining $\sigma_a^2 = \frac{\mathbf{a}' \mathbf{a}}{\theta}$ where, as mentioned, $\theta \sim \chi_{b-1}^2$.

iv.) If the preselected values for $\boldsymbol{\lambda}$ are now used together with the first simulated values $\mu^{(1)}$, $\sigma_\varepsilon^{2(1)}$ and $\sigma_a^{2(1)}$, the vector $\mathbf{a}^{(1)}$ can be drawn using the normal distribution given by

$$p(\mathbf{a} | \mu, \sigma_\varepsilon^2, \sigma_a^2, H, \mathbf{y}) \sim N \left[\left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right)^{-1} \left(\frac{1}{\sigma_\varepsilon^2} Z' H \mathbf{y}^* \right), \left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right)^{-1} \right]$$

derived in Theorem 5.6.2.

v.) These simulated values $\mu^{(1)}$, $\sigma_\varepsilon^{2(1)}$, $\sigma_a^{2(1)}$ and the vector $\mathbf{a}^{(1)}$ are then used to draw the $\lambda_{ij}^{(1)}$ ($i = 1, \dots, b$, $j = 1, \dots, k$) values used to set up the matrix H defined in equation 5.2.6. These $\lambda_{ij}^{(1)}$ values can be simulated using the conditional posterior density $p(\lambda_{ij} | \mu, \sigma_\varepsilon^2, \sigma_a^2, \mathbf{a}, H, \mathbf{y})$ by simulating $w = \lambda_{ij} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{ij} - \mu - a_i)^2 \right]$ from a chi - square distribution with $\nu + 1$ degrees of freedom, and, then by obtaining $\lambda_{ij} = \frac{w}{\left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{ij} - \mu - a_i)^2 \right]}$ where $w \sim \chi_{\nu+1}^2$ as mentioned.

This completes one iteration of the Gibbs sampler, and as a result, the first simulated values for μ , σ_ε^2 , σ_a^2 , \mathbf{a} and $\boldsymbol{\lambda}$ or in other words, $\mu^{(1)}$, $\sigma_\varepsilon^{2(1)}$, $\sigma_a^{2(1)}$, $\mathbf{a}^{(1)}$ and $\boldsymbol{\lambda}^{(1)}$ respectively were obtained. These values are then used as initial values for the second iteration of

¹Remember all the λ_{ij} values ($i = 1, \dots, b$, $j = 1, \dots, k$) down the main diagonal of H can for example be taken as chi - square random values with $\nu = 5$ degrees of freedom.

the Gibbs sampling algorithm. For the burn - in period, np values are simulated and every p^{th} value retained, meaning that we will have n retained, simulated values. Of these, the first $n - 1$ retained values are disregarded, and the values simulated and retained on the np^{th} iteration are used as starting values for the Gibbs sampler which is used to simulate the random effects and variance components. The Gibbs sampler is then run $\tilde{\ell}_p$ times to obtain the estimated marginal posterior densities (estimated unconditional posterior densities) $(\mu|\mathbf{y})$, $(\sigma_\varepsilon^2|\mathbf{y})$, $(\sigma_a^2|\mathbf{y})$, $(\mathbf{a}|\mathbf{y})$ and $(\boldsymbol{\lambda}|\mathbf{y})$ using the same method as described above.

5.8 A Prior Distribution for the Degrees of Freedom ν

In the previous sections, the Bayesian approach to variance component - and random effects estimation for the random effects model given in equation 5.2.7 was considered for the case where the degrees of freedom ν was taken as a fixed value, and, prior distributions were defined for μ , σ_ε^2 , σ_a^2 , \mathbf{a} and $\boldsymbol{\lambda}$ to obtain the joint prior distribution given in equation 5.3.1.

The random effects model given in equation 5.2.7 will now be reconsidered for the case where the degrees of freedom ν is not held fixed. This implies that a prior distribution for the degrees of freedom ν will also have to be specified.

As prior distribution for the degrees of freedom ν , it was suggested by Sahu, Dey and Branco (2003) that a truncated ($\nu > 2$) exponential distribution given by

$$p(\nu) = \xi \exp(-\xi\nu) \quad \text{for } \nu > 2 \quad (5.8.1)$$

be used with parameter $\xi = 0.1$. This truncation ensures the finiteness of the mean and variance of the associated t - error distribution. For further details on this, see Sahu, Dey and Branco (2003).

Also, since the parameter $\xi = 0.1$, it implies that the expected value of ν is given by

$$E(\nu) = \frac{1}{\xi} = 10.$$

The joint prior distribution of the parameters $(\mu, \sigma_\varepsilon^2, \sigma_a^2, \mathbf{a}, \boldsymbol{\lambda}, \nu)$ is then obtained by multiplying the joint prior distribution of the parameters $(\mu, \sigma_\varepsilon^2, \sigma_a^2, \mathbf{a}, \boldsymbol{\lambda})$ given by equation 5.3.1 with the prior distribution of ν given by equation 5.8.1. The joint prior distribution of the parameters $(\mu, \sigma_\varepsilon^2, \sigma_a^2, \mathbf{a}, \boldsymbol{\lambda}, \nu)$ is then given by

$$p(\mu, \sigma_\varepsilon^2, \sigma_a^2, \mathbf{a}, \boldsymbol{\lambda}, \nu) \propto \sigma_\varepsilon^{-2} \left(\frac{1}{2\pi\sigma_a^2} \right)^{\frac{1}{2}b} \exp\left(\frac{1}{2\pi\sigma_a^2} \mathbf{a}' \mathbf{a} \right) \\ \frac{\nu^{\frac{1}{2}b} \nu^{bk}}{2^{\frac{\nu}{2}bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left(-\frac{1}{2}\nu \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} \right) \xi \exp(-\xi\nu). \quad (5.8.2)$$

To obtain the joint posterior distribution, as mentioned, the likelihood function given by equation 5.2.9 has to be multiplied with the joint prior distribution given by equation 5.8.2. The joint posterior density is then given by

$$p(\mu, \sigma_\varepsilon^2, \sigma_a^2, \mathbf{a}, \boldsymbol{\lambda}, \nu | \mathbf{y}) \propto \\ \left(\frac{1}{2\pi\sigma_\varepsilon^2} \right)^{\frac{1}{2}bk} \left| H \right|^{\frac{1}{2}} \exp\left\{ \frac{1}{2\pi\sigma_\varepsilon^2} (\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H (\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a}) \right\} \sigma_\varepsilon^{-2} \left(\frac{1}{2\pi\sigma_a^2} \right)^{\frac{1}{2}b} \exp\left\{ \frac{1}{2\pi\sigma_a^2} \mathbf{a}' \mathbf{a} \right\} \\ \frac{\nu^{\frac{1}{2}b} \nu^{bk}}{2^{\frac{\nu}{2}bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \left\{ \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu-1} \right\} \exp\left(-\frac{1}{2}\nu \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} \right) \xi \exp(-\xi\nu) \quad (5.8.3)$$

where H is defined as in equation 5.2.6 and \mathbf{y} , \mathbf{j} , \mathbf{u} and Z defined as in equation 5.2.8.

From the joint posterior distribution given in equation 5.8.3, the conditional posterior distribution of ν can now be obtained.

Theorem 5.8.1

For the joint posterior distribution given in equation 5.8.3 the conditional posterior distribution of $\nu|\mathbf{y}, \lambda_{ij}$ is given by

$$p(\nu|\mathbf{y}, \lambda_{ij}) \propto \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{\nu}{2}bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu} \exp\left\{ \nu \left(-\frac{1}{2} \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} + \xi \right) \right\} \quad \text{for } \nu > 2 \quad (5.8.4)$$

where \mathbf{y} is defined as in equation 5.2.8.

The conditional posterior distribution of $\nu|\mathbf{y}, \lambda_{ij}$, is not in the general form of a known continuous distribution, and as a result, cannot be simulated directly.

Proof

The proof of Theorem 5.8.1 is given in Appendix C.

As mentioned, equation 5.8.4 does not correspond to any common distribution, hence only the kernel of the density is available. Geweke (2005) used a metropolis within Gibbs step to generate random numbers from equation 5.8.4. As candidate density he used a univariate normal distribution, with mean at the mode $\hat{\nu}$ of $p(\nu|\mathbf{y}, \boldsymbol{\lambda})$ and precision equal to $-d^2 \log p(\nu|\mathbf{y}, \boldsymbol{\lambda}) \Big|_{d\nu^2} \Big|_{\nu = \hat{\nu}}$. This method is also used in the Bayesian Analysis Computation and Communication (BACC) software for models with student t - distributions. It is however recommended that the rejection sampling method as discussed by Rice (1995) be used whereby a known distribution is fitted over the unknown distribution given in equation 5.8.4. In order for the rejection method to be computationally efficient, it was decided to use as candidate function also a normal distribution. As will be seen later, the normal distribution fitted $p(\nu|\mathbf{y}, \boldsymbol{\lambda})$ so well that ν could be simulate directly from it.

Remember, to simulate ν from this normal distribution, the Gibbs sampling procedure is run in the same way as described earlier, with the only difference being that ν is now also drawn using this mentioned normal distribution. It must be noted however, that

since the posterior density of ν is now continuous, a drawn value of ν may not always be a whole number. Since ν represents the degrees of freedom, the simulated value for ν therefore has to be rounded off to the nearest whole number before substitution into the Gibbs sampling algorithm discussed earlier. Also, remember to start the burn - in period by still specifying an initial value for the degrees of freedom ν , as mentioned in step i.) of the Gibbs sampling algorithm, on the first iteration. The degrees of freedom is then simulated using the normal distribution mentioned in the next section. This simulated degrees of freedom value ν then simply replaces the fixed degrees of freedom value ν in step i.) of the Gibbs sampling algorithm from the second iteration onwards.

5.9 Results and Discussions

Recall that the data provided in Table 5.1 comes from a new manufacturing process and each row of Table 5.1 represents two measurements of iron concentration. Also recall that the two iron concentrations measured on the same part were determined by emission spectroscopy. If a part had a measurement of under 225 ppm of iron, it was considered to be acceptable (*Wilson, Hamada and Xu, 2004*).

Since the full conditional posterior distributions of the unknown parameters have been determined, the Gibbs sampler can now be implemented to obtain the estimated marginal posterior distributions of the random effects and variance components. MATHWORKS MATLAB will be used to do the simulations for the Gibbs sampler.

The iteration process for the burn - in period was started by specifying initial values for $\sigma_\varepsilon^{2(0)} = m_1 = 1721.3285$ and $\sigma_a^{2(0)} = \frac{1}{k}(m_2 - m_1) = 1693.6653$. Also, with these preselected values, $\mathbf{a}^{(0)}$ was determined as follows: For part 1, the average were obtained for measurement 1 and measurement 2. The average of all the observations, $\mu^{(0)}$, were then subtracted from this average. Therefore a_1 for example was determined as $a_1 =$

$\left(\frac{206+258}{2}\right) - 175.37 = 56.63$. This was also repeated for the remaining 23 parts. The preselected values used for $\mathbf{a}^{(0)}$ were determined and is given by

$$\mathbf{a}^{(0)} = \begin{bmatrix} 56.63 & 13.63 & -1.87 & 19.63 & -18.22 & 33.93 & 55.53 & 18.78 & 24.03 & -36.22 & 35.78 \\ -59.22 & -24.62 & -46.72 & -38.12 & -12.62 & -36.12 & 43.13 & 17.63 & -22.32 & -94.97 \\ 150.13 & -53.22 & -24.62 \end{bmatrix}'.$$

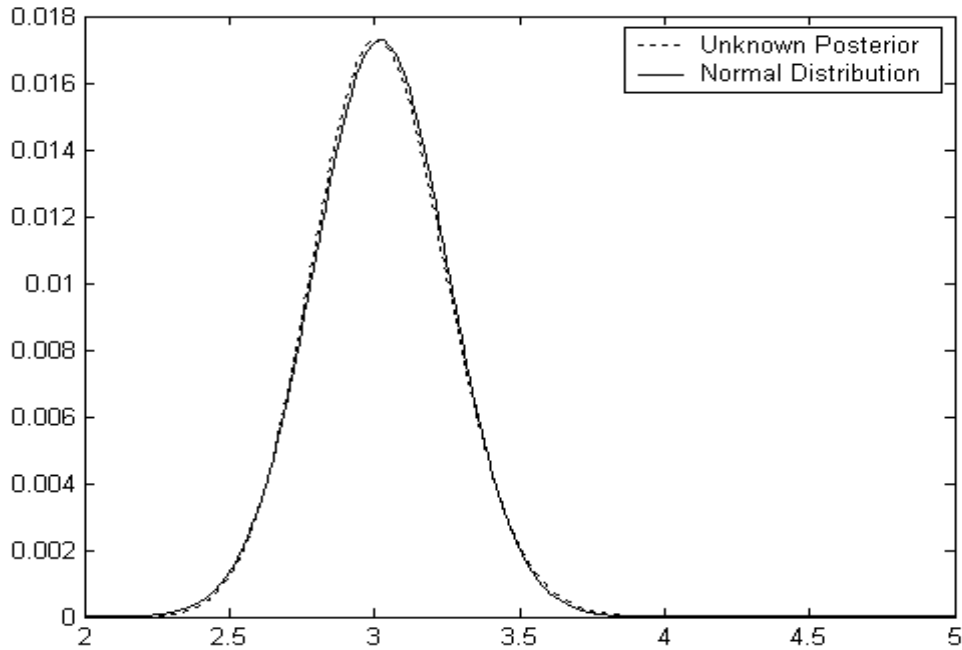
Also, λ was taken as a vector of 48 chi - square random values with $\nu = 5$ degrees of freedom each. These preselected values were substituted into the full conditional posterior densities of the unknown parameters to draw $\nu^{(1)}$ (if ν was simulated²), $\mu^{(1)}$, $\sigma_\varepsilon^{2(1)}$, $\sigma_a^{2(1)}$, $\mathbf{a}^{(1)}$ and $\lambda^{(1)}$. The full conditional posterior distributions were updated after every iteration. The results from the burn - in period were obtained after running the procedure 30000 times. Every 10^{th} value for ν , μ , σ_ε^2 , σ_a^2 , \mathbf{a} and λ were stored, meaning that 3000 values were stored for each of the unknown parameters. Of these 3000 stored combinations, the first 2999 stored combinations were disregarded and the values stored on the 30000th iteration, i.e. $\nu^{(3000)}$, $\mu^{(3000)}$, $\sigma_\varepsilon^{2(3000)}$, $\sigma_a^{2(3000)}$, $\mathbf{a}^{(3000)}$ and $\lambda^{(3000)}$, were used as starting values for the Gibbs sampling procedure.

The new starting values obtained from the burn - in period were then substituted into the full conditional posterior distributions and the Gibbs sampling procedure was run $\tilde{\ell}p = 100000$ times. Again, every $p = 10^{th}$ sample was saved, meaning that the total number of samples saved was $\tilde{\ell} = 10000$. Also, the full conditional posterior distributions were updated after every iteration. These results obtained from the Gibbs sampler were then used to obtain the histograms of the estimated marginal posterior distributions $p(\nu|\mathbf{y})$, $p(\mu|\mathbf{y})$, $p(\sigma_\varepsilon^2|\mathbf{y})$, $p(\sigma_a^2|\mathbf{y})$ and for example $p(\lambda_{20}|\mathbf{y})$ ³. Estimates from the marginal posterior densities of functions of the variance components such as the total variance $\sigma_\varepsilon^2 + \sigma_a^2$ and variance ratio $\frac{\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + \sigma_a^2)}$ could also easily be obtained. These estimated marginal posterior distributions are displayed in Figures 5.9.2 - 5.9.7.

² ν was not simulated in the case of the joint posterior distribution given by equation 5.4.1. The Gibbs sampler was then run using fixed values for ν , for example $\nu = 5$ or $\nu = 10$ etc.

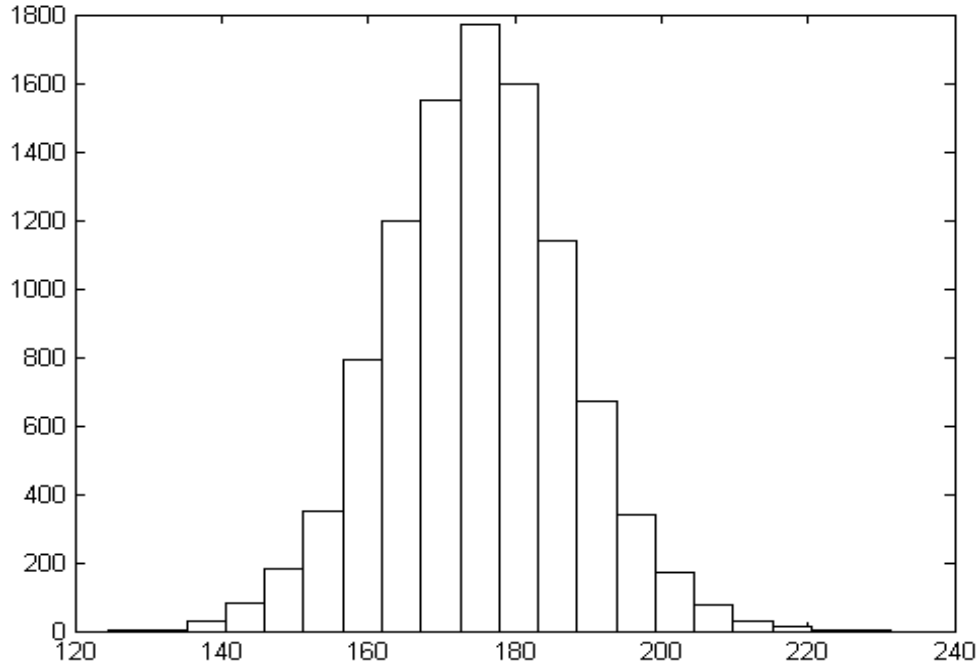
³The marginal posterior distributions for $p(\lambda_i|\mathbf{y}) \forall i = 1, \dots, 24$ can also easily be obtained. $i = 20$ was selected for illustrative purpose only.

Figure 5.9.1: Unknown Conditional Posterior Distribution of the Degrees of Freedom ν and a Normal Distribution.



Different starting values for the variance components were also considered for the iteration process for the burn - in period. For example, for $\sigma_\varepsilon^{2(0)}$, 500 and 2500 were used, while 400 and 2400 were used for $\sigma_a^{2(0)}$. Also, instead of simulating 48 different χ_5^2 values for λ_{ij} ($i = 1, \dots, 24, j = 1, 2$), all λ'_{ij} s were considered to be equal to 5, the expected value of this χ_5^2 distribution. The estimated marginal posterior distributions obtained from the Gibbs sampler were however for all practical purposes the same as those illustrated.

From Figure 5.9.1 it can be seen that the conditional posterior distribution of ν appears to be symmetrical and bell shaped. As mentioned, it was decided to fit a normal distribution over it, with mean and variance equal to $E(\nu|\boldsymbol{\lambda}, \mathbf{y})$ and $Var(\nu|\boldsymbol{\lambda}, \mathbf{y}) = E[(\nu|\boldsymbol{\lambda}, \mathbf{y})^2] - E[(\nu|\boldsymbol{\lambda}, \mathbf{y})]^2$ respectively. Remember also, that as mentioned, this normal distribution (illustrated by the solid line in Figure 5.9.1) fitted $p(\nu|\mathbf{y}, \boldsymbol{\lambda})$ so well, that as a result, the rejection sampling method as proposed by Rice (1995) was not used and ν was simulated directly from this normal distribution.

Figure 5.9.2: Histogram of the Estimated Marginal Posterior Distribution of μ .

From Figure 5.9.1 it is clear that the posterior mode is equal to 3, indicating that the most frequently used degrees of freedom to simulate the variance components and random effects with, was 3. The low number of degrees of freedom is also an indication that a student t - distribution will fit the measurement errors better than the normal distribution.

As expected, the histogram of the estimated marginal posterior distribution of μ (Figure 5.9.2), obtained using the student t - measurement error model, is symmetrical due to the fact that the conditional posterior distribution of μ (equation 5.6.1) is normal. Figure 5.9.2 also compares well with the posterior distributions of the production mean obtained using the normal multiplicative measurement error model, the log - normal measurement error model and the censored data multiplicative measurement error model which were illustrated in Figure 2 of Wilson et. al. (2004).

It should be noted that the histogram of the estimated marginal posterior distribution of σ_{ϵ}^2 is usually symmetrical or fairly symmetrical due to the high number of degrees

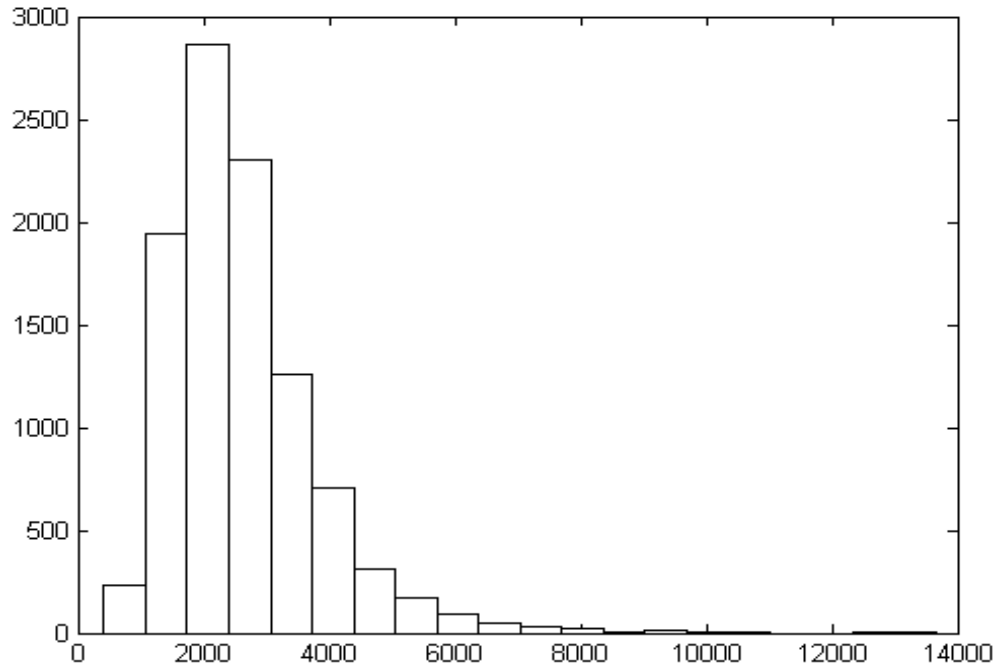
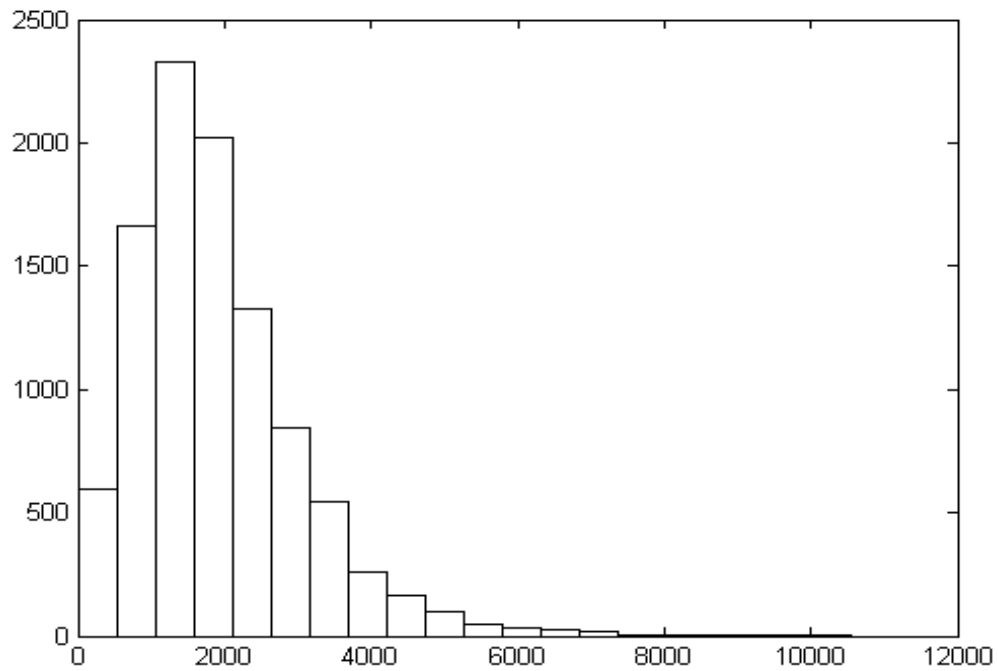
Figure 5.9.3: Histogram of the Estimated Marginal Posterior Distribution of σ_ε^2 .**Figure 5.9.4:** Histogram of the Estimated Marginal Posterior Distribution of σ_a^2 .

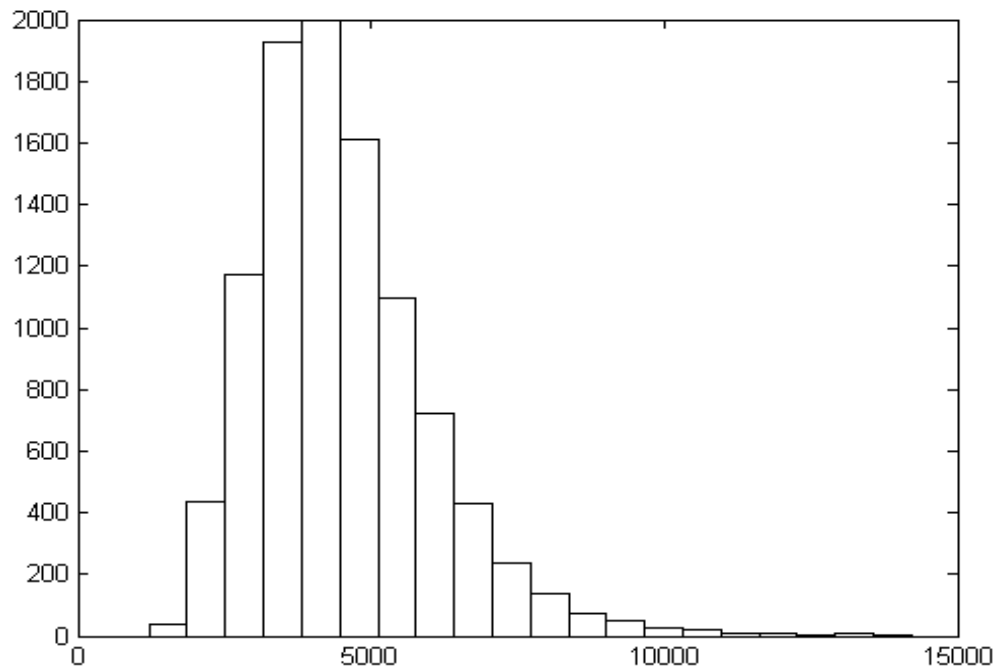
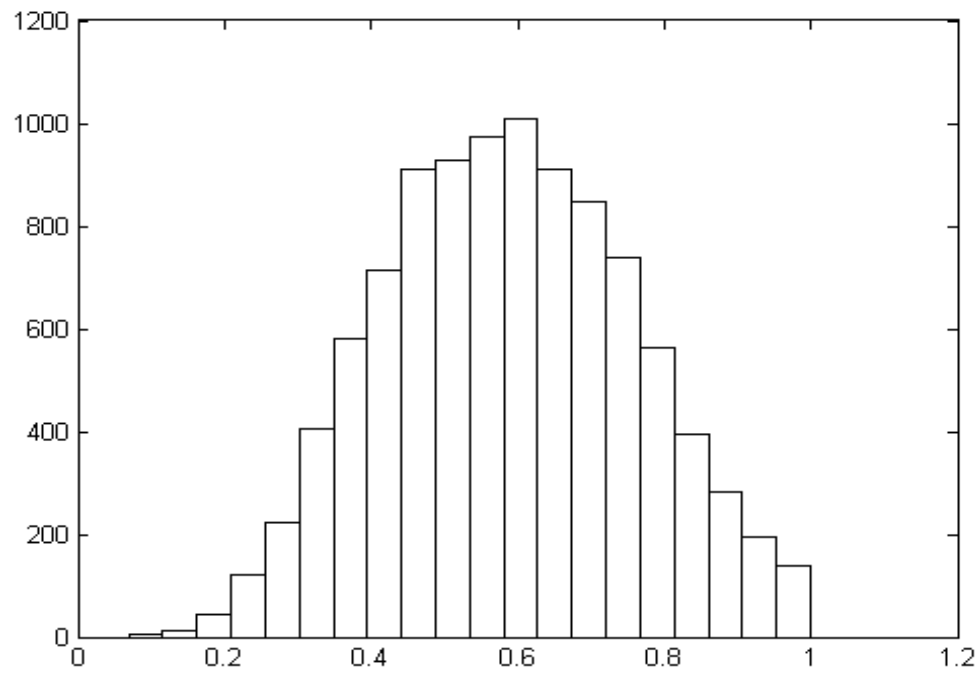
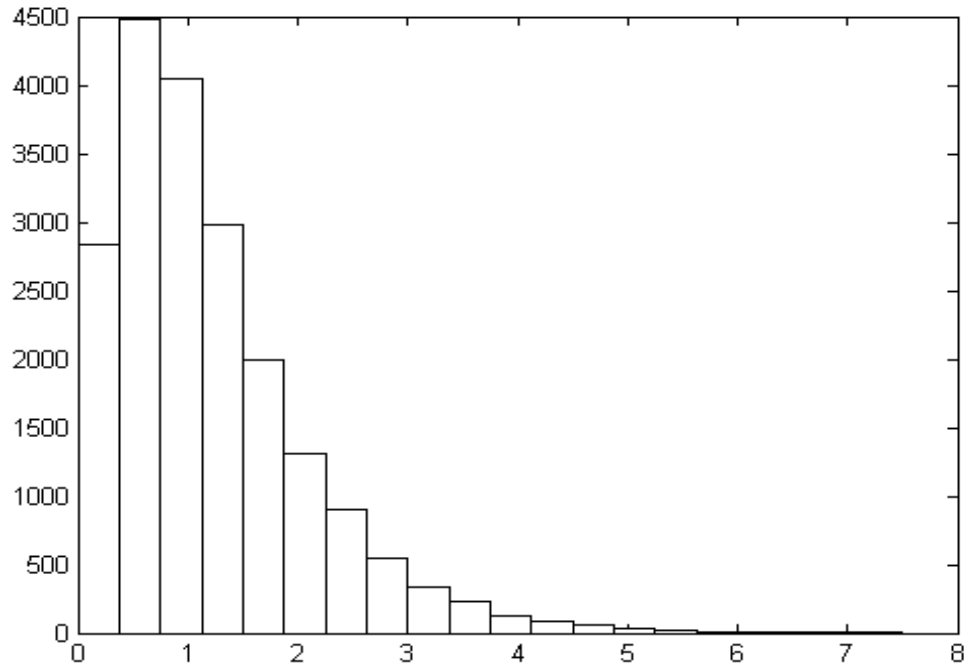
Figure 5.9.5: Histogram of the Estimated Marginal Posterior Distribution of $(\sigma_\varepsilon^2 + \sigma_a^2)$.**Figure 5.9.6:** Histogram of the Estimated Marginal Posterior Distribution of $\frac{\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + \sigma_a^2)}$.

Figure 5.9.7: Histogram of the Estimated Marginal Posterior Distribution of λ for the 20th Day.



of freedom associated with σ_ε^2 . If Figure 5.9.3 is however considered, it can be seen that the histogram is relatively positively skewed. The reason for this is the low number of degrees of freedom associated with σ_ε^2 i.e. $b(k-1) = 24$ which is due to the small number of 24 selected parts with only 2 measurement per part.

The histogram of the estimated marginal posterior distribution of σ_a^2 depicted in Figure 5.9.4 is similar to the graphs of the posterior distributions of the production variance displayed by Wilson et. al. (2004) (Figure 2) for the normal -, multiplicative -, and log - normal measurement error cases.

For comparative purposes the above simulations were also repeated for cases where the degrees of freedom were held fixed at 3, 5, 10, 15, 20, 25 and 30 and for the case where the residuals were considered to be random $N(0, \sigma_\varepsilon^2)$ variables. In Table 5.2 the estimated posterior median values for the variance components, functions of the variance components and λ_{20} are given.

Table 5.2: Estimated Posterior Median Values for Variance Components, Functions of Variance Components and λ_{20} .

Posterior Median for	ν Simulated	$\nu = 3$	$\nu = 5$	$\nu = 10$	$\nu = 15$
μ	174.9342	175.1402	175.0101	175.1528	175.2785
σ_ε^2	2394.9	2393.5	2168.9	2024.5	1935.9
σ_a^2	1678.4	1676.4	1666.9	1699.3	1720.5
$\sigma_\varepsilon^2 + \sigma_a^2$	4272.4	4310.1	4011.3	3863.4	3802.5
$\frac{\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + \sigma_a^2)}$	0.5849	0.5855	0.5598	0.5390	0.5228
λ_{20}	0.9930	1.0052	0.9865	0.9936	0.9953

Posterior Median for	$\nu = 20$	$\nu = 25$	$\nu = 30$	$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$
μ	175.3668	175.4160	175.1960	175.1925
σ_ε^2	1911.7	1893.5	1902.2	1823.3
σ_a^2	1716.1	1720.2	1719.8	1730.7
$\sigma_\varepsilon^2 + \sigma_a^2$	3739.5	3728.0	3720.9	3659.7
$\frac{\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + \sigma_a^2)}$	0.5218	0.5171	0.5172	0.5071
λ_{20}	0.9954	0.9976	0.9996	—

Comparing the results presented in Table 5.2 reveal that the target value μ , remains more or less constant for all cases mentioned. The same applies to the variance component measuring the between parts variation, σ_a^2 . This would be expected since neither μ nor σ_a^2 directly depends on the degrees of freedom ν . Even though the simulation of λ_{ij} ($i = 1, \dots, b, j = 1, \dots, k$) depends on the degrees of freedom, the posterior median values for λ_{20} , remained approximately the same for all the cases mentioned in Table 5.2. Also, recall that when ν was simulated, the estimated posterior mode of the estimated marginal posterior distribution of ν was equal to 3, indicating that the most frequently used degrees of freedom was 3. From Table 5.2 it is therefore evident that as the degrees of freedom increase and the student t - distributions thus

approaches a normal distribution, the estimated posterior median values for σ_ε^2 decrease and approach the estimated posterior median value of σ_ε^2 for the case where neither ν nor λ is present i.e. where it is assumed that the residuals follow a $N(0, \sigma_\varepsilon^2)$ distribution. The same can be seen for the estimated posterior median values for the total variance, $(\sigma_\varepsilon^2 + \sigma_a^2)$, and the variance ratio, $\frac{\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + \sigma_a^2)}$. This is also expected since both these quantities are functions of the error variance component, σ_ε^2 .

By considering Table 5.3, it can be seen that the estimated interval width for the location parameter μ decreases generally as the degrees of freedom increase and the student t - distribution thus approach a normal distribution. This is to be expected since the student t - distribution has heavier tails than a normal distribution. The same can be seen for σ_ε^2 , $(\sigma_\varepsilon^2 + \sigma_a^2)$, $\frac{\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + \sigma_a^2)}$ and λ_{20} . Although the interval width also decrease slightly for σ_a^2 , this decrease is not as dramatic as for σ_ε^2 .

Table 5.3: 95% Equal Tail Credibility Intervals and Interval Widths of Quantities of Interest.

Quantity of Interest	ν - simulated		$\nu = 3$		$\nu = 5$	
	95% CI	Width	95% CI	Width	95% CI	Width
μ	149.6802		150.8202		150.919	
	—		—		—	
	200.6703	50.99	200.0941	49.2738	199.0883	48.1693
σ_ε^2	1035.0		1080.3		1031.8	
	—		—		—	
	5338.7	4303.7	5587.4	4507.1	4542.1	3510.3
σ_a^2	536.5785		298.4185		543.4245	
	—		—		—	
	5298.0	4761.42	4662.8	4364.4	5105.8	4562.38
$\sigma_\varepsilon^2 + \sigma_a^2$	2467.6		2334.2		2424.2	
	—		—		—	
	8622.6	6155	8141.0	5806.8	7820.6	5396.4
$\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_a^2}$	0.2434		0.2650		0.2376	
	—		—		—	
	0.8661	0.6227	0.9268	0.6618	0.8557	0.6181
λ_{20}	0.1288		0.1453		0.2289	
	—		—		—	
	3.4817	3.3529	3.4058	3.2604	2.7097	2.4808

Quantity of Interest	$\nu = 10$		$\nu = 15$		$\nu = 20$	
	95% CI	Width	95% CI	Width	95% CI	Width
μ	151.5970		152.217		152.0222	
	–		–		–	
	198.7091	47.1121	197.9371	45.72	197.8950	45.873
σ_ε^2	1021.7		1038.09		1036.1	
	–		–		–	
	3905.2	2883.5	3688.0	2649.1	3558.4	2522.3
σ_a^2	556.6503		568.2349		567.6030	
	–		–		–	
	4984.9	4428.3	4965.0	4396.8	5039.9	4472.297
$\sigma_\varepsilon^2 + \sigma_a^2$	2422.4		2449.1		2444.7	
	–		–		–	
	7209.7	4787.3	7036.9	4587.8	7062.6	4617.9
$\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_a^2}$	0.2341		0.2302		0.2319	
	–		–		–	
	0.8407	0.6066	0.8303	0.6	0.8336	0.6017
λ_{20}	0.3652		0.4468		0.5026	
	–		–		–	
	2.1343	1.7691	1.8771	1.4303	1.7338	1.2312

Quantity of Interest	$\nu = 25$		$\nu = 30$		$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$	
	95% CI	Width	95% CI	Width	95% CI	Width
μ	152.3915		151.5411		152.69	
	–		–		–	
	198.48	46.09	197.669	46.13	198.099	45.41
σ_ε^2	1050.1		1039.6		1058.9	
	–		–		–	
	3528.0	2477.9	3467.5	2427.9	3292.3	2233.4
σ_a^2	553.369		581.535		574.825	
	–		–		–	
	4860.2	4306.831	4971.8	4390.27	4790.7	4215.88
$\sigma_\varepsilon^2 + \sigma_a^2$	2463.4		2475.6		2475.1	
	–		–		–	
	6962.3	4498.9	7027.2	4551.6	6773.9	4298.8
$\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_a^2}$	0.232		0.2274		0.2325	
	–		–		–	
	0.83	0.5980	0.8367	0.5993	0.8224	0.5899
λ_{20}	0.5432		0.5729		–	
	–		–		–	
	1.6625	1.1193	1.6027	1.0598	–	–

In the next section, tolerance intervals will be determined for the balanced one - way random effects model with student t - distributed measurement errors.

5.10 Tolerance Intervals

It was mentioned in previous chapters that in any production process, designers will specify tolerances for various characteristics. These characteristics are based on considerations of requirements for fit or function, in use, or in subsequent levels of assembly. The dimensions within which a produced part should fall in order to be acceptable, is a typical example (*Easterling, Johnson, Bement and Nachtsheim, 1991*). To protect against measurement error and to keep the production facility on its toes, designers sometimes specify tolerance limits with an interval width less than the width of the true required tolerance limits. Since these ad hoc tolerances may impose undue costs due to scrap or rework, it is desirable to take a more systematic look at the determination of tolerances, taking measurement error into account (*Easterling, Johnson, Bement and Nachtsheim, 1991*).

Based on this information, and, since the Bayesian model has been specified and the variance components and random effects have been determined, the three important research questions proposed by Wolfinger (1998) and stated in Chapter 1, can now be investigated for the data given in Table 5.1.

These three questions, and many similar ones, can be addressed by the three Bayesian tolerance intervals proposed by Wolfinger (1998) which allow inference about the quantiles of a probability distribution that is assumed to adequately describe a process (*Wolfinger, 1998*).

To reiterate, all three kinds of tolerance intervals can take the following forms: lower limit (t_ℓ, ∞) , an upper limit (∞, t_u) , or a two - sided limit (t_ℓ, t_u) . For further details about

confidence intervals and tolerance limits, see previous chapters as well as Hahn and Meeker (1991) and Wolfinger (1998).

A simulation based approach will now be presented for determining Bayesian tolerance intervals for the one - way variance component model given in equation 5.2.8 with student t - distributed measurement errors. The procedure will then be applied to the data given in Table 5.1. Since theorems proved for determining the α - expectation tolerance interval will also be used for the determination of the (α, δ) tolerance intervals, the α - expectation tolerance interval will be discussed first.

5.10.1 α - Expectation Tolerance Intervals

According to Wolfinger (1998), the α - expectation tolerance interval addresses research question 2 and focuses on prediction of one or a few future observations from the process.

By using the results given in equations 5.6.1 - 5.6.5, as well as equation 5.8.4, the following theorems can now be proved.

Theorem 5.10.1.1

The predictive distribution of the average of k^* future measurements $(y_{f1}, y_{f2}, \dots, y_{fk^*})$ for a new or unknown part, given the variance components σ_ε^2 and σ_a^2 , λ_{fj^*} ($j^* = 1, \dots, k^*$) and μ , is normally distributed with mean

$$E(\bar{y}_f | \mu, \sigma_\varepsilon^2, \sigma_a^2, \lambda_{f1}, \dots, \lambda_{fk^*}) = \mu \quad (5.10.1)$$

and variance

$$Var(\bar{y}_f | \mu, \sigma_\varepsilon^2, \sigma_a^2, \lambda_{f1}, \dots, \lambda_{fk^*}) = \frac{\sigma_\varepsilon^2}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{fj^*}} + \sigma_a^2 \quad (5.10.2)$$

Proof

The proof of Theorem 5.10.1.1 is given in Appendix C.

Theorem 5.10.1.2

The predictive distribution of the average of k^* future measurements $(\tilde{y}_{i1}, \tilde{y}_{i2}, \dots, \tilde{y}_{ik^*})$ for a specific part (i^{th} part) or a new part similar to the given specific part, given the variance components σ_ε^2 and σ_a^2 , and, $\lambda_{i11}, \lambda_{i12}, \dots, \lambda_{i1k^*}, \lambda_{i21}, \lambda_{i22}, \dots, \lambda_{i2k^*}$ as well as μ , is normally distributed with mean

$$E(\bar{\tilde{y}}_i | \mu, a_i, \lambda_{i11}, \dots, \lambda_{i2k^*}, \sigma_\varepsilon^2, \sigma_a^2) = \mu + a_i \quad (5.10.3)$$

and variance

$$Var(\bar{\tilde{y}}_i | \mu, a_i, \lambda_{i11}, \dots, \lambda_{i2k^*}, \sigma_\varepsilon^2, \sigma_a^2) = \frac{\sigma_\varepsilon^2}{k^{*2}} \left\{ p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right\} \quad (5.10.4)$$

where $p = 0.5$.

Proof

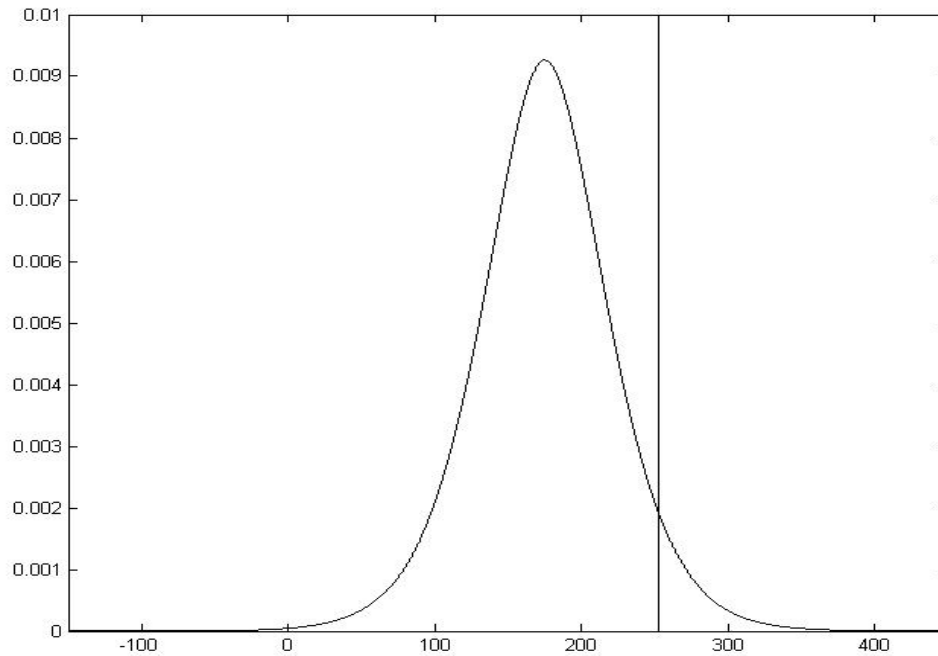
The proof of Theorem 5.10.1.2 is given in Appendix C.

To be able to provide comparative results with results obtained by Wilson et. al. (2004), all future posterior distributions and tolerance intervals will be based on a $N(\mu, \sigma_a^2)$ distribution which represents the variation amongst the parts. Figure 5.10.1 represents the unconditional predictive parts distribution x , which conditional on μ , σ_ε^2 , σ_a^2 and indirectly conditional on \mathbf{a} and $\boldsymbol{\lambda}$, is distributed $N(\mu, \sigma_a^2)$.

The Bayesian simulation procedure used for obtaining Figure 5.10.1 was performed in the following way.

- i.) By using the Gibbs sampling procedure discussed in section 5.7, the variance components and random effects were generated from the joint posterior distribution in the case where the degrees of freedom was simulated.

Figure 5.10.1: Estimated Unconditional Predictive Part Distribution, x , if ν is Simulated. Determined for the Iron Data Given in Table 5.1.



95% Upper Prediction Limit: 254.5517

This was also done for the cases where the degrees of freedom were held fixed.

- ii.) For each of the $\tilde{\ell} = 10000$ sets of simulated values for μ , σ_ε^2 , σ_a^2 , \mathbf{a} , $\boldsymbol{\lambda}$ and ν (if ν is also simulated), the conditional predictive part distribution, which is normally distributed with mean μ and variance σ_a^2 , was determined.
- iii.) Using the Rao Blackwell argument discussed in section 2.5 (see Gelfand and Smith, 1991), the unconditional predictive part distribution was obtained by averaging the conditional predictive part distributions over the $\tilde{\ell} = 10000$ repetitions.

For comparative purposes, the median values and 95% upper prediction limits for the predictive part distribution, for $\nu = 3$, $\nu = 5$, $\nu = 10$, $\nu = 15$, $\nu = 20$, $\nu = 25$, $\nu = 30$, ν simulated and the case where $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$ are given in Table 5.4.

Table 5.4: Median Values and 95% Upper Prediction Limits of the Predictive Part Distribution Determined for the Iron Data Given in Table 5.1.

Degrees of Freedom	Median	95% Upper Prediction Limit
ν simulated	174.5121	254.5517
$\nu = 3$	175.3703	253.7655
$\nu = 5$	175.1837	252.9174
$\nu = 10$	175.9202	253.4124
$\nu = 15$	175.1280	252.7596
$\nu = 20$	175.0276	252.6984
$\nu = 25$	175.4042	253.3415
$\nu = 30$	175.8456	254.0012
$\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$	175.0278	254.8190

For illustrative purposes, using results obtained if the degrees of freedom ν is simulated, it follows that the process is in control if 95% or more of the parts have iron concentration measurements that fall below 254.5517 ppm.

5.10.2 One - Sided (α, δ) Tolerance Intervals

Assuming that a process is in a state of control, research question 1 can be addressed by the (α, δ) tolerance intervals. As mentioned, these (α, δ) tolerance intervals are typically applied in cases requiring long - run prediction about numerous observations from this in - control process, and, inference is based on the actual quantiles of the assumed underlying probability distribution (Wolfinger, 1998). Based on an available sample of measurements, manufacturers often use these (α, δ) tolerance intervals to predict the future performance of a product (Wolfinger, 1998).

To construct one - sided (α, δ) tolerance intervals, the estimated marginal posterior distribution of q^* must be obtained. In this case, it represents the estimated marginal posterior distribution of the $(1 - \alpha)^{th}$ quantile of the

$$N \left\{ (\mu + a_i), \frac{\sigma_\varepsilon^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1 - p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right] \right\} \text{ distribution derived in Theorem 5.10.1.2.}$$

The average of future data of the i^{th} part or a part similar to the i^{th} part is described by this distribution. The Bayesian simulation procedure for obtaining the posterior distribution of q^* can be performed as follows:

i.) Use the Gibbs sampling procedure described in section 5.7 to generate the variance components σ_ε^2 and σ_a^2 , μ , λ and the random effects \mathbf{a} (as well as ν if ν is simulated) from the joint posterior distribution for the cases where ν were held fixed and for the case where ν was simulated.

ii.) For the retained Gibbs simulated values of μ , σ_ε^2 , σ_a^2 , λ and \mathbf{a} (as well as ν if ν is simulated), determine

$$q^* = (\mu + a_i) - z_\alpha \left\{ \frac{\sigma_\varepsilon^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1 - p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right] \right\}^{\frac{1}{2}}.$$

Remember, since $k = 2$ measurements were taken on the same part, q^* needs to be determined twice, once for each λ_{ij} for $j = 1, 2$.

iii.) Repeat steps i.) and ii.) $\tilde{\ell} = 10000$ times to obtain the histogram of the estimated marginal posterior distribution of the $(1 - \alpha)^{th}$ quantile of the normal distribution derived in Theorem 5.10.1.2.

For comparative purposes, the histogram of the estimated marginal posterior distribution of the 0.95^{th} quantile of the part distribution, is displayed in Figure 5.10.2 for the data given in Table 5.1. The histogram displayed in Figure 5.10.2 was obtained for the case where ν was simulated, by calculating

$$q_p^* = \mu + 1.645\sigma_a$$

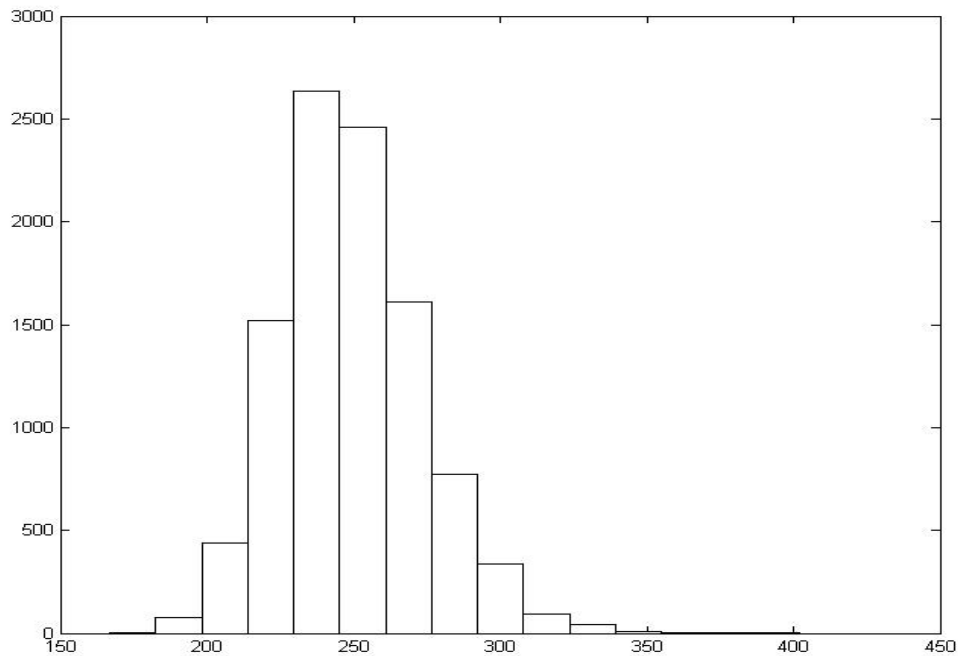
using the Gibbs sampled values for μ and σ_a^2 . The (0.95, 0.95) one - sided upper tolerance bound can then easily be obtained by ranking the simulated q_p^* values in order of magnitude and obtaining the 95th percentile of these ranked simulated values. The (0.95, 0.95) one - sided upper tolerance bounds for $\nu = 3, \nu = 5, \nu = 10, \nu = 15, \nu = 20, \nu = 25, \nu = 30, \nu$ simulated and the case where $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$, are given for comparative purposes in Table 5.5.

Table 5.5: 95% Upper Credibility Limits of the 0.95th Quantiles of the Part Distributions Determined for the Iron Data Given in Table 5.1.

Case	Upper Credibility Limits
ν simulated	291.9719
$\nu = 3$	291.4412
$\nu = 5$	290.0369
$\nu = 10$	289.5633
$\nu = 15$	288.5418
$\nu = 20$	289.1931
$\nu = 25$	288.7837
$\nu = 30$	287.7826
$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$	287.8199

From Table 5.5 it can be seen that the upper 95% credibility limit for the normally distributed errors is only somewhat smaller than the corresponding 95% credibility limits for the student t - distributed measurement errors with simulated degrees of freedom and degrees of freedom less than 25. Also, the histogram depicted in Figure 5.10.2 compares well with the posterior distributions of the 0.95 quantile of the part distribution (especially the normal and log - normal measurement error models) given in Figure 3 on page 200 of Wilson et. al. (2004). According to Wilson, Hamada and Xu (2004), the 95% upper tolerance bound to contain 95% of the population is 286.7 for the normal model, 277.6 for the log - normal model and 305.3 for the censored normal model.

Figure 5.10.2: Histogram of the Estimated Marginal Posterior Distribution of the 0.95th Quantile of the Part Distribution for the Data Given in Table 5.1.



Upper (0.95, 0.95) One - Sided Tolerance Limit: 291.9719

These authors also mentioned that the upper tolerance bound for the censored normal model will be higher, since the distribution for this model is somewhat wider due to the additional uncertainty introduced when exact measurements are not made. If the upper one - sided (0.95, 0.95) tolerance limits for the true part measurement are compared to these results obtained by Wilson, Hamada and Xu (2004), it can be seen that for the case assuming that ν is simulated, 291.9719 compares well with the above mentioned results by Wilson, et.al. (2004). Also, by considering Table 5.5, it can be seen that the results obtained for the different degrees of freedom used, also compare well with these results. For illustrative purposes using the case where ν was simulated, the upper one - sided (0.95, 0.95) tolerance limit equals to 291.9719 can be interpreted as follows: 95% of true emission spectroscopy values of iron in parts per million will fall below 291.9719 with probability 0.95.

Also, the histograms of the estimated marginal posterior distributions of

$$1. \quad q_{\ell}^* = (\mu + a_i) - 1.96 \left\{ \frac{\sigma_{\varepsilon}^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right] \right\}^{\frac{1}{2}} \text{ and}$$

$$2. \quad q_u^* = (\mu + a_i) + 1.96 \left\{ \frac{\sigma_{\varepsilon}^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right] \right\}^{\frac{1}{2}},$$

which in this case represent the $\left[\frac{(1 \pm 0.95)}{2} \right]^{th}$ quantiles of the $N \left\{ (\mu + a_i), \frac{\sigma_{\varepsilon}^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right] \right\}$ distribution derived in Theorem 5.10.2, as well as

$$3. \quad q_{\ell}^* = \mu - 1.96 \left\{ \frac{\sigma_{\varepsilon}^2}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{fj^*}} + \sigma_a^2 \right\}^{\frac{1}{2}} \text{ and}$$

$$4. \quad q_u^* = \mu + 1.96 \left\{ \frac{\sigma_{\varepsilon}^2}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{fj^*}} + \sigma_a^2 \right\}^{\frac{1}{2}},$$

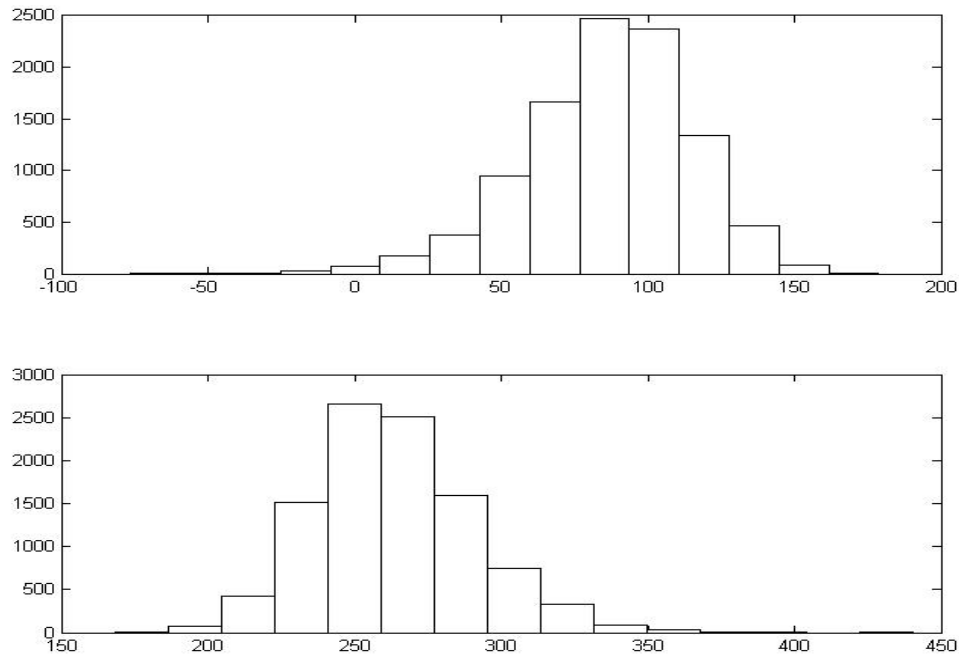
which in this case represent the $\left[\frac{(1 \pm 0.95)}{2} \right]^{th}$ quantiles of the $N \left(\mu, \frac{\sigma_{\varepsilon}^2}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{fj^*}} + \sigma_a^2 \right)$ distribution derived in Theorem 5.10.1.1, can also easily be determined by using steps i.) to iii.) described for the simulation procedure. Remember that for 1. and 2. above, q_{ℓ}^* and q_u^* need to be determined twice, once for each λ_{ij} for $j = 1, 2$. Also, for 3. and 4. above, λ_{fj^*} ($j^* = 1, \dots, k^*$) are simulated from the prior distribution of λ , since a future observation for any new or unknown part is simulated.

For illustrative purposes, Figure 5.10.3, displays the histograms of the estimated marginal posterior distributions of 5. and 6. below, where

$$5. \quad q_{\ell}^* = \mu - 1.96 \left\{ \sigma_a^2 \right\}^{\frac{1}{2}} \text{ and}$$

$$6. \quad q_u^* = \mu + 1.96 \left\{ \sigma_a^2 \right\}^{\frac{1}{2}},$$

Figure 5.10.3: Histograms of the Estimated Marginal Posterior Distributions of the $\left[\frac{(1 \pm 0.95)}{2}\right]^{th}$ Quantiles of the Part Distribution for the Data Given in Table 5.1.



which in this case represent respectively the $\left(\frac{1-0.95}{2}\right)^{th}$ and $\left(\frac{1+0.95}{2}\right)^{th}$ quantiles of the part distribution for the iron data given in Table 5.1.

5.10.3 Two - Sided (α, δ) Tolerance Intervals

It was already mentioned that the construction of two - sided (α, δ) tolerance intervals are more complex than the construction of one - sided (α, δ) tolerance intervals. Since the posterior correlation between q_ℓ^* and q_u^* determined in section 5.10.2 is not equal to 1, the q_ℓ^* and q_u^* values can also not just be combined for determining a valid two - sided (α, δ) tolerance interval for the student t - distributed measurement error model. Instead, the method as proposed by Wolfinger (1998) should also be used.

It is therefore suggested (see Wolfinger, 1998) that a valid two - sided (α, δ) tolerance

interval for the student t - distributed measurement error model can be determined by computing the two quantiles

$$q_l^* = \mu - z_{\frac{1+\alpha}{2}} \left\{ \sigma_a^2 \right\}^{\frac{1}{2}} \text{ and}$$

$$q_u^* = \mu + z_{\frac{1+\alpha}{2}} \left\{ \sigma_a^2 \right\}^{\frac{1}{2}} .$$

These (q_l^*, q_u^*) pairs then form a sample from the bivariate posterior distribution of the $\left[\frac{(1-\alpha)}{2} \right]^{th}$ and $\left[\frac{(1+\alpha)}{2} \right]^{th}$ quantiles of the true part distribution.

In order to obtain a valid two - sided (α, δ) tolerance interval that is one - dimensional and symmetric about the posterior mean, Wolfinger (1998) suggested to first form a scatter plot of q_l^* versus q_u^* , with q_l^* plotted on the vertical axis. Then as mentioned previously, proceed by constructing the reference line given by

$$q_l^* = -q_u^* + 2\bar{y}_{..} .$$

Two additional lines are then drawn, one parallel to each axis and intersecting on the reference line. This intersecting point is then slid along the reference line until $100(1-\delta)\%$ of the (q_l^*, q_u^*) pairs are contained in the half - rectangle opening towards the lower right portion of the graph. The coordinates of the resulting intersection point form a two - sided (α, δ) tolerance interval of the desired form. In Figure 5.10.4 this procedure is graphically illustrated for the construction of a valid two - sided $(0.90, 0.95)$ tolerance interval for the part distribution of the data given in Table 5.1 for the case where ν was simulated.

In Table 5.6, the two - sided $(0.90, 0.95)$ tolerance intervals determined for the part distribution of the data given in Table 5.1, are given for $\nu = 3, \nu = 5, \nu = 10, \nu = 15, \nu = 20, \nu = 25, \nu = 30, \nu$ simulated and the case where $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$.

Figure 5.10.4: Constructing a Two - Sided (0.90, 0.95) Tolerance Interval for the Part Distribution of the Data Given in Table 5.1.

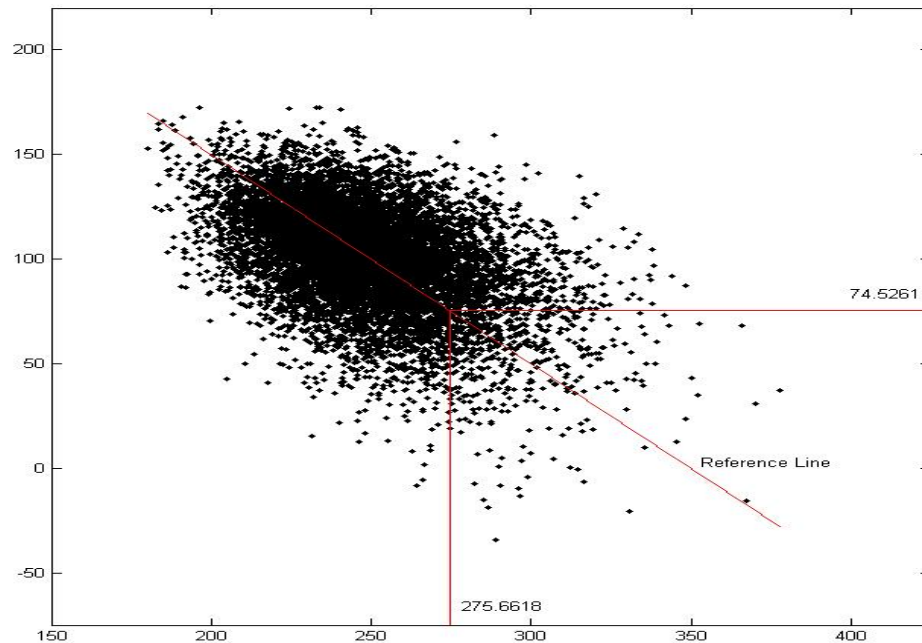


Table 5.6: Two - Sided (0.90, 0.95) Tolerance Intervals for Different Cases for the Data Given in Table 5.1.

Case	Average of q Unknown or Future Parts	Interval Width
ν simulated	74.5261 – 275.6618	201.1357
$\nu = 3$	75.5734 – 274.4092	198.8358
$\nu = 5$	75.5011 – 274.2497	198.7486
$\nu = 10$	76.5824 – 273.9850	197.4025
$\nu = 15$	76.8208 – 274.1172	197.2964
$\nu = 20$	76.8048 – 273.4648	196.6600
$\nu = 25$	77.5064 – 273.2710	195.7646
$\nu = 30$	76.6002 – 273.2897	196.6895
$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$	78.4079 – 272.0621	193.6542

For illustrative purposes, the two - sided (0.90, 0.95) tolerance interval for the part dis-

tribution with ν simulated, given by [74.5261 – 275.6618], can be interpreted as follows: 90% of emission spectroscopy true values of iron in parts per million will fall in the interval [74.5261, 275.6618] with probability 0.95.

Although not given here, two - sided (0.90, 0.95) tolerance intervals can also be constructed for the quantiles of the $N\left(\mu + a_i, \frac{\sigma_\varepsilon^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right]\right)$ distribution derived in Theorem 5.10.1.2 using 1. and 2. in section 5.10.2, or for the quantiles of the $N\left(\mu, \frac{\sigma_\varepsilon^2}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{fj^*}} + \sigma_a^2\right)$ distribution derived in Theorem 5.10.1.1 using 3. and 4. given in section 5.10.2.

5.10.4 Fixed - in - Advanced Tolerance Intervals

As mentioned before, fixed - in - advance tolerance intervals invert the prediction problem by considering the content of predetermined bounds, and according to Wolfinger (1998), answer research question 3 mentioned in Chapter 1.

To determine the content of a fixed - in - advance tolerance interval, the posterior distribution of the content has to be determined. For example, suppose an upper fixed - in - advance limit s is specified for data assumed to arise from a new batch. Then for the average of k^* future measurements on the i^{th} part or future part similar to the i^{th} part, compute

$$c^* = 1 - \Phi \left[\frac{s - (\mu + a_i)}{\sqrt{\frac{\sigma_\varepsilon^2}{k^{*2}} \left[p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right]}} \right] \quad (5.10.5)$$

where c^* has to be determined k times, once for each λ_{ij} for $j = 1, \dots, k$. Also, for the true part measurements, compute

$$c^* = 1 - \Phi \left[\frac{s - \mu}{\sqrt{\sigma_a^2}} \right]. \quad (5.10.6)$$

It must be noted that for both of the above cases, $\Phi[\cdot]$ represents the standard normal cumulative distribution function. As mentioned previously, the simulated c^* values represent a sample from the posterior distributions of the content of the interval $[s, \infty]$.

To illustrate, suppose that a lower fixed - in - advance limit of $s = 225$ ppm is specified for the data given in Table 5.1. This limit is selected, since according to Wilson, Hamada and Xu (2004), a part is considered to be acceptable if it has under 225 ppm of iron. By selecting $s = 225$ as lower fixed - in - advance limit, the content of the interval $(225, \infty)$ is determined for each observation in the sample of simulated parameter values. In other words, the posterior distribution of the proportion of parts produced by the process that is not acceptable, is estimated.

Using equations 5.10.6, the histogram of the sample of simulated c^* values from the posterior distribution of the content of the interval $[225, \infty]$ for the part distribution, with ν simulated, is displayed in Figure 5.10.5. Figure 5.10.5 therefore represents the estimated posterior distribution of the content above the preselected specification limit $s = 225$ ppm.

From Figure 5.10.5 it can be seen that the posterior contents are positively skewed and has a posterior median equals to 0.1268. This means that if the degrees of freedom ν is simulated, on average 12.68% unacceptable parts will be produced by the manufacturing process with probability 0.95.

For comparative purposes, the posterior median values and 95% equal tail credibility intervals for the content of the interval $[225, \infty]$ for a fixed - in - advance lower limit $s = 225$ ppm of iron, for the part distribution are given in Table 5.7 for the cases where $\nu = 3, \nu = 5, \nu = 10, \nu = 15, \nu = 20, \nu = 25, \nu = 30, \nu$ simulated and the case where $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$.

Figure 5.10.5: Histogram of the Estimated Posterior Distribution of the Content of the Interval $[225, \infty]$ for a Fixed - in - Advance Lower Limit $s = 225$ ppm of Iron for the Part Distribution with ν Simulated. Determined for the Data Given in Table 5.1.

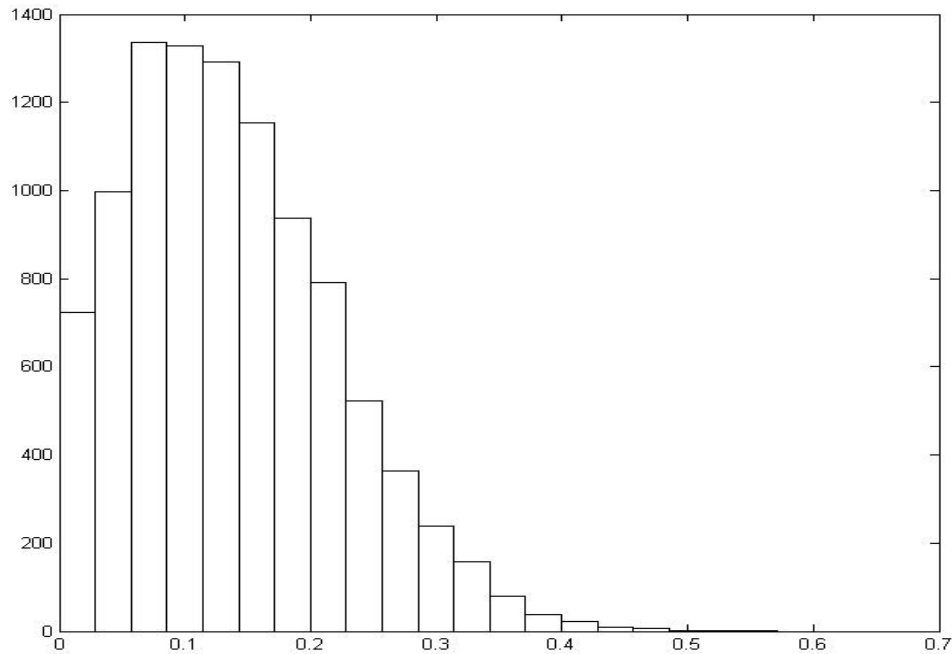


Table 5.7: Posterior Median Values and 95% Equal Tail Credibility Intervals of the Content of the Interval $[225, \infty]$ for a Fixed - in - Advanced Lower Limit $s = 225$ ppm of Iron for Different Cases for the Data Given in Table 5.1.

Case	Median	95% Equal Tail Credibility Interval	Interval Width
ν simulated	0.1268	0.0106 – 0.3237	0.3131
$\nu = 3$	0.1264	0.0091 – 0.3236	0.3145
$\nu = 5$	0.1267	0.0099 – 0.3146	0.3047
$\nu = 10$	0.1283	0.0122 – 0.3160	0.3039
$\nu = 15$	0.1303	0.0138 – 0.3104	0.2967
$\nu = 20$	0.1290	0.0133 – 0.3133	0.3001
$\nu = 25$	0.12.96	0.0134 – 0.3144	0.3011
$\nu = 30$	0.1279	0.0148 – 0.3077	0.2928
$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$	0.1286	0.0123 – 0.3096	0.2973

From Table 5.7 it can be seen that the 95% credibility intervals of the content of the interval $[225, \infty]$ for a fixed - in - advance lower limit $s = 225$ ppm of iron for the part distribution tend to be narrower and the posterior median values appear to be slightly higher as the degrees of freedom increase. Although these mentioned median values given in Table 5.7 generally appear to increase as the degrees of freedom increase, they are also for all practical purposes the same. Also, using equation 5.10.5, the fixed - in - advance tolerance interval for the content of the interval $[225, \infty]$ for the average of k^* measurements on the i^{th} part or a similar future part can be determined, although it is not given here.

5.11 Checking for Outliers

In the previous sections, the assumption of Gaussian errors was relaxed in the direction of the student t - family to accommodate for the possibility of outlying part measurements. If present, these possible outlying part measurements generated by the balanced one - way random effects model given in equation 5.2.7 will have unexpectedly large random errors $(\varepsilon'_{ij}s)$, and as a result, are thus considered outliers, since Chaloner and Brant (1988) defined outliers in linear models as observations with surprisingly large realized absolute errors.

The identification of outliers is an important problem, since according to Barnett and Lewis (1994), in almost every real data set observations will be found which differ so much from the other observations that some abnormal source of error, not contemplated in the theory, can be inferred. Although surveys from 18th century statistical literature have indicated awareness of the outlier problem, it was not until the 1850's that first attempts to develop statistically objective methods for dealing with outliers were reported (Meiring, van der Merwe and Viljoen, 1998). Even though since then there were many contributors investigating the topic, it was not until the early 1960's

that the first mathematical attempts to treat the outlier problem were published by Srikantan (1961) and Ferguson (1961).

Later on, authors such as Box and Tiao (1968), Abraham and Box (1978), Guttman, Dutter and Freeman (1978) and Freeman (1980) defined outliers as arising from a separate, expanded model. These authors all employed a Bayesian approach to outlier detection. The method proposed by Chaloner and Brant (1988) however defined outliers as arising from the model under consideration, rather than arising from an expanded model as previously proposed.

Guttman (1973) also presented a Bayesian approach to the identification of a single outlier in multivariate normal distributions, while Weisberg (1985) utilized standardized residuals based on a student t - distribution. Weisberg (1985) used the Bonferroni inequality to provide critical values. Geisser (1987) on the other hand, suggested a Bayesian predictive method based on the predictive distribution $p(y_i|y_{(i)})$ which he recommended for regression problems where $p(y_i|y_{(i)})$ had different scale factors. Geisser (1987) showed algebraically that in the case of linear models using non-informative prior distributions, his proposed predictive discordancy diagnostics were closely related to standardized residuals. He obtained p - values from the outlier tests based on comparing these external standardized residuals to their student t - distribution, without using the Bonferroni inequality. Other related more general conditional predictive discordancy diagnostics were also suggested and discussed by Geisser (1987). For more information, see Geisser (1980), (1987), (1988a) and (1988b). Kass and Raftery (1995) also proposed a Bayesian method for outlier detection based on prior and posterior odds, while Varbanov (1996), used Markov chain Monte Carlo simulation to obtain posterior probabilities used for declaring the i^{th} observation an outlier.

As was mentioned, the principle aim of the remainder of this section is the detection of outlying part measurements utilizing the mentioned student t - distributed measurement error model.

According to Wakefield et. al. (1994) the scale parameter λ_{ij} or $\log_{10}(\lambda_{ij})$ serve as a good indicator to detect outlying part measurements, since there is a λ_{ij} for each measurement on each part. The prior expectation of $\log_{10}(\lambda_{ij})$ is 0, so that a $\log_{10}(\lambda_{ij})$ -value substantially below zero, indicates that the j^{th} measurement on the i^{th} part is likely to be an outlier (*van der Merwe and du Plessis, 1996*). Interval estimates of $\log_{10}(\lambda_{ij})$ can also easily be constructed.

In Tables 5.8 and 5.9 the 90% equal tail credibility intervals of λ_{ij} and $\log_{10}(\lambda_{ij})$ (for $j = 1, 2$) are given for the j^{th} measurement on the i^{th} part for the case where ν was simulated. By considering both Tables 5.8 and 5.9, it can be seen that the first measurement of part 22 can be considered an outlying observation, since both the upper and lower limits of the 90% equal tail credibility interval of $\log_{10}(\lambda_{i=22j=1})$ is below zero. As a result, zero is therefore not contained in the interval. This is to be expected since the first measurements of part 22 i.e. 439.5, is substantially larger than the rest of the observations given in Table 5.1. All other 90% equal tail credibility intervals for $\log_{10}(\lambda_{ij})$ (for $j = 1, 2$) covered zero, resulting in none of the other measurements to be considered as possible outliers.

Table 5.8: 90% Equal Tail Credibility Intervals for λ_{i1} and $\log_{10}(\lambda_{i1})$ for the Case where ν was Simulated for the First Measurement of the Data Given in Table 5.1.

Part Number	λ_{i1}	$\log_{10}(\lambda_{i1})$
1	0.2104 – 2.9213	–0.677 – 0.4656
2	0.2140 – 2.9630	–0.6696 – 0.4717
3	0.2047 – 3.0019	–0.6889 – 0.4774
4	0.1998 – 2.971	–0.6994 – 0.4729
5	0.2041 – 2.9791	–0.6902 – 0.4741
6	0.2091 – 3.0294	–0.6796 – 0.4814
7	0.1704 – 2.6145	–0.7685 – 0.4174
8	0.2034 – 2.9875	–0.6916 – 0.4753
9	0.1987 – 2.8872	–0.7018 – 0.4605
10	0.2053 – 2.9108	–0.6876 – 0.464
11	0.1782 – 2.751	–0.7491 – 0.4395
12	0.1786 – 2.577	–0.7481 – 0.4111
13	0.1248 – 2.1969	–0.9038 – 0.3418
14	0.1984 – 2.9595	–0.7025 – 0.4712
15	0.2047 – 2.9528	–0.6889 – 0.4702
16	0.2095 – 2.9632	–0.6788 – 0.4718
17	0.1985 – 2.9585	–0.7022 – 0.4711
18	0.2024 – 3.0077	–0.6938 – 0.4782
19	0.2079 – 2.9697	–0.6821 – 0.4727
20	0.1934 – 2.8861	–0.7135 – 0.4603
21	0.1584 – 2.5484	–0.8002 – 0.4063
22	0.0337 – 0.8379	–1.4724 – –0.0768
23	0.191 – 2.928	–0.719 – 0.4666
24	0.1383 – 2.3124	–0.8592 – 0.3641

Table 5.9: 90% Equal Tail Credibility Intervals for λ_{i2} and $\log_{10}(\lambda_{i2})$ for the Case where ν was Simulated for the Second Measurement of the Data Given in Table 5.1.

Part Number	λ_{i2}	$\log_{10}(\lambda_{i2})$
1	0.14726 – 2.5009	–0.8309 – 0.3981
2	0.2109 – 2.9867	–0.6759 – 0.4752
3	0.2046 – 3.0226	–0.6891 – 0.4804
4	0.2045 – 3.0174	–0.6893 – 0.4796
5	0.2000 – 2.8625	–0.699 – 0.4567
6	0.1834 – 2.7852	–0.7366 – 0.4449
7	0.1942 – 2.9645	–0.7118 – 0.4704
8	0.2072 – 2.9645	–0.6836 – 0.472
9	0.2124 – 3.0061	–0.6728 – 0.478
10	0.2037 – 2.9831	–0.691 – 0.4747
11	0.2073 – 2.8716	–0.6834 – 0.4581
12	0.2115 – 2.8525	–0.6747 – 0.4552
13	0.1623 – 2.7152	–0.7897 – 0.4338
14	0.1847 – 2.8386	–0.7335 – 0.4531
15	0.1951 – 2.8532	–0.7097 – 0.4553
16	0.2081 – 3.0224	–0.6817 – 0.4809
17	0.1996 – 2.9745	–0.6998 – 0.4734
18	0.1896 – 2.8209	–0.7222 – 0.4504
19	0.1993 – 2.9732	–0.7005 – 0.4732
20	0.2183 – 2.9522	–0.6609 – 0.4701
21	0.1686 – 2.6414	–0.7731 – 0.4218
22	0.1292 – 2.5478	–0.8887 – 0.4062
23	0.1767 – 2.6822	–0.7528 – 0.4285
24	0.1571 – 2.6094	–0.8038 – 0.4156

5.12 Appendix C

Definition of the Student t - Distribution

Let Z be a standard normal random variable and let χ_ν^2 be an independent chi - square random variable with ν degrees of freedom. The random variable

$$t = \frac{Z}{\sqrt{\frac{\chi_\nu^2}{\nu}}}$$

is then said to follow a student t - distribution with ν degrees of freedom and density function given by:

$$f(t) = \frac{\Gamma\left[\frac{\nu+1}{2}\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \text{ for } -\infty < t < \infty.$$

(Milton and Arnold, 1990)

Proof of Theorem 5.2.1

If $x \sim N(\mu, \sigma^2)$ then it follows from a normal density that

$$f(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right].$$

Therefore, if $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ it follows that

$$f(\varepsilon|\sigma_\varepsilon^2) = \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2\right]$$

and

$$\begin{aligned} f(\varepsilon|\sigma_\varepsilon^2) &= \prod_{i=1}^b \prod_{j=1}^k \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{\varepsilon_{ij}}{\sigma_\varepsilon}\right)^2\right] \\ &= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{bk}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^b \sum_{j=1}^k \left(\frac{\varepsilon_{ij}}{\sigma_\varepsilon}\right)^2\right] \\ &= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{bk}{2}} \exp\left[-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^b \sum_{j=1}^k \varepsilon_{ij}^2\right]. \end{aligned}$$

Now, if $\varepsilon_{ij} \sim N\left(0, \frac{\sigma_\varepsilon^2}{\lambda_{ij}}\right)$,

then

$$\begin{aligned} f(\varepsilon_{ij} | \sigma_\varepsilon^2, \lambda_{ij}) &= \left(\frac{1}{2\pi \frac{\sigma_\varepsilon^2}{\lambda_{ij}}} \right)^{\frac{1}{2}} \exp \left[-\frac{1}{2} \left(\frac{\varepsilon_{ij}}{\sqrt{\frac{\sigma_\varepsilon^2}{\lambda_{ij}}}} \right)^2 \right] \\ &= \left(\frac{1}{2\pi \sigma_\varepsilon^2} \right)^{\frac{1}{2}} \lambda_{ij}^{\frac{1}{2}} \exp \left[-\frac{1}{2\sigma_\varepsilon^2} \lambda_{ij} \varepsilon_{ij}^2 \right] \end{aligned}$$

and

$$\begin{aligned} f(\boldsymbol{\varepsilon} | \boldsymbol{\lambda}, \sigma_\varepsilon^2) &= \prod_{i=1}^b \prod_{j=1}^k \left(\frac{\lambda_{ij}}{2\pi \sigma_\varepsilon^2} \right)^{\frac{1}{2}} \exp \left[-\frac{1}{2} \left(\frac{\varepsilon_{ij}}{\sqrt{\frac{\sigma_\varepsilon^2}{\lambda_{ij}}}} \right)^2 \right] \\ &= \left(\frac{1}{2\pi \sigma_\varepsilon^2} \right)^{\frac{bk}{2}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}} \exp \left[-\frac{1}{2} \sum_{i=1}^b \sum_{j=1}^k \left(\frac{\varepsilon_{ij} \sqrt{\lambda_{ij}}}{\sigma_\varepsilon} \right)^2 \right] \\ &= \left(\frac{1}{2\pi \sigma_\varepsilon^2} \right)^{\frac{bk}{2}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}} \exp \left[-\frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} \varepsilon_{ij}^2 \right]. \end{aligned}$$

Proof of Theorem 5.2.2

Since it is assumed that $\nu \lambda_{ij} \sim \chi_\nu^2$ it is known that

$$p(\nu \lambda_{ij}) = \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} (\nu \lambda_{ij})^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu \lambda_{ij}}.$$

Let $y = \nu \lambda_{ij}$, then

$$p(y) = \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} y^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}y} \quad \text{for } y > 0, x > 0. \quad (5.12.1)$$

Also, let $x = \frac{y}{\nu}$,

then $y = \nu x$.

To obtain the distribution of x , the following transformation from y to x has to be made.

To do the transformation, first obtain the jacobian of the transformation from y to x which is given by

$$J(y \rightarrow x) = \frac{\partial y}{\partial x} = \nu,$$

therefore

$$\begin{aligned} p(x) &= \frac{1}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \nu x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu x} \left| J(y \rightarrow x) \right| \\ &= \frac{\nu^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu x} \nu \\ &= \frac{\nu^{\frac{1}{2}(\nu-2)} \nu}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu x} \\ &= \frac{\nu^{\frac{1}{2}(\nu-2+2)}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu x} \\ &= \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu x} \quad \text{for } x > 0. \end{aligned} \tag{5.12.2}$$

Since $x = \frac{y}{\nu}$ and $y = \nu \lambda_{ij}$,

it can easily be shown that

$$x = \frac{\nu \lambda_{ij}}{\nu} = \lambda_{ij}.$$

Therefore, substitute $x = \lambda_{ij}$ in 5.12.2 and get

$$p(\lambda_{ij}) = \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \lambda_{ij}^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu \lambda_{ij}} \quad \text{for } \lambda_{ij} > 0.$$

Proof of Theorem 5.2.3

From equation 5.2.1, the conditional density of $\varepsilon_{ij} | \lambda_{ij}, \sigma_{\varepsilon}^2$ is given by

$$p(\varepsilon_{ij} | \lambda_{ij}, \sigma_{\varepsilon}^2) = \left(\frac{1}{2\pi\sigma_{\varepsilon}^2} \right)^{\frac{1}{2}} \lambda_{ij}^{\frac{1}{2}} \exp\left\{ -\frac{1}{2\sigma_{\varepsilon}^2} \lambda_{ij} \varepsilon_{ij}^2 \right\}.$$

Now, the conditional density of $\varepsilon_{ij} | \sigma_{\varepsilon}^2$ can be written as

$$p(\varepsilon_{ij} | \sigma_{\varepsilon}^2) = \int_0^{\infty} p(\varepsilon_{ij} | \lambda_{ij}, \sigma_{\varepsilon}^2) p(\lambda_{ij}) d\lambda_{ij}.$$

From equation 5.2.3, it follows that $p(\lambda_{ij}) = \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu \lambda_{ij}}$.

Therefore

$$\begin{aligned}
p(\varepsilon_{ij}|\sigma_\varepsilon^2) &= \int_0^\infty \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \lambda_{ij}^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \lambda_{ij} \varepsilon_{ij}^2\right) \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \lambda_{ij}^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu\lambda_{ij}} \partial\lambda_{ij} \\
&= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^\infty \lambda_{ij}^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \lambda_{ij} \varepsilon_{ij}^2\right) \lambda_{ij}^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}\nu\lambda_{ij}} \partial\lambda_{ij} \\
&= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^\infty \lambda_{ij}^{\frac{1}{2}+\frac{1}{2}\nu-1} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \lambda_{ij} \varepsilon_{ij}^2 - \frac{1}{2}\nu\lambda_{ij}\right) \partial\lambda_{ij} \\
&= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^\infty \lambda_{ij}^{\frac{1}{2}(\nu+1)-1} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_\varepsilon^2} \lambda_{ij} \varepsilon_{ij}^2 + \nu\lambda_{ij}\right)\right] \partial\lambda_{ij} \\
&= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^\infty \lambda_{ij}^{\frac{1}{2}(\nu+1)-1} \exp\left[-\frac{1}{2}\lambda_{ij}\left(\frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2} + \nu\right)\right] \partial\lambda_{ij}.
\end{aligned}$$

Consider the integral

$$\int_0^\infty \lambda_{ij}^{\frac{1}{2}(\nu+1)-1} \exp\left[-\frac{1}{2}\lambda_{ij}\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)\right] \partial\lambda_{ij}, \text{ and,}$$

$$\text{let } y = \frac{1}{2}\lambda_{ij}\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right).$$

It then follows that

$$\lambda_{ij} = 2y\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}.$$

The jacobian of the transformation from λ_{ij} to y is

$$J(\lambda_{ij} \rightarrow y) = \frac{\partial\lambda_{ij}}{\partial y} = 2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}, \text{ and}$$

the integral can therefore be written as

$$\begin{aligned}
&\int_0^\infty \left[2y\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}\right]^{\frac{1}{2}(\nu+1)-1} \exp\left[-y\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)\right] \cdot 2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1} \partial y \\
&= \left[2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}\right]^{\frac{1}{2}(\nu+1)-1} \cdot 2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1} \int_0^\infty y^{\frac{1}{2}(\nu+1)-1} e^{-y} \partial y. \tag{5.12.3}
\end{aligned}$$

Since $\int_0^\infty y^{\frac{1}{2}(\nu+1)-1} e^{-y} dy = \Gamma\left[\frac{1}{2}(\nu+1)\right]$, equation 5.12.3 can be written as

$$\begin{aligned} & \left[2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}\right]^{\frac{1}{2}(\nu+1)-1} \cdot 2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1} \Gamma\left[\frac{1}{2}(\nu+1)\right] \\ &= \left[2\left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-1}\right]^{\frac{1}{2}(\nu+1)} \Gamma\left[\frac{1}{2}(\nu+1)\right] \\ &= 2^{\frac{1}{2}(1+\nu)} \left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}(\nu+1)} \Gamma\left[\frac{1}{2}(\nu+1)\right]. \end{aligned}$$

Therefore, the conditional density of $\varepsilon_{ij}|\sigma_\varepsilon^2$ is given by

$$\begin{aligned} p(\varepsilon_{ij}|\sigma_\varepsilon^2) &= \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} 2^{\frac{1}{2}(\nu+1)} \left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}(\nu+1)} \Gamma\left[\frac{1}{2}(\nu+1)\right] \\ &= \nu^{\frac{1}{2}\nu} \Gamma\left[\frac{1}{2}(\nu+1)\right] \left(\frac{1}{2\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} 2^{\frac{1}{2}(\nu+1)} \left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}(\nu+1)} \\ &= \nu^{\frac{1}{2}\nu} \cdot \Gamma\left[\frac{1}{2}(\nu+1)\right] \cdot \left(\frac{1}{2}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{\pi\sigma_\varepsilon^2}\right)^{\frac{1}{2}} \cdot \frac{1}{2^{\frac{\nu}{2}}} \cdot \frac{1}{\Gamma(\frac{\nu}{2})} \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{\nu}{2}} \cdot \left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}(\nu+1)} \\ &= \frac{\nu^{\frac{1}{2}\nu} \Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi\sigma_\varepsilon^2)^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2}\right)^{-\frac{1}{2}(\nu+1)} \quad \text{for } -\infty < \varepsilon_{ij} < \infty. \end{aligned}$$

To show that equation 5.2.4 is in the general form of a univariate t -distribution, consider the following:

Since $(\varepsilon_{ij}|\lambda_{ij}, \sigma_\varepsilon^2) \sim N(0, \frac{\sigma_\varepsilon^2}{\lambda_{ij}})$, and if the standardized case where $\sigma_\varepsilon^2 = 1$ (See definition of student t -distribution) is considered, then

$$(\varepsilon_{ij}|\lambda_{ij}, 1) \sim N(0, \frac{1}{\lambda_{ij}}).$$

Therefore

$$\begin{aligned} p(\varepsilon_{ij}|\sigma_\varepsilon^2 = 1) &= \frac{\nu^{\frac{1}{2}\nu} \Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi \cdot 1)^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \left(\nu + \frac{\varepsilon_{ij}^2}{1}\right)^{-\frac{1}{2}(\nu+1)} \\ &= \frac{\nu^{\frac{1}{2}\nu} \Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi)^{\frac{1}{2}} \Gamma(\frac{\nu}{2})} \left(\nu + \varepsilon_{ij}^2\right)^{-\frac{1}{2}(\nu+1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\nu^{\frac{1}{2}} \nu \Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi)^{\frac{1}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left[\left(1 + \frac{\varepsilon_{ij}^2}{\nu}\right) \nu \right]^{-\frac{1}{2}(\nu+1)} \\
&= \frac{\nu^{\frac{1}{2}} \nu \Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi)^{\frac{1}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left[1 + \frac{\varepsilon_{ij}^2}{\nu} \right]^{-\frac{1}{2}(\nu+1)} \nu^{-\frac{1}{2}(\nu+1)} \\
&= \frac{\Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi\nu)^{\frac{1}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left[1 + \frac{\varepsilon_{ij}^2}{\nu} \right]^{-\frac{1}{2}(\nu+1)} \quad \text{for } -\infty < \varepsilon_{ij} < \infty
\end{aligned}$$

which is in the standardized form of a univariate t - distribution (See definition of the student t - distribution).

The joint density of the errors is therefore given by

$$\begin{aligned}
p(\boldsymbol{\varepsilon} | \sigma_\varepsilon^2) &= \prod_{i=1}^b \prod_{j=1}^k p(\varepsilon_{ij} | \sigma_\varepsilon^2) \\
&= \prod_{i=1}^b \prod_{j=1}^k \frac{\nu^{\frac{1}{2}} \nu \Gamma\left[\frac{1}{2}(\nu+1)\right]}{(\pi\sigma_\varepsilon^2)^{\frac{1}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left(\nu + \frac{\varepsilon_{ij}^2}{\sigma_\varepsilon^2} \right)^{-\frac{1}{2}(\nu+1)} \\
&= \frac{\nu^{\frac{1}{2}} \nu^{bk} \left\{ \Gamma\left[\frac{1}{2}(\nu+1)\right] \right\}^{bk}}{(\pi\sigma_\varepsilon^2)^{\frac{1}{2}bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \left\{ \nu + \frac{1}{\sigma_\varepsilon^2} \varepsilon_{ij}^2 \right\}^{-\frac{1}{2}(\nu+1)}
\end{aligned}$$

which is in the general form of a multivariate t - distribution.

Proof of Theorem 5.6.1

To obtain the conditional posterior distribution of μ , only consider the parts in the joint posterior distribution given in equation 5.4.1 that contains μ , and then complete the square with respect to μ .

Therefore consider

$$\begin{aligned}
&\frac{1}{\sigma_\varepsilon^2} (\mathbf{y} - \mu \mathbf{j} - Z \mathbf{a})' H (\mathbf{y} - \mu \mathbf{j} - Z \mathbf{a}) \\
&= \frac{1}{\sigma_\varepsilon^2} (\tilde{\mathbf{y}} - \mu \mathbf{j})' H (\tilde{\mathbf{y}} - \mu \mathbf{j}) \quad \text{where } \tilde{\mathbf{y}} = \mathbf{y} - Z \mathbf{a}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma_\varepsilon^2} \left(\tilde{\mathbf{y}}' H \tilde{\mathbf{y}} - \tilde{\mathbf{y}}' H \mu \mathbf{j} - (\mu \mathbf{j})' H \tilde{\mathbf{y}} + (\mu \mathbf{j})' H \mu \mathbf{j} \right) \\
&= \frac{1}{\sigma_\varepsilon^2} \left(\tilde{\mathbf{y}}' H \tilde{\mathbf{y}} - 2\mu \mathbf{j}' H \tilde{\mathbf{y}} + \mu^2 \mathbf{j}' H \mathbf{j} \right).
\end{aligned}$$

Now consider

$$\begin{aligned}
&\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \mu^2 - 2\mu \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \\
&= \left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]' \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \mu^2 \left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right] \\
&\quad - \left[\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]' \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \left[\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right] \\
&= \left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]' \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right] \\
&\quad - \left[\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]^2 \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1}
\end{aligned}$$

since $\left[\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]$ is a scalar.

It therefore follows that since $\left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]$ is also a scalar,

$$\begin{aligned}
&\left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]' \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right] \\
&\quad - \left[\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]^2 \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \\
&= \left[\mu - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]^2 \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} - \left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \left[\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}) \right]^2.
\end{aligned}$$

From this it thus follows that the conditional posterior distribution of μ is given by

$$p(\mu | \mathbf{y}, \sigma_\varepsilon^2, H, \mathbf{a}) \sim N \left(\left(\frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H \mathbf{j} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} \mathbf{j}' H (\mathbf{y} - Z\mathbf{a}), \sigma_\varepsilon^2 (\mathbf{j}' H \mathbf{j})^{-1} \right)$$

Proof of Theorem 5.6.2

To obtain the conditional posterior distribution of the vector of random effects \mathbf{a} , only consider the parts in the joint posterior distribution given in equation 5.4.1 that contains \mathbf{a} , and, then complete the square with respect to \mathbf{a} .

Therefore consider

$$\begin{aligned}
& \frac{1}{\sigma_\varepsilon^2} (\mathbf{y} - \mu \mathbf{j} - Z\mathbf{a})' H (\mathbf{y} - \mu \mathbf{j} - Z\mathbf{a}) + \frac{1}{\sigma_a^2} \mathbf{a}' \mathbf{a} \\
&= \frac{1}{\sigma_\varepsilon^2} (\mathbf{y}^* - Z\mathbf{a})' H (\mathbf{y}^* - Z\mathbf{a}) + \frac{1}{\sigma_a^2} \mathbf{a}' \mathbf{a} \quad \text{where } \mathbf{y}^* = \mathbf{y} - \mu \mathbf{j} \\
&= \frac{1}{\sigma_\varepsilon^2} (\mathbf{y}^{*'} H \mathbf{y}^* - \mathbf{y}^{*'} H Z \mathbf{a} - \mathbf{a}' Z' H \mathbf{y}^* + \mathbf{a}' Z' H Z \mathbf{a}) + \frac{1}{\sigma_a^2} \mathbf{a}' \mathbf{a} \\
&= \frac{1}{\sigma_\varepsilon^2} \mathbf{y}^{*'} H \mathbf{y}^* - \frac{1}{\sigma_\varepsilon^2} \mathbf{y}^{*'} H Z \mathbf{a} - \frac{1}{\sigma_\varepsilon^2} \mathbf{a}' Z' H Z \mathbf{y}^* + \frac{1}{\sigma_\varepsilon^2} \mathbf{a}' Z' H Z \mathbf{a} + \frac{1}{\sigma_a^2} \mathbf{a}' \mathbf{a} \\
&= \frac{1}{\sigma_\varepsilon^2} \mathbf{y}^{*'} H \mathbf{y}^* - 2 \frac{1}{\sigma_\varepsilon^2} \mathbf{a}' Z' H \mathbf{y}^* + \frac{1}{\sigma_\varepsilon^2} \mathbf{a}' Z' H Z \mathbf{a} + \frac{1}{\sigma_a^2} \mathbf{a}' \mathbf{a} \\
&= \mathbf{a}' \left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right) \mathbf{a} - 2 \frac{1}{\sigma_\varepsilon^2} \mathbf{a}' Z' H \mathbf{y}^* + \frac{1}{\sigma_\varepsilon^2} \mathbf{y}^{*'} H \mathbf{y}^* \\
&= \left[\mathbf{a} - \left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right)^{-1} \left(\frac{1}{\sigma_\varepsilon^2} Z' H \mathbf{y}^* \right) \right]' \left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right) \left[\mathbf{a} - \left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right)^{-1} \left(\frac{1}{\sigma_\varepsilon^2} Z' H \mathbf{y}^* \right) \right].
\end{aligned}$$

From the above, it follows that the conditional posterior distribution of \mathbf{a} is given by:

$$p(\mathbf{a} | \mu, \sigma_\varepsilon^2, \sigma_a^2, H, \mathbf{y}) \sim N \left(\left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right)^{-1} \left(\frac{1}{\sigma_\varepsilon^2} Z' H \mathbf{y}^* \right), \left(\frac{1}{\sigma_\varepsilon^2} Z' H Z + \frac{1}{\sigma_a^2} I_b \right)^{-1} \right)$$

Proof of Theorem 5.6.3

To obtain the conditional posterior distribution of the error variance component σ_ε^2 , only consider the parts in the joint posterior distribution given in equation 5.4.1 that contains σ_ε^2 .

Therefore, the conditional posterior distribution of σ_ε^2 is given by:

$$\begin{aligned} p(\sigma_\varepsilon^2 | \mu, \mathbf{a}, \sigma_a^2, H, \mathbf{y}) &\propto \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{1}{2}bk} \sigma_\varepsilon^{-2} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})\right) \\ &= \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{1}{2}bk+1} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})\right) \\ &= \left(\frac{1}{\sigma_\varepsilon^2}\right)^{\frac{1}{2}(bk+2)} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})' H(\mathbf{y} - \mu\mathbf{j} - Z\mathbf{a})\right) \end{aligned}$$

This is in the general form of an inverse gamma distribution

Proof of Theorem 5.6.4

To obtain the conditional posterior distribution of the parts variance component σ_a^2 , only consider the parts in the joint posterior distribution given in equation 5.4.1 that contains σ_a^2 .

Therefore the conditional posterior distribution of σ_a^2 is given by:

$$p(\sigma_a^2 | \mathbf{a}, \mathbf{y}) \propto (\sigma_a^2)^{-\frac{1}{2}b} \exp\left(-\frac{1}{2\sigma_a^2} \mathbf{a}' \mathbf{a}\right)$$

This conditional posterior density of σ_a^2 is also in the general form of an inverse gamma distribution.

Proof of Theorem 5.6.5

The conditional posterior distribution of λ_{ij} (for $i = 1, \dots, b, j = 1, \dots, k$) is given by:

$$\begin{aligned}
p(\lambda_{ij} | \mathbf{y}, \mathbf{a}, \mu, \sigma_\varepsilon^2, \sigma_a^2) &\propto \lambda_{ij}^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \lambda_{ij} (y_{ij} - \mu - a_i)^2\right) \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left(-\frac{1}{2}\nu\lambda_{ij}\right) \\
&= \lambda_{ij}^{\frac{1}{2}} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \lambda_{ij} (y_{ij} - \mu - a_i)^2 - \frac{1}{2}\nu\lambda_{ij}\right) \\
&= \lambda_{ij}^{\frac{1}{2}} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left\{-\frac{1}{2}\lambda_{ij} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{ij} - \mu - a_i)^2\right]\right\} \\
&= \lambda_{ij}^{\frac{1}{2}\nu-1+\frac{1}{2}} \exp\left\{-\frac{1}{2}\lambda_{ij} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{ij} - \mu - a_i)^2\right]\right\} \\
&= \lambda_{ij}^{\frac{1}{2}\nu-\frac{1}{2}} \exp\left\{-\frac{1}{2}\lambda_{ij} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{ij} - \mu - a_i)^2\right]\right\} \\
&= \lambda_{ij}^{\frac{1}{2}(\nu+1)-1} \exp\left\{-\frac{1}{2}\lambda_{ij} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{ij} - \mu - a_i)^2\right]\right\}.
\end{aligned}$$

This conditional posterior distribution of λ_{ij} is in the general form of a gamma distribution.

Proof of Theorem 5.8.1

To obtain the conditional posterior distribution of ν , only consider the parts in the joint posterior distribution given in equation 5.8.3 that contains ν . Therefore, the conditional posterior distribution of ν , is given by

$$\begin{aligned}
p(\nu | \mathbf{y}, \lambda_{ij}) &\propto \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{1}{2}\nu} \left[\Gamma\left(\frac{\nu}{2}\right)\right]} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left\{-\frac{1}{2}\nu\lambda_{ij}\right\} \cdot \xi \exp(-\xi\nu) \\
&= \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{1}{2}\nu} \left[\Gamma\left(\frac{\nu}{2}\right)\right]} \xi \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left\{-\frac{1}{2}\nu\lambda_{ij} - \xi\nu\right\} \\
&= \frac{\nu^{\frac{1}{2}\nu}\xi}{2^{\frac{1}{2}\nu} \left[\Gamma\left(\frac{\nu}{2}\right)\right]} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left\{\nu\left(-\frac{1}{2}\lambda_{ij} - \xi\right)\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
 p(\nu|\mathbf{y}, \boldsymbol{\lambda}) &\propto \prod_{i=1}^b \prod_{j=1}^k \frac{\nu^{\frac{1}{2}\nu} \xi}{2^{\frac{1}{2}\nu} \left[\Gamma\left(\frac{\nu}{2}\right) \right]} \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left\{-\nu\left(\frac{1}{2}\lambda_{ij} + \xi\right)\right\} \\
 &= \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{1}{2}\nu bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \xi \lambda_{ij}^{\frac{1}{2}\nu-1} \exp\left\{-\nu\left(\frac{1}{2}\sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} + \xi\right)\right\} \\
 &= \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{1}{2}\nu bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \xi \lambda_{ij}^{\frac{1}{2}\nu} \lambda_{ij}^{-1} \exp\left\{-\nu\left(\frac{1}{2}\sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} + \xi\right)\right\} \\
 &\propto \frac{\nu^{\frac{1}{2}\nu bk}}{2^{\frac{1}{2}\nu bk} \left[\Gamma\left(\frac{\nu}{2}\right) \right]^{bk}} \prod_{i=1}^b \prod_{j=1}^k \lambda_{ij}^{\frac{1}{2}\nu} \exp\left\{-\nu\left(\frac{1}{2}\sum_{i=1}^b \sum_{j=1}^k \lambda_{ij} + \xi\right)\right\}
 \end{aligned}$$

which is in the general form of an unknown distribution.

Proof of Theorem 5.10.1.1

Future measurements are generate by the same model as defined in equation 5.2.7,

i.e.

$$y_{fj} = \mu + a_f + \varepsilon_{fj^*} \text{ for } j^* = 1, 2, \dots, k^*$$

when $\varepsilon_{ij}|\lambda_{fj^*} \sim N(0, \frac{\sigma_\varepsilon^2}{\lambda_{fj^*}})$.

From this, it follows that $\bar{y}_f = \frac{1}{k^*} \sum_{j=1}^{k^*} y_{fj^*}$ is normally distributed with mean

$$E(\bar{y}_f|\mu, a_f, \sigma_\varepsilon^2, \sigma_a^2, \lambda_{f1}, \dots, \lambda_{fk^*}) = \mu + a_f$$

and

$$\begin{aligned}
 Var(\bar{y}_f|\mu, a_f, \sigma_\varepsilon^2, \sigma_a^2, \lambda_{f1}, \dots, \lambda_{fk^*}) &= \frac{1}{k^{*2}} \sum_{j=1}^{k^*} Var(y_{fj^*}) \\
 &= \frac{1}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{\sigma_\varepsilon^2}{\lambda_{fj^*}} \\
 &= \frac{\sigma_\varepsilon^2}{k^{*2}} \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{fj^*}}.
 \end{aligned}$$

Since $a_f \sim N(0, \sigma_a^2)$ it follows that

$$E(\bar{y}_f | \mu, a_f, \sigma_\varepsilon^2, \sigma_a^2, \lambda_{f1}, \dots, \lambda_{fk^*}) = \mu$$

and

$$Var(\bar{y}_f | \mu, \sigma_\varepsilon^2, \sigma_a^2, \lambda_{f1}, \dots, \lambda_{fk^*}) = \frac{\sigma_\varepsilon^2}{k^{*2}} \sum_{j=1}^{k^*} \frac{1}{\lambda_{fj^*}} + \sigma_a^2$$

when

$$p(\lambda_{fj^*}) = \frac{\nu^{\frac{1}{2}\nu}}{2^{\frac{\nu}{2}}} \lambda_{fj^*}^{\frac{1}{2}\nu-1} \exp(-\frac{1}{2}\lambda_{fj^*}\nu) \quad (\text{See the prior distribution of } \lambda_{ij} \text{ in section 5.3}).$$

Proof of Theorem 5.10.1.2

Future measurements for the i^{th} part are generated by the model

$$\tilde{y}_{ij^*} = \mu + a_i + \varepsilon_{ij^*} \quad \text{for } j^* = 1, \dots, k^*$$

where

$$\varepsilon_{ij^*} \sim pN(0, \frac{\sigma_\varepsilon^2}{\lambda_{i1j^*}}) + (1-p)N(0, \frac{\sigma_\varepsilon^2}{\lambda_{i2j^*}}) \quad \text{for } j^* = 1, \dots, k^* .$$

From this it follows that $\bar{\tilde{y}}_i = \frac{1}{k^*} \sum_{j^*=1}^{k^*} \tilde{y}_{ij^*}$ is normally distributed with mean

$$E(\bar{\tilde{y}}_i | \mu, a_i, \lambda_{i11}, \dots, \lambda_{i2k^*}, \sigma_\varepsilon^2, \sigma_a^2) = \mu + a_i$$

and variance

$$Var(\bar{\tilde{y}}_i | \mu, a_i, \lambda_{i11}, \dots, \lambda_{i2k^*}, \sigma_\varepsilon^2, \sigma_a^2) = \frac{\sigma_\varepsilon^2}{k^{*2}} \left\{ p \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + (1-p) \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right\} .$$

For this specific case $p = \frac{1}{2}$, therefore

$$Var(\bar{\tilde{y}}_i | \mu, a_i, \lambda_{i11}, \dots, \lambda_{i2k^*}, \sigma_\varepsilon^2, \sigma_a^2) = \frac{\sigma_\varepsilon^2}{2k^{*2}} \left\{ \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i1j^*}} + \sum_{j^*=1}^{k^*} \frac{1}{\lambda_{i2j^*}} \right\} .$$

Also

$$p(\lambda_{i1} | \mathbf{y}, \mathbf{a}, \mu, \sigma_\varepsilon^2, \sigma_a^2) \propto \lambda_{i1}^{\frac{1}{2}(\nu+1)-1} \exp \left\{ -\frac{1}{2} \lambda_{i1} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{i1} - \mu - a_i)^2 \right] \right\}$$

and

$$p(\lambda_{i2} | \mathbf{y}, \mathbf{a}, \mu, \sigma_\varepsilon^2, \sigma_a^2) \propto \lambda_{i2}^{\frac{1}{2}(\nu+1)-1} \exp \left\{ -\frac{1}{2} \lambda_{i2} \left[\nu + \frac{1}{\sigma_\varepsilon^2} (y_{i2} - \mu - a_i)^2 \right] \right\}.$$

See equation 5.6.5.

Chapter 6

Two - Factor Nested Random Effects Model

In this chapter, the unique and flexible Bayesian simulation method for determining tolerance intervals originally proposed by Wolfinger (1998) and discussed in Chapters 3 and 4 will be adapted to derive Bayesian tolerance intervals for a balanced two - factor nested random effects model. The proposed Bayesian method will be illustrated using data obtained from Laubscher (1996), originally collected at SANS Fibres (Pty.) Ltd. South - Africa. In addition, it will also be shown that the proposed non - informative prior distribution is a probability matching prior for the α^{th} quantile of the distribution of averages of k future packages with r samples per packages from any new or unknown day. It will also be shown that a proposed prior distribution for the content of the fixed - in - advance tolerance interval, is also a probability matching prior.

Parts of this chapter have been published in Test. For more details see van der Merwe and Hugo (2007). The example and proposed model have also been adopted for use by Krishnamoorthy and Mathew (2009) for publication in their book on statistical tolerance regions. For more details see Krishnamoorthy and Mathew (2009), Chapter 11, sections 11.3 and 11.4.

6.1 Introduction

It was mentioned in Chapter 1 that in order to support the economic objectives and profitability of a firm, more tools are needed by quality engineers to cope with the rapidly changing manufacturing environment and to meet the intense international competition (*Black Nembhard and Valverde - Ventura, 2003*). Statistical quality control has therefore become the new fashion and a very important requirement of modern process improvement techniques (<http://www.quality-one.com>). As a result, well known companies such as Motorola, Allied Signal and General Electric have moved to make use of for example, the six sigma (6σ) quality improvement strategy (<http://www.quality-one.com>).

The logic behind the six sigma principle is relatively simple, since it is known that a large group of any population (process, analysis, etc.) cluster around the middle and form what is referred to as a bell shaped curve. Six sigma represents six standard deviations (sigma is the Greek letter used to represent the population standard deviation) from the middle of this bell shaped curve, three standard deviations above and three standard deviations below the middle. Also, from the empirical rule of statistics it is well known that six standard deviations, three each side of the mean, cover approximately 99.73% of the population (<http://www.freequality.org/documents/six-Sigma/> and http://www.en.wikipedia.org/wiki/69-95-99.7_rule).

Six sigma is a term coined by Motorola and emphasizes the improvement of a process for the purpose of reducing variability and thus making general improvements (<http://www.quality-one.com>). In general, the six sigma methodology provides the techniques and tools to improve the capability and reduce the defects in any process and is instrumental in the construction of control charts, a primary tool of statistical process control (<http://www.sixsigmatutorial.com/Six-Sigma/Six-Sigma-Tutorial.aspx>).

Statistical process control (SPC) is one of the most powerful tools used for a continuous process improvement scheme (*Reneau and Kinsel, 2001*). Statistical techniques

are used in SPC to measure and analyze the variation in a process by means of as mentioned in chapter 1, a pair of control charts - one for the average and one for the variation, each with its own 3σ control limits (*Laubscher, 1996*). Since these standard Shewhart variable control charts only allow for within sample variation, different models allowing for more sources of variation may provide more satisfactory results.

Since the quality of a manufactured item is a function of the sources of variation in the manufacturing process, estimating variance components can present a method for evaluating the observed process variation. In this chapter, a situation is therefore analyzed where replicate observations are obtained from a number of random samples taken daily from a process. This allows for substantial day - to - day and within day variation.

According to Laubscher (1996), a typical observations (the t^{th} where $t = 1, \dots, r$) from the j^{th} ($j = 1, \dots, k$) package sampled on day i ($i = 1, \dots, b$) using the sampling procedure described above, is denoted by y_{ijt} . Laubscher (1996) also suggested that a classical analysis of variance model providing for an overall target value, for main and possibly interaction effects as well as some unexplained residual variation, could be used to explain the variation observed in a response such as y_{ijt} . Both "packages" and "days" can be considered random factors with packages hierarchically nested within days (*Laubscher, 1996*). A two - way nested ANOVA is therefore suggested, since this model allows for day - to - day and package - to - package variation (*Neter, Wasserman and Kutner, 1985, and Laubscher, 1996*).

Once the appropriate variance components model has been specified, the manufacturing process's performance can be assessed in order to identify possible problems. The identification of these possible problems is in line with the principles of sound data analysis, since further progress on the path of continuous quality improvement can only be expected once these problems have been identified (*Laubscher, 1996*). The procedure for evaluating the appropriateness of the selected variance components model and its subsequent application for statistical process control using Shewhart

variable control charts, is summarized in Laubscher (1996). For more details see Laubscher (1996). As mentioned in Chapter 1, the Bayesian approach however also serves as a possible alternative approach to be utilized for statistical process control by answering the three research questions proposed by Wolfinger (1998).

For both conceptual and practical reasons, hierarchical models are central to modern Bayesian statistics and full Bayesian analyses of hierarchical models have been considered by various authors including Hill (1965), Tiao and Tan (1965), Stone and Springer (1965), Portnoy (1971), Box and Tiao (1973), Carlin and Louis (1996), van Dyk and Meng (2001), Gelman et. al. (2003) and Browne and Draper (2006) (*Gelman, 2006*). The three research questions proposed by Wolfinger (1998) can be answered by obtaining the three different Bayesian tolerance intervals, also proposed by Wolfinger (1998) and described in Chapter 1. These tolerance intervals thus provide a full Bayesian solution to quality management using statistical process control.

Although some research have been done, there do not appear to be many papers on quality control from a Bayesian point of view. The Bayesian approach to controlling processes for the production of only small numbers of items has been developed by Woodward and Naylor (1993), while Arnold (1990), developed an economic \bar{X} - chart for the joint control of the means of independent quality characteristics (*van der Merwe and Hugo, 2007*). The first application of the Bayesian paradigm to tolerance intervals specifically, was due to Aitchison (1964, 1966) and Aitchison and Dunsmore (1975). These authors also presented arguments in favour of the Bayesian approach as apposed to classical frequentist methods (*van der Merwe and Hugo, 2007*).

In the subsequent sections, the Bayesian approach to variance component and tolerance interval estimation will be discussed for the balanced two - factor nested random effects model. The proposed methods will be illustrated using a data set originally presented in a paper by Prof. Nico F. Laubscher (see Laubscher, 1996) and is from a continuous process for sampling spun yarn at SANS Fibers (Pty.) Ltd. South Africa. For the data, variance components, credibility intervals, (α, δ) one - and two - sided toler-

ance intervals, α - expectation tolerance intervals and fixed - in - advance tolerance intervals will be determined.

6.2 The Data

In 1995 certain statistical process control procedures had to be set up in a new plant of the Bellville works of SANS Fibres (Pty.) Ltd. South Africa, a company manufacturing continuous filament polyester and nylon yarns. It was a continuous process on which various physical properties of the continuously manufactured synthetic yarn had to be monitored. Daily samples of $k = 8$ packages of yarn were sampled per machine and various physical properties of the yarn were replicated in the laboratory by analyzing $r = 5$ samples per package. For illustrative purposes, a section of the data is provided in Table 6.1. The data given in Table 6.1 represents the yarn property of extension (the percentage increase in length before breaking). The complete data set is given in Table D1 in Appendix D. Packages with yarn sampled for the first $b = 15$ days of January 1995 were selected as review data to determine statistical process control limits for future use in monitoring the process. The data used have been made available by Prof. Nico F. Laubscher, Company Statistician at the time of data collection at SANS Fibres (Pty.) Ltd., with permission of SANS Fibres (Pty.) Ltd.

Table 6.1: The Physical Property of "Extension" of a Synthetic Yarn, Measured over the First 15 Consecutive Days of January 1995.

Day	Packages	Extension	Daily Average	
1	1	20.30 *		
1	2	20.06		
1	3	20.48		
1	4	19.22		
1	5	19.16		
1	6	19.94		
1	7	20.69		
1	8	19.02		19.86
.	.	.		
.	.	.		
.	.	.		
10	1	20.99		
10	2	20.53		
10	3	19.37		
10	4	19.93		
10	5	19.31		
10	6	19.86		
10	7	19.00		
10	8	19.81		19.85 **
15	1	22.48		
15	2	21.65		
15	3	22.00		
15	4	22.71		
15	5	22.28		
15	6	21.38		
15	7	21.35		
15	8	22.43	22.04	

* Average of 5 samples selected from the first packages of day 1.

** Daily average of 8 packages for day 10.

It is clear from the example that the variation observed could possibly be explained by several components such as a "between" days (day - to - day) component, a "within" days (packages - to - packages) component and a residual component. The latter may consist of several lurking components such as sampling variation as well as sample - to - sample (within packages) variation and experimental error (*Laubscher, 1996*). These sources of variation should be incorporated into a suitable model. In this chapter, the emphasis will be placed on the proper analysis of a feasible model for the situation where nested sampling occurs. The results can readily be extended to

more complicated analysis of variance type of models within this context (e.g. more factors with more involved nesting).

To start analyzing the data it is important to know what it means when it is said that a process is in control. In this respect, Laubscher (1996) quoted Shewhart (1931) as follows: *"... a phenomenon will be said to be controlled when, through the use of past experience, we can predict, at least within limits, how the phenomenon may be expected to vary in future. Here it is understood that prediction within limits means that we can state, at least approximately, the probability that the observed phenomenon will fall within the given limits."*

How does one use "past experience" to be in a position to claim (with associated probability) that a process response is within predicted limits? This is only possible by using past data as guidance on the selection of a model and to derive the prediction formula and the associated probability for predictions from that model (Laubscher, 1996). This means that the choice of model for process control is vital and without it no serious statistical analysis can be undertaken (Laubscher, 1996). Since the packages were nested within days, the model chosen was as mentioned previously, the balanced two - factor nested random effects model.

The flexibility and unique features of the Bayesian simulation method for the construction of tolerance intervals, originally proposed by Wolfinger (1998), will in the subsequent sections be extended for determining Bayesian tolerance intervals in the case of this mentioned balanced two - factor nested random effects model.

6.3 The Balanced Two - Factor Nested Random Effects Model

As mentioned, since the data given in Table 6.1 constitute replicate observations obtained from a number of random packages of spun yarn sampled daily per machine, with the random factor “packages” hierarchically nested within the random factor “days”, it was suggested by Laubscher (1996) that a balanced two - factor nested random effects model be used to analyze the data. This balanced two - factor nested random effects model is given by

$$y_{ijt} = \mu + d_i + p_{ij} + \varepsilon_{ijt} \quad (6.3.1)$$

where $(i = 1, \dots, b)$, $(j = 1, \dots, k)$, $(t = 1, \dots, r)$, y_{ijt} represents the observations (percentage increase in length of synthetic yarn before breaking), μ is a common location parameter (the grand mean), d_i , p_{ij} and ε_{ijt} are three different kinds of random effects (days, packages within days and residual). It is further assumed that the random effects $(d_i, p_{ij}, \varepsilon_{ijt})$ are all independent and that $d_i \sim N(0, \sigma_d^2)$, $p_{ij} \sim N(0, \sigma_p^2)$ and $\varepsilon_{ijt} \sim N(0, \sigma_\varepsilon^2)$. The three different random effects can be interpreted as follows: d_i (for $i = 1, \dots, b$) represents the days effect, p_{ij} (for $i = 1, \dots, b$, $j = 1, \dots, k$) represents the packages within days effects and ε_{ijt} (for $i = 1, \dots, b$, $j = 1, \dots, k$, $t = 1, \dots, r$) represents the measurement error of the t^{th} measurement on the j^{th} package sampled on day i .

The balanced two - factor nested random effects model given in equation 6.3.1 can also be written in matrix notation as follows:

$$\mathbf{y} = \mu \mathbf{X} + Z_1 \mathbf{u}_1 + Z_2 \mathbf{u}_2 + \boldsymbol{\varepsilon} \quad (6.3.2)$$

where

$$\mathbf{y} = \begin{bmatrix} y_{111} & \cdots & y_{11r} & y_{121} & \cdots & y_{12r} & \cdots & y_{1kr} & \cdots & y_{bkr} \end{bmatrix}' ,$$

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}' (1 \times bkr) ,$$

μ represents the grand mean,

$$\mathbf{u}_1 = \begin{bmatrix} d_1 & d_2 & \cdots & d_b \end{bmatrix}' (1 \times b) ,$$

$$\mathbf{u}_2 = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} & \cdots & p_{bk} \end{bmatrix}' (1 \times bk) ,$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \cdots & \varepsilon_{11r} & \varepsilon_{121} & \cdots & \varepsilon_{12r} & \cdots & \varepsilon_{1kr} & \cdots & \varepsilon_{bkr} \end{bmatrix}' (1 \times bkr) ,$$

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1_{(kr)} & 0_{(kr)} & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1_{(kr)} & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1_{(kr)} \end{bmatrix} , \text{ and}$$

$$Z_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1_{(r)} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1_{(r)} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1_{(r)} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1_{(r)} \end{bmatrix}.$$

Also

$$\nu_1 = bk(r-1), \quad \nu_2 = b(k-1), \quad \nu_3 = (b-1),$$

$$y_{ij.} = \sum_{t=1}^r y_{ijt}, \quad y_{i..} = \sum_{j=1}^k \sum_{t=1}^r y_{ijt},$$

$$y_{...} = \sum_{i=1}^b \sum_{j=1}^k \sum_{t=1}^r y_{ijt}, \quad \bar{y}_{ij.} = \frac{1}{r} y_{ij.},$$

$$\bar{y}_{i..} = \frac{1}{kr} y_{i..}, \quad \bar{y}_{...} = \frac{1}{bkr} y_{...},$$

$$\nu_1 m_1 = \sum_{t=1}^r \sum_{j=1}^k \sum_{i=1}^b (y_{ijt} - \bar{y}_{ij.})^2 = \sum_{t=1}^r \sum_{j=1}^k \sum_{i=1}^b y_{ijt}^2 - \frac{1}{r} \sum_{i=1}^b \sum_{j=1}^k \bar{y}_{ij.}^2,$$

$$\nu_2 m_2 = r \sum_{i=1}^b \sum_{j=1}^k (\bar{y}_{ij.} - \bar{y}_{i..})^2 = \frac{1}{r} \sum_{i=1}^b \sum_{j=1}^k y_{ij.}^2 - \frac{1}{rk} \sum_{i=1}^b y_{i..}^2, \text{ and}$$

$$\nu_3 m_3 = \frac{1}{rk} \sum_{i=1}^b y_{i..}^2 - \frac{1}{bkr} y_{...}^2 = rk \sum_{i=1}^b (\bar{y}_{i..} - \bar{y}_{...})^2.$$

Remember also, $\nu_1 m_1$ represents the sum of squares for error (*SSE*), $\nu_2 m_2$ represents the sum of squares of packages within days (*SSP*) and $\nu_3 m_3$ represents the sum of squares for days (*SSD*).

Consider again the balanced two - factor nested random effects model given in equation 6.3.2, i.e.

$$\mathbf{y} = \mu \mathbf{X} + Z_1 \mathbf{u}_1 + Z_2 \mathbf{u}_2 + \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon}$, \mathbf{u}_2 and \mathbf{u}_1 are independently normally distributed as $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma_\varepsilon^2 I_{bkr})$, $\mathbf{u}_2 \sim N(\mathbf{0}, \sigma_p^2 I_{bk})$ and $\mathbf{u}_1 \sim N(\mathbf{0}, \sigma_d^2 I_b)$.

6.4 The Bayesian Method

Since the balanced two - factor nested random effects model given in equation 6.3.2 has been specified, the Bayesian method for variance component estimation will now be discussed.

6.4.1 The Likelihood Function

In matrix notation, the likelihood function of the balanced two - factor nested random effects model given in equation 6.3.2, is given by

$$\begin{aligned} L(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1, \mathbf{u}_2 | \mathbf{y}) \\ \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}bkr} \exp \left\{ -\frac{1}{2\sigma_\varepsilon^2} (\mathbf{y} - \mu \mathbf{X} - Z_1 \mathbf{u}_1 - Z_2 \mathbf{u}_2)' (\mathbf{y} - \mu \mathbf{X} - Z_1 \mathbf{u}_1 - Z_2 \mathbf{u}_2) \right\} \\ (\sigma_p^2)^{-\frac{1}{2}bk} \exp \left\{ -\frac{1}{2\sigma_p^2} \mathbf{u}_2' \mathbf{u}_2 \right\} (\sigma_d^2)^{-\frac{1}{2}b} \exp \left\{ -\frac{1}{2\sigma_d^2} \mathbf{u}_1' \mathbf{u}_1 \right\}. \end{aligned} \quad (6.4.1)$$

Equation 6.4.1, as mentioned, is known as the likelihood function and can be regarded as the function through which the data \mathbf{y} modifies prior knowledge of the unknown parameters (*Box and Tiao, 1973*).

Now, by using the results given in Box and Tiao (1973), the integrated likelihood function is given by

$$\begin{aligned}
& L(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 | \mathbf{y}) \\
& \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}\nu_1} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}\nu_2} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+1)} \exp \left\{ -\frac{1}{2} \left[\frac{kr \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right] \right\} \\
& \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}\nu_1} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}\nu_2} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+1)} \\
& \quad \exp \left\{ -\frac{1}{2} \left[\frac{bkr(\bar{y}_{...} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_3 m_3}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right] \right\}. \quad (6.4.2)
\end{aligned}$$

The parameters $\theta_0 = \sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2$, $\theta_1 = \sigma_\varepsilon^2 + r\sigma_p^2$ and $\theta_2 = \sigma_\varepsilon^2$, are subject to the constraint

$$C : 0 < \theta_2 < \theta_1 < \theta_0 < \infty$$

to ensure that only positive variance components are obtained.

6.4.2 The Prior Distribution

It was mentioned in Chapters 1 and 2 that the determination of reasonable non - informative priors in multiparameter problems are not easy. The determination of reasonable non - informative prior distributions is however important, since the use of prior information is one of the main advantages of the Bayesian approach.

For parameters about which not much is known beyond the data included in the analysis, non - informative prior distributions intend to allow Bayesian inference (*Gelman,*

2006). Over the years, various justifications and interpretations of non - informative prior distribution have been proposed (Gelman, 2006). Some of these mentioned by Gelman (2006), included invariance (Jeffreys', 1961), maximum entropy (Jaynes, 1983) and agreement with classical estimators (Box and Tiao, 1973 and Meng and Zaslavsky, 2002). Bernardo (1979) also considered non - informative priors as reference models to be used in place of proper, informative prior distributions as a standard of comparison or starting point (Gelman, 2006).

In a study reviewing the extensive literature in the course of comparing Bayesian and non - Bayesian inference for hierarchical models, Browne and Draper (2006) also considered different prior distributions for the concerned variance parameters (see also Gelman, 2006). For a simple two - level model with group level effects a_j where $a_j \sim N(0, \sigma_a^2)$ for $j = 1, \dots, k$, Gelman (2006) mentioned that various non - informative prior distributions have been suggested for the parameter σ_a^2 . Some of these include an improper uniform density on σ_a (Gelman et.al., 2003), proper distributions such as $p(\sigma_a^2) \sim IG(0.001, 0.001)$ (Spiegelhalter et.al. 1994, 2003), and distributions that depended on the data - level variance (Box and Tiao, 1973). It is important to note also that Gelman (2006) mentioned that especially for models where the number of groups k is small, or where the group - level variance σ_a^2 is close to zero, the choice of a non - informative prior distribution can have a dramatic effect on the inference. Various other prior distributions have also been suggested which do not form part of the scope of this research. For more information, see Gelman (2006).

For the balanced two - factor nested random effects model given in equation 6.3.1, it was decided to follow Box and Tiao (1973) and also use the non - informative prior distribution given by

$$p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1}. \quad (6.4.3)$$

In a later section it will be shown that the predictive density of the average of k^* future packages with r^* samples per package from a new or unknown day, given the

variance components, follows a normal distribution given by

$$\bar{y}_f^* | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 \sim N\left(\mu, \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*}\right)$$

where y_{fjt}^* represents a new observation generated by the model $y_{fjt}^* = \mu + d_f + p_{ij} + \varepsilon_{fjt}$ with f representing a new or unknown day, $j = 1, \dots, k^*$ and $t = 1, \dots, r^*$, $\bar{y}_{fj}^* = \frac{1}{r^*} \sum_{t=1}^{r^*} y_{fjt}^*$ and $\bar{y}_{f..}^* = \frac{1}{k^*} \sum_{j=1}^{k^*} \bar{y}_{fj}^*$.

If $k^* = k$ and $r^* = r$ as defined in equation 6.3.1, it follows that the α^{th} quantile of the above normal distribution given by

$$\bar{y}_{f..} | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 \sim N\left(\mu, \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr}\right)$$

can be written as

$$q = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr}} \quad (6.4.4)$$

where z_α denotes the $100(\alpha)^{th}$ percentile of a standard normal distribution. It will now be proved that the non - informative prior distribution given in equation 6.4.3 is also a probability matching prior for the α^{th} quantile given in equation 6.4.4.

Theorem 6.4.2.1

For the balanced two - factor nested random effects model given in equation 6.3.1, the non - informative prior distribution given by

$$p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1}$$

is a probability matching prior for

$$q = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr}},$$

the α^{th} quantile of the normal distribution given by

$$\bar{y}_{f..} | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 \sim N\left(\mu, \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr}\right).$$

Proof

The proof of Theorem 6.4.2.1 is given in Appendix D.

6.4.3 The Joint Posterior Distribution

The joint posterior distribution for the balanced two - factor nested random effects model given in equation 6.3.1 can be determined by multiplying the likelihood function given in equation 6.4.1 with the non - informative prior distribution given in equation 6.4.3. The joint posterior distribution for the unknown parameters is then given by

$$\begin{aligned} & p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1, \mathbf{u}_2 | \mathbf{y}) \\ & \propto L(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1, \mathbf{u}_2 | \mathbf{y}) p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \\ & \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}bkr} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 - Z_2\mathbf{u}_2)'(\mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 - Z_2\mathbf{u}_2)\right\} \times \left(\frac{1}{\sigma_p^2}\right)^{-\frac{1}{2}bk} \exp\left\{-\frac{1}{2\sigma_p^2}\mathbf{u}_2'\mathbf{u}_2\right\} \\ & \quad \times \left(\frac{1}{\sigma_d^2}\right)^{-\frac{1}{2}b} \exp\left\{-\frac{1}{2\sigma_d^2}\mathbf{u}_1'\mathbf{u}_1\right\} \times (\sigma_\varepsilon^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1} \end{aligned} \quad (6.4.5)$$

where \mathbf{y} , \mathbf{X} , μ , \mathbf{u}_1 , \mathbf{u}_2 , ε , Z_1 , Z_2 are defined as given in equation 6.3.2. Also, remember $\mathbf{u}_1 \sim N(\mathbf{0}, \sigma_d^2 I_b)$, $\mathbf{u}_2 \sim N(\mathbf{0}, \sigma_p^2 I_{bk})$ and $\varepsilon \sim N(\mathbf{0}, \sigma_\varepsilon^2 I_{bkr})$.

From the joint posterior distribution given in equation 6.4.5, the conditional posterior distributions of the unknown parameters and variance components can now be determined.

6.4.4 The Conditional Posterior Distributions

Since the joint posterior distribution of the unknown parameters and variance components has been determined, the following theorems can now be proved.

Theorem 6.4.4.1

For the two - factor nested random effects model given in equation 6.3.2, the conditional posterior distribution of $\mathbf{u}_2 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1, \mathbf{y}$ is normally distributed with

$$E(\mathbf{u}_2 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1, \mathbf{y}) = D^{-1} \mathbf{C}, \text{ and}$$

$$Var(\mathbf{u}_2 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1, \mathbf{y}) = D^{-1}$$

where

$$D^{-1} = \left(\frac{\sigma_\varepsilon^2 \sigma_p^2}{\sigma_\varepsilon^2 + r \sigma_p^2} \right) I_{bk} ,$$

$\mathbf{C} = Z_2' \tilde{\mathbf{y}}$ where $\tilde{\mathbf{y}} = \mathbf{y} - \mu \mathbf{X} - Z_1 \mathbf{u}_1$, and, where $\mathbf{y}, \mathbf{X}, \mathbf{u}_1, \mathbf{u}_2, Z_1$ and Z_2 are defined as in 6.3.2.

Proof

The proof of Theorem 6.4.4.1 is given in Appendix D.

Theorem 6.4.4.2

The conditional posterior distribution of $d_i | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ (for $i = 1, \dots, b$) for the balanced two - factor nested random effects model follows a normal distribution with

$$E(d_i | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{rk\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{i..}$$

and

$$Var(d_i | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} .$$

It therefore follows that

$$p(d_i | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) \sim N\left(\frac{rk\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{i...}, \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}\right) \text{ for } i = 1, \dots, b. \quad (6.4.6)$$

Proof

The proof of Theorem 6.4.4.2 is given in Appendix D.

Theorem 6.4.4.3

For the balanced two - factor nested random effects model given in equation 6.3.2, the conditional posterior distribution of $\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ is normally distributed with

$$E(\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \bar{y}_{...}$$

and variance

$$Var(\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}$$

i.e.

$$p(\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) \sim N\left(\bar{y}_{...}, \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}\right). \quad (6.4.7)$$

Proof

The proof of Theorem 6.4.4.3 is given in Appendix D.

Theorem 6.4.4.4

The joint posterior distribution of the variance components $\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2$, for the balanced two - factor nested random effects model given in equation 6.3.2, is given by

$$p(\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}(\nu_2+2)} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+2)} \\ \times \exp\left\{-\frac{1}{2}\left[\frac{\nu_3 m_3}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2}\right]\right\} \quad (6.4.8)$$

subject to the constraint stating that

$$0 < \theta_2 < \theta_1 < \theta_0 < \infty \quad (6.4.9)$$

where $\theta_2 = \sigma_\varepsilon^2$, $\theta_1 = \sigma_\varepsilon^2 + r\sigma_p^2$, and $\theta_0 = \sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2$.

Proof

The proof of Theorem 6.4.4.4 is given in Appendix D.

Theorem 6.4.4.5

For the balanced two - factor nested random effects model given in equation 6.3.2, the conditional posterior distribution of $(\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ (for $i = 1, \dots, b$) is normally distributed with mean given by

$$E\left\{(\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} = \frac{kr\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{i..} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{...}$$

and variance given by

$$Var\left\{(\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} = \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \left[\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + bkr\sigma_d^2}{bkr} \right].$$

Therefore,

$$p\left((\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right) \sim N\left(\frac{kr\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{i..} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{...}, \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \left[\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + bkr\sigma_d^2}{bkr} \right]\right). \quad (6.4.10)$$

Proof

The proof of Theorem 6.4.4.5 is given in Appendix D.

6.4.5 Marginal Posterior Distributions

Similar to the balanced one - way random effects model discussed in Chapter 3, analytical derivations of unconditional posterior densities of the unknown parameters μ , σ_ε^2 , σ_p^2 , σ_d^2 , \mathbf{u}_1 , \mathbf{u}_2 and posterior densities of quantiles in order to construct tolerance intervals, appear to be a daunting task, due to the complexity of the balanced two - factor nested random effects model. It was therefore decided to extend the straightforward Bayesian simulation method proposed by Wolfinger (1998) for the balanced one - way random effects model, in order to also estimate marginal posterior distributions of unknown parameters and quantiles for the balanced two - factor nested random effects model. The proposed Bayesian method will be illustrated for the balanced two - factor nested random effects model using the data provided in Table 6.1 (the complete data set is provided in Table D1 as part of Appendix D.)

Similar to what was illustrated in Chapters 2, 3 and 4, Markov chain Monte Carlo (MCMC) simulation will also be used to obtain random samples from the joint posterior distribution of the unknown model parameters using a computer random number generator. Remember, these simulated samples will also represent samples from the marginal posterior distributions of the unknown model parameters and variance components, equivalent to what was indicated in Chapter 3.

Estimated marginal posterior distributions of the unknown parameters σ_ε^2 , σ_p^2 , σ_d^2 and μ , are simulated as follows:

a.) Simulation of the variance components σ_ε^2 , σ_p^2 and σ_d^2 .

If the constraint given in equation 6.4.9 did not apply, the posterior distributions of $\theta_2 = \sigma_\varepsilon^2$, $\theta_1 = \sigma_\varepsilon^2 + r\sigma_p^2$ and $\theta_0 = \sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2$ would be independent, each proportional to an inverse gamma distribution. That is

$$p(\theta_2|\mathbf{y}) \propto (\theta_2)^{-\frac{1}{2}(\nu_1+2)} \exp\left(-\frac{1}{2} \frac{\nu_1 m_1}{\theta_2}\right),$$

$$p(\theta_1|\mathbf{y}) \propto (\theta_1)^{-\frac{1}{2}(\nu_2+2)} \exp\left(-\frac{1}{2} \frac{\nu_2 m_2}{\theta_1}\right), \text{ and}$$

$$p(\theta_0|\mathbf{y}) \propto (\theta_0)^{-\frac{1}{2}(\nu_3+2)} \exp\left(-\frac{1}{2} \frac{\nu_3 m_3}{\theta_0}\right)$$

where $\frac{\nu_1 m_1}{\theta_2} \sim \chi_{\nu_1}^2$,

$$\frac{\nu_2 m_2}{\theta_1} \sim \chi_{\nu_2}^2, \text{ and}$$

$$\frac{\nu_3 m_3}{\theta_0} \sim \chi_{\nu_3}^2,$$

with the joint posterior distribution of θ_2 , θ_1 and θ_0 , being the product of these three inverse gamma distributions. Note however that the restrictions given in equation 6.4.9 do apply. Nevertheless, using a rejection sampling procedure (see Guttman and Menzefricke, 2003), it is straight forward to generate samples from the joint posterior distribution of σ_ε^2 , σ_p^2 and σ_d^2 . This is done as follows:

- i.) Simulate $\frac{\nu_1 m_1}{\theta_2}$ from a chi - square distribution with ν_1 degrees of freedom. Then determine σ_ε^2 by calculating $\theta_2 = \sigma_\varepsilon^2 = \frac{\nu_1 m_1}{\chi_{\nu_1}^2}$, where for the data given in Table 6.1, $b = 15$, $k = 8$, $r = 5$ and $\nu_1 = bk(r - 1) = 480$. Also

$$\nu_1 m_1 = \sum_{t=1}^r \sum_{j=1}^k \sum_{i=1}^b (y_{ijt} - \bar{y}_{ij.})^2 = 390.6720.$$

- ii.) Simulate $\frac{\nu_2 m_2}{\theta_1}$ from a chi - square distribution with ν_2 degrees of freedom and determine $\theta_1 = (\sigma_\varepsilon^2 + r\sigma_p^2) = \frac{\nu_2 m_2}{\chi_{\nu_2}^2}$, where for the data given in Table 6.1, $\nu_2 = b(k - 1) = 105$ and $\nu_2 m_2 = r \sum_{i=1}^b \sum_{j=1}^k (\bar{y}_{ij.} - \bar{y}_{i..})^2 = 132.6570$.

$$\text{Then determine } \sigma_p^2 = \frac{(\theta_1 - \sigma_\varepsilon^2)}{r}.$$

- iii.) Simulate $\frac{\nu_3 m_3}{\theta_0}$ from a chi - square distribution with ν_3 degrees of freedom and determine $\theta_0 = (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2) = \frac{\nu_3 m_3}{\chi_{\nu_3}^2}$, where $\nu_3 = (b - 1) = 14$,

$$\nu_3 m_3 = rk \sum_{i=1}^b (\bar{y}_{i..} - \bar{y}_{...})^2 = 5(8)[(19.86 - 20.96)^2 + \dots + (22.04 - 20.96)^2] = 395.4342$$

$$\text{and } \bar{y}_{...} = \frac{1}{bkr} \sum_{t=1}^r \sum_{j=1}^k \sum_{i=1}^b y_{ijt} = 20.96 \text{ for the data provided in Table 6.1.}$$

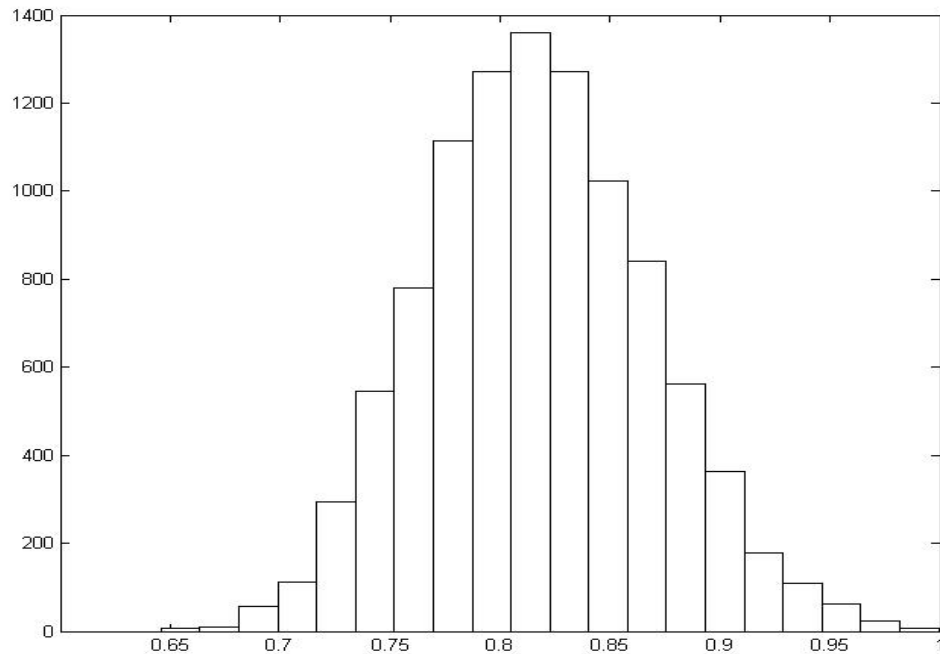
$$\text{Then determine } \sigma_d^2 = \frac{(\theta_0 - \sigma_\varepsilon^2 - r\sigma_p^2)}{kr}.$$

- iv.) If the constraint given in equation 6.4.9 is met, retain the set of simulated variance components. A set of variance components not obeying the restricted parameter space given in equation 6.4.9, should be disregarded as some of the simulated variance components may be negative.
- v.) Repeat steps i.) to iv.) for example $\tilde{\ell} = 10000$ times, retaining only the permissible sets of variance components. Steps i.) to iv.) can also be repeated until for example $\tilde{\ell} = 10000$ permissible sets of variance components are obtained.

In Figures 6.4.1, 6.4.2 and 6.4.3, histograms of the estimated marginal posterior distributions of the variance components $p(\sigma_\varepsilon^2|\mathbf{y})$, $p(\sigma_p^2|\mathbf{y})$ and $p(\sigma_d^2|\mathbf{y})$, are illustrated. Posterior medians and 95% equal tail credibility intervals are also given.

As is often the case, the posterior distributions of σ_ε^2 is symmetrical or fairly symmetrical. The reason for this is the large number of degrees of freedom $\nu_1 = bk(r - 1) = 480$ associated with the residual (error) variance. This is also the case for σ_p^2 where $\nu_2 = b(k - 1) = 105$. The posterior distribution of σ_d^2 on the other hand, is quite skew, and, the 95% credibility interval wide, which is a definite indication of the uncertainty associated with the between - days variance. From Figure 6.4.3 it is also clear that there is large "day - to - day" (between - days variation). The number of degrees of freedom associated with σ_d^2 is also quite small ($\nu_3 = b - 1 = 14$). The 95% equal tail credibility intervals for the variance components σ_ε^2 , σ_p^2 and σ_d^2 were obtained by finding the 2.5th and 97.5th percentiles of the respective ranked simulated variance component values.

Figure 6.4.1: Histogram of the Estimated Marginal Posterior Distribution of σ_ε^2 - Error Variance.



Median: 0.8156
95% Credibility Interval: [0.7212, 0.9229]

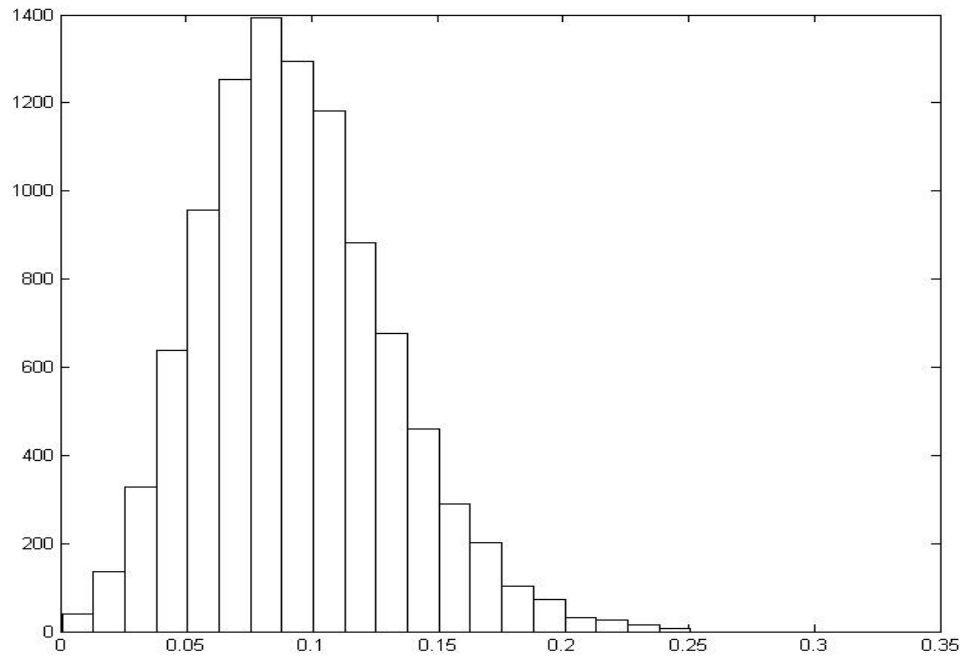
b.) Simulation of the target value μ .

The estimated marginal posterior distribution of the target value μ can be simulated as follows:

Substitute each of the simulated and retained sets of variance components σ_ε^2 , σ_p^2 and σ_d^2 into equation 6.4.7. Then draw a value μ from this normal distribution given in equation 6.4.7. There will therefore be one simulated μ value for each set of retained simulated variance components. This resulting set of simulated μ values can then be used to plot a histogram. This histogram will represent the estimated marginal posterior distribution of $\mu|y$.

Values for the target value μ can also be determined by only simulating $\theta_0 = (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)$ and substituting the simulated θ_0 values into equation 6.4.7. For each simulated θ_0 , a value μ is then simulated from equation 6.4.7. As was seen in Chapter 4, this would

Figure 6.4.2: Histogram of the Estimated Marginal Posterior Distribution of σ_p^2 - Variance Between Packages Within Days.

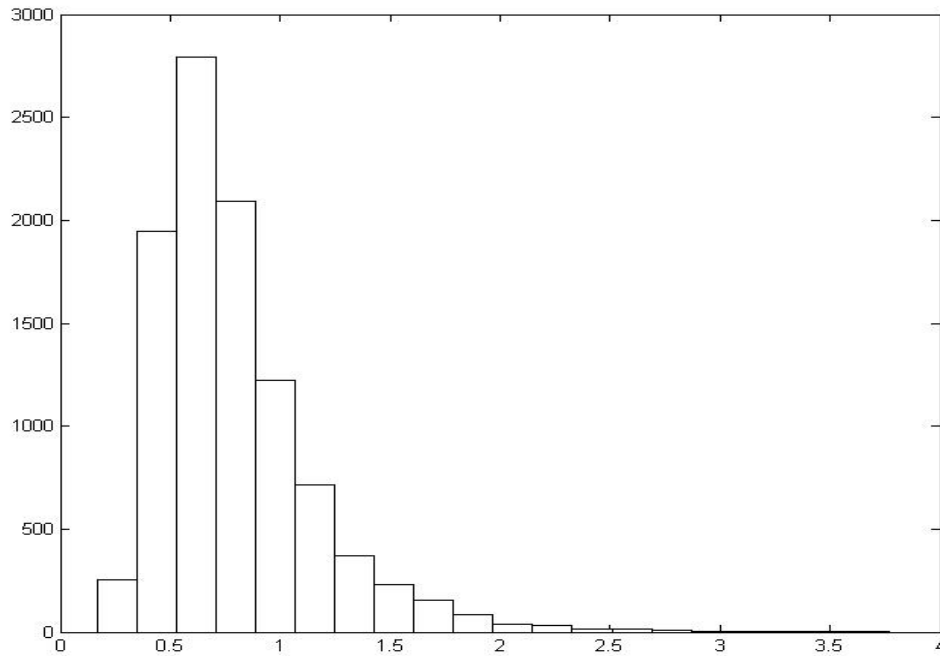


Median: 0.0903
95% Credibility Interval: [0.0303, 0.1761]

be the preferred method, since only θ_0 is simulated, and as a result, it is not necessary to disregard simulated sets of variance components not satisfying the constraint stated in equation 6.4.9. Although not given here, the marginal posterior distribution of $\mu|\mathbf{y}$ can also be determined using the Rao Blackwell method described in section 2.5. For each simulated θ_0 , the normal distribution given in equation 6.4.7 is drawn. If this process is for example repeated $\tilde{\ell} = 10000$ times, the estimated marginal posterior distribution of $\mu|\mathbf{y}$ would be represented by the average distribution of these $\tilde{\ell} = 10000$ conditional posterior distributions of $\mu|\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$.

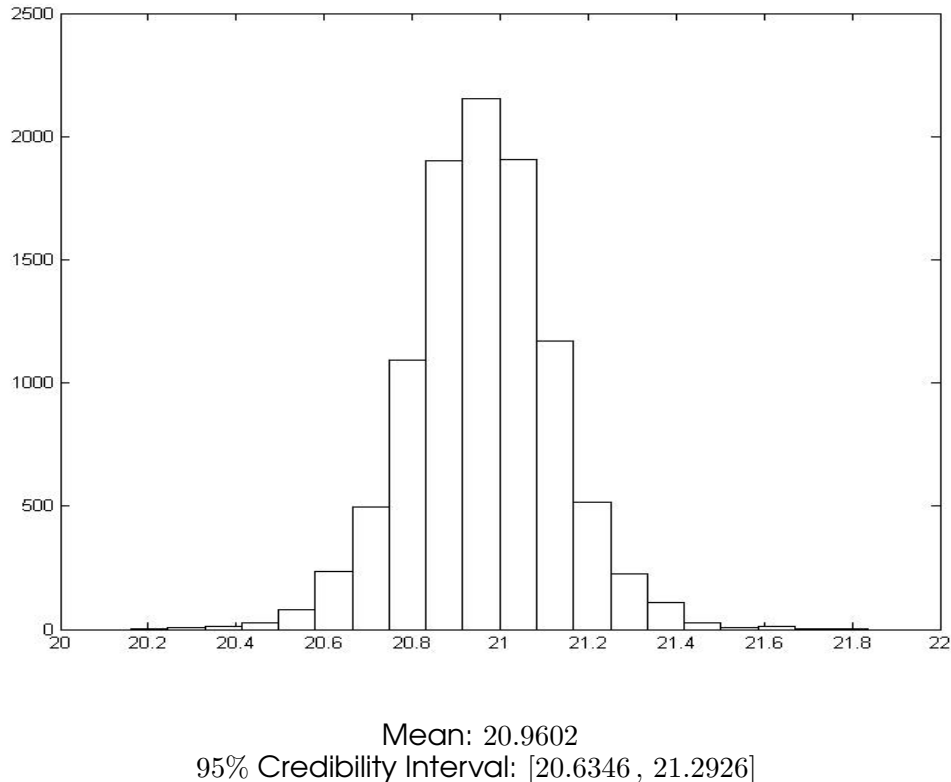
In Figure 6.4.4 the histogram of the estimated marginal posterior distribution of $\mu|\mathbf{y}$ is depicted for the data given in Table 6.1 with $\bar{y}_{\dots} = 20.96$ as mentioned. The posterior mean and 95% equal tail credibility interval is also given. The 95% credibility interval was obtained by determining the 2.5th and 97.5th percentiles of the ranked simulated μ values.

Figure 6.4.3: Histogram of the Estimated Marginal Posterior Distribution of σ_d^2 - Variance Between Days.



Median: 0.7097
95% Credibility Interval: [0.3496 , 1.7298]

Although not given here, the estimated marginal posterior distributions of the day effects d_i (for $i = 1, \dots, b$) can also easily be obtained. This can be done by simulating the separate variance components σ_ε^2 , σ_p^2 and σ_d^2 , and only retaining the sets satisfying the constraint stated in equation 6.4.9. The simulated retained sets of variance components are then substituted into equation 6.4.6 and then a normal distribution is drawn for each d_i ($i = 1, \dots, b$). If the process is for example repeated $\tilde{\ell} = 10000$ times, the marginal posterior distribution of each d_i ($i = 1, \dots, b$) can easily be obtained using the Rao Blackwell method discussed in section 2.5.

Figure 6.4.4: Histogram of the Estimated Marginal Posterior Distribution of μ .

6.5 Tolerance Intervals

The tolerance interval problem has been well investigated in the case of the one - way random effects model (*Fonseca et. al., 2007*). Limited results are however available for more general mixed and random effects models (*Fonseca, et. al., 2007*). Assuming that the variance components were known, Bagui, Bhaumik and Parnes (1996) attempted to construct upper one - sided tolerance limits for general unbalanced random effects models with more than two variance components (see also Krishnamoorthy and Mathew, 2009). It was however pointed out by Smith (2002) and Krishnamoorthy and Mathew (2009) that these resulting tolerance limits have actual confidence levels quite different from the assumed nominal level. Under a two - way crossed classification model with interaction, random effects and with unbalanced

data, Smith (2002) also addressed the computation of one - sided tolerance limits (see also Krishnamoorthy and Mathew, 2009). Fonseca et. al. (2007), also derived tolerance intervals for the two - way nested model with mixed effects or random effects. These authors used the generalized confidence interval idea by Krishnamoorthy and Mathew (2004) and the tolerance interval idea by Weerahandi (1993) in order to develop their mentioned tolerance intervals. Krishnamoorthy and Mathew (2009) also derived one - sided and two - sided tolerance intervals in a very general setting applicable to mixed and random effects models with balanced data.

The purpose of the remainder of this chapter is to present Bayesian tolerance intervals for the balanced two - factor nested random effects model given in equation 6.3.2. Using Bayesian simulation, the procedure will be applied to the data given in Table 6.1. Since theorems proved for determining the α - expectation tolerance interval will also be used for determining the one - and two - sided (α, δ) tolerance intervals, the α - expectation tolerance interval will be discussed first.

6.5.1 α - Expectation Tolerance Intervals

As mentioned, according to Wolfinger (1998), research question 2 mentioned in Chapter 1 is addressed by the α - expectation tolerance interval, since these intervals focus on prediction of one or a few future observations from a process.

By using the results proved in Theorems 6.4.4.2, 6.4.4.3 and 6.4.4.5, the following theorems can now be proved.

Theorem 6.5.1.1

The predictive density of the average of k^* future packages with r^* samples per packages from a specific day (i^{th} day) given the variance components, is normally distributed with mean

$$E(\bar{y}_{i..}^* | \mathbf{y}, \sigma_d^2, \sigma_p^2, \sigma_\varepsilon^2) = \frac{kr\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{i..} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{...} \quad (6.5.1)$$

and variance

$$Var(\bar{y}_{i..}^* | \mathbf{y}, \sigma_d^2, \sigma_p^2, \sigma_\varepsilon^2) = \frac{\sigma_\varepsilon^2 + r^*\sigma_p^2}{k^*r^*} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \left\{ \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + bkr\sigma_d^2}{bkr} \right\}. \quad (6.5.2)$$

Proof

The proof of Theorem 6.5.1.1 is given in Appendix D.

Theorem 6.5.1.2

The predictive density of the average of k^* future packages with r^* samples per package from a new unknown day, given the variance components, is normally distributed with mean

$$E(\bar{y}_{f..}^* | \mathbf{y}, \sigma_d^2, \sigma_p^2, \sigma_\varepsilon^2) = \bar{y}_{...} \quad (6.5.3)$$

and variance

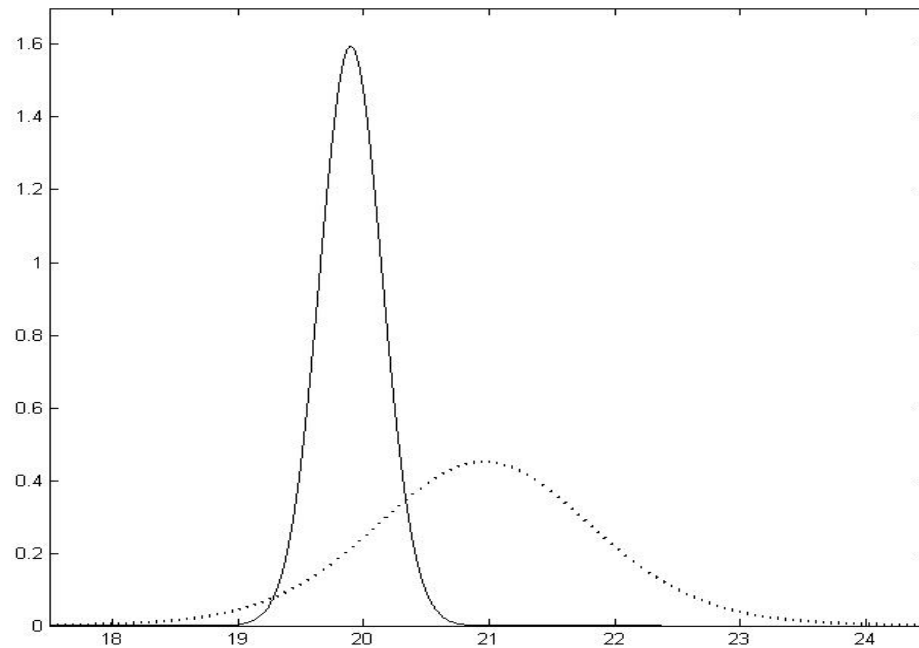
$$Var(\bar{y}_{f..}^* | \mathbf{y}, \sigma_d^2, \sigma_p^2, \sigma_\varepsilon^2) = \frac{\sigma_\varepsilon^2 + r^*\sigma_p^2 + k^*r^*\sigma_d^2}{k^*r^*} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}. \quad (6.5.4)$$

Proof

The proof of Theorem 6.5.1.2 is given in Appendix D.

Figure 6.5.1: Estimated Unconditional Predictive Distributions:

- Estimated Unconditional Predictive Distribution of the Average of k^* Packages, r^* Replicates of Any Future Day $p(\bar{y}_{f..}^* | \mathbf{y})$. For $k^* = 8$ and $r^* = 5$.
- Estimated Unconditional Predictive Distribution of the Average of k^* Packages, r^* Replicates of the i^{th} Day (Specific Day - 10th Day). For $k^* = 8$, $r^* = 5$ and $i = 10$.



95% Credibility Interval for $p(\bar{y}_{i..}^* | \mathbf{y})$: [19.41, 20.40]
 95% Credibility Interval for $p(\bar{y}_{f..}^* | \mathbf{y})$: [19.09, 22.82]

For illustrative purposes, the estimated unconditional predictive distributions of $\bar{y}_{i..}^*$ ($i = 10$) i.e. for the tenth day, and $\bar{y}_{f..}^*$ i.e. a new or unknown day, for $k^* = 8$ packages per day and $r^* = 5$ samples per package are depicted in Figure 6.5.1. The 95% equal tail credibility intervals are given in Table 6.2 and are the so called Bayesian 0.95 - expectation tolerance limits. Figure 6.5.1 was obtained for the spun yarn data given in Table 6.1.

The Bayesian simulation procedure for obtaining Figure 6.5.1 was performed in the following way:

- i.) By using the rejection sampling procedure as described in section 6.4.5, the variance components were generated from their joint posterior distribution.
- ii.) For each set of simulated values $(\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2)$, the conditional predictive distributions $p(\bar{y}_{i..}^* | \sigma_d^2, \sigma_p^2, \sigma_\varepsilon^2, \mathbf{y})$ and $p(\bar{y}_{f..}^* | \sigma_d^2, \sigma_p^2, \sigma_\varepsilon^2, \mathbf{y})$ are normally distributed (see Theorems 6.5.1.1 and 6.5.1.2). Substitute the simulated set of variance components into the two normal distributions given in Theorems 6.5.1.1 and 6.5.1.2, and, draw the two normal distributions.
- iii.) Steps i.) and ii.) were repeated $\tilde{\ell}$ times. As mentioned for this example, $\tilde{\ell}$ was taken as 10000. Using the Rao Blackwell argument (see Gelfand and Smith 1991), the estimated unconditional predictive distributions $p(\bar{y}_{i..}^* | \mathbf{y})$ and $p(\bar{y}_{f..}^* | \mathbf{y})$ were obtained by averaging the conditional predictive distributions over the $\tilde{\ell}$ repetitions.

As described above, the predictive distributions were obtained through Monte Carlo simulations where independent samples were obtained from the joint posterior distribution.

For comparative purposes, the estimated mean values and 95% equal tail credibility intervals for both $p(\bar{y}_{i..}^* | \mathbf{y})$ (for $i = 10$) and $p(\bar{y}_{f..}^* | \mathbf{y})$ are given in Table 6.2.

Table 6.2: Estimated Mean Values and 95% Equal Tail Credibility Intervals for the Two Estimated Unconditional Predictive Distributions Depicted in Figure 6.5.1. Obtained for the Spun Yarn Data Given in Table 6.1.

Day	Mean	95% Equal Tail Credibility Interval
$p(\bar{y}_{i..}^* \mathbf{y})$ (for $i = 10$)	19.90	[19.41 , 20.40]
$p(\bar{y}_{f..}^* \mathbf{y})$	20.96	[19.09 , 22.82]

Remember, as mentioned, the 95% equal tail credibility intervals given in Table 6.2, respectively represent the Bayesian 0.95 - expectation tolerance intervals for both $p(\bar{y}_{i..}^* | \mathbf{y})$ (for $i = 10$) and $p(\bar{y}_{f..}^* | \mathbf{y})$.

The Bayesian 0.95 - expectation tolerance interval (95% credibility interval) for $\bar{y}_{f..}^*$, the average of $k^* = 8$ future packages with $r^* = 5$ samples per package, sampled on any unknown or future day, is equal to [19.09 , 22.82]. Laubscher (1996) calculated

$$S\bar{y}_{f..}^* = \sqrt{\frac{1}{b-1} \sum_{i=1}^b (\bar{y}_{i..} - \bar{y}_{...})^2} = \sqrt{\frac{m_3}{k^* r^*}} = 0.8403$$

and used a bias correcting constant $C_4 = 0.9823$. The classical 95% limits are given by $\bar{y}_{...} \pm t_{0.025; b-1} \left(\frac{S\bar{y}_{f..}^*}{C_4} \right)$. Now $t_{0.025; 14} = 2.145$, $\bar{y}_{...} = 20.96$ and the classical limits are therefore equal to [19.12 , 22.79], which are for all practical purposes equal to the Bayesian 0.95 - expectation tolerance limits.

Since the 95% Bayesian prediction interval for the 10th day is [19.41 , 20.40], it follows that if further sample means are obtained for the 10th day or for any future day similar to the 10th day from, $k^* = 8$ packages and $r^* = 5$ samples per packages, the process is in control if 95% or more of these sample means fall within these specified Bayesian 0.95 - expectation tolerance interval limits.

From this comparison, the question may also arise as to what kind of frequentist coverage properties the Bayesian interval under the probability matching prior given in

equation 6.4.3 will have. This was investigated briefly by simulating 1000 data sets with the same structure as the spun yard data given in Table 6.1. The estimated confidence coefficient was calculated as the percentage of intervals that contained the true predictive value. The coverage percentage in the case of the 95% prediction interval was 95.2%. For the simulation study, $\bar{y}_{f..}^*$ was simulated from the following normal distribution:

$$\bar{y}_{f..}^* | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y} \sim N(20.96, 0.706085)$$

where

$$\bar{y}_{f..}^* | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y} \sim N\left(\mu, \frac{\sigma_\varepsilon^2}{k^*r^*} + \frac{\sigma_p^2}{k^*} + \sigma_d^2\right)$$

with $k^* = 8$, $r^* = 5$ and since the 1000 simulated data sets have the same structure as the data given in Table 6.1, $\sigma_\varepsilon^2 = 0.8139$, $\sigma_p^2 = 0.0899$, $\sigma_d^2 = 0.6745$ and $\mu = 20.96$.

These values for σ_ε^2 , σ_p^2 and σ_d^2 can be calculated as follows: In section 6.4.5 it was shown that $\nu_1 m_1 = 390.6720$, $\nu_2 m_2 = 132.6570$ and $\nu_3 m_3 = 395.4343$.

Also, since $\nu_1 = bk(r - 1) = 15(8)(5 - 1) = 480$, $\nu_2 = b(k - 1) = 15(8 - 1) = 105$ and $\nu_3 = (b - 1) = 15 - 1 = 14$ it can therefore be shown that

$$m_1 = \frac{\nu_1 m_1}{\nu_1} = \frac{390.6720}{480} = 0.8139,$$

$$m_2 = \frac{\nu_2 m_2}{\nu_2} = \frac{132.6570}{105} = 1.2634, \text{ and}$$

$$m_3 = \frac{\nu_3 m_3}{\nu_3} = \frac{395.4343}{14} = 28.2453.$$

Now, $E(m_1) = \theta_2 = \sigma_\varepsilon^2$ (see Laubscher, 1996) and therefore σ_ε^2 was taken to be equal to 0.8139. Similarly

$$E(m_2) = \theta_1 = (\sigma_\varepsilon^2 + r\sigma_p^2) = 1.2634.$$

Therefore

$$\sigma_p^2 = \frac{1}{r}(\theta_1 - \sigma_\varepsilon^2) = \frac{1}{5}(1.2634 - 0.8139) = 0.0899.$$

Also, $E(m_3) = \theta_0 = (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2) = 28.2453$.

Therefore

$$\sigma_d^2 = \frac{1}{kr}(\theta_0 - \sigma_\varepsilon^2 - r\sigma_p^2) = \frac{1}{8(5)}[28.2453 - 0.8139 - 5(0.0899)] = \frac{1}{40}[28.2453 - 0.8139 - 0.4495] = 0.6745.$$

6.5.2 One - Sided (α, δ) Tolerance Intervals

Since it was proved in Theorem 6.5.1.2 that

$$\bar{y}_{f..}^* | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y} \sim N\left(\mu, \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*}\right),$$

for the construction of lower one - sided (α, δ) tolerance limits, the estimated marginal posterior density of q^* must therefore be obtained, which in this case represents the $(1 - \alpha)^{th}$ quantile of the $N\left(\mu, \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*}\right)$ distribution.

This distribution describes the averages of future data from new or unknown days. The Bayesian simulation procedure for obtaining the posterior distribution of q^* can be performed in the following way:

- i.) The rejection sampling procedure described in section 6.4.5 can be used to generate the variance components. In Chapter 4 it was however also mentioned that if the separate variance components are not needed, it is preferable to simulate functions of the variance components. For the balanced two - way nested random effects model given in equation 6.3.2, it is not necessary to check that the constraint given in equation 6.4.9 is met. The implication of this is that no set of simulated variance components will be disregarded. For the balanced two - factor nested random effects model given in equation 6.3.2, $\theta_0 = (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)$ can therefore be simulated directly using the method described in section 6.4.5.

- ii.) Substitute the simulated $\theta_0 = (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)$ into the normal distribution given by

$$\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y} \sim N\left(\bar{y}_{\dots}, \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}\right)$$

and simulate a value for μ .

- iii.) If $k^* = k$ and $r^* = r$, then substitute the simulated values for μ and θ_0 into the equation for q given by

$$q = \mu - z_\alpha \left\{ \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr} \right\}$$

where z_α represents the α^{th} quantile of a standard normal distribution.

If either $k^* \neq k$ or $r^* \neq r$ or both $k^* \neq k$ and $r^* \neq r$, then simulate the separate variance components and μ as described in section 6.4.5 ensuring that the constraint given in equation 6.4.9 is met. If this constraint is not met, disregard this simulated set of variance components and μ . Substitute the retained simulated variance components σ_ε^2 , σ_p^2 and σ_d^2 together with the simulated target value μ into the equation for q^* given by

$$q^* = \mu - z_\alpha \left\{ \frac{\sigma_\varepsilon^2 + r^*\sigma_p^2 + k^*r^*\sigma_d^2}{k^*r^*} \right\}$$

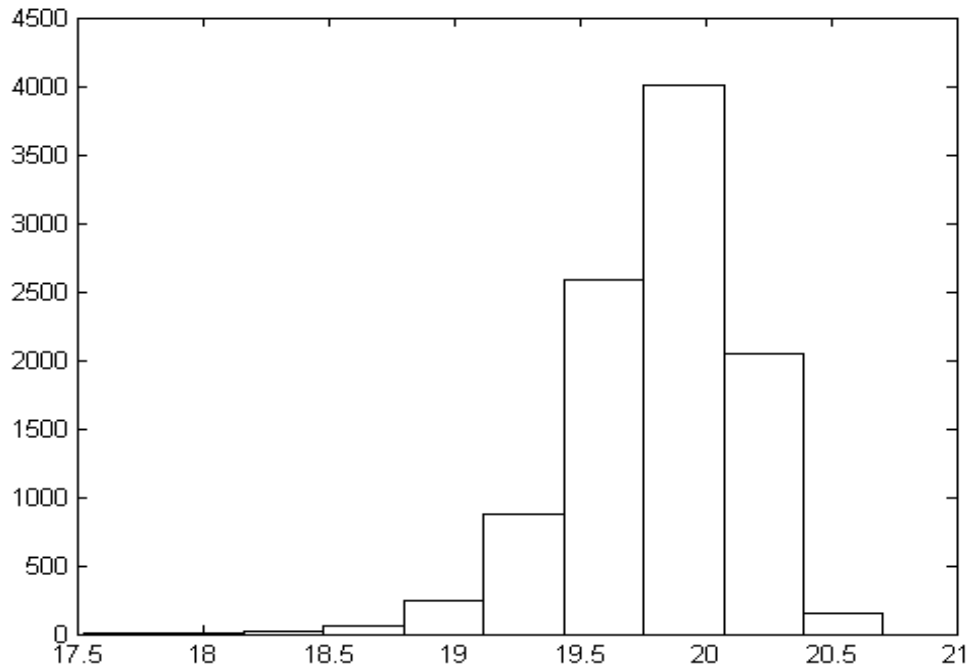
where z_α represents the α^{th} quantile of a standard normal distribution.

- iv.) Repeat steps i.) to iii.) for example $\tilde{\ell} = 10000$ times and draw the histogram. This histogram represents the estimated marginal posterior distribution of $q | \mathbf{y}$.

Although not given here, the Rao Blackwell method described in section 2.5 can also be used to obtain the estimated marginal posterior distributions of the $(1 - \alpha)^{th}$ quantile of the $N\left(\mu, \frac{\sigma_\varepsilon^2 + r^*\sigma_p^2 + k^*r^*\sigma_d^2}{k^*r^*}\right)$ distribution. It must also be noted that upper one - sided (α, δ) tolerance limits can also be constructed for q_u^* or q_u by first determining estimated marginal posterior distributions for

$$q_u^* = \mu + z_\alpha \left\{ \frac{\sigma_\varepsilon^2 + r^*\sigma_p^2 + k^*r^*\sigma_d^2}{k^*r^*} \right\} \text{ and } q_u = \mu + z_\alpha \left\{ \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr} \right\},$$

Figure 6.5.2: Histogram of the Estimated Marginal Posterior Distribution of the $(1 - 0.90)^{th}$ Quantile q .



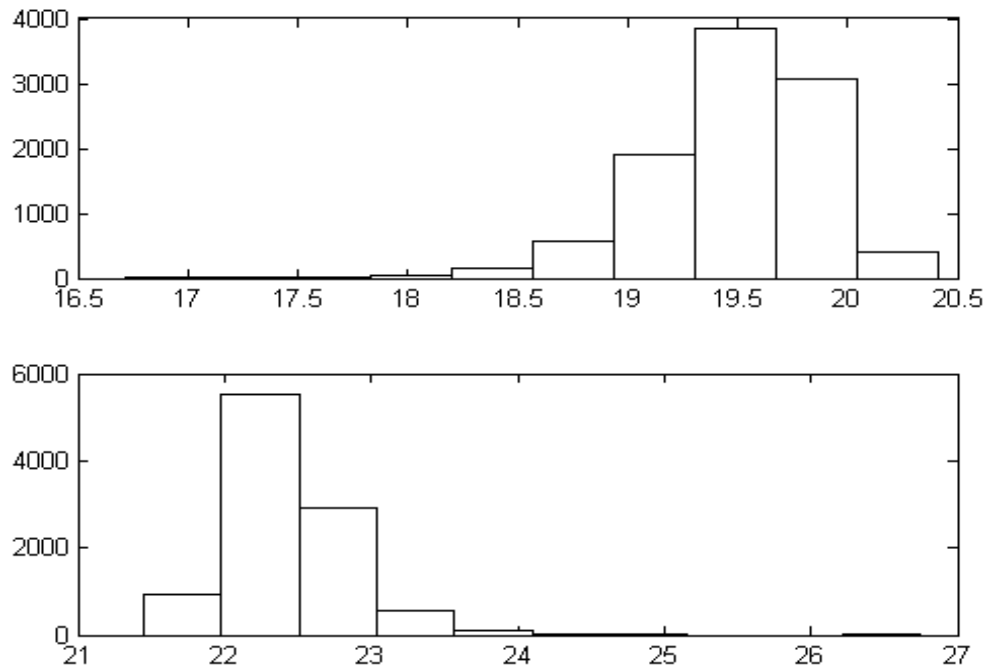
(0.90, 0.95) Lower One - Sided Tolerance Limit: 19.2311

where q_u^* and q_u represent the α^{th} quantiles of the $N\left(\mu, \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*}\right)$ and $N\left(\mu, \frac{\sigma_\varepsilon^2 + r \sigma_p^2 + k r \sigma_d^2}{k r}\right)$ distributions respectively.

Figure 6.5.2 plots the histogram of the estimated marginal posterior distribution of q , which in this case represents the $(1 - 0.90)^{th}$ quantile (or tenth percentile) of the normal distribution given by $N\left(\mu, \frac{\sigma_\varepsilon^2 + r \sigma_p^2 + k r \sigma_d^2}{k r}\right)$.

For the histogram depicted in Figure 6.5.2, the lower one - sided (0.90, 0.95) tolerance limit is equal to 19.2311. The value 19.2311 therefore represents the estimated $(1 - 0.95)^{th}$ quantile (or fifth percentile) of the marginal posterior distribution of q , thus indicating the Bayesian "B - basis" (0.90, 0.95) lower tolerance limit. The Bayesian "B - basis" lower (0.90, 0.95) tolerance limit equals to 19.2311 represents the extension value of which 90% of the average extensions of $k = 8$ future packages with $r = 5$ samples per package from a new or unknown day will be greater than with probability 0.95.

Figure 6.5.3: Histograms of the Estimated Marginal Posterior Distributions of the $\left[\frac{(1\pm 0.90)}{2}\right]^{th}$ Quantiles q_l and q_u for the Data Given in Table 6.1.



In Figure 6.5.3 the histograms of the estimated marginal posterior distributions of

$$\text{a.) } q_l = \mu - 1.645 \left\{ \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr} \right\}^{\frac{1}{2}} \text{ and}$$

$$\text{b.) } q_u = \mu + 1.645 \left\{ \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr} \right\}^{\frac{1}{2}},$$

representing the $\left[\frac{(1\pm 0.90)}{2}\right]^{th}$ quantiles, are displayed.

The 95% credibility interval of the estimated marginal posterior distribution of q_l is equal to $[18.6112, 20.0924]$, while the 95% credibility interval of the estimated marginal posterior distribution of q_u is equal to $[21.8179, 23.3036]$.

6.5.3 Two - Sided (α, δ) Tolerance Intervals

As mentioned by Wolfinger (1998) and also in the previous chapters, the construction of two - sided (α, δ) tolerance intervals is slightly more complex. The simple procedure of computing upper and lower limits separately and then combining them is not precisely valid, since as mentioned, the two quantiles do not have a posterior correlation of 1. According to Wolfinger (1998), one way of constructing a valid two - sided (α, δ) tolerance interval, is to begin by computing the two quantiles \tilde{q}_l^* and \tilde{q}_u^* given by

$$\tilde{q}_l^* = \mu - z_{[(1+\alpha)/2]} \left\{ \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*} \right\}^{\frac{1}{2}}, \quad (6.5.5)$$

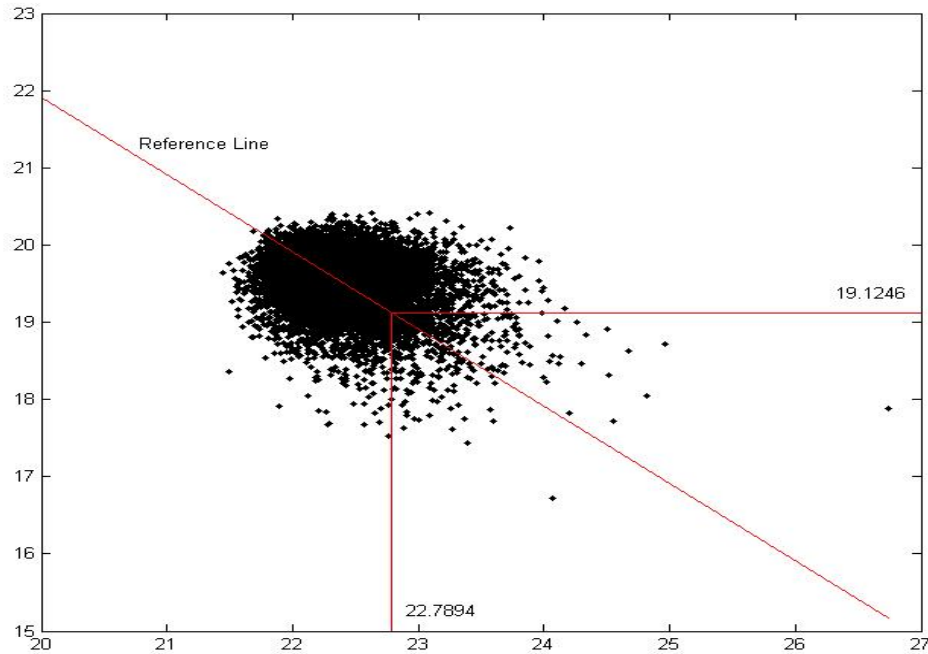
and

$$\tilde{q}_u^* = \mu + z_{[(1+\alpha)/2]} \left\{ \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*} \right\}^{\frac{1}{2}}, \quad (6.5.6)$$

where $z_{\frac{1+\alpha}{2}}$ represents the $\left(\frac{1+\alpha}{2}\right)^{th}$ quantile of a standard normal distribution.

For the balanced two - factor nested random effects model given in equation 6.3.2, the pairs $(\tilde{q}_l^*, \tilde{q}_u^*)$ form a sample from the bivariate posterior distribution of the $\left(\frac{1-\alpha}{2}\right)^{th}$ and $\left(\frac{1+\alpha}{2}\right)^{th}$ quantiles (Wolfinger, 1998). Bayesian confidence regions can be obtained for these bivariate samples, but as mentioned, are difficult to use in practice since they are two - dimensional ellipsoids. It was already mentioned that Wolfinger (1998) succeeded in constructing a two - sided interval that is one - dimensional and symmetric about the mean. To obtain such an interval, first form a scatterplot of \tilde{q}_l^* versus \tilde{q}_u^* , with \tilde{q}_l^* plotted on the vertical axis, and \tilde{q}_u^* plotted on the horizontal axis. Then, as mentioned, construct the reference line given by $\tilde{q}_l^* = -\tilde{q}_u^* + 2\bar{y}_{...}$, and draw two additional lines, one parallel to each axis and intersecting on the reference line. Slide this intersection point along the reference line until $100(1 - \delta)\%$ of the observations are contained in the half - rectangle opening towards the lower right portion of the graph. The coordinates of the resulting intersection point form a two - sided tolerance interval of the desired form. This procedure is illustrated graphically in Figure 6.5.4 for the spun yarn data given in Table 6.1.

Figure 6.5.4: Construction of a Two - Sided $(0.90, 0.95)$ Tolerance Interval in the Case of $\bar{y}_{f..}^*$ for the Spun Yarn Data Given in Table 6.1.



The $(\frac{1-\alpha}{2})^{th}$ and $(\frac{1+\alpha}{2})^{th}$ quantiles given by \tilde{q}_l^* and \tilde{q}_u^* respectively, which are needed for the construction of the scatter plot and subsequent two - sided $(0.90, 0.95)$ tolerance interval depicted in Figure 6.5.4, can be determined as follows:

- i.) Simulate the variance components σ_ε^2 , σ_p^2 and σ_d^2 as well as the target value μ , or, simulate θ_0 and the target value μ using the method described for simulating q^* or q for the one - sided (α, δ) tolerance limit described in section 6.5.2.
- ii.) Substitute either the simulated variance components and the target value μ or the simulated θ_0 and target value μ into both equations 6.5.5 and 6.5.6 to obtain a $(\tilde{q}_l^*, \tilde{q}_u^*)$ pair.

- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times and draw the scatter plot. Proceed to construct the two - sided (0.90, 0.95) tolerance interval using the method described above.

From Figure 6.5.4 it can be seen that the estimated two - sided (0.90, 0.95) tolerance interval is equal to [19.1246 , 22.7894]. This estimated Bayesian two - sided (0.90, 0.95) tolerance interval can be interpreted as follows: If $k = 8$ future packages of spun yarn with $r = 5$ samples per packages from a new or unknown day are selected, 90% of the average extensions will fall in the interval [19.1246 , 22.7894] with probability 0.95.

The one - and two - sided (α, δ) tolerance intervals described above, apply to averages of observations from new or unknown days. If inference is desired for the average of future observations from a specific day, say the i^{th} day, then the marginal posterior distribution of the $(1 - \alpha)^{th}$ quantiles of the $N\left(\mu + d_i, \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{k^*r^*}\right)$ distribution must be estimated. Remember also that it was shown in Theorem 6.4.4.5 that the conditional posterior distribution of $(\mu + d_i)$, given the variance components, is normally distributed with mean equals to

$$E\left((\mu + d_i) | \mathbf{y}, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2\right) = \frac{k r \sigma_d^2}{\sigma_\varepsilon^2 + r \sigma_p^2 + k r \sigma_d^2} \bar{y}_{i..} + \frac{\sigma_\varepsilon^2 + r \sigma_p^2}{\sigma_\varepsilon^2 + r \sigma_p^2 + k r \sigma_d^2} \bar{y} \dots$$

and variance equals to

$$Var\left((\mu + d_i) | \mathbf{y}, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2\right) = \frac{\sigma_\varepsilon^2 + r \sigma_p^2}{\sigma_\varepsilon^2 + r \sigma_p^2 + k r \sigma_d^2} \left\{ \frac{\sigma_\varepsilon^2 + r \sigma_p^2 + b k r \sigma_d^2}{b k r} \right\}.$$

6.5.4 Fixed - in - Advance Tolerance Intervals

Finally, to determine the content of a fixed - in - advance tolerance interval, the posterior distribution of this content has to be determined. For example, suppose that a lower fixed - in - advance limit of s is specified for data assumed to arise from a new

batch. Therefore, for the balanced two - factor nested random effects model given in equation 6.3.2, compute the content c^* of the interval $[s, \infty]$ given by

$$c^* = 1 - \Phi \left[\frac{(s - \mu)(k^* r^*)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2)^{\frac{1}{2}}} \right] \quad (6.5.7)$$

where $\Phi[\cdot]$ represents the standard normal cumulative distribution function which is given by

$$\Phi(s^*) = \int_{-\infty}^{s^*} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}z^2\right) dz$$

where

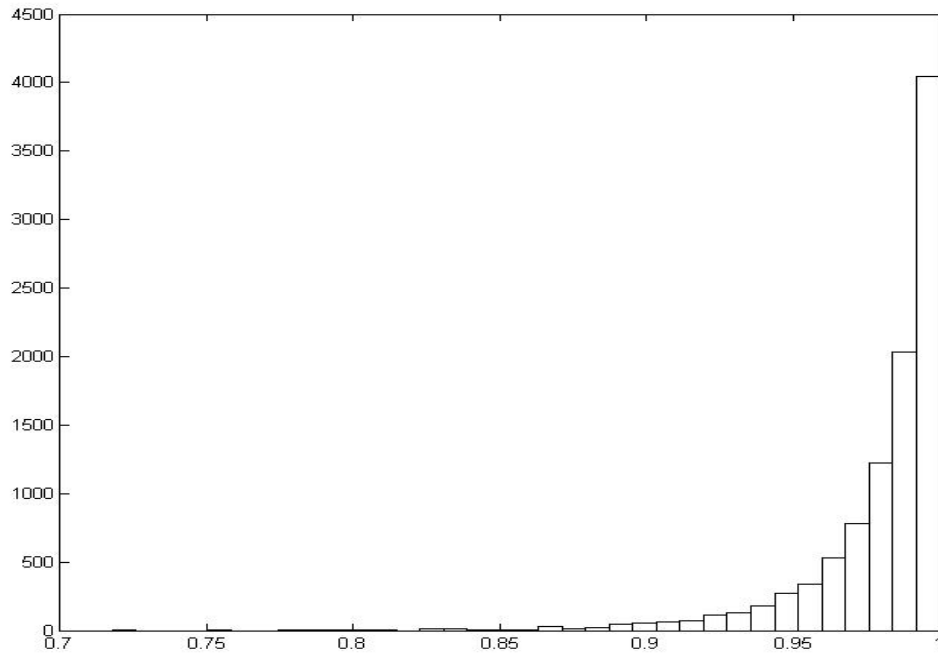
$$s^* = (s - \mu) \left\{ \frac{(\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2)}{(k^* r^*)} \right\}^{-\frac{1}{2}}.$$

Thus, a content c^* is found for each average of future observations from a new or unknown day, and these c^* values therefore form a sample from the posterior distribution of this content.

To determine a fixed - in - advance tolerance interval for the content c^* of the interval $[s, \infty]$, the following steps can be followed.

- i.) Simulate the variance components σ_ε^2 , σ_p^2 and σ_d^2 as well as the target value μ , or, simulate θ_0 and the target value μ using the method described for simulating q^* or q for the one - sided (α, δ) tolerance interval described in section 6.5.2.
- ii.) Depending on the situations mentioned in section 6.5.2, either substitute the simulated variance components and the target value μ , or the simulated θ_0 and the target value μ , into equation 6.5.7 to obtain a content c^* .
- iii.) Repeat steps i.) and ii.) for example $\tilde{\ell} = 10000$ times and draw a histogram. This histogram represents the estimated posterior distribution of the content c^* above the preselected specification limit s .

Figure 6.5.5: Histogram of the Estimated Posterior Content of the Interval $[19.0, \infty]$ in the Case of $\bar{y}_{f..}^*$, Determined for the Spun Yarn Data Given in Table 6.1.



95% Fixed - in - Advance Tolerance Interval: $[0.9083, 0.9995]$

A $100(\alpha)\%$ equal tail credibility interval can also easily be determined for the content c^* of the interval $[s, \infty]$ by ranking the sample of c^* values in order of magnitude and then finding the $100(\frac{1-\alpha}{2})^{th}$ and $100(\frac{1+\alpha}{2})^{th}$ percentiles of the ranked simulated c^* values. Remember, this $100(\alpha)\%$ equal tail credibility interval represents the fixed - in - advance tolerance interval for the content of the interval $[s, \infty]$ i.e. the content above a pre-selected specification limit s .

For illustrative purposes, the histogram of the estimated posterior content of the interval $[19.0, \infty]$ for a fixed - in - advance lower specification limit $s = 19.0$, is given in Figure 6.5.5 for the spun yarn data given in Table 6.1. It must be mentioned that this lower specification limit $s = 19.0$ was selected solely for illustrative purposes.

From Figure 6.5.5 it can be seen that the histogram of the estimated posterior contents c^* of the interval $[19.0, \infty]$ is negatively skewed and varies from 0.7 to 1 with a median equals to 0.9885. The $100(0.95)\%$ Bayesian fixed - in - advance tolerance interval, is

equal to $[0.9083, 0.9995]$ and can be interpreted as follows: For the spun yarn data given in Table 6.1, if $k^* = k = 8$ packages with $r^* = r = 5$ samples per package from a new or unknown day are selected, between 90.83% and 99.95% of the average extensions will fall in the interval $[19.0, \infty]$ with probability 0.95.

In section 4.6.4, a probability matching prior was proposed for the fixed - in - advance tolerance interval for averages of observations from new or unknown batches in the case of the balanced one - way random effects model. If $k^* = k$ and $r^* = r$ (as defined in equations 6.3.1 and 6.3.2), it can also be shown that for the average of k packages with r samples per packages from a new or unknown day, the prior distributions

$$\pi_a(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}$$

and

$$\pi_b(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{-\frac{1}{2}}$$

are probability matching priors for the contents c of the interval $[s, \infty]$ where c is given by

$$c = 1 - \Phi \left[\frac{(s-\mu)(kr)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}} \right] \text{ and}$$

$\Phi[\cdot]$ represents a standard normal cumulative distribution function.

The following theorem can now be proved.

Theorem 6.5.4.1

For the average of $k^* = k$ packages with $r^* = r$ samples per package from a new or unknown day, the prior distributions given by

$$\pi_a(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}} \quad (6.5.8)$$

and

$$\pi_b(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{-\frac{1}{2}} \quad (6.5.9)$$

are probability matching priors for the contents c of the interval $[s, \infty]$ given by

$$c = 1 - \Phi \left[\frac{(s - \mu)(kr)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}} \right], \quad (6.5.10)$$

where $\Phi[\cdot]$ represents a standard normal cumulative distribution function.

Proof

The proof of Theorem 6.5.4.1 is given in Appendix D.

Although not given here, for the probability matching prior given in equation 6.5.9 for example, the fixed - in - advance tolerance interval can also be determined for the content c given in equation 6.5.10 using the weighted Monte Carlo (sampling - importance resampling) method.

For this probability matching prior given in equation 6.5.9, the proposal distribution is therefore given by

$$P_R(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}(\nu_2+2)} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+3)} \\ \exp \left\{ -\frac{1}{2} \left[\frac{bkr(\bar{y}_{..} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_3 m_3}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right] \right\},$$

while the target distribution is given by

$$P_m(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}(\nu_2+2)} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+4)} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{-\frac{1}{2}} \\ \exp \left\{ -\frac{1}{2} \left[\frac{bkr(\bar{y}_{..} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_3 m_3}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right] \right\}.$$

Also, the resulting normalized weights for $l = 1, 2, \dots, \tilde{\ell}$ is given by

$$W_l = \frac{(\sigma_\varepsilon^{2(\ell)} + r\sigma_p^{2(\ell)} + kr\sigma_d^{2(\ell)})^{-\frac{1}{2}} \left\{ 1 + \frac{(s-\mu^{(\ell)})^2 kr}{2(\sigma_\varepsilon^{2(\ell)} + r\sigma_p^{2(\ell)} + kr\sigma_d^{2(\ell)})} \right\}^{-\frac{1}{2}}}{\sum_{l=1}^{\tilde{\ell}} (\sigma_\varepsilon^{2(\ell)} + r\sigma_p^{2(\ell)} + kr\sigma_d^{2(\ell)})^{-\frac{1}{2}} \left\{ 1 + \frac{(s-\mu^{(\ell)})^2 kr}{2(\sigma_\varepsilon^{2(\ell)} + r\sigma_p^{2(\ell)} + kr\sigma_d^{2(\ell)})} \right\}^{-\frac{1}{2}}}.$$

The simulation procedure is further similarly applied to the weighted Monte Carlo (sampling - importance resampling) methods described in both sections 2.6.4 and 4.6.4.1.

The situation is somewhat more complex for unbalanced mixed linear models. Given the variance components, the fixed effects, random effects and predictive densities are still normally distributed. It is therefore recommended that the best way to derive a joint prior for the variance components is to use the method of Wolfinger and Kass (2000). These authors calculated the square root of the determinant of the Fisher information matrix, i.e. Jeffreys' rule, where the calculations are based on the likelihood of the variance components alone. The resulting prior therefore becomes a special case of that of Berger and Bernardo (1992c).

Since the posterior distribution of the variance components will not be proportional to the product of inverse gamma distribution, MCMC procedures should be used to generate samples from the joint posterior of the variance components. The independence chain algorithm (*Tierney, 1994*) is one such procedure that can be used.

6.6 Appendix D

Table D1: The Physical Property of "Extension" of a Synthetic Yarn, Measured over 15 Consecutive Days of January, 1995. Eight Packages of Yarn were Sampled Each Day and Each Measurement Represents the Average of Five Replicate Samples per Package. Data Collected by Prof. N.F. Laubscher at SANS Fibres (Pty.) Ltd.

Day	Package	Extension	Daily Average
1	1	20.30	
1	2	20.06	
1	3	20.48	
1	4	19.22	
1	5	19.16	
1	6	19.94	
1	7	20.69	
1	8	19.02	19.86
2	1	21.91	
2	2	21.68	
2	3	22.09	
2	4	22.39	
2	5	21.56	
2	6	22.38	
2	7	21.82	
2	8	21.90	21.97
3	1	22.29	
3	2	21.19	
3	3	21.90	
3	4	21.70	
3	5	21.69	
3	6	21.65	
3	7	21.85	
3	8	21.50	21.72

Table D1: Continued

Day	Package	Extension	Daily Average
4	1	20.86	
4	2	21.12	
4	3	20.20	
4	4	20.36	
4	5	20.64	
4	6	19.85	
4	7	19.72	
4	8	20.33	20.39
5	1	20.48	
5	2	20.43	
5	3	20.07	
5	4	19.69	
5	5	20.60	
5	6	20.49	
5	7	20.46	
5	8	19.74	20.25
6	1	21.79	
6	2	21.65	
6	3	21.75	
6	4	21.94	
6	5	21.77	
6	6	22.17	
6	7	21.67	
6	8	21.51	21.78

Table D1: Continued

Day	Package	Extension	Daily Average
7	1	21.52	
7	2	20.44	
7	3	20.88	
7	4	20.13	
7	5	20.64	
7	6	20.58	
7	7	20.50	
7	8	20.71	20.68
8	1	22.09	
8	2	20.71	
8	3	20.68	
8	4	20.67	
8	5	20.67	
8	6	20.98	
8	7	20.94	
8	8	21.13	20.98
9	1	19.19	
9	2	19.84	
9	3	19.95	
9	4	20.25	
9	5	21.60	
9	6	19.65	
9	7	20.77	
9	8	20.37	20.20

Table D1: Continued

Day	Package	Extension	Daily Average
10	1	20.99	
10	2	20.53	
10	3	19.37	
10	4	19.93	
10	5	19.31	
10	6	19.86	
10	7	19.00	
10	8	19.81	19.85
11	1	21.47	
11	2	20.42	
11	3	20.61	
11	4	19.91	
11	5	19.86	
11	6	20.57	
11	7	20.61	
11	8	19.37	20.35
12	1	19.22	
12	2	20.59	
12	3	20.68	
12	4	20.23	
12	5	20.93	
12	6	20.98	
12	7	20.69	
12	8	20.46	20.47

Table D1: Continued

Day	Package	Extension	Daily Average
13	1	21.58	
13	2	22.27	
13	3	21.52	
13	4	22.62	
13	5	21.73	
13	6	21.80	
13	7	22.00	
13	8	21.53	21.88
14	1	22.31	
14	2	21.84	
14	3	22.29	
14	4	22.38	
14	5	22.40	
14	6	21.60	
14	7	22.14	
14	8	20.94	21.99
15	1	22.48	
15	2	21.65	
15	3	22.00	
15	4	22.71	
15	5	22.28	
15	6	21.38	
15	7	21.35	
15	8	22.43	22.04

Proof of Theorem 6.4.2.1

Firstly we will briefly review the theory of the probability matching prior. Datta and Ghosh (1995) derived the differential equation that a prior must satisfy if the posterior probability of a one sided credibility interval for a parametric function and its frequentist probability agree up to $O(n^{-1})$ where n is the sample size. They proved that the agreement between the posterior probability and the frequentist probability holds if and only if

$$\sum_{\alpha=1}^m \frac{\partial}{\partial \theta_{\alpha}} \{ \eta_{\alpha}(\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) \} = 0 ,$$

where $p(\boldsymbol{\theta})$ is the probability matching prior for $\boldsymbol{\theta}$, the vector of unknown parameters.

Also

$$\nabla_t = \left[\frac{\partial}{\partial \theta_1} t(\boldsymbol{\theta}), \dots, \frac{\partial}{\partial \theta_m} t(\boldsymbol{\theta}) \right]'$$

and

$$\eta(\boldsymbol{\theta}) = \frac{F^{-1}(\boldsymbol{\theta}) \nabla_t(\boldsymbol{\theta})}{\sqrt{\nabla_t'(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) \nabla_t(\boldsymbol{\theta})}} = [\eta_1(\boldsymbol{\theta}), \dots, \eta_m(\boldsymbol{\theta})]'$$

It is clear that $\eta'(\boldsymbol{\theta}) F(\boldsymbol{\theta}) \eta(\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta}$ where $F^{-1}(\boldsymbol{\theta})$ is the inverse of $F(\boldsymbol{\theta})$, the Fisher Information matrix of $\boldsymbol{\theta}$. Also, $t(\boldsymbol{\theta})$ is the parameter of interest.

For the balanced two - factor nested random effects model given in equation 6.3.1, it was mentioned that the integrated likelihood function was given by

$$L(\mu, \sigma_{\varepsilon}^2, \sigma_p^2, \sigma_d^2 | \mathbf{y}) \propto (\sigma_{\varepsilon}^2)^{-\frac{1}{2}\nu_1} (\sigma_{\varepsilon}^2 + \sigma_p^2)^{-\frac{1}{2}\nu_2} (\sigma_{\varepsilon}^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+1)}$$

$$\exp \left\{ -\frac{1}{2} \left[\frac{kr \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_{\varepsilon}^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_{\varepsilon}^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_{\varepsilon}^2} \right] \right\} .$$

Now

$$\begin{aligned} \ln(L) = & -\frac{\nu_1}{2} \ln(\sigma_\varepsilon^2) - \frac{\nu_2}{2} \ln(\sigma_\varepsilon^2 + r\sigma_p^2) - \frac{(\nu_3+1)}{2} \ln(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2) - \frac{kr \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \\ & - \frac{\nu_2 m_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)} - \frac{\nu_1 m_1}{2\sigma_\varepsilon^2} . \end{aligned}$$

For the two - factor nested random effects model, the Fisher Information matrix is given by

$$F(\theta, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial \ln(L)}{\partial \mu} &= \frac{2kr \sum_{i=1}^b (\bar{y}_{i..} - \mu)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \\ &= \frac{2kr \sum_{i=1}^b \bar{y}_{i..} - 2krb\mu}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} . \end{aligned}$$

Also

$$\frac{\partial^2 \ln(L)}{(\partial \mu)^2} = \frac{-2krb}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} .$$

$$\begin{aligned} \text{Therefore } F_{11} &= -E \left\{ \frac{\partial^2 \ln(L)}{(\partial \mu)^2} \right\} \\ &= \frac{bkr}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \\ &= \frac{(\nu_3+1)kr}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} . \end{aligned}$$

Similarly

$$\frac{\partial \ln(L)}{\partial \sigma_\varepsilon^2} = -\frac{\nu_1}{2\sigma_\varepsilon^2} - \frac{\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)} - \frac{(\nu_3+1)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{kr \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} + \frac{\nu_2 m_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{\nu_1 m_1}{2(\sigma_\varepsilon^2)^2} .$$

Therefore

$$\frac{\partial^2 \ln(L)}{(\partial \sigma_\varepsilon^2)^2} = \frac{\nu_1}{2} \left(\frac{1}{\sigma_\varepsilon^2} \right)^2 + \frac{\nu_2}{2} \left(\frac{1}{\sigma_\varepsilon^2 + r\sigma_p^2} \right)^2 + \frac{(\nu_3+1)}{2} \left(\frac{1}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \right)^2 - \frac{kr \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^3} - \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)^3} - \frac{\nu_1 m_1}{(\sigma_\varepsilon^2)^3} .$$

F_{22} is therefore equals to

$$F_{22} = -E \left[\frac{\partial^2 \ln(L)}{(\partial \sigma_\varepsilon^2)^2} \right]$$

$$= -\frac{\nu_1}{2(\sigma_\varepsilon^2)^2} - \frac{\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} - \frac{(\nu_3+1)}{2} \left(\frac{1}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \right)^2 + \frac{(kr)(\frac{1}{kr})b(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} - \frac{\nu_2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2)^3} - \frac{\nu_1\sigma_\varepsilon^2}{(\sigma_\varepsilon^2)^3}.$$

$$\text{Therefore } F_{22} = \frac{1}{2} \left\{ \frac{\nu_1}{(\sigma_\varepsilon^2)^2} + \frac{\nu_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right\},$$

and

$$\frac{\partial \ln(L)}{\partial \sigma_p^2} = -\frac{\nu_2}{2} \frac{r}{(\sigma_\varepsilon^2 + r\sigma_p^2)} - \frac{(\nu_3+1)}{2} \frac{r}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{kr^2 \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} + \frac{r\nu_2 m_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}.$$

Therefore

$$\frac{\partial^2 \ln(L)}{(\partial \sigma_p^2)^2} = \frac{\nu_2}{2} \frac{r^2}{(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)}{2} \frac{r^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} - \frac{kr^3 \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^3} - \frac{r^2 \nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)^3}.$$

Now

$$F_{33} = -E \left[\frac{\partial^2 \ln(L)}{(\partial \sigma_p^2)^2} \right]$$

$$= -\frac{\nu_2}{2} \frac{r^2}{(\sigma_\varepsilon^2 + r\sigma_p^2)^2} - \frac{(\nu_3+1)}{2} \frac{r^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} + \frac{kr^3 b (\frac{1}{kr})(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} + \frac{r^2 \nu_2 (\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2)^3}, \text{ and}$$

$$\text{therefore } F_{33} = \frac{\nu_2 r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2}.$$

Similarly

$$\frac{\partial \ln(L)}{\partial \sigma_d^2} = -\frac{rk(\nu_3+1)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{(kr)^2 \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2}.$$

Therefore

$$\frac{\partial^2 \ln(L)}{(\partial \sigma_d^2)^2} = \frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} - \frac{(kr)^3 \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^3}.$$

Now

$$\begin{aligned}
F_{44} &= -E \left[\frac{\partial^2 \ln(L)}{(\partial \sigma_d^2)^2} \right] \\
&= -\frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} + \frac{(kr)^3 b(\frac{1}{kr})(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)}{(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^3} \\
&= \frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} .
\end{aligned}$$

Also $F_{12} = F_{21} = 0$, $F_{13} = F_{31} = 0$, and $F_{14} = F_{41} = 0$.

Now

$$\frac{\partial^2 \ln(L)}{\partial \sigma_p^2 \partial \sigma_\varepsilon^2} = \frac{\nu_2}{2} \frac{r}{(\sigma_\varepsilon^2+r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} - \frac{kr^2 \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^3} - \frac{r\nu_2 m_2}{(\sigma_\varepsilon^2+r\sigma_p^2)^3} .$$

Therefore

$$\begin{aligned}
F_{32} &= -E \left[\frac{\partial^2 \ln(L)}{\partial \sigma_p^2 \partial \sigma_\varepsilon^2} \right] \\
&= -\frac{\nu_2 r}{2(\sigma_\varepsilon^2+r\sigma_p^2)^2} - \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} + \frac{rb(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2}{(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^3} + \frac{r\nu_2(\sigma_\varepsilon^2+r\sigma_p^2)}{(\sigma_\varepsilon^2+r\sigma_p^2)^3} \\
&= \frac{r\nu_2}{2(\sigma_\varepsilon^2+r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} = F_{23} .
\end{aligned}$$

Also

$$\frac{\partial^2 \ln(L)}{\partial \sigma_d^2 \partial \sigma_\varepsilon^2} = \frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} - \frac{(kr)^2 \sum_{i=1}^b (\bar{y}_{i..} - \mu)^2}{(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^3} .$$

Therefore

$$F_{42} = -E \left[\frac{\partial^2 \ln(L)}{\partial \sigma_d^2 \partial \sigma_\varepsilon^2} \right] = \frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} = F_{24} , \text{ and}$$

similarly

$$F_{43} = -E \left[\frac{\partial^2 \ln(L)}{\partial \sigma_d^2 \partial \sigma_p^2} \right] = \frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2+r\sigma_p^2+kr\sigma_d^2)^2} = F_{34} .$$

Therefore, the Fisher information matrix can be written as

$$F = \begin{bmatrix} F_{11} & 0 & 0 & 0 \\ 0 & F_{22} & F_{23} & F_{24} \\ 0 & F_{32} & F_{33} & F_{34} \\ 0 & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

$$\text{with } F^* = \begin{bmatrix} F_{22} & F_{23} & F_{24} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{bmatrix}.$$

$$\text{Now, it is clear that } F^{-1} = \begin{bmatrix} F^{11} & 0 & 0 & 0 \\ 0 & F^{22} & F^{23} & F^{24} \\ 0 & F^{32} & F^{33} & F^{34} \\ 0 & F^{42} & F^{43} & F^{44} \end{bmatrix}$$

$$\begin{aligned} \text{with } F^{11} &= (F_{11})^{-1} = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{(\nu_3+1)kr} \\ &= \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr} \end{aligned}$$

$$\text{Also, } F^{*-1} = \frac{1}{|F^*|} \begin{bmatrix} F_{22} & F_{23} & F_{24} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{bmatrix}$$

where $|F^*|$

$$= (-1)^{1+1} [F_{33}F_{44} - F_{43}^2] + (-1)^{1+2} [F_{32}F_{44} - F_{42}F_{34}] + (-1)^{1+3} [F_{32}F_{43} - F_{42}F_{33}]$$

$$\begin{aligned} &= \left\{ \left[\frac{\nu_2 r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right]^2 \right\} \\ &\quad - \left\{ \left[\frac{r\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \right\} \\ &\quad + \left\{ \left[\frac{r\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{\nu_2 r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \right\} \\ &= \frac{\nu_1 \nu_2 (\nu_3+1) k^2 r^4}{8(\sigma_\varepsilon^2)^2 (\sigma_\varepsilon^2 + r\sigma_p^2)^2 (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2}. \end{aligned}$$

Also

$$\begin{aligned}
f_{22} &= (-1)^{1+1} \left[F_{33}F_{44} - F_{43}^2 \right] \\
&= \left[\frac{\nu_2 r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right]^2 \\
&= \frac{\nu_2(\nu_3+1)k^2 r^4}{4(\sigma_\varepsilon^2 + r\sigma_p^2)^2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} .
\end{aligned}$$

$$\begin{aligned}
f_{23} &= (-1)^{1+2} \left[F_{32}F_{44} - F_{42}F_{34} \right] \\
&= - \left\{ \left[\frac{r\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \right\} \\
&= - \frac{k^2 r^3 \nu_2 (\nu_3+1)}{4(\sigma_\varepsilon^2 + r\sigma_p^2)^2 (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} .
\end{aligned}$$

$$\begin{aligned}
f_{24} &= (-1)^{1+3} \left[F_{32}F_{43} - F_{42}F_{33} \right] \\
&= \left[\frac{r\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{\nu_2 r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f_{33} &= (-1)^{2+2} \left[F_{22}F_{44} - F_{42}^2 \right] \\
&= \left[\frac{\nu_1}{2(\sigma_\varepsilon^2)^2} + \frac{\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)(kr)^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] - \left[\frac{(\nu_3+1)^2(kr)^2}{4(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^4} \right] \\
&= \frac{\nu_1(\nu_3+1)k^2 r^2}{4(\sigma_\varepsilon^2)^2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} + \frac{\nu_2(\nu_3+1)k^2 r^2}{4(\sigma_\varepsilon^2 + r\sigma_p^2)^2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} .
\end{aligned}$$

$$\begin{aligned}
f_{34} &= (-1)^{2+3} \left[F_{22}F_{43} - F_{42}F_{23} \right] \\
&= - \left\{ \left[\frac{\nu_1}{2(\sigma_\varepsilon^2)^2} + \frac{\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{(\nu_3+1)kr^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \right. \\
&\quad \left. - \left[\frac{(\nu_3+1)(kr)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{r\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \right\} \\
&= - \frac{\nu_1(\nu_3+1)kr^2}{4(\sigma_\varepsilon^2)^2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} .
\end{aligned}$$

$$\begin{aligned}
f_{44} &= (-1)^{3+3} \left[F_{22} F_{33} - F_{32}^2 \right] \\
&= \left[\frac{\nu_1}{2(\sigma_\varepsilon^2)^2} + \frac{\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \left[\frac{\nu_2 r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r^2}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right] \\
&\quad - \left[\frac{r\nu_2}{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{(\nu_3+1)r}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \right]^2 \\
&= \frac{\nu_1 \nu_2 r^2}{4(\sigma_\varepsilon^2)^2 (\sigma_\varepsilon^2 + r\sigma_p^2)^2} + \frac{\nu_1 (\nu_3+1) r^2}{4(\sigma_\varepsilon^2)^2 (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} .
\end{aligned}$$

Therefore

$$F^{11} = \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{bkr} .$$

$$F^{22} = \frac{f_{22}}{|F^*|} = \frac{2(\sigma_\varepsilon^2)^2}{\nu_1} = \frac{2(\sigma_\varepsilon^2)^2}{bk(r-1)} .$$

$$F^{23} = \frac{f_{23}}{|F^*|} = \frac{-2(\sigma_\varepsilon^2)^2}{\nu_1 r} = \frac{-2(\sigma_\varepsilon^2)^2}{bkr(r-1)} .$$

$$F^{24} = F^{12} = 0 .$$

$$F^{33} = \frac{f_{33}}{|F^*|} = \frac{-2 \left\{ \nu_1 (\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2 (\sigma_\varepsilon^2)^2 \right\}}{\nu_1 \nu_2 r^2} .$$

$$F^{34} = \frac{f_{34}}{|F^*|} = \frac{-2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 kr^2} .$$

$$F^{44} = \frac{f_{44}}{|F^*|} = \frac{-2 \left\{ \nu_2 (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2 + (\nu_3+1) (\sigma_\varepsilon^2 + r\sigma_p^2)^2 \right\}}{\nu_2 (\nu_3+1) k^2 r^2} .$$

Now define $t(\boldsymbol{\theta}) = \mu + z_\alpha \sqrt{\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{kr}}$,

therefore $t(\boldsymbol{\theta}) = \mu + z_\alpha \left(\frac{1}{kr}\right)^{\frac{1}{2}} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}$.

It therefore follows that

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} = 1 ,$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} = z_\alpha \left(\frac{1}{kr}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} ,$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} = z_\alpha \left(\frac{r}{k}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} , \text{ and}$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} = z_\alpha (kr)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}}.$$

Now

$$\nabla'_t(\boldsymbol{\theta}) = \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} \right] \text{ and}$$

$$\begin{aligned} \nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta}) &= \left\{ \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} F^{11} \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{22} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{32} \right] \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{23} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{33} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{43} \right] \right. \\ &\quad \left. \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{34} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{44} \right] \right\}. \end{aligned}$$

Also

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} F^{11} = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}, \text{ and}$$

$$\begin{aligned} &\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{22} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{32} \\ &= z_\alpha \left(\frac{1}{kr}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{2(\sigma_\varepsilon^2)^2}{bkr(r-1)}\right) + z_\alpha \left(\frac{r}{k}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{-2(\sigma_\varepsilon^2)^2}{bkr(r-1)}\right) \\ &= 0. \end{aligned}$$

$$\text{Now } \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{23} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{33} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{43}$$

$$\begin{aligned} &= z_\alpha \left(\frac{1}{kr}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{-2(\sigma_\varepsilon^2)^2}{bkr(r-1)}\right) + z_\alpha \left(\frac{r}{k}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{-2\{\nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2(\sigma_\varepsilon^2)^2\}}{\nu_1\nu_2r^2}\right) \\ &\quad + z_\alpha (kr)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{-2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2kr^2}\right) \\ &= z_\alpha (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1} \left\{ \frac{-b(k-1)(\sigma_\varepsilon^2) + bk(r-1)(\sigma_\varepsilon^2 + r\sigma_p^2)}{b^2k^{\frac{3}{2}}r^{\frac{3}{2}}(k-1)(r-1)} + \frac{b(k-1)(\sigma_\varepsilon^2) - bk(r-1)(\sigma_\varepsilon^2 + r\sigma_p^2)}{b^2k^{\frac{3}{2}}r^{\frac{3}{2}}(k-1)(r-1)} \right\} \\ &= 0. \end{aligned}$$

And

$$\begin{aligned}
& \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{34} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{44} \\
&= z_\alpha \left(\frac{r}{k}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{-2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 kr^2}\right) \\
&\quad + z_\alpha (rk)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{2\{\nu_2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2 + (\nu_3 + 1)(\sigma_\varepsilon^2 + r\sigma_p^2)^2\}}{\nu_2(\nu_3 + 1)k^2 r^2}\right) \\
&= z_\alpha (rk)^{\frac{1}{2}} \left(\frac{1}{2}\right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left(\frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2}{(\nu_3 + 1)k^2 r^2}\right) \\
&= \frac{z_\alpha (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{3}{2}}}{bk^{\frac{3}{2}} r^{\frac{3}{2}}}.
\end{aligned}$$

Therefore

$$\nabla'_t(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) = \left[\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr} \quad 0 \quad 0 \quad \frac{z_\alpha (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{3}{2}}}{bk^{\frac{3}{2}} r^{\frac{3}{2}}} \right]$$

and

$$\begin{aligned}
\nabla'_t(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) \nabla_t(\boldsymbol{\theta}) &= \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr} + z_\alpha^2 \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{2bkr} \\
&= \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr} \left[1 + \frac{1}{2} z_\alpha^2 \right].
\end{aligned}$$

Therefore

$$\sqrt{\nabla'_t(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) \nabla_t(\boldsymbol{\theta})} = \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{(bkr)^{\frac{1}{2}}} \left[1 + \frac{1}{2} z_\alpha^2 \right]^{\frac{1}{2}}.$$

Now

$$\begin{aligned}
\boldsymbol{\eta}'(\boldsymbol{\theta}) &= \frac{\nabla'_t(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta})}{\sqrt{\nabla'_t(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) \nabla_t(\boldsymbol{\theta})}} \\
&= \left[\eta_1(\boldsymbol{\theta}) \quad \eta_2(\boldsymbol{\theta}) \quad \eta_3(\boldsymbol{\theta}) \quad \eta_4(\boldsymbol{\theta}) \right]
\end{aligned}$$

where

$$\begin{aligned}\eta_1(\boldsymbol{\theta}) &= \frac{\frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{(bkr)}}{\frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{(bkr)^{\frac{1}{2}}} \left[1 + \frac{1}{2}z_\alpha^2\right]^{\frac{1}{2}}} \\ &= \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{(bkr)^{\frac{1}{2}}} \left[1 + \frac{1}{2}z_\alpha^2\right]^{-\frac{1}{2}},\end{aligned}$$

$\eta_2(\boldsymbol{\theta}) = \eta_3(\boldsymbol{\theta}) = 0$, and

$$\begin{aligned}\eta_4(\boldsymbol{\theta}) &= \frac{\frac{z_\alpha(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{3}{2}}}{bk^{\frac{3}{2}}r^{\frac{3}{2}}}}{\frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{b^{\frac{1}{2}}k^{\frac{1}{2}}r^{\frac{1}{2}}} \left[1 + \frac{1}{2}z_\alpha^2\right]^{\frac{1}{2}}} \\ &= \frac{z_\alpha(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{b^{\frac{1}{2}}kr} \left[1 + \frac{1}{2}z_\alpha^2\right]^{-\frac{1}{2}}.\end{aligned}$$

The prior distribution $p(\boldsymbol{\theta}) = p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2)$ is a probability matching prior if the following differential equation is satisfied:

$$\frac{\partial\{\eta_1(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\mu} + \frac{\partial\{\eta_2(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_\varepsilon^2} + \frac{\partial\{\eta_3(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_p^2} + \frac{\partial\{\eta_4(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_d^2} = 0.$$

Now if $p(\boldsymbol{\theta}) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1}$

then

$$\frac{\partial\{\eta_1(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\mu} = 0 \text{ (since it does not contain } \mu\text{)}$$

$$\frac{\partial\{\eta_2(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_\varepsilon^2} = 0 \text{ (since } \eta_2(\boldsymbol{\theta}) = 0\text{),}$$

$$\frac{\partial\{\eta_3(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_p^2} = 0 \text{ (since } \eta_3(\boldsymbol{\theta}) = 0\text{), and}$$

$$\frac{\partial\{\eta_4(\boldsymbol{\theta})p(\boldsymbol{\theta})\}}{\partial\sigma_d^2} = \frac{\partial\left\{\frac{z_\alpha\sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} \left[1 + \frac{1}{2}z_\alpha^2\right]^{-\frac{1}{2}}}{b^{\frac{1}{2}}kr}\right\}}{\partial\sigma_d^2} = 0 \text{ (since it does not contain } \sigma_d^2\text{).}$$

It therefore follows that the prior distribution

$$p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1}$$

given by equation 6.4.3 is a probability matching prior for

$$q = \mu + z_\alpha \sqrt{\frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{kr}},$$

the α^{th} quantile of the normal distribution given by

$$\bar{y}_{f..} | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 \sim N\left(\mu, \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{kr}\right).$$

Proof of Theorem 6.4.4.1

To determine the conditional posterior distribution of $\mathbf{u}_2 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1$, only consider the terms in the joint posterior distribution given in equation 6.4.5 that contain \mathbf{u}_2 , therefore only consider the exponent

$$\exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 - Z_2\mathbf{u}_2)'(\mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 - Z_2\mathbf{u}_2) + \frac{1}{\sigma_p^2}\mathbf{u}_2'\mathbf{u}_2\right]\right\}.$$

Now complete the square with respect to \mathbf{u}_2 .

Therefore

$$\begin{aligned} s &= \frac{1}{\sigma_\varepsilon^2}(\mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 - Z_2\mathbf{u}_2)'(\mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 - Z_2\mathbf{u}_2) + \frac{1}{\sigma_p^2}\mathbf{u}_2'\mathbf{u}_2 \\ &= \frac{1}{\sigma_\varepsilon^2}(\tilde{\mathbf{y}} - Z_2\mathbf{u}_2)'(\tilde{\mathbf{y}} - Z_2\mathbf{u}_2) + \frac{1}{\sigma_p^2}\mathbf{u}_2'\mathbf{u}_2 \end{aligned}$$

where $\tilde{\mathbf{y}} = \mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1$.

Therefore

$$s = \frac{1}{\sigma_\varepsilon^2}(\tilde{\mathbf{y}}'\tilde{\mathbf{y}} - 2\mathbf{u}_2'Z_2'\tilde{\mathbf{y}} + \mathbf{u}_2'Z_2'Z_2\mathbf{u}_2) + \frac{1}{\sigma_p^2}\mathbf{u}_2'\mathbf{u}_2$$

Now consider

$$\tilde{s} = \frac{1}{\sigma_\varepsilon^2}\left[\mathbf{u}_2'Z_2'Z_2\mathbf{u}_2 - 2\mathbf{u}_2'Z_2'\tilde{\mathbf{y}}\right] + \frac{1}{\sigma_p^2}\mathbf{u}_2'\mathbf{u}_2$$

$$\begin{aligned}
&= \mathbf{u}'_2 \left[\frac{1}{\sigma_\varepsilon^2} Z'_2 Z_2 - \frac{1}{\sigma_p^2} I_{bk} \right] \mathbf{u}_2 - 2 \mathbf{u}'_2 \frac{1}{\sigma_\varepsilon^2} Z'_2 \tilde{\mathbf{y}} \\
&= \mathbf{u}'_2 D \mathbf{u}_2 - 2 \mathbf{u}'_2 \mathbf{C}
\end{aligned}$$

where $D = \left[\frac{1}{\sigma_\varepsilon^2} Z'_2 Z_2 + \frac{1}{\sigma_p^2} I_{bk} \right]$, $\mathbf{C} = \frac{1}{\sigma_\varepsilon^2} Z'_2 \tilde{\mathbf{y}}$, and $\tilde{\mathbf{y}} = \mathbf{y} - \mu \mathbf{X} - Z_1 \mathbf{u}_1$.

Therefore

$$\tilde{s} = (\mathbf{u}_2 - D^{-1} \mathbf{C})' D (\mathbf{u}_2 - D^{-1} \mathbf{C}) - \mathbf{C} D^{-1} \mathbf{C}.$$

From \tilde{s} it therefore follows that

$$\mathbf{u}_2 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1 \sim N(D^{-1} \mathbf{C}, D^{-1})$$

where

$$\begin{aligned}
D^{-1} &= \left[\frac{1}{\sigma_\varepsilon^2} Z'_2 Z_2 + \frac{1}{\sigma_p^2} I_{bk} \right]^{-1} \\
&= \left[\frac{r}{\sigma_\varepsilon^2} I_{bk} + \frac{1}{\sigma_p^2} I_{bk} \right]^{-1} \\
&= \left[\frac{r}{\sigma_\varepsilon^2} + \frac{1}{\sigma_p^2} \right]^{-1} I_{bk} \\
&= \left[\frac{r\sigma_p^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2 \sigma_p^2} \right]^{-1} I_{bk} \\
&= \left[\frac{\sigma_\varepsilon^2 \sigma_p^2}{r\sigma_p^2 + \sigma_\varepsilon^2} \right] I_{bk}.
\end{aligned}$$

Also

$$\begin{aligned}
D^{-1} \mathbf{C} &= \left[\frac{\sigma_\varepsilon^2 \sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2} \right] \frac{1}{\sigma_\varepsilon^2} Z'_2 [\mathbf{y} - \mu \mathbf{X} - Z_1 \mathbf{u}_1] \\
&= \left[\frac{r\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2} \right] \left\{ \begin{bmatrix} \bar{y}_{11.} \\ \bar{y}_{12.} \\ \vdots \\ \bar{y}_{bk.} \end{bmatrix} - (\mu \mathbf{X} + \frac{1}{k} Z'_2 Z_1 \mathbf{u}_1) \right\},
\end{aligned}$$

$$\text{and } |D|^{-\frac{1}{2}} = \left(\frac{\sigma_\varepsilon^2 \sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2} \right)^{\frac{1}{2} bk}.$$

Proof of Theorem 6.4.4.2

In order to obtain the conditional posterior distribution of $\mathbf{u}_1 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2$, the joint posterior distribution has to be integrated with respect to \mathbf{u}_2 (i.e. integrate \mathbf{u}_2 out of the joint posterior distribution), therefore

$$p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1 | \mathbf{y}) \propto \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left[\mathbf{u}_2 - \left(\frac{1}{\sigma_\varepsilon^2} Z_2' Z_2 + \frac{1}{\sigma_p^2} I_{bk} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} Z_2' \tilde{\mathbf{y}} \right]' \left[\frac{1}{\sigma_\varepsilon^2} Z_2' Z_2 + \frac{1}{\sigma_p^2} I_{bk} \right] \right. \\ \left. \left[\mathbf{u}_2 - \left(\frac{1}{\sigma_\varepsilon^2} Z_2' Z_2 + \frac{1}{\sigma_p^2} I_{bk} \right)^{-1} \frac{1}{\sigma_\varepsilon^2} Z_2' \tilde{\mathbf{y}} \right] \right\} d\mathbf{u}_2 \\ \propto |D|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \left[\frac{1}{\sigma_\varepsilon^2} \tilde{\mathbf{y}}' \tilde{\mathbf{y}} - \mathbf{C}' D^{-1} \mathbf{C} \right] \right\}$$

where D , $\tilde{\mathbf{y}}$, \mathbf{C} is defined as in Theorem 6.4.4.1.

Therefore, after integrating \mathbf{u}_2 out, the joint posterior distribution $p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1 | \mathbf{y})$ can be written as

$$p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{u}_1 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}bk(r-1)} \left(\frac{1}{\sigma_p^2} \right)^{\frac{1}{2}b} \left(\frac{1}{\sigma_\varepsilon^2 + r\sigma_p^2} \right)^{\frac{1}{2}bk} (\sigma_\varepsilon^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} \\ (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1} \times \exp \left\{ -\frac{1}{2\sigma_\varepsilon^2} \tilde{\mathbf{y}}' \tilde{\mathbf{y}} + \frac{1}{2} \mathbf{C}' D^{-1} \mathbf{C} - \frac{1}{2\sigma_d^2} \mathbf{u}_1' \mathbf{u}_1 \right\}. \quad (6.6.1)$$

By only considering the parts in the joint posterior distribution that contains \mathbf{u}_1 , the conditional posterior distribution of \mathbf{u}_1 given μ , and the variance components, can be written as

$$p(\mathbf{u}_1 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) \propto \exp \left\{ -\frac{1}{2} \left[\frac{1}{\sigma_\varepsilon^2} \tilde{\mathbf{y}}' \tilde{\mathbf{y}} + \frac{1}{\sigma_d^2} \mathbf{u}_1' \mathbf{u}_1 - \mathbf{C}' D^{-1} \mathbf{C} \right] \right\}.$$

Now, complete the square with respect to \mathbf{u}_1 in the exponent. Therefore, first consider

$$\frac{1}{2} \mathbf{C}' D^{-1} \mathbf{C} = \frac{1}{2} \left(\frac{1}{\sigma_\varepsilon^2} \right)^2 \tilde{\mathbf{y}}' Z_2 \left(\frac{\sigma_\varepsilon^2 \sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2} \right) Z_2' \tilde{\mathbf{y}} \\ = \frac{\sigma_p^2}{2\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)} \tilde{\mathbf{y}}' Z_2 Z_2' \tilde{\mathbf{y}}.$$

$$\text{Also, } \tilde{\mathbf{y}} = \mathbf{y} - \mu\mathbf{X} - Z_1\mathbf{u}_1 = \tilde{\tilde{\mathbf{y}}} - Z_1\mathbf{u}_1$$

$$\text{where } \tilde{\tilde{\mathbf{y}}} = \mathbf{y} - \mu\mathbf{X} .$$

Now consider the exponent

$$\begin{aligned} & -\frac{1}{2\sigma_\varepsilon^2}\tilde{\mathbf{y}}'\tilde{\mathbf{y}} + \frac{1}{2}\mathbf{C}'D^{-1}\mathbf{C} - \frac{1}{2\sigma_d^2}\mathbf{u}'_1\mathbf{u}_1 \\ &= -\frac{1}{2\sigma_\varepsilon^2}(\tilde{\tilde{\mathbf{y}}} - Z_1\mathbf{u}_1)'\left[I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right](\tilde{\tilde{\mathbf{y}}} - Z_1\mathbf{u}_1) - \frac{\mathbf{u}'_1\mathbf{u}_1}{2\sigma_d^2} \\ &= -\frac{1}{2\sigma_\varepsilon^2}\left\{\tilde{\tilde{\mathbf{y}}}'\left[I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right]\tilde{\tilde{\mathbf{y}}} - 2\mathbf{u}'_1Z_1'\left[I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right]\tilde{\tilde{\mathbf{y}}}\right. \\ &\quad \left.+ \mathbf{u}'_1Z_1'\left[I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right]Z_1\mathbf{u}_1\right\} - \frac{1}{2\sigma_d^2}\mathbf{u}'_1\mathbf{u}_1 . \end{aligned}$$

$$\text{Let } S = \mathbf{u}'_1\left\{\frac{1}{\sigma_\varepsilon^2}Z_1'\left(I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right)Z_1 + \frac{1}{2\sigma_d^2}I_b\right\}\mathbf{u}_1 - 2\mathbf{u}'_1\frac{1}{\sigma_\varepsilon^2}Z_1'\left(I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right)\tilde{\tilde{\mathbf{y}}} .$$

$$\text{Also, let } D = \frac{1}{\sigma_\varepsilon^2}Z_1'Z_1 - \frac{1}{\sigma_\varepsilon^2}\left[\frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}\right]Z_1'Z_2Z_2'Z_1 + \frac{1}{\sigma_d^2}I_b , \text{ and}$$

$$\mathbf{C} = \frac{1}{\sigma_\varepsilon^2}Z_1'\left(I_{bkr} - \frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}Z_2Z_2'\right)\tilde{\tilde{\mathbf{y}}} .$$

Therefore

$$s = (\mathbf{u}_1 - D^{-1}\mathbf{C})'D(\mathbf{u}_1 - D^{-1}\mathbf{C}) - \mathbf{C}'D^{-1}\mathbf{C} .$$

Now

$$D = \frac{rk}{\sigma_\varepsilon^2}I_b - \frac{kr^2}{\sigma_\varepsilon^2}\frac{\sigma_p^2}{(\sigma_\varepsilon^2+r\sigma_p^2)}I_b + \frac{1}{\sigma_d^2}I_b .$$

Therefore

$$\begin{aligned} D &= \left(\frac{rk}{\sigma_\varepsilon^2} - \frac{kr^2\sigma_p^2}{\sigma_\varepsilon^2(\sigma_\varepsilon^2+r\sigma_p^2)} + \frac{1}{\sigma_d^2}\right)I_b \\ &= \left(\frac{rk(\sigma_\varepsilon^2+r\sigma_p^2)\sigma_d^2 - kr^2\sigma_p^2\sigma_d^2 + \sigma_\varepsilon^2(\sigma_\varepsilon^2+r\sigma_p^2)}{\sigma_\varepsilon^2(\sigma_\varepsilon^2+r\sigma_p^2)\sigma_d^2}\right)I_b \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{rk\sigma_\varepsilon^2\sigma_d^2 + r^2k\sigma_p^2\sigma_d^2 - kr^2\sigma_p^2\sigma_d^2 + (\sigma_\varepsilon^2)^2 + r\sigma_\varepsilon^2\sigma_p^2}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)\sigma_d^2} \right) I_b \\
&= \left(\frac{rk\sigma_\varepsilon^2\sigma_d^2 + (\sigma_\varepsilon^2)^2 + r\sigma_\varepsilon^2\sigma_p^2}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)\sigma_d^2} \right) I_b \\
&= \frac{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)\sigma_d^2} I_b \\
&= \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)} I_b .
\end{aligned}$$

Therefore

$$Var(\mathbf{u}_1 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) = D^{-1} = \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} I_b .$$

Also

$$\begin{aligned}
\mathbf{C} &= \frac{1}{\sigma_\varepsilon^2} Z_1' \left(I_{bkr} - \frac{\sigma_p^2}{\sigma_\varepsilon^2 + r\sigma_p^2} Z_2 Z_2' \right) \tilde{\mathbf{y}} \\
&= \frac{1}{\sigma_\varepsilon^2} Z_1' \tilde{\mathbf{y}} - \frac{\sigma_p^2}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)} Z_1' Z_2 Z_2' \tilde{\mathbf{y}} \\
&= \frac{1}{\sigma_\varepsilon^2} Z_1' (\mathbf{y} - \mu \mathbf{X}) - \frac{\sigma_p^2}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)} \begin{bmatrix} r & r & r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r & r & r \end{bmatrix}_{(b \times bk)} \cdot Z_2' (\mathbf{y} - \mu \mathbf{X}) \\
&= \frac{1}{\sigma_\varepsilon^2} \left\{ \begin{bmatrix} \bar{y}_{1..} \\ \bar{y}_{2..} \\ \vdots \\ \bar{y}_{b..} \end{bmatrix} - rk\mu \mathbf{X} \right\} - \frac{\sigma_p^2}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)} \begin{bmatrix} r & r & r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r & r & r \end{bmatrix}_{(b \times bk)} \cdot \left\{ \begin{bmatrix} \bar{y}_{11.} \\ \bar{y}_{12.} \\ \vdots \\ \bar{y}_{bk.} \end{bmatrix} - r\mu \mathbf{X} \right\} \\
&= \frac{rk}{\sigma_\varepsilon^2} \left\{ \begin{bmatrix} \bar{y}_{1..} \\ \bar{y}_{2..} \\ \vdots \\ \bar{y}_{b..} \end{bmatrix} - \mu \mathbf{X} \right\} - \frac{\sigma_p^2 r^2 k}{\sigma_\varepsilon^2(\sigma_\varepsilon^2 + r\sigma_p^2)} \left\{ \begin{bmatrix} \bar{y}_{1..} \\ \bar{y}_{2..} \\ \vdots \\ \bar{y}_{b..} \end{bmatrix} - \mu \mathbf{X} \right\}
\end{aligned}$$

$$= \frac{rk}{\sigma_\varepsilon^2 + r\sigma_p^2} \begin{bmatrix} \bar{y}_{1..} - \mu \\ \bar{y}_{2..} - \mu \\ \vdots \\ \bar{y}_{b..} - \mu \end{bmatrix}.$$

Therefore

$$E(\mathbf{u}_1 | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = D^{-1}\mathbf{C}$$

$$= \frac{rk\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \begin{bmatrix} \bar{y}_{1..} - \mu \\ \bar{y}_{2..} - \mu \\ \vdots \\ \bar{y}_{b..} - \mu \end{bmatrix}.$$

Since $\mathbf{u}_1 = \begin{bmatrix} d_1 & d_2 & \dots & d_b \end{bmatrix}'$ it follows that

$$E(d_i | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{rk\sigma_d^2}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \bar{y}_{i..} \text{ (since } E(\mu) = 0\text{), and}$$

$$\text{Var}(d_i | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}.$$

Proof of Theorem 6.4.4.3

To obtain the conditional posterior distribution of μ given the variance components, first integrate \mathbf{u}_1 out of the joint posterior distribution given in equation 6.6.1. It can be shown that after integrating \mathbf{u}_1 out of equation 6.6.1, the joint posterior distribution can be written as

$$p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2 | \mathbf{y}) \propto (\sigma_\varepsilon^2)^{-\frac{1}{2}\nu_1 - 1} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}\nu_2 - 1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3 + 1) - 1} \\ \times \exp \left\{ -\frac{1}{2} \left[\frac{bkr(\bar{y}_{...} - \mu)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_3 m_3}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2} \right] \right\} \quad (6.6.2)$$

(See also Box and Tiao p.278).

From equation 6.6.2, it immediately follows that

$$E(\mu|\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \bar{y}_{\dots}, \text{ and}$$

$$Var(\mu|\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{bkr}.$$

Proof of Theorem 6.4.4.4

To obtain the joint posterior distribution of the variance components, integrate μ out of equation 6.6.2.

Therefore

$$\begin{aligned} p(\sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2|\mathbf{y}) &= \int_{-\infty}^{\infty} p(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2|\mathbf{y})d\mu \\ &\propto (\sigma_\varepsilon^2)^{-\frac{1}{2}(\nu_1+2)}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-\frac{1}{2}(\nu_2+2)}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}(\nu_3+2)} \\ &\quad \times \exp\left\{-\frac{1}{2}\left[\frac{\nu_3 m_3}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{\nu_2 m_2}{(\sigma_\varepsilon^2 + r\sigma_p^2)} + \frac{\nu_1 m_1}{\sigma_\varepsilon^2}\right]\right\}. \end{aligned}$$

Proof of Theorem 6.4.4.5

In Theorem 6.4.4.2 it was proved that

$d_i|\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ followed a normal distribution with mean

$$E(d_i|\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{kr\sigma_d^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}(\bar{y}_{i..} - \mu)$$

and variance

$$Var(d_i|\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}.$$

Therefore

$$\begin{aligned} E\left\{(\mu + d_i) \mid \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} &= \frac{kr\sigma_d^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}(\bar{y}_{i..} - \mu) + \mu \\ &= \frac{kr\sigma_d^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}\bar{y}_{i..} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}\mu \end{aligned}$$

and

$$Var\left\{(\mu + d_i) \mid \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} = \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}.$$

Also, $(\mu + d_i) \mid \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ follows a normal distribution.

Since $\mu \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ is normally distributed with mean

$$E(\mu \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \bar{y}_{...}, \text{ and}$$

variance

$$Var(\mu \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr},$$

it follows that $(\mu + d_i) \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ is also normally distributed with mean equals to

$$E\left\{(\mu + d_i) \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} = \frac{rk\sigma_d^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}\bar{y}_{i..} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}\bar{y}_{...}$$

and variance

$$\begin{aligned} Var\left\{(\mu + d_i) \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} &= \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} Var(\mu \mid \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) \\ &= \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2} \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{bkr} \\ &= \frac{\sigma_d^2(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{bkr(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \\ &= \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \left[\sigma_d^2 + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{bkr} \right] \\ &= \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2} \left[\frac{\sigma_\varepsilon^2 + r\sigma_p^2 + bkr\sigma_d^2}{bkr} \right]. \end{aligned}$$

Proof of Theorem 6.5.1.1

The new observations are generated by the model $y_{ijt}^* = \mu + d_i + p_{ij} + \varepsilon_{ijt}$, $j = 1, \dots, k^*$ and $t = 1, \dots, r^*$

where $d_i \sim N(0, \sigma_d^2)$, $p_{ij} \sim N(0, \sigma_p^2)$ and $\varepsilon_{ijt} \sim N(0, \sigma_\varepsilon^2)$.

Now $\bar{y}_{ij.}^* = \frac{1}{r^*} \sum_{t=1}^{r^*} y_{ijt}^*$ and $\bar{y}_{ij.}^* | \mu, d_i, p_{ij}, \sigma_\varepsilon^2 \sim N(\mu + d_i + p_{ij}, \frac{\sigma_\varepsilon^2}{r^*})$.

Also $\bar{y}_{ij.}^* | \mu, d_i, \sigma_\varepsilon^2, \sigma_p^2 \sim N(\mu + d_i, \frac{\sigma_\varepsilon^2}{r^*} + \sigma_p^2)$, since $p_{ij} \sim N(0, \sigma_p^2)$.

Further, $\bar{y}_{i..}^* = \frac{1}{k^*} \sum_{j=1}^{k^*} \bar{y}_{ij.}^*$, and, therefore

$$\bar{y}_{i..}^* | \mu, d_i, \sigma_\varepsilon^2, \sigma_p^2 \sim N(\mu + d_i, \frac{\sigma_\varepsilon^2}{k^* r^*} + \frac{\sigma_p^2}{k^*}).$$

It was shown in Theorem 6.4.4.5 that the posterior distribution of $(\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ followed a normal distribution with mean

$$E\left\{(\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} = \frac{kr\sigma_d^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \bar{y}_{i..} + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \bar{y} \dots$$

and variance

$$Var\left\{(\mu + d_i) | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right\} = \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \left\{ \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + bkr\sigma_d^2)}{bkr} \right\}.$$

It therefore follows that

$\bar{y}_{i..}^* | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}$ is normally distributed with mean

$$E\left(\bar{y}_{i..}^* | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right) = \frac{kr\sigma_d^2}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \bar{y}_{i..} + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \bar{y} \dots$$

and variance

$$Var\left(\bar{y}_{i..}^* | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}\right) = \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2}{k^* r^*} + \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \left\{ \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + bkr\sigma_d^2}{bkr} \right\}.$$

Proof of Theorem 6.5.1.2

In Theorem 6.5.1.1 it was assumed that d_i is known and it was therefore shown that

$$p(\bar{y}_{i..}^* | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) \sim N\left(\mu + d_i, \frac{\sigma_\varepsilon^2}{k^* r^*} + \frac{\sigma_p^2}{k^*}\right).$$

Also, $d_i \sim N(0, \sigma_d^2)$.

Since d_i is not known (the day is unknown), it has to be integrated out. Also, the unknown day will be denoted by f .

Therefore

$$p(\bar{y}_{f..}^* | \mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) \sim N\left(\mu, \frac{\sigma_\varepsilon^2}{k^* r^*} + \frac{\sigma_p^2}{k^*} + \sigma_d^2\right).$$

According to Theorem 6.4.4.3, the posterior distribution of μ given the variance components is normally distributed with mean

$$E(\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \bar{y}_{...}$$

and variance

$$Var(\mu | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}.$$

It therefore follows that the predictive distribution of the mean of k^* future packages with r^* observations per package from a new or unknown day, is normally distributed with mean

$$E(\bar{y}_{f..}^* | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) = \bar{y}_{...}$$

and variance

$$\begin{aligned} Var(\bar{y}_{f..}^* | \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2, \mathbf{y}) &= \frac{\sigma_\varepsilon^2}{k^* r^*} + \frac{\sigma_p^2}{k^*} + \sigma_d^2 + \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr} \\ &= \frac{\sigma_\varepsilon^2 + r^* \sigma_p^2 + k^* r^* \sigma_d^2}{k^* r^*} + \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr}. \end{aligned}$$

Proof of Theorem 6.5.4.1**a.) For the prior distribution given in equation 6.5.8**

For the prior distribution given by

$$\pi_a(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}$$

for the balanced two - factor nested random effects model given in equation 6.3.1, determine the Fisher information matrix and its inverse using the method described in the proof of Theorem 6.4.2.1.

Now, define

$$t(\boldsymbol{\theta}) = c = 1 - \Phi \left[\frac{(s-\mu)(kr)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}} \right] = 1 - \Phi \left[\boldsymbol{\theta} \right]$$

where

$$\Phi \left[\boldsymbol{\theta} \right] = \Phi \left[\frac{(s-\mu)(kr)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}} \right].$$

Further

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} = f(\boldsymbol{\theta}) \frac{(kr)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}},$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} = f(\boldsymbol{\theta}) \left(\frac{kr}{2} \right)^{\frac{1}{2}} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} (s - \mu),$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} = f(\boldsymbol{\theta}) \left(\frac{k^{\frac{1}{2}} r^{\frac{3}{2}}}{2} \right) (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} (s - \mu), \text{ and}$$

$$\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} = f(\boldsymbol{\theta}) \frac{(kr)^{\frac{3}{2}}}{2} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} (s - \mu)$$

where

$$f(\boldsymbol{\theta}) = \frac{e^{-\frac{1}{2}\boldsymbol{\theta}^2}}{\sqrt{2\pi}}.$$

Now

$$\nabla'_t(\boldsymbol{\theta}) = \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} \quad \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} \right] \text{ and}$$

$$\begin{aligned} & \nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta}) \\ &= \left[\frac{\partial t(\boldsymbol{\theta})}{\partial \mu} F^{11} \left\{ \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{22} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{32} \right\} \left\{ \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{23} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{33} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{43} \right\} \left\{ \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{34} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{44} \right\} \right] \end{aligned}$$

where

$$F^{11} = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{(\nu_3 + 1)kr} = \frac{\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2}{bkr},$$

$$F^{22} = \frac{2(\sigma_\varepsilon^2)^2}{\nu_1} = \frac{2(\sigma_\varepsilon^2)^2}{bk(r-1)},$$

$$F^{32} = \frac{-2(\sigma_\varepsilon^2)^2}{\nu_1 k} = \frac{-2(\sigma_\varepsilon^2)^2}{bkr(r-1)},$$

$$F^{33} = \frac{2\{\nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2(\sigma_\varepsilon^2)^2\}}{\nu_1 \nu_2 r^2},$$

$$F^{34} = \frac{-2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 kr^2},$$

$$F^{44} = \frac{2\{\nu_2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2 + (\nu_3 + 1)(\sigma_\varepsilon^2 + r\sigma_p^2)^2\}}{\nu_2(\nu_3 + 1)k^2 r^2}, \text{ and}$$

$$F^{12} = F^{21} = 0, F^{13} = F^{31} = 0, F^{14} = F^{41} = 0, F^{24} = F^{42} = 0.$$

Also,

$$\begin{aligned} \frac{\partial t(\boldsymbol{\theta})}{\partial \mu} F^{11} &= f(\boldsymbol{\theta}) \frac{(kr)^{\frac{1}{2}}}{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}} \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)}{bkr} \\ &= f(\boldsymbol{\theta}) \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{b(kr)^{\frac{1}{2}}}, \text{ and} \end{aligned}$$

$$\begin{aligned} & \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{22} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{32} \\ &= f(\boldsymbol{\theta}) \frac{(kr)^{\frac{1}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \frac{2(\sigma_\varepsilon^2)^2}{bk(r-1)} - f(\boldsymbol{\theta}) \frac{k^{\frac{1}{2}} r^{\frac{3}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \frac{2(\sigma_\varepsilon^2)^2}{bkr(r-1)} \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned}
& \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_\varepsilon^2} F^{23} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{33} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{43} \\
&= -f(\boldsymbol{\theta}) \frac{(kr)^{\frac{1}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \frac{2(\sigma_\varepsilon^2)^2}{bkr(r-1)} \\
&\quad + f(\boldsymbol{\theta}) \frac{k^{\frac{1}{2}} r^{\frac{3}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \frac{2\left\{ \nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2(\sigma_\varepsilon^2)^2 \right\}}{\nu_1 \nu_2 r^2} \\
&\quad - f(\boldsymbol{\theta}) \frac{(kr)^{\frac{3}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \frac{2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 kr^2} \\
&= f(\boldsymbol{\theta}) (kr)^{\frac{1}{2}} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \left\{ \frac{-(\sigma_\varepsilon^2)^2}{bkr(r-1)} + \frac{\nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2(\sigma_\varepsilon^2)^2}{\nu_1 \nu_2 r} - \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 r} \right\} \\
&= f(\boldsymbol{\theta}) (kr)^{\frac{1}{2}} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \left\{ \frac{-(\sigma_\varepsilon^2)^2}{\nu_1 r} + \frac{\nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2(\sigma_\varepsilon^2)^2}{\nu_1 \nu_2 r} - \frac{(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 r} \right\} \\
&= f(\boldsymbol{\theta}) (kr)^{\frac{1}{2}} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} \left\{ \frac{-\nu_2(\sigma_\varepsilon^2)^2 + \nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2 + \nu_2(\sigma_\varepsilon^2)^2 - \nu_1(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_1 \nu_2 r} \right\} \\
&= 0.
\end{aligned}$$

Now, $\frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_p^2} F^{34} + \frac{\partial t(\boldsymbol{\theta})}{\partial \sigma_d^2} F^{44}$

$$\begin{aligned}
&= f(\boldsymbol{\theta}) \frac{k^{\frac{1}{2}} r^{\frac{3}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \left\{ \frac{-2(\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 kr^2} \right\} \\
&\quad + f(\boldsymbol{\theta}) \frac{(kr)^{\frac{3}{2}}}{2} \left(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2 \right)^{-\frac{3}{2}} (s - \mu) \left\{ \frac{2\left[\nu_2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2 + b(\sigma_\varepsilon^2 + r\sigma_p^2)^2 \right]}{\nu_2 bk^2 r^2} \right\} \\
&= -f(\boldsymbol{\theta}) \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} (s - \mu) (\sigma_\varepsilon^2 + r\sigma_p^2)^2}{\nu_2 k^{\frac{1}{2}} r^{\frac{1}{2}}} + f(\boldsymbol{\theta}) \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} (s - \mu) \left\{ \nu_2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^2 + b(\sigma_\varepsilon^2 + r\sigma_p^2)^2 \right\}}{\nu_2 bk^{\frac{1}{2}} r^{\frac{1}{2}}} \\
&= f(\boldsymbol{\theta}) \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}} (s - \mu)}{bk^{\frac{1}{2}} r^{\frac{1}{2}}}.
\end{aligned}$$

Therefore

$$\nabla'_t(\boldsymbol{\theta}) F^{-1}(\boldsymbol{\theta}) = f(\boldsymbol{\theta}) \begin{bmatrix} \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{b(kr)^{\frac{1}{2}}} & 0 & 0 & \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}} (s - \mu)}{b(kr)^{\frac{1}{2}}} \end{bmatrix}.$$

From this, it follows that

$$\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta}) = f^2(\boldsymbol{\theta}) \left\{ \frac{1}{b} + \frac{kr(s-\mu)^2}{2b(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}.$$

Therefore

$$\sqrt{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})} = f(\boldsymbol{\theta}) \frac{1}{b^{\frac{1}{2}}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}.$$

Now

$$\eta(\boldsymbol{\theta}) = \frac{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})}{\sqrt{\nabla'_t(\boldsymbol{\theta})F^{-1}(\boldsymbol{\theta})\nabla_t(\boldsymbol{\theta})}} = \begin{bmatrix} \eta_1(\boldsymbol{\theta}) & \eta_2(\boldsymbol{\theta}) & \eta_3(\boldsymbol{\theta}) & \eta_4(\boldsymbol{\theta}) \end{bmatrix}$$

where

$$\eta_1(\boldsymbol{\theta}) = \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{(bkr)^{\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}},$$

$$\eta_2(\boldsymbol{\theta}) = \eta_3(\boldsymbol{\theta}) = 0, \text{ and}$$

$$\eta_4(\boldsymbol{\theta}) = \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}(s-\mu)}{(bkr)^{\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}}.$$

For the prior distribution $\pi_a(\boldsymbol{\theta})$ to be a probability matching prior, the differential equation

$$\frac{\partial}{\partial \mu} \left\{ \eta_1(\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \right\} + \frac{\partial}{\partial \sigma_\varepsilon^2} \left\{ \eta_2(\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \right\} + \frac{\partial}{\partial \sigma_p^2} \left\{ \eta_3(\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \right\} + \frac{\partial}{\partial \sigma_d^2} \left\{ \eta_4(\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \right\} = 0 \quad (6.6.3)$$

must be satisfied.

Therefore, the prior distribution

$$\pi_a(\boldsymbol{\theta}) = \pi_a(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2}(\sigma_\varepsilon^2 + r\sigma_p^2)^{-1}(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{1}{2}} \left[1 + \frac{(s-\mu)kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{\frac{1}{2}}$$

will be a probability matching prior since

$$\frac{\partial}{\partial \mu} \left\{ \eta_1(\boldsymbol{\theta})\pi_a(\boldsymbol{\theta}) \right\} = 0,$$

$$\frac{\partial}{\partial \sigma_\varepsilon^2} \left\{ \eta_2(\boldsymbol{\theta})\pi_a(\boldsymbol{\theta}) \right\} = 0,$$

$$\frac{\partial}{\partial \sigma_p^2} \left\{ \eta_3(\boldsymbol{\theta})\pi_a(\boldsymbol{\theta}) \right\} = 0, \text{ and}$$

$$\frac{\partial}{\partial \sigma_d^2} \left\{ \eta_4(\boldsymbol{\theta}) \pi_a(\boldsymbol{\theta}) \right\} = 0 .$$

The differential equation 6.6.3 is therefore satisfied.

b.) For the prior distribution given in equation 6.5.9

The prior distribution

$$\pi_b(\boldsymbol{\theta}) = \pi_b(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-\frac{1}{2}}$$

will also be a probability matching prior, since

$$\begin{aligned} \eta_1(\boldsymbol{\theta}) \pi_b(\boldsymbol{\theta}) &= \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}}}{(bkr)^{\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}} \left\{ \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-\frac{1}{2}} \right\} \\ &= \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1} (bkr)^{-\frac{1}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-1} . \end{aligned}$$

Also,

$$\begin{aligned} \frac{\partial}{\partial \mu} \left\{ \eta_1(\boldsymbol{\theta}) \pi_b(\boldsymbol{\theta}) \right\} &= \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1} (bkr)^{-\frac{1}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-2} \frac{2(s-\mu)}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \\ &= \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-2} (bkr)^{-\frac{1}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-2} (s-\mu) kr . \end{aligned} \tag{6.6.4}$$

Now, $\eta_4(\boldsymbol{\theta}) \pi_b(\boldsymbol{\theta})$

$$\begin{aligned} &= \frac{(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{\frac{1}{2}} (s-\mu)}{(bkr)^{\frac{1}{2}} \left\{ 1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right\}^{\frac{1}{2}}} \left\{ \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-\frac{1}{2}} \right\} \\ &= \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-1} (bkr)^{-\frac{1}{2}} \left[1 + \frac{(s-\mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-1} (s-\mu) \\ &= \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (bkr)^{-\frac{1}{2}} \left[(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2) + \frac{1}{2}(s-\mu)^2 kr \right]^{-1} (s-\mu) . \end{aligned}$$

Also

$$\begin{aligned}
& \frac{\partial}{\partial \sigma_d^2} \left\{ \eta_4(\boldsymbol{\theta}) \pi_b(\boldsymbol{\theta}) \right\} \\
&= -\sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (bkr)^{-\frac{1}{2}} \left[(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2) + \frac{1}{2}(s - \mu)kr \right]^{-2} kr(s - \mu) \\
&= -\sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-2} (bkr)^{-\frac{1}{2}} \left[1 + \frac{(s - \mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-2} (s - \mu)kr.
\end{aligned} \tag{6.6.5}$$

Furthermore $\eta_2(\boldsymbol{\theta}) = \eta_3(\boldsymbol{\theta}) = 0$.

Now, although equation 6.6.5 is negative, both equations 6.6.4 and 6.6.5 have the same absolute value. The differential equation given in equation 6.6.3 will there also be satisfied indicating that

$$\pi_b(\boldsymbol{\theta}) = \pi_b(\mu, \sigma_\varepsilon^2, \sigma_p^2, \sigma_d^2) \propto \sigma_\varepsilon^{-2} (\sigma_\varepsilon^2 + r\sigma_p^2)^{-1} (\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)^{-\frac{3}{2}} \left[1 + \frac{(s - \mu)^2 kr}{2(\sigma_\varepsilon^2 + r\sigma_p^2 + kr\sigma_d^2)} \right]^{-\frac{1}{2}}$$

is also a probability matching prior for the contents c of the interval $[s, \infty]$.

Chapter 7

Conclusions and Future Research

7.1 Conclusions

In this thesis, a full Bayesian solution to variance component and tolerance interval estimation were provided for various variance component models. By applying the simulation based approach originally presented by Wolfinger (1998) for determining Bayesian tolerance intervals, the flexible and unique features of the Bayesian simulation method were illustrated, since it could easily be applied to models with one or several variance components.

More specifically, in Chapter 2, exact and estimated marginal posterior distributions were provided for the location and variance parameters of a univariate normal model. In addition, exact moments were also derived for the α^{th} quantile q of a $N(\mu, \sigma_\varepsilon^2)$ distribution and the difference between two α quantiles. For this model, tolerance intervals were determined using two Bayesian simulation methods. It was also shown that the Jeffreys' independence prior was both a reference - and probability matching prior for the α^{th} quantile q of a $N(\mu, \sigma_\varepsilon^2)$ distribution, and, that the proposed prior distribution for the content of the fixed - in - advance tolerance interval was also a probability matching prior. The posterior density of the content could easily be obtained using

two Bayesian simulation methods which provided equivalent results. For the univariate normal model, it was shown that two or more α quantiles could be compared with relative ease. A simulation study also revealed that the two Bayesian multiple comparisons procedures for comparing more than two 0.95 quantiles performed well across the range of selected sample sizes, since the percentage differences indicated for both methods (for each sample size) were approximately 5%, thereby meeting the frequentist property.

In Chapter 3, the Bayesian simulation method for obtaining estimated marginal posterior distributions of unknown model parameters and quantiles for obtaining tolerance intervals originally proposed by Wolfinger (1998) for the balanced one - way random effects model, were reviewed. The specific model reviewed used a non - informative prior distribution and was successfully implemented using an example where medicinal tablets were manufactured in small batches. Following Chapter 3, the theory and results originally presented by Wolfinger (1998) were extended in Chapter 4, to include the estimation of marginal posterior distributions of quantiles and subsequent tolerance intervals for averages of observations from new or unknown batches. A reference - and probability matching prior have been derived for the α^{th} quantile of the distribution of averages of observations from new or unknown batches, which is then used to obtain the (α, δ) one - and two - sided tolerance intervals. Also, a probability matching prior has been derived for the content of the fixed - in - advance tolerance interval. Using two simulation methods (method 1 simulated a function of the variance components i.e. $(\sigma_{\epsilon}^2 + k\sigma_a^2)$ while method 2 simulated the variance components separately and retained only the simulated pairs that met the condition $(\sigma_{\epsilon}^2 + k\sigma_a^2) > \sigma_{\epsilon}^2$) an extensive numerical experiment was performed to investigate the frequentist properties of Bayesian inference based on the non - informative priors. The simulation results substantiated what was expected. The second method was conservative for low values of the intraclass correlation ρ , with large average interval lengths, particularly for small values of b and k . The coverage of the first method on the other hand was near

the nominal confidence of 0.95 uniformly across the range of ρ , b and k values. For large values of ρ , the coverage probabilities, average interval lengths and standard deviations of the two methods were close to each other. As in Chapter 3, the medicinal tablets data was used to illustrate the unique features and flexibility of the Bayesian simulation method for obtaining variance components and tolerance intervals.

Although the standard one - way variance components model has been studied widely, few authors have investigated the model in cases where the measurement error model do not have the standard $N(0, \sigma_{\varepsilon}^2)$ form. In Chapter 5 it was shown that the balanced one - way random effects model can be extended to the case where the measurement error model has a student t - distributional form. The Bayesian approach had several advantages for this type of problem, since a prior distribution could for example either not be specified, or be specified for the degrees of freedom. This provided the flexibility to fit non - standard measurement system models and generate analyses quickly, using in this case an adaptive Monte Carlo technique known as the Gibbs sampler. Similar to ordinary Monte Carlo techniques, the Gibbs sampler is notable for its ease of implementation to a wide variety of models. The conceptual simplicity of the approach, together with the use of MATHWORKS MATLAB, made it easy to write or change the necessary programs to handle different estimating problems. Competitive results when compared to results obtained by Wilson, Hamada and Xu (2004), could therefore be obtained easily for tolerance intervals using the iron data given in Table 5.1, without using sophisticated numerical techniques. The ultimate value of the Gibbs sampler can therefore be found in its practical potential. Although it was pointed out that Gibbs sampling produce dependent draws from the joint posterior density, Wilson, Hamada and Xu (2004) and others indicated that dependence between draws can be reduced by only retaining every p^{th} draw. For the data given in Table 5.1, every 10^{th} draw was retained. Following the estimation of the tolerance intervals, the student - t distributed measurement error model was also successfully used in the identification of a possible outlying observation. In this chapter, the flexibility of

Bayesian simulation methods for estimating variance components and tolerance intervals were therefore illustrated for the balanced one - way random effects model with student t - distributed measurement errors.

In Chapter 6, the simulation - based approach for determining Bayesian tolerance intervals originally presented by Wolfinger (1998) for the balanced one - way random effects model, were extended from the two variance component random effects model to the three variance component balanced two - way nested random effects model. If $k^* = k$ and $r^* = r$, a probability matching prior has also successfully been derived for the α^{th} quantiles of the distribution of the average of k^* future packages with r^* samples per package from a new or unknown day. Similarly, if $k^* = k$ and $r^* = r$, it was shown that two prior distributions proposed for the content of a fixed - in - advance tolerance interval for the distribution of the average of k^* packages with r^* samples per package from a new or unknown day, were both also meeting the probability matching criteria. It was also shown that the Bayesian 0.95 - expectation tolerance interval obtained for the spun yarn data given in Table 6.1, were for all practical purposes equal to the classical 95% interval limits determined by Laubscher (1996). Under the probability matching prior given in equation 6.4.3, the frequentist coverage properties for the 95% prediction interval were also met, since the coverage percentage for the Bayesian 95% predictions interval was equal to 95.2%. For the above balanced two - factor nested random effects model, variance components and tolerance intervals were obtained using Monte Carlo simulation.

On a general note, it was also illustrated that the use of the Bayesian simulation method for obtaining posterior distributions of quantiles and subsequent tolerance intervals, provided final results in terms of common statistics and histograms, thus providing a straightforward means of communication to investigators. Also notable is the fact that the same analysis strategy can be used for the estimation of all three kinds of tolerance intervals. In contrast, the frequentist analysis, as pointed out by Wolfinger (1998), differs depending on the kind of tolerance interval and model under consideration, and,

can become quite complex, even for balanced one - way random effects models.

7.2 Future Research

In this thesis, the estimation of variance components and tolerance intervals have been discussed for the balanced univariate normal model, the balanced one - way random effects model with standard and non - standard measurement errors and the balanced two - factor nested random effects model. For some models, reference and probability matching priors have also been derived for quantiles and the content of predetermined bounds. The work is ongoing and several problems still need to be addressed. For example, the Bayesian simulation method discussed for the balanced two - factor nested random effects model can be extended to include models with more random effects and interaction as well as higher order models with unbalanced data sets. Tolerance intervals can in future research also be estimated for higher order models with non - standard measurement errors. Further work also needs to be done on the development of procedures for comparing two or more α quantiles using models with non - standard measurement errors or higher order models. According to Krishnamoorthy and Mathew (2009), the problem of constructing tolerance intervals for discrete distributions have not received much attention, and as a result, the construction of Bayesian tolerance intervals based on discrete distributions, such as binomial or poisson distributions, also provide a topic for future research. The Dirichlet process priors can in future research also be used to provide non - parametric Bayes estimates for the random effects and variance components. As also suggested by Wolfinger (1998), applying the Bayesian simulation method to process capability analysis, by considering for example process capability indexes in mixed model settings, also provide a promising topic for future research.

It is hoped that the Bayesian simulation methods and other information provided in this thesis will significantly contribute towards variance component estimation and the

determination of tolerance intervals, since it is an important ingredient in the design and production of high quality, high reliability production processes.

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List of Algorithms

Selective Algorithms from Chapter 2

2.1 Simulation of the Variance Component and the Mean: Population 1

```
clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%i=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
i=10000;

%S=input('WHAT IS THE STANDARD DEVIATION OF THE SAMPLE? ... ');
S=0.000986;

%BARX=input('WHAT IS THE MEAN OF THE SAMPLE? ... ');
BARX=0.0070;

%n=input('GIVE THE NUMBER OF MEASUREMENTS (i.e. THE SAMPLE SIZE) ... ');
n=36;

v=(n-1);
VSSQR=(v*(S^2));

SIG=[];
SQRTSIG=[];
MU=[];

for k=1:i
    k
    % Simulate variance
    Z1=randn([v,1]);
    Z2=(Z1.^2);
    Z=sum(Z2);
    SIG1=VSSQR/Z;
    SQRTSIG1=(VSSQR/Z)^(0.5);
    SQRTSIG=[SQRTSIG SQRTSIG1];
    SIG=[SIG SIG1];

    % Simulate mean
    Z3=randn([1,1]);
    Z4=sqrt(SIG1/n);
    MU1=Z3*Z4 + BARX;
    MU=[MU MU1];
end;

figure(1)
hist(SIG,20)

figure(2)
hist(MU,20)

A=sort(SIG);
LLSIG=A(250)
ULSIG=A(9750)

B=sort(MU);
LLMU=B(250)
ULMU=B(9750)

save C:\Varcomp SQRTSIG SIG MU
```

2.2 Simulation of the Variance Components and means: Population 1 and 2

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%i=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
i=10000;

%S1=input('WHAT IS THE STANDARD DEVIATION OF THE FIRST SAMPLE? ... ');
S1=0.000986;

%S2=input('WHAT IS THE STANDARD DEVIATION OF THE SECOND SAMPLE? ... ');
S2=0.000981;

%BARX1=input('WHAT IS THE MEAN OF THE FIRST SAMPLE? ... ');
BARX1=0.0070;

%BARX2=input('WHAT IS THE MEAN OF THE SECOND SAMPLE? ... ');
BARX2=0.0058;

%n1=input('GIVE THE NUMBER OF MEASUREMENTS OF THE FIRST SAMPLE ... ');
n1=36;

%n2=input('GIVE THE NUMBER OF MEASUREMENTS OF THE SECOND SAMPLE ... ');
n2=27;

v1=(n1-1);
v2=(n2-1);

VSSQR1=(v1*(S1^2));
VSSQR2=(v2*(S2^2));

SIG1=[];
SIG2=[];
SQRTSIG1=[];
SQRTSIG2=[];
MU1=[];
MU2=[];

for k=1:i
    k

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Simulations for population 1

    %Simulate the variance
    Z11=randn([v1,1]);
    Z21=(Z11.^2);
    Z1=sum(Z21);
    SIG11=VSSQR1/Z1;
    SQRTSIG11=(VSSQR1/Z1)^(0.5);
    SQRTSIG1=[SQRTSIG1 SQRTSIG11];
    SIG1=[SIG1 SIG11];

    %Simulate the mean
    Z31=randn([1,1]);
    Z41=sqrt(SIG11/n1);
    MU11=Z31*Z41 + BARX1;
    MU1=[MU1 MU11];

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % Simulations for population 2

    %Simulate the variance
    Z12=randn([v2,1]);
    Z22=(Z12.^2);
    Z2=sum(Z22);
    SIG12=VSSQR2/Z2;

```

```
SQRTSIG12=(VSSQR2/Z2)^(0.5);
SQRTSIG2=[SQRTSIG2 SQRTSIG12];
SIG2=[SIG2 SIG12];

% Simulate the mean
Z32=randn([1,1]);
Z42=sqrt(SIG12/n2);
MU12=Z32*Z42 + BARX2;
MU2=[MU2 MU12];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

end;

figure(1)
hist(SIG1)

figure(2)
hist(MU1)

figure(3)
hist(SIG2)

figure(4)
hist(MU2)

A1=sort(SIG1);
LLSIG1=A1(250)
ULSIG1=A1(9750)

B1=sort(MU1);
LLMU1=B1(250)
ULMU1=B1(9750)

A2=sort(SIG2);
LLSIG2=A2(250)
ULSIG2=A2(9750)

B2=sort(MU2);
LLMU2=B2(250)
ULMU2=B2(9750)

save C:\Varcomp2 SQRTSIG1 SIG1 MU1 SQRTSIG2 SIG2 MU2
```

2.3 Simulation of Unconditional Posterior Distributions ($\mu | y$) and ($\sigma_\epsilon^2 | y$)

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

load c:\Varcomp %Simulated variance components

%i=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
i=10000;

%S=input('WHAT IS THE STANDARD DEVIATION OF THE SAMPLE? ... ');
S=0.000986;

%BARX=input('WHAT IS THE MEAN OF THE SAMPLE? ... ');
BARY=0.0070;

%n=input('GIVE THE NUMBER OF MEASUREMENTS (i.e. THE SAMPLE SIZE) ... ');
n=36;

v=(n-1);

VSSQR=(v*(S^2));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Estimated unconditional posterior density of MU (MU/y): Rao Blackwell Method

X1=[];
A=[];
KK=0;

B=BARY-0.001:0.00001:BARY+0.001;

D=[zeros(1,(max(size(B))))];

for i=1:max(size(SIG))
    i

    SIG1=SIG(i);
    GEMID=BARY;
    VARIAN=SIG1/n;
    A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));
    X11=(randn(1,1)*sqrt(VARIAN))+GEMID;

    D=[D+A1];
    X1=[X1 X11];
    KK=KK+1;

end;

figure(1)
E1=D'/KK;
G1=E1/(sum(D));
plot(B,E1);
%plot(B,G1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Unconditional Posterior Density of MU using the Student-t distribution

TDIST=[];

T1=((VSSQR/2)^(v/2));
T2=1/gamma(v/2);
T3=((T1*T2)*(n^0.5))/((2*pi)^0.5);

```



```

T4=gamma(n/2);
for U=BARY-0.001:0.00001:BARY+0.001;
    T5=n*((U-GEMID)^2);
    T6=(2/(T5+VSSQR))^(n/2);
    tdist=T3*T4*T6;
    TDIST=[TDIST,tdist];
end;
UU=BARY-0.001:0.00001:BARY+0.001;
figure(2)
plot(UU,TDIST);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Unconditional posterior distribution of the variance: Inverse Gamma Distribution
INGAM=[];
G1=((VSSQR/2)^(v/2));
G2=1/gamma(v/2);
for IG=0:0.000000001:0.000003
    G3=IG^((-0.5)*(n+1));
    G4=exp((-0.5*VSSQR)/IG);
    ingam=G1*G2*G3*G4;
    INGAM=[INGAM ingam];
end;
IGG=0:0.000000001:0.000003;
figure(3)
plot(IGG,INGAM);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
SX1=sort(X1);
POINT=mean(X1)
LOWER=SX1(250)
UPPER=SX1(9750)
%save c:\Postdist X1 E1 G1 TDIST INGAM POINT LOWER UPPER

```

2.4 Simulation of Marginal Posterior Distribution of q

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Plot the distribution of the quantile q by using the Rau Blackwell Method.

load c:\Varcomp

X1=[];
A=[];
Q=[];
KK=0;
ZB=1.645;
BARX=0.0070;
n=36;

B=BARX-0.004:0.00001:BARX+0.0005; %For lower limit
%B=BARX-0.0005:0.00001:BARX+0.004; %For upper limit

D=[zeros(1,(max(size(B))))];

for i=1:max(size(SIG))
    i

    mu1=MU(i);
    SIG1=SIG(i);

    Q1=mu1-(ZB*(sqrt(SIG1))); %For lower limit
    %Q1=mu1+(ZB*(sqrt(SIG1))); %For upper limit

    Q=[Q Q1];

    GEMID=BARX-(ZB*sqrt(SIG1)); %For lower limit
    %GEMID=BARX+(ZB*sqrt(SIG1)); %For upper limit

    VARIAN=SIG1/n;
    A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));
    X11=(randn(1,1)*sqrt(VARIAN))+GEMID;

    D=[D+A1];
    X1=[X1 X11];
    KK=KK+1;

end;

figure(1)
E1=D'/KK;
G1=E1/(sum(D));
plot(B,E1);

figure(2)
hist(Q,20)

esort=sort(Q);
perce1 = prctile(esort,2.5)
perce2 = prctile(esort,97.5)
perce3 = prctile(esort,5)
%perce3 = prctile(esort,95)

save c:\Betgam1L X1 E1 G1 Q B
%save c:\Betgam1U X1 E1 G1 Q B

```

2.5 Simulation of the Predictive Distribution

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

load c:\Varcomp

%i=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
i=10000;

%S=input('WHAT IS THE STANDARD DEVIATION OF THE SAMPLE? ... ');
S=0.000986;

%BARX=input('WHAT IS THE MEAN OF THE SAMPLE? ... ');
BARY=0.0070;

%n=input('GIVE THE NUMBER OF MEASUREMENTS (i.e. THE SAMPLE SIZE) ... ');
n=36;

v=(n-1);
VSSQR=(v*(S^2));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Plot the alpha expectation tolerance interval for Yf using the Rao Blackwell method
%i.e N(MU, SIG)

X1=[];
A=[];
KK=0;

B=BARY-0.004:0.00001:BARY+0.004;

D=[zeros(1,(max(size(B))))];

for i=1:max(size(SIG))
    i

    SIG1=SIG(i);
    GEMID=MU(i);

    VARIAN=SIG1;
    A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));
    X11=(randn(1,1)*sqrt(VARIAN))+GEMID;

    D=[D+A1];
    X1=[X1 X11];
    KK=KK+1;

end;

figure(1)
E1=D'/KK;
G1=E1/(sum(D));
plot(B,E1);
%plot(B,G1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Plot the alpha expectation tolerance interval for Yf using the Rao Blackwell
%method i.e N(BARY, ((n+1)/n)*SIG)

XX=[];
AA=[];
KKK=0;

B1=BARY-0.004:0.00001:BARY+0.004;

D1=[zeros(1,(max(size(B1))))];

for j=1:max(size(SIG))
    j

```

```

    SIG11=SIG(j);
    GEMID1=BARY;

    VARIAN1=SIG11;
    AA1=(exp(-0.5*((B1-GEMID1).^2)/VARIAN1))/(sqrt(2*pi)*sqrt(VARIAN1));
    XX1=(randn(1,1)*sqrt(VARIAN1))+GEMID1;

    D1=[D1+AA1];
    XX=[XX XX1];
    KKK=KKK+1;

end;

figure(2)
EE1=D1'/KKK;
GG1=EE1/(sum(D1));
plot(B1,EE1);
%plot(B1,GG1);

SX1=sort(X1);
SXX=sort(XX);

POINT1=mean(X1)
esort1=sort(X1);
perce11 = prctile(esort1,2.5)
perce21 = prctile(esort1,97.5)

POINT2=mean(XX)
esort2=sort(XX);
perce12 = prctile(esort2,2.5)
perce22 = prctile(esort2,97.5)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
%Plot the predictive density of yf using the Student-t distribution

TDIST=[];

T1=(VSSQR/2)^(v/2);
T2=1/gamma(v/2);
T3=((T1*T2)*(n^0.5))/(((2*pi)^0.5)*((n+1)^0.5));
T4=gamma(n/2);

for U=BARY-0.004:0.00001:BARY+0.004;

    T5=(n/(n+1))*((U-GEMID)^2);
    T6=(2/(T5+VSSQR))^(n/2);
    tdist=T3*T4*T6;
    TDIST=[TDIST,tdist];

end;

UU=BARY-0.004:0.00001:BARY+0.004;

figure(3)
plot(UU,TDIST);

save c:\Alphaexpt X1 E1 G1 XX EE1 GG1

```

2.6 Simulation of q_l and q_u

```

clear
clc

randn('seed',sum(100*clock));

%Determine QL and QU and plot the histograms

load c:\Varcomp

Q=[];
QL=[];
QU=[];
KK=0;

for i=1:max(size(MU))
    i
        t1=MU(i);
        s1=SIG(i);
        GEMID=t1;
        VARIAN=s1;
        %Q1=GEMID-(1.282*(sqrt(VARIAN)));
        %Q1=GEMID+(1.282*(sqrt(VARIAN)));
        %QU1=GEMID+(1.645*(sqrt(VARIAN)));
        Q1=GEMID+(1.645*(sqrt(VARIAN)));
        %Q1=GEMID-(1.645*(sqrt(VARIAN)));
        QL1=GEMID-(1.96*(sqrt(VARIAN)));
        QU1=GEMID+(1.96*(sqrt(VARIAN)));

        Q=[Q Q1];
        QL=[QL QL1];
        QU=[QU QU1];
        KK=KK+1;

end;

gamma1=0.95;
%gamma1=0.9;
m=max(size(Q))

figure(1);
hist(Q,20)
BARQ=median(VECQ)
esort=sort(Q);
perce1 = prctile(esort,2.5)
perce2 = prctile(esort,97.5)
perce3 = prctile(esort,95)

%Lower limit
vec=QL;
figure(2);
subplot(2,1,1);
hist(QL);
VECQL=sort(QL);
CIQL=[VECQL(1,round(m*((1-gamma1)/2))) VECQL(1,round(m*(1-((1-gamma1)/2))))]
BARQL=median(VECQL)

%Upper limit
vec=QU;
subplot(2,1,2);
hist(QU);
VECQU=sort(vec);
CIQU=[VECQU(1,round(m*((1-gamma1)/2))) VECQU(1,round(m*(1-((1-gamma1)/2))))]
BARQU=median(VECQU)
BHat=mean(MU)

save C:\Adt_2 Q QL QU CIQ BARQ CIQL BARQL CIQU BARQU BHat

```

2.7 Construction of a Two - Sided (α, δ) Tolerance Interval

```

clear;
clc;

load C:\Varcomp
load C:\Adt_2

%Determines the Two sided tolerance interval and plot a scatter plot

Q2=[QL;QU]';
BHat=mean(MU);
gamma1=0.95;
m=max(size(QU));
p=round(((1-gamma1)*m))
p1=round(((1-gamma1)*m)-1)

%Reference lyn
x1=min(QU);
max1=max(QU);
min1=min(QU);
y1=-x1+2*BHat;
y2=-max1+2*BHat;
X=[x1 max1];Y=[y1 y2];
figure(1)
plot(QU,QL,'.')
hold on
plot(X,Y);

%Check the number of points
interval=max1:-0.00001:x1;
i=length(interval);
j=0;
st=0;

while (st<=p1)&(j<i),
    j=j+1;
    y11=-interval(1,j)+2*BHat;
    X1=[interval(1,j),min(QU)+0.003];
    Y1=[y11,y11];

    X2=[interval(1,j),interval(1,j)];
    Y2=[y11,min1-0.005];

    O=find(Q2(:,1)<=y11 & Q2(:,2)>=interval(1,j));
    st=size(O);
end

st
plot(X1,Y1)
hold on
plot(X2,Y2)
hold on
X=[X(1,1) X1(1,1)];
Y=[Y(1,1) Y1(1,1)];
[X1(1,1) Y1(1,1)]
Width=X1(1,1)-Y1(1,1)
text(9.3*10^-3,3.25*10^-3,'0.0093');
text(10.8*10^-3,4.8*10^-3,'0.0047');
text(10.35*10^-3,3.70*10^-3,'Reference Line');

```

2.8 Simulation of the Content of the Fixed - in - Advance Tolerance Interval

```

clear
clc

%Determines the Content of a Fixed in advance tolerance interval
%S=0.009

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

load c:\Varcomp

S=input('WHAT IS THE VALUE OF THE PRESELECTED LIMIT S? ... ');

CUM_STD=[];           %Percentage less than S
CUM_STD1=[];         %Percentage greater than S

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:max(size(SIG))
    i

    GEMID=MU(i);
    VARIAN=SIG(i);

    Z1=(S-GEMID)/sqrt(VARIAN);

    V=Z1;

    if Z1<=0
        cons=(1/sqrt(2*pi));
        h=V/1500;

        if h<0

            h=-h;
            k=-10:h:V;
            l=exp((-1/2)*(k.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

        else

            k=-10:h:V;
            l=exp((-1/2)*(k.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

        end;

    else

        cons=(1/sqrt(2*pi));
        h=V/1500;
        k=-10:h:V;
        l=exp((-1/2)*(k.^2));
        std_norm=cons*l;
        cum_std=h*(sum(std_norm));
        cum_std1=1-cum_std;

    end;

    CUM_STD=[CUM_STD cum_std];           %Percentage of values less than S
    CUM_STD1=[CUM_STD1 cum_std1];       %Percentage of values greater than S

end;

figure(1)
hist(CUM_STD,20)

```

```
figure(2)
hist(CUM_STD1,20)

SCUM_STD=sort(CUM_STD);
SCUM_STD1=sort(CUM_STD1);

PER1=median(CUM_STD)

LOWER1=SCUM_STD(250)
UPPER1=SCUM_STD(9750)

PER2=median(CUM_STD1)

LOWER2=SCUM_STD1(250)
UPPER2=SCUM_STD1(9750)

%save c:\Fixint CUM_STD CUM_STD1 PER1 LOWER1 UPPER1 PER2 LOWER2 UPPER2
```


2.9 Algorithm for Determining the Fixed - in - advance Tolerance Interval using the Weighted Monte Carlo Method

```

clear
clc
format long

randn('seed',sum(100*clock));

%Determine the Fixed in advance interval using importance sampling
%S = 0.009

load C:\Varcomp

THETA=[];
Z=[];
CUM_STD=[];
CUM_STD1=[];
W=[];
IIL=[];
IIU=[];
POSLL=0;
POSUL=0;
POSLL1=[];
POSUL1=[];

S=input('WHAT IS S FOR THE FIXED IN ADVANCE TOLERANCE INTERVAL? ... ');

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

for i=1:max(size(MU))
    i
        GEMID=MU(i);
        VARIAN=SIG(i);
        Z11=(S-GEMID)/sqrt(VARIAN);

        V=Z11;

        if Z11<=0
            cons=(1/sqrt(2*pi));
            h=V/1500;

            if h<0

                h=-h;
                kk=-10:h:V;
                l=exp((-1/2)*(kk.^2));
                std_norm=cons*l;
                cum_std=h*(sum(std_norm));
                cum_std1=1-cum_std;

            else

                kk=-10:h:V;
                l=exp((-1/2)*(kk.^2));
                std_norm=cons*l;
                cum_std=h*(sum(std_norm));
                cum_std1=1-cum_std;

            end;

        else

            cons=(1/sqrt(2*pi));
            h=V/1500;
            kk=-10:h:V;
            l=exp((-1/2)*(kk.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

        end;
    end;
end;

```

```

W1=SIG(i)^(-0.5);
W2=(S-MU(i))^2;
W3=W2/(2*SIG(i));
W4=(1+W3)^(-0.5);
W5=W1*W4;
W=[W W5];

Z=[Z Z11];
CUM_STD=[CUM_STD cum_std];
CUM_STD1=[CUM_STD1 cum_std1];

end;

WEIGHTS=W./sum(W);
CW1=[CUM_STD1;WEIGHTS];
%CW1=[CUM_STD;WEIGHTS];
CW=CW1';
CWS=sortrows(CW,1);

C1=CWS(:,1);
C2=CWS(:,2);

for iil=1:max(size(C2))
    iil

    POSLL=POSLL+C2(iil);

    if POSLL <= 0.025

        IIL=[IIL iil];
        POSLL1=[POSLL1 POSLL];

    end;

    if POSLL <= 0.975

        IIU=[IIU iil];
        POSUL1=[POSUL1 POSLL];

    end;

end;

end;

disp('Importance Sampling')
ll=max(size(IIL))
mll=max(POSLL1)

ul=max(size(IIU))
mul=max(POSUL1)

LOWERLIMIT=C1(ll)
UPPERLIMIT=C1(ul)

disp('Ordinary Monte Carlo')
LOWER1=C1(250)
UPPER1=C1(9750)

%figure(1)
%hist(CUM_STD,30);

figure(1)
hist(CUM_STD1,30);

```

2.10 Simulation of the Posterior Distribution of The Difference Between Two α Quantiles

```

clear
clc

randn('seed',sum(100*clock));

%Determine and plot the difference between two alpha delta quantiles

load c:\Varcomp2

%BARX1=input('WHAT IS THE MEAN OF THE FIRST SAMPLE? ... ');
BARX1=0.0070;

%BARX2=input('WHAT IS THE MEAN OF THE SECOND SAMPLE? ... ');
BARX2=0.0058;

%n1=input('GIVE THE SAMPLE SIZE OF THE FIRST SAMPLE ... ');
n1=36;

%n2=input('GIVE THE SAMPLE SIZE OF THE SECOND SAMPLE ... ');
n2=27;

ZB=1.645;

BARY=BARX1-BARX2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Histogram of the difference between two quantiles

Q=[];

for i=1:max(size(MU1))
    i

    t1=MU1(i);
    t2=MU2(i);
    s1=SQRTSIG1(i);
    s2=SQRTSIG2(i);

    GEMID=t1-t2;
    STDDIF=s1-s2;

    Q1=GEMID+(ZB*STDDIF);
    Q=[Q Q1];

end;

gamma1=0.95;
%gamma1=0.9;
m=max(size(Q));

figure(1);
hist(Q,39)

VECQ=sort(Q);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Unconditional posterior density of the difference between two means using
%the Rao Blackwell method

X1=[];
A=[];
KK=0;

B=BARY-0.001:0.00001:BARY+0.001;
D=[zeros(1,(max(size(B))))];

for j=1:max(size(MU1))
    j

    SIG11=SIG1(j);
    SIG22=SIG2(j);

```

```

GEMID=BARY;
VARIAN=(SIG11/n1)+(SIG22/n2);

A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));
X11=(randn(1,1)*sqrt(VARIAN))+GEMID;

D=[D+A1];
X1=[X1 X11];
KK=KK+1;

end;

figure(2)
E1=D'/KK;
G1=E1/(sum(D));
plot(B,E1);
%plot(B,G1);

SORTDIFF=sort(X1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Unconditional posterior density of the difference between two Quantiles
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

XX=[];
AA=[];
KKK=0;

B1=BARY-0.002:0.00001:BARY+0.00175;
D1=[zeros(1,(max(size(B1))))];

for jj=1:max(size(MU1))
    jj

    SIG111=SIG1(jj);
    SIG222=SIG2(jj);

    GEMID1=BARY+(ZB*(sqrt(SIG111)-sqrt(SIG222)));
    VARIAN1=(SIG111/n1)+(SIG222/n2);

    AA1=(exp(-0.5*((B1-GEMID1).^2)/VARIAN1))/(sqrt(2*pi)*sqrt(VARIAN1));
    XX1=(randn(1,1)*sqrt(VARIAN1))+GEMID1;

    D1=[D1+AA1];
    XX=[XX XX1];
    KKK=KKK+1;

end;

figure(3)
EE1=D1'/KKK;
GG1=EE1/(sum(D1));
plot(B1,EE1);
%plot(B1,GG1);

SORTDIFF1=sort(XX);
BARQ=mean(VEQC)
CIQ=[VEQC(1,round(m*((1-gamma1)/2))) VECQ(1,round(m*(1-((1-gamma1)/2))))]
MEANDIFF=mean(SORTDIFF)
LDIFF=SORTDIFF(250)
UDIFF=SORTDIFF(9750)
MEANDIFF1=mean(SORTDIFF1)
LDIFF1=SORTDIFF1(250)
UDIFF1=SORTDIFF1(9750)

figure(4)
hist(Q,39)
hold on
plot(B1,EE1)

save C:\Adt_2_2 Q CIQ BARQ X1 E1 G1 XX EE1 GG1

```

2.11 Simulation Study: Multiple Comparisons using Pairwise Differences

```

clear
clc

randn('seed',sum(100*clock));

%Multiple comparisons using pairwise differences

%j=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
j=10000;
%i=input('HOW MANY SIMULATIONS PER P-VALUE CALCULATION DO YOU WANT TO RUN ... ');
i=1000;
%STD=input('WHAT IS THE STANDARD DEVIATION OF THE SAMPLES? ... ');
STD=0.00012;
%MU=input('WHAT IS THE MEAN OF THE SAMPLES? ... ');
MU=0.00045;
%n1=input('GIVE THE NUMBER OF MEASUREMENTS (i.e. THE SAMPLE SIZE) OF THE SAMPLES ... ');
n1=20;

n2=n1;
n3=n1;

ZB=1.645;

SIG=(STD^2);
SIGG=SIG/n1;

v1=(n1-1);
v2=(n2-1);
v3=(n3-1);

BARXS1=[];
BARXS2=[];
BARXS3=[];

VSSQRS1=[];
VSSQRS2=[];
VSSQRS3=[];

PER=0;

for kk=1:j
    kk
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %Simulation of first data set.

    ZD11=randn([1,1]);
    BARX1=MU+(ZD11*sqrt(SIGG));
    ZD121=randn([v1,1]);
    ZD12=sum(ZD121.^2);
    VSSQR1=SIG*ZD12;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %Simulation of second data set.

    ZD21=randn([1,1]);
    BARX2=MU+(ZD21*sqrt(SIGG));
    ZD221=randn([v2,1]);
    ZD22=sum(ZD221.^2);
    VSSQR2=SIG*ZD22;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %Simulation of third data set.

    ZD31=randn([1,1]);
    BARX3=MU+(ZD31*sqrt(SIGG));
    ZD321=randn([v3,1]);

```

```

ZD32=sum(ZD321.^2);
VSSQR3=SIG*ZD32;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Determine the expected value of q1/y

EQ11=(ZB*(VSSQR1^0.5));
EQ12=(gamma((v1-1)/2))/((2^0.5)*(gamma(v1/2)));
EQ1=BARX1+(EQ11*EQ12);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Determine the expected value of q2/y

EQ21=(ZB*(VSSQR2^0.5));
EQ22=(gamma((v2-1)/2))/((2^0.5)*(gamma(v2/2)));
EQ2=BARX2+(EQ21*EQ22);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Determine the expected value of q3/y

EQ31=(ZB*(VSSQR3^0.5));
EQ32=(gamma((v3-1)/2))/((2^0.5)*(gamma(v3/2)));
EQ3=BARX3+(EQ31*EQ32);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

T1=[];
PER1=0;

for k=1:i

    % Simulation of Sigma Squared for population 1

    Z1=randn([v1,1]);
    Z2=(Z1.^2);
    Z3=sum(Z2);
    SIG11=VSSQR1/Z3;
    SQRTSIG11=(SIG11^(0.5));

    % Simulate q for population 1

    Z11=randn([1,1]);
    Z12=sqrt(SIG11/n1);
    Q111=(Z11*Z12) + (BARX1+(ZB*SQRTSIG11));
    Q11=Q111-EQ1;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Simulation of Sigma Squared for population 2

    Z4=randn([v2,1]);
    Z5=(Z4.^2);
    Z6=sum(Z5);
    SIG22=VSSQR2/Z6;
    SQRTSIG22=(SIG22^(0.5));

    % Simulate q for population 2

    Z21=randn([1,1]);
    Z22=sqrt(SIG22/n2);
    Q221=(Z21*Z22) + (BARX2+(ZB*SQRTSIG22));
    Q22=Q221-EQ2;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Simulation of Sigma Squared for population 3

    Z7=randn([v3,1]);

```

```

Z8=(Z7.^2);
Z9=sum(Z8);
SIG33=VSSQR3/Z9;
SQRTSIG33=(SIG33^(0.5));

% Simulate q for population 3

Z31=randn([1,1]);
Z32=sqrt(SIG33/n3);
Q331=(Z31*Z32) + (BARX3+(ZB*SQRTSIG33));
Q33=Q331-EQ3;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

QQ11=[Q11 Q22 Q33];
QQ1=sort(QQ11);

T11=QQ1(3)-QQ1(1);
T1=[T1 T11];

end;

T=sort(T1);
UL=T(i*0.95);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
DIFF1=abs(EQ1-EQ2);
DIFF2=abs(EQ1-EQ3);
DIFF3=abs(EQ2-EQ3);

if DIFF1>=UL
    PER11=1;
else
    PER11=0;
end;

if DIFF2>=UL
    PER22=1;
else
    PER22=0;
end;

if DIFF3>=UL
    PER33=1;
else
    PER33=0;
end;

PER1=PER11+PER22+PER33;

if PER1>0
    PER=PER+1;
end;

BARXS1=[BARXS1 BARX1];
BARXS2=[BARXS2 BARX2];
BARXS3=[BARXS3 BARX3];
VSSQRS1=[VSSQRS1 VSSQR1];
VSSQRS2=[VSSQRS2 VSSQR2];
VSSQRS3=[VSSQRS3 VSSQR3];

end;

PERS=(PER/j)*100

%save C:\Pairsim20 BARXS1 BARXS2 BARXS3 VSSQRS1 VSSQRS2 VSSQRS3 PERS

```



```

k

% Simulation of Sigma Squared for population 1

Z1=randn([v1,1]);
Z2=(Z1.^2);
Z3=sum(Z2);
SIG11=VSSQR1/Z3;
SQRTSIG11=(SIG11^(0.5));

% Simulate q for population 1

Z11=randn([1,1]);
Z12=sqrt(SIG11/n1);
Q111=(Z11*Z12) + (BARX1+(ZB*SQRTSIG11));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Simulation of Sigma Squared for population 2

Z4=randn([v2,1]);
Z5=(Z4.^2);
Z6=sum(Z5);
SIG22=VSSQR2/Z6;
SQRTSIG22=(SIG22^(0.5));

% Simulate q for population 2

Z21=randn([1,1]);
Z22=sqrt(SIG22/n2);
Q221=(Z21*Z22) + (BARX2+(ZB*SQRTSIG22));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Simulation of Sigma Squared for population 3

Z7=randn([v3,1]);
Z8=(Z7.^2);
Z9=sum(Z8);
SIG33=VSSQR3/Z9;
SQRTSIG33=(SIG33^(0.5));

% Simulate q for population 3

Z31=randn([1,1]);
Z32=sqrt(SIG33/n3);
Q331=(Z31*Z32) + (BARX3+(ZB*SQRTSIG33));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

THETA1=[Q111
        Q221
        Q331];

ETHETA=[EQ1
        EQ2
        EQ3];

THETA=THETA1-ETHETA;

VARTHETA=[VARQ1 0 0
          0 VARQ2 0
          0 0 VARQ3];

CON11=(L1'*THETA)^2;
CON12=(L2'*THETA)^2;

CON21=L1'*VARTHETA*L1;

```

```

CON22=L2'*VARTHETA*L2;

T311=CON11/CON21;
T312=CON12/CON22;

T31=[T311 T312];
T3=max(T31);
T1=[T1 T3];

end;

T=sort(T1);

%TT=T(4)
TT=T(i*0.95);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Perform Simultaneous Credibility Intervals Test
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Test Between Quantiles 1 and 2

LL1=(L1'*ETHETA)-(sqrt(CON21*TT));
UL1=(L1'*ETHETA)+(sqrt(CON21*TT));

if LL1 <= 0 && UL1 >= 0
    PER11=0;
else
    PER11=1;
end;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Test Between Quantiles 2 and 3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

LL2=(L2'*ETHETA)-(sqrt(CON22*TT));
UL2=(L2'*ETHETA)+(sqrt(CON22*TT));

if LL2 <= 0 && UL2 >= 0
    PER22=0;
else
    PER22=1;
end;

PER1=PER11+PER22;

if PER1 > 0
    PER=PER+1;
end;

BARXS1=[BARXS1 BARX1];
BARXS2=[BARXS2 BARX2];
BARXS3=[BARXS3 BARX3];
VSSQRS1=[VSSQRS1 VSSQR1];
VSSQRS2=[VSSQRS2 VSSQR2];
VSSQRS3=[VSSQRS3 VSSQR3];

end;

PERS=(PER/j)*100

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
save C:\Credsim BARXS1 BARXS2 BARXS3 VSSQRS1 VSSQRS2 VSSQRS3 PERS

```

Simulation Algorithm for Chapter 3

```

%Random Effects Model
%-----
format
clc;
clear;
randn('seed',sum(100*clock));
b = 15;
k = 10;
v1 = b*(k-1);
v2 = b-1;
dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

Ybar = mean(dataY,2);
YbarBig = mean(Ybar);

%-----
%Calculate v1m1 & v2m2

v1m1 = 0;
v2m2 = 0;
for i = 1:b;
    for j = 1:k;
        v1m1 = v1m1 + (dataY(i,j) - Ybar(i))^2;
    end
end
for i = 1:b;
    v2m2 = v2m2 + k*(Ybar(i) - YbarBig)^2;
end

%-----
v1m1
v2m2
%-----

deltaE = [ ];
delta12 = [ ];
deltaA = [ ];
U = [ ];
var2 = [ ];
for i = 1:10000
    i
    z1 = randn(1,v1);
    chi2v1 = z1*z1';
    deltaE1 = v1m1/chi2v1;
    z2 = randn(1,v2);
    chi2v2 = z2*z2';
    delta121 = v2m2/chi2v2;
    deltaA1 = (delta121 - deltaE1)/k;
    z3 = randn(1,1);
    var1 = (deltaE1 + k*deltaA1)/(k*b);
    U1 = (z3*sqrt(var1)) + YbarBig;
    if delta121 > deltaE1

```

```

        delta12 = [delta12 delta121];
        deltaE = [deltaE deltaE1];
        deltaA = [deltaA deltaA1];
        U = [U U1];
        var2 = [var2 var1];
    end
end

varmed = median(var2);
x1 = 150.35:0.00001:150.65;
fx1 = normpdf(x1, YbarBig, sqrt(varmed));
%plot(x1, fx1)

%-----
%Random effects

x2 = -0.31:0.001:0.35;
fxA1a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(1)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA1a = fxA1a + normpdf(x2, ea, sqrt(va));
end
fxA1 = fxA1a/length(deltaE);

fxA2a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(2)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA2a = fxA2a + normpdf(x2, ea, sqrt(va));
end
fxA2 = fxA2a/length(deltaE);

fxA3a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(3)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA3a = fxA3a + normpdf(x2, ea, sqrt(va));
end
fxA3 = fxA3a/length(deltaE);

fxA4a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(4)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA4a = fxA4a + normpdf(x2, ea, sqrt(va));
end
fxA4 = fxA4a/length(deltaE);

fxA5a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(5)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA5a = fxA5a + normpdf(x2, ea, sqrt(va));
end
fxA5 = fxA5a/length(deltaE);

fxA6a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(6)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA6a = fxA6a + normpdf(x2, ea, sqrt(va));
end
fxA6 = fxA6a/length(deltaE);

fxA7a = 0;

```

```

for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(7)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA7a = fxA7a + normpdf(x2,ea,sqrt(va));
end
fxA7 = fxA7a/length(deltaE);

fxA8a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(8)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA8a = fxA8a + normpdf(x2,ea,sqrt(va));
end
fxA8 = fxA8a/length(deltaE);

fxA9a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(9)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA9a = fxA9a + normpdf(x2,ea,sqrt(va));
end
fxA9 = fxA9a/length(deltaE);

fxA10a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(10)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA10a = fxA10a + normpdf(x2,ea,sqrt(va));
end
fxA10 = fxA10a/length(deltaE);

fxA11a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(11)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA11a = fxA11a + normpdf(x2,ea,sqrt(va));
end
fxA11 = fxA11a/length(deltaE);

fxA12a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(12)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA12a = fxA12a + normpdf(x2,ea,sqrt(va));
end
fxA12 = fxA12a/length(deltaE);

fxA13a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(13)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA13a = fxA13a + normpdf(x2,ea,sqrt(va));
end
fxA13 = fxA13a/length(deltaE);

fxA14a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(14)-YbarBig);
    va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
    fxA14a = fxA14a + normpdf(x2,ea,sqrt(va));
end
fxA14 = fxA14a/length(deltaE);

fxA15a = 0;
for j = 1:length(deltaE)
    ea = ((k*deltaA(j))/(deltaE(j)+k*deltaA(j)))*(Ybar(5)-YbarBig);

```

```

        va = (deltaA(j)/(deltaE(j)+k*deltaA(j)))*(deltaE(j)+((k/b)*deltaA(j)));
        fxA15a = fxA15a + normpdf(x2,ea,sqrt(va));
    end
    fxA15 = fxA15a/length(deltaE);

%-----
%Plots similar to Wolfinger (1998) p25 plots

figure(1)
nbins = 25;
[count, binvals] = hist(deltaE, nbins);
ncount = 100*count/sum(count);
bar(binvals, ncount, 'barwidth', 1);
xlabel('deltaE');
ylabel('Percentage');
esort=sort(deltaE);
perce1 = prctile(esort,2.5)
perce2 = prctile(esort,97.5)

figure(2)
nbins = 25;
[count, binvals] = hist(deltaA, nbins);
ncount = 100*count/sum(count);
bar(binvals, ncount, 'barwidth', 1);
xlabel('deltaA');
ylabel('Percentage');
asort=sort(deltaA);
perca1 = prctile(asort,2.5)
perca2 = prctile(asort,97.5)

figure(3)
nbins = linspace(150.35,150.65,30);
[count, binvals] = hist(U, nbins);
ncount = 100*count/sum(count);
bar(binvals, ncount, 'barwidth', 1);
xlabel('MU');
ylabel('Percentage');
hold on;
plot(x1,fx1,'-k');
usort=sort(U);
percu1 = prctile(usort,2.5)
percu2 = prctile(usort,97.5)

figure(4)
plot(x2,fxA1);
xlabel('Random Effects');
ylabel('Percentage');
hold on;
plot(x2,fxA2);
plot(x2,fxA3);
plot(x2,fxA4);
plot(x2,fxA5);
plot(x2,fxA6);
plot(x2,fxA7);
plot(x2,fxA8);
plot(x2,fxA9);
plot(x2,fxA10);
plot(x2,fxA11);
plot(x2,fxA12);
plot(x2,fxA13);
plot(x2,fxA14);
plot(x2,fxA15);

%-----
%Plot similar to Wolfinger (1998) p28 plot
%alfa expectation tolerance interval

```

```

Yf = [ ];
var5 = [ ];
for i = 1:length(deltaE)
    z4 = randn(1,1);
    var3 = (deltaE(i) + k*deltaA(i))/(k*b);
    var4 = deltaE(i) + deltaA(i);
    UU = U(i);
    Yf1 = z4*sqrt(var4) + UU;
    Yf = [Yf Yf1];
    var5 = [var5 var4];
end

figure(5)
x3 = 150:0.0001:151;
varmed = median(var5);
YfDensity = normpdf(x3,mean(U),sqrt(varmed));
YfSort = sort(Yf);
disp('The Lower and Upper Tolerance Limits for the Predictive Density is ...')
perc1 = prctile(YfSort,2.5)
perc2 = prctile(YfSort,97.5)

yvals = 0:0.005:12;
nbins = linspace(150,151,28);
[count, binvals] = hist(Yf, nbins);
ncount = 100*count/sum(count);
bar(binvals, ncount, 'barwidth', 1);
xlabel('Future Y');
ylabel('Percentage');
hold on;
%plot(x3,1200*YfDensity,'-k');
plot(x3,3.6*YfDensity,'-k');
plot(perc1,yvals,'-k');
plot(perc2,yvals,'-k');

%-----
%alfa expectation tolerance interval (Method 2)

Yfm2 = [ ];
varm2 = [ ];
fxaem21 = 0;
xm2 = 149.8:0.0001:151.4;
yvalsm2 = 0:0.0005:3;

for i = 1:length(deltaE)
    zm2 = randn(1,1);
    varm21 = ((deltaE(i) + k*deltaA(i))/(k*b)) + (deltaE(i) + deltaA(i));
    avgm2 = YbarBig;
    Yfm21 = zm2*sqrt(varm21) + YbarBig;
    Yfm2 = [Yfm2 Yfm21];
    varm2 = [varm2 varm21];
    fxaem21 = fxaem21 + normpdf(xm2,avgm2,sqrt(varm21));
end

fxaem2 = fxaem21/length(deltaE);
Yfm2Sort = sort(Yfm2);
disp('The Lower and Upper Tolerance Limits for the Predictive Density Using Method 2 is ...')
percm21 = prctile(Yfm2Sort,2.5)
percm22 = prctile(Yfm2Sort,97.5)
perctest = prctile(Yfm2Sort,5)

figure(6)
plot(xm2,fxaem2);
hold on;
plot(percm21,yvalsm2,'-k');

```



```

plot(percm22,yvalsm2,'-k');

%-----
%Plot similar to Wolfinger (1998) p27 plot: (1-alpha)th Quantile

q = [ ];
for i = 1:length(deltaE)
    var3 = (deltaE(i) + k*deltaA(i))/(k*b);
    UU = U(i);
    q1 = UU - 1.282*sqrt(deltaE(i) + deltaA(i));
    q = [q q1];
end
qsort = sort(q);
disp('The Lower Limit for q is ...')
perc3 = prctile(qsort,5)

figure(7)
yvals = 0:0.005:14;
nbins = linspace(150.15,150.42,28);
[count, binvals] = hist(q, nbins);
ncount = 100*count/sum(count);
bar(binvals, ncount, 'barwidth', 1);
xlabel('5th Percentile');
ylabel('Percentage');
hold on;
plot(perc3,yvals,'-k');

%(1-alpha)th quantile using Method 2 and the Rao Blackwell Method.

qnm = [ ];
fxqnm=0;
xnm = 150.1:0.0001:150.5;
for i = 1:length(deltaE)
    varnm = (deltaE(i) + k*deltaA(i))/(k*b);
    UNM = YbarBig - 1.282*sqrt(deltaE(i) + deltaA(i));
    qnm11 = randn(1,1);
    qnm1 = (qnm11*sqrt(varnm))+UNM;
    qnm = [qnm qnm1];
    fxqnm = fxqnm + normpdf(xnm,UNM,sqrt(varnm));
end
fxqnm1 = fxqnm/length(deltaE);

qsortnm = sort(qnm);
disp('The Lower Limit for q is ...')
percnm = prctile(qsortnm,5)

figure(8)
yvals = 0:0.005:14;
nbins = linspace(150.15,150.42,28);
[countnm, binvals] = hist(qnm, nbins);
ncountnm = 100*countnm/sum(countnm);
bar(binvals, ncountnm, 'barwidth', 1);
xlabel('5th Percentile');
ylabel('Percentage');
hold on;
plot(percnm,yvals,'-k');

figure(9)
plot(xnm,fxqnm1);

%-----
%Scatter Plot. Construction of a two-sided (alpha, delta) tolerance
%interval, see Wolfinger p28

QU = [ ];
QL = [ ];

```

```

for i = 1:length(deltaE)
    var3 = (deltaE(i) + k*deltaA(i))/(k*b);
    UU = U(i);
    ql = UU - 1.96*sqrt(deltaE(i) + deltaA(i));
    qu = UU + 1.96*sqrt(deltaE(i) + deltaA(i));
    QL = [QL ql];
    QU = [QU qu];
end

Q2=[QL;QU]';
BHat=mean(U);
gamma1=0.95;
m=max(size(QU));
p=round(((1-gamma1)*m))
p1=round(((1-gamma1)*m)-1)

x1=min(QU);
max1=max(QU);
min1=min(QU);
y1=-x1+2*BHat;
y2=-max1+2*BHat;

X=[(x1-0.5) max1];Y=[(y1+0.5) y2];
figure(10)
plot(QU,QL, '. ')
hold on
plot(X,Y);

%Check the number of points
interval=max1:-0.01:x1;
i=length(interval);
j=0;
st=0;
%gamma1=0.95;
while (st<=p1)&(j<i),
    j=j+1;
    y11=-interval(1,j)+2*BHat;
    X1=[interval(1,j),min(QU)+1];
    %X1=[interval(1,j),min(QU)+18];
    Y1=[y11,y11];
    %plot(X1,Y1);

    X2=[interval(1,j),interval(1,j)];
    Y2=[y11,min1-1];
    %Y2=[y1,min1-22];

    O=find(Q2(:,1)<=y11 & Q2(:,2)>=interval(1,j));
    st=size(O);
end
st
plot(X1,Y1)
hold on
plot(X2,Y2)
hold on
disp('The Two Sided Tolerance Interval Is ...')
X=[X(1,1) X1(1,1)];
Y=[Y(1,1) Y1(1,1)];
[X1(1,1) Y1(1,1)]
text(151.01,150.23,'150.2482');
text(150.78,149.96,'150.7676');
text(150.6,150.44,'QL = -QU+2BHat')
xlabel('0.95th Quantile');
ylabel('0.05th Quantile');

%-----
%Plot similar to Wolfinger (1998)p29 plot

```

```
c = [ ];
for i = 1:length(deltaE)
    var3 = (deltaE(i) + k*deltaA(i))/(k*b);
    UU = U(i);
    temp = (150.30 - UU)/sqrt(deltaE(i) + deltaA(i));
    %c1 = 1 - normcdf(temp,0,1);
    c1 = normcdf(temp,0,1);
    c = [c c1];
end

AA = sort(c);
disp('The Fixed in Advanced Tolerance Interval for S = 150.30 IS ...')
fsort=AA;
percf1 = prctile(fsort,2.5)
percf2 = prctile(fsort,97.5)

figure(11)
%nbins = linspace(0.7,1,35);
nbins = linspace(0,0.3,35);
[count, binvals] = hist(c, nbins);
ncount = 100*count/sum(count);
bar(binvals, ncount, 'barwidth', 1);
xlabel('Content');
ylabel('Percentage');
size(deltaA)
```

Selective Algorithms from Chapter 4

4.1 Simulation Study: Results Given in Table 4.1. Obtained using Method 1

```

clc
clear

%Simulation Study Method 1

randn('seed',sum(100*clock));

rho = [0 0.1 0.25 0.5 0.75 0.85 0.95];
sigma_a2 = rho;
sigma_e2 = 1 - sigma_a2;
I = 3;
J = 2;
b = I;
k = J;
v1 = b*(k-1);
v2 = b-1;

%simulation of v1m1,v2m2,y_
SI = [];
AVG = [];
COUNT = [];

for i = 1:7
    Ii = [];
    count = 0;

    for j = 1:1000
        j

        x1 = randn(1,v1);
        tao = sum(x1.^2);
        v1m1 = tao*sigma_e2(i);
        x = randn(1,v2);
        delta =sum(x.^2);
        v2m2 = delta*(sigma_e2(i) + k*sigma_a2(i));
        Vary_ = (sigma_e2(i) + k*sigma_a2(i))/(b*k);
        Y_ = sqrt(Vary_)*randn(1,1);
        S_12 = [];

        for z = 1:1000
            x1 = randn(1,v1);
            I = sum(x1.^2);
            sigma_e = v1m1/I;
            x = randn(1,v2);
            delta =sum(x.^2);
            sigma_12 = v2m2/delta;
            S_12 = [S_12 sigma_12];
        end

        Q = [];

        for l = 1:1000
            Emu = Y_;
            Varmu = (S_12(l))/(b*k);
            mu = sqrt(Varmu)*randn(1,1)+Emu;
            q = mu + 1.282*sqrt((S_12(l))/k);
            Q = [Q q];
        end

        temp= sort(Q);
        temp2 = temp(950);
        temp3 = temp(1);
        leng = temp2 - temp3;
        Ii = [Ii leng];

        if temp3 < (1.282*sqrt(((1-rho(i))+k*rho(i))/k));
            if temp2 > (1.282*sqrt(((1-rho(i))+k*rho(i))/k));
                count = count + 1;
            end
        end
    end
end

```

```
end

avg = sum(Ii)/1000;
si = sqrt(1/999*(sum(Ii.^2) - (sum(Ii))^2/1000));
SI = [SI si];
AVG = [AVG avg];
COUNT = [COUNT count];

end

data1 = [COUNT(1)/1000 AVG(1) SI(1) COUNT(2)/1000 AVG(2) SI(2)]'
data2 = [COUNT(3)/1000 AVG(3) SI(3) COUNT(4)/1000 AVG(4) SI(4)]'
data3 = [COUNT(5)/1000 AVG(5) SI(5) COUNT(6)/1000 AVG(6) SI(6)]'
data4 = [COUNT(7)/1000 AVG(7) SI(7)]'
```

4.2 Simulation Study: Results Given in Table 4.1. Obtained using Method 2

```

clc
clear

%Simulation study Method 2

randn('seed',sum(100*clock));

rho = [0 0.1 0.25 0.5 0.75 0.85 0.95];
sigma_a2 = rho;
sigma_e2 = 1 - sigma_a2;
I = 3;
J = 2;
b = I;
k = J;
v1 = b*(k-1);
v2 = b-1;

%simulation of v1m1,v2m2,y_
SI = [];
AVG = [];
COUNT = [];

for i = 1:7
    Ii = [];
    count = 0;

    for j = 1:1000
        j

        x1 = randn(1,v1);
        tao = sum(x1.^2);
        v1m1 = tao*sigma_e2(i);
        x = randn(1,v2);
        delta =sum(x.^2);
        v2m2 = delta*(sigma_e2(i) + k*sigma_a2(i));
        Vary_ = (sigma_e2(i) + k*sigma_a2(i))/(b*k);
        y_ = sqrt(Vary_)*randn(1,1);
        S_E = [];
        S_A = [];

        for z = 1:40000000
            x1 = randn(1,v1);
            I = sum(x1.^2);
            sigma_e = v1m1/I;
            x = randn(1,v2);
            delta =sum(x.^2);
            sigma_12 = v2m2/delta;
            sigma_a = (sigma_12-sigma_e)/k;

            if sigma_a > 0
                S_A = [S_A sigma_12];
            end

            if length(S_A) == 1000
                break
            end

        end

        Q = [];

        for l = 1:1000
            Emu = y_;
            Varmu = (S_A(l))/(b*k);
            mu = sqrt(Varmu)*randn(1,1)+Emu;
            q = mu + 1.282*sqrt((S_A(l))/k);
            Q = [Q q];
        end

        temp= sort(Q);
        temp2 = temp(950);
    end
end

```

```
temp3 = temp(1);
leng = temp2 - temp3;
Ii = [Ii leng];

if temp3 < (1.282*sqrt(((1-rho(i))+k*rho(i))/k));
    if temp2 > (1.282*sqrt(((1-rho(i))+k*rho(i))/k));
        count = count + 1;
    end
end

end

end

avg = sum(Ii)/1000;
si = sqrt(1/999*(sum(Ii.^2) - (sum(Ii))^2/1000));
SI = [SI si]
AVG = [AVG avg]
COUNT = [COUNT count]

end

data1 = [COUNT(1)/1000 AVG(1) SI(1) COUNT(2)/1000 AVG(2) SI(2)]'
data2 = [COUNT(3)/1000 AVG(3) SI(3) COUNT(4)/1000 AVG(4) SI(4)]'
data3 = [COUNT(5)/1000 AVG(5) SI(5) COUNT(6)/1000 AVG(6) SI(6)]'
data4 = [COUNT(7)/1000 AVG(7) SI(7)]'
```

4.3 Simulation of Variance Components and Mean. Obtained using Method 1

```

clear
clc

randn('seed',sum(100*clock));

%Chapter 4 Similation of variance components for Random Effects model using Method 1

%ii=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
ii=10000;

format
b = 15;
k = 10;
V1 = b*(k-1);
V2 = b-1;
dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

Ybar = mean(dataY,2);
YbarBig = mean(Ybar);

%-----
%Calculate v1m1 & v2m2

v1m1 = 0;
v2m2 = 0;
for i = 1:b;
    for j = 1:k;
        v1m1 = v1m1 + (dataY(i,j) - Ybar(i))^2;
    end
end
for i = 1:b;
    v2m2 = v2m2 + k*(Ybar(i) - YbarBig)^2;
end

%-----
V1M1=v1m1
V2M2=v2m2
%-----

SIG12=[];
MU=[];

T1=mean(dataY,2);
Y=mean(T1)

for p=1:ii
    % p

    % Simulate sigma squared 12 ((SigmaE Sqr) + k (SigmaA Sqr))
    Z1=randn([V2,1]);
    Z2=(Z1.^2);
    Z=sum(Z2);

```



```
S12=V2M2/Z;  
SIG12=[SIG12 S12];  
  
% Simulate Mu  
Z3=randn([1,1]);  
Z4=sqrt(S12/(b*k));  
MU1=Z3*Z4 + Y;  
MU=[MU MU1];  
  
end;  
  
figure(1)  
hist(MU,20)  
  
figure(2)  
hist(SIG12,20)  
  
save C:\Compres1 SIG12 MU
```

4.4 Simulation of Variance Components and Mean. Obtained using Method 2

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Simulation of variance components for Random Effects Model. Using Method 2.

%ii=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
ii=10000;

format
b = 15;
k = 10;
V1 = b*(k-1);
V2 = b-1;
dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

Ybar = mean(dataY,2);
YbarBig = mean(Ybar);

%-----
%Calculate v1m1 & v2m2

v1m1 = 0;
v2m2 = 0;
for i = 1:b;
    for j = 1:k;
        v1m1 = v1m1 + (dataY(i,j) - Ybar(i))^2;
    end
end
for i = 1:b;
    v2m2 = v2m2 + k*(Ybar(i) - YbarBig)^2;
end

%-----
V1M1=v1m1
V2M2=v2m2
%-----

SIG11=[];
SIG12=[];
SIG22=[];
MU=[];
I=[];

T1=mean(dataY,2);
Y=mean(T1)

for p=1:ii
    p

```

```

% Simulate SigmaE Sqr

Z1=randn([V1,1]);
Z2=(Z1.^2);
R=sum(Z2);
S11=(V1M1/R);
S1=S11;

% Simulate (SigmaE Sqr) + k (SigmaA Sqr)

Z3=randn([V2,1]);
Z4=(Z3.^2);
Z=sum(Z4);
S12=V2M2/Z;

% Simulate SigmaA Sqr

S22=(S12-S11)/k;
S2=S22;

% Simulate Mu

Z5=randn([1,1]);
Z6=sqrt((S11+(k*S22))/(b*k));
MU1=Z5*Z6 + Y;

if S12 > S11
    if S1 > 0
        if S2 > 0

            SIG11=[SIG11 S11];
            SIG12=[SIG12 S12];
            SIG22=[SIG22 S22];
            MU=[MU MU1];
            I=[I p];

        end;
    end;
end;

save C:\Compres2 SIG11 SIG12 SIG22 MU I

```

4.5 Simulation of Marginal Posterior Distribution of q . Obtained using Method 1

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Plot the distribution of q for the drugs data by using the Rau
%Blackwell Method. Method 1

load c:\Compres1

dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

T1=mean(dataY,2);
Y=mean(T1);

X1=[];
KK=0;
ZB=1.645;
b=15;
k=10;

B=Y-0.5:0.01:Y+0.1;

D=[zeros(1,(max(size(B))))];

for i=1:max(size(SIG12))
    i

    mu1=MU(i);
    s12=SIG12(i);

    GEMID=Y-(ZB*sqrt(s12/k));
    VARIAN=s12/(b*k);
    A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));
    X11=(randn(1,1)*sqrt(VARIAN))+GEMID;

    D=[D+A1];
    X1=[X1 X11];
    KK=KK+1;

end;

figure(1)
E1=D'/KK;
G1=E1/(sum(D));
plot(B,E1);

%save c:\Resmeth1 X1 E1 G1

```

4.6 Simulation of Marginal Posterior Distribution of q . Obtained using Method 2

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Plot the distribution of q for the drugs data by using the Rau
%Blackwell Method. Method 2

load c:\Compres2

dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

T1=mean(dataY,2);
Y=mean(T1);

X2=[];
A=[];
KK=0;
ZB=1.645;
b=15;
k=10;

B=Y-0.5:0.01:Y+0.1;

D=[zeros(1,(max(size(B))))];

for i=1:max(size(I))
    i

    mu1=MU(i);
    s11=SIG11(i);
    s22=SIG22(i);

    GEMID=Y-(ZB*sqrt((s11+(k*s22))/k));
    VARIAN=(s11+(k*s22))/(b*k);
    A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));
    X22=(randn(1,1)*sqrt(VARIAN))+GEMID;

    D=[D+A1];
    X2=[X2 X22];
    KK=KK+1;

end;

E2=D'/KK;
G2=E2/(sum(D));
plot(B,E2);

%save c:\Resmeth2 X2 E2 G2

```

4.7 Simulation of the Predictive Distribution and Construction of the Two - Sided (α, δ) Tolerance Interval

```

clc;
clear;

randn('seed', sum(100*clock));

%Averages of observations from new or unknown batches. Plot the predictive density
%and construction of the alfa delta two sided tolerance interval.

%-----
format
b = 15;
k = 10;
v1 = b*(k-1);
v2 = b-1;
dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

Ybar = mean(dataY,2);
YbarBig = mean(Ybar);

%-----
%calculate v1m1 & v2m2

v1m1 = 0;
v2m2 = 0;
for i = 1:b;
    for j = 1:k;
        v1m1 = v1m1 + (dataY(i,j) - Ybar(i))^2;
    end
end
for i = 1:b;
    v2m2 = v2m2 + k*(Ybar(i) - YbarBig)^2;
end

%-----
v1m1
v2m2
%-----

delta12 = [ ];
U = [ ];
var12 = [ ];

for i = 1:10000
    i
    z1 = randn(1,v2);
    chi2v2 = z1*z1';
    delta121 = v2m2/chi2v2;
    z2 = randn(1,1);
    var1 = (delta121)/(k*b);
    var121=(delta121)/(k);
    U1 = (z2*sqrt(var1)) + YbarBig;

    delta12 = [delta12 delta121];
    U = [U U1];
    var12 = [var12 var121];
end

```

```

%-----
%Predictive density. Method 1

Yf = [ ];
var5 = [ ];
fxael = 0;
xml = 150:0.0001:151;
yvalsm1 = 0:0.0005:4;

for i = 1:length(U)
    z4 = randn(1,1);
    var3 = (delta12(i))/(k);
    UU = U(i);
    Yf1 = z4*sqrt(var3) + UU;
    Yf = [Yf Yf1];
    var5 = [var5 var3];
    fxael = fxael + normpdf(xml,UU,sqrt(var3));
end

fxae = fxael/length(delta12);
YfSort = sort(Yf);
disp('Lower and Upper Tolerance Limits for the Predictive Density Using Method 1')
percm11 = prctile(YfSort,2.5)
percm12 = prctile(YfSort,97.5)
perctest = prctile(YfSort,5)

figure(1)
plot(xml,fxae);
hold on;
plot(percm11,yvalsm1,'-k');
plot(percm12,yvalsm1,'-k');

%-----
%Construction of the two sided alfa delta tolerance interval

QU = [ ];
QL = [ ];
for i = 1:length(delta12)
    var3 = (delta12(i))/(k);
    UU = U(i);
    ql = UU - 1.96*sqrt(var3);
    qu = UU + 1.96*sqrt(var3);
    QL = [QL ql];
    QU = [QU qu];
end

Q2=[QL;QU]';
BHat=mean(U);
gammal=0.95;
m=max(size(QU));
p=round(((1-gammal)*m))
p1=round(((1-gammal)*m)-1)

%Reference lyn
x1=min(QU);
max1=max(QU);
min1=min(QU);
y1=-x1+2*BHat;
y2=-max1+2*BHat;

X=[(x1-0.5) max1];Y=[(y1+0.5) y2];
figure(2)
plot(QU,QL, '.')
hold on
plot(X,Y);

%Check the number of points
interval=max1:-0.01:x1;
i=length(interval);

```

```

j=0;
st=0;
while (st<=p1)&(j<i),
    j=j+1;
    y11=-interval(1,j)+2*BHat;
    X1=[interval(1,j),min(QU)+1];
    %X1=[interval(1,j),min(QU)+18];
    Y1=[y11,y11];
    %plot(X1,Y1);

    X2=[interval(1,j),interval(1,j)];
    Y2=[y11,min1-1];

    O=find(Q2(:,1)<=y11 & Q2(:,2)>=interval(1,j));
    st=size(O);
end
st
plot(X1,Y1)
hold on
plot(X2,Y2)
hold on
disp('The Two Sided Tolerance Interval Is ...')
X=[X(1,1) X1(1,1)];
Y=[Y(1,1) Y1(1,1)];
[X1(1,1) Y1(1,1)]
text(151.01,150.23,'150.2482');
text(150.78,149.96,'150.7676');
text(150.6,150.44,'QL = -QU+2BHat')
xlabel('0.95th Quantile');
ylabel('0.05th Quantile');
%-----

```


4.8 Simulation of the Content of the Fixed - in - Advance Tolerance Interval

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Determine the content of the Fixed in advance tolerance interval

load c:\Compres1

dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

T1=mean(dataY,2);
Y=mean(T1);

THETA=[];
Z=[];
CUM_STD=[];
CUM_STD1=[];
b=15;
k=10;

%S=input('WHAT IS THE PRESELECTED LIMIT S? ... ');
S=150.30;

for i=1:max(size(MU))
    i

    GEMID=MU(i);
    VARIAN=SIG12(i)/k;
    Z11=(S-GEMID)/sqrt(VARIAN);

    V=Z11;

    if Z11<=0
        cons=(1/sqrt(2*pi));
        h=V/1500;

        if h<0

            h=-h;
            kk=-10:h:V;
            l=exp((-1/2)*(kk.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

        else

            kk=-10:h:V;
            l=exp((-1/2)*(kk.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

        end;

    end;

```

```
else
    cons=(1/sqrt(2*pi));
    h=V/1500;
    kk=-10:h:V;
    l=exp((-1/2)*(kk.^2));
    std_norm=cons*l;
    cum_std=h*(sum(std_norm));
    cum_std1=1-cum_std;

end;

Z=[Z Z11];
CUM_STD=[CUM_STD cum_std];      %Percentage of values less than S
CUM_STD1=[CUM_STD1 cum_std1];  %Percentage of values greater than S

end;

figure(1)
hist(CUM_STD,30);
AA=sort(CUM_STD);
LOWERF=AA(250)
UPPERF=AA(9750)

figure(2)
hist(CUM_STD1,30);
```

4.9 Algorithm for Determining the Fixed - in - advance Tolerance Interval using the Weighted Monte Carlo Method

```

clear
clc
format long

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Determine the content of the Fixed in advance tolerance interval
%using the weighted Monte Carlo method

load c:\Compres1

dataY = [150.52 150.39 150.31 150.49 150.47 150.67 150.17 150.45 150.42 150.37
150.35 150.47 150.72 150.56 150.53 150.62 150.60 150.52 150.51 150.63
150.48 150.79 150.63 150.46 150.71 150.67 150.70 150.48 150.48 150.58
150.41 150.45 150.40 150.33 150.24 150.39 150.28 150.36 150.27 150.33
150.58 150.54 150.30 150.54 150.50 150.32 150.58 150.46 150.41 150.49
150.49 150.83 150.66 150.63 150.72 150.79 150.64 150.62 150.71 150.73
150.33 150.44 150.48 150.34 150.50 150.42 150.37 150.54 150.39 150.52
150.39 150.52 150.35 150.52 150.47 150.54 150.51 150.37 150.54 150.53
150.64 150.78 150.51 150.69 150.51 150.47 150.60 150.50 150.69 150.72
150.61 150.49 150.60 150.50 150.68 150.56 150.59 150.73 150.62 150.62
150.48 150.25 150.49 150.43 150.40 150.44 150.31 150.36 150.30 150.40
150.35 150.41 150.36 150.39 150.34 150.37 150.51 150.32 150.25 150.32
150.54 150.67 150.57 150.45 150.57 150.48 150.39 150.38 150.67 150.42
150.41 150.54 150.57 150.73 150.47 150.72 150.49 150.66 150.58
150.60 150.45 150.66 150.72 150.45 150.51 150.69 150.62 150.55 150.45];

T1=mean(dataY,2);
Y=mean(T1);
THETA=[];
Z=[];
CUM_STD=[];
CUM_STD1=[];
W=[];
b=15;
k=10;
IIL=[];
IIU=[];
POSLL=0;
POSUL=0;
POSLL1=[];
POSUL1=[];
%S=input('WHAT IS THE PRESELECTED LIMIT S? ... ');
S=150.30;

for i=1:max(size(MU))
    i
    GEMID=MU(i);
    VARIAN=SIG12(i)/k;
    Z11=(S-GEMID)/sqrt(VARIAN);
    V=Z11;

    if Z11<=0
        cons=(1/sqrt(2*pi));
        h=V/1500;

        if h<0

            h=-h;
            kk=-10:h:V;
            l=exp((-1/2)*(kk.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

        else

            kk=-10:h:V;
            l=exp((-1/2)*(kk.^2));
            std_norm=cons*l;

```

```

        cum_std=h*(sum(std_norm));
        cum_std1=1-cum_std;

    end;

else

    cons=(1/sqrt(2*pi));
    h=V/1500;
    kk=-10:h:V;
    l=exp((-1/2)*(kk.^2));
    std_norm=cons*l;
    cum_std=h*(sum(std_norm));
    cum_std1=1-cum_std;

end;

W1=SIG12(i)^(-0.5);
W2=(1+((k*(S-MU(i))^2)/(2*(SIG12(i)))))^(-0.5);
W3=W1*W2;
W=[W W3];
Z=[Z Z11];
CUM_STD=[CUM_STD cum_std];
CUM_STD1=[CUM_STD1 cum_std1];

end;

WEIGHTS=W./sum(W);
%CW1=[CUM_STD1;WEIGHTS];
CW1=[CUM_STD;WEIGHTS];
CW=CW1';
CWS=sortrows(CW,1);

C1=CWS(:,1);
C2=CWS(:,2);

for iil=1:max(size(C2))
    iil

    POSLL=POSLL+C2(iil);

    if POSLL <= 0.025

        IIL=[IIL iil];
        POSLL1=[POSLL1 POSLL];

    end;

    if POSLL <= 0.975

        IIU=[IIU iil];
        POSUL1=[POSUL1 POSLL];

    end;

end;

disp('Weighted Monte Carlo 95% Credibility Interval of the content')
ll=max(size(IIL))
mll=max(POSLL1)
ul=max(size(IIU))
mul=max(POSUL1)

%Importance Sampling Interval
LOWERLIMIT=C1(ll)
UPPERLIMIT=C1(ul)

%Ordinary Monte Carlo Interval
disp('Ordinary Monte Carlo 95% Credibility Interval of the content')
LOWER1=C1(250)
UPPER1=C1(9750)

```



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LAMBDA1= [5
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```

5.2 Plot the Unknown Distribution of ν and the Normal Distribution

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Plot unknown ditribution of the degrees of freedom and the normal
%distribution.
%The subprogram vectors has to be included. See below.

Vectors

Ytr=[206.0000 258.0000
181.0000 197.0000
185.0000 162.0000
195.0000 195.0000
170.5000 143.8000
193.8000 224.8000
244.8000 217.0000
191.5000 196.8000
209.3000 189.5000
134.5000 143.8000
223.8000 195.5000
103.0000 129.3000
99.7000 201.8000
137.5000 119.8000
144.5000 130.0000
159.0000 166.5000
140.5000 138.0000
207.0000 230.0000
195.5000 190.5000
142.3000 163.8000
74.3000 86.5000
439.5000 211.5000
130.3000 114.0000
99.7000 201.8000];

THETA1=175.37;
SIG11=1721.3285;
SIG22=1693.6653;
I=24;
J=2;
v=15;
xsi=0.1;

LAMBDA1=[];
f=[];
V=[];
GG=[];
ZZZ=[];
XX=[];
NU=[];

Lam = sum(randn(v,I*J).^2);
H = sparse(1:I*J,1:I*J,Lam);
rows1 = 1:I*J;
cols1 = ones(J,1) * [1:I];
Z = (sparse(rows1,cols1,ones(I*J,1),I*J,I));

for k=1:100
    k

    for kk=1:4

        % Simulate Lambda

        LAMBDA1=[];

        for i = 1:I
            for j = 1:J
                Lam(i,j) = sum(randn(v+1,1).^2) / (v + (1/SIG11)*(Ytr(i,j) - THETA1-U1(i))^2);
            end
        end
    end
end

```



```

        LAMBDA1=[LAMBDA1 Lam(i,j)];
    end
end

%Simulation of v

f=[];
V=[];

for vv=2.01:0.01:5.0;
    cons1=vv^vv;
    cons2=vv^I;
    cons3=2^vv;
    cons4=2^I;
    cons5=(gamma(vv/2))^(I*J);
    cons6=cons2/(cons4*cons5);
    cons=(cons1/cons3)*cons6;
    LAMB=prod(LAMBDA1.^0.5*vv);
    E1=LAMBDA1;
    E2=(sum(E1)+xsi)/2;
    E3=-vv*E2;
    E=exp(E3);
    F=cons*LAMB*E;
    f=[f,F];
    V=[V,vv];
end;

F=sum(f);
FF=f/F;
MEANV=V*FF';
VV=V.^2;
VARV=(VV*FF')-(MEANV^2);
LAMBDA1=[LAMBDA1 LAMBDA1'];

ZZ=randn([1,1]);
NU1=round((sqrt(VARV)*ZZ)+MEANV);

end;

NU=[NU NU1];

end;

for xx=2.01:0.01:5.0
    ZZ1=(1/(sqrt(2*pi*VARV)))*exp(-0.5*((xx-MEANV)/sqrt(VARV))^2);
    ZZZ=[ZZZ,ZZ1];
    XX=[XX,xx];
end;

zz=sum(ZZZ);
STAND=ZZZ/zz;

figure(1)
plot(V,FF);

figure(2)
plot(V,FF,'--');
hold on
plot(XX,STAND);

```

5.3 Algorithm for Implementing the Gibbs Sampler: Burn - in Period

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Do simulations for burn-in period for Gibbs sampling for errors
%assumed to follow a t-distribution. Degrees of freedom simulated.
%Subprogram Vectors has to be included. See below

Vectors

Ytr=[206.0000 258.0000
      181.0000 197.0000
      185.0000 162.0000
      195.0000 195.0000
      170.5000 143.8000
      193.8000 224.8000
      244.8000 217.0000
      191.5000 196.8000
      209.3000 189.5000
      134.5000 143.8000
      223.8000 195.5000
      103.0000 129.3000
      99.7000 201.8000
      137.5000 119.8000
      144.5000 130.0000
      159.0000 166.5000
      140.5000 138.0000
      207.0000 230.0000
      195.5000 190.5000
      142.3000 163.8000
      74.3000 86.5000
      439.5000 211.5000
      130.3000 114.0000
      99.7000 201.8000];

SIG11=1721.3285;
SIG22=1693.6653;
I=24;
J=2;
v=5;
xsi=0.1;
THETAB=[];
SIG1B=[];
SIG2B=[];
UB=[];
LAMBDA B=[];
f=[];
V=[];
NU=[];

Lam = sum(randn(v,I*J).^2);
H = sparse(1:I*J,1:I*J,Lam);
rows1 = 1:I*J;
cols1 = ones(J,1) * [1:I];
Z = (sparse(rows1,cols1,ones(I*J,1),I*J,I));

for k=1:3000
    k

    for kk=1:10

        % Simulation of Theta
        Z1=randn([1,1]);
        VT=(inv(P'*H*P))*SIG11;
        MT=(inv(P'*H*P))*(P'*H*(Y-Z*U1));
        THETA1=((sqrt(VT))*Z1)+MT;

        % Simulate Sigma 1 Squared
        Z2=randn([I*J,1]);

```

```

T2=sum(Z2.^2);
T2A=(Y-(THETA1*P)-(Z*U1))'*H*(Y-(THETA1*P)-(Z*U1));
SIG11=(T2A/T2);

% Simulate Sigma 2 Squared
Z3=randn([I-2,1]);
T3=sum(Z3.^2);
T3A=U1'*U1;
SIG22=(T3A/T3);

% Simulate U
Z4=randn([I,1]);
VU=inv(((1/SIG11)*Z'*H*Z)+((1/SIG22)*eye(I)));
MU=VU*((1/SIG11)*Z'*H*(Y-(THETA1*P)));
U1=(sqrtm(VU))*Z4+MU;

% Simulate Lambda
LAMBDA1=[];

for i = 1:I
    for j = 1:J
        Lam(i,j) = sum(randn(v+1,1).^2) / (v + (1/SIG11)*(Ytr(i,j) - THETA1-U1(i))^2);
        LAMBDA1=[LAMBDA1 Lam(i,j)];
    end
end

H = sparse(1:I*J,1:I*J,Lam(1:I*J)');

%Simulation of v
f=[];
V=[];

for vv=2.01:0.01:4.5;
    cons1=vv^vv;
    cons2=vv^I;
    cons3=2^vv;
    cons4=2^I;
    cons5=(gamma(vv/2))^(I*J);
    cons6=cons2/(cons4*cons5);
    cons=(cons1/cons3)*cons6;
    LAMB=prod(LAMBDA1.^0.5*vv);
    E1=LAMBDA1;
    E2=(sum(E1)+xsi)/2;
    E3=-vv*E2;
    E=exp(E3);
    F=cons*LAMB*E;
    f=[f,F];
    V=[V,vv];
end;

F=sum(f);
FF=f/F;
MEANV=V*FF';
VV=V.^2;
VARV=(VV*FF')-(MEANV^2);
ZZ=randn([1,1]);
NU1=round((sqrt(VARV)*ZZ)+MEANV);
v=NU1;

end;

THETAB=[THETAB THETA1];
SIG1B=[SIG1B SIG11];
SIG2B=[SIG2B SIG22];
UB=[UB U1];
LAMBDA1B=[LAMBDA1B LAMBDA1'];
NU=[NU NU1];

end;

save c:\Burnrev THETAB SIG1B SIG2B UB LAMBDA1B NU

```

5.4 Algorithm for Implementing the Gibbs Sampler

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

% Do Simulations for Gibbs sampling for errors assumed
% to follow a t-Distribution. Degrees of freedom simulated
% Subprogram Vectors has to be included. See below

Vectors

load c:\Burnrev

Ytr=[206.0000 258.0000
      181.0000 197.0000
      185.0000 162.0000
      195.0000 195.0000
      170.5000 143.8000
      193.8000 224.8000
      244.8000 217.0000
      191.5000 196.8000
      209.3000 189.5000
      134.5000 143.8000
      223.8000 195.5000
      103.0000 129.3000
      99.7000 201.8000
      137.5000 119.8000
      144.5000 130.0000
      159.0000 166.5000
      140.5000 138.0000
      207.0000 230.0000
      195.5000 190.5000
      142.3000 163.8000
      74.3000 86.5000
      439.5000 211.5000
      130.3000 114.0000
      99.7000 201.8000];

B=max(size(THETAB));
THETA1=THETAB(B);
SIG11=SIG1B(B);
SIG22=SIG2B(B);
U1=UB(:,B:B);
LAMBDA1=LAMBDAB(:,B:B);
I=24;
J=2;
v=NU(B);
xsi=0.1;
THETA=[];
SIG1=[];
SIG2=[];
U=[];
LAMBDA=[];
f=[];
V=[];
NUS=[];

Lam = sum(randn(v,I*J).^2);
H = sparse(1:I*J,1:I*J,Lam);
rows1 = 1:I*J;
cols1 = ones(J,1) * [1:I];
Z = (sparse(rows1,cols1,ones(I*J,1),I*J,I));

for k=1:10000
    k
    for kk=1:10

        % Simulation of Mu
        Z1=randn([1,1]);
        VT=(inv(P'*H*P))*SIG11;

```

```

MT=(inv(P'*H*P))*(P'*H*(Y-Z*U1));
THETA1=((sqrt(VT))*Z1)+MT;

% Simulate Sigma E Squared
Z2=randn([I*J,1]);
T2=sum(Z2.^2);
T2A=(Y-(THETA1*P)-(Z*U1))'*H*(Y-(THETA1*P)-(Z*U1));
SIG11=(T2A/T2);

% Simulate Sigma A Squared
Z3=randn([I-2,1]);
T3=sum(Z3.^2);
T3A=U1'*U1;
SIG22=(T3A/T3);

% Simulate A
Z4=randn([I,1]);
VU=inv(((1/SIG11)*Z'*H*Z)+((1/SIG22)*eye(I)));
MU=VU*((1/SIG11)*Z'*H*(Y-(THETA1*P)));
U1=(sqrtm(VU))*Z4)+MU;

% Simulate Lambda
LAMBDA1=[];

for i = 1:I
    for j = 1:J
        Lam(i,j) = sum(randn(v+1,1).^2) / (v + (1/SIG11)*(Ytr(i,j) - THETA1-U1(i))^2);
        LAMBDA1=[LAMBDA1 Lam(i,j)];
    end
end

H = sparse(1:I*J,1:I*J,Lam(1:I*J)');

%Simulation of v
f=[];
V=[];

for vv=2.01:0.01:4.5;
    cons1=vv^vv;
    cons2=vv^I;
    cons3=2^vv;
    cons4=2^I;
    cons5=(gamma(vv/2))^ (I*J);
    cons6=cons2/(cons4*cons5);
    cons=(cons1/cons3)*cons6;
    LAMB=prod(LAMBDA1.^0.5*vv);
    E1=LAMBDA1;
    E2=(sum(E1)+xsi)/2;
    E3=-vv*E2;
    E=exp(E3);
    F=cons*LAMB*E;
    f=[f,F];
    V=[V,vv];
end;

F=sum(f);
FF=f/F;
MEANV=V*FF';
VV=V.^2;
VARV=(VV*FF')-(MEANV^2);
ZZ=randn([1,1]);
NU1=round((sqrt(VARV)*ZZ)+MEANV);
v=NU1;

end;

THETA=[THETA THETA1];
SIG1=[SIG1 SIG11];
SIG2=[SIG2 SIG22];
U=[U U1];
LAMBDA=[LAMBDA LAMBDA1'];

```

```

    NUS=[NUS NU1];

end;

disp('GIVE THE NUMBER OF THE PART FOR WHICH YOU WANT TO OBTAIN THE HISTOGRAM');
part=input('OF THE LAMBDA'S? ... ');

SIGRATIO=[];
LL=[];

figure(1)
hist(THETA,20)
disp('MEAN RESULTS');
mean(THETA)
sort1=sort(THETA);
percml = prctile(sort1,2.5)
percmu = prctile(sort1,97.5)
widthm = percmu - percml

figure(2)
hist(SIG1,20)
disp('SIG1 RESULTS');
median(SIG1)
sort2=sort(SIG1);
percs1l = prctile(sort2,2.5)
percs1u = prctile(sort2,97.5)
widths1 = percs1u - percs1l

figure(3)
hist(SIG2,20)
disp('SIG2 RESULTS');
median(SIG2)
sort3=sort(SIG2);
percs2l = prctile(sort3,2.5)
percs2u = prctile(sort3,97.5)
widths2 = percs2u - percs2l

figure(4)
SIGT=SIG1+SIG2;
hist(SIGT,20)
disp('SIGT RESULTS');
median(SIGT)
sort4=sort(SIGT);
percstl = prctile(sort4,2.5)
percstu = prctile(sort4,97.5)
widthst = percstu - percstl

for ii=1:max(size(SIG1))
    SIGRATIO1=SIG1(ii)/SIGT(ii);
    SIGRATIO=[SIGRATIO SIGRATIO1];
end;

figure(5)
hist(SIGRATIO,20)
disp('SIGRATIO RESULTS');
median(SIGRATIO)
sort5=sort(SIGRATIO);
perc5srl = prctile(sort5,2.5)
perc5sru = prctile(sort5,97.5)
width5sr = perc5sru - perc5srl

s=1:2:min(size(LAMBDA));
ss=2:2:min(size(LAMBDA));

for jj=1:max(size(LAMBDA))
    l1=LAMBDA(:,jj);
    L1=l1(s);
    L2=l1(ss);
    LL1=[L1(part) L2(part)];
    LL=[LL LL1];
end;

```

```
L=LL(:);  
  
figure(6)  
hist(L,20)  
disp('LAMBDA RESULTS');  
median(L)  
sort6=sort(L);  
percsll = prctile(sort6,2.5)  
percslu = prctile(sort6,97.5)  
widthl = percslu - percsll  
  
save c:\Gibbrev THETA SIG1 SIG2 U LAMBDA NUS SIGT SGRATIO L
```

5.5 Simulation of Predictive Part Distribution, α - Quantile q and Construction of Two - Sided (α, δ) Tolerance Interval

```

clear
clc

%Determine the predictive part distribution. Also Determine one and
%two - sided (Alpha,Delta) tolerance intervals

load c:\Gibbrev

X= [];
K=0;
QL1= [];
QU1= [];
KK=0;
QUO= [];
QLT= [];
QUT= [];

Ytr= [206.0000 258.0000
      181.0000 197.0000
      185.0000 162.0000
      195.0000 195.0000
      170.5000 143.8000
      193.8000 224.8000
      244.8000 217.0000
      191.5000 196.8000
      209.3000 189.5000
      134.5000 143.8000
      223.8000 195.5000
      103.0000 129.3000
      99.7000 201.8000
      137.5000 119.8000
      144.5000 130.0000
      159.0000 166.5000
      140.5000 138.0000
      207.0000 230.0000
      195.5000 190.5000
      142.3000 163.8000
      74.3000 86.5000
      439.5000 211.5000
      130.3000 114.0000
      99.7000 201.8000];

T1=mean(Ytr);
Y=mean(T1);
%THETA=THETAB;
%SIG2=SIG2B;

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Determine the alpha expectation tolerance interval

B=mean(Y)-400:0.01:mean(Y)+400;
D=[zeros(1,(max(size(B))))];

Yfm2 = [ ];
varm2 = [ ];
fxaem21 = 0;
xm2 = mean(Y)-400:0.01:mean(Y)+400;
yvalsm2 = 0:0.000005:0.01;

for i=1:max(size(THETA))
    i

    t1=THETA(i);
    s2=SIG2(i);

    GEMID=t1;
    VARIAN=s2;
    A1=(exp(-0.5*((B-GEMID).^2)/VARIAN))/(sqrt(2*pi)*sqrt(VARIAN));

```



```

X1=(randn(1,1)*sqrt(VARIAN))+GEMID;

D=[D+A1];
X=[X X1];
K=K+1;

zm2 = randn(1,1);
%varm21 = ((deltaE(i) + k*deltaA(i))/(k*b)) + (deltaE(i) + deltaA(i));
%avgm2 = YbarBig;
avgm2=t1;
varm21=s2;
Yfm21 = zm2*sqrt(varm21) + avgm2;
Yfm2 = [Yfm2 Yfm21];
varm2 = [varm2 varm21];
fxaem21 = fxaem21 + normpdf(xm2,avgm2,sqrt(varm21));

quo=t1+(1.645*sqrt(s2));
qlt=t1-(1.645*sqrt(s2));
qut=t1+(1.645*sqrt(s2));
QL11=GEMID-(1.96*(sqrt(VARIAN)));
QU11=GEMID+(1.96*(sqrt(VARIAN)));

QUO=[QUO quo];
QL1=[QL1 QL11];
QU1=[QU1 QU11];
QLT=[QLT qlt];
QUT=[QUT qut];

KK=KK+1;

end;

figure(1)
E=D'/K;
G=E/(sum(D));
plot(B,E);

fxaem2 = fxaem21/length(THETA);
Yfm2Sort = sort(Yfm2);
disp('The Lower and Upper Tolerance Limits for the Predictive Density is ...')
AVG=median(Yfm2Sort)
percm21 = prctile(Yfm2Sort,2.5)
percm22 = prctile(Yfm2Sort,97.5)
perctest1 = prctile(Yfm2Sort,5)
disp('Upper Prediction Limit')
perctest2 = prctile(Yfm2Sort,95)

figure(2)
plot(xm2,fxaem2);
hold on;
%plot(percm21,yvalsm2,'-k');
%plot(percm22,yvalsm2,'-k');
%plot(perctest1,yvalsm2,'-k');
plot(perctest2,yvalsm2,'-k');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Determine the alpha quantile

gamma1=0.95;
m=max(size(QUO))

vec=QUO;
figure(3);
hist(QUO,15);
VECQ=sort(QUO);
disp('95% Upper Credibility Limits for q')
CIQ=[VECQ(1,round(m*((1-gamma1)/2))) VECQ(1,round(m*(1-((1-gamma1)/2))))]
BARQ=median(VECQ)
UCIQ=prctile(VECQ,95)

%One sided tolerance interval lower limit

```

```

vec=QL1;
figure(4);
    subplot(2,1,1);
    hist(QL1,15);
VECQL=sort(QL1);
%CQQL=[VECQL(1,round(m*((1-gamma1)/2))) VECQL(1,round(m*(1-((1-gamma1)/2))))]
%BARQL=median(VECQL)

%One sided tolerance interval upper limit
vec=QU1;
    subplot(2,1,2);
    hist(QU1,15);
VECQU=sort(vec);
%CQU=[VECQU(1,round(m*((1-gamma1)/2))) VECQU(1,round(m*(1-((1-gamma1)/2))))]
%BARQU=median(VECQU)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Determine the two sided alpha delta tolerance interval

QU = QUT;
QL = QLT;

disp('Two-sided (0.90,0.95) tolerance interval')
Q2=[QL;QU]';
BHat=mean(THETA);
gamma1=0.95;
m=max(size(QU));
p=round((1-gamma1)*m)
p1=round((1-gamma1)*m)-1

x1=min(QU);
max1=max(QU);
min1=min(QU);
y1=-x1+2*BHat;
y2=-max1+2*BHat;
X=[x1 max1];Y=[y1 y2];
figure(5)
plot(QU,QL,'.')
hold on
plot(X,Y);

%Check the number of points
interval=max1:-0.1:x1;
i=length(interval);
j=0;
st=0;
while (st<=p1)&(j<i),
    j=j+1;
    y11=-interval(1,j)+2*BHat;
    X1=[interval(1,j),min(QU)+350];
    Y1=[y11,y11];

    X2=[interval(1,j),interval(1,j)];
    Y2=[y11,min1-350];

    O=find(Q2(:,1)<=y11 & Q2(:,2)>=interval(1,j));
    st=size(O);
end
st
plot(X1,Y1)
hold on
plot(X2,Y2)
hold on
X=[X(1,1) X1(1,1)];
Y=[Y(1,1) Y1(1,1)];
[X1(1,1) Y1(1,1)]
%text(200,0,'273.6197')
%text(350,100,'77.4677')
text(150.6,150.44,'Reference Line')
width = X1(1,1)-Y1(1,1)

```

5.6 Simulation of the Content of the Fixed - in - Advance Tolerance Interval

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Determine the content of a fixed-in-advance tolerance interval

load c:\Gibbrev

Ytr=[206.0000 258.0000
     181.0000 197.0000
     185.0000 162.0000
     195.0000 195.0000
     170.5000 143.8000
     193.8000 224.8000
     244.8000 217.0000
     191.5000 196.8000
     209.3000 189.5000
     134.5000 143.8000
     223.8000 195.5000
     103.0000 129.3000
     99.7000 201.8000
     137.5000 119.8000
     144.5000 130.0000
     159.0000 166.5000
     140.5000 138.0000
     207.0000 230.0000
     195.5000 190.5000
     142.3000 163.8000
     74.3000 86.5000
     439.5000 211.5000
     130.3000 114.0000
     99.7000 201.8000];

T1=mean(Ytr);
Y=mean(T1);

%nu1=input('WHAT IS THE DEGREES OF FREEDOM NU? ...');

S=input('WHAT IS S THE PRESELECTED LIMIT S? ...');

CUM_STD=[];
CUM_STD1=[];

for i=1:max(size(THETA))
    i

    t1=THETA(i);
    %nu1=NUS(i);
    s1=SIG1(i);
    s2=SIG2(i);

    GEMID=t1;
    VARIAN=s2;

    Z1=(S-GEMID)/sqrt(VARIAN);

    V=Z1;

    if Z1<=0
        cons=(1/sqrt(2*pi));
        h=V/1500;

        if h<0

            h=-h;
            k=-10:h:V;
            l=exp((-1/2)*(k.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));

```

```

        cum_std1=1-cum_std;
        %p11=cum_std;

    else

        k=-10:h:V;
        l=exp((-1/2)*(k.^2));
        std_norm=cons*l;
        cum_std=h*(sum(std_norm));
        cum_std1=1-cum_std;
        %p11=cum_std;

    end;

else

    cons=(1/sqrt(2*pi));
    h=V/1500;
    k=-10:h:V;
    l=exp((-1/2)*(k.^2));
    std_norm=cons*l;
    cum_std=h*(sum(std_norm));
    cum_std1=1-cum_std;
    %p11=1-cum_std;

end;

CUM_STD=[CUM_STD cum_std];           %Percentage of values less than S
CUM_STD1=[CUM_STD1 cum_std1];       %Percentage of values greater than S
%p1=[p1 p11];

end;

%end;

figure(1)
hist(CUM_STD1,20)
AVG = median(CUM_STD1)
LL = prctile(CUM_STD1,2.5)
UL = prctile(CUM_STD1,97.5)
WIDTH = UL-LL

```

5.7 Algorithm for Determining Outlying Part Measurements

```

clear
clc
% Do test for outliers using the lambdas for each part.
% If 0 falls in the interval, the observation is not an outlier.

load C:\Gibbrev

LREP1=[];
LREP2=[];

s=1:2:min(size(LAMBDA));
ss=2:2:min(size(LAMBDA));

for j=1:max(size(LAMBDA))
    l1=LAMBDA(:,j);
    L1=l1(s);
    L2=l1(ss);
    LREP1=[LREP1 L1];
    LREP2=[LREP2 L2];
end;

LOWER1=[];
UPPER1=[];
WIDTH1=[];
LOGLOWER1=[];
LOGUPPER1=[];
LOWER2=[];
UPPER2=[];
WIDTH2=[];
LOGLOWER2=[];
LOGUPPER2=[];

for amore1=1:min(size(LREP1))
    p1=LREP1(amore1,:);
    [Y1 i1]=sort(p1);
    j1=sort(i1);
    number1=max(size(p1));
    UL1=prctile(p1,95);
    LL1=prctile(p1,5);
    wydte1=UL1-LL1;
    LOGLL1=log10(LL1);
    LOGUL1=log10(UL1);
    LOWER1=[LOWER1 LL1];
    UPPER1=[UPPER1 UL1];
    WIDTH1=[WIDTH1 wydte1];
    LOGLOWER1=[LOGLOWER1 LOGLL1];
    LOGUPPER1=[LOGUPPER1 LOGUL1];
end;

Part1=[LOWER1;UPPER1]';
Logpart1=[LOGLOWER1;LOGUPPER1]';

for amore2=1:min(size(LREP2))
    p2=LREP2(amore2,:);
    [Y2 i2]=sort(p2);
    j2=sort(i2);
    number2=max(size(p2));
    UL2=prctile(p2,95);
    LL2=prctile(p2,5);
    wydte2=UL2-LL2;
    LOGLL2=log10(LL2);
    LOGUL2=log10(UL2);
    LOWER2=[LOWER2 LL2];
    UPPER2=[UPPER2 UL2];
    WIDTH2=[WIDTH2 wydte2];
    LOGLOWER2=[LOGLOWER2 LOGLL2];
    LOGUPPER2=[LOGUPPER2 LOGUL2];
end;

Part2=[LOWER2;UPPER2]';
Logpart2=[LOGLOWER2;LOGUPPER2]';

```

Selective Algorithms from Chapter 6

6.1 Simulation of Variance Components and the Mean

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Simulation of variance components for Hierarchical Model

%k=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
k=10000;

%V1M1=input('GIVE THE VALUE FOR V1M1 (SUM OF SQUARES FOR ERROR) ... ');
V1M1=390.6720;

%V2M2=input('GIVE THE VALUE FOR V2M2 (SUM OF SQUARES FOR PACKAGES) ... ');
V2M2=132.6570;

%V3M3=input('GIVE THE VALUE FOR V3M3 (SUM OF SQUARES FOR DAYS) ... ');
V3M3=395.4342;

%dae=input('GIVE THE NUMBER OF DAYS ... ');
dae=15;

%pac=input('GIVE THE NUMBER OF PACKAGES ... ');
pac=8;

%rep=input('GIVE THE NUMBER OF REPLICATIONS ... ');
rep=5;

%GEMID=input('GIVE THE OVERALL MEAN OF THE DATA ... ');
GEMID=20.96;

V1=dae*pac*(rep-1);
V2=dae*(pac-1);
V3=(dae-1);

SIG11=[];
SIG12=[];
SIG13=[];
SIG1=[];
SIG2=[];
SIG3=[];
MU=[];
VARU=[];
I=[];

for i=1:k
    i

    %Simulate sigma squared e
    Z1=randn([V1,1]);
    Z2=(Z1.^2);
    R=sum(Z2);
    S11=(V1M1/R);
    S1=S11;

    %Simulate Theta 1 and ultimately Sigma squared p
    Z3=randn([V2,1]);
    Z4=(Z3.^2);
    P=sum(Z4);
    S12=V2M2/P;
    S2=(S12-S1)/rep;

    %Simulate Theta 0 and ultimately Sigma squared d
    Z5=randn([V3,1]);
    Z6=(Z5.^2);
    D=sum(Z6);
    S13=V3M3/D;
    S3=(S13-S1-(rep*S2))/(pac*rep);

    %Simulate the target value MU

```

```

Z7=randn([1,1]);
VARU1=(S1+(rep*S2)*(pac*rep*S3))/(dae*pac*rep);
MU1=(Z7*sqrt(VARU1))+GEMID;

if S13 > S12 > S11
  if S1 > 0
    if S2 > 0
      if S3 > 0

        SIG11=[SIG11 S11];
        SIG12=[SIG12 S12];
        SIG13=[SIG13 S13];
        SIG1=[SIG1 S1];
        SIG2=[SIG2 S2];
        SIG3=[SIG3 S3];
        MU=[MU MU1];
        VARU=[VARU VARU1];
        I=[I i];

      end;
    end;
  end;
end;

figure(1)
hist(SIG1,20)
asort=sort(SIG1);
SIG1L = prctile(asort,2.5)
SIG1U = prctile(asort,97.5)

figure(2)
hist(SIG2,20)
bsort=sort(SIG2);
SIG2L = prctile(bsort,2.5)
SIG2U = prctile(bsort,97.5)

figure(3)
hist(SIG3,20)
csort=sort(SIG3);
SIG3L = prctile(csort,2.5)
SIG3U = prctile(csort,97.5)

figure(4)
hist(MU,20)
AVG=mean(MU)
dsort=sort(MU);
MUL = prctile(dsort,2.5)
MUU = prctile(dsort,97.5)

save c:\Hiersim SIG11 SIG12 SIG13 SIG1 SIG2 SIG3 I MU VARU
max(size(I))      %Number of positive combinations retained

```

6.2 Simulation of Predictive Part Distributions for a Specific Day and Any Day in Future

```

clear
clc

randn('seed',sum(100*clock));

%Determine predictive densities for: A pecific day in future and any day in future

load c:\Hiersim

%GEMID=input('What is the mean of the data set? ... ');
GEMID=20.96;

%GEMID1=input('What is the mean of the observations for day 10? ... ');
GEMID1=19.85;

%dae=input('GIVE THE NUMBER OF DAYS ... ');
dae=15;

%pac=input('GIVE THE NUMBER OF PACKAGES ... ');
pac=8;

%rep=input('GIVE THE NUMBER OF REPLICATIONS ... ');
rep=5;

%pac1=input('FOR FUTURE PREDICTIONS, HOW MANY PACKAGES WILL BE DRAWN? ... ');
pac1=8;

%rep1=input('FOR FUTURE PREDICTIONS, HOW MANY REPLICATIONS PER PACKAGE? ... ');
rep1=5;

B=15.5:0.001:26.5;
D=[zeros(1,(max(size(B))))];
D1=[zeros(1,(max(size(B))))];
D2=[zeros(1,(max(size(B))))];

XX=[];
XX1=[];
XX2=[];

for j=1:max(size(I))
    j
    S1=SIG1(j);
    S2=SIG2(j);
    S3=SIG3(j);

    %Specific day in futute
    t1=((pac*rep*S3)/(S1+(rep*S2)+(pac*rep*S3))*GEMID1);
    t2=((S1+(rep*S2))/(S1+(rep*S2)+(pac*rep*S3))*GEMID);
    t=t1+t2;
    q11=((S1+(rep1*S2))/(pac1*rep1))+((S1+(rep*S2))/(S1+(rep*S2)+(pac*rep*S3)));
    q12=((dae*pac*rep*S3)+S1+(rep*S2))/(dae*pac*rep);
    q=q11*q12;
    A=exp((-1/2)*((B-t).^2/q))/(sqrt(2*pi)*sqrt(q));
    z1=randn(1,1);
    X=(z1*sqrt(q))+t;
    D=[D+A];
    XX=[XX X];

    %Any day in future
    t2=GEMID;
    q21=((S1+(rep1*S2)+(pac1*rep1*S3))/(pac1*rep1));
    q22=((S1+(rep*S2)+(pac*rep*S3))/(dae*pac*rep));
    q2=q21+q22;
    A2=exp((-1/2)*((B-t2).^2/q2))/(sqrt(2*pi)*sqrt(q2));
    z12=randn(1,1);
    X2=(z12*sqrt(q2))+t2;
    D2=[D2+A2];
    XX2=[XX2 X2];
end

```



```
E=D'/max(size(I));  
E2=D2'/max(size(I));
```

```
figure(1)  
plot(B,E)  
hold on  
plot(B,E2)
```

```
sorta=sort(XX);  
Lower1=prctile(sorta,2.5)  
Upper1=prctile(sorta,97.5)  
MEAN1=mean(sorta)
```

```
sortb=sort(XX2);  
Lower2=prctile(sortb,2.5)  
Upper2=prctile(sortb,97.5)  
MEAN2=mean(sortb)
```

6.3 Simulation Study: Coverage of Predictive Distribution

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Simulation study: Determine coverage of predictive density

%K=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
K=1000;

%GEMID=input('WHAT IS THE MEAN OF THE DATA SET? ... ');
GEMID=20.96;

%dae=input('GIVE THE NUMBER OF DAYS ... ');
dae=15;

%pac=input('GIVE THE NUMBER OF PACKAGES ... ');
pac=8;

%rep=input('GIVE THE NUMBER OF REPLICATIONS ... ');
rep=5;

%pac1=input('FOR FUTURE PREDICTIONS, HOW MANY PACKAGES WILL BE DRAWN? ... ');
pac1=8;

%repl=input('FOR FUTURE PREDICTIONS, HOW MANY REPLICATIONS PER PACKAGE? ... ');
repl=5;

%SI1=input('GIVE THE KNOWN VALUE FOR SIGMA SQUARED 1 ... ');
SI1=0.8139;
%SI2=input('GIVE THE KNOWN VALUE FOR SIGMA SQUARED 2 ... ');
SI2=0.0899;
%SI3=input('GIVE THE KNOWN VALUE FOR SIGMA SQUARED 3 ... ');
SI3=0.6745;

V1=dae*pac*(rep-1);
V2=dae*(pac-1);
V3=(dae-1);
s11=SI1;
s12=SI1+(rep*SI2);
s123=SI1+(rep*SI2)+(pac*rep*SI3);

YSTRS=[];
GEMID1=[];
LOWER=[];
UPPER=[];
C1=[];

for i1=1:K
    i1

    XX2=[];
    THETA0=[];
    J=[];
    %D2=[zeros(1,(max(size(B))))];

    %Simulate a value for Ys*..
    n1=randn(1,1);
    var1=((SI1+(repl*SI2)+(pac1*repl*SI3))/(pac1*repl));
    %var1=0.706085;
    ystrs=(n1*sqrt(var1))+GEMID;

    %Simulate values for MEAN, V1M1, V2M2, V3M3

    n22=randn(1,1);
    var2=((SI1+(rep*SI2)+(pac*rep*SI3))/(dae*pac*rep));
    gemid1=(n22*sqrt(var2))+GEMID;

    n4=randn([V3,1]);
    N4=(n4.^2);

```

```

M3=sum(N4);
V3M3=sil23*M3;

for j1=1:1000

    Z5=randn([V3,1]);
    Z6=(Z5.^2);
    D=sum(Z6);
    THETA01=V3M3/D;
    THETA0=[THETA0 THETA01];
    J=[J j1];

end;

for h=1:max(size(J))

    THETA011=THETA0(h);
    t2=gemid1;
    q2=THETA011*((1/(pac*rep))+(1/(dae*pac*rep)));
    z12=randn(1,1);
    X2=(z12*sqrt(q2))+t2;
    XX2=[XX2 X2];

end;

sortXX2=sort(XX2);
LL=prctile(sortXX2,2.5);
UL=prctile(sortXX2,97.5);

if LL < ystrs
    if ystrs < UL
        C1=[C1 il];
    end;
end;

YSTRS=[YSTRS ystrs];
GEMID1=[GEMID1 gemid1];
LOWER=[LOWER LL];
UPPER=[UPPER UL];
WIDTH=UPPER-LOWER;

end;

disp('THE AVERAGE UPPER LIMIT OF THE 95% CREDIBILITY INTERVAL IS')
disp('')
up=mean(UPPER)
disp('');
disp('THE AVERAGE LOWER LIMIT OF THE 95% CREDIBILITY INTERVAL IS')
disp('')
lo=mean(LOWER)
disp('THE AVERAGE INTERVAL WIDTH IS')
disp('')
width=mean(WIDTH)
disp('THE COVERAGE OF THE INTERVAL IS')
disp('')
coverage=(max(size(C1))/K)*100

```

6.4 Simulation of the α -Quantile q

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Alpha quantile

%k=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
k=10000;

%e=input('GIVE THE VALUE FOR V1M1 (SUM OF SQUARES FOR ERROR) ... ');
e=390.6720;

%p=input('GIVE THE VALUE FOR V2M2 (SUM OF SQUARES FOR PACKAGES) ... ');
p=132.6570;

%d=input('GIVE THE VALUE FOR V3M3 (SUM OF SQUARES FOR DAYS) ... ');
d=395.4342;

%dae=input('GIVE THE NUMBER OF DAYS ... ');
dae=15;

%pac=input('GIVE THE NUMBER OF PACKAGES ... ');
pac=8;

%rep=input('GIVE THE NUMBER OF REPLICATIONS ... ');
rep=5;

%GEMID=input('GIVE THE OVERALL MEAN FOR THE DATA ... ');
GEMID=20.96;

%pac1=input('FOR FUTURE PREDICTIONS, HOW MANY PACKAGES WILL BE DRAWN? ... ');
pac1=8;

%rep1=input('FOR FUTURE PREDICTIONS, HOW MANY REPLICATIONS PER PACKAGE? ... ');
rep1=5;

%GEMID1=input('WHAT IS THE MEAN OF THE SPECIFIC DAY YOU WANT TO PREDICT? ... ');
GEMID1=19.85;

V1M1=e;
V2M2=p;
V3M3=d;
V1=dae*pac*(rep-1);
V2=dae*(pac-1);
V3=(dae-1);

SIG11=[];
SIG12=[];
SIG13=[];
SIG1=[];
SIG2=[];
SIG3=[];
I=[];
THETA=[];
Q=[];
QU=[];
QL=[];
%QR=[];

for i=1:k
    i

    %Simulate sigma squared 1
    Z1=randn([V1,1]);
    Z2=(Z1.^2);
    R=sum(Z2);
    S11=(V1M1/R);
    S1=S11;

```

```

%Simulate sigma squared 12 and ultimately Sigma squared 2
Z3=randn([V2,1]);
Z4=(Z3.^2);
P=sum(Z4);
S12=V2M2/P;
S2=(S12-S1)/rep;

%Simulate sigma squared 13 and ultimately Sigma squared 3
Z5=randn([V3,1]);
Z6=(Z5.^2);
D=sum(Z6);
S13=V3M3/D;
S3=(S13-S1-(rep*S2))/(pac*rep);

if S13 > S12
    if S12 > S11
        if S1 >= 0
            if S2 >= 0
                if S3 >= 0

                    n22=randn(1,1);
                    var2=((S1+(rep*S2)+(pac*rep*S3))/(dae*pac*rep));
                    THETA1=(n22*sqrt(var2))+GEMID;
                    q2=((S1+(rep1*S2)+(pac1*rep1*S3))/(pac1*rep1));
                    Q1=THETA1-(1.282*(sqrt(q2)));
                    QL1=THETA1-(1.645*(sqrt(q2)));
                    QU1=THETA1+(1.645*(sqrt(q2)));

                    SIG11=[SIG11 S11];
                    SIG12=[SIG12 S12];
                    SIG13=[SIG13 S13];
                    SIG1=[SIG1 S1];
                    SIG2=[SIG2 S2];
                    SIG3=[SIG3 S3];
                    I=[I i];
                    THETA=[THETA THETA1];
                    Q=[Q Q1];
                    QL=[QL QL1];
                    QU=[QU QU1];

                end;
            end;
        end;
    end;
end;

end;

m=max(size(I))

gammal=0.95;
%gammal=0.9;
m=max(size(I))

%One sided tolerance interval
vec=Q;
[n,xout]=hist(vec);
dis=xout(9)-xout(8);
opp=sum(n*dis);
std=n/opp;
figure(1);
hist(Q);
Q=sort(Q);
CI=[Q(1,round(m*((1-gammal)/2))) Q(1,round(m*(1-((1-gammal)/2))))]
disp('THE 95% LOWER TOLERANCE LIMIT IS ... ')
LOWER=prctile(Q,5)

%One sided tolerance interval lower limit
vec=QL;
[n,xout]=hist(vec);
dis=xout(9)-xout(8);

```

```
opp=sum(n*dis);
std=n/opp;
figure(2);
subplot(2,1,1);
hist(QL);
QL=sort(vec);
CI=[QL(1,round(m*((1-gamma1)/2))) QL(1,round(m*(1-((1-gamma1)/2))))]

%One sided tolerance interval upper limit
vec=QU;
[n,xout]=hist(vec);
dis=xout(9)-xout(8);
opp=sum(n*dis);
std=n/opp;
subplot(2,1,2);
hist(QU);
QU=sort(vec);
CI=[QU(1,round(m*((1-gamma1)/2))) QU(1,round(m*(1-((1-gamma1)/2))))]
Q2=[QL;QU]';
BHat=mean(THETA)

save C:\Hiertoll1 SIG11 SIG12 SIG13 SIG1 SIG2 SIG3 THETA Q QL QU I
```

6.5 Construction of the (α, δ) Two - Sided Tolerance Interval

```

clear;
clc;

%Two sided tolerance interval

load c:\Hiertoll

Q2=[QL;QU]';
BHat=mean(THETA);
gammal=0.95;
m=max(size(I));
p=round((1-gammal)*m);
p1=round((1-gammal)*m)-1;

%Reference lyn
x1=(max(QU))-min(QU);x2=max(QU);
max1=x2;min1=min(QU);
y1=-x1+2*BHat;
y2=-max1+2*BHat;
X=[x1 max1];Y=[y1 y2];
figure(1)
plot(QU,QL,'.')
hold on
plot(X,Y);
%Check the number of points
interval=max1:-0.005:x1;
i=length(interval);
j=0;
st=0;
%gammal=0.95;
while (st<=p1)&(j<i),
    j=j+1;
    y1=-interval(1,j)+2*BHat;
    X1=[interval(1,j),min(QU)+10];
    Y1=[y1,y1];
    X2=[interval(1,j),interval(1,j)];
    Y2=[y1,min1-22];
    O=find(Q2(:,1)<=y1 & Q2(:,2)>=interval(1,j));
    st=size(O);
end
st
plot(X1,Y1)
hold on
plot(X2,Y2)
hold on
X=[X(1,1) X1(1,1)];
Y=[Y(1,1) Y1(1,1)];
[X1(1,1) Y1(1,1)]
text(21.54,20.54,'19.1246');
text(21.45,20.42,'22.7894');
text(21.3,20.67,'Reference Line');

```

6.6 Simulation of the Content of the Fixed - in - Advance Tolerance Interval

```

clear
clc

randn('seed',sum(100*clock));
rand('seed',sum(100*clock));

%Determine the content of the fixed in advanced tolerance intervals for a given limit
S

%K=input('HOW MANY SIMULATIONS DO YOU WANT TO RUN ... ');
K=10000;

%e=input('GIVE THE VALUE FOR V1M1 (SUM OF SQUARES FOR ERROR) ... ');
e=390.6720;

%p=input('GIVE THE VALUE FOR V2M2 (SUM OF SQUARES FOR PACKAGES) ... ');
p=132.6570;

%d=input('GIVE THE VALUE FOR V3M3 (SUM OF SQUARES FOR DAYS) ... ');
d=395.4342;

%dae=input('GIVE THE NUMBER OF DAYS ... ');
dae=15;

%pac=input('GIVE THE NUMBER OF PACKAGES ... ');
pac=8;

%rep=input('GIVE THE NUMBER OF REPLICATIONS ... ');
rep=5;

%GEMID=input('GIVE THE OVERALL MEAN FOR THE DATA ... ');
GEMID=20.96;

%pac1=input('FOR FUTURE PREDICTIONS, HOW MANY PACKAGES WILL BE DRAWN? ... ');
pac1=8;

%repl=input('FOR FUTURE PREDICTIONS, HOW MANY REPLICATIONS PER PACKAGE? ... ');
repl=5;

%GEMID1=input('WHAT IS THE MEAN OF THE SPECIFIC DAY YOU WANT TO PREDICT? ... ');
GEMID1=19.85;

%S=input('WHAT IS S FOR THE FIXED IN ADVANCE TOLERANCE INTERVAL? ... ');
S=19.0;

V1M1=e;
V2M2=p;
V3M3=d;
V1=dae*pac*(rep-1);
V2=dae*(pac-1);
V3=(dae-1);

SIG11=[];
SIG12=[];
SIG13=[];
SIG1=[];
SIG2=[];
SIG3=[];
I=[];
THETA=[];
Z=[];
CUM_STD=[];
CUM_STD1=[];

for i=1:K
    i

    %Simulate sigma squared e
    Z1=randn([V1,1]);
    Z2=(Z1.^2);
    R=sum(Z2);

```



```

S11=(V1M1/R);
S1=S11;

% Simulate sigma squared 12 and ultimately Sigma squared p
Z3=randn([V2,1]);
Z4=(Z3.^2);
P=sum(Z4);
S12=V2M2/P;
S2=(S12-S1)/rep;

% Simulate sigma squared 13 and ultimately Sigma squared d
Z5=randn([V3,1]);
Z6=(Z5.^2);
D=sum(Z6);
S13=V3M3/D;
S3=(S13-S1-(rep*S2))/(pac*rep);

if S13 > S12
  if S12 > S11
    if S1 >= 0
      if S2 >= 0
        if S3 >= 0

          n22=randn(1,1);
          var2=((S1+(rep*S2)+(pac*rep*S3))/(dae*pac*rep));
          THETA1=(n22*sqrt(var2))+GEMID;
          Z11=((S-THETA1)*(sqrt((pac1*rep1)))/(sqrt(S1+(rep1*S2)+(pac1*rep1*S3)));

          V=Z11;

          if Z11<=0
            cons=(1/sqrt(2*pi));
            h=V/1500;

            if h<0

              h=-h;
              k=-10:h:V;
              l=exp((-1/2)*(k.^2));
              std_norm=cons*l;
              cum_std=h*(sum(std_norm));
              cum_std1=1-cum_std;

            else

              k=-10:h:V;
              l=exp((-1/2)*(k.^2));
              std_norm=cons*l;
              cum_std=h*(sum(std_norm));
              cum_std1=1-cum_std;

            end;

          else

            cons=(1/sqrt(2*pi));
            h=V/1500;
            k=-10:h:V;
            l=exp((-1/2)*(k.^2));
            std_norm=cons*l;
            cum_std=h*(sum(std_norm));
            cum_std1=1-cum_std;

          end;

          SIG11=[SIG11 S11];
          SIG12=[SIG12 S12];
          SIG13=[SIG13 S13];
          SIG1=[SIG1 S1];
          SIG2=[SIG2 S2];
          SIG3=[SIG3 S3];

```

```
        I=[I i];
        THETA=[THETA THETA1];
        Z=[Z Z11];
        CUM_STD=[CUM_STD cum_std];
        CUM_STD1=[CUM_STD1 cum_std1];

        end;
    end;
end;
end;
end;

end;

figure(1)
hist(CUM_STD,35)

figure(2)
hist(CUM_STD1,35)

m=max(size(I))
sorta=sort(CUM_STD);
Lower1=prctile(sorta,2.5)
Upper1=prctile(sorta,97.5)
MEDIAN1=median(sorta)

sortb=sort(CUM_STD1);
Lower2=prctile(sortb,2.5)
Upper2=prctile(sortb,97.5)
MEDIAN2=median(sortb)
```