MODELLING OF GROUNDWATER FLOW WITHIN A LEAKY AQUIFER WITH FRACTAL-FRACTIONAL DIFFERENTIAL OPERATORS

Mahatima Gandi Khoza

Submitted in fulfilment of the requirements for the degree

Magister Scientiae in Geohydrology

In the

Faculty of Natural and Agricultural Sciences

(Institute for Groundwater Studies)

At the

University of the Free State

Supervisor: Prof. Abdon Atangana

September 2020

DECLARATION

I Mahatima Gandi Khoza, hereby declare that the thesis is my work and it has never been submitted to any Institution. I hereby submit my dissertation in fulfilment of the requirement for Magister Science at Free State University, Institute for Groundwater Studies, Faculty of Natural and Agricultural Science, Bloemfontein. This thesis is my own work, I state that all the correct sources have been cited correctly and referenced.

I furthermore *cede* copyright of the dissertation and its contents in favour of the University of the Free State.

In addition, the following article has been submitted and is under review:

Khoza, M. G. and Atangana, A., 2020. Modelling groundwater flow within a leaky aquifer with fractal-fractional differential operators.

ACKNOWLEDGMENTS

First and foremost, I am proud of the effort, devotion, and hard work I have put into this dissertation. I would also like to express my gratitude to the Almighty God for the opportunity, good health, wisdom, and strength towards the completion of my dissertation.

I also take this opportunity to express my gratitude and deepest appreciation to my supervisor, Prof Abdon Atangana for not giving up on me, all the motivation and guidance I received until this moment of completion. This dissertation would have remained a dream if not for your wise words, patience, immense dedication, and knowledge. God bless you.

I would like to extend my heartfelt gratitude to the Department of Water and Sanitation for generously funding this study. I am deeply thankful because without funding this dissertation would have not come to fruition. In addition, I would also like to appreciate Ms. Nobulali Mtikitiki for her wise words, support, and encouragement. Your kindness remains overwhelming, I am truly humbled, this dissertation was not going to be possible without you, I thank you.

I also extend my appreciation to my family, especially my father for the support, guidance, and believing in my vision. You bring the best out of me. To my sister Olympia, you made me realize that I can break through anything that I believe in. Thank you for your support and encouragement. Lastly, to all my friends who supported me through the difficult times, your effort did not go unnoticed, I am in awe.

I specially dedicate this dissertation in memory of my mother Guelheminah Gomez Khoza, this one is for to you, I wish you had lived long enough to see your dominant breed. “Você sempre estará em meu coração. obrigado por me fazer o homem forte que sou hoje”.

Obrigado

ABSTRACT

The major leading problem in groundwater modelling is to produce a suitable model for leaky aquifers. An attempt to answer this, researchers devoted their focus to propose new models for complex real-world physical problems within a leaky aquifer. Their proposed models include the non-local fractional differentiation operator with kernel power law, fractional differentiation and integral with exponential decay, and fractional differentiation and integral with the generalized Mittag-Leffler function. However, for these operators, none of these models were suitable to model anomalous real-world physical problems. In this study, the main purpose of this project is to develop an operator that is suitable to capture groundwater flow within a leaky aquifer, and this operator will also aim to attract non-local problems that display at the same time. The solution to this problem, this project will introduce a new powerful operator called fractal-fractional differentiation and integral which have been awarded by many researchers with both self-similar and memory effects. For solution, we firstly derive solution using Hantush extended equation with respect to time and space. We also extend by employing the Predictor-Corrector method and Atangana Baleanu derivative to obtain numerical solution for non-linear differentiation and integral, and the exact numerical solution for groundwater flow within a leaky aquifer. We further presented the special solution, uniqueness and also the stability analysis. While conducting the research, our finding also indicated that fractal-fractional operators depict real-world physical problems to capture groundwater flow within a leaky aquifer than classical equation.

Keywords: Hantush Model, Fractal-fractional operator, Leaky aquifer, Predictor- Correct method, Atangana Baleanu (AB) derivative, Non-local, Non-linear equations, Self-similar and Memory effects.

LIST OF GREEK NOTATIONS

λ	Lambda
τ	Tau
δ	Zeta
∂	Partial differential
Γ	Gamma
ρ	rho
μ	Mu
Σ	Sigma
Δ	Delta
γ	Gamma
ψ	Psi
Φ	Phi
ε	Epsilon
σ	Sigma
ADE	Advection-dispersion equation
D	Dispersion Coefficient
v	Seepage velocity
R	Retardation factor
t	Time
e	Exponent
1-D	One-dimensional
f	Function
ϵ	Element of
C	Concentration
cos	Cosine
sin	Sine
α	alpha
β	beta
τ	tau
θ	theta
σ	sigma
Σ	Sum
ω	omega
ξ	xi
T	Transmissivity
S	Storativity
t	Time
Ss	Specific storage
Sy	Specific yield
K	Hydraulic conductivity
h	Hydraulic head
r	Radial distance
Q	Discharge

CONTI…OF GREEK NOTATIONS	
q	Darcy flux
n	Porosity
s	Drawdown
v	Velocity
f	Function
b	Aquifer thickness
π	Pie
$\int$	Definite Integral
$t\alpha$	Fractal
∂t	Time derivative
H	Hydraulic head
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
lim	Limit
$z \rightarrow (z\char94 2)+c$	Mandelbrot set
Π	Products
CFFD	Caputo-Fabrizio fractional derivative
CFD	Caputo fractional derivative
PCM	Predictor-Corrector method
EM	Euler's method
ALPM	Langrage polynomial method

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1:
INTRODUCTION

1. BACKGROUND

Water is an organic, transparent close to colorless, tasteless, and odorless chemical substance that mainly covers the earth's atmosphere. Most living organism mainly depends on water to survive. For this reason, mankind has always been concerned about the vulnerability and pollution of water occurring on the surface and subsurface geological formations. In nature, water exists in three forms, namely; gaseous state, solid-state, and liquid state. Water relocates in a continuous water cycle of evapotranspiration (evaporation and transpiration), precipitation, condensation, and runoff reaching the sea. The major water volume in the world is mainly dominated in the ocean which is approximately cover 75% of the world's face (Boonstra and Kselik, 2002) though seawater neither suitable for drinking, industrial, and household purpose. For this reason, humans have always dependent only on freshwater. Globally, water can coexist in two divided types namely; surface water and subsurface water which is groundwater (Atangana, 2016). Surface water includes water that is occurring in rivers, lakes, streams, and oceans while groundwater is water that is occurring in geological formations on earth. Recent studies conducted by Boonstra and Kselik (2002) indicate that approximately 97% of portable water exists in geological formation below the earth's atmosphere. This evidence indicates that groundwater can be explored for water supply and can also be utilized with a connection with surface water.

Groundwater occurs in a geological rock formation called an aquifer. An aquifer is a body of permeable rock that formation that can transmit water or fluid. An aquifer can be classified in four different forms namely; unconfined aquifer, confined aquifer, leaky aquifer, and perched aquifers (Cherry *et al.,* 2004). In most cases, leaky aquifers are more common than purely confined and unconfined aquifers. Confined aquifers and unconfined aquifers when explored can later be classified as leaky aquifer depending on the hydraulic properties of the aquitard (Amanda and Atangana, 2018). In general, in an environmental system for leaky aquifers, the confining layers of the aquifer are occasionally impermeable. Therefore, when the water is being abstracted from the aquifer, the water is not only withdrawn from the aquifer itself but is also withdrawn from the overlying and underlying layers of the aquifer (Kruseman and De Ridder, 1990). Atangana (2016) indicated that it is most probable for leaky aquifers to occur in deep sedimentary basins in a complex aquifer system as indicated in **(Figure 1)**

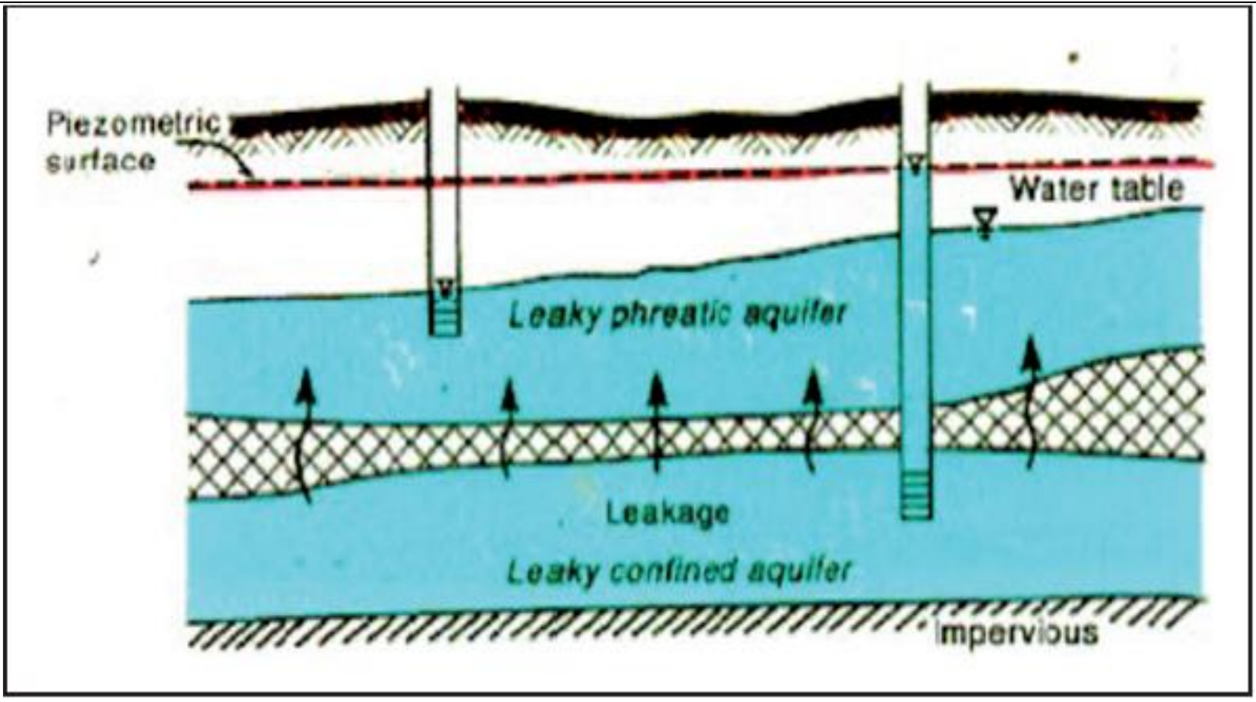

Figure 1: Graphical representation of piezometric surface and water table within a leaky aquifer (Atangana, 2016)

In most cases, one of the leading problems in groundwater modelling is to provide a suitable or correct model to be used for groundwater flow movement in a geological formation. To develop a model for an example; mass, space, and time are measurable objects and forms part of the numerical basis of mathematics. For this matter, the question arises about whether it is most possible to build a mathematical algorithm based on human reasoning and observation that can describe a natural phenomenon? This problem will lead to the derivation of mathematical formula for groundwater flow in a geological formation aquifer. In principle, this mathematical approach could further be applied to study future behavior of various conditions in a given aquifer condition (Atangana, 2016). Furthermore, Hantush was the first scientist to describe the movement of groundwater flow in leaky aquifers using a mathematical formula which was described by Theis (1935), the equation is as indicated below

$$\frac{S}{T}\partial_t\big(p(r,t)\big) = \partial^2\big(p(r,t)\big) + \frac{1}{r}\partial_r\big(p(r,t)\big) + \frac{\big(p(r,t)\big)}{\lambda^2} \tag{1.1}$$

$$\lambda^2 = \frac{B}{K'} \tag{1.2}$$

Initial boundary condition

$$p(r,0) = p_0, \lim_{r\to\infty} p(r,t) = p_0 \tag{1.3}$$

$$Q = 2\pi r_a kd\partial_r(p(r_a, t) \tag{1.4}$$

Where $\big(p(r,t)\big)$ is the change in the water level or drawdown; S defined as storage of the aquifer; T is the transmissivity of the aquifer, K' is the hydraulic conductivity of the main aquifer; B is the thickness of the main confining layers of the aquifer and Q is the discharging rate of the pump.

Atangana (2016) specified that Hantush groundwater flow equation for leaky aquifers is used by many hydrogeologists, however, it is worth nothing, the model cannot satisfy better results due to inconsistency and uncertainty associated with the aquifer, non-local operators, frequency, and the model does not take into account the complexity of the groundwater movement in leaky underground formations. Nonetheless, to accurately enhanced the model that accommodates properties such as frequency, non-local operators, power law, and other modeling tools fractal-fractional operators have to be introduced. The concept of fractal-fractional derivatives and the applications to the field of groundwater has been rewarded and revised by a number of researchers in the past with accurate results (Atangana, 2016). To be exact, non-integer derivatives have produced excellent results to describe the physical system in the world of engineering and science than the usual ordered integer ones. If we transform the concept of leaky aquifer suggested by Hantush to fractal-fractional operator order as:

$$\frac{S}{T}\partial_t^{\alpha}\big(p(r,t)\big) = \partial_{rr}^2\big(p(r,t)\big) + \frac{1}{r}\partial_r\big(p(r,t)\big) + \frac{\big(p(r,t)\big)}{\lambda^2} \tag{1.4}$$

$$0 < \alpha \leq 1$$

Where ∂_t^{α} is defined as the fractional derivative according to Caputo, ∂_t^{α} is defined as

$${}_0^C D_x^{\alpha}\big(g(x)\big) = \frac{1}{\Gamma(p-\alpha)}\int_0^x (x-t)^{p-\alpha-1}\frac{d^p g(t)}{dt^p}dt \tag{1.5}$$

$$p - 1 < \alpha \leq p$$

According to literature, it shows no sign of analytical solution for equation (1.5). In this study, we will consider the work that was presented by Hantush on leaky aquifers and propose analytical solutions by applying a special operator called fractal-fractional derivative.

1.1 CONCEPT OF LEAKY AQUIFERS

Leaky aquifers are known as aquifers that are bounded between confining layers of low permeability. In nature, the aquifer is bounded between the upper and lower limits, with possibly an aquitard boundary at the surface and aquiclude at lower limits. An aquitard is defined as a geological formation that is permeable enough to transmit water in significant quantities when observed over a larger area but its permeability is not sufficient to justify production wells being placed on it, and while an aquiclude is a geological formation that does not transmit fluid. Leaky aquifers are also known as semi-confined aquifers. To add, leaky aquifers arise through vertical leakage taking place due to head difference. To understand the concept of a leaky aquifer, Hantush and Jacob (1955) developed a model which is based on unsteady radial flow for leaky aquifer characterized by the following equation below

$$\frac{\partial^2 h}{\partial^2 r} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{e}{T} = \frac{S}{T}\frac{\partial h}{\partial t} \tag{1.6}$$

Where r is defined as the radial distance from the pumping well and e is the rate of vertical leakage

Hantush and Jacob (1955) conducted a study on leaky a study on leaky aquifers and presented that when leaky aquifers are pumped, the water is removed from both the saturated (wet) overlying aquitard and the aquifer. Due to pumping, a decrease in the piezometric head will occur in the aquifer and this will create a hydraulic gradient in the aquitard resulting in groundwater to move vertically in the aquifer. The hydraulic gradient created within the aquitard is proportional to change in the water table and the piezometric head. On the other hand, Steady-state flow can also be achieved in leaky aquifers through recharging the semi-pervious layer (Hantush and Jacob, 1955). With this said, the discharge rate withdrawn from the aquifer due to pumping is equal to the recharging rate of the vertical flow in the aquifer. With these results, equilibrium will be achieved assuming that the water table is constant. The assumption made by Hantush and Jacob (1955) indicated that hydraulic conductivity is valid if the aquifer exceeds the aquitard, such that the thickness of the aquifer is greater than the thickness of the aquitard (Zhan *et al*., 2003). However, Amanda and Atangana (2018) indicated that the aquifer storage is determined by the overlying aquitard within the leaky aquifer. For aquifer storage, Hantush and Jacob (1955) used both thin and thick aquitard overlying the aquifer within the semi-confined aquifer. Storage was neglected in the thin aquitard while it was considered for the thick aquitard. The results indicated that vertical leakage within the thick aquitards differed as compared to thin aquitards. Therefore, this caused more water to come through the thin aquitard.

Hantush and Jacob conducted an investigation and they indicated that partial differential is usually carried up for radial flow in elastic artesian. Their results showed that vertical leakage is proportional to drawdown. Initially, when the water that is pumped from the elastic storage is proportional to time, these causes more water to leak through the aquitard. Hantush and Jacob in their model took into account the water from aquitard that arises either from storage in the aquitard and or the over/underlying aquifer. Hantush and Jacob (1955) model were built based on unsteady state aquifer in a fully penetrating leaky aquifer under the assumption of homogeneous and constant pumping rate. The model was developed based on the assumption that the main aquifer underlies beneath an aquiclude and the aquitard is overlain by an unconfined aquifer. Later on, Hantush modified the model for leaky aquifer and included the effects of storage capacity where water is abstracted from both the leaky aquifer and the main aquifer. In their study, they noticed that when water is being pumped it produces a cone of depression resulting in vertical leakage into the aquifer. Therefore, the vertical leakage will cause a steady-state to occur because the aquifer will begin to recharge from the top. Moreover, when steady-state is achieved the drawdown in the aquifer with leakage from an aquitard is relative to the hydraulic gradient in all the aquitards.

1.1.1 Hantush and Jacob assumptions for fully penetrating leaky aquifers

- Assumed that the aquifer is leaky and has an infinite apparent radius extent,
- The aquifer has a uniform thickness, isotropic and homogeneous, isotropic. The thickness of the aquifer is not influenced over the area by pumping,
- Initially, the surface of influence is horizontal prior to pumping,
- The water from the well is extracted at a constant rate,
- The well is penetrated fully,
- The water discharged from storage is removed instantly with a decline in the head
- The diameter of the borehole is small such that borehole storage is negligible,
- Seepage over the aquitard layer occurs vertically.

Later on, Hantush and Jacob (1955) presented a solution for leaky aquifer as

$$s = \frac{Q}{4\pi T} W(u, \frac{r}{B}) \tag{1.7}$$

And U is given as

$$u = \frac{r^2 S}{4Tt} \tag{1.8}$$

Where $W(u, \frac{r}{B})$ the well function of the semi-confined aquifer, Q is the rate of pumping in (L^3/day), T is transmissivity (L^2/day), S is well storativity (dimensionless), t is the time elapsed since pumping started, r is the radial distance between the pumping well and observation borehole, s is the drawdown (L) and B is the seepage factor

$$B = \sqrt{\frac{Tb'}{K'}} \tag{1.9}$$

Where b' aquifer thickness in (m) is, K' is the aquitard hydraulic conductivity (L/day). Neumann and Witherspoon 1969 described a mathematical equation for the groundwater flow in the unconfined aquifer for pumping well and they considered storage. Their method determines the hydraulic features of aquitard at small scales of extracting time when the drawdown is neglected in the superimposing aquifer. The Neumann and Witherspoon (1969) method were constructed based on the theory of slightly leaky aquifers were the drawdown in the pumped aquifer is assumed by the Theis equation. The method indicates that flow due to pumping is vertical in the aquitard and horizontal within the layer (Streltrova, 1973). Hantush and Jacob (1955) stated that until a steady state is achieved the water removed from the borehole will continue to leak through the aquitard and these will result in a decrease in storage from the pumped aquifer and aquitard. Therefore, for this assumption, Hantush and Jacob developed methods to analyze unsteady-state flow. These four available methods include; 1) Walton curve-fitting method, 2) Hantush inflection point method which both neglect aquifer storage, 3) Hantush curve-fitting method, and 4) Neumann & Witherspoon ration method which both takes into account storage. Abdal and Ramadurgiah indicated that aquifer storage is assumed to extend laterally to infinity and the flow in aquifer is then considered to be governed by Jacob's theory of linear leakage. Later on, they assumed borehole to be constant and neglected well losses. Chen and Chang (2002) conducted a similar study and indicated that content head parameters are used to determine hydraulic conductivity and storage coefficient.

1.2 FRACTIONAL DERIVATIVE APPROACH

Within the branch of mathematical analysis that studies the several diverse possibilities defining real numbers and complex powers numbers of differential operator D. Fractal differentiation and integration were introduced to connect fractal and fractional calculus to envisage a complex system in the real world. Different types of fractal and fractional differential operators exist to predict real-world physical problems. This includes Riemann-Liouville fractional derivative, Caputo fractional derivative, and Atangana-Baleanu which is considered as mostly applied fractal derivatives for a complex system (Atangana and Bildik, 2013). Fractal-fractional differentiation and integration are distinguished based on their names and association. Riemann-Liouville fractional integration, Hadamard fractional integration, and Atangana-Baleanu fractional integration are associated with fractional integration while Riemann-Liouville fractional derivative, Caputo fractional derivative and Riez derivative associated with fractional derivative. The well-known fractal-fractional derivatives are obtainable below.

1.2.1 Fractional Integrals

1.2.1.1 Riemann-Liouville integral

$$ {}_{a}^{RL}D_{w}^{-\alpha} f(w) = {}_{a}^{RL}I_{w}^{\alpha} f(w) = \frac{1}{\Gamma(\alpha)} \int_{a}^{w} (w-\tau)^{\alpha-1} f(\tau) d\tau \tag{1.10} $$

$$ {}_{a}^{RL}D_{w}^{-\alpha} f(w) = {}_{\alpha}^{RL}I_{w}^{\alpha} f(w) = \frac{1}{\Gamma(\alpha)} \int_{w}^{b} (\tau - w)^{\alpha-1} f(\tau) d\tau \tag{1.11} $$

Where $w > a$, and can later be valid is $w < b$.

1.2.1.2 Atangana-Baleanu fractional integral

$$ {}_{a}^{AB}D_{t}^{-\alpha} h(t) = {}_{a}^{AB}I_{t}^{\alpha} h(t) = \frac{1-\alpha}{AB(\alpha)} h(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} h(\tau) \tag{1.12a} $$

Where $AB(\alpha)$ is a stabilization function such that when $AB(0) = AB(1)$

$$ AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \tag{1.12b} $$

1.2.2 Fractional derivative

1.2.2.1 Riemann-Liouville fractional derivative

$$ {}_{a}D_{t}^{\alpha} f(t) = \frac{d^n}{dt^n} \; {}_{a}D_{t}^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n} \; {}_{a}I_{t}^{n-\alpha} f(t) \tag{1.13} $$

$$ {}_{t}D_{b}^{\alpha} f(t) = \frac{d^n}{dt^n} \; {}_{t}D_{b}^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n} \; {}_{t}I_{b}^{n-\alpha} f(t) \tag{1.14} $$

This deferential operator is computed using n order derivatives and $(n - \alpha)$ order integrals, α is obtained and $n > \alpha$. The derivative is defined by using upper and lower limits

1.2.2.2 Caputo fractional derivative

$$ {}_{a}^{c}D_{t}^{\alpha} f(w) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{w} \frac{f^{(n)}(\tau) d\tau}{(w-\tau)^{\alpha+1-n}} \tag{1.15} $$

Then the Caputo fractional derivative defined as

$$ D^{v} f(w) = \frac{1}{\Gamma(n-v)} \int_{0}^{w} (w-u)^{(n-v-1)} f^{n} u du \tag{1.16} $$

Where $(n - 1) < v < n$

This derivative is advantageous because of the nil constant and the Laplace Transformation. Furthermore, Caputo fractional derivative of dispersed is defined as

$$ {}_{a}^{b}D^{v} p(s) = \int_{a}^{b} s(v) \left[D^{(v)} p(s) \right] dv = \int_{a}^{b} \left[\frac{s(v)}{\Gamma(1-v)} \int_{0}^{w} (s-u)^{-v} p'(u) du \right] dv \tag{1.17} $$

Where $s(v)$ is defined as the weight function and indicates the mathematical memory of Caputo

1.2.2.3 Atangana-Baleanu derivative

$$ {}^{ABC}_{a}D^{t}_{\alpha}f(t) = \frac{AB(\alpha)}{1-\alpha}\int_{\alpha}^{t} f'\,(\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right)d\tau \tag{1.18} $$

And the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is defined as:

$$ {}^{ABC}_{a}D^{t}_{\alpha}f(t) = \frac{AB(\alpha)}{1-\alpha}\frac{d}{dt}\int_{\alpha}^{t} f\,(\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right)d\tau \tag{1.19} $$

1.2.3 Advantage of fractal derivative and fractional derivatives

- Fractal and fractional derivatives are suitable for solving real-world physical problems.
- The operators apply power law to model elastic and homogenous media.
- The operators have the ability to enables both initial conditions in simple definitions and has the ability to present standard initial boundary conditions.
- The derivatives have the ability to model chaos, linear and nonlinear, non-equilibrium and complex phenomena.
- The derivative can be used to model heat flow.
- Fractional integrals are considered as non-local. This non-locality has the ability to account for the memory effects.
- Fractional derivatives apply exponential law which is most likely applied in heterogeneous media and fractures occurring at different scales.
- AB fractional operator employs both non-local and non-singular kernels.

1.2.4 Limitation of fractal derivative and fractional derivatives

- Riemann-Liouville has drawbacks in modelling real-world problems unlike other derivatives.
- It is difficult to model the real-world problems using Caputo and Fabrizio, exponential decal law, power law because these laws assume real-world problems are either exponential or decaying which can be very complex with real-world physical problems.
- With Riemann-Liouville derivative the description of non-integer must be clearly defined because if all the requirement is not provided the application will remain complex.
- Caputo derivatives are only applicable for differentiable functions even though is most popular.
- Singular kernel operators had the ability to limit their applications when applied in physical real word problems. The heterogeneity at altered scales cannot be accounted.

- Caputo and Fabrizio fractional derivative although is regarded as non-singular, however, the operator is measured as non-local.

1.3 PROBLEM STATEMENT

The major water volume of the world is mainly dominated in oceans, approximately 75% though seawater is not suitable for drinking, industrial, and household use. For this reasoning, human beings' existences have always depended on freshwater supply. To this point, freshwater only coexists in two phases namely; surface water that exists in oceans, lakes, rivers and streams, and subsurface water that occurs in subsurface geological formations of the earth. Supporting available evidence from Booustra and Kselik (2002) indicates that approximately 97% of portable water supply exists in the subsurface geological formation and this water can be used in conjunction with surface water. However, in most cases, one of the leading problems in groundwater modelling is to provide a suitable or correct model to be used particularly in leaky aquifer geological formation. In the previous studies conducted by researchers, Pythagoras was the first scientist to realise that human ideology and observation is a powerful tool and can be used as a mathematical tool to model real physical problems. Later on, Darcy firstly formulated an equation to define the flow of fluid in a porous geological formation. Darcy formulated the mathematical equation based on experimental results from the lab. However, Darcy law can only model fluid flow within a homogeneous laminar flow but cannot explain the complex flow occurring in heterogeneous media. In the early 18^{th}, Sir Leibniz and Isaac Newton introduced the concept of laws of motion to describe physical problems. It is important to note that one of the leading challenges with Newton's laws of motion is to formulate a mathematical algorithm that will explain the physical observation. It appears that Newton's laws of motion could not explain the complexity of natural problems and non-local problems. In an attempt to answer this, researchers devoted their focus in proposing new models for complex real-world problems. These models that were suggested include the first non-local fractional differentiation operator with a kernel power law, fractional differentiation and integral with exponential decay which was proposed by Caputo and Fabrizio, fractional differential and integral with the generalized Mittag-Leffler function which was suggested proposed by (Atangana and Mekkaoui, 2020). For each of these operators and their association were proposed and suitable to model anomalous problems. However recent studies indicated that these operators and their associations could not model real physical world problems. For example, one can discover that the real-world problem can either exhibit power law, exponential decay law, and self-similar behaviour or even more complex one such as self-similar and crossover (Atangana and Mekkaoui, 2020). None of the above can models can be used for this purpose. Due to the complexity of real-world problems, this study will propose a new operator scheme that is convolution with all the functions called fractal-fractional differentiation and integral with the kernel.

This operator will aim to attract more non-local problems that display fractal behaviour in leaky aquifers. Lastly, this new approach that will be introduced will be able to capture the flow, randomness, and memory effects of homogeneity/heterogeneity aquifer in porous media.

1.4 AIM AND OBJECTIVES

The purpose of the study is to develop an operator for groundwater modelling that will capture flow within a leaky aquifer, and this operator will also aim to attract more non-local problems that display at the same time fractal behaviour. The study includes objectives, namely:

Objectives

- Revising the model for groundwater flow in leaky aquifers.
- Reviewing the flow through advection and dispersion to delineate storage within a leaky aquifer.
- Capturing the flow in leaky aquifer using the Hantush groundwater model.
- Modifying the existing mathematical equation describing flow within a leaky aquifer with the fractal-fractional operator.
- Analysing via the corresponding Predictor-corrector method.
- Solving the modified model using a special numerical scheme called Atangana Baleanu fractal-fractional operator.
- Comparing the predictor-corrector method with Atangana Baleanu fractional operator.

1.5 RESEARCH METHODOLOGY

To achieve the thesis, we adopt the following steps

- Develop a conceptual model for underground flow in a leaky aquifer.
- Apply fractal-fractional operator to derive a new equation within a leaky aquifer and solve the equation using a numerical method.
- Apply the Predictor-Corrector method and Atangana Baleanu fractional operator to discretize the model to get the numerically exact solution.
- Use the Von Neumann stability method to determine the balance and stability of the numerical scheme.
- Develop numerical algorithm code that will be inserted into Matlab software.
- Simulate the discretized numerical code and compare the simulation.

1.6 DISSERTATION OUTLINE

Chapter 1 provides a brief inside background about different types of groundwater flow models, the different types of an aquifer, and entails more on the leaky aquifer and their behaviour. This chapter also touches on the review of Hantush groundwater flow models. We also provide limitations of the available models on the leaky aquifer. Besides, to enhance the available models, fractal-fractional calculus and their application are introduced to this chapter. Chapter 2 is a literature review, this chapter entails prior studies conducted for a leaky aquifer. Derivation of alternatives methods and numerical solutions to the leaky aquifers are provided. Furthermore, chapter 3 provides the conceptual model and numerical solution for groundwater flow in a leaky aquifer. The concept of fractal-fractional operators is applied to develop 2 equations. Moreover, chapter 4 covers the reviews on the Predictor-Corrector method and AB derivative method. These two methods are suggested to discretize the groundwater model that is developed. These two numerical schemes aim to solve local non-local natural problems within the leaky aquifer. The concept of Partial Differential Equations is introduced in this chapter accommodate the errors in the model. Finally, chapter 5 and 6 entails the Von Neumann stability analysis, numerical coding algorithm, and numerical simulations.

1.7 RESEARCH FRAMEWORK

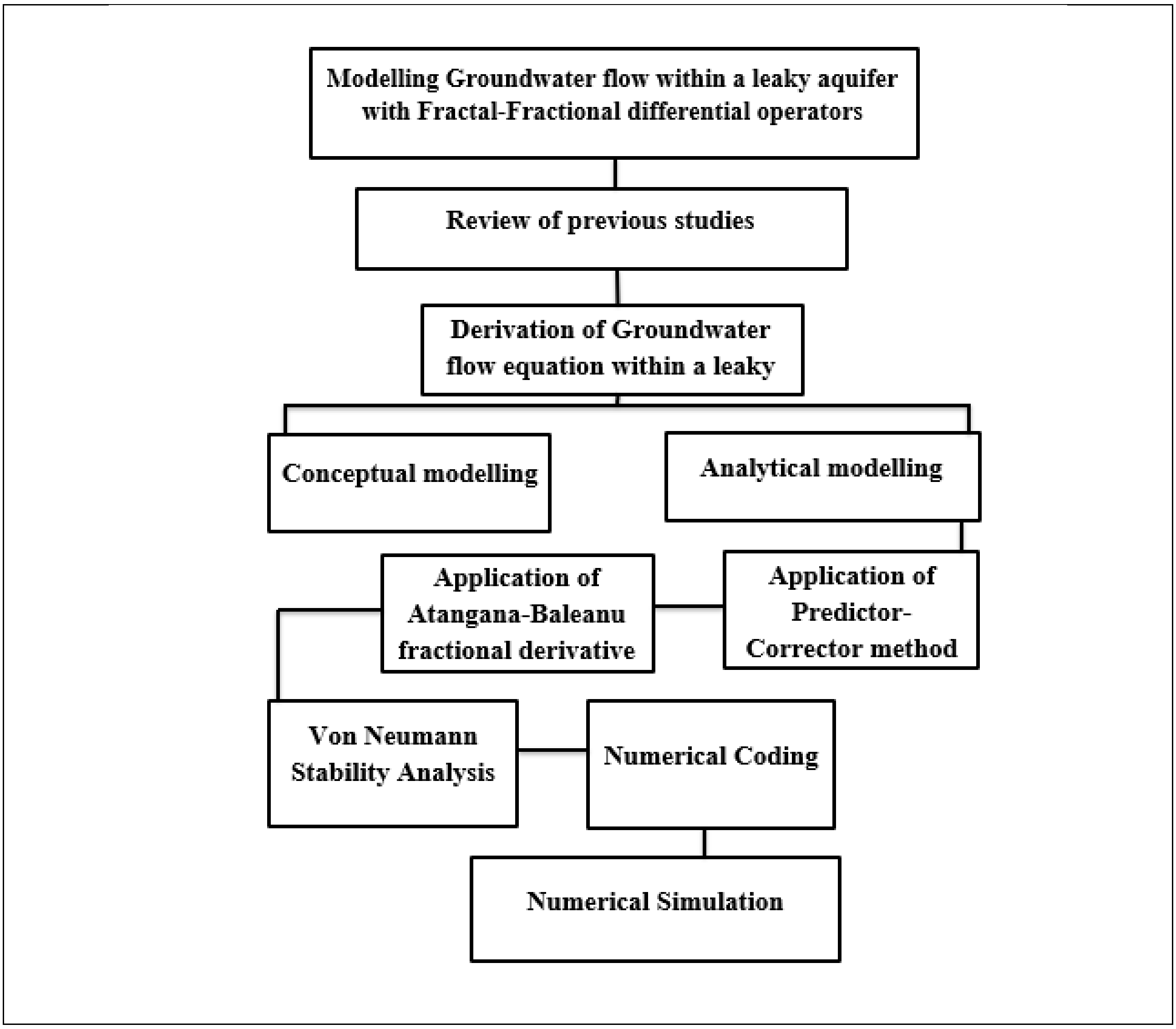

Figure 2: Research Framework of the study.

CHAPTER 2:
LITERATURE REVIEW

2.1 BACKGROUND

2.1.1 Groundwater flow

Groundwater flow equation is the mathematical relationship that defines the flow of water beneath the subsurface geological formation in the aquifer (Kruseman and De Ridder, 1990). Groundwater flows at different rates that are defined by mathematical equations that are associated with diffusion equations and Laplace equation in the aquifer. Within the aquifer, different types of flow occur such as transient flow, steady-state, and unsteady state. Transient flow in groundwater is associated with the diffusion equation while the steady-state is associated with the Laplace equation. Furthermore, the groundwater flow equation is derived under certain assumptions such that the properties of the medium are considered to be constant. Mass balance is the most practical example done to show the relation of water flowing in and out of a system. This relation is stated by a consecutive equation called Darcy's law in which is describes in laminar flow.

2.1.2 Groundwater flow models

Groundwater models are computer representations of the movement of groundwater flow. Groundwater models are used to guess and simulate the conditions of the aquifer. In a natural environment, groundwater models represent the flow, and sometimes groundwater models are used for quality aspects of groundwater. Groundwater models are used to also predict the outcomes movements of pollutants and contaminates. These movements of contaminates and pollution are described by a mechanism mainly associated with adsorption, advection, biological process, dispersion, and ion exchange (Amanda and Atangana, 2018).

Groundwater models are commonly used for various water management associated with hydrological effects in urban areas. They can also be used to calculate hydrological effects and variations in groundwater irrigation development and groundwater extraction within the aquifer. Groundwater flow is a model by four well-known methods which are based on mathematical groundwater flow equations called differential equation. These differential equations are based on approximations method by means of numerical analysis. These models are called conceptual models, analytical models, numerical models, and physical analog models (Rushton, 2003). These numerical models are resolved using numerical codes such as MODFLOW, OpenGeoSys, Matlab, etc.

Analytical models are usually applied in shallow groundwater and these require hydrological inputs such as initial boundary conditions. Consequently, the analytical model can incorporate a small system as compared to the numerical solution which is very complex (Yen and Yang, 2013). In the past, Theis conducted research on groundwater flow with the attempt of applying mathematical equations using conservation law. Theis derived a mathematical equation to capture the flow in a confined aquifer. The Theis equation is a profound solution for groundwater flow models. However, Hantush modified the Theis equation by adding a leakage factor in Theis groundwater equation (Zhan et al., 2006). Ehligg and Halepaska (1976) used the Hantush model in the unconfined aquifer considered the effect of leakage to joint similar boundaries structured using the Boulton model to simulate the flow in the unconfined aquifer (Malama *et al.*, 2007). Pogany, Baricz, and Rudas conducted a study on partial Kratzel function to determine the difference between leaky aquifer temperature, and thermonuclear integrals. Furthermore, they transformed the leaky aquifer integrals and suggested that more numerical analysis must be conducted because of complex numerical analysis (Rudas et al., 2015).

Yavus (1976) conducted a study on leaky aquifers using rheological properties. He connected the compressional mechanism with the rheological consecutive equation of leaky aquifer. In the study, he realized that due to pumping the thickness of fine-grained aquifer beds alters due to compaction and compression. Moench derived the solution for pseudo-2D flow in a semi-confined aquifer with the assumption that the aquifer is fully penetrated and the underlying and overlying beds are semi-confined. However, the results were unable to indicate quasi-3D (Stehfest, 1970). Consequently, Sepulveda modified Moench finding using Laplace transformation and presented 3D flow in storage semi-confined aquifers.

Hantush later modified the theory of leaky aquifers and developed an analytical and numerical solution using Laplace space and asymptotic expansion at early and late time. Kuhlman later on added on Hantush findings by using Laplace transformation to build a model for transient flow simulation. Schroth and Narasimham (1997) used a numerical model to interpret data for leaky aquifers. Their results indicated that transient data is demonstrated using an integrated numerical model. These integrated numerical models contribute to decrease the assumptions made in leaky aquifers and therefore deduce complex systems. Recent studies by Meester *et al*. (2004) analyzed groundwater flow in horizontally stratified layers. His results indicated that horizontal anisotropy occurs at a greater scale parallel to the strike direction of faults and fractures leaking in stratigraphy. Furthermore, the approach by Meester, Hemker, and Berry indicated exact solutions but failed to suggest a good solution in stratified aquifers. Later, Hemker and Maas (1987) suggested a solution for steady-state flow in isotropic aquifer using eigenvalue. His method was based on redefining partially fixed constraints using eigenvalues. However, new studies indicate that Bekker and Hemker derived a numerically exact solution for horizontal anisotropic stratified confined layers. Bruggenian modified the Hemker model primarily by introducing a steady-state and unsteady-state solution in semi-confined aquifers.

Dupuit introduced the existence of error discharge on multiple leaky aquifers. Hemker improved the Dupuit solution with the approach of using eigenvalues to describe the flow in multiple leaky aquifers. Hemker applied the analytical and the numerical solution in transient flow to describe a numerically exact solution for confined and leaky aquifers. Moreover, Hemker considered time and distance to calculate drawdown in a multi-layered aquifer. Streltrova associated leakage aquifers with a vertical component in an unconfined aquifer. Adding, Zhang and Brian (2006) suggested an analytical solution to determine the rate and volume of leakage within the leaky aquifer. Their assumption was based on constant discharge rate; aquifer is fully penetrated; constant drawdown, and neglecting aquifer storage. According to Zhang and Brian (2006), when the leaky aquifer is pumped at a constant rate the entire volume within the aquifer-aquitard is denoted in forms of integrals and can be estimated using a numerical solution such as Matlab software.

Rai *et al*. (2006) used the Boussinesq equation to estimate groundwater flow in response to the water table when the aquifer recharging and discharging instantly. Bansal (2016) suggested that the calculations of seepage can be made using different conditions in confined and unconfined aquifers. With that said, Bansal (2016) established a 2D analytical model using the Boussinesq equation in an anisotropic unconfined aquifer. He later applied Fourier transformation to determine the hydraulic head of the aquifer. Jiang (2006) conducted an investigation of transient radial flow in confined-leaky aquifers to determine the analytical solution. In the study, he took into account the conservation of mass, linear momentum, and the drawdown in the late time of the aquifer. The results suggested that a vertical increase in the movement of groundwater flow in radial flow and leakage occurs vertically rather than horizontal (Jiang, 2016).

Yen and Yang (2009) establish a model to determine the drawdown resulting in a constant pumping rate. The results indicated that the skin effect of the well plays a significant role in the movement of water, either positive or negative skin effect. Positive skin promotes the release of water in storage while negative skin reduces the permeability of the well. Consequently, positive skin with inverse proportion in thickness affects the drawdown. Hund-der *et al*. (2013) modified Yen and Yang work and concluded that the drawdown in semi-aquifer is influenced by indirect results during the early time in pumping which has an effect on water discharge at late time.

Atangana and Goufo (2015) described the movement of groundwater flow occurring in a particular geological formation. They additionally derived a mathematical equation for a local variable to further improve the groundwater model for leaky aquifers (Atangana and Goufo, 2015). Their model was based on the complexity of leaky aquifer variation and heterogeneity in the leaky aquifer. Atangana and Goufo introduced these concepts of fractional derivatives to solve complex and chaotic problems. However, the concept of fractional derivatives has also some limitations, for example, it is difficult to model the trap of water accurately in a matrix (Atangana and Goufo, 2015). Nevertheless, they derived a mathematical equation to describe an exact numerical solution to explain the chaotic system.

Atangana and Bildik modified the Theis equation by introducing the time factor and using fractional derivatives to model the complexity of groundwater flow. Atangana and Bildik realize that their solution due to pumping only accommodate constant rate and does not accommodate chaotic aquifers, and therefore suggested a new model for a leaky aquifer. They are many models that exist to determine groundwater flow in linear aquifers. Nonetheless accommodates non-linear cases, therefore a numerical scheme is needed to obtain an approximation. Recent studies by Djida *et al.* (2016) suggested the concept of the fractional differential equation using the Mittag-Leffler function as kernel and associated it with fractional integrals to model groundwater flow in leaky aquifers. They derived the equation of groundwater flow using non-local and non-singular kernel. They formulated the mathematical equation based on the time-fractional derivative using the Mittag-Leffler function. Moreover, their model is based on numerical and analytical solutions and is exclusively centered on reality. Lastly, the numerical scheme to derive the new model is applied.

2.1.3 Derivation of the mathematical equation using Darcy's law

Groundwater flow equations are derived using Darcy's law and mass conservation balance (Amanda and Atangana, 2018). Darcy's law defines the flow of fluid within a porous media. The law was expressed by Henry Darcy through conducting experimental results on the fluid flow of water within sand beds (Darcy, 1856). Darcy's law and mass conservation balance are related to the groundwater flow equation. Darcy developed the model to describe flow fluid in porous media and named it a controlled volume fluid mechanism. The flow mechanism was based on the initial boundary condition Figure **3** and the mass balance equation.

$$Q = -KA\frac{dh}{dl} \tag{2.1}$$

$$\frac{dh}{dl} = i \tag{2.2}$$

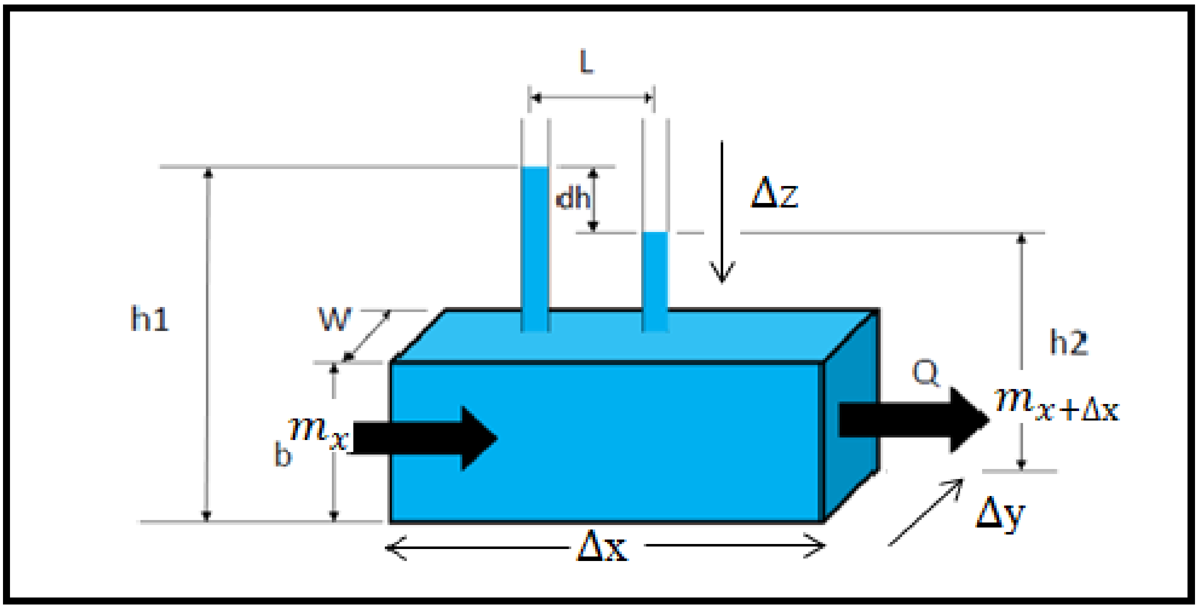

Figure 3: A unit volume of the saturated aquifer in porous geological formation (Amanda and Atangana, 2018)

Where Q is discharge rate (m^3/d), A cross-sectional area of flow (m^2), $dh = h_1 - h_2$ in (m); W aquifer width (m); K is the hydraulic conductivity (m/d); $\frac{dh}{dl}$ hydraulic gradient; L is the length of flow (m) and b is the aquifer thickness (m). Now defining parameter x, y and z

Let

$$\begin{aligned} \Delta x &= m_1 \\ \Delta y &= m_2 \\ \Delta z &= m_3 \end{aligned} \tag{2.3}$$

Therefore

$$m_x = (\rho_w q(x))\Delta y \Delta z \tag{2.4}$$

$$m_{x+m_1} = (\rho_w q(x + m_1))m_2 m_3 \tag{2.5}$$

$$m_{y+m_2} = (\rho_w q(y + m_2))m_1 m_3 \tag{2.6}$$

$$m_y = (\rho_w q(y))m_1 m_3 \tag{2.7}$$

$$m_z = (\rho_w q(z))m_2 m_1 \tag{2.8}$$

$$m_{z+m_3} = (\rho_w q(z + m_3))m_1 m_3 \tag{2.9}$$

Substitute equation into mass balance equation

$$\frac{\partial m}{\partial t} = mass\ in - mass\ out \tag{2.10}$$

$$\frac{\partial m}{\partial t} = \left(m_x - m_{x+m_1}\right) + \left(m_y - m_{y+m_2}\right) + \left(m_z - m_{z+m_3}\right) \tag{2.11}$$

And therefore, density is defined as:

$$mass(m) = \emptyset \rho_w v_w \tag{2.12}$$

$$\frac{\partial m}{\partial t} = \frac{\partial(\emptyset \rho_w v_w)}{\partial t} \tag{2.13}$$

Substitution

$$\frac{\partial(\emptyset \rho_w v_w)}{\partial t} = (\rho_w q(x))m_2 m_3 - \left(\rho_w q(x+m_1)\right)m_2 m_3 + \left(\rho_w q(y)\right)m_1 m_3 - \left(\rho_w q(y+m_2)\right)m_1 m_3 + \left(\rho_w q(z)\right)m_1 m_2 - (\rho_w q(z+m_3))m_1 m_2 \tag{2.14}$$

Since

$$\emptyset = \frac{v_w}{v} \tag{2.15}$$

And

$$\frac{\partial m}{\partial t} = m_1 m_2 m_3 \frac{\partial\left(\frac{v_w}{v}\rho_w\right)}{\partial t} \tag{2.16}$$

Now we divide equation (2.16) with $\Delta x \Delta y \Delta z$ to give rise to the new equation, direct substitution

$$\frac{\partial m}{\partial t} = m_1 m_2 m_3 \frac{\partial\left(\frac{v_w}{v}\rho_w\right)}{m_1 m_2 m_3 \partial t} \tag{2.17}$$

$$\frac{\partial m}{\partial t} = \frac{(\rho_w q(x)) - \left(\rho_w q(x+m_1)\right)}{m_1} + \frac{\left(\rho_w q(y)\right) - \left(\rho_w q(y+m_2)\right)}{m_2} + \frac{\left(\rho_w q(z)\right) - (\rho_w q(z+m_3))}{m_3} \tag{2.18}$$

Hantush assumption does not take into account the effect of density, nevertheless, now divide the equation with density (ρ_w). The new equation will result in:

$$\frac{\partial\left(\frac{v_w}{v}\rho_w\right)}{\rho_w \partial t} = \frac{(q(x)) - \left(q(x+m_1)\right)}{m_1} + \frac{\left(q(y)\right) - \left(q(y+m_2)\right)}{m_2} + \frac{\left(q(z)\right) - (q(z+m_3))}{m_3} \tag{2.19}$$

Now the limit of $m_1 m_2 m_3$ approaches 0, the new differential equation will be transformed to first the using the first principle

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x)}{\Delta x} \tag{2.20}$$

If we can revert to Darcy's law $m_1 m_2 m_3$ parameters can be defined in terms of specific discharge $q_x, q_z, and\ q_z$, now we finally arise to a new equation.

$$q_x = \frac{Q}{A} = K_x \frac{\partial h}{\partial x} \tag{2.21}$$

$$q_y = \frac{Q}{A} = K_y \frac{\partial h}{\partial y} \tag{2.22}$$

$$q_z = \frac{Q}{A} = K_z \frac{\partial h}{\partial z} \tag{2.23}$$

Now

$$\frac{\partial \left(\frac{v_w}{v} \rho_w\right)}{\rho_w \partial t} = \frac{q_x}{\partial x} + \frac{q_y}{\partial y} + \frac{q_z}{\partial z} \tag{2.24}$$

Taking a factor on the left side

$$\frac{1}{\rho_w} \left(\frac{\partial \frac{v_w}{v}}{\partial t} + \frac{\partial (\rho_w)}{\partial t} \right) = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \tag{2.25}$$

Now defining the compression on the water in the aquifer, it gives rise to:

$$\frac{1}{\rho_w} \frac{\partial \rho_w}{\partial t} = \emptyset \beta \rho_w g \frac{\partial h}{\partial t} \tag{2.26}$$

Substitution of the water compression equation will give rise to Hassanizadeh (1986) equation

$$\emptyset \beta \rho_w g \frac{\partial h}{\partial t} + \alpha \rho_w g \frac{\partial h}{\partial t} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \tag{2.27}$$

Hantush derived a mathematical equation that describes the movement of groundwater in leaky aquifer based on the model that was proposed by Theis (1935) which has initial boundary conditions. The initial boundary conditions and assumptions are listed at the beginning of chapter 1. The Hantush equation is proposed below:

$$\frac{T}{S} \partial_r \big(p(r,t)\big) = \partial_{rr}^2 \big(p(r,t)\big) + \frac{1}{r} \partial_r \big(p(r,t)\big) + \frac{p(r,t)}{\lambda^2} \tag{2.28}$$

$$\lambda^2 = \frac{B}{K'} \tag{2.29}$$

Initial boundary conditions

$$p(r,0) = p, \lim_{r\to\infty} p(r,t) = p_0, \qquad Q = 2\pi r_b K d \partial_r (p(r_b,t)) \tag{2.30}$$

Where T is the transmissivity; S is the specific storage of aquifer; $p(r,t)$ is the drawdown or variation in water level; Q is discharged pumping rate; K and K' are the hydraulic conductivity of the confining layer and, B and d are the thickness of the confining layers of the aquifer (Atangana, 2016). Kruseman and De Ridder (1990) indicated that when water is pumped on a leaky aquifer, water is not only withdrawn in the aquifer itself but also from the superimposing and primary layers. Atangana realized that the Hantush equation has a limitation, therefore introduced the concept of fractal derivatives.

2.1.3.1 Limitations for Hantush equation

- Can only be applied where physicals laws such as Darcy law, Fick's law, and Fourier law are not applicable
- Does not apply in heterogeneous aquifers
- Has limitation for initial boundary conditions in cylindrical equation (Chen, 2005)
- Properties of the hydraulic head of the aquifer and aquitard are temporally invariable
- Applicable in uniform thickness
- Applicable to physical laws that account Euclidean geometry
- The effects of aquifer storage are neglected, uses transient flow and the leakage is proportional to drawdown
- Not applicable in self-similar aquifers
- The aquifer is homogeneous, isotropic with aquifer occurring in the horizontal plane

2.1.4 Derivation via an alternative method for the analytical exact solution

An alternative approach by means of deriving an analytical solution based on an estimated solution using time-fractional to derive the groundwater flow equation in leaky aquifers (Atangana, 2016). The equation was developed with the approach of a partial differential equation called the Boltzmann transformation (Zauderer, 1985). Atangana (2016) expresses $t_0 < t$ with the equation suggested below.

$$y_0 = \frac{Sr^2}{4T(t-t_0)} \tag{2.31}$$

In the derivation of the groundwater flow equation, Atangana (2016) consider the following function

$$s(r,t) = \frac{h}{(t-t_0)} E_\alpha \left[-y_0 - \frac{r^2}{4By_0} \right] \tag{2.32}$$

Where h denotes parameter, h can be derived from using the initial boundary conditions. Water in the leaky aquifer can be determined by drilling well prior to pumping. Moreover, Atangana (2016) also indicated that boreholes have ratio given as r_b and withdrawal of water from a borehole can be determined by the equation below:

$$Q_0\Delta t_0 = 4\pi hT \tag{2.33}$$

Therefore, h can be determined from the equation by rearranging the equation above, making h subject of the formula. Now the drawdown can denote as time interval as

$$s(r,t) = \frac{Q_0\Delta t_0}{4\pi T(t-t_0)} E_\alpha\left[-y_0 - \frac{r^2}{4\lambda^2 y_0}\right] \tag{2.34}$$

Assuming that the water is being withdrawn instantaneously from the leaky aquifer (hence m-times) such that the change Δt_i, $t_{r+1} = t_r + \Delta t_r$ where $(r = 0, \ldots.m)$ given that the fractional differential groundwater is linear, then the drawdown will be represented as $t > t_m$ at any time indicated by the equation below:

$$s(r,y) = \frac{1}{4\pi T}\sum_{j=0}^{\infty} \frac{Q_j\Delta t_j}{(t-t_j)} E_\alpha\left[-y_j - \frac{r^2}{4\lambda^2 y_j}\right] \tag{2.35}$$

Now introducing an integral into the summation such that, the change occurs in a very short period of time. The equation will be given as follows:

$$s(r,y) = \frac{1}{4\pi T}\int_{t_0}^{t} \frac{Q(y)d\omega}{(t-\omega)} E_\alpha\left[-y_j - \frac{r^2}{4\lambda^2 y_j}\right] d\omega \tag{2.36}$$

Making use of the Boltzmann equation, the new equation will be given as

$$s(r,y) = \frac{1}{4\pi T}\int_{y}^{\infty} \frac{Q(x)}{x} E_\alpha\left[-x - \frac{r^2}{4\lambda^2 x}\right] dx \tag{2.37}$$

Therefore, equation (2.37) is regarded as the practical solution for fractional flow in leaky aquifer space and time. However, if we neglect the space and time, the new equation will be given as:

$$s(r,y) = \frac{1}{4\pi T}\int_{y}^{\infty} \frac{1}{x} E_\alpha\left[-x - \frac{r^2}{4\lambda^2 x}\right] dx \tag{2.38}$$

2.2 FRACTAL DERIVATIVE APPROACH

This chapter will introduce the concept of fractional differentiation and will aim to focus on enhancing a new method that will assist in problems associated with groundwater flow within leaky aquifers. The notion of differentiation and integrals is advantageous because it accounts for physical problems that cannot be captured stochastic approach and these include viscoelasticity in a different scale and memory effects. Also, Mirza and Vieru (2016) indicated that the advanced development of fractional calculus has presented great results within the groundwater flow system. If we revert to chapter 1, the common fractional derivatives applied in the field of science and engineering include Riemann-Liouville, Atangana Baleanu, Caputo, and Caputo-Fabrizio. This section will also focus on the properties of fractional operators.

2.3 FRACTIONAL DERIVATIVE APPROACH

Within the branch of mathematical analysis that studies the several different possibilities defining real numbers and complex powers numbers of differential operator D. Fractal differentiation and integration were introduced to connect fractal and fractional calculus to predict complex systems in the real world. Fractal-fractional differentiation and integration are distinguished based on their names and association. Riemann-Liouville fractional integration, Hadamard fractional integration, and Atangana-Baleanu fractional integration are associated with fractional integration while Riemann-Liouville fractional derivative, Caputo fractional derivative and Riez derivative associated with fractional derivative

2.3.1 Fractional derivatives and Integrals

2.3.1.1 Riemann-Liouville integral

This theory includes periodic boundary conditions that repeat themselves after a period which are defined as the Weyl integral. The Riemann-Liouville integral can also be defined in the Fourier series by assuming that the Fourier constant coefficient is zero. Furthermore, the derivative is mostly in linear fractional differentiation (Gladkina *et al*., 2018; Antony *et al*., 2006). Atangana and Kilicman (2016) also indicated that the appropriate for practice with Laplace transformation, however, insignificant is argued.

$$ {}_{\alpha}^{BL}D_t^{\alpha} f(t) = \frac{d^n}{dt^n} D^{-(m-\alpha)} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{\alpha}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau \tag{2.39} $$

Riemann-Liouville fractional integral, $f(t) \in w^1([0,b],\mathbb{R}^m); b$

$$ {}_{\alpha}^{BL}i_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{2.40} $$

Where Γ is defined as Euler gamma function

2.3.1.2 Caputo Fractional Derivative

This fractional derivative is mostly applied for linear differential equations and is constructed using power law (Gladkina *et al*., 2018; Anatoly *et al*., 2006; Stefan *et al*., 1993). In most cases, Caputo fractional is explicatable more than Riemann-Liouville because of its properties to improve the description of boundary conditions (Kavvas *et al*., 2017; Podlubny, 1998). The CFD account for memory effects but has a deficiency in accuracy due to the singular kernel (Gómez-Aguilar, *et al.*, 2016; Caputo and Fabrizio, 2015). Moreover, CFD has presented outstanding results with the use of Laplace transformation in time and space components. For instance, Atangana and Kiliçman (2013) derived an analytical solution CFD in the classical hydrodynamic advection-dispersion equation with respect to time and space.

Definition: let $b > 0, \mathrm{f} \in H^1(0, \mathrm{b})$ and $0 < \alpha > 1$, given a function $g(t)$ then CFD is given by

$$ {}_{0}^{c}D_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g^{'}(\tau) d\tau. \tag{2.41} $$

2.3.1.3 Caputo-Fabrizio Fractional Derivative

This CFFD was proposed by Caputo and Fabrizio to account for limitations presented by the aforementioned Riemann-Liouville and Caputo derivatives. The derivative is constructed convolution of 1st order derivative and exponential function. The CFFD is a non-singular kernel and is well known for its ability to account for heterogeneity medium at different scales (Al-Salti *et al*., 2016; Caputo and Fabrizio, 2016). Furthermore, CFFD accounts for Laplace transformation and contains Fourie transformation in it (Shan and Khan, 2016). To verify these abilities, Alkahtani and Atangana (2016) applied CFFD in shallow surface water to determine the wave movement at different scales.

Definition of CFFD: supposed that $g \in H^1(\mathrm{a}, \mathrm{b}), \mathrm{b} > \mathrm{a}, \alpha \in [0,1]$, then CFFD is given by

$$ {}_{0}^{CF}D_t^{\alpha} g(t) = \frac{M(\alpha)}{1-\alpha} \int_{\alpha}^{t} (g'(\tau)) exp\left[-\alpha \frac{t-\alpha}{1-\alpha}\right] d\tau \tag{2.42} $$

$$ {}_{0}^{CF}D_t^{\alpha} g(t) = \frac{M(\alpha)}{1-\alpha} \int_{\alpha}^{t} \big(g(t) - g(r)\big) exp\left[-\alpha \frac{t-\alpha}{1-\alpha}\right] d\tau \tag{2.43} $$

2.3.1.4 Atangana-Baleanu Fractional Derivative

Recent studies conducted by Atangana and Baleanu introduced the generalized Mittag-Leffler function to develop a new formula called fractional derivative that incorporates nonsingular kernel and non-local physical problems (Atangana and Baleanu, 2016). The AB derivative can model complexity in heterogeneous medium, viscoelasticity, and accounts for memory effects (Atangana and Baleanu, 2016; Atangana and Alkahtani, 2016) to model real-world physical problems. AB uses the Laplace transformation to define initial boundary conditions. Atangana and Alkahtani (2016) applied AB derivative in the unconfined aquifer to account for complexity. Furthermore, their study includes the application of Mittag-Leffler function which enhanced better the description of world physical problems. To add, they applied Laplace transformation and Crank-Nicolson to obtain analytical-numerical solution respectively. Furthermore, Djida *et al.* (2016) applied AB fractional integral within leaky aquifer to determine the groundwater flow. Lastly, Gómez-Aguilar *et al*. (2016) applied AB fractional derivative to study electromagnetic waves in a dielectric.

AB definition: supposed that$g \in H^1(\mathrm{b,c}), \mathrm{c} > \mathrm{a}, \alpha \in [0,1]$ with AB fractional derivative Riemann-Liouville sense, the derivative is given by

$$ {}^{ABR}_{\ \ \alpha}D_t^{\alpha} g(t) = \frac{d}{dt}\frac{B(\alpha)}{1-\alpha}\int_{\alpha}^{t} g(\tau)E_{\alpha}\left[-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau, \qquad 0 < \alpha < 1 \tag{2.44} $$

Definition: supposed that $g \in H^1(\mathrm{b,c}), \mathrm{c} > \mathrm{a}, \alpha \in [0,1]$ AB derivative with Caputo sense is given by

$$ {}^{ABR}_{\ \ \alpha}D_t^{\alpha} g(t) = \frac{B(\alpha)}{1-\alpha}\int_{\alpha}^{t} g'(\tau)E_{\alpha}\left[-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau, \qquad 0 < \alpha < 1 \tag{2.45} $$

The AB fractional integral with order α function is given by

$$ {}^{AB}_{\ \alpha}D_t^{\alpha} g(t) = \frac{1-\alpha}{B(\alpha)}g(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\alpha}^{t} g(\tau)(t-\tau)d\tau \tag{2.46} $$

Equation (2.46) is connected with the new fractional derivative with the non-local kernel. If α is 0, then the initial function is recovered and when α is 1, we obtain an ordinary equation. Furthermore, The Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty}\frac{t^m}{\Gamma(\alpha m+\beta)}, \qquad \alpha > 0, \beta > 0 \tag{2.47} $$

Where $\Gamma(x)$ is the gamma function, when $\beta = 1$, it is also be given as $E_{\alpha}(t) = E_{\alpha,1}(t)$ and for $\alpha = 0$, equation (4.47) will be equals to Taylor expansion.

The concept of fractional calculus enhances modelling techniques to groundwater flow and accounts for randomness, memory effects, heterogeneity, and viscoelasticity in a porous medium. This will employ the approach to modify the existing groundwater flow equation that will capture water in a complex groundwater structure. The new convolution of now locality kernel permits better analysis of memory within structure and media and diverse scales (Atangana and Baleanu, 2016).

2.3.2 Properties of new derivative

In this section, we introduce the Laplace transformation in relation to the derivatives. If we transform both sides of the AB derivative equation, equation (2.44) and (2.45) to the following equation

$$ {}^{ABC}_{0}D_x^{\alpha}\big(s(x)\big) = {}^{C}_{0}D_x^{\alpha}\big(s(x)\big) \tag{2.48}$$

Apply Laplace transformation

$$\mathcal{L}\left\{{}^{ABC}_{0}D_x^{\alpha}\big(s(x)\big)\right\}(p) = \frac{p^{\alpha}\mathcal{L}\{s(x)\}p}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{B(\alpha)}{1-\alpha} \tag{2.49}$$

And

$$\mathcal{L}\left\{{}^{ABC}_{0}D_x^{\alpha}\big(s(x)\big)\right\}(p) = \frac{p^{\alpha}\mathcal{L}\{s(x)\}p - p^{\alpha-1}s(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{B(\alpha)}{1-\alpha} \tag{250}$$

Given that $g \in H^1\ (b,c), c > b, \alpha \in [0,1]$ therefore, this relation will be obtained

$$ {}^{ABC}_{0}D_x^{\alpha}\big(s(t)\big) = {}^{ABR}_{0}D_x^{\alpha}\big(s(x)\big) + H(t) \tag{2.51}$$

Now apply Laplace in equation (2.51)

$$\mathcal{L}\left\{{}^{ABC}_{0}D_x^{\alpha}\big(s(x)\big)\right\}(p) = \left(\frac{p^{\alpha}\mathcal{L}\{s(x)\}p}{p^{\alpha}+\frac{\alpha}{1-\alpha}} - \frac{-p^{\alpha-1}s(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\right)\left(\frac{B(\alpha)}{1-\alpha}\right) \tag{2.52}$$

Followed by equation (2.49), we obtain

$$\mathcal{L}\left\{{}^{ABC}_{0}D_x^{\alpha}\big(s(x)\big)\right\}(p) = \mathcal{L}\left\{{}^{ABR}_{0}D_x^{\alpha}\big(s(x)\big)\right\}(p)\frac{p^{\alpha}\mathcal{L}\{s(x)\}p}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{B(\alpha)}{1-\alpha} \tag{2.53}$$

Now applying inverse Laplace transformation in equation (2.53) we obtain

$$ {}^{ABC}_{0}D_x^{\alpha}\big(s(x)\big) = {}^{ABR}_{0}D_x^{\alpha}\big(s(x)\big) - \frac{B(\alpha)}{1-\alpha}s(0)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}x^{\alpha}\right) \tag{2.54}$$

Atangana and Baleanu (2016) applied fractional integrals to heat transfer to analyse experimental in cylindrical heterogeneous shells like pipe in complex system. They described the rate of time through heat conduction materials at different scales. Their results indicated that head transfer is determined from the internal radius and the external radius.

2.3.3 Applications of fractal derivatives

In the framework of fractional derivatives, there are major fractional derivatives and integral operators which are mainly; Caputo-Fabrizio fractional derivatives which is associated with exponential decay, Riemann-Liouville and Caputo fractional-order derivatives which is the convolution of power law with first derivative, and lastly the Atangana Baleanu derivative and fractional operator which is convolution of Mittag-Leffler function. These three fractional operators have been applied several forms with success. In recent times, fractal-fractional operators have been applied in science and engineering, control theory, groundwater flow and Geo-hydrology, biological processes, finance system, viscoelasticity, fluid diffusion, wave propagation, rheology and chaotic processes and other several with great success.

CHAPTER 3:
GROUNDWATER DERIVATION EQUATION

3.1 INTRODUCTION

In the history of all branches of STEM, the conceptual approach of using the non-local operator-differential equation was not taken into account because of the complex mathematics. Recent studies have indicated that researchers have suggested three operators within these field. This includes the power law, exponential decay law, and the generalized Mittag-Leffler law (Alkahtani, 2016). Researchers categorized these operators and their association according to names; these names include the Riemann-Liouville and the Caputo-fractional operator associated with power law or non-local operator and kernel type; Caputo Fabrizio associated with non-singular local type, and Atangana-Baleanu associated with both non-singular type and non-local type (Atangana, 2016). Atangana noticed that the kernel Mittag-Leffler function is mostly universal than other proposed models such as exponential decay function and power law. With this said, Atangana then suggested that both Caputo-Fabrizio and Riemann-Liouville are distinct cases for Atangana-Baleanu fractional operators (Atangana and Dumitru, 2016). Consequently, some researchers recommended the general kernel operator must be applied but they realized that kernel type operator is not practical. The condition of Caputo type has difficulties in local derivative, either power law, exponential decay function, or Mittag-Leffler function. Nonetheless, researchers realized that the notion of the local differential operator is not appropriate and cannot model real-world complications as it was specified prior in explored papers. Real-world problems are very complex, for instance, one can consider modelling real-world problems with the structure of applied mathematics using fractal derivatives. One will realize that there's an unusual type of derivative that exists known as a fractal derivative. A modified fractal derivative is variably scaled according to t^{a} (Atangana, 2016). These derivatives are usually applied to model physical problems defined by classical laws such as Fick's law, Darcy's law, and Fourier's law. These problems are constructed on Euclidean geometry and are not valid to media of non-integral fractals that can model nature problems such as turbulence, porous media, and some other aspect characterised by fractals (Atangana, 2016). The concept of fractal-fractional derivatives and the applications to the field of groundwater have shown excellent results in the past, and they have been rewarded and revised by a number of researchers with accurate results (Atangana, 2016) In this chapter a new operator called fractal derivative and fractional integral will be introduced. This operator will aim to capture water within a leaker aquifer and other non-local natural problems that exhibit fractal behavior. In the chapter, the conceptual model and analytical model will be developed.

3.1.1 Conceptual model

A conceptual model is an illustration of a structure, composition, understanding, or simulation that represents a model. It is also usually a set of models, some may be physical concepts, for instance, modelling a toy which may be assembled and may be made to work as an object that it characterizes. Conceptual models are ideas that represent physical real-world physical problems. In this chapter, a conceptual model is represented in **Figure 4**. In this picture, groundwater is withdrawn from a leaky aquifer for irrigation and household purpose.

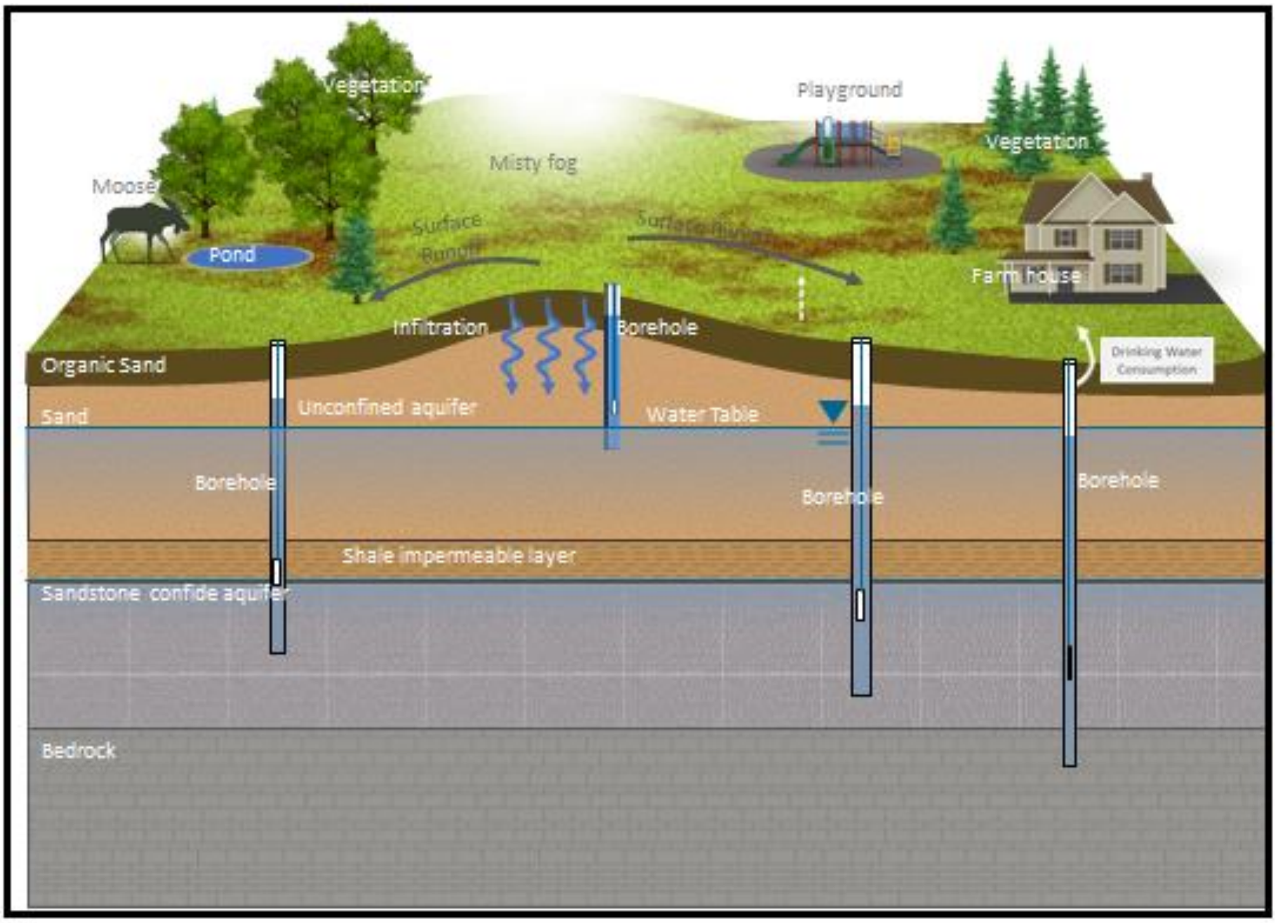

Figure 4: Conceptual model of rainwater recharging the aquifer through infiltration. The water from the leaky aquifer is being abstracted with boreholes for irrigation and drinking for a household.

3.1.2 Analytical model

Analytical models are usually defined as the mathematical model or expressions that are closely related to the solution. For instance, the solution to the equation describes the physical real-world problem in a mathematical expression. Analytical models are also used to describe the complex real-world problem. In this chapter, an analytical model that also describes the conceptual will be introduced Figure 1: Graphical representation of piezometric surface and water table within a leaky aquiferin (**Figure 5**)

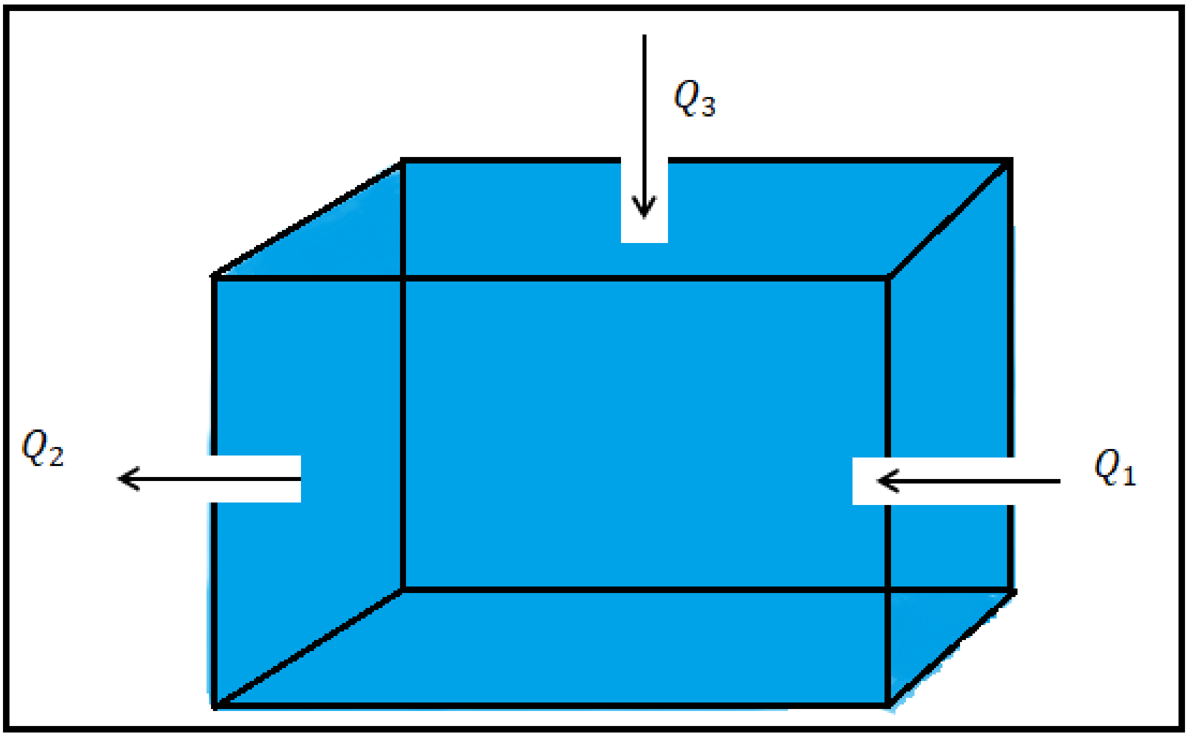

Figure 5: Saturated geological formation aquifer simulating different rates within an aquifer. Q_1 represent water recharging the aquifer, Q_2 is the water discharging in the aquifer, and Q_3 is the vertical leakage

3.2 DERIVATION OF NEW GROUNDWATER FLOW EQUATION WITHIN LEAKY AQUIFER WITH THE EFFECTS OF STORAGE

3.2.1 New Groundwater equation

To determine the storage in leaky aquifer equation in a saturated geological formation, one must consider (**Figure 5**) which presents different volumes of water leaning and entering the aquifer system. Groundwater flow in an aquifer arises in two traditions, either through advection or dispersion. Advection defines the mass movement of solutes in subsurface water while dispersion describes the mixing and distribution that results from molecular diffusion and variation in velocity in which water exchanges at altered scales. The water within a porous medium does not travel at an average speed and also varies in direction. Water particles can flow within porous media and different paths. These can further be explained using the groundwater flow equation that includes the storage equation is given below.

$$\frac{\partial v}{\partial t} = Q_1 + Q_3 - Q_2 \tag{3.1}$$

Where Q_3 defines the leakage factor from the main aquifer, Q_1 is the water recharging the aquifer, and Q_2 defines the discharged water or water moving out of the aquifer and $\frac{\partial v}{\partial t}$ signifies the storage of the aquifer.

$$\frac{\partial v}{\partial t} = S(2\pi r)dr\frac{\partial h}{\partial t} \tag{3.2}$$

Now

$$Q_1 = 2\pi(r+\Delta r)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K \tag{3.3}$$

$$Q_2 = 2\pi rb\left(\frac{\partial h}{\partial r}\right)K \tag{3.4}$$

$$Q_3 = 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb \tag{3.5}$$

Substitution

$$S(2\pi r)dr\frac{\partial h}{\partial t} = 2\pi(r+dr)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K + 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb - 2\pi rb\left(\frac{\partial h}{\partial r}\right)K \tag{3.6}$$

Now divide by $2\pi r\Delta r$

$$\frac{S(2\pi r)dr\frac{\partial h}{\partial t}}{2\pi r\Delta r} = \frac{2\pi(r+dr)b\left[\left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}\Delta r + \frac{\partial h}{\partial r}\right]K}{2\pi r\Delta r} + \frac{2\pi rdr\frac{h(r,t)}{\lambda^2}Kb}{2\pi r\Delta r} + \frac{2\pi rb\left(\frac{\partial h}{\partial r}\right)K}{2\pi r\Delta r} \tag{3.7}$$

$$S\frac{\partial h}{\partial t} = \frac{(r+dr)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K}{r\Delta r} + \frac{h(r,t)}{\lambda^2}Kb - \frac{b\left(\frac{\partial h}{\partial r}\right)K}{\Delta r} \tag{3.8}$$

Many geohydrologist uses this equation to determine aquifer parameters can. From literature, transmissivity is given by

$$T = Kb \tag{3.9}$$

$$S\frac{\partial h}{\partial t} = T\frac{(r+dr)\left[\left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}\Delta r + \frac{\partial h}{\partial r}\right]}{r\Delta r} + T\frac{h(r,t)}{\lambda^2} - T\frac{\left(\frac{\partial h}{\partial r}\right)}{\Delta r} \tag{3.10}$$

Now divide by T, and $\frac{r+\Delta r}{r\Delta r}$, can be simplified as $\frac{1}{\Delta r}+\frac{1}{r}$:

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{(r+dr)\left[\left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}\Delta r + \frac{\partial h}{\partial r}\right]}{r\Delta r} + \frac{h(r,t)}{\lambda^2} - \frac{\left(\frac{\partial h}{\partial r}\right)}{\Delta r} \tag{3.11}$$

$$\frac{\partial h}{\partial t}\frac{S}{T} = \left(\frac{1}{r}+\frac{1}{\Delta r}\right)\left[\left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}\Delta r + \frac{\partial h}{\partial r}\right] + \frac{h(r,t)}{\lambda^2} - \frac{\left(\frac{\partial h}{\partial r}\right)}{\Delta r} \tag{3.12}$$

Factorization on both sides

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r} + \frac{\partial h}{\partial r}\frac{1}{\Delta r} + \frac{1}{r}\left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}\Delta r + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{h(r,t)}{\lambda^2} - \frac{1}{\Delta r}\left(\frac{\partial h}{\partial r}\right) \tag{3.13}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r} + \frac{1}{r}\left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}dr + \frac{\partial h}{\partial r}\frac{1}{r} + \frac{h(r,t)}{\lambda^2} \tag{3.14}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r} + \left(\frac{\partial h}{\partial r}\right)\frac{1}{r}\frac{\partial}{\partial r}dr + \frac{\partial h}{\partial r}\frac{1}{r} + \frac{h(r,t)}{\lambda^2} \tag{3.15}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial h}{\partial r}\right)\frac{\partial}{\partial r}\left(1 + \frac{\Delta r}{r}\right) + \frac{\partial h}{\partial r}\frac{1}{r} + \frac{h(r,t)}{\lambda^2} \tag{3.16}$$

The Eq. (3.16) includes mathematical formula and more terms within a leaky aquifer groundwater flow.

3.2.2 Fractional derivative approach

Fractured rock aquifers are very difficult due to heterogeneity in fractures which causes the groundwater flow and storage to be very complex. Nevertheless, a mathematical equation will be derived that takes into account the complexity of fractures and can be applied to define the storage effect in a leaky aquifer (Amanda and Atangana, 2018). Moreover, the introduction of the α fractional derivative will be applied to determine the groundwater flow equation when water is pumped out of the aquifer within a shorter period within the leaky aquifer.

$$\frac{\partial v}{\partial t} = {Q_1}^{\alpha} + {Q_3}^{\alpha} - Q_2 \tag{3.17}$$

The new discharge rate $Q_1,\ Q_3, and\ Q_2$ will be defined in terms of fractals α

$${Q_1}^{\alpha} = 2\pi(r + \Delta r)b\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r + \frac{\partial h}{\partial r^{\alpha}}\right)K \tag{3.18}$$

$${Q_2}^{\alpha} = 2\pi r b\left(\frac{\partial h}{\partial r^{\alpha}}\right)K \tag{3.19}$$

$$Q_3 = 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb \tag{3.20}$$

Now the substitution of equation $Q_1,\ Q_3, and\ Q_2$ into the original equation

$$S(2\pi r)dr\frac{\partial h}{\partial t} = 2\pi(r + \Delta r)b\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r + \frac{\partial h}{\partial r^{\alpha}}\right)K + 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb - 2\pi r b\left(\frac{\partial h}{\partial r^{\alpha}}\right)K \tag{3.21}$$

Therefore, we divide by $2\pi r\Delta r$ in both side of the equation

$$\frac{S(2\pi r)dr\frac{\partial h}{\partial t}}{2\pi r\Delta r} = \frac{2\pi(r + \Delta r)b\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r + \frac{\partial h}{\partial r^{\alpha}}\right)K}{2\pi r\Delta r} + \frac{2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb}{2\pi r\Delta r} - \frac{2\pi r b\left(\frac{\partial h}{\partial r^{\alpha}}\right)K}{2\pi r\Delta r} \tag{3.22}$$

$$S\frac{\partial h}{\partial t}=\frac{(r+\Delta r)b\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r+\frac{\partial h}{\partial r^{\alpha}}\right)K}{r\Delta r}+\frac{h(r,t)}{\lambda^2}Kb-\frac{b\left(\frac{\partial h}{\partial r^{\alpha}}\right)K}{\Delta r} \tag{3.23}$$

Since $T=Kb$, therefore

$$\frac{S\partial h}{\partial t}=T\frac{(r+\Delta r)\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r+\frac{\partial h}{\partial r^{\alpha}}\right)}{r\Delta r}+T\frac{h(r,t)}{\lambda^2}-T\frac{\left(\frac{\partial h}{\partial r^{\alpha}}\right)}{\Delta r} \tag{3.24}$$

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{(r+\Delta r)\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r+\frac{\partial h}{\partial r^{\alpha}}\right)}{r\Delta r}+\frac{h(r,t)}{\lambda^2}-\frac{\left(\frac{\partial h}{\partial r^{\alpha}}\right)}{\Delta r} \tag{3.24}$$

Now $\frac{r+\Delta r}{r\Delta r}$, can be represented as $\frac{1}{\Delta r}+\frac{1}{r}$:

$$\frac{S}{T}\frac{\partial h}{\partial t}=\left(\frac{1}{dr}+\frac{1}{r}\right)\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r+\frac{\partial h}{\partial r^{\alpha}}\right)+\frac{h(r,t)}{\lambda^2}-\frac{\left(\frac{\partial h}{\partial r^{\alpha}}\right)}{\Delta r} \tag{3.26}$$

In simple form the new equation

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)+\frac{1}{r}\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r+\frac{1}{r}\frac{\partial h}{\partial r^{\alpha}}+\frac{h(r,t)}{\lambda^2} \tag{3.27}$$

Simplification

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\left(1+\frac{\Delta r}{r}\right)+\frac{1}{r}\frac{\partial h}{\partial r^{\alpha}}+\frac{h(r,t)}{\lambda^2} \tag{3.28}$$

3.2.3 Water discharging in the aquifer

Now fractal derivative will be considered to a normal leaky aquifer, where water in the aquifer is flowing normally. The fractional derivative was used to define water leaving the aquifer in equation (3.28), however, now the water in the aquifer is assumed to be inflow.

$$\frac{\partial v}{\partial t}=Q_1+Q_3-Q_2 \tag{3.29}$$

Where $Q_1, Q_2, and\ Q_3$ can be defined in simple terms as:

$${Q_1}^{\alpha}=2\pi(r+\Delta r)b\left(\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r+\frac{\partial h}{\partial r^{\alpha}}\right)K \tag{3.30}$$

$$Q_2 = 2\pi rb\left(\frac{\partial h}{\partial r}\right)K \tag{3.31}$$

$$Q_3 = 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb \tag{3.32}$$

From equation (3.31) and (3.32) you can notice that the fractional derivative is not applied. Now we substitute the equations (3.30), (3.31), and (3.32) to the original equation.

$$S(2\pi r)\Delta r\frac{\partial h}{\partial t} = 2\pi(r+\Delta r)b\left(\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\Delta r + \frac{\partial h}{\partial r^\alpha}\right)K + 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb - 2\pi rb\left(\frac{\partial h}{\partial r}\right)K \tag{3.32}$$

Now divide $2\pi rdr$

$$S\frac{\partial h}{\partial t} = \frac{(r+\Delta r)b\left(\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\Delta r + \frac{\partial h}{\partial r^\alpha}\right)K}{r\Delta r} + \frac{h(r,t)}{\lambda^2}K - b\left(\frac{\partial h}{\partial r\Delta r}\right)K \tag{3.33}$$

Since we know that: $T = Kb$, therefore

$$S\frac{\partial h}{\partial t} = \frac{T(r+\Delta r)\left(\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\Delta r + \frac{\partial h}{\partial r^\alpha}\right)}{r\Delta r} + \frac{h(r,t)}{\lambda^2}T - \left(\frac{\partial h}{\partial r\Delta r}\right)T \tag{3.34}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{(r+\Delta r)\left(\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\Delta r + \frac{\partial h}{\partial r^\alpha}\right)}{r\Delta r} + \frac{h(r,t)}{\lambda^2} + \left(\frac{\partial h}{\partial r\Delta r}\right) \tag{3.35}$$

Simplification

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{1}{\Delta r} + \frac{1}{r}\right)\left(\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\Delta r + \frac{\partial h}{\partial r^\alpha}\right) + \frac{h(r,t)}{\lambda^2} - \left(\frac{\partial h}{\partial r\Delta r}\right) \tag{3.36}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right) + \frac{\partial h}{\partial r^\alpha}\frac{1}{\Delta r} + \frac{1}{r}\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\Delta r + \frac{1}{r}\frac{\partial h}{\partial r^\alpha} + \frac{h(r,t)}{\lambda^2} - \left(\frac{\partial h}{\partial r\Delta r}\right) \tag{3.37}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{\partial h}{\partial r^\alpha}\left(\frac{1}{\Delta r} + \frac{1}{r}\right) + \frac{h(r,t)}{\lambda^2} - \frac{\partial h}{\partial r\Delta r} \tag{3.38}$$

This equation is considered to have more benefits as compared to the one determined using classical operators and the estimation given by Hantush. This equation has more benefits and is more accurate than the suggested equation (3.16).

CHAPTER 4
DERIVATION OF NUMERICAL SOLUTIONS

4.1 INTRODUCTION

Due to challenges of groundwater flow encountered when the water is captured in time and space, the approach of a numerical solution with fractal-fractional operators also becomes a challenging task. Therefore, developing a numerical algorithm that will account for accuracy, stability, computation, and simplicity will be efficient. Thus, the different methods reported in the literature for groundwater flow within leaky aquifer with fractal-fractional operators will come to play. This chapter will aim to develop a new numerical exact approach to capture groundwater flow within a leaky aquifer by incorporating the concept of fractal-fractional operator. This concept will be achieved by applying the Predictor-Correct method and Atangana-Baleanu (AB) derivatives.

This chapter also presents a numerical approximation of the Predictor-Correct method and Atangana-Baleanu (AB) derivatives applications to groundwater modelling. Later on, a numerical solution will also be provided.

4.2 NUMERICAL APPROXIMATION

4.2.1 Predictor corrector method

In general, the Predictor-Corrector method (PCM) is a system that is used to assimilate ordinary differential equations (ODEs) that determines unknown differential equations for a given function. PCM is usually assembled using Euler's method (EM). This EM is mostly applied to determine 1st order differential equation assuming that the initial rate is provided. EM is an explicit method that transforms the 1st ODE to the Runge –Kutta method. The global error in EM is proportional to the step size while the local error in EM is proportional to the square size. In summary, EM is used to hypothesis more complex methods. EM methods can be constructed either by Taylor expansion such that y function is given by

$$y(t_0 + h) = y(t_0) + ht'(t_0) + \frac{1}{2}h^2 y''(t_0) + O(h^3) \tag{4.1}$$

The differential equation states that $y' = f(t, y)$. Suppose that equation (4.1) is substituted in the Taylor expansion such that only the first order derivative is considered, the Euler method arises. Therefore, Taylor's expansion will arise below to consider the errors encountered by the Euler method. Another method related derivation is to account for forward finite difference equation which substitutes for the derivative.

$$h'(t_0) \approx \frac{h(t_0 + p) - y(t_0)}{p} \tag{4.2}$$

In the differential equation $h' = f(t, y)$ produces the EM. Similar calculations either lead to a backward method and central method. Finally, when one considers integrating ODE from integral t_0 to $t_0 + p$ and apply the fundamental formula of calculus to results to

$$h(t_0 + p) - h(t_0) = \int_{t_0}^{t_0+p} f\big(t, y(t)\big)dt. \tag{4.3}$$

Now when applying the left-hand rectangle method, new approximate of the integral is

$$\int_{t_0}^{t_0+p} f\big(t, h(t)\big)dt \approx pf(t_0, h(t_0)). \tag{4.44}$$

Combining both equation (4.2) and (4.3) finally leads for the Euler method.

4.2.1.1 Trapezoidal rule

Trapezoidal is a mathematical technique applied numerical analysis to definite integrals for approximations.

$$\int_{i}^{t} f(x)dx. \tag{4.5}$$

The trapezoidal rule is based on approximation under the function$f(x)$, hence it calculates the area under the curve for a given function as follows

$$\int_{i}^{t} f(x)dx \approx (t - i)\,.\frac{f(i) + f(t)}{2}\ . \tag{4.6}$$

The trapezoidal rule may be obtained by combining both the left and right Riemann sums

4.2.1.2 Predictor-corrector method approximation for resolving ODEs

In numerical solution for ODEs, a PDM uses an explicit method to guess the next step and apply an implicit method for the numerically exact solution. Implicit method describing something that is implied or understood though not directly expressed. In mathematical terms something that is not expressed directly in terms of independent variables. A fully implicit FD approximation has head values are scaled to $t + 1$ are considered to represent the spatial derivative. PDM known as Heun's method can be constructed from the Euler method when applying an explicit method and the trapezoidal rule can also be constructed from the Euler method using the implicit method. When considering different equations such as.

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{4.7}$$

Now the 1st Predictor-Corrector when the value of y_i, guess will start to calculates the initial values at $\tilde{y}_{i+1}$ via the Euler method

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{4.8}$$

The next step improved by the initial guess using the trapezoidal rule,

$$y_{i+1} = y_i + \frac{1}{2} h(f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})). \tag{4.9}$$

Now, this value can be used for the next predictor step. Predictor-Corrector is applied in different methods. Equation (4.9) is referred to as Predict–Evaluate–Correct–Evaluate (PECE). Predictor Evaluate the corrector can evaluate a function step by step. Furthermore, PECE can be applied more than once to achieve approximation close to a solution. Lastly, when PDM is run more than twice it produces PECECE.

$$\tilde{y}_{i+1} = y_i + hf(t_i, y_i), \tag{4.10}$$

$$\hat{y}_{i+1} = y_i + \frac{1}{2} h(f(t_i, \tilde{y}_i) + f(t_{i+1}, \tilde{y}_{i+1})). \tag{4.11}$$

$$y_{i+1} = y_i + \frac{1}{2} h(f(t_i, y_i) + f(t_{i+1}, \hat{y}_{i+1})). \tag{4.12}$$

4.2.2 Adams-Bashforth Method (AB)

They are different methods applied for a numerical solution in linear multistep methods for ODE, more especially to problems with initial values.

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{4.14}$$

Adams-Moulton and Adams-Bashforth are methods usually to describe the linear multistep method. Both methods are constructed and approximated using integral polynomial interval defined at (t_n, t_{n+1}). These two methods are based on explicit and implicit theorems, the Adams-Bashforth based on explicit while Adams-Moulton method is defined by implicit type. To solve ODEs, firstly PDM algorithm is applied in two steps. Firstly, to predict the approximation of the preferred quantity using the implicit method and secondly to refine the initial approximation by means of applying an explicit method. The methods are based on the Langrage polynomial method (LPM). The LPM is generally used for verification in theoretical argumentation.

4.2.2.1 Derivation of two-step Adams-Bashforth method using polynomial interpolation

$$h(t_0+g)-h(t_0)=\int_{t_0}^{t_0+g} f\big(t,y(t)\big)dt. \tag{4.15}$$

Set

$$A=\int_{t_n}^{t_{n+1}} y'(t)dt=\int_{t_n}^{t_{n+1}} f\big(t,y(t)\big)dt\,. \tag{4.16}$$

The value of A is determined by using interpolation method of polynomial $P(t)$ such that the estimation is given by $f\big(t,y(t)\big)$. This interpolation of polynomial is approximated using Langrage from

$$L(g)\coloneqq\sum_{j=0}^{k} y_j\, l_j(g) \tag{4.17}$$

Where

$$l_j(g)\coloneqq\prod_{0\le m\le k.m\ne j}\frac{d-x_m}{d_j-x_m} \tag{4.18}$$

Then the interpolation can be

$$P(t)=f(t_n,y_n)\frac{t-t_{n-1}}{t_n-t_{n-1}}+f(t_{n-1},y_{n-1})\frac{t-t_n}{t_{n-1}-t_n} \tag{4.19}$$

Thus, equation becomes

$$\begin{aligned}A&=\int_{t_n}^{t_{n+1}} f\big(t,y(t)\big)dt\\ &\approx\int_{t_n}^{t_{n+1}} P(t)dt=\int_{t_n}^{t_{n+1}} f(t_n,y_n)\frac{t-t_{n-1}}{t_n-t_{n-1}}+f(t_{n-1},y_{n-1})\frac{t-t_n}{t_{n-1}-t_n}dt\end{aligned} \tag{4.20}$$

Integration and simplifying, the right-hand side of the equation (4.20) becomes

$$\frac{1}{2}(t_n+t_{n+1})\big(f(t_n,y_n)-f(t_{n-1},y_{n-1})\big)-t_{n-1}f(t_n,y_n)+t_nf(t_{n-1},y_{n-1}) \tag{4.21}$$

$$=\frac{1}{2}(t_n+t_{n+1}-2t_{n-1})f(t_n,y_n)+\frac{1}{2}(2t_n-t_{n+1}-t_n)f(t_{n-1},y_{n-1}) \tag{4.22}$$

Since t_{n-1},t_n and t_{n+1} are equally spaced, then $t_n-t_{n-1}=t_{n+1}-t_n=h.$ Therefore, the value of A is

$$A=\frac{3}{2}hf\big(t_n,y_n\big)-\frac{1}{2}hf(t_{n-1},y_{n-1}) \tag{4.23}$$

Now substituting back to equation (4.23) yields

$$y(t_{n+1}) \approx y(t_n) + \frac{3}{2}hf(t_n, y_n) - \frac{1}{2}hf(t_{n-1}, y_{n-1}) \tag{4.24}$$

Thus, the equation $y_{n+1} = y_n + \frac{3}{2}hf(t_n, y_n) - \frac{1}{2}hf(t_{n-1}, y_{n-1})$ is the two step Adams – Bashforth method.

4.2.3 Atangana-Baleanu Derivative in Caputo sense (ABC)

Due to the complications of groundwater flow within a leak aquifer, there is a gap that has to be enclosed by introducing the application of operators that will account for the complexity of the aquifer. Recent studies by Mekkaoui *et al*. (2020) suggested fractal-fractional operator which is convolution and is based on generalized Mittag-Leffler function. Furthermore, this new fractal-fractional differential operator is called Atangana-Baleanu derivative (AB) and it includes both non-local and non-singular kernel functions. AB derivative has been revised by other researchers and it presented excellent results in resolving real-world physical problems. Therefore, this chapter will aim to apply the benefits of AB derivative to resolve real-world problems.

The new AB derivative is given by

Definition: Lest assume that g(x) is continuous on an open interval (a, b), then the fractal-fractional integration of g(x) with order α, and having Mittag-Leffler type kernel is assumed by

$${}^{\mathrm{PPM}}D_{0,t}^{\alpha,\beta}\big(g(x)\big) = \frac{\alpha\beta}{\mathrm{AB}(\alpha)\Gamma(\alpha)}\int_0^x r^{\beta-1}\,(g(r)(t-r)^{\alpha-1}dr + \frac{\beta(1-\alpha)x^{\beta-1}g(x)}{AB(\alpha)} \tag{4.25}$$

Given the leaky aquifer equation

$$\frac{S}{T}\frac{\partial h(r,t)}{\partial t} = \frac{\partial h(r,t)}{r\partial r} + \frac{\partial^2 h(r,t)}{\partial r^2}\left(1 + \frac{\Delta r}{r}\right) + \frac{h(r,t)}{\lambda^2} \tag{4.26}$$

Let

$$\frac{\partial h}{\partial t} = f(r,t,h) = g(t,h(t)) \tag{4.27}$$

Therefore

$${}^{ff}_{\;0}D_t^{\alpha,\beta}h(t) = g(t,h(t)) \tag{4.28}$$

$$h(0) = h_0,$$

Now if we can apply the derivative of AB fractal-fractional to equation (4.28), the equation will be given by

$$h(t) - h(0) = \frac{1-\alpha}{AB(\alpha)}\beta t^{\beta-1}g(t,h(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t r^{\beta-1}(t-r)^{\alpha-1}\,g\big(r,h(r)\big)dr. \tag{4.29}$$

Initial conditions

$$h(t) = h_0 + \frac{1-\alpha}{AB(\alpha)} G\big(t, h(t)\big) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{i=0}^{m} \int_0^t G\big(r, h(r)\big)\,(t-r)^{\alpha-1} dr \tag{4.30}$$

Where at $t_{m+1} = (m+1)\Delta t$ we have

$$\begin{aligned} h(t_{m+1}) = h_0 &+ \frac{1-\alpha}{AB(\alpha)} \beta t_{m+1}^{\beta-1} g\big(t_{m+1}, h(t_{m+1})\big) \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1}-r)^{\alpha-1} g(r, h(r)) dr. \end{aligned} \tag{4.31}$$

The function $g(r, h(r))$ is based on approximation and can be represented as newton polynomial which is given by

$$\begin{aligned} P_2(r) = g(t_{i+1}, h_{i+1}) &+ \frac{g\big(t_{i+1,} h_{i+1}\big) - 2g(t_i, h_i) + g(t_{t-1,} h_{i-1})}{2f^2} \left(1 \frac{r_{i+1} - t_i}{r_i}\right)(r - t_{i+1})(r - t_i) \\ &+ \frac{1}{r_i} \frac{g\big(t_{i+1,} h_{i+1}\big) - g(t_{i,} h_i)}{f} (r - t_{i+1}) + \frac{g(r_{i,} t_i)}{\lambda^2} \end{aligned} \tag{4.32}$$

Now if we substitute polynomial back into the equation

$$\begin{aligned} h_{m+1} = h_0 &+ \frac{1-\alpha}{AB(\alpha)} \beta t_{m+1}^{\beta-1} g\big(t_{m+1,} h_{m+1}^{p}\big) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \\ &\times \sum_{i=0}^{m} \left\{ \begin{array}{c} g(t_{i+1}, h_{i+1}) k_{1,i}^{\alpha,\beta} + \\ \dfrac{g\big(t_{i+1,} h_{i+1}\big) - g\big(t_{i,} h_i\big)}{f} k_{2,i}^{\alpha,\beta} + \\ \dfrac{g\big(t_{i+1,} h_{i+1}\big) - 2g(t_i, h_i) + g(t_{t-1,} h_{i-1})}{2f^2} k_{3,i}^{\alpha,\beta} \end{array} \right\} \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} g\big(t_{m+1,} h_{m+1}^{p}\big) k_{1,i}^{\alpha,\beta} \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{g\big(t_{m+1,} h_{m+1}^{p}\big) - g(t_{m,} h_m)}{f} k_{2,i}^{\alpha,\beta} \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{g\big(t_{m+1,} h_{m+1}^{p}\big) - 2g(t_m, h_m) + g(t_{m-1,} h_{m-1})}{2f^2} k_{3,i}^{\alpha,\beta}. \end{aligned} \tag{4.33}$$

$$k_{1,i}^{\alpha,\beta} = \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1}\, dr \tag{4.34}$$

$$k_{2,i}^{\alpha,\beta} = \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1} (r - t_{i+1})\, dr \tag{4.35}$$

$$k_{3,i}^{\alpha,\beta} = \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1}(r - t_{i+1})(r - t_i)dr \tag{4.36}$$

Here

$$h_{m+1}^{p} = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} g(t_m, h_m) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1} g(r, h(r))dr \tag{4.37}$$

Also, we can write it as

$$h_{m+1}^{p} = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} g(t_m, h_m) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m} g(t_i, h_i)\int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1}\, dr \tag{4.38}$$

And

$$h_{m+1}^{p} = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} g(t_m, h_m) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m} g(t_i, h_i) k_{1,i}^{\alpha,\beta} \tag{4.39}$$

We write the first iteration as

$$h_1 = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta Q^{\beta-1} g(t_0, h_0) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m} g(t_0, h_0) k_{0,0}^{\alpha,\beta} \tag{4.40}$$

Where

$$k_{0,0}^{\alpha,\beta} = \frac{\Gamma(1+\beta)\Gamma(\alpha)Q^{\alpha+\beta-1}}{\beta\Gamma(\alpha+\beta)} \tag{4.42}$$

4.3 NUMERICAL SOLUTION

4.3.1 Application of predictor corrector method in normal leaky aquifer

Initially we have introduced the groundwater flow equation at the beginning of this thesis. Furthermore, the Predictor-Corrector method will be introduced to modify the equation by Hantush presented in Chapter 3. Now, the Predictor method equation will be applied to the equation in leaky aquifers. This Predictor-Corrector method is applied to discretization the equation and account for the complexity of the leaky aquifers.

4.3.1.1 Numerical solution within a normal leaky aquifer

$$\frac{S}{T}\frac{\partial h(r,t)}{\partial t}=\frac{\partial h(r,t)}{r\partial r}+\frac{\partial^2 h(r,t)}{\partial r^2}\left(1+\frac{\Delta r}{r}\right)+\frac{h(r,t)}{\lambda^2} \tag{4.43}$$

$$\frac{\partial h(r,t)}{\partial t}=f(h(r,t),t) \tag{4.44}$$

According to the method

$$y_p^{n+1}=y_n+hf(t_n,y_n) \tag{4.45}$$

$$y^{n+1}=y_n+\frac{h}{2}(f(t_n,y_n)+f(t_{n+1},y_p^{n+1}\ ) \tag{4.46}$$

$$f\big((r,t),h(r,t)\big)=\frac{\partial h(r,t)}{r\partial r}+\frac{\partial^2 h(r,t)}{\partial r^2}+\frac{h(r,t)}{\lambda^2} \tag{4.47}$$

$$f(r_i,t_n,h(r_i,t_n)=\frac{T}{S}\left(\frac{\partial h(r_i,t_n)}{r_i\partial r}+\frac{\partial^2 h(r_i,t_n)}{\partial r^2}+\frac{h(r_i,t_n)}{\lambda^2}\right) \tag{4.48}$$

Where

$$\frac{\partial h(r_i,t_n)}{\partial r}=\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r} \tag{4.49a}$$

$$\frac{\partial^2 h(r,t)}{\partial r^2}=\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2} \tag{4.49b}$$

Now substitute back equation (4.49a) and (4.49b) into (4.48)

$$\begin{aligned}f(r_i,t_n,h(r_i,t_n)\\&=\frac{T}{S}\left(\frac{1}{r_i}\left(\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\right)\right.\\&\left.+\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)+\frac{h(r_i,t_n)}{\lambda^2}\right)\end{aligned} \tag{4.50}$$

According to the equation (4.45), we have

$$\begin{aligned}y_{p,i}^{n+1}=y_i^n+h\frac{T}{S}&\left[\frac{1(}{r_i}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n))}{2\Delta r}\right.\\&\left.+\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)+\frac{h(r_i,t_n)}{\lambda^2}\right]\end{aligned} \tag{4.51}$$

Now the new predictor is given by, according to equation (4.46)

$$y_i^{n+1} = y_i^n + \frac{hT}{2S}\left\{\left[\frac{1}{r_i}\frac{(h(r_{i+1},t_n) - h(r_{i-1},t_n))}{2\Delta r} + \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}\right] + \left[\frac{1}{r_i}\frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r} + \frac{h_p(r_i,t_{n+1})}{\lambda^2} + \frac{h_p(r_{i+1},t_{n+1}) - 2h_p(r_i,t_{n+1}) + h_p(r_{i-1},t_{n+1})}{(\Delta r)^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right)\right]\right\} \tag{4.52}$$

Applying predictor corrector

$$y_i{}^{p,n+1} = h_i{}^n + hf(r_i, t_n, h_i^n) \tag{4.53}$$

$$y_i^{n+1} = h_i{}^n + \frac{h}{2}\left[f(r_i, t_n, h_i^n) + f(r_i, t_n, h_i^{p,n+1})\right] \tag{4.54}$$

Substitute equation (4.50) into equation (4.53) and (4.54)

$$y_i{}^{p,n+1} = h_i{}^n + h\frac{T}{S}\left[\frac{1}{r_i}\frac{(h(r_{i+1},t_n) - h(r_{i-1},t_n))}{2\Delta r} + \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}\right] \tag{4.55}$$

$$y_i^{n+1} = h_i{}^n + \frac{hT}{2S}\left\{\left[\frac{1}{r_i}\frac{(h(r_{i+1},t_n) - h(r_{i-1},t_n))}{2\Delta r} + \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}\right] + \left[\frac{1}{r_i}\frac{(h(r_{i+1},t_n) - h(r_{i-1},t_n))}{2\Delta r} + \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}, h_i^{p,n+1}\right]\right\} \tag{4.56}$$

4.3.1.2 Numerical solution derived from leaky aquifer: water flowing in and out of the leaky aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\frac{\partial h}{\partial r^\alpha} + \frac{h(r,t)}{\lambda^2} \tag{4.57}$$

$$\frac{\partial h(r,t)}{\partial t} = f(h(r,t), t) \tag{4.58}$$

The equation above is a double derivative and can further be simplified below.

$$\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)=\frac{\partial}{\partial r}\left(\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right)\frac{r^{1-\alpha}}{\alpha}=\frac{\partial^2 h(r,t)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2}+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha} \tag{4.59}$$

In time is given by

$$f(r,t),h(r,t)=\frac{T}{S}\left[\frac{\partial^2 h(r,t)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2}+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right)+\frac{1}{r}\frac{\partial h(r,t)}{\partial r^\alpha}+\frac{h(r,t)}{\lambda^2}\right] \tag{4.60}$$

Given in space

$$f(r_i,t_n,h(r_i,t_n)=\frac{T}{S}\left[\frac{\partial^2 h(r_i,t_n)}{\partial r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_i)}{\partial r}{r_i}^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right)+\frac{1}{r_i}\frac{\partial h(r_i,t_i)}{\partial r^\alpha}+\frac{h(r_i,t_n)}{\lambda^2}\right] \tag{4.61}$$

Now

$$\begin{aligned}f(r_i,t_n,h(r_i,t_n)&=\frac{T}{S}\left[\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-r_i}{r_i}\right)\right]\\&+\frac{T}{S}\left[\frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\left(1+\frac{r_{i+1}-r_i}{r_i}\right){r_i}^{1-2\alpha}\right]\\&++\frac{T}{S}\left[\frac{1}{r_i}\frac{{r_i}^{1-\alpha}}{\alpha}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}+\frac{h(r_i,t_i)}{\lambda^2}\right]\end{aligned} \tag{4.62}$$

According to the method:

$$y_p^{n+1}=y_n+hf(t_n,y_n) \tag{4.63}$$

$$y^{n+1}=y_n+\frac{h}{2}(f(t_n,y_n)+f(t_{n+1},y_p^{n+1}\ ) \tag{4.64}$$

Therefore, substitute equation (4.62) into (4.63) and (4.64)

$$\begin{aligned}y_{p,i}^{n+1}=y_i^n&+h\frac{T}{S}\left[\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-r_i}{r_i}\right)\right]\\&+h\frac{T}{S}\left[\frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\left(1+\frac{r_{i+1}-r_i}{r_i}\right){r_i}^{1-2\alpha}\right]\\&+h\frac{T}{S}\left[\frac{1}{r_i}\frac{{r_i}^{1-\alpha}}{\alpha}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}+\frac{h(r_i,t_i)}{\lambda^2}\right]\end{aligned} \tag{4.65}$$

Now the predictor method is given by:

$$
\begin{aligned}
y_i^{n+1} = y_i^n &+ \frac{hT}{2S}\left[\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-r_i}{r_i}\right)\right] \\
&+ \frac{hT}{2S}\left[\frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\left(1+\frac{r_{i+1}-r_i}{r_i}\right){r_i}^{1-2\alpha}\right] \\
&+ \frac{hT}{2S}\left[\frac{1}{r_i}\frac{{r_i}^{1-\alpha}}{\alpha}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}+\frac{h(r_i,t_n)}{\lambda^2}\right] \\
&+ \frac{hT}{2S}\left[\frac{h_p(r_{i+1},t_{n+1})-2h_p(r_i,t_{n+1})+h_p(r_{i-1},t_{n-1})}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-r_i}{r_i}\right)\right] \\
&+ \frac{hT}{2S}\left[\frac{(1-\alpha)}{\alpha^2}\frac{h_p(r_{i+1},t_{n+1})-h_p(r_{i-1},t_n)}{2\Delta r}\left(1+\frac{r_{i+1}-r_i}{r_i}\right){r_i}^{1-2\alpha}\right] \\
&+ \frac{hT}{2S}\left[\frac{1}{r_i}\frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_p(r_{i+1},t_{n+1})-h_p(r_{i-1},t_n)}{2\Delta r}+\frac{h(r_i,t_{n+1})}{\lambda^2}\right]
\end{aligned} \tag{4.66}
$$

New predictor corrector

$$
{y_i}^{p,n+1} = {h_i}^n + hf(r_i,t_n,h_i^n) \tag{4.67}
$$

$$
y_i^{n+1} = {h_i}^n + \frac{h}{2}\left[f(r_i,t_n,h_i^n)+f(r_i,t_n,h_i^{p,n+1})\right] \tag{4.68}
$$

Substitute equation (4.62) into (4.67) and (4.68)

$$
\begin{aligned}
{y_i}^{p,n+1} = {h_i}^n + \frac{hT}{S}\Bigg[&\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-r_i}{r_i}\right) \\
&+ \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\left(1+\frac{r_{i+1}-r_i}{r_i}\right){r_i}^{1-2\alpha} \\
&+ \frac{1}{r_i}\frac{{r_i}^{1-\alpha}}{\alpha}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}+\frac{h(r_i,t_i)}{\lambda^2}\Bigg]
\end{aligned} \tag{4.49}
$$

$$h_i{}^{n+1} = h_i{}^n + \frac{h}{2}\frac{T}{S}\Bigg\{\Bigg[\frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - r_i}{r_i}\right) \tag{4.70}$$

$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\left(1 + \frac{r_{i+1} - r_i}{r_i}\right) r_i{}^{1-2\alpha}$$

$$+ \frac{1}{r_i}\frac{r_i{}^{1-\alpha}}{\alpha}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r} + \frac{h(r_i,t_i)}{\lambda^2}\Bigg]$$

$$+ \Bigg[\frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - r_i}{r_i}\right)$$

$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\left(1 + \frac{r_{i+1} - r_i}{r_i}\right) r_i{}^{1-2\alpha}$$

$$+ \frac{1}{r_i}\frac{r_i{}^{1-\alpha}}{\alpha}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r} + \frac{h(r_i,t_n)}{\lambda^2}, h_i{}^{p,n+1}\Bigg]\Bigg\}$$

4.3.1.3 Numerical solution derived from leaky aquifer: water flowing in and out of the leaky aquifer normal aquifer, fractal derivative considered only on inflow

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\frac{\partial h}{\partial r^\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r}\right) + \frac{h(r,t)}{\lambda^2} - \frac{\partial h}{\partial r \Delta r} \tag{4.71}$$

Given in time

$$\frac{\partial h}{\partial t} = \frac{T}{S}\Bigg[\left(\frac{\partial^2 h(r,t)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2} + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r} r^{1-2\alpha}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{\partial h(r,t)}{\partial r^\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r}\right) + \frac{h(r,t)}{\lambda^2} - \frac{\partial h}{\partial r \Delta r}\Bigg] \tag{4.72}$$

In space

$$f(r_i, t_n, h(r_i,t_n)) = \frac{T}{S}\Bigg[\frac{\partial^2 h(r_i,t_n)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1 + \frac{\Delta r}{r_i}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r_i,t_n)}{\partial r} r_i{}^{1-2\alpha}\left(1 + \frac{\Delta r}{r_i}\right) + \frac{\partial h(r_i,t_n)}{\partial r^\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2} - \frac{\partial h(r_i,t_n)}{\partial r \Delta r}\Bigg] \tag{4.73}$$

Now

$$f(r_i,t_n,h(r_i,t_n) = \frac{T}{S}\left[\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}r_i^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right)+\frac{h(r_i,t_n)}{\lambda^2} - \frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\frac{1}{\Delta r}\right] \tag{4.74}$$

According to the method, substitute equation (4.74) into (4.63) and (4.64)

$$y_{p,i}^{n+1} = y_i^n + h\frac{T}{S}\left[\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}r_i^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right)+\frac{h(r_i,t_n)}{\lambda^2} - \frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\frac{1}{\Delta r}\right] \tag{4.75}$$

The predictor correct method

$$y_i^{n+1} = y_i^n + \frac{hT}{2S}\left\{\left[\frac{h(r_{i+1},t_n)-2h(r_i,t_n)+h(r_{i-1},t_n)}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}r_i^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right)+\frac{h(r_i,t_n)}{\lambda^2} - \frac{h(r_{i+1},t_n)-h(r_{i-1},t_n)}{2\Delta r}\frac{1}{\Delta r}\right] + \left[\frac{h_p(r_{i+1},t_{n+1})-2h_p(r_i,t_{n+1})+h_p(r_{i-1},t_{n+1})}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{h_p(r_{i+1},t_{n+1})-h_p(r_{i-1},t_n)}{2\Delta r}{r_i}^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right) + \frac{h_p(r_{i+1},t_{n+1})-h_p(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right)+\frac{h(r_i,t_{n+1})}{\lambda^2} - \frac{h_p(r_{i+1},t_{n+1})-h_p(r_{i-1},t_n)}{2\Delta r}\left(\frac{1}{\Delta r}\right)\right]\right\} \tag{4.76}$$

Recall that

$$y_i{}^{p,n+1} = h_i{}^n + hf(r_i, t_n, h_i^n) \tag{4.77}$$

$$y_i^{n+1} = h_i{}^n + \frac{h}{2}\left[f(r_i, t_n, h_i^n) + f(r_i, t_n, h_i^{p,n+1})\right] \tag{4.78}$$

Now substitute equation (4.73) into equation (4.77)

$$\begin{aligned} y_i{}^{p,n+1} = h_i{}^n + h\frac{T}{S}\Bigg[& \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2}\frac{r_i{}^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) \\ & + \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}r_i{}^{1-2\alpha}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) \\ & + \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}\frac{r_i{}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h(r_i, t_n)}{\lambda^2} \\ & - \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}\frac{1}{\Delta r}\Bigg] \end{aligned} \tag{4.79}$$

Substitute equation (4.79) into equation (4.78)

$$\begin{aligned} y_i^{n+1} = h_i{}^n + \frac{h}{2}\frac{T}{S}\Bigg\{\Bigg[& \frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2}\frac{r_i{}^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) \\ & + \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}r_i{}^{1-2\alpha}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) \\ & + \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}\frac{r_i{}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h(r_i, t_n)}{\lambda^2} \\ & - \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}\frac{1}{\Delta r}\Bigg] \\ & + \Bigg[\frac{h(r_{i+1}, t_n) - 2h(r_i, t_n) + h(r_{i-1}, t_n)}{\Delta r^2}\frac{r_i{}^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) \\ & + \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}r_i{}^{1-2\alpha}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) \\ & + \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}\frac{r_i{}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h(r_i, t_n)}{\lambda^2} \\ & - \frac{h(r_{i+1}, t_n) - h(r_{i-1}, t_n)}{2\Delta r}\frac{1}{\Delta r}, h_i{}^{p,n+1}\Bigg]\Bigg\} \end{aligned} \tag{4.80}$$

4.3.2 Application of Atangana-Baleanu Derivative in Caputo sense (ABC)

$$ {}^{\mathrm{PPM}}D_{0,t}^{\alpha,\beta}\big(g(x)\big) = \frac{\alpha\beta}{\mathrm{AB}(\alpha)\Gamma(\alpha)} \int_0^x r^{\beta-1}\,(g(r)(t-r)^{\alpha-1}dr + \frac{\beta(1-\alpha)x^{\beta-1}g(x)}{AB(\alpha)} \tag{4.81} $$

4.3.2.1 Numerical solution within a normal leaky aquifer

Given the leaky aquifer equation

$$ \frac{S}{T}\frac{\partial h(r,t)}{\partial t} = \frac{\partial h(r,t)}{r\partial r} + \frac{\partial^2 h(r,t)}{\partial r^2}\left(1+\frac{\Delta r}{r}\right) + \frac{h(r,t)}{\lambda^2} \tag{4.82} $$

Let $\frac{\partial h}{\partial t} = f(r,t,h) = g(t,h(t))$

Therefore

$$ \begin{gathered} {}_0^{ff}D_t^{\alpha,\beta} h(t) = g(t,h(t)) \\ h(0) = h_0, \end{gathered} \tag{4.83} $$

From section 4.2.3, replace the following

$$ h_{m+1} = h_{(r,m+1)} \tag{4.84} $$

$$ g\big(t_{m+1,}h_{m+1}^{p}\big) = F(r,t_{m+1,}h(r,t_{m+1})) \tag{4.85} $$

$$ g\big(t_{i+1,}h_{i+1}\big) = F(r,t_{i+1,}h(r,t_{i+1})) \tag{4.86} $$

$$ g\big(t_{m-1,}h_{m-1}\big) = F(r,t_{m-1,}h(r,t_{m-1})) \tag{4.87} $$

Now if we can apply the derivative of Atangana Baleanu fractal-fractional to equation. Integrating equation (4.83) will give rise to

$$ h(t) - h(0) = \frac{1-\alpha}{AB(\alpha)}\beta t^{\beta-1}F(t,h(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t r^{\beta-1}(t-r)^{\alpha-1}\,F\big(r,h(r)\big)dr. \tag{4.88} $$

Initial conditions

$$ h(t) = h_0 + \frac{1-\alpha}{AB(\alpha)}F\big(t,h(t)\big) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_0^t F\big(r,h(r)\big)\,(t-r)^{\alpha-1}dr \tag{4.89} $$

Where at $t_{m+1} = (r,m+1)\Delta t$ we have

$$h(t_{r,m+1}) = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} F\big(r, t_{m+1}, h(r, t_{m+1})\big) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1}-r)^{\alpha-1} F(r, h(r))dr. \tag{4.90}$$

The function $F(r, h(r))$ is based on approximation and can be represented as newton polynomial which is given by

$$P_2(r) = F(r, t_{i+1}, h(r, t_{i+1}) + \frac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - 2F(t_i, h_i) + F(r, t_{t-1,} h(r, t_{i-1}))}{2f^2}\left(1\frac{r_{i+1}-t_i}{r_i}\right)(r - t_{i+1})(r-t_i) + \frac{1}{r_i}\frac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - F(t_{i,} h_i)}{f}(r - t_{i+1}) + \frac{F(r_{i,} t_i)}{\lambda^2} \tag{4.91}$$

Now if we substitute polynomial into the equation

$$\begin{aligned} &h(r_i, t_{m+1}) \\ &= h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} F\big(r, t_{m+1,} h^p(r, t_{m+1})\big) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \\ &\times \sum_{i=0}^{m} \left\{ \begin{array}{c} F(r, t_{i+1}, h(r, t_{i+1}))k_{1,i}^{\alpha,\beta} + \\ \dfrac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - F\big(t_{i,} h_i\big)}{f} k_{2,i}^{\alpha,\beta} + \\ \dfrac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - 2F(t_i, h_i) + F(r, t_{i-1,} h(r, t_{i-1}))}{2f^2} k_{3,i}^{\alpha,\beta} \end{array} \right\} \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} F\big(r, t_{m+1,} h^p(r, t_{m+1}\big) k_{1,i}^{\alpha,\beta} \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{F\big(r, t_{m+1,} h^p(r, t_{m+1}\big) - F(t_{m,} h_m)}{f} k_{2,i}^{\alpha,\beta} \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{F\big(r, t_{m+1,} h^p(r, t_{m+1})\big) - 2F(t_m, h_m) + F(r, t_{m-1,} h(r, t_{m-1}))}{2f^2} k_{3,i}^{\alpha,\beta}. \end{aligned} \tag{4.92}$$

For more, the identities are given by

$$k_{1,i}^{\alpha,\beta} = \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1}\, dr \tag{4.93}$$

$$k_{2,i}^{\alpha,\beta} = \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1}(r - t_{i+1})\, dr \tag{4.94}$$

$$k_{3,i}^{\alpha,\beta} = \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1}(r - t_{i+1})(r - t_i) dr \tag{4.95}$$

Here

$$h_{m+1}^{p}=h_0+\frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}g(t_m,h_m)$$
$$+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}r^{\beta-1}\times(t_{m+1}-r)^{\alpha-1}g(r,h(r))dr \tag{4.96}$$

Also, we can write it as

$$h_{m+1}^{p}=h_0+\frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}F(t_m,h_m)$$
$$+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}F(t_{i,}h_i)\int_{t_i}^{t_{i+1}}r^{\beta-1}\times(t_{m+1}-r)^{\alpha-1}\,dr \tag{4.97}$$

And

$$h_{m+1}^{p}=h_0+\frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}F(t_m,h_m)+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}F(t_{i,}h_i)k_{1,i}^{\alpha,\beta} \tag{4.98}$$

We write the first iteration as

$$h_1=h_0+\frac{1-\alpha}{AB(\alpha)}\beta Q^{\beta-1}F(t_0,h_0)+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}F(t_{0,}h_0)k_{0,0}^{\alpha,\beta} \tag{4.99}$$

Where

$$k_{0,0}^{\alpha,\beta}=\frac{\Gamma(1+\beta)\Gamma(\alpha)Q^{\alpha+\beta-1}}{\beta\Gamma(\alpha+\beta)} \tag{4.100}$$

Since we know that

$$F(r,t_{m+1},h^p(r,t_{m+1}))=F(r,t,h(r,t))=\frac{T}{S}\left[\frac{\partial h(r,t)}{r\partial r}+\frac{\partial^2 h(r,t)}{\partial r^2}\left(1+\frac{\Delta r}{r}\right)+\frac{h(r,t)}{\lambda^2}\right] \tag{4.101}$$

This can also be given as

$$F(r,t_{m+1},h^p(r,t_{m+1}))=\frac{T}{S}\left[\frac{\partial h^p(r,t_{m+1})}{r\partial r}+\frac{\partial^2 h(r,t_{m+1})}{\partial r^2}\left(1+\frac{\Delta r}{r}\right)+\frac{h^p(r,t_{m+1})}{\lambda^2}\right] \tag{4.102}$$

and

$$F(r_i,t_{m+1},h^p(r_i,t_{m+1}))$$
$$=\frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}+\frac{h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{p,m+1}}{\lambda^2}\right] \tag{4.103}$$

$$F\big(r_i,t_{m-1,}h(r_i,t_{m-1})\big)=\frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^{m-1}-h_{i-1}^{m-1}}{2\Delta r}+\frac{h_{i+1}^{m-1}-2h_i^{m-1}+h_{i-1}^{m-1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{m-1}}{\lambda^2}\right] \tag{4.104}$$

Now substitute equation (4.103) and (4.104) into equation (4.92)

`

$$
\begin{aligned}
h(r_i, t_{m+1}) = h(r_i, 0) & \qquad (4.105)\\
&+ \frac{1-\alpha}{AB(\alpha)} \beta t_{m+1}^{\beta-1} \frac{T}{S} \left[\frac{1}{r_i} \frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r} \right.\\
&\left. + \frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) + \frac{h_i^{p,m+1}}{\lambda^2} \right] + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\\
&\times \sum_{i=0}^{m} \left\{ \begin{matrix} F(r, t_{i+1}, h(r, t_{i+1})) k_{1,i}^{\alpha,\beta} + \\ \frac{F\left(r, t_{i+1,} h(r, t_{i+1})\right) - F\left(t_{i,} h_i\right)}{f} k_{2,i}^{\alpha,\beta} + \\ \frac{F\left(r, t_{i+1,} h(r, t_{i+1})\right) - 2F(t_i, h_i) + F(r, t_{i-1,} h(r, t_{i-1}))}{2f^2} k_{3,i}^{\alpha,\beta} \end{matrix} \right\}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{T}{S} \left[\frac{1}{r_i} \frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r} + \frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) \right.\\
&\left. + \frac{h_i^{p,m+1}}{\lambda^2} \right] k_{1,i}^{\alpha,\beta}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \left[\frac{T}{Sf} \left(\frac{1}{r_i} \frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r} \right. \right.\\
&\left. \left. + \frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) + \frac{h_i^{p,m+1}}{\lambda^2} \right) - \frac{F\left(t_{m,} h_m\right)}{f} \right] k_{2,i}^{\alpha,\beta}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \left[\frac{T}{2Sf^2} \left(\frac{1}{r_i} \frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r} \right. \right.\\
&+ \frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) + \frac{h_i^{p,m+1}}{\lambda^2} + \frac{1}{r_i} \frac{h_{i+1}^{m-1} - h_{i-1}^{m-1}}{2\Delta r}\\
&\left. \left. + \frac{h_{i+1}^{m-1} - 2h_i^{m-1} + h_{i-1}^{m-1}}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) + \frac{h_i^{m-1}}{\lambda^2} \right) - \frac{F(t_m, h_m)}{f^2} \right] k_{3,i}^{\alpha,\beta}.
\end{aligned}
$$

Now let

$$
\frac{1-\alpha}{AB(\alpha)} \beta t_{m+1}^{\beta-1} = a, \qquad \frac{1}{r_i \Delta r} = b, \qquad \frac{1}{(\Delta r)^2} \left(1 + \frac{\Delta r}{r_i}\right) = c, \qquad \frac{1}{\lambda^2} = d
$$

$$
\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} = e, \qquad \frac{1}{f^2} = g, \qquad \frac{T}{S} = k, \qquad k_{1,i}^{\alpha,\beta} = a_1, \qquad \frac{1}{f} = g_1
$$

$$
k_{2,i}^{\alpha,\beta} = a_2, \qquad k_{3,i}^{\alpha,\beta} = a_3
$$

$$
e \times \sum_{i=0}^{m} \left\{ \begin{matrix} F(r, t_{i+1}, h(r, t_{i+1})) k_{1,i}^{\alpha,\beta} + \\ \frac{F\left(r, t_{i+1,} h(r, t_{i+1})\right) - F\left(t_{i,} h_i\right)}{f} k_{2,i}^{\alpha,\beta} + \\ \frac{F\left(r, t_{i+1,} h(r, t_{i+1})\right) - 2F(t_i, h_i) + F(r, t_{i-1,} h(r, t_{i-1}))}{2f^2} k_{3,i}^{\alpha,\beta} \end{matrix} \right\} = D
$$

Substitution

$$h(r_i, t_{m+1}) = h(r_i, 0) \tag{4.106}$$
$$+ ak\left[\frac{1}{2}b\left(h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}\right) + c\left(h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}\right) + dh_i^{p,m+1} +\right]$$
$$+ D$$
$$+ ek\left[\frac{1}{2}b\left(h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}\right) + c\left(h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}\right) + dh_i^{p,m+1}\right]a_1$$
$$+ e\left[kg_1\left(\frac{1}{2}b\left(h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}\right) + c\left(h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}\right)\right.\right.$$
$$\left.\left. + dh_i^{p,m+1}\right) - dg_1F\left(t_{m,}h_m\right)\right]a_2$$
$$+ e\left[kg\frac{1}{2}\left(\frac{1}{2}b\left(h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}\right) + c\left(h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}\right)\right.\right.$$
$$\left. + dh_i^{p,m+1} + b\frac{1}{2}(h_{i+1}^{m-1} - h_{i-1}^{m-1}) + c(h_{i+1}^{m-1} - 2h_i^{m-1} + h_{i-1}^{m-1}) + dh_i^{m-1}\right)$$
$$\left. - dg_1F\left(t_{m,}h_m\right)\right]a_3$$

Grouping

$$h(r_i, t_{m+1}) = h_i^m\left(1 - edg_1a_2 - edg_1a_3\frac{1}{2}\right) + h_{i+1}^{m-1}\left[e\left(b\frac{1}{4} + c\frac{1}{2}\right)\right] \tag{4.107}$$
$$+ h_i^{m-1}ekg\left(-c + d\frac{1}{2}\right) + h_{i-1}^{m-1}ekg\left(-b\frac{1}{4} + c\frac{1}{2}\right)$$
$$+ h_{i+1}^{p,m+1}\left[ak\left(b\frac{1}{2} + c\right) + ek\left(b\frac{1}{2} + c\right)a_1 + e\left(kg_1\left(b\frac{1}{2} + c\right)\right)a_2\right.$$
$$\left. + e\left(kg\frac{1}{2}\left(b\frac{1}{2} + c\right)\right)a_3\right]$$
$$+ h_{i-1}^{p,m+1}\left[ak\left(-b\frac{1}{2} + c\right) + ek\left(-b\frac{1}{2} + c\right)a_1 + e\left(kg_1\left(-b\frac{1}{2} + c\right)a_2\right)\right.$$
$$\left. + e\left(kg\frac{1}{2}\left(-b\frac{1}{2} + c\right)a_3\right)\right]$$
$$+ h_i^{p,m+1}\left[ak(-2c + d) + ek(-2c + d)a_1 + e\left(kg_1(-2c + d)\right)a_2\right.$$
$$\left. + e\left(kg\frac{1}{2}(-2c + d)\right)\right]$$

Now

$$h(r_i, t_{m+1}) = h_i^m b_1 + D + h_{i+1}^{m-1}b_2 + h_i^{m-1}b_3 + h_{i-1}^{m-1}b_4 + h_{i+1}^{p,m+1}b_5 + h_{i-1}^{p,m+1}b_6 \tag{4.108}$$
$$+ h_i^{p,m+1}b_7$$

4.3.2.2 Numerical solution derived from leaky aquifer: water flowing in and out of the leaky aquifer

Given the leaky aquifer equation

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\frac{\partial h}{\partial r^{\alpha}} + \frac{h(r,t)}{\lambda^2} \tag{4.109}$$

And

$$\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right) = \frac{\partial}{\partial r}\left(\frac{\partial h(r,t)}{\partial r}\frac{r^{1-\alpha}}{\alpha}\right)\frac{r^{1-\alpha}}{\alpha} = \frac{\partial^2 h(r,t)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2} + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha} \tag{4.110}$$

Let $\frac{\partial h}{\partial t} = f(r,t,h) = g(t,h(t))$

$$ {}_{0}^{ff}D_t^{\alpha,\beta}h(t) = g\big(t,h(t)\big) \tag{4.111}$$

$$h(0) = h_0,$$

Now if we can apply the derivative of Atangana Baleanu fractal-fractional to the equation. Integrating equation (4.83) will give rise to

$$h(t) - h(0) = \frac{1-\alpha}{AB(\alpha)}\beta t^{\beta-1}F(t,h(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t r^{\beta-1}(t-r)^{\alpha-1}\,F\big(r,h(r)\big)dr. \tag{4.112}$$

Initial conditions

$$h(t) = h_0 + \frac{1-\alpha}{AB(\alpha)}F\big(t,h(t)\big) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_0^t F\big(r,h(r)\big)\,(t-r)^{\alpha-1}dr \tag{4.113}$$

Where at $t_{m+1} = (r, m+1)\Delta t$ we have

$$h(t_{r,m+1}) = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}F\big(r,t_{m+1},h(r,t_{m+1})\big)$$
$$+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} r^{\beta-1}\times(t_{m+1}-r)^{\alpha-1}F(r,h(r))dr. \tag{4.114}$$

The function $F(r,h(r))$ is based on approximation and can be represented as newton polynomial which is given by

$$
\begin{aligned}
P_2(r) = F(r, t_{i+1}, h(r, t_{i+1}) \\
&+ \frac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - 2F(t_i, h_i) + F(r, t_{t-1,} h(r, t_{i-1}))}{2f^2} \frac{r^{2-2\alpha}}{\alpha^2}\Big(1 \\
&+ \frac{r_{i+1} - t_i}{r_i}\Big)(r - t_{i+1})(r - t_i) \\
&+ \frac{1}{r_i}\frac{{r_i}^{1-\alpha}}{\alpha}\frac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - F(t_{i,} h_i)}{f}(r - t_{i+1}) \\
&+ \frac{(1-\alpha)}{\alpha^2}\frac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - F(t_{i,} h_i)}{f}\Big(1 + \frac{r_{i+1} - r_i}{r_i}\Big){r_i}^{1-2\alpha}(r - t_{i+1}) \\
&+ \frac{F(r_{i,} t_i)}{\lambda^2}
\end{aligned}
\tag{4.115}
$$

Now if we substitute polynomial into the equation

$$
\begin{aligned}
&h_{r,m+1} \\
&= h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} F\big(r, t_{m+1,} h^p(r, t_{m+1})\big) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \\
&\times \sum_{i=0}^{m} \left\{ \begin{array}{c} F(r, t_{i+1}, h(r, t_{i+1})) k_{1,i}^{\alpha,\beta} + \\ \dfrac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - F\big(t_{i,} h_i\big)}{f} k_{2,i}^{\alpha,\beta} + \\ \dfrac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - 2F(t_i, h_i) + F(r, t_{i-1,} h(r, t_{i-1}))}{2f^2} k_{3,i}^{\alpha,\beta} \end{array} \right\} \\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} F\big(r, t_{m+1,} h^p(r, t_{m+1}\big) k_{1,i}^{\alpha,\beta} \\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{F\big(r, t_{m+1,} h^p(r, t_{m+1}\big) - F(t_{m,} h_m)}{f} k_{2,i}^{\alpha,\beta} \\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{F\big(r, t_{m+1,} h^p(r, t_{m+1})\big) - 2F(t_m, h_m) + F(r, t_{m-1,} h(r, t_{m-1}))}{2f^2} k_{3,i}^{\alpha,\beta}.
\end{aligned}
\tag{4.116}
$$

Here

$$
\begin{aligned}
h_{m+1}^p = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} g(t_m, h_m) \\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1} g(r, h(r)) dr
\end{aligned}
\tag{4.117}
$$

Since

$$
F(r, t_{m+1}, h^p(r, t_{m+1}) = F(r, t, h(r, t) = \frac{T}{S}\left[\frac{\partial}{\partial r^\alpha}\left(\frac{\partial h}{\partial r^\alpha}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\frac{\partial h}{\partial r^\alpha} + \frac{h(r,t)}{\lambda^2}\right] \tag{4.118}
$$

This can also be given as

$$F(r, t_{m+1}, h^p(r, t_{m+1})$$
$$= \frac{T}{S}\left[\frac{(1-\alpha)}{\alpha^2}\frac{\partial h^p(r, t_{m+1})}{\partial r} r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right) + \frac{r^{2-2\alpha}}{\alpha^2}\frac{\partial^2 h(r, t_{m+1})}{\partial r^2}\left(1+\frac{\Delta r}{r}\right) + \frac{r^{1-\alpha}}{\alpha}\frac{\partial h^p(r, t_{m+1})}{r\partial r} + \frac{h^p(r, t_{m+1})}{\lambda^2}\right] \tag{4.119}$$

and

$$F(r_i, t_{m+1}, h^p(r_i, t_{m+1})$$
$$= \frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\frac{r^{1-\alpha}}{\alpha} + \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r} r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right) + \frac{r^{2-2\alpha}}{\alpha^2}\frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right) + \frac{h_i^{p,m+1}}{\lambda^2}\right] \tag{4.120}$$

Now substitute equation (4.119) and (4.120) into equation (4.92)

$$
\begin{aligned}
h(r_i,t_{m+1}) = h(r_i,0) & \qquad (4.121)\\
&+\frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}\frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}\frac{r^{1-\alpha}}{\alpha}\right.\\
&+\frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right)\\
&\left.+\frac{r^{2-2\alpha}}{\alpha^2}\frac{h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{p,m+1}}{\lambda^2}\right]+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\\
&\times\sum_{i=0}^{m}\left\{\begin{array}{c}F(r,t_{i+1},h(r,t_{i+1}))k_{1,i}^{\alpha,\beta}+\\ \dfrac{F\big(r,t_{i+1,}h(r,t_{i+1})\big)-F\big(t_{i,}h_i\big)}{f}k_{2,i}^{\alpha,\beta}+\\ \dfrac{F\big(r,t_{i+1,}h(r,t_{i+1})\big)-2F(t_i,h_i)+F(r,t_{i-1,}h(r,t_{i-1}))}{2f^2}k_{3,i}^{\alpha,\beta}\end{array}\right\}\\
&+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\frac{T}{S}\left[\frac{1}{r_i}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}\frac{r^{1-\alpha}}{\alpha}\right.\\
&+\frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right)\\
&\left.+\frac{r^{2-2\alpha}}{\alpha^2}\frac{h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{p,m+1}}{\lambda^2}\right]k_{1,i}^{\alpha,\beta}\\
&+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\left[\frac{T}{Sf}\left(\frac{1}{r_i}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}\frac{r^{1-\alpha}}{\alpha}\right.\right.\\
&+\frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right)\\
&\left.\left.+\frac{r^{2-2\alpha}}{\alpha^2}\frac{h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{p,m+1}}{\lambda^2}\right)-\frac{F\big(t_{m,}h_m\big)}{f}\right]k_{2,i}^{\alpha,\beta}\\
&+\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\left[\frac{T}{Sf^2}\left(\frac{1}{r_i}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}\frac{r^{1-\alpha}}{\alpha}\right.\right.\\
&+\frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right)\\
&+\frac{r^{2-2\alpha}}{\alpha^2}\frac{h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{p,m+1}}{\lambda^2}\\
&+\frac{1}{r_i}\frac{h_{i+1}^{m-1}-h_{i-1}^{m-1}}{2\Delta r}\frac{r^{1-\alpha}}{\alpha}+\frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{m-1}-h_{i-1}^{m-1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r}\right)\\
&\left.\left.+\frac{r^{2-2\alpha}}{\alpha^2}\frac{h_{i+1}^{m-1}-2h_i^{m-1}+h_{i-1}^{m-1}}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)+\frac{h_i^{m-1}}{\lambda^2}\right)-\frac{2F(t_m,h_m)}{2f^2}\right]k_{3,i}^{\alpha,\beta}
\end{aligned}
$$

Now let

$$\frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}=a,\qquad \frac{1}{r_i\Delta r}\frac{r^{1-\alpha}}{\alpha}=b,\qquad \frac{(1-\alpha)}{\alpha^2}\frac{1}{\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right)=c,\qquad \frac{1}{\lambda^2}=d$$

$$\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}=e,\qquad \frac{1}{f^2}=g,\qquad \frac{T}{S}=k,\qquad k_{1,i}^{\alpha,\beta}=a_1,\qquad \frac{1}{f}=g_1$$

$$k_{2,i}^{\alpha,\beta}=a_2,\qquad k_{3,i}^{\alpha,\beta}=a_3,\qquad \frac{r^{2-2\alpha}}{\alpha^2}\frac{1}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)=s$$

$$e\times\sum_{i=0}^{m}\left\{\begin{array}{c}F(r,t_{i+1},h(r,t_{i+1}))k_{1,i}^{\alpha,\beta}+\\ \dfrac{F\big(r,t_{i+1,}h(r,t_{i+1})\big)-F\big(t_{i,}h_i\big)}{f}k_{2,i}^{\alpha,\beta}+\\ \dfrac{F\big(r,t_{i+1,}h(r,t_{i+1})\big)-2F(t_i,h_i)+F(r,t_{i-1,}h(r,t_{i-1}))}{2f^2}k_{3,i}^{\alpha,\beta}\end{array}\right\}=D$$

Substitution

$$\begin{aligned}h(r_i,t_{m+1})&=h(r_i,0)\\&+ak\left[\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\right.\\&\left.+s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}+\right]+D\\&+ek\left[\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\right.\\&\left.+s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}+\right]a_1\\&+e\left[kg_1\left(\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\right.\right.\\&\left.\left.+s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}\right)-g_1F\big(t_{m,}h_m\big)\right]a_2\\&+e\left[kg\left(\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\right.\right.\\&+s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}+\frac{1}{2}b(h_{i+1}^{m-1}-h_{i-1}^{m-1})\\&\left.+\frac{1}{2}c(h_{i+1}^{m-1}-h_{i-1}^{m-1})+s(h_{i+1}^{m-1}-2h_i^{m-1}+h_{i-1}^{m-1})+dh_i^{m-1}\right)\\&\left.-gF(t_m,h_m)\right]a_3\end{aligned}\tag{4.122}$$

Grouping

$$h(r_i,t_{m+1}) = h_i^m(1-eg_1a_2-ega_3) + h_{i+1}^{m-1}\left[ekga_3\left(\frac{1}{2}b+\frac{1}{2}c+s\right)\right] \tag{4.123}$$
$$+ h_i^{m-1}[ekga_3(-2s+d)] + h_{i-1}^{m-1}\left[ekga_3\left(-\frac{1}{2}b-\frac{1}{2}c+s\right)\right]$$
$$+ h_{i+1}^{p,m+1}\left[ak\left(\frac{1}{2}b-\frac{1}{2}c+s\right)+eka_1\left(\frac{1}{2}b-\frac{1}{2}c+s\right)\right.$$
$$\left.+ekg_1a_2\left(\frac{1}{2}b-\frac{1}{2}c+s\right)+ekga_3\left(\frac{1}{2}b-\frac{1}{2}c+s\right)\right]+D$$
$$+ h_{i-1}^{p,m+1}\left[ak\left(-\frac{1}{2}b-\frac{1}{2}c+s\right)+eka_1\left(-\frac{1}{2}b-\frac{1}{2}c+s\right)\right.$$
$$\left.+ekg_1a_2\left(-\frac{1}{2}b-\frac{1}{2}c+s\right)+ekga_3\left(-\frac{1}{2}b-\frac{1}{2}c+s\right)\right]$$
$$+ h_i^{p,m+1}[ak(-2s+d)+eka_1(-2s+d)+ekg_1a_2(-2s+d)$$
$$+ekga_3(-2s+d)]$$

Now

$$h(r_i,t_{m+1}) = h_i^m b_1 + D + h_{i+1}^{m-1}b_2 + h_i^{m-1}b_3 + h_{i-1}^{m-1}b_4 + h_{i+1}^{p,m+1}b_5 + h_{i-1}^{p,m+1}b_6 + h_i^{p,m+1}b_7 \tag{4.124}$$

4.3.2.3 Numerical solution derived from leaky aquifer: water flowing in and out of the leaky aquifer normal aquifer, fractal derivative considered only on the inflow

Given the leaky aquifer equation

$$\frac{S}{T}\frac{\partial h(r,t)}{\partial t} = \left(\frac{\partial^2 h(r,t)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2}+\frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r}r^{1-2\alpha}\right)\left(1+\frac{\Delta r}{r}\right)+\frac{\partial h(r,t)}{\partial r^\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r}\right)+\frac{h(r,t)}{\lambda^2}-\frac{\partial h}{\partial r\Delta r} \tag{4.125}$$

Let $\frac{\partial h}{\partial t} = f(r,t,h) = g(t,h(t))$

Therefore

$${}_0^{ff}D_t^{\alpha,\beta}h(t) = g\big(t,h(t)\big) \tag{4.126}$$
$$h(0) = h_0,$$

From section 4.2.3, replace the following

$$h_{m+1} = h_{(r,m+1)} \tag{4.127}$$

$$g\big(t_{m+1,}h_{m+1}^p\big) = F(r,t_{m+1,}h(r,t_{m+1})) \tag{4.128}$$

$$g\left(t_{i+1,}h_{i+1}\right) = F(r, t_{i+1,}h(r, t_{i+1})) \tag{4.129}$$

$$g\left(t_{m-1,}h_{m-1}\right) = F(r, t_{m-1,}h(r, t_{m-1})) \tag{4.130}$$

Now if we can apply the derivative of Atangana Baleanu fractal-fractional to equation. Integrating equation (4.83) will give rise to

$$h(t) - h(0) = \frac{1-\alpha}{AB(\alpha)}\beta t^{\beta-1}F(t, h(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t r^{\beta-1}(t-r)^{\alpha-1}F\left(r, h(r)\right)dr. \tag{4.131}$$

Initial conditions

$$h(t) = h_0 + \frac{1-\alpha}{AB(\alpha)}F\left(t, h(t)\right) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_0^t F\left(r, h(r)\right)(t-r)^{\alpha-1}dr \tag{4.132}$$

Where at $t_{m+1} = (r, m+1)\Delta t$ we have

$$\begin{aligned} h(t_{r,m+1}) = h_0 &+ \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}F\left(r, t_{m+1}, h(r, t_{m+1})\right) \\ &+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1}-r)^{\alpha-1}F(r, h(r))dr. \end{aligned} \tag{4.133}$$

The function $F(r, h(r))$ is based on approximation and can be represented as newton polynomial which is given by

$$\begin{aligned} P_2(r) = F(r, t_{i+1}, h(r, t_{i+1}) \\ &+ \frac{F\left(r, t_{i+1,}h(r, t_{i+1})\right) - 2F(t_i, h_i) + F(r, t_{t-1,}h(r, t_{i-1}))}{2f^2}\left(1\frac{r_{i+1}-t_i}{r_i}\right)(r \\ &- t_{i+1})(r-t_i) + \frac{1}{r_i}\frac{F\left(r, t_{i+1,}h(r, t_{i+1})\right) - F(t_{i,}h_i)}{f}(r-t_{i+1}) + \frac{F(r_{i,}t_i)}{\lambda^2} \end{aligned} \tag{4.134}$$

Now if we substitute polynomial into the equation

$$
\begin{aligned}
&h_{r,m+1} && (4.135)\\
&= h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} F\big(r, t_{m+1,} h^p(r, t_{m+1})\big) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\\
&\times \sum_{i=0}^{m} \left\{ \begin{array}{c} F(r, t_{i+1}, h(r, t_{i+1}))k_{1,i}^{\alpha,\beta} + \\ \dfrac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - F\big(t_{i,} h_i\big)}{f} k_{2,i}^{\alpha,\beta} + \\ \dfrac{F\big(r, t_{i+1,} h(r, t_{i+1})\big) - 2F(t_i, h_i) + F(r, t_{i-1,} h(r, t_{i-1}))}{2f^2} k_{3,i}^{\alpha,\beta} \end{array} \right\}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} F\big(r, t_{m+1,} h^p(r, t_{m+1}\big) k_{1,i}^{\alpha,\beta}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{F\big(r, t_{m+1,} h^p(r, t_{m+1}\big) - F(t_{m,} h_m)}{f} k_{2,i}^{\alpha,\beta}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{F\big(r, t_{m+1,} h^p(r, t_{m+1})\big) - 2F(t_m, h_m) + F(r, t_{m-1,} h(r, t_{m-1}))}{2f^2} k_{3,i}^{\alpha,\beta}.
\end{aligned}
$$

For more, the identities are given by

Here

$$
\begin{aligned}
h_{m+1}^p = h_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1} g(t_m, h_m)& \\
+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} r^{\beta-1} \times (t_{m+1} - r)^{\alpha-1} g(r, h(r)) dr& \qquad (4.136)
\end{aligned}
$$

Since we know that

$$
\begin{aligned}
F(r, t_{m+1}, h^p(r, t_{m+1}) &= F(r, t, h(r, t) \qquad (4.137)\\
&= \frac{T}{S}\left[\left(\frac{\partial^2 h(r,t)}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2} + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h(r,t)}{\partial r} r^{1-2\alpha}\right)\left(1 + \frac{\Delta r}{r}\right)\right.\\
&\left.+ \frac{\partial h(r,t)}{\partial r^\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r}\right) + \frac{h(r,t)}{\lambda^2} - \frac{\partial h}{\partial r \Delta r}\right]
\end{aligned}
$$

This can also be given as

$$
\begin{aligned}
&F(r, t_{m+1}, h^p(r, t_{m+1})) \qquad (4.138)\\
&= \frac{T}{S}\left[\frac{\partial^2 h(r, t_{m+1})}{\partial r^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1 + \frac{\Delta r}{r}\right) + \frac{(1-\alpha)}{\alpha^2}\frac{\partial h^p(r, t_{m+1})}{\partial r} r^{1-2\alpha}\left(1 + \frac{\Delta r}{r}\right)\right.\\
&\left.+ \frac{{r_i}^{1-\alpha}}{\alpha}\frac{\partial h^p(r, t_{m+1})}{\partial r}\left(\frac{1}{\Delta r} - \frac{1}{r}\right) + \frac{h^p(r, t_{m+1})}{\lambda^2} - \frac{\partial h^p(r, t_{m+1})}{\partial r \Delta r}\right]
\end{aligned}
$$

And

$$F(r_i, t_{m+1}, h^p(r_i, t_{m+1}) \tag{4.139}$$
$$= \frac{T}{S}\left[\frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1 + \frac{\Delta r}{r_i}\right)\right.$$
$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r} r^{1-2\alpha}\left(1 + \frac{\Delta r}{r_i}\right)$$
$$\left. + \frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h_i^{p,m+1}}{\lambda^2} - \left(\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\right)\right]$$

$$F\left(r_i, t_{m-1,} h(r_i, t_{m-1}\right) \tag{4.140}$$
$$= \frac{T}{S}\left[\frac{h_{i+1}^{m-1} - 2h_i^{m-1} + h_{i-1}^{m-1}}{(\Delta r)^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1 + \frac{\Delta r}{r_i}\right)\right.$$
$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{m-1} - h_{i-1}^{m-1}}{2\Delta r} r^{1-2\alpha}\left(1 + \frac{\Delta r}{r}\right) + \frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_{i+1}^{m-1} - h_{i-1}^{m-1}}{2\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r}\right)$$
$$\left. + \frac{h_i^{m+1}}{\lambda^2} - \frac{h_{i+1}^{m-1} - h_{i-1}^{m-1}}{2\Delta r}\right]$$

Now substitute equation (4.139) and (4.140) into equation (4.92)

`

$$
\begin{aligned}
&h(r_i, t_{m+1}) \qquad (4.141)\\
&= h(r_i, 0)\\
&+ \frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}\frac{T}{S}\left[\frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{\Delta r}{r_i}\right)\right.\\
&+ \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right) + \frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right)\\
&\left.+ \frac{h_i^{p,m+1}}{\lambda^2} - \left(\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\right)\right] + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\\
&\times \sum_{i=0}^{m}\left\{\begin{array}{c} F(r, t_{i+1}, h(r, t_{i+1}))k_{1,i}^{\alpha,\beta} + \\ \dfrac{F\left(r, t_{i+1,} h(r, t_{i+1})\right) - F\left(t_{i,} h_i\right)}{f}k_{2,i}^{\alpha,\beta} + \\ \dfrac{F\left(r, t_{i+1,} h(r, t_{i+1})\right) - 2F(t_i, h_i) + F(r, t_{i-1,} h(r, t_{i-1}))}{2f^2}k_{3,i}^{\alpha,\beta} \end{array}\right\}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\frac{T}{S}\left[\frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{\Delta r}{r_i}\right)\right.\\
&+ \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right) + \frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right)\\
&\left.+ \frac{h_i^{p,m+1}}{\lambda^2} - \left(\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\right)\right]k_{1,i}^{\alpha,\beta}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\left[\frac{T}{Sf}\left(\frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{\Delta r}{r_i}\right)\right.\right.\\
&+ \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right) + \frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right)\\
&\left.\left.+ \frac{h_i^{p,m+1}}{\lambda^2} - \left(\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\right)\right) - \frac{F\left(t_{m,} h_m\right)}{f}\right]k_{2,i}^{\alpha,\beta}\\
&+ \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\left[\frac{T}{Sf^2}\left(\frac{h_{i+1}^{p,m+1} - 2h_i^{p,m+1} + h_{i-1}^{p,m+1}}{(\Delta r)^2}\frac{r^{2-2\alpha}}{\alpha^2}\left(1+\frac{\Delta r}{r_i}\right)\right.\right.\\
&+ \frac{(1-\alpha)}{\alpha^2}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right) + \frac{{r_i}^{1-\alpha}}{\alpha}\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right)\\
&\left.\left.+ \frac{h_i^{p,m+1}}{\lambda^2} - \left(\frac{h_{i+1}^{p,m+1} - h_{i-1}^{p,m+1}}{2\Delta r}\right) + F\left(r, t_{m-1,} h(r, t_{m-1})\right)\right) - \frac{2F(t_m, h_m)}{2f^2}\right]k_{3,i}^{\alpha,\beta}.
\end{aligned}
$$

Now let

$$\frac{1-\alpha}{AB(\alpha)}\beta t_{m+1}^{\beta-1}=a, \qquad \frac{1}{r_i\Delta r}\frac{r^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right)=b, \qquad \frac{(1-\alpha)}{\alpha^2}\frac{1}{\Delta r}r^{1-2\alpha}\left(1+\frac{\Delta r}{r_i}\right)=c, \qquad \frac{1}{\lambda^2}=d$$

$$\frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}=e, \qquad \frac{1}{f^2}=g, \qquad \frac{T}{S}=k, \qquad k_{1,i}^{\alpha,\beta}=a_1, \qquad \frac{1}{f}=g_1$$

$$k_{2,i}^{\alpha,\beta}=a_2, \qquad k_{3,i}^{\alpha,\beta}=a_3, \qquad \frac{r^{2-2\alpha}}{\alpha^2}\frac{1}{(\Delta r)^2}\left(1+\frac{\Delta r}{r_i}\right)=s, \qquad \frac{1}{\Delta r}$$

$$e\times\sum_{i=0}^{m}\left\{\begin{array}{c} F(r,t_{i+1},h(r,t_{i+1}))k_{1,i}^{\alpha,\beta}+ \\ \dfrac{F\big(r,t_{i+1,}h(r,t_{i+1})\big)-F\big(t_{i,}h_i\big)}{f}k_{2,i}^{\alpha,\beta}+ \\ \dfrac{F\big(r,t_{i+1,}h(r,t_{i+1})\big)-2F(t_i,h_i)+F(r,t_{i-1,}h(r,t_{i-1}))}{2f^2}k_{3,i}^{\alpha,\beta} \end{array}\right\}=D$$

Substitution

$$\begin{aligned} h(r_i,t_{m+1}) &= h(r_i,0) \\ &+ak\Big[s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big) \\ &+\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}-w\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\Big]+D \\ &+ek\Big[s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big) \\ &+\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}-w\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\Big]a_1 \\ &+e\Big[kg_1\Big(s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big) \\ &+\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}-w\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)\Big) \\ &-g_1F\big(t_{m,}h_m\big)\Big]a_2 \\ &+e\Big[kg\Big(s\big(h_{i+1}^{p,m+1}-2h_i^{p,m+1}+h_{i-1}^{p,m+1}\big)+\frac{1}{2}c\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big) \\ &+\frac{1}{2}b\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big)+dh_i^{p,m+1}-w\big(h_{i+1}^{p,m+1}-h_{i-1}^{p,m+1}\big) \\ &+s(h_{i+1}^{m-1}-2h_i^{m-1}+h_{i-1}^{m-1})+\frac{1}{2}c(h_{i+1}^{m-1}-h_{i-1}^{m-1})+\frac{1}{2}b(h_{i+1}^{m-1}-h_{i-1}^{m-1}) \\ &+dh_i^{m-1}-w(h_{i+1}^{m-1}-h_{i-1}^{m-1})\Big)-gF(t_m,h_m)\Big]a_3 \end{aligned} \tag{4.142}$$

Grouping

$$
\begin{aligned}
h(r_i, t_{m+1}) = {} & h_i^m[(1 - eg_1a_2 - ega_3)] + h_{i+1}^{m-1}\left[ekga_3\left(s + \frac{1}{2}c + \frac{1}{2}b - w\right)\right] \\
& + h_i^{m-1}[ekga_3(-2s + d)] + h_{i-1}^{m-1}\left[ekga_3\left(s - \frac{1}{2}c - \frac{1}{2}b + w\right)\right] \\
& + h_{i+1}^{p,m+1}\left[ak\left(s + \frac{1}{2}c + \frac{1}{2}b - w\right) + eka_1\left(s + \frac{1}{2}c + \frac{1}{2}b - w\right)\right. \\
& \left. + ekg_1a_2\left(s + \frac{1}{2}c + \frac{1}{2}b - w\right) + ekga_3\left(s + \frac{1}{2}c + \frac{1}{2}b - w\right)\right] + D \\
& + h_{i-1}^{p,m+1}\left[ak\left(s - \frac{1}{2}c - \frac{1}{2}b + w\right) + eka_1\left(s - \frac{1}{2}c - \frac{1}{2}b + w\right)\right. \\
& \left. + ekg_1a_2\left(s - \frac{1}{2}c - \frac{1}{2}b + w\right) + ekga_3\left(s - \frac{1}{2}c - \frac{1}{2}b + w\right)\right] \\
& + h_i^{p,m+1}[ak(-2s + d) + eka_1(-2s + d) + ekg_1a_2(-2s + d) \\
& + ekga_3(-2s + d)]
\end{aligned} \tag{4.143}
$$

Now

$$
\begin{aligned}
h(r_i, t_{m+1}) = {} & h_i^m b_1 + D + h_{i+1}^{m-1} b_2 + h_i^{m-1} b_3 + h_{i-1}^{m-1} b_4 + h_{i+1}^{p,m+1} b_5 + h_{i-1}^{p,m+1} b_6 \\
& + h_i^{p,m+1} b_7
\end{aligned} \tag{4.143}
$$

CHAPTER 5:
VON NEUMANN STABILITY TEST ANALYSIS

5.1 INTRODUCTION

In most circumstances of science and engineering, Von Neumann's analysis is used as a numerical method to check the stability and finite differential scheme, particularly in linear partial differential equations. Von Neumann stability test is also referred to as Fourier stability analysis. Von Neumann stability is mostly applied in explicit time integration. In Von Neumann stability testing, the numerical scheme accounts for numerical errors. The finite numerical algorithm calculates errors and uncertainly checks their stability step at the time. An impartial stable scheme is when the errors endure constant as computation continues. If the errors continue, errors will show a pattern of decaying trend and eventually will drop out, and therefore the finite numerical algorithm is considered to be stable. Consequently, when the errors increase with time and space the numerical scheme is considered to be unstable. In Von Neumann stability analysis for problems that are time dependant, the numerical method will produce a boundary solution when it encounters a numerical solution differential equation that is bounded. Stability analysis can be very challenging to non-linear equations (Smith, 1985). Von Neumann stability test can be applied in different circumstances. More precisely, the method provides more detailed and good guess at restrictions.

5.2 ILLUSTRATION OF VON NEUMANN STABILITY TEST METHOD

To demonstrate the method, we consider one-dimension heat equation. The method will be based on decay numerical errors based on Fourier series.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{5.1}$$

When the equation is defined at interval ***L***, the discretized equation will then be defined as

$$u_j{}^{n+1} = u_j^n + r\left(u_{j+1}^n - 2u_j^n - u_{j-1}^n\right) \tag{5.2}$$

Where by r is defined as

$$r = \frac{\alpha \Delta t}{(\Delta x)^2} \tag{5.3}$$

When the approximation $u(x,t)$ has been defined on discrete close interval of PDE on the grid, it gives a product solution of u_j^n. Therefore, round off errors ϵ_j^n will also be defined at the interval as

$$\varepsilon_j^n = N_j^n - u_j^n \tag{5.4}$$

We can recall that $u_j^n$is the analytical solution that has been discretized in equation (5.2) the round off errors was not included while in the PDE of N_j^n round-off error was included. Therefore, the numerical exact solution of equation (5.1) must fulfil the exact discretized equation such that all the errors defined by ε_j^n also fulfil the conditions of discretized equation (Anderson, 1994). Now supposed that N_j^n satisfy discretized equation too (applicable in the laboratory), therefore ε_j^{n+1} will be defined as

$$\varepsilon_j^{n+1} = \varepsilon_j^n + r\left(\varepsilon_{j+1}^n - 2\varepsilon_j^n - \varepsilon_{j-1}^n\right) \tag{5.5}$$

Both equation (5.2) and (5.5) show the same numerical decay or growth and the both also have same numerical errors. In linear PDE with sporadic boundary, the distinctive change of numerical error may possibly be extended in a finite Fourier series within the interval L and is defined as:

$$\varepsilon(x) = \sum_{p=1}^{M} A_p exp^{ik_p x} \tag{5.6}$$

Where

$$\begin{cases} k_p = \dfrac{\pi p}{L} \\ \quad and \\ M = \dfrac{L}{\Delta x} \end{cases} \tag{5.7}$$

k_m is the wavelength where $p = 1,2,3, \ldots \ldots, M$; A is the amplitude and $A_p$is the error in the amplitude which depends on the time; and L is defend as the spatial interval. Since the error in the amplitude tend to be either proportionally or decay with respect to time, hence it practical to assume that the error $\varepsilon(x,t)$ function will diverge with exponentially with time, therefore the error will be defined as

$$\varepsilon(x,t) = \sum_{p=1}^{M} e^{bt} e^{ik_p x} \tag{5.8}$$

Where b is a constant in the error. Subsequently, linear differential equations are considered to be linear and it is therefore to assume that the growth or decay in error is typing given as

$$\varepsilon_p(x,t) = e^{bt} e^{ik_p x} \tag{5.9}$$

In general, errors are characterised by the algorithm in equation (5.9) and hence errors with no loss can be studied using the equation. Von Neumann analysis assumes the following

$$S_{j-1}^{m}(x,t) = \hat{\varepsilon}_m e^{ik_p(x-\Delta x)} \tag{5.10}$$

$$S_{j-1}^{m-1}(x,t) = \hat{\varepsilon}_{m-1} e^{ik_p(x-\Delta x)} \tag{5.11}$$

$$S_{j}^{m-1}(x,t) = \hat{\varepsilon}_{m-1} e^{ik_p x} \tag{5.12}$$

$$S_{j}^{m+1}(x,t) = \hat{\varepsilon}_{m+1} e^{ik_p x} \tag{5.13}$$

$$S_{j+1}^{m-1}(x,t) = \hat{\varepsilon}_{m-1} e^{ik_p(x+\Delta x)} \tag{5.14}$$

$$S_{j}^{m}(x,t) = \hat{\varepsilon}_m e^{ik_p x} \tag{5.15}$$

$$S_{j+1}^{m}(x,t) = \hat{\varepsilon}_m e^{ik_p(x+\Delta x)} \tag{5.16}$$

$$S_{j+1}^{m+1}(x,t) = \hat{\varepsilon}_{m+1} e^{ik_p(x+\Delta x)} \tag{5.17}$$

For simplification

$$e^{b\Delta t} = 1 + r(e^{ik_p\Delta x} + e^{-ik_p\Delta x} - 2) \tag{5.18}$$

Now using identities

$$\sin\left(\frac{k_p\Delta x}{2}\right) = \frac{e^{ik_p\Delta x/2} + e^{-ik_p\Delta x/2}}{2i} \tag{5.19}$$

And

$$\sin^2\left(\frac{k_p\Delta x}{2}\right) = -\frac{e^{ik_p\Delta x} + e^{-ik_p\Delta x} - 2}{2} \tag{5.20}$$

Equation (5.18) can also be expressed as

$$e^{b\Delta t} = 1 - 4r\sin^2\left(\frac{k_p\Delta x}{2}\right) \tag{5.21}$$

Implication factor

$$G = \frac{\varepsilon_j^{n+1}}{\varepsilon_j^{n}} \tag{5.22}$$

Thus, satisfactory condition and basic for the error propagation to be remain constrained at $|G| \leq 1$.
Now substitution

$$G = \frac{e^{b(t-\Delta t)}e^{ik_p x}}{e^{bt}e^{ik_p x}} = e^{b\Delta t} \tag{5.23}$$

Now thus, the stability condition is given by

$$\left|1 - 4r\sin^2\left(\frac{k_p \Delta x}{2}\right)\right| \leq 1 \tag{5.24}$$

Note that absolute $4r\sin^2\left(\frac{k_p \Delta x}{2}\right)$ is continuously positive, such that the above equation is given by.

$$4r\sin^2\left(\frac{k_p \Delta x}{2}\right) \leq 2 \tag{5.25}$$

For the equation to satisfy conditions for all p and thus values of $\sin^2\left(\frac{k_p \Delta x}{2}\right)$, we have

$$r = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{5.26}$$

Equation above is known stability condition for finite differential method that is applied as numerical scheme for resolving heat equation in 1D. The numerical scheme state that "for a known value of Δx, the acceptable values of Δt must be insignificant sufficient such that they fulfil equation (5.25)

5.2.1 Application of Von Neumann Stability test into Predictor-Corrector method

If we can recall equation 4.40 in chapter 4, $f\left(r_i, t_n, h_i^{p,n+1}\right)$ can also be given as

$$f\left(r_i, t_n, h_i^{p,n+1}\right) \tag{5.27}$$

$$= \frac{T}{S}\left[\frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)\right.$$

$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}{r_i}^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)$$

$$+ \frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}$$

$$\left. - \frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\frac{1}{\Delta r}, {h_i}^{p,n+1}\right]$$

$$= \frac{T}{S}\left[\frac{h_p(r_{i+1},t_{n+1}) - 2h_p(r_i,t_{n+1}) + h_p(r_{i-1},t_{n+1})}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)\right.$$

$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r}{r_i}^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)$$

$$+ \frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right) + \frac{h(r_i,t_{n+1})}{\lambda^2}$$

$$\left. - \frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r}\left(\frac{1}{\Delta r}\right)\right]$$

Also

$$f\left(r_i, t_n, h_i^{p,n+1}\right) \tag{5.28}$$

$$= \frac{T}{S}\left[\frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)\right.$$

$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}{r_i}^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)$$

$$+ \frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}$$

$$\left. - \frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\frac{1}{\Delta r}, {h_i}^{p,n+1}\right]$$

$$= \frac{T}{S}\left[\frac{h_p(r_{i+1},t_{n+1}) - 2h_p(r_i,t_{n+1}) + h_p(r_{i-1},t_{n+1})}{\Delta r^2}\frac{{r_i}^{2-2\alpha}}{\alpha^2}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)\right.$$

$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r}{r_i}^{1-2\alpha}\left(1+\frac{r_{i+1}-t_n}{r_i}\right)$$

$$+ \frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r}\frac{{r_i}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r}-\frac{1}{r_i}\right) + \frac{h(r_i,t_{n+1})}{\lambda^2}$$

$$\left. - \frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n)}{2\Delta r}\left(\frac{1}{\Delta r}\right)\right]$$

Now predictor can be given as

$$y_i^{n+1} = h_i{}^n + \frac{h}{2}\frac{T}{S}\Bigg\{\Bigg[\frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\frac{r_i{}^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right)$$
$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r} r_i{}^{1-2\alpha}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right)$$
$$+ \frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\frac{r_i{}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h(r_i,t_n)}{\lambda^2}$$
$$- \frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r}\frac{1}{\Delta r}\Bigg]$$
$$+ \Bigg[\frac{h_p(r_{i+1},t_{n+1}) - 2h_p(r_i,t_{n+1}) + h_p(r_{i-1},t_{n+1})}{\Delta r^2}\frac{r_i{}^{2-2\alpha}}{\alpha^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right)$$
$$+ \frac{(1-\alpha)}{\alpha^2}\frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_{n+1})}{2\Delta r} r_i{}^{1-2\alpha}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right)$$
$$+ \frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_{n+1})}{2\Delta r}\frac{r_i{}^{1-\alpha}}{\alpha}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) + \frac{h(r_i,t_{n+1})}{\lambda^2}$$
$$- \frac{h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_{n+1})}{2\Delta r}\left(\frac{1}{\Delta r}\right)\Bigg]\Bigg\} \tag{4.29}$$

Let

$$\frac{hT}{S} = a, \quad \frac{r_i^{2-2\alpha}}{\Delta r^2\alpha^2}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) = b, \quad \frac{(1-\alpha)}{\alpha^2}\frac{r_i^{1-2\alpha}}{2\Delta r}\left(1 + \frac{r_{i+1} - t_n}{r_i}\right) = c,$$
$$\frac{r_i^{1-2\alpha}}{2\alpha\Delta r}\left(\frac{1}{\Delta r} - \frac{1}{r_i}\right) = d, \quad \frac{1}{2\Delta r}\frac{1}{\Delta r} = g, \quad \frac{1}{\lambda^2} = f$$

Substitution

$$y_i^{n+1} = h_i^n + \frac{1}{2}a\{[(h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n))b + (h(r_{i+1},t_n) - h(r_{i-1},t_n))c$$
$$+ (h(r_{i+1},t_n) - h(r_{i-1},t_n))d - (h(r_{i+1},t_n) - (h(r_{i-1},t_n))g + h(r_i,t_n))f]$$
$$+ [(h_p(r_{i+1},t_{n+1}) - 2h_p(r_i,t_{n+1}) + h_p(r_{i-1},t_{n+1}))b + h_p(r_{i+1},t_{n+1})$$
$$- h_p(r_{i-1},t_n)c + (h_p(r_{i+1},t_{n+1}) - h_p(r_{i-1},t_n))d - (h_p(r_{i+1},t_{n+1})$$
$$- h_p(r_{i-1},t_n))g + h_p(r_i,t_{n+1})f]\} \tag{5.30}$$

Simplification

$$y_i^{n+1} = h_i^n\left[\left(1 - ab + \frac{1}{2}af\right)\right] + h_{i+1}^n\left[\left(\frac{1}{2}ab + \frac{1}{2}ac + \frac{1}{2}ad - \frac{1}{2}ag\right)\right]$$
$$+ h_{i-1}^n\left[\left(\frac{1}{2}ab - \frac{1}{2}ac - \frac{1}{2}ad + \frac{1}{2}ag\right)\right]$$
$$+ h_{i+1}^{p,n+1}\left[\left(\frac{1}{2}ab + \frac{1}{2}ac + \frac{1}{2}ad - \frac{1}{2}ag\right)\right] + h_i^{p,n+1}\left[\left(-ab + \frac{1}{2}af\right)\right]$$
$$+ h_{i-1}^{p,n}\left[\left(-\frac{1}{2}ac - \frac{1}{2}ad + \frac{1}{2}ag\right)\right] + h_{i-1}^{p,n+1}\left[\frac{1}{2}ab\right] \tag{5.31}$$

simplification

$$h_i^{n+1} = h_i^n\left[\frac{1}{2}(2-2ab+af)\right] + h_{i+1}^n\left[\frac{1}{2}a(b+c+d-g)\right] + h_{i-1}^n\left[\frac{1}{2}a(b-c-d+g)\right] + h_{i+1}^{p,n+1}\left[\frac{1}{2}a(b+c+d-g)\right] + h_i^{p,n+1}\left[\frac{1}{2}a(-2b+f)\right] + h_{i-1}^{p,n}\left[\frac{1}{2}a(-c-d+g)\right] + h_{i-1}^{p,n+1}\left[\frac{1}{2}ab\right] \quad (5.32)$$

Now

$$h_i^{n+1} = h_i^n U_1 + h_{i+1}^n U_2 + h_{i-1}^n U_3 + h_{i+1}^{p,n+1} U_4 + h_i^{p,n+1} U_5 + h_{i-1}^{p,n} U_6 + h_{i-1}^{p,n+1} U_7 \quad (5.33)$$

Substitute stability equation

$$\hat{\varepsilon}_{n+1} e^{ik_p x} = \hat{\varepsilon}_n e^{ik_p x} U_1 + U_2 \hat{\varepsilon}_n e^{ik_p(x+\Delta x)} + U_3 \hat{\varepsilon}_n e^{ik_p(x-\Delta x)} + \hat{\varepsilon}_{n+1} e^{ik_p(x+\Delta x)} U_4 + \hat{\varepsilon}_{n+1} e^{ik_p x} U_5 + \hat{\varepsilon}_n e^{ik_p(x-\Delta x)} U_6 + \hat{\varepsilon}_{n+1} e^{ik_p(x-\Delta x)} U_7 \quad (5.34)$$

Now divide equation (5.34) throughout by $e^{ik_p x}$

$$\hat{\varepsilon}_{n+1} = \hat{\varepsilon}_n U_1 + U_2 \hat{\varepsilon}_n e^{ik_p \Delta x} + U_3 \hat{\varepsilon}_n e^{-ik_p \Delta x} + \hat{\varepsilon}_{n+1} e^{ik_p \Delta x} U_4 + \hat{\varepsilon}_{n+1} U_5 + \hat{\varepsilon}_n e^{-ik_p \Delta x} U_6 + \hat{\varepsilon}_{n+1} e^{-ik_p \Delta x} U_7 \quad (5.35)$$

Suppose that

$$e^{ik_p \Delta x} = \cos(k_p \Delta x) + isin(k_p \Delta x)$$
$$e^{-ik_p \Delta x} = \cos(k_p \Delta x) - isin(k_p \Delta x) \quad (5.36)$$

Substitution

$$\hat{\varepsilon}_{n+1} = \hat{\varepsilon}_n U_1 + U_2[\cos(k_p \Delta x) + isin(k_p \Delta x)]\hat{\varepsilon}_n + U_3[\cos(k_p \Delta x) - isin(k_p \Delta x)]\hat{\varepsilon}_n + \hat{\varepsilon}_{n+1} U_4[\cos(k_p \Delta x) + isin(k_p \Delta x)] + \hat{\varepsilon}_{n+1} U_5 + \hat{\varepsilon}_n U_6[\cos(k_p \Delta x) - isin(k_p \Delta x)] + \hat{\varepsilon}_{n+1} U_7[(\cos(k_p \Delta x) - isin(k_p \Delta x))] \quad (5.37)$$

Grouping and factorizing

$$\hat{\varepsilon}_{n+1}\left[1 - U_4\left(\cos(k_p \Delta x) + isin(k_p \Delta x)\right) - U_5 - U_7\left(\cos(k_p \Delta x) - isin(k_p \Delta x)\right)\right] = \hat{\varepsilon}_n\left[1 + U_2\left(\cos(k_p \Delta x) + isin(k_p \Delta x)\right) + U_3\left(\cos(k_p \Delta x) - isin(k_p \Delta x)\right) + U_6\left(\cos(k_p \Delta x) - isin(k_p \Delta x)\right)\right] \quad (5.38)$$

Supposed that

$$A = \left[1 + (-U_4 - U_7)\left(\cos(k_p \Delta x) + \cos(k_p \Delta x)\right) + (-U_4 - U_7)\left(isin(k_p \Delta x) - isin(k_p \Delta x)\right) - U_5\right] \quad (5.39)$$

$$A_1 = (-U_4 - U_7)\big(\cos(k_p\Delta x) + \cos(k_p\Delta x)\big) \tag{5.40}$$

$$A_2 = (-U_4 - U_7)\left(isin(k_p\Delta x) - isin(k_p\Delta x)\right) \tag{5.41}$$

$$A = [1 + A_1 + A_2 - U_5] \tag{5.42}$$

And

$$B = \Big[1 + (U_2 + U_3 + U_6)\big(\cos(k_p\Delta x) + \cos(k_p\Delta x) + \cos(k_p\Delta x)\big) + (U_2 + U_3 + U_6)\left(-isin(k_p\Delta x) - isin(k_p\Delta x) - isin(k_p\Delta x)\right)\Big] \tag{5.43}$$

$$B_1 = (U_2 + U_3 + U_6)\big(\cos(k_p\Delta x) + \cos(k_p\Delta x) + \cos(k_p\Delta x)\big) \tag{5.44}$$

$$B_2 = (U_2 + U_3 + U_6)\left(-isin(k_p\Delta x) - isin(k_p\Delta x) - isin(k_p\Delta x)\right) \tag{5.45}$$

$$B = [1 + B_1 + B_2] \tag{5.46}$$

The equation above can also be represented as

$$\hat{\varepsilon}_{n+1}A = \hat{\varepsilon}_n B \tag{5.47}$$

When $n = 0$

$$\hat{\varepsilon}_n A = \hat{\varepsilon}_0 B \tag{5.48}$$

Rearranging

$$\frac{\hat{\varepsilon}_n}{\hat{\varepsilon}_0} = \frac{B}{A} \tag{5.49}$$

$$\frac{\hat{\varepsilon}_n}{\hat{\varepsilon}_0} = \frac{[1 + B_1 + B_2]}{[1 + A_1 + A_2 - U_5]} \tag{5.50}$$

Thus, stability condition

$$\left|\frac{\hat{\varepsilon}_n}{\hat{\varepsilon}_0} < 1\right| \tag{5.51}$$

Which mean that

$$\left|\frac{B}{A} < 1\right| \tag{5.52}$$

Now if we apply the complex number and absolute value by assuming that $r = c + ix\ and\ |s| = \sqrt{c^2 + d^2}$.

$$\left|\frac{B}{A}\right| = \sqrt{\frac{(1+B_1)^2 + {B_2}^2}{(1+A_1)^2 + (A_2 - U_5)^2}} \tag{5.53}$$

Can also represent the equation above as

$$\frac{1+B_1+B_2}{1+A_1+A_2-U_5} = \sqrt{\frac{(1+B_1)^2 + {B_2}^2}{(1+A_1)^2 + (A_2 - U_5)^2}} \tag{5.54}$$

Thus

$$\left|\frac{B}{A}\right| = \sqrt{\frac{(1+B_1)^2 + {B_2}^2}{(1+A_1)^2 + (A_2 - U_5)^2}} < 1 \tag{5.55}$$

Therefore, we infer that $|\hat{\varepsilon}_n| < |\hat{\varepsilon}_0|$

$$\sqrt{\frac{(1+B_1)^2 + {B_2}^2}{(1+A_1)^2 + (A_2 - U_5)^2}} < 1 \tag{5.56}$$

Therefore, the stability of this equation is conditionally stable. However, the stability for equation (4.56) and equation (4.70) will be stable as equation (5.33) because the same operator.

5.2.2 Application of Von Neumann Stability test in AB derivative

If we can recall equation (4.108) in chapter 4

$$h(r_i, t_{m+1}) = h_i^m b_1 + D + h_{i+1}^{m-1} b_2 + h_i^{m-1} b_3 + h_{i-1}^{m-1} b_4 + h_{i+1}^{p,m+1} b_5 + h_{i-1}^{p,m+1} b_6 + h_i^{p,m+1} b_7 \tag{5.57}$$

Applying Von Neuman Stability analysis for equation

$$\hat{\varepsilon}_{m+1} e^{ik_p x} = \hat{\varepsilon}_m e^{ik_p x} b_1 + D + b_2 \hat{\varepsilon}_{m-1} e^{ik_p (x+\Delta x)} + b_3 \hat{\varepsilon}_{m-1} e^{ik_p x} + b_4 \hat{\varepsilon}_{m-1} e^{ik_p (x-\Delta x)} + b_5 \hat{\varepsilon}_{m+1} e^{ik_p (x+\Delta x)} + b_6 \hat{\varepsilon}_{m+1} e^{ik_p (x-\Delta x)} + b_7 \hat{\varepsilon}_{m+1} e^{ik_p x} \tag{5.58}$$

Now divide by $e^{ik_p x}$

$$\hat{\varepsilon}_{m+1} = \hat{\varepsilon}_m b_1 + \frac{D}{e^{ik_p x}} + b_2 \hat{\varepsilon}_{m-1} e^{ik_p \Delta x} + b_3 \hat{\varepsilon}_{m-1} + b_4 \hat{\varepsilon}_{m-1} e^{-ik_p \Delta x} + b_5 \hat{\varepsilon}_{m+1} e^{ik_p \Delta x} + b_6 \hat{\varepsilon}_{m+1} e^{-ik_p \Delta x} + b_7 \hat{\varepsilon}_{m+1} \tag{5.59}$$

Grouping like terms

$$\hat{\varepsilon}_{m+1} = \hat{\varepsilon}_m b_1 + \frac{D}{e^{ik_p x}} + \left(e^{ik_p \Delta x} b_2 + b_3 + b_4 e^{-ik_p \Delta x}\right)\hat{\varepsilon}_{m-1} + \hat{\varepsilon}_{m+1}\left(b_5 e^{ik_p \Delta x} + b_6 e^{-ik_p \Delta x} + b_7\right) \tag{5.60}$$

$$\left(-e^{ik_p\Delta x}b_2 - b_3 - b_4 e^{-ik_p\Delta x}\right)\hat{\varepsilon}_{m-1} + \hat{\varepsilon}_{m+1}\left(1 - b_5 e^{ik_p\Delta x} - b_6 e^{-ik_p\Delta x} - b_7\right) = \hat{\varepsilon}_m b_1 + \frac{D}{e^{ik_p x}} \tag{5.61}$$

Suppose that

$$e^{ik_p\Delta x} = \cos(k_p\Delta x) + isin(k_p\Delta x) \tag{5.62}$$

$$e^{-ik_p\Delta x} = \cos(k_p\Delta x) - isin(k_p\Delta x) \tag{5.63}$$

Substitution

$$\begin{aligned}&\left(-\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right)b_2 - b_3\right.\\ &\quad \left. - b_4\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right)\right)\hat{\varepsilon}_{m-1} + \widehat{\varepsilon}_{m+1}\left(1 - b_5\left(\cos(k_p\Delta x)\right.\right.\\ &\quad \left.\left. + isin(k_p\Delta x)\right) - b_6\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)\\ &\quad = \hat{\varepsilon}_m b_1 + \frac{D}{\cos(k_p\Delta x) + isin(k_p\Delta x)}\end{aligned} \tag{5.64}$$

Supposed that

$$A = \left(-\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right)b_2 - b_3 - b_4\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right)\right) \tag{5.65}$$

$$B = \left(1 - b_5\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right) - b_6\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right) \tag{5.66}$$

The equation above can be given as

$$A\hat{\varepsilon}_{m-1} + B\hat{\varepsilon}_{m+1} = \hat{\varepsilon}_m b_1 + \frac{D}{\cos(k_p\Delta x) + isin(k_p\Delta x)} \tag{5.67}$$

When $m = 0$, then

$$B\hat{\varepsilon}_1 = \hat{\varepsilon}_0 b_1 \tag{5.68}$$

$$\hat{\varepsilon}_0 = \frac{1}{b_1}\left(\left(1 - b_5\left(cos(k_p\Delta x) + isin(k_p\Delta x)\right) - b_6\left(cos(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)\hat{\varepsilon}_1\right) \tag{5.69}$$

Rearranging

$$\frac{\hat{\varepsilon}_0}{\hat{\varepsilon}_1} = \frac{1}{b_1}\left(\left(1 - b_5\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right) - b_6\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)\right) \tag{5.70}$$

According to stability conditions

$$\left|\frac{\hat{\varepsilon}_0}{\hat{\varepsilon}_1}\right| < 1 \tag{5.71}$$

Which can also be indicated as

$$\left|\frac{\hat{\varepsilon}_0}{\hat{\varepsilon}_1}\right| = B < 1 \tag{5.72}$$

$$B < 1$$

Recall that

$$B = \left(1 - b_5\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right) - b_6\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right) \tag{5.73}$$

Therefore, now apply complex numbers by assuming that $r = c + ix\ and\ |s| = \sqrt{c^2 + d^2}$

$$|B| = \sqrt{\left(1 + (b_6 - b_5)\left(\cos(k_p\Delta x + k_p\Delta x)\right)\right)^2 + \left((b_6 - b_5)\left(isin(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)^2} \tag{5.74}$$

Equation can also be indicated as

$$\left|\left(1 - b_5\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right) - b_6\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)\right| = \sqrt{\left(1 + (b_6 - b_5)\left(\cos(k_p\Delta x + k_p\Delta x)\right)\right)^2 + \left((b_6 - b_5)\left(isin(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)^2} \tag{5.75}$$

Thus

$$|B| = \sqrt{\left(1 + (b_6 - b_5)\left(\cos(k_p\Delta x + k_p\Delta x)\right)\right)^2 + \left((b_6 - b_5)\left(isin(k_p\Delta x) - isin(k_p\Delta x)\right) - b_7\right)^2} < 1 \tag{5.76}$$

Now we conclude $|\hat{\varepsilon}_1| < |\hat{\varepsilon}_0|$

Recall that A can also be give as

$$A = \left(-\left(\cos(k_p\Delta x) + isin(k_p\Delta x)\right) b_2 - b_3 - b_4\left(\cos(k_p\Delta x) - isin(k_p\Delta x)\right)\right) \tag{5.77}$$

And also

$$|A| = \sqrt{\left(-\left(\cos(k_p\Delta x) + \cos(k_p\Delta x)\right)(b_2 - b_4) - b_3\right)^2 + \left((b_2 - b_4)\left(isin(k_p\Delta x) - isin(k_p\Delta x)\right)\right)^2} \tag{5.78}$$

Similar

$$\left|\left(-\left(\cos(k_p\Delta x)+isin(k_p\Delta x)\right)b_2-b_3-b_4\left(\cos(k_p\Delta x)-isin(k_p\Delta x)\right)\right)\right| \tag{5.79}$$

$$=\sqrt{\left(-(\cos(k_p\Delta x)+\cos(k_p\Delta x))(b_2-b_4)-b_3\right)^2+\left((b_2-b_4)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)\right)^2}$$

When $m > 0$

$$|\hat{\varepsilon}_1| < |\hat{\varepsilon}_0| \to \frac{|\hat{\varepsilon}_0|}{|\hat{\varepsilon}_1|} < 1 \tag{5.80}$$

Applying the assumptions now

$$|\hat{\varepsilon}_0| < |\hat{\varepsilon}_{m+1}| \to \frac{|\hat{\varepsilon}_0|}{|\hat{\varepsilon}_{m+1}|} < 1 \tag{5.81}$$

Thus

$$|\hat{\varepsilon}_m| = |A\hat{\varepsilon}_{m-1} + B\hat{\varepsilon}_{m+1}| \tag{5.82}$$

$$\begin{aligned}|\hat{\varepsilon}_m| = \Big|&\left(-\left(\cos(k_p\Delta x)+isin(k_p\Delta x)\right)b_2-b_3-b_4\left(\cos(k_p\Delta x)-isin(k_p\Delta x)\right)\right)\hat{\varepsilon}_{m-1}\\ &+\left(1-b_5\left(\cos(k_p\Delta x)+isin(k_p\Delta x)\right)-b_6\left(\cos(k_p\Delta x)-isin(k_p\Delta x)\right)\right.\\ &\left.-b_7\right)\hat{\varepsilon}_{m+1}\Big|\end{aligned} \tag{5.83}$$

Meaning

$$|\hat{\varepsilon}_0| < |\hat{\varepsilon}_{m+1}| \to \frac{|\hat{\varepsilon}_0|}{|\hat{\varepsilon}_{m+1}|} < 1 \tag{5.84}$$

$$\begin{aligned}&|\hat{\varepsilon}_m|\\ &\le |\hat{\varepsilon}_{m-1}|\sqrt{\left(-(\cos(k_p\Delta x)+\cos(k_p\Delta x))(b_2-b_4)-b_3\right)^2+\left((b_2-b_4)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)\right)^2}\\ &+|\hat{\varepsilon}_{m+1}|\sqrt{\left(1+(b_6-b_5)\left(\cos(k_p\Delta x+k_p\Delta x)\right)\right)^2+\left((b_6-b_5)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)-b_7\right)^2}\end{aligned} \tag{5.85}$$

Now inductions hypothesis

$$|\hat{\varepsilon}_m| < 0 \; and \; |\hat{\varepsilon}_{m-1}| < |\hat{\varepsilon}_0| \tag{5.86}$$

$$|\hat{\varepsilon}_m| < |A||\hat{\varepsilon}_{m+1}| + |B||\hat{\varepsilon}_{m+1}| \tag{5.87}$$

$$|\hat{\varepsilon}_m| < |\hat{\varepsilon}_{m+1}|\left(\sqrt{\left(-\left(\cos(k_p\Delta x)+\cos(k_p\Delta x)\right)(b_2-b_4)-b_3\right)^2+\left((b_2-b_4)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)\right)^2}\right) + |\hat{\varepsilon}_{m+1}|\left(\sqrt{\left(1+(b_6-b_5)\left(\cos(k_p\Delta x+k_p\Delta x)\right)\right)^2+\left((b_6-b_5)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)-b_7\right)^2}\right) \quad (5.89)$$

Common factor

$$|\hat{\varepsilon}_m| < |\hat{\varepsilon}_{m+1}|\left(\sqrt{\left(-\left(\cos(k_p\Delta x)+\cos(k_p\Delta x)\right)(b_2-b_4)-b_3\right)^2+\left((b_2-b_4)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)\right)^2} + \sqrt{\left(1+(b_6-b_5)\left(\cos(k_p\Delta x+k_p\Delta x)\right)\right)^2+\left((b_6-b_5)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)-b_7\right)^2}\right) \quad (5.90)$$

Further simplification

$$\frac{|\hat{\varepsilon}_0|}{|\hat{\varepsilon}_{m+1}|} < |A| + |\mathrm{B}| \quad (5.91)$$

$$\frac{|\hat{\varepsilon}_0|}{|\hat{\varepsilon}_{m+1}|} = \sqrt{\left(-\left(\cos(k_p\Delta x)+\cos(k_p\Delta x)\right)(b_2-b_4)-b_3\right)^2+\left((b_2-b_4)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)\right)^2} + \sqrt{\left(1+(b_6-b_5)\left(\cos(k_p\Delta x+k_p\Delta x)\right)\right)^2+\left((b_6-b_5)\left(isin(k_p\Delta x)-isin(k_p\Delta x)\right)-b_7\right)^2} \quad (5.92)$$

Recall that

$$\frac{|\hat{\varepsilon}_0|}{|\hat{\varepsilon}_{m+1}|} < 1 \quad (5.93)$$

This also indicates that

$$|A| + |\mathrm{B}| < 1 \quad (9.94)$$

$$
\begin{aligned}
&\left|\left(-\left(\cos\left(k_p\Delta x\right) + \cos\left(k_p\Delta x\right)\right)(b_2 - b_4) - b_3\right)^2\right. \\
&\qquad\qquad \left.+ \left((b_2 - b_4)\left(isin\left(k_p\Delta x\right) - isin\left(k_p\Delta x\right)\right)\right)^2\right| \\
&\qquad\qquad + \left|\left(1 + (b_6 - b_5)\left(\cos\left(k_p\Delta x + k_p\Delta x\right)\right)\right)^2\right. \\
&\qquad\qquad \left.+ \left((b_6 - b_5)\left(isin\left(k_p\Delta x\right) - isin\left(k_p\Delta x\right)\right) - b_7\right)^2\right| < 1
\end{aligned} \tag{5.95}
$$

Alternatively

$$
\begin{aligned}
&\sqrt{\left(-\left(\cos\left(k_p\Delta x\right) + \cos\left(k_p\Delta x\right)\right)(b_2 - b_4) - b_3\right)^2 + \left((b_2 - b_4)\left(isin\left(k_p\Delta x\right) - isin\left(k_p\Delta x\right)\right)\right)^2} \\
&+ \sqrt{\left(1 + (b_6 - b_5)\left(\cos\left(k_p\Delta x + k_p\Delta x\right)\right)\right)^2 + \left((b_6 - b_5)\left(isin\left(k_p\Delta x\right) - isin\left(k_p\Delta x\right)\right) - b_7\right)^2} \\
&< 1
\end{aligned} \tag{5.96}
$$

Therefore, we infer that our numerical solution is conditionally stable. To add, the stability of equation (4.124) and equation (4.143) will also be stable because of the same operator with equation (4.108)

CHAPTER 6:
NUMERICAL SIMULATIONS

6.1 INTRODUCTION

When modelling real world physical problem, one needs to consider three steps examination, this includes observation, predictions, and analysis. Furthermore, mathematical models are also done to convent the conceptual analysis into mathematical equation. To approach this analysis, one needs to initially perform the conceptual model, and analytical modelling which will therefore provide an exact solution. For non-linear, the analytical methods will be converted to numerical methods. Numerical methods are a model that provides an insight numerical approximation to the problem. In the previous chapters (chapter 4 and 5) we presented the stability analysis and numerical solution to groundwater flow within a leaky aquifer with fractal-fractional derivates. The stability indicated accurate results.

In this section, we present the numerical solution for the suggested numerical scheme in chapter 4 using Predictor-Corrector method and AB derivate to capture the flow within a leaky aquifer. To achieve this, we firstly present mathematical numerical coding scheme and simulate the numerical solution using a software called MATLAB. Furthermore, we simulate the numerical solution at different alpha values. The simulation depicts the change in water level with respective to their contour plots. The aquifer is simulated in two dimension and three dimensions. Furthermore, the initials boundaries that represent the model are indicated in the appendix. Lastly, numerical simulations are given in different lambda and different velocities. The lambda values range between 0,5 and 1.5 while the velocity values ranges between 0.91 and 0.95.

6.2 NUMERICAL SIMULATIONS

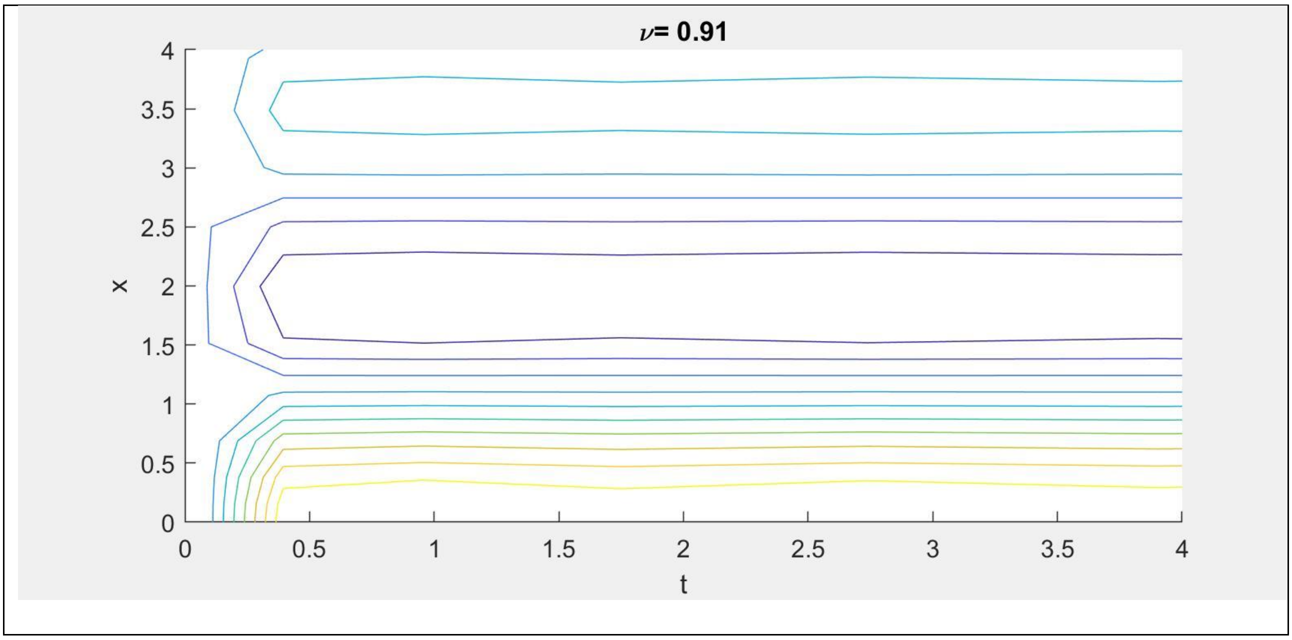

Figure 6: 2-Dimensional counter plot simulations for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 0.5 and velocity is 0.91

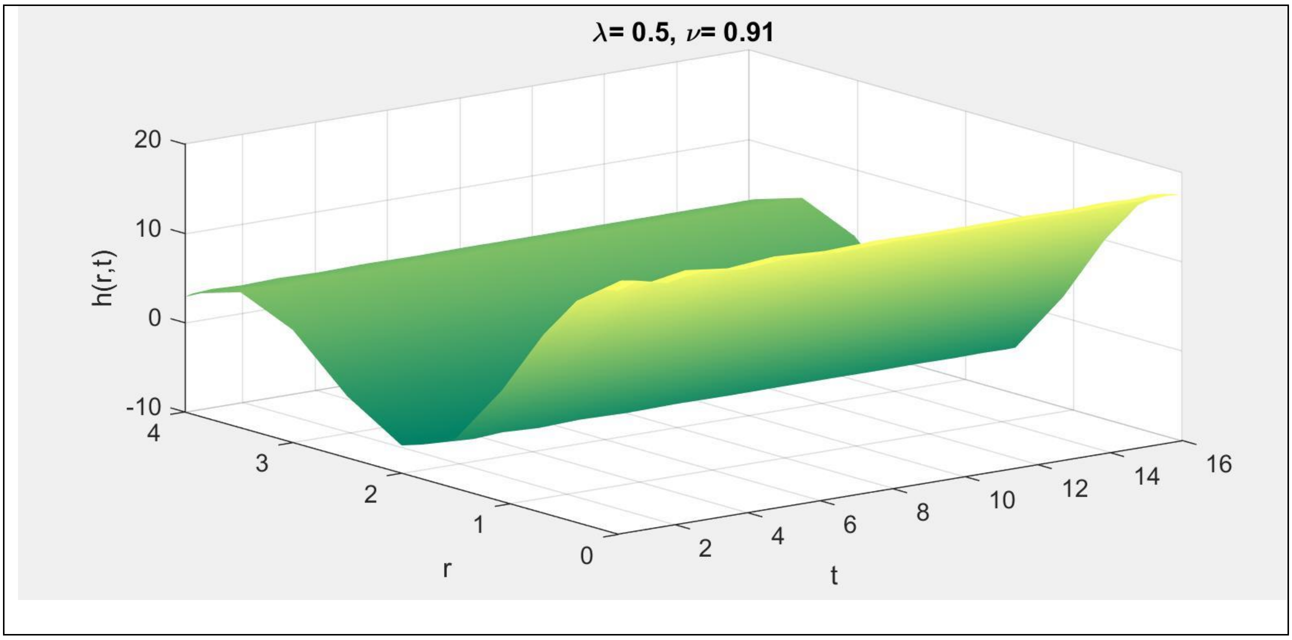

Figure 7: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is only 0.5 and velocity is 0.91

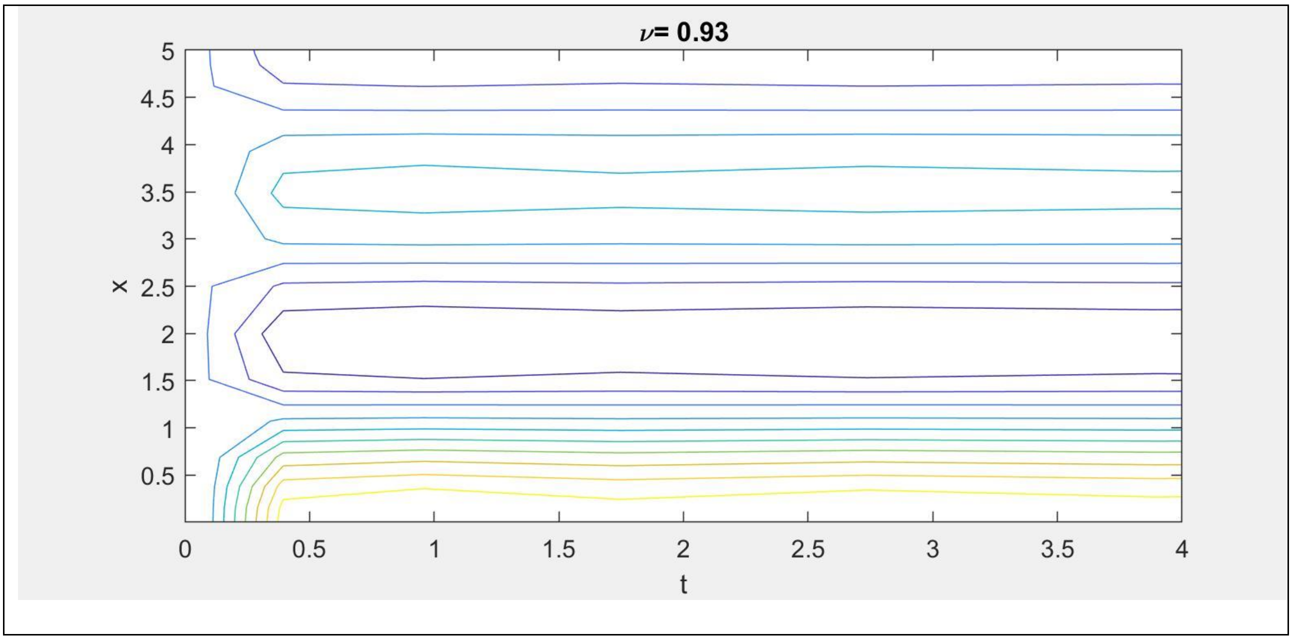

Figure 8: 2-Dimensional counter plot simulations for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 0.5 and velocity is 0.93

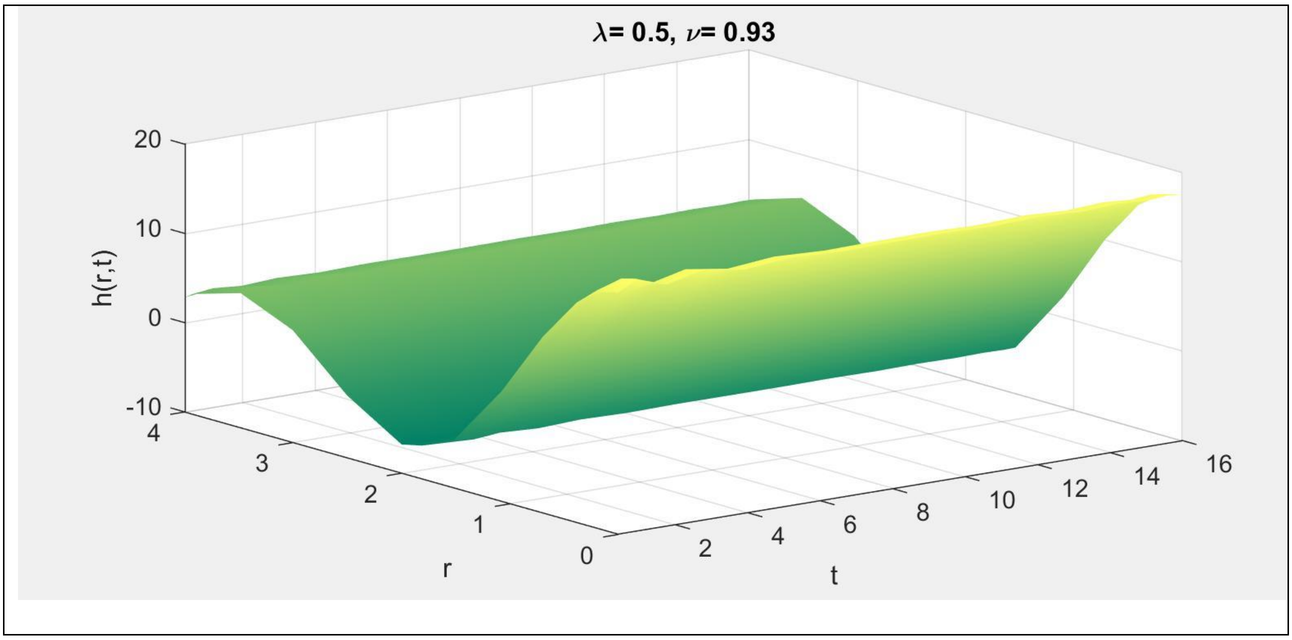

Figure 9: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 0.5 and velocity is 0.93

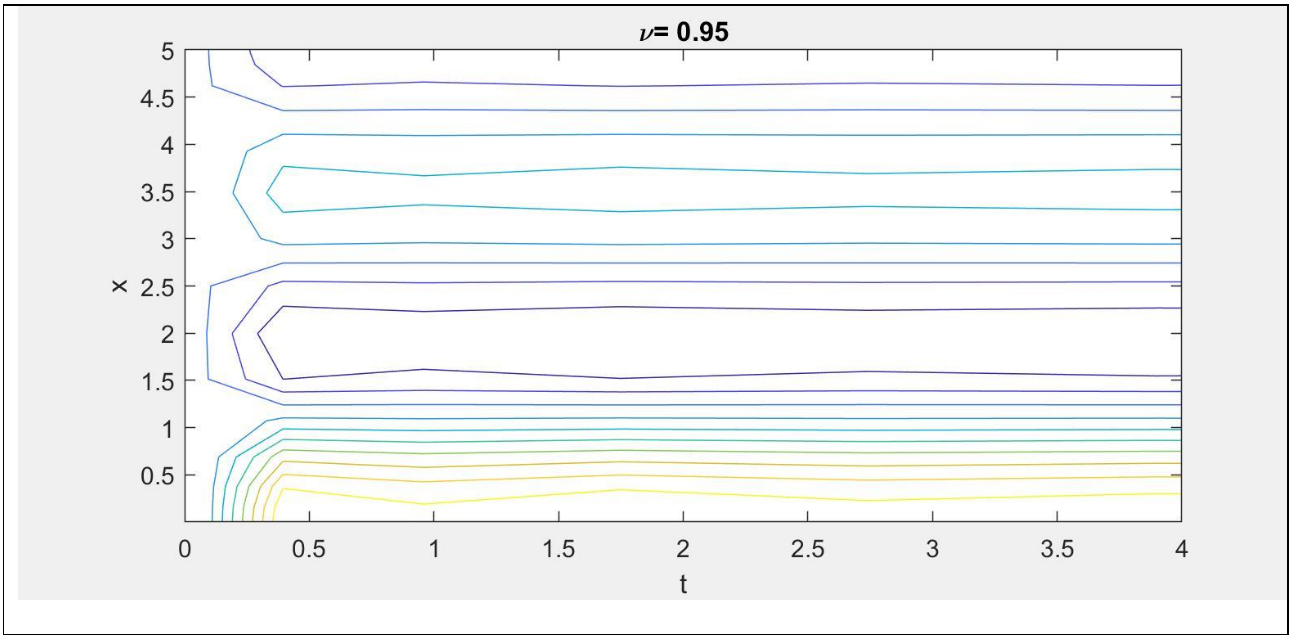

Figure 10: 2**-Dimensional counter plot simulations for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is constant and velocity is 0.95**

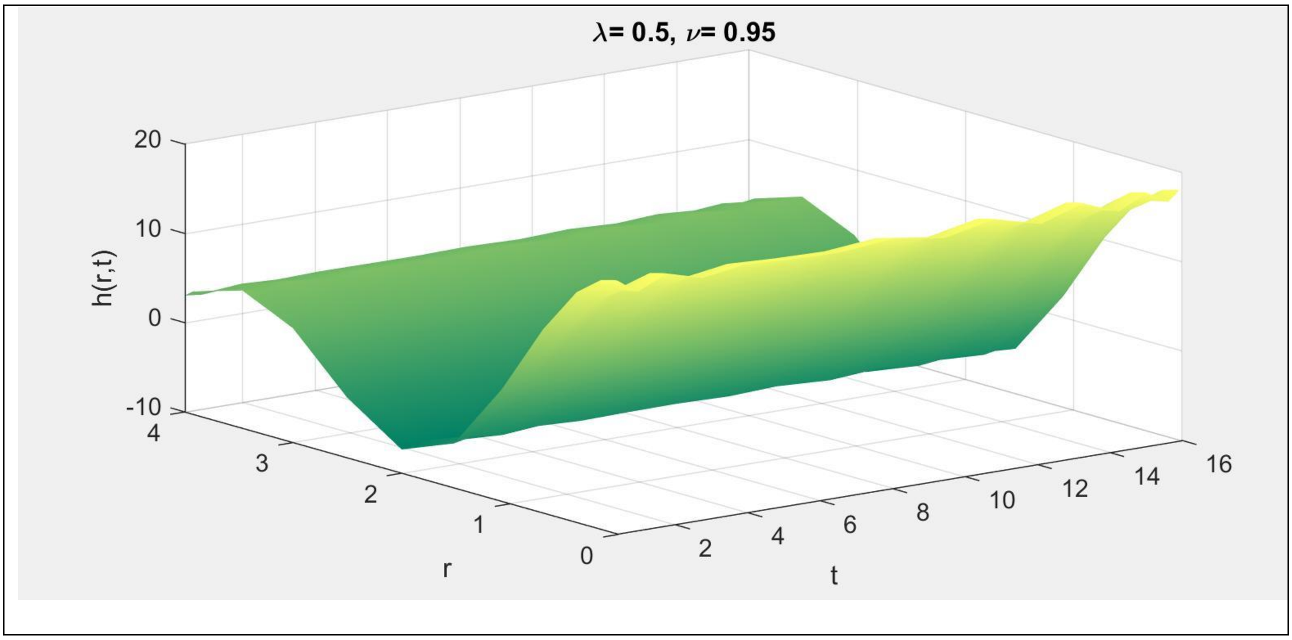

Figure 11: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 0.5 and velocity is 0.95

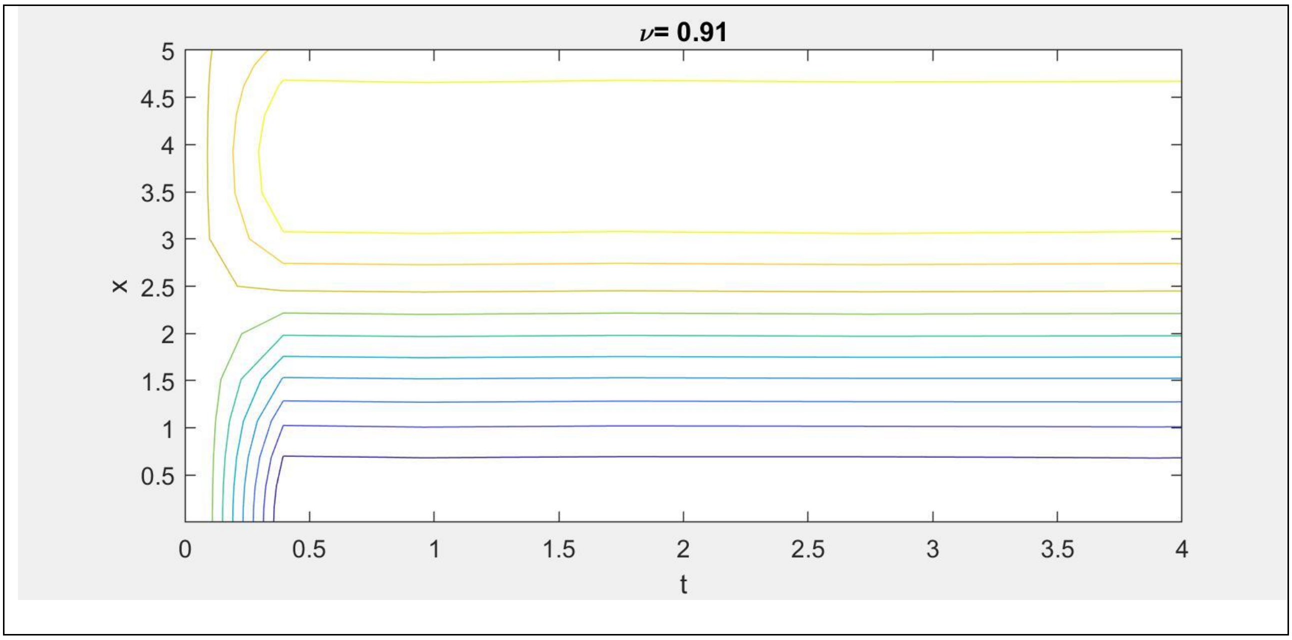

Figure 12: 2-Dimensional counter plot simulations for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1 and velocity is 0.91

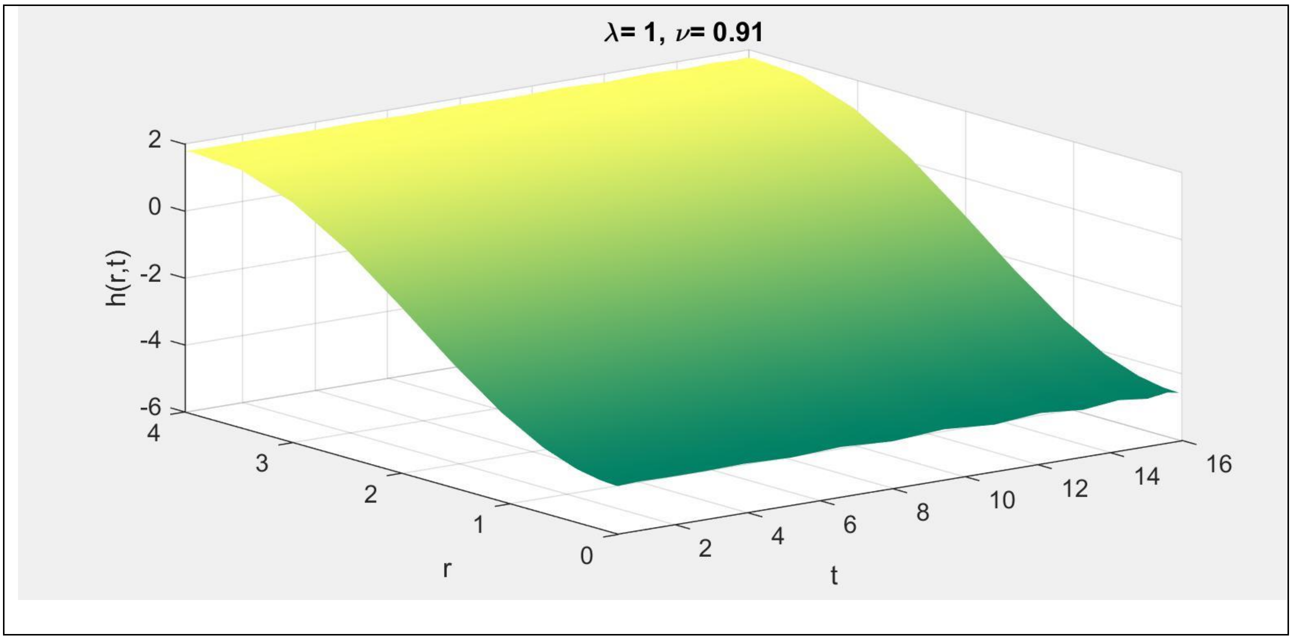

Figure 13: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 0.5 and velocity is 0.91

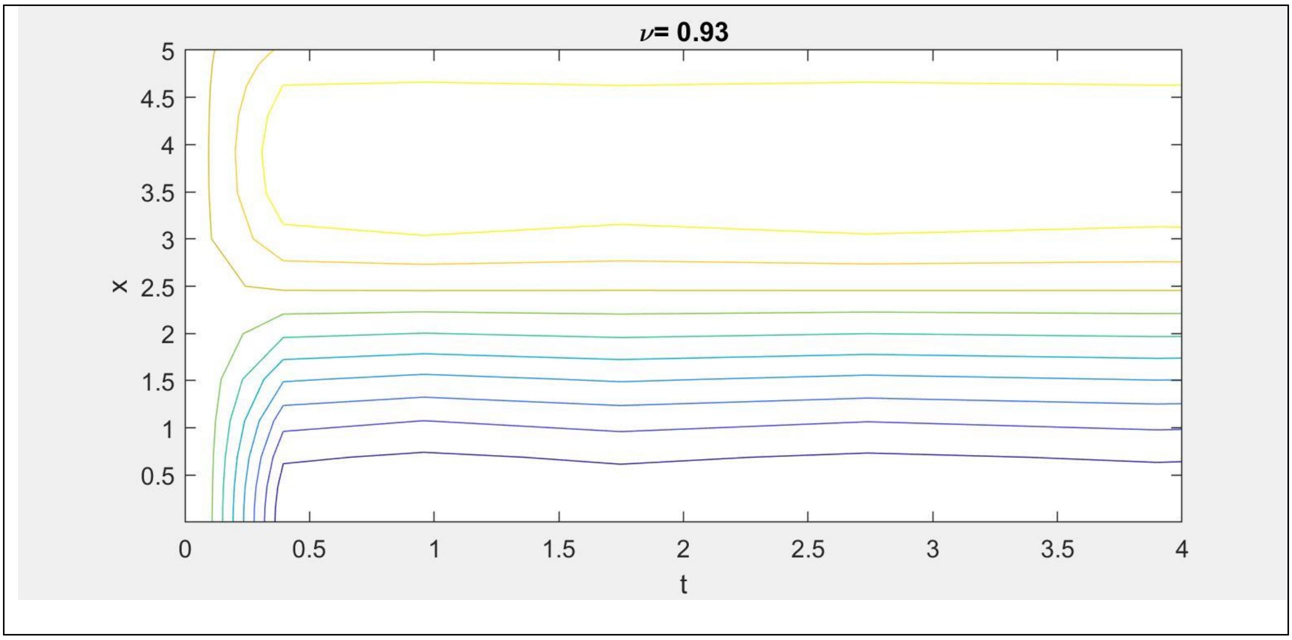

Figure 14: 2-Dimensional counter plot simulations for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1 and velocity is 0.93

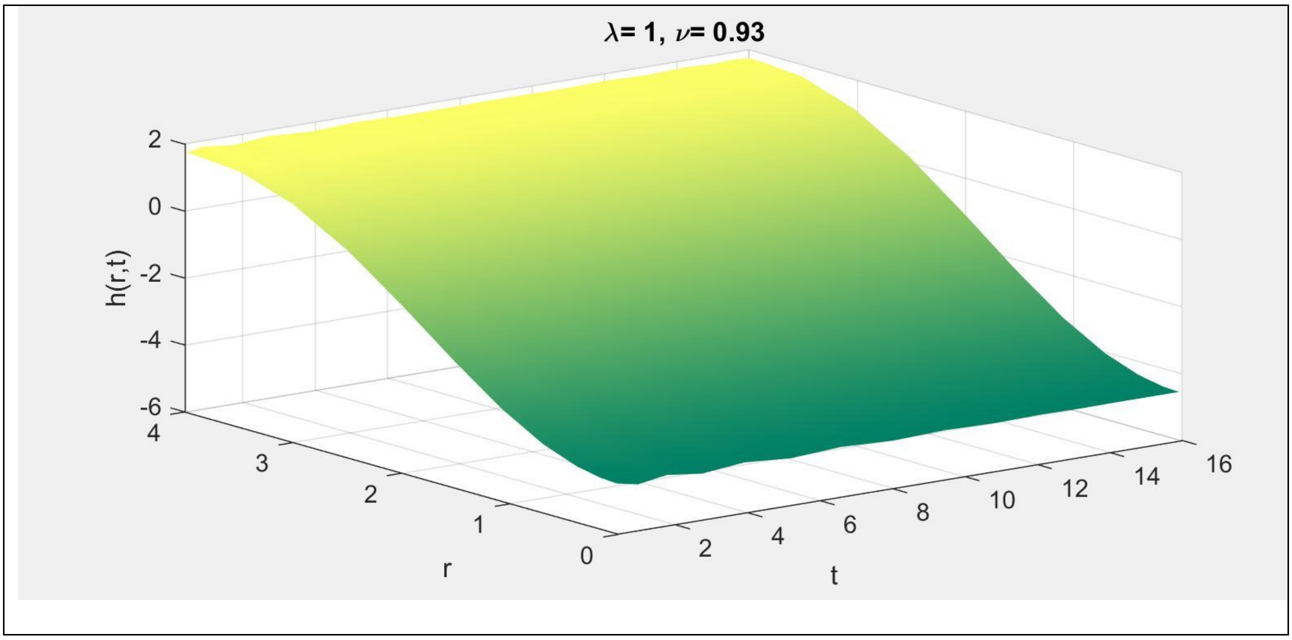

Figure 15: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1 and velocity is 0.93

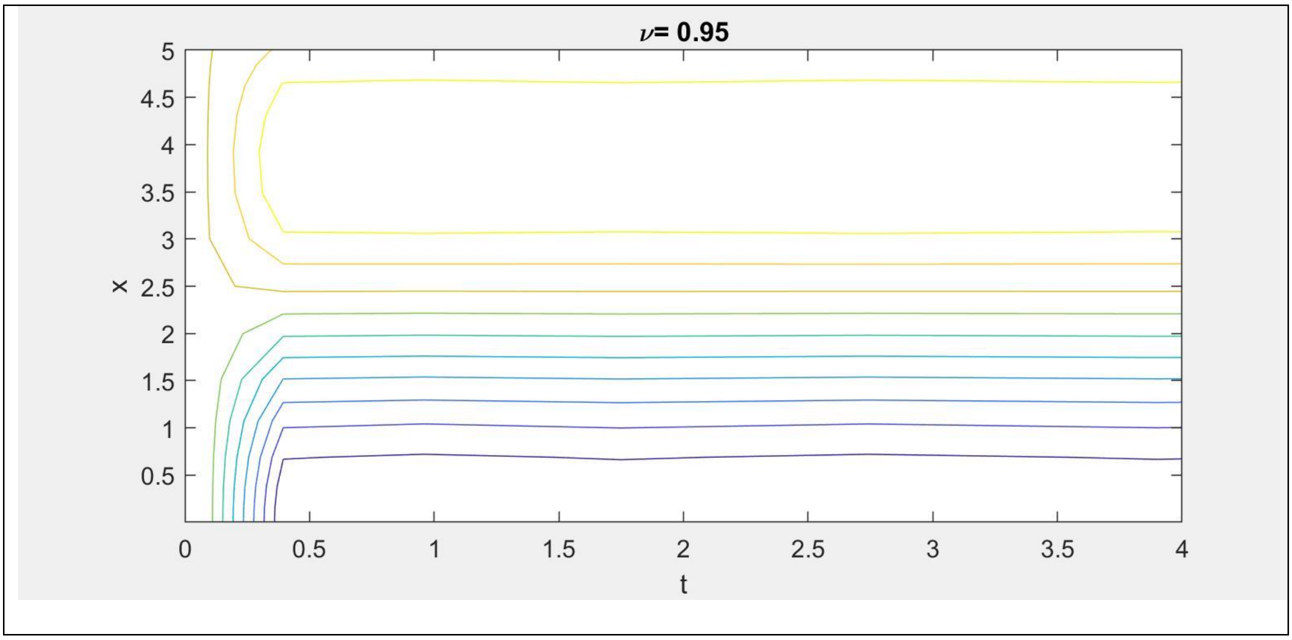

Figure 16: 2-Dimensional counter plot simulations for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1 and velocity is 0.95

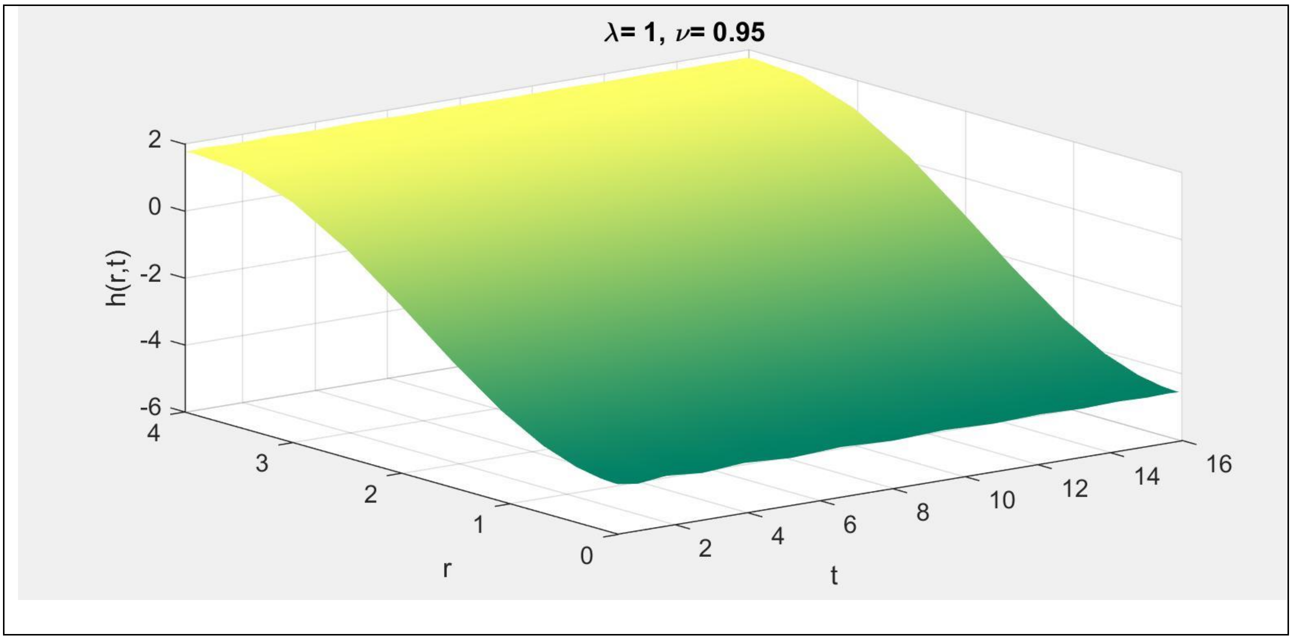

Figure 17: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1 and velocity is 0.95

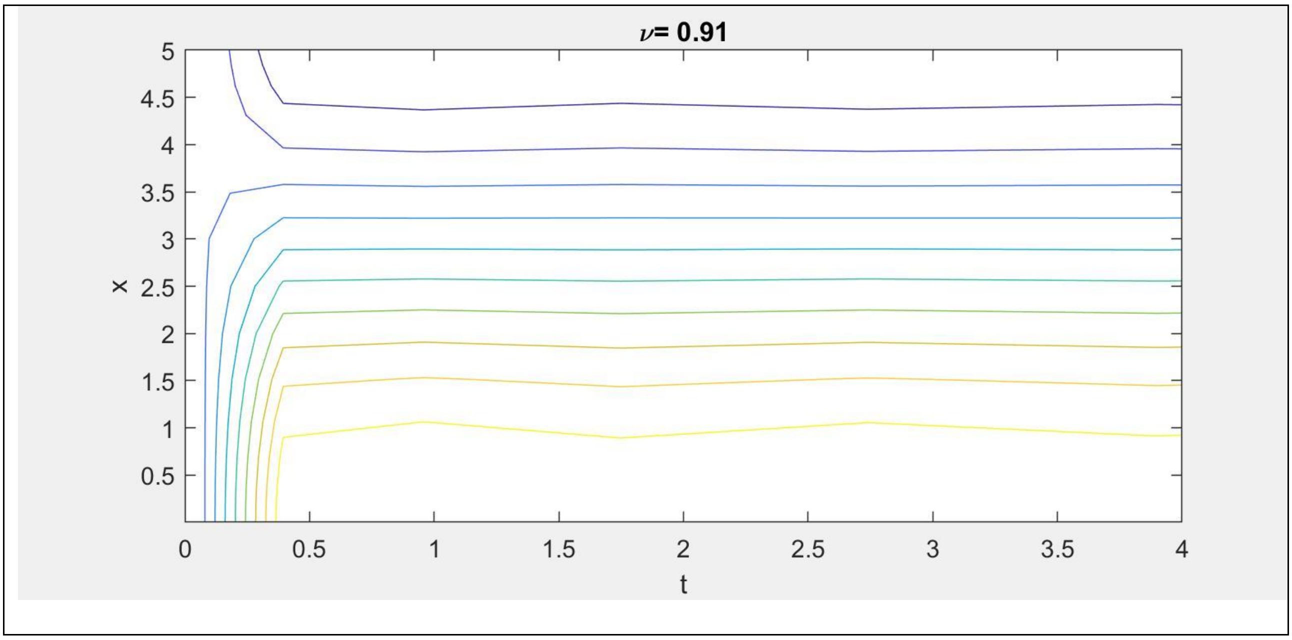

Figure 18: 2-Dimensional counter plot simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1.5 and velocity is 0.91

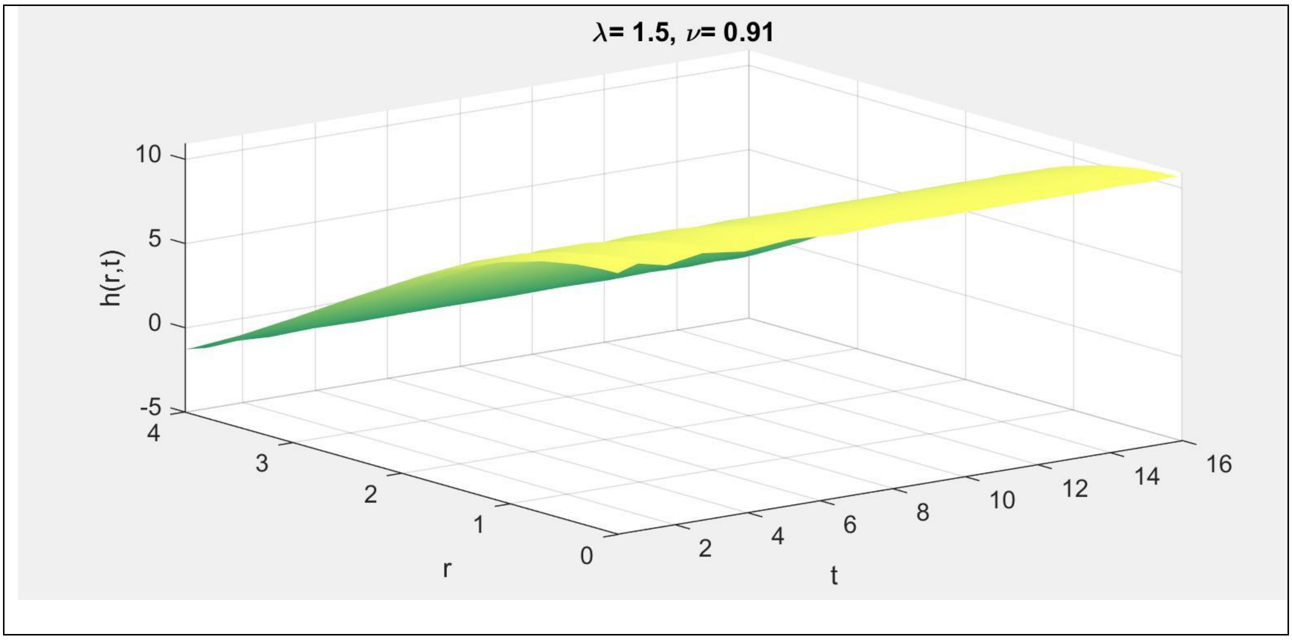

Figure 19: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1.5 and velocity is 0.91

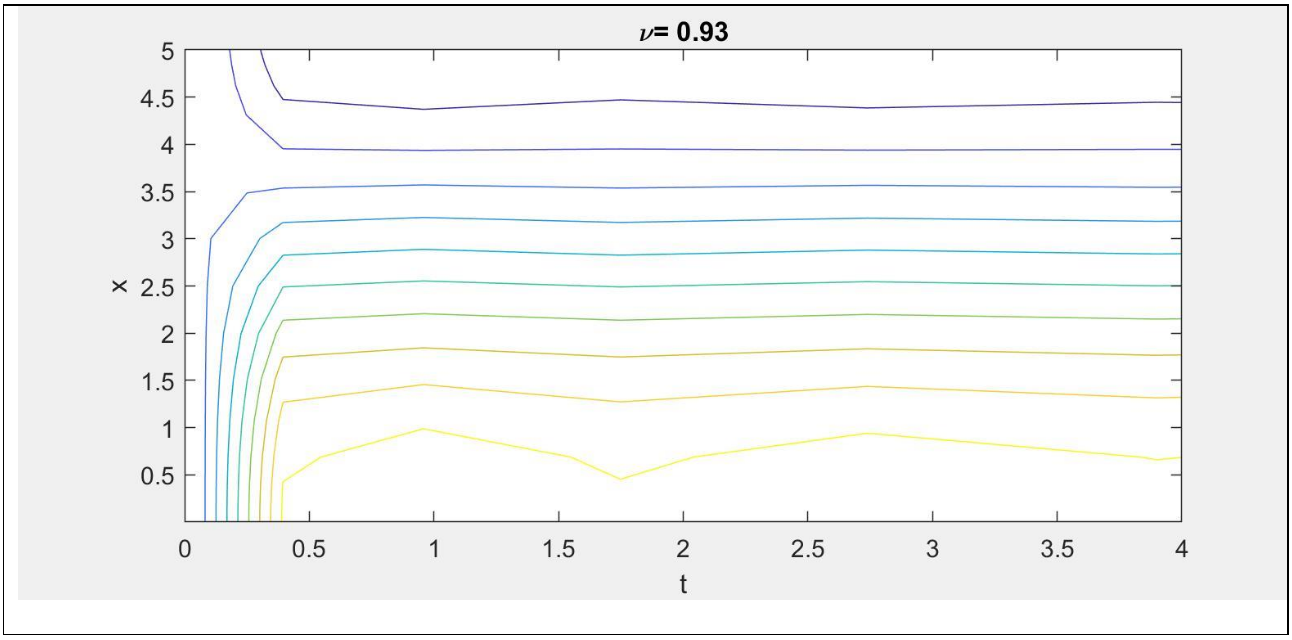

Figure 20: 2-Dimensional counter plot simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1.5 and velocity is 0.93

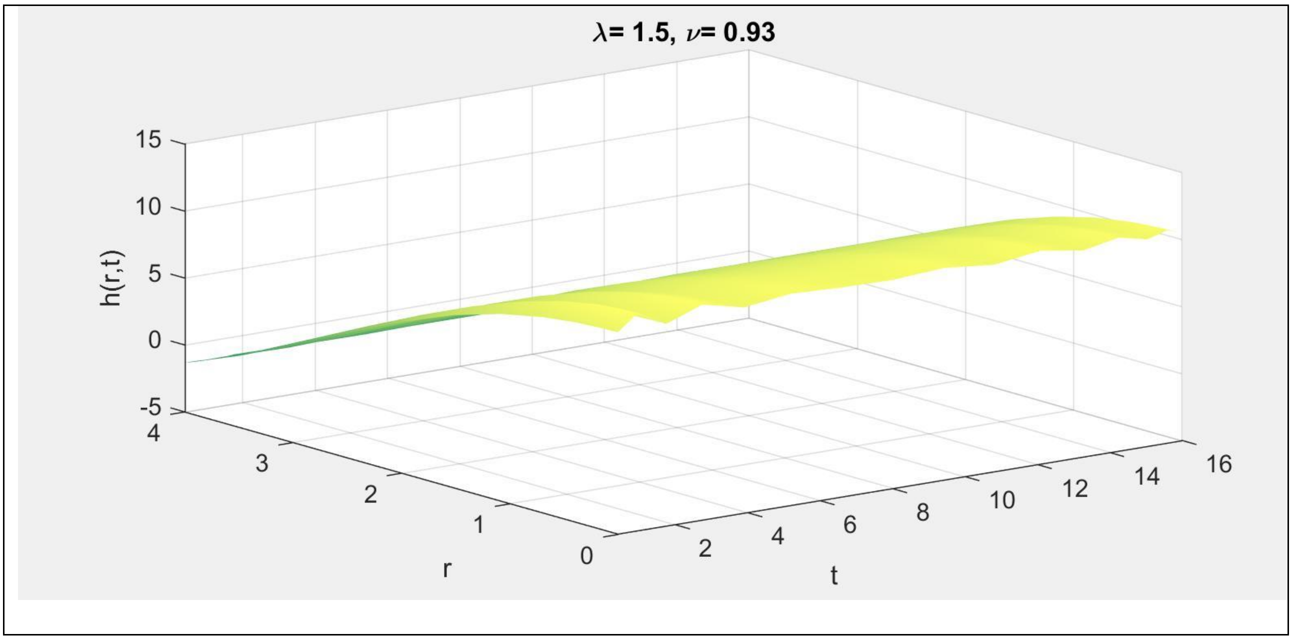

Figure 21: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1.5 and velocity is 0.93

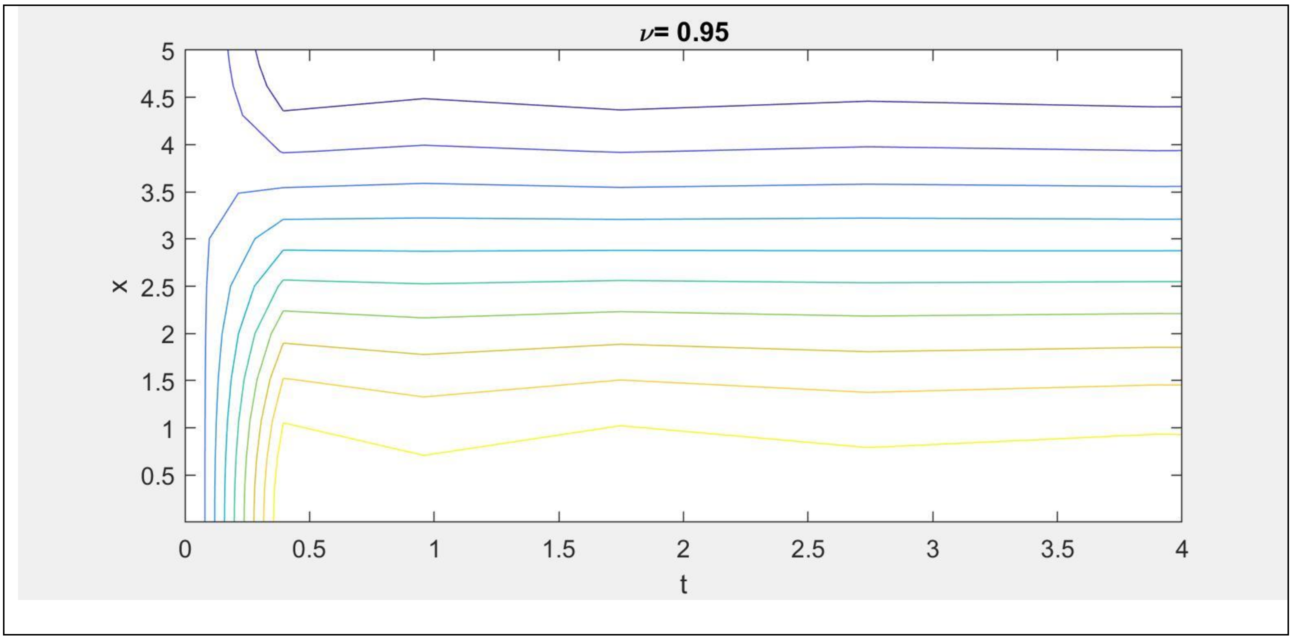

Figure 22: 2-Dimensional counter plot simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1.5 and velocity is 0.95

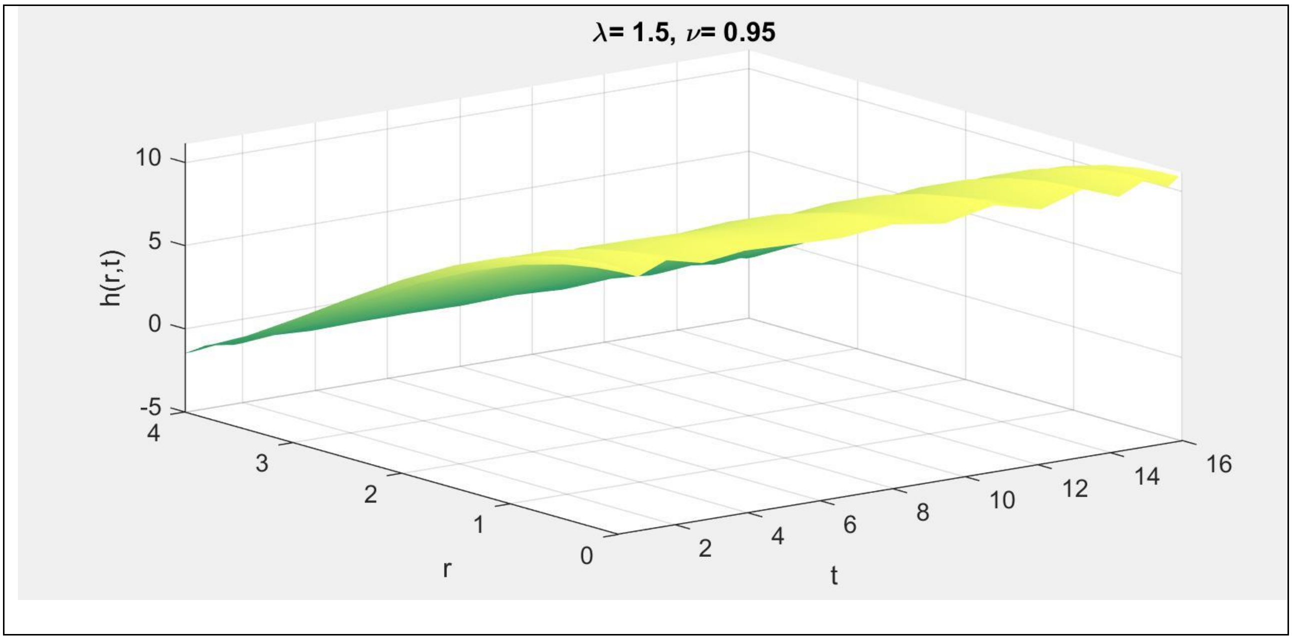

Figure 23: Numerical simulation for groundwater flow with fractal-fractal operator within a leaky aquifer when λ is 1.5 and velocity is 0.95

6.3 DISCUSSION

The results have been presented in section 6.1 above. This numerical simulation presented above represents the real-world physical behaviour system, and hydraulic head of leaky aquifer system with respect to time and space. This also shows that the occurrence of heterogeneity within a leaky aquifer with time and space has a huge influence on groundwater flow velocity in the aquifer. In figure 6-11 we observe gradual groundwater and symmetric flow with time while the concentration of groundwater flow tends to change pattern in figure 12-22. This rapid change of groundwater flow is due to the high and low values of lambda though the velocity of the groundwater flow is also changing. This means that the increase and decreasing scale of lambda values has huge impact on groundwater flow. Furthermore, the water level is high with low values of lambda and low with high values of lambda which is expected in the field for fractured heterogeneity network. The simulations from figure 6-22 also indicate non probability distribution when changing the lambda values and flow velocity. This is due to occurring of heterogeneity in the groundwater leaky aquifer system which is also influenced by the groundwater velocity. The figures above also depict a normal distribution within the same values of lambda scale but non-normal distribution at different lambda scales with time and space.

The counter plot figures also show the cone of depression due to the effect of fractal-fractional dimension with time and space. The numerical simulation with high counter plot has low lambda values as compared numerical simulations with low counter plot which have higher values of lambda. This also indicate that groundwater flow velocity has a huge impact on the cone of depression. Furthermore, crossover is captured and also visible in figure 18-22. This crossover was captured using a special operator called Atangana-Baleanu fractal derivative and this crossover could not be captured when using classical equations. The AB derivatives in the simulations also indicate a leaky aquifer that fractal-fractional order that replicates itself depending on the lambda values. If we revert to Hantush, it indicates that classical differential operators can only be applied to homogeneous aquifer system. Nevertheless, in reality groundwater aquifers are characterized heterogeneity.

CONCLUSION

The concept of developing a model for groundwater flow equations within a leaky aquifer still remain a tricky challenge in understanding geological formations. In the past, scientist have devoted their time to derive groundwater flow equations using mathematical tools and operations that describes the migration of groundwater within a leaky aquifer. Most of the methods were successfully only based on homogeneity and local problems but could not describe randomness, accuracy, complexity, self-similarities and non-local problems. Nevertheless, Hantush described a mathematical groundwater flow equation for movement of water within a leaky aquifer, however the equation could not account for heterogeneity, complexity, and also could not predict the behaviour of water level. To enhance to model for leaky aquifers, (Atangana and Qureshi, 2020) revealed that the concept fractal-fractional calculus operators gives better predictions to model groundwater flow than using integer derivatives. With this revelation in mind, fractal-fractional operators were introduced in this thesis to capture these scenarios. Due to complexity of real world-physical problems, a new numerical solution was needed to capture flow within a leaky aquifer. In this dissertation, we employ the methodology of Predictor-Corrector and Atangana-Baleanu derivatives to generate a new numerical scheme to capture non-local and other related groundwater problems encountered within a leaky aquifer. Therefore, the concept of fractal-fractional operators is appropriate to model complex different type of geological formation. For this method, detail stability testing using Von Neuman and also the uniqueness for distinct solution has been presented. Lastly, numerical simulations and contour plot simulations have been presented.

REFERENCES

Alkahtani, B.S.T. (2016). "*Chua's circuit model with Atangana-Baleanu derivative with fractal order*". Chaos, Solutions & Fractals. **89**, 547-551.

Algahtani, O.J.J. (2016). "*Comparing the Atangana-Baleanu and Caputo-Fabrizio derivative with fractional order: Allen Cahn model*". Chaos, Solutions & Fractals. **89**,552-559.

Al-Salti, N., Karimov, E. & Sadarangani, K. (2016). *On a Differential Equation with Caputo-Fabrizio Fractional Derivative of Order 1 < β ≤ 2 and Application to Mass-spring-damper System*. Research Gate (2016). 1-11.

Amanda, R. and Atangana, A. (2018). *Derivation of a groundwater flow model within leaky and self-similar aquifers: Beyond Hantush model*. Chaos, Solitons & Fractals. **116**,414-423.

Atangana, A., 2017. *Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system*. Chaos, solitons & fractals. **102**, 396-406.

Atangana, A. and Qureshi, S. (2020). Fractal-fractional differentiation for the modeling and mathematical analysis of nonlinear diarrhea transmission dynamics under the use of real data. *Chaos, Solitons & Fractals*, *136*, p.109812.

Atangana, A. (2016). *Exact solution of the time-fractional groundwater flow equation within a leaky aquifer equation*. Journal of Vibration and control. 1749-1759.

Atangana, A. & Alqahtani, R. T. (2016). *Numerical Approximation of the Space-Time Caputo-Fabrizio Fractional Derivative and Application to Groundwater Pollution*. *Advances in* Difference Equations 2016. **156**, 1-13.

Atangana, A. & Baleanu, D. (2016). *New Fractional Derivatives with Non-Local and Non-Singular Kernel: Theory and Application to Heat Transfer Model*. Thermal Science. **20** (2), 763-769.

Atangana, A. & Bildik, N. (2013). *The Use of Fractional Order Derivative to Predict the Groundwater Flow*. Mathematical Problems in Engineering 2013. **543026**, 1-9.

Atangana, A. & Goufo, E. (2015). *A model of the groundwater flowing within a leaky aquifer using the concept of local variable order derivative*. Journal of nonlinear science and applications. 763-775.

Atangana, A. and Kilicman, A. (2013). *Analytical solutions of the space-time fractional derivative of advection dispersion equation*. Mathematical Problems in Engineering. **13**, 9.

Atangana, A. and Koca, I. (2017). *New direction in fractional differentiation*. Math. Nat. Sci.. **1**, 18-25.

Atangana, A. and Koca, I. (2016). *"Chaos in simple nonlinear system with Atangana-Baleanu derivates with fractional order"*. Chaos, Solutions & Fractals. **89**: 447-454.

Atangana, A. and Vermeulen, P.D. (2014). *Analytical solutions of a space-time fractional derivative of groundwater flow equation*. Abstract and Applied Analysis. **14**, 11.

Atkinson, Kendall, A. (1989). *An Introduction to Numerical Analysis.* 2nd edn. New York: John Wiley & Sons.

Ayvaz, M.T. and Gurarslan, G. (2018). September. *Identification of the aquifer parameters from pumping test data by using a hybrid optimization approach*. In HIC 2018. 13th International Conference on Hydro informatics. 147-154. EasyChair.

Bansal, R. (2016). *Analytical solution of linearized Boussinesq equation in isotropic medium with leaky base*. International journal of Engineering. 107-111.

Benson, D.A., Wheatcraft, S.W. and Meerschaert, M.M. (2000). *Application of a fractional advection-dispersion equation*. Water Resources Research. **36**(6), 1403–1412.

Botha, J.F. and Cloot, A.H. (2006). *A generalised groundwater flow equation using the concept of non-integer order derivatives*. Water SA. **32**(1), 1–7.

Boonstra, J. and Kselik, R.A.L. (2002). *Software for Aquifer Test Evaluation, International Institute for Land Reclamation and Improvement*. Wageningen, The Netherlands: International Institute for Land Reclamation and Improvement.

Caputo, M. (1967). *"Linear model of dissipation whose Q is almost frequently independent-II"*. Geophysical Journal International. **13**(5), 529-539.

Caputo, M. & Fabrizio, M. (2015). *A New Definition of Fractional Derivative without Singular Kernel*. Prog. Fract. Differ. Appl. **1** (2), 73-85.

Chen, C. and Chang, C. (2002). *Use of cumulative volume of constant-head injection test to estimate aquifer parameters with skin effects: filed experiment and data analysis*. Water resource research. **38** (5), 1056 10.1029/2001WR000300.

Chen, W. (2005). *Time-Space fabric underlying anomalous diffusion*. Chaos solution and fractals, 923-929.

Cherry, A., Parker, B. and Borchardt, A. (2004). *Role of aquitards in the protection of aquifers from contamination. USA*: Awwa Research Foundation.

Djida, J. D., Area, I. & Atangana, A. (2016). New Numerical Scheme of Atangana-Baleanu Fractional Integral: An Application to Groundwater Flow within Leaky Aquifer. 1-12.

Ehligg, C. and Halepaska, J. (1976). *A numerical study of confined-unconfined aquifers including the effects of delayed yield and leakage*. Water resource research, 1175-1183.

Gladkina, A., Shchedrin, G., Al Kawaja, U. & Carr, L. D. (2018). *Expansion of Fractional Derivatives in Terms of an Integer Derivative Series: Physical and Numerical Applications*. 1-18.

Gnitchogna Batogna, R. and Atangana, A. (2017). *New Two Step Laplace Adam-Bashforth Method for Integer an Non integer Order Partial Differential Equations*. arXiv preprint arXiv:1708.01417.

Gómez-Aguilar, J. F., Escobar-Jiménez, R. F., López-López, M. G., & Alvarado-Martínez, V. M. (2016). *Atangana-Baleanu Fractional Derivative Applied to Electromagnetic Waves in Dielectric Media*. Journal of Electromagnetic Waves and Applications. **30** (15), 1937-1952.

Gómez-Aguilar, J. F., Torres, L., Yépez-Martínez, H., Baleanu, D., Reyes, J. M. & Sosa, I. O. (2016). *Fractional Liénard Type Model of a Pipeline within the Fractional Derivative without Singular Kernel*. Advances in Difference Equation. **173**, 1-13.

Hantush, M. and Jacob, C. (1955). *Non-Steady radial flow in infinite leaky aquifer*. **36** (1): 95-100.

Hadamard, J. (1892). *"Essai sur l'étude des fonctions dennées par leur déveleppement de Taylor"*. Journal of Pure and Applied Mathematics. **4**(8), 101-186.

Hassanizadeh, S.M. (1986). *Derivation of basic equations of mass transport in porous media*. Advance in Water Resources. 196-206.

Hemker, C. and Maas, C. (1987). *Unsteady flow to wells in layered ad fissured aquifer system*. Journal of hydrology. 231-249.

Hermann, R. (2014). Fractal Calculus: An Introduction for Physicists, 2nd Edn. New Jersey: World Scientific.

Hund-Der, Y., Chia-Chin, Y. and Shaw-Yang, Y. (2013). *Semi-analytical solution of groundwater flow in a leaky aquifer system subject to bending effect*. Journal of hydrology. 395-402.

Jiang, Li. (2006). *Transient radial movement of confined leaky aquifer due to variable well flow rates*. Journal of hydrology. 542-553.

Kilbas, A., Srivastava, H.M. and Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier Science Vol. 204 of North-Holland Mathematics Studies.

Kuhlman, K. (2008). *Laplace transform analytical element method for transient groundwater flow simulation*. University of Arizona. Leaky aquifer. Retrieved from hhtp://content.alterra.wur.nl/Internet/webdocs/ilripublicaties/publicaties/Pub47.

Kutaganpola, U.N. (2014). *"A New Approach To Generalized Fractional Derivatives" (PDF)*. Bulletin of Mathematical Analysis and Applications. **6** (4), 1-15.

Kruseman, G.P. and De Ridder, N.A. (1990). *Analysis and Evaluation of Pumping Test Data*. 2nd edn. Wageningen, The Netherlands: International Institute for Land Reclamation and Improvement.

Mahantane, M. L. and Atangana, A., (2019). *A New Method for Modeling Groundwater Flow Problems: Fractional-Stochastic Modeling*. 9-13.

Mainardi, F., Mura, A. and Pagnini, G. (2009). *The M-Wright function in time-fractional diffusion processes: a tutorial survey*. Int J Differ Eq 2010:29 Article ID 104505. doi: 10.1155/2010/104505.

Mekkaoui, T., Atangana, A. and Araz, S.İ. (2020). *Predictor–corrector for non-linear differential and integral equation with fractal–fractional operators*. Engineering with Computers. 1-10.

Malama, B., Kuhlman, K. and Baarrash, W. (2007). *Semi-analytical solution for flow in leaky unconfined aquifer-aquitard systems*. Journal of hydrology. 59-68.

Mirza, I. A. & Vieru, D. (2017). *Fundamental Solutions to Advection–Diffusion Equation with Time-Fractional Caputo–Fabrizio Derivative*. Computers & Mathematics with Applications. **73** (1), 1-10.

Mishra, P.K. and Kuhlman, K.L. (2013). *Unconfined aquifer flow theory: from Dupuit to present*. In Advances in Hydrogeology. 185-202.

Moench, A. (1985). *Transient flow to large-diameter well in an aquifer with storative semi-confined layers*. Water resource. 1121-1131.

Neuman, S. and Witherspoon, A. (1969). *Theory of flow in a confined two-aquifer system*. Water resource research. **5** (4), 803-816.

Rai, N., Manglik, A. and Singh, S. (2006). *Water table fluctuation owing to time varying, recharge, pumping and leakage*. Journal of hydrology. 350-358.

Rudas, I., Barics, A. and Pogany, K. (2015). *Incomplete kratzel function model of leaky aquifer and alike functions*. IEEE 10th International Symposium and Applied Computation Intelligence and Informatics, At Timisoara, Romania.

Schroth, B. and Narasimhan, N. (1997). *Application of a numerical model in interpretation of a leaky aquifer test*. Groundwater. 371-375.

Shan, N. A. & Khan, I. (2016). *Heat Transfer Analysis in a Second Grade Fluid over and Oscillating Vertical Plate Using Fractional Caputo–Fabrizio Derivatives*. Eur. Phys. J. C. **76** (362): 1-11.

Smith, G.D. (1985). *Numerical Solution of Partial Differential Equations: Finite Difference Methods*. 3rd ed. Oxford university press. 67-68.

Stehfest, H. (1970). *Numerical inversion of Laplace transformation*. Communications ACM. 47-49.

Streltsova, D. (1973). *On the leakage assumption applied to equation of groundwater flow*. Journal of hydrology. 237-253.

Theis, C. (1935). *The relation between the lowering of the piezometric surface and rate duration of discharge of a well using groundwater storage*. Transaction, American Geophysical Union. 519-524.

Yavuz, C. (1976). *Mathematical modelling of leaky aquifers with rheological properties*. Turkey.

Yen, D. and Yang, S. (2009). *Radial groundwater flow to a finite diameter well in a leaky confined aquifer with a finite thickness skin*. Hydrological processes. 3382-3390.

Yen, D. and Yang, Y. (2003). *Recent advances in modelling of well hydraulics*. Advances in water Resources. 27-51.

Zauderer, E. (1985). *Partial differential equations of applied mathematics*. 2nd Edn. New York: John Wiley and sons.

Zhan, H. and Bian, A. (2006). *A method of calculating pumping induced leakage*. Journal of Hydrology. 659-667.

Zhan, H. and Park, E. (2003). *Horizontal well hydraulics in leaky aquifers*. Journal of hydrology. 129-146.

APPENDIX A

```
#NUMERICAL CODING
function [ F ] = equations ( U )
global N M  P_RN P_TM  D_x1 D_x2 D_tnu q  SS TT QQ lam
  F = zeros (M+2,N+1);
 for m = 2:M+1
for n = 2:N
F(m,n) = (SS/TT)*P_TM(:,m)' *D_tnu' *U* P_RN(:,n) ...
- q(m,n)*( P_TM (:,m)' *U* D_x1 * P_RN (:,n))-( P_TM (:,m)' *U* D_x2 * P_RN (:,n))-...
( P_TM (:,m)' *U*P_RN (:,n))/lam^2;
end
end
% boundary conditions in t
for n = 1:N+1
% Dirichlet
F(1,n) = P_TM (: ,1)'*U* P_RN (:,n)  ; %u(x,0)
F(m+2,n) = P_TM (: ,1)'*U* P_RN (:,n);%u(x,T)
end
% boundary conditions in x
for m = 1:M+1
  %F(m ,1) = P_TM (:,m)'*U* P_RN (: ,1) - u(m ,1) ;%u(0,t)
  F(m ,1) = P_TM (:,m)'*U*D_x1* P_RN (: ,1) -(-QQ/(2*pi*TT));%u_x(0,t)
%F(m,N +1) = P_TM (:,m)'*U* P_RN (:,N +1)  ;%u(L,t)
F(m,N +1) = P_TM (:,m)'*U*D_x1* P_RN (:,N +1)  ;%u_x(L,t)
end
end
```