MATM 9100
Dual Closure Operators on a Category and their Applications

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Submitted in accordance with the requirements for the degree of

Philosophiae Doctor

in the

Faculty of Natural and Agricultural Sciences

Department of Mathematics and Applied Mathematics

at the

University of the Free State

Bloemfontein

South Africa

January 2020

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Declaration:

1. I, Renier Stefan Jansen (2008064590), declare that the thesis that I herewith submit for the Doctoral Degree in Mathematics at the University of the Free State, is my independent work, and that I have not previously submitted it for a qualification at another institution of higher education.

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Acknowledgements:

Firstly, I would like to thank my supervisor, Prof. David B. Holgate, for being willing to supervise this project from a long distance. I would also like to thank him for all his advice and time spent on this dissertation. I would also like to thank him for leaving me to my own devices to explore the world of category theory.

I would also like to thank my co-supervisor, Prof. Johan H. Meyer, for all his guidance, ideas and effort to get me on the right track. He also made a fair contribution to improve the readability of this dissertation. He also made it possible to do a few research visits to Prof. David Holgate and Prof. Maria Clementino.

Another person I would like to thank is Prof. Maria Manuel Clementino. I would like to thank her for all her inputs and the fact that she made time for me when I visited her for two weeks. It was a tremendous help and I truly appreciate it.

Also, thank you to Dr. Ando Razafindrakoto. Not only for all his suggestions, remarks and counterexamples to some ideas, but also for always listening when I encountered difficulties.

I offer a word of thanks to the Mathematics and Applied Mathematics Department at the University of the Free State. Not only for all their support during the past few years, but for all the great lecturers from undergraduate as well. In particular, I would like to thank Dr. Ben-Eben de Klerk and Dr. Hubertus W. Bargenda. They were not only the two people that sparked my interest in category theory in the first place, but these are two people I have had some of the most interesting conversations with.

I would like to thank my family for all their support. In particular, my mom, Mariechen, for all her encouragement and absolute devotion to helping wherever she could. My father, Mornay, for all his encouragement and eagerness to understand this project.

Lastly, I would also like to acknowledge the National Research Foundation's (NRF) contribution to three years of this project.

Contents

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.

∼ H. Poincaré (1845-1912)

1 Introduction and Synopsis

This dissertation is divided into three chapters. Chapter 2 is merely a reflection on the background of factorisation structures on categories. This includes pre-factorisation structures and orthogonal factorisation structures for morphisms. Our main focus will be on factorisation structures for sources, hence some results from the literature are quoted.

Chapter 3 starts off with a discussion of different notions of constant morphisms. A fairly general notion of constant morphism is selected, but in order for this to be more fruitful, some assumptions are added. Note that our notion of constant morphism is self-dual, but since some of our additional assumptions are not, the whole of the chapter is not. This notion of constant morphisms introduced is equivalent to the notion of constant subcategories as introduced in [19].

For a fixed notion of constant morphism there is a relatively old topic of investigation in categorical topology, such as in [41] or [2], that is, to study generalised subcategories of connected, respectively, disconnected or separated objects. These correspondences are usually referred to as the *Herrlich-Preuß-Arhangel'skii-Wiegandt(HPAW)* correspondence and there is a Galois correspondence between the classes of connected-and disconnected objects. In abelian or normal categories, a similar study was done and this provides the basis for generalised torsion and torsion-free theories. See [21], [51] and [52] for an exposition. From a categorical point of view these classes of torsion or connected objects may be viewed as a left constant subcategory. In a similar manner the classes of disconnected or torsion-free objects may be viewed as right constant subcategories. We then obtain a Galois correspondence between the left and right constant subcategories as one would expect.

The next part of the chapter deals with some properties of right constant subcategories. Some of the properties we study highly depend on some extra properties of the constant subcategory, but in the case we're interested in, every right constant subcategory turns out to be $\mathscr{E}$-reflective.

The last part of the chapter investigates the left constant subcategories. Since most parts of this dissertation assumes a factorisation structure for sources and the constant subcategories depend on this. The theory of left and right constant subcategories is not self dual. The goal is to determine whether the left constant subcategories will be $\mathscr{M}$-coreflective. Since we would not like to restrict ourselves, we don't add a lot of assumptions on the notion of constant morphism. As it turns out, left-constant subcategories are not generally $\mathscr{M}$-coreflective. A generalised notion of coreflectiveness is developed. Under some mild assumptions, it then follows that the left-constant subcategories satisfy this notion.

Chapter 4 deals with dual closure operators. This is the categorical dual of closure operators as studied in [25], [27] and [49] to mention a few. Note that this is not the same as interior operators, as interior operators are the order dual of closure operators.

One particular topic that was studied for closure operators is to factorise the HPAW correspondence through other Galois correspondences between subcategories and closure operators. Since dual closure operators are the categorical dual of closure operators, it is to be expected that such factorisations exist. Some authors have also studied subcategories of generalised connected and disconnected objects via closure operators.

Dual closure operators were probably first studied by D. Dikranjan and W. Tholen. They published an extensive article ([26]) on dual closure operators with applications to left and right constant subcategories, and there are also some applications to pre-radicals. In particular, Dikranjan and Tholen showed that their HPAW correspondence factors through two other Galois correspondences between subcategories and dual closure operators. One of the aims for this chapter is to do a similar thing for our notion of constant morphism. Note that our notion is quite different from the one in [26] and these constructions don't generally coincide.

The first section introduces dual closure operators with some basic properties. Even though a lot of the results in this section are dual to ones for closure operators, the goal is to familiarise the reader with them. If one is already familiar with dual closure operators, then this section may be skipped. As far as possible, we tried to use the same terminology as that in [26].

A lot of credit should be given to [26] as this was a type of model for this chapter. In some instances, constant subcategories may be viewed as a generalisation of their notion of constant morphism. Hence, there are cases in which this chapter can be viewed as a generalisation of some parts of their article. We start by constructing two subcategories from a dual closure operator and a constant subcategory. These subcategories are very similar to the ones constructed in [26], but are essentially different. We also use reflective subcategories and arbitrary subcategories to construct dual closure operators. Both of these approaches provide us with Galois correspondences between arbitrary subcategories, respectively reflective subcategories, and all dual closure operators of $\mathscr{E}$ in $\mathbb{A}$. It is also shown that our HPAW correspondence factors through the Galois correspondences mentioned.

We then study a few variations of these constructions and show that they are all different. A necessary and sufficient condition for the various constructions to coincide is also provided.
This dissertation was written with the idea in mind that it ought to be considered as a self contained document for anyone with basic knowledge of category theory. Of course, some of the most basic notions of category theory is assumed, but some are defined for completeness sake. Our notation is very similar to that used in [1]. Unless stated otherwise, we will always assume that any subcategory is a full subcategory.

Furthermore, some well known results are cited and others are assumed in proofs. However, there are some propositions that are well known, but for which the author could not always find a reference. There are also other fairly obvious results which might be proved elsewhere, but it is proved here nonetheless. A fair literature study was done to avoid this, but some results may be found as exercises in textbooks.

Most of the proofs were also written with the idea that gaps are not left as an exercise for the reader. This is the main reason why some of the proofs are rather detailed. It is sometimes the case that a shorter proof can be found with an extra assumption. Some remarks about these are made throughout the thesis.

A mathematician is a device for turning coffee into theorems.

∼ Paul Erdös (1913-1996)

2 Basic Properties of Factorisation Structures

2.1 Factorisation structures for morphisms

Remark 2.1: This first chapter discusses some topics on factorisation structures. This should all rather be well-known, but it is included for completeness as there are many different types. If one is familiar with pre-factorisation structures, orthogonal factorisation structures and factorisation structures for sources or sinks, then this chapter may be skipped.

Definition 2.2: Orthogonal morphisms
([6]) Let e and m be morphisms in a category $\mathbb{A}$. Then, we say that e **is orthogonal to** m **(in** $\mathbb{A}$**)**, denoted by $e \perp m$, if for each commutative square $\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$ in $\mathbb{A}$, there exists a unique morphism d such that $de = f$ and $md = g$, i.e., such that $\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \arrow[dl, dashed, "!d"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$ commutes. The morphism d is often called the **diagonal morphism or the diagonal fill in** for the diagram.
Let $\mathscr{H}$ be a class of morphisms. We write $\mathscr{H}^{\uparrow}$ for the class of morphisms which is orthogonal to each $h \in \mathscr{H}$. To denote a particular morphism f of $\mathscr{H}^{\uparrow}$, we write $f \perp \mathscr{H}$. We also denote the class of all morphisms for which every member is orthogonal to each $h \in \mathscr{H}$ by $\mathscr{H}^{\downarrow}$. Dual to $f \perp \mathscr{H}$, we write $\mathscr{H} \perp f$ if $f \in \mathscr{H}^{\downarrow}$. Explicitly, this is

$$\mathscr{H}^{\uparrow} = \{f \in Mor(\mathbb{A}) \mid f \perp h \text{ for each } h \in \mathscr{H}\}$$

and

$$\mathscr{H}^{\downarrow} = \{f \in Mor(\mathbb{A}) \mid h \perp f \text{ for each } h \in \mathscr{H}\}.$$

Remark 2.3: The assignments as in Definition 2.2 clearly defines two endomaps $(-)^{\uparrow}$ and $(-)^{\downarrow}$ from all subclasses of morphisms of $\mathbb{A}$. We are particularly interested in considering $\mathscr{H}^{\uparrow\downarrow}$ and $\mathscr{H}^{\downarrow\uparrow}$ for classes of morphisms $\mathscr{H}$.
We will denote all subclasses of Mor($\mathbb{A}$) by $Sub(Mor(\mathbb{A}))$ with respect to inclusion.

Proposition 2.4: ([6, 2.1]) Let $\mathbb{A}$ be a category and $\mathscr{H}$ and $\mathscr{K}$ be classes of morphisms. Then, the following hold:

(a) $\mathscr{H} \subset \mathscr{K}$ implies that $\mathscr{K}^{\downarrow} \subset \mathscr{H}^{\downarrow}$,
(b) $\mathscr{H} \subset \mathscr{K}$ implies that $\mathscr{K}^{\uparrow} \subset \mathscr{H}^{\uparrow}$,
(c) $\mathscr{H} \subset \mathscr{H}^{\uparrow\downarrow}$,
(d) $\mathscr{H} \subset \mathscr{H}^{\downarrow\uparrow}$,
(e) $\mathscr{H}^{\uparrow\downarrow\uparrow} = \mathscr{H}^{\uparrow}$,
(f) $\mathscr{H}^{\downarrow\uparrow\downarrow} = \mathscr{H}^{\downarrow}$.

Definition 2.5: Prefactorisation system
A **prefactorisation system** $\mathscr{F}$ **on a category** $\mathbb{A}$ is a pair $(\mathscr{E}, \mathscr{M})$ where $\mathscr{E}$ and $\mathscr{M}$ are classes of morphisms for which $\mathscr{E} = \mathscr{M}^{\uparrow}$ and $\mathscr{M} = \mathscr{E}^{\downarrow}$.

Proposition 2.6: ([6, 2.2, 2.4]) Let $(\mathscr{E}, \mathscr{M})$ be a prefactorisation system on $\mathbb{A}$. Then, the following hold:

(a) $\mathscr{E} \cap \mathscr{M} = Iso(\mathbb{A})$,
(b) $\mathscr{M}$ and $\mathscr{E}$ are closed under composition,
(c) $\mathscr{M}$ is closed under pullbacks,

(*d*) $\mathscr{M}$ is closed under products,

(*e*) $fg \in \mathscr{M}$ and ($f \in \mathscr{M}$ or f is a monomorphism) implies that $g \in \mathscr{M}$,

(*f*) $\mathscr{M}$ is closed under multiple pullbacks,

(*g*) Let $D, D' : \mathbb{I} \rightrightarrows \mathbb{A}$ be diagrams and $\alpha : D \to D'$ a natural transformation with $\alpha_i \in \mathscr{M}$ for each $i \in Ob(\mathbb{I})$. Then, if $(L, (\ell_i)_{i \in Ob(\mathbb{I})})$ and $(L', (\ell'_i)_{i \in Ob(\mathbb{I})})$ are limits of D and D' respectively, then the unique morphism $\alpha : L \to L'$ induced by the limit is a member of $\mathscr{M}$.

Definition 2.7: Factorisation Structures for morphisms
([1]) Let $\mathbb{A}$ be a category, and $\mathscr{E}$ and $\mathscr{M}$ be classes of $\mathbb{A}$-morphisms that is closed under composition with isomorphisms. The pair $(\mathscr{E}, \mathscr{M})$ is called a factorisation structure for morphisms on $\mathbb{A}$ or $\mathbb{A}$ is said to be **$(\mathscr{E}, \mathscr{M})$-structured** or $(\mathscr{E}, \mathscr{M})$ is an **orthogonal factorisation structure of** $\mathbb{A}$, provided the following conditions hold:

(*Fact*) $\mathbb{A}$ has the **$(\mathscr{E}, \mathscr{M})$-factorisation property**, i.e., every $\mathbb{A}$-morphism $f : X \to Y$ is $(\mathscr{E}, \mathscr{M})$-**factorisable**, i.e., there exists an element $e \in \mathscr{E}$ and $m \in \mathscr{M}$ such that $f = m \circ e$, so that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e"'] & Y \\ M \arrow[ur, "m"'] & \end{tikzcd}$$

commutes.

In such a case, the pair (e, m) is called an **$(\mathscr{E}, \mathscr{M})$-factorisation of** f.

(*Diag*) $\mathscr{E} = \mathscr{M}^{\uparrow}$ and $\mathscr{M} = \mathscr{E}^{\downarrow}$.

Sometimes we will abuse notation and simply say that $m \circ e$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of f. In order to simplify the notation, we will sometimes omit the symbol " $\circ$ " for composition of morphisms and only use it whenever confusion could arise.

Remark 2.8: Every factorisation structure for morphisms is a prefactorisation structure and the converse holds if every morphism has an $(\mathscr{E}, \mathscr{M})$-factorisation. See [6, 2.10] for more information on this.

The *Role of the Diagonalisation Property* is to ensure that factorisations for a morphism are unique up to isomorphism: Indeed, suppose $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured and let $f : X \to Z$ be an $\mathbb{A}$-morphism with (e, m) and (e', m') two $(\mathscr{E}, \mathscr{M})$-factorisations of f . Then, each square in the diagram

$$\begin{tikzcd} Y \arrow[d, "m"'] & X \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, "e'"] & Y \arrow[d, "m"] \arrow[dl, dashed, "!\bar{d}"] \\ Z & Y' \arrow[l, "m'"] \arrow[r, "m'"'] \arrow[ul, dashed, "!d'"] & Z \end{tikzcd}$$

commutes for unique morphisms d' and $\bar{d}$.
Note that $id_{Y'}$ is the unique morphism such that

$$\begin{tikzcd} X \arrow[r, "e'"] \arrow[d, "e'"'] & Y' \arrow[d, "m'"] \\ Y' \arrow[r, "m'"'] \arrow[ur, equal, "id_{Y'}"] & Z \end{tikzcd}$$

commutes. Since $e' = \bar{d}e = \bar{d}d'e'$ and $m' = md' = m'\bar{d}d'$, it follows that $\bar{d}d' = id_{Y'}$. In a similar manner it follows that $d'\bar{d} = id_Y$. Therefore d' and $\bar{d}$ are isomorphisms and inverses of each other. Therefore, e' and m' is a composition of an isomorphism with e and m respectively.

It's important to notice that $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured if and only if $\mathbb{A}^{op}$ is $(\mathscr{M}, \mathscr{E})$-structured.

Sometimes we will assume that $\mathscr{E}$ is a class of epimorphisms and $\mathscr{M}$ is a class of monomorphisms, but this need not be the case.

We say that $\mathbb{A}$ is $(\mathscr{E}, -)$-structured, respectively $(-, \mathscr{M})$-structured, if there exists a class $\mathscr{M}$, respectively a class $\mathscr{E}$, of $\mathbb{A}$-morphisms such that $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured.

Example 2.9:

(a) Every category $\mathbb{A}$ is $(Iso(\mathbb{A}), Mor(\mathbb{A}))$ and $(Mor(\mathbb{A}), Iso(\mathbb{A}))$-structured, where the factorisation of a morphism f is given by $f = f \circ id$ and $f = id \circ f$ respectively. These are called the **trivial factorisation structures** for $\mathbb{A}$.

(b) $\mathbb{Set}$ is an $(Epi, Mono)$-structured category.

(c) Many familiar constructs, for e.g.: $\mathbb{Set}$, $\mathbb{Vec}$, $\mathbb{Grp}$ and $\mathbb{Mon}$ are all $(RegEpi, Mono)$-structured.

(d) $\mathbb{Top}$ and $\mathbb{Rel}$ ([33]) both have an illegitimate conglomerate of factorisation structures for morphisms. $(Epi, RegMono)$, $(RegEpi, Mono)$ and $(dense, closed\ embedding)$ are all factorisation structures on $\mathbb{Top}$, whereas $(Epi, Mono)$ is not.

Proposition 2.10: ([1, 14.7]) Let $\mathscr{E}$ and $\mathscr{M}$ be classes of $\mathbb{A}$-morphisms. Then $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured if and only if the following conditions hold:

(i) $Iso(\mathbb{A}) \subset \mathscr{E} \cap \mathscr{M}$,

(ii) $\mathscr{E}$ and $\mathscr{M}$ are closed under composition,

(iii) $\mathbb{A}$ has the $(\mathscr{E}, \mathscr{M})$-factorisation property and factorisations are unique in the sense that if $m \circ e = f = m' \circ e'$ are two $(\mathscr{E}, \mathscr{M})$-factorisations of f, then there is a unique isomorphism h such that the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e'"'] & B \arrow[d, "m"] \arrow[dl, dashed, "!h"'] \\ C \arrow[r, "m'"'] & D \end{tikzcd}$$

commutes.

Lemma 2.11: ([1, 14.5]) Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured and let $e \in \mathscr{E}$ and $m \in \mathscr{M}$.

If

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "id"'] & \bullet \arrow[d, "m"] \arrow[dl, "d"'] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

commutes, then e is an isomorphism and $f \in \mathscr{M}$.

Proposition 2.12: ([1, 14.10]) Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured. Then the following hold:

(a) If $\mathscr{E} \subset Epi(\mathbb{A})$, then $ExtrMono(\mathbb{A}) \subset \mathscr{M}$,

(b) If $\mathbb{A}$ is also $(Epi(\mathbb{A}), Mono(\mathbb{A}))$-structured, then:

  (i) $Epi(\mathbb{A}) \subset \mathscr{E}$ implies that $\mathscr{M} \subset ExtrMono(\mathbb{A})$ and

  (ii) $Epi(\mathbb{A}) = \mathscr{E}$ implies that $\mathscr{M} = ExtrMono(\mathbb{A})$.

Proposition 2.13: ([1]) Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured and have finite products. Then, the following are equivalent:

(1) $\mathscr{E} \subset Epi(\mathbb{A})$,

(2) $ExtrMono(\mathbb{A}) \subset \mathscr{M}$,

(3) $Sect(\mathbb{A}) \subset \mathscr{M}$,

(4) for each $\mathbb{A}$-object A, **the diagonal morphism** $\Delta_A := \langle id_A, id_A \rangle : A \to A \times A$ is a member of $\mathscr{M}$,

(5) $fg \in \mathscr{M}$ implies $g \in \mathscr{M}$,

(6) $fe \in \mathscr{M}$ and $e \in \mathscr{E}$ implies that e is an isomorphism,

(7) $\mathscr{M} = \{f \in Mor(\mathbb{A}) \mid f = ge \text{ and } e \in \mathscr{E} \Rightarrow e \in Iso(\mathbb{A})\}$.

2.2 Sources and sinks

Definition 2.14: Sources and sinks

A pair of the form $(A, (f_i : A \to A_i)_I)$ is **an ($\mathbb{A}$)-source (at A, or from A to $(A_i)_I$)** if and only if A is an $\mathbb{A}$-object, I an index class, $(A_i)_I$ is a family of $\mathbb{A}$-objects and $(f_i : A \to A_i)_I$ a family of $\mathbb{A}$-morphisms with common domain A.

A pair of the form $((f_i : A_i \to A)_I, A)$ is **an ($\mathbb{A}$)-sink (at A or from $(A_i)_I$ to A)** if and only if A is an $\mathbb{A}$-object, I an index class, $(A_i)_I$ is a family of $\mathbb{A}$-objects and $(f_i : A_i \to A)_I$ a family of $\mathbb{A}$-morphisms with common codomain.

Remark 2.15: Let $(A, (f_i : A \to A_i)_I)$ be a source.

(a) For a source at A, we simply write $(A, f_i)_I$ or $(A \xrightarrow{f_i} A_i)_I$ or $(f_i)_I$. The **domain** of a source at A is A and the **codomain** is the family $(A_i)_I$. We use similar notation for sinks.
The index class I of a source $(A, f_i)_I$ may be a proper class or a set.

(b) If $I = \varnothing$, then the source $(f_i : A \to A_i)_I$ may be identified with A.

(c) We say that the source is **set-indexed** or **small**, respectively **finite** if and only if I is a set, respectively a finite set.

(d) Any morphism $f : A \to B$ can be viewed as a source $(f : A \to B)$. A source with index set I such that I is a singleton, will be referred to as a one-source. We will sometimes abuse notation and denote a one-source as a morphism.

(e) A source $(A, f_i)_I$ at A is an **all-source** at A if and only if for each $\mathbb{A}$-morphism $f : A \to B$, there is a $j \in I$ such that $f = f_j$.
For any $\mathbb{A}$-morphism $f : A \to B$ and $\mathbb{A}$-source $(g : B \to B_i)_I$ at B, the **composite of** f **and** $(g_i)_I$ is the $\mathbb{A}$-source $(g_i \circ f : A \to B_i)$ at A, written $(g_i)_I \circ f$. Dual notions and phrases applies to sinks.

Definition 2.16: Mono-source, Epi-sink
A source $(f_i : A \to A_i)_I$ in $\mathbb{A}$ is a **mono-source** if and only if it's **left cancellative**, i.e., whenever $g, h : B \rightrightarrows A$ are $\mathbb{A}$-morphisms such that $(f_i)_I \circ g = (f_i)_I \circ h$, (this is $f_i \circ g = f_i \circ h$ for all $i \in I$), then $g = h$.

Dually: A sink $(f_i : A_i \to A)_I$ in $\mathbb{A}$ is an **epi-sink** if and only if it's **right cancellative**, i.e., whenever $g, h : A \rightrightarrows B$ are $\mathbb{A}$-morphisms such that $g \circ (f_i)_I = h \circ (f_i)_I$ (this is $g \circ f_i = h \circ f_i$ for all $i \in I$), then $g = h$.

Remark 2.17: The empty source $(A, \varnothing)$ at A in $\mathbb{A}$ is a mono-source if and only if for all $B \in Ob\mathbb{A}$, $|\mathbb{A}(B, A)| \leq 1$. This is the case, for if there is a $B \in Ob\mathbb{A}$ with $g, h : B \rightrightarrows A$ distinct morphisms, then the composites $\varnothing \circ g$ and $\varnothing \circ h$ are both the empty source at B. Therefore, not a mono-source. If it's a mono-source and $B \in Ob\mathbb{A}$, either $\mathbb{A}(B, A)$ is empty, or not. If it's empty, there is nothing to show. If not, then there can be at most one member, otherwise it's not a mono-source.
An $\mathbb{A}$-morphism $f : A \to B$ is a monomorphism if and only if $(f : A \to B)$ is a mono-source.

Example 2.18:

(a) A source $(A, f_i)_I$ is a mono-source in $\mathbb{Set}$ if and only if $(A, f_i)_I$ is **point-separating**, i.e., whenever $a \neq b \in A$, then there is an $i \in I$ such that $f_i(a) \neq f_i(b)$. E.g., for any set-indexed family of sets $(X_i)_I$, the source of projections $(\prod_{i \in I} X_i :\xrightarrow{\pi_i} X_i)_I$ is point-separating.

(b) A sink $(f_i, A)_I$ in $\mathbb{Set}$ is an epi-sink if and only if $(f_i : A_i \to A)_I$ is **covering**, i.e., for each $a \in A$, there is a $j \in I$ and an element $b \in A_j$ such that $f_j(b) = a$, or equivalently $\bigcup_I f_i[A_i] = A$. E.g., for any set-indexed family $(X_i)_I$ of sets, the family of injections $(X_i \hookrightarrow \bigcup_I X_i)_I$ is covering.

(c) In each construct, each point-separating source and each covering sink is a mono-source, respectively an epi-sink. This is the case because faithful functors reflect mono-sources and epi-sinks.

(d) In $\mathbb{Vec}_{\mathbb{R}}$, $\mathbb{Pos}$ and $\mathbb{Top}$, all epi-sinks are covering.

(e) Note that the converse of (c) is not generally true. To see that an epi-sink need not be covering, consider the category $\mathbb{CRng}$ of commutative rings with ring homomorphisms. The inclusion morphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi-morphism(or rather epi-sink), but of course the map is not surjective.

Definition 2.19: Extremal Mono-source
A mono-source $(m_i : X \to X_i)_I$ is said to be **extremal** or an **extremal mono-source**, provided that whenever $(m_i)_I$ factors as $(m_i)_I = (f_i)_i \circ e$ where e is an epimorphism, then e is an isomorphism. We denote the collection of all extremal mono-sources of $\mathbb{A}$ by $ExtrMonoSource(\mathbb{A})$.

Definition 2.20: Orthogonal sources and morphisms
Let e be an $\mathbb{A}$-morphism and $(m_i)_I$ be an $\mathbb{A}$-source with codomain $(X_i)_I$. Let $(g_i)_I$ be any $\mathbb{A}$-source with the same codomain as $(m_i)_I$ and let f be any $\mathbb{A}$-morphism. Then, e is said to be **orthogonal to** $(m_i)_I$, denoted by $e \perp (m_i)_I$, provided that if

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & Y \arrow[d, "g_i"] \\ X \arrow[r, "m_i"'] & X_i \end{tikzcd}$$

commutes for each $i \in I$, there exists a unique morphism $d : Y \to X$ such that $de = f$ and $m_i d = g_i$ for each $i \in I$. The morphism d is then called a **diagonal morphism** or a **diagonal fill-in** for the particular diagram.
If $\mathscr{H}$ is a class of morphisms, we can define $\mathscr{H}^{\downarrow}_{\mathcal{S}}$ to be the conglomerate of all sources $(f_i)_I$ for which $h \perp (f_i)_I$ for each $h \in \mathscr{H}$. Similarly, for a conglomerate $\mathbb{H}$ of sources, we can define $\mathbb{H}^{\uparrow}_{\mathcal{S}}$ to be the class of all morphisms g for which $g \perp (m_i)_I$ for each $(m_i)_I$ in $\mathbb{H}$.

We can then compose the maps $(-)^{\downarrow}_{\mathcal{S}}$ and $(-)^{\uparrow}_{\mathcal{S}}$ between subclasses of morphisms and subconglomerates of $\mathbb{A}$-sources with respect to inclusion to obtain similar results as in Proposition 2.4.

Definition 2.21: Prefactorisation structure for sources
A prefactorisation structure $\mathbb{F}$ for sources is a pair $(\mathscr{E}, \mathbb{M})$ with $\mathscr{E}$ a class of morphisms and $\mathbb{M}$ a conglomerate of sources such that $\mathscr{E} = \mathbb{M}^{\uparrow}_{\mathcal{S}}$ and $\mathbb{M} = \mathscr{E}^{\downarrow}_{\mathcal{S}}$.

Definition 2.22: $(\mathscr{E}, \mathbb{M})$-category
Let $\mathbb{A}$ be a category and $\mathbb{F} = (\mathscr{E}, \mathbb{M})$ a prefactorisation structure for sources. Then, $\mathbb{A}$ is said to be an $(\mathscr{E}, \mathbb{M})$**-category** provided that it has $(\mathscr{E}, \mathbb{M})$**-factorisations for sources**, i.e., for each $\mathbb{A}$-source $(A, (f_i)_I)$, there exists a morphism $e : A \to M$ in $\mathscr{E}$ and a source $(M, (m_i)_I)$ in $\mathbb{M}$ such that for each $i \in I$, it's the case that $m_i e = f_i$.

Remark 2.23: Suppose that $\mathcal{S} := (f_i : A \to A_i)_I$ is a source at A in $\mathbb{A}$ and for each $i \in I$, $\mathcal{S}_i := (f_{j_i} : A_i \to A_{j_i})_{J(i)}$ is a source at A_i. Then, the composition $(\mathcal{S}_i)_I \circ \mathcal{S}$ of $\mathcal{S}$ and $(\mathcal{S}_i)_I$ is the source $(f_{j_i} \circ f_i : A \to A_i \to A_{j_i})_{i \in I, j \in J(i)}$ at A to $(A_{j_i})_{i \in I, j \in J(i)}$.
Note that in case $\mathcal{S}$ is a source with index class consisting of a singleton, say $\{0\}$, then, $\mathcal{S}_0$ is a single source. If $\mathcal{S} = (f : A \to B)$ and $\mathcal{S}_0 = (f_i : B \to B_i)_I$, then we will only write $\mathcal{S}_0 \circ f$ or $(f_i)_I f$ or even $(f_i f)_I$, instead of the more cumbersome $\mathcal{S}_0 \circ (f)_0$.

Example 2.24:

(a) $\mathbb{Set}$ is an (Epi, Mono-source)-category. If $(f_i : A \to A_i)_I$ is a source of maps, define a relation $\sim$ on A by $x \sim y$ if and only if for each $i \in I$: $f_i(x) = f_i(y)$. Then, let $e : A \to A/\sim$ be the projection map to the equivalence classes. Define for each $i \in I$ a map $m_i : A/\sim \longrightarrow A_i$ by $m_i([x]_\sim) = f_i(x)$. It can easily be shown that m_i is a well-defined map and $(m_i)_I$ a mono-source such that $(m_i)_I \circ e = (f_i)_I$. The diagonalisation property is also easily established.

(b) The categories $\mathbb{Grp}$, $\mathbb{Ab}$ and $\mathbb{Vec}_{\mathbb{R}}$ are all (RegEpi, Mono-source)-categories. The factorisation of a source $(f_i : A \to A_i)_I$ is established via the canonical morphism $e : A \to A/K$, where $K = \bigcap_I Ker(f_i)$. The source part $(m_i)_I$ of the factorisation is the unique source such that $m_i \circ e = f_i$ for each $i \in I$.

(c) For $\mathbb{Top}$ various constructions can be made, but if the one on $\mathbb{Set}$ is replicated as in (a), then there are at least two natural choices which occur. One can choose either the initial topology on $A/\sim$ with respect to the source $(m_i)_I$ or the final topology with respect to the projection map e.

Proposition 2.25: If $(\mathscr{E}, \mathbb{M})$ is a prefactorisation structure for sources on a category $\mathbb{A}$, then both $\mathscr{E}$ and $\mathbb{M}$ are closed under composition with isomorphisms, i.e.,

(i) whenever $e : X \to Y$ is in $\mathscr{E}$ and $h : Y \to Z$ and $k : W \to X$ are isomorphisms, then $hek : W \to Z$ is a member of $\mathscr{E}$.

(ii) whenever $(m_i : X \to X_i)_I$ is a source in $\mathbb{M}$ and $f : Y \to X$ is an isomorphism, then $(m_i)_I \circ f$ is a member of $\mathbb{M}$.

Proof : (i) Let I be an index class such that

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "f"'] & X \arrow[r, "e"] & Y \arrow[r, "h"] & Z \arrow[d, "g_i"] \\ A \arrow[rrr, "m_i"'] & & & Y_i \end{tikzcd}$$

commutes for each $i \in I$, where $(m_i)_I$ is a source in $\mathbb{M}$ and $e \in \mathscr{E}$ and h and k are isomorphisms. Then, $(g_i)_I \circ h \circ e = (m_i)_I \circ f \circ k^{-1}$ and since e is in $\mathscr{E}$ and $(m_i)_I$ in $\mathbb{M}$, there is a unique morphism $s : Y \to A$ such that $se = fk^{-1}$ and $m_i \circ s = g_i h$ for each $i \in I$. We assert that $d := sh^{-1}$ is the unique morphism such that $dhek = f$ and $m_i d = g_i$ for each $i \in I$. To see this, note that $dhek = sh^{-1}hek = sek = fk^{-1}k = f$ and for each i, $m_i d = m_i sh^{-1} = g_i hh^{-1} = g_i$. Suppose d' is any morphism such that it's a diagonal morphism for the diagram above. Then, $t := d'h$ is a morphism such that for each $i \in I$: $m_i t = m_i d'h = g_i h$ and $te = d'he = d'hekk^{-1} = fk^{-1}$. By uniqueness of s, this implies that $t = d'h = s$ and thus $d' = sh^{-1} = d$. It follows that hek is a member of $\mathscr{E}$.

(ii) Let $(m_i : X \to X_i)_I$ be an $\mathbb{M}$-source and $f : Y \to X$ be an $\mathbb{A}$-iso. Let $e \in \mathscr{E}$ and suppose that

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[d, "z"'] & & B \arrow[d, "g_i"] \\ Y \arrow[r, "f"'] & X \arrow[r, "m_i"'] & X_i \end{tikzcd}$$

commutes for each $i \in I$. Since $\mathcal{S}$ and e are respective members of $\mathbb{M}$ and $\mathscr{E}$, it follows that there is a unique morphism $p : B \to X$ such that $pe = fz$ and $m_i p = g_i$ for each $i \in I$. We assert that $d := f^{-1}p$ is the unique morphism such that $m_i f d = g_i$ for each $i \in I$ and $de = z$. It's easy to see that, for any i, $m_i fd = m_i ff^{-1}p = m_i p = g_i$ and $de = f^{-1}pe = f^{-1}fz = z$. If d' was another morphism with this property, then $n := fd'$ is a morphism such that $m_i n = g_i$ and $ne = fz$. This implies that $n = fd' = p$ and consequently, $d' = f^{-1}p = d$. It's then clear that $(m_i)_I \circ f$ is a member of $\mathbb{M}$. $\square$

Proposition 2.26: ([1, 15.4]) Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category, then $\mathscr{E}$ is a class of epimorphisms.

Proposition 2.27: ([1, 15.5]) If $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then the following hold:

(i) $(\mathscr{E}, \mathbb{M})$-factorisations are essentially unique,

(ii) $\mathscr{E} \subset Epi(\mathbb{A})$ and $ExtrMonoSource(\mathbb{A}) \subset \mathbb{M}$,

(iii) $\mathscr{E} \cap \mathbb{M} = Iso(\mathbb{A})$,

(iv) each of $\mathscr{E}$ and $\mathbb{M}$ is closed under composition,

(v) if $f \circ g \in \mathscr{E}$ and $g \in Epi(\mathbb{A})$, then $f \in \mathscr{E}$,

(vi) if $f \circ g \in \mathscr{E}$ and f is a section in $\mathbb{A}$, then $g \in \mathscr{E}$,

(vii) if $(\mathcal{S}_i)_I \circ \mathcal{S} \in \mathbb{M}$, then $\mathcal{S} \in \mathbb{M}$,

$(viii)$ if a subsource of $\mathcal{S}$ belongs to $\mathbb{M}$, then so does $\mathcal{S}$.

Definition 2.28: Partial order on the conglomerate of all prefactorisation structures
Let $\mathbb{A}$ be a category and let $\mathrm{PreFact}(\mathbb{A})$ and $\mathrm{PreFact}_{\mathcal{S}}(\mathbb{A})$ denote the conglomerate of all prefactorisation structures for morphisms and sources respectively. Then we can define the following relations:

(*i*) $(\mathscr{E},\mathscr{M}) \leq (\mathscr{E}',\mathscr{M}')$ in $\mathrm{PreFact}(\mathbb{A})$ if and only if $\mathscr{E} \subset \mathscr{E}'$ and $\mathscr{M}' \subset \mathscr{M}$ and

(*ii*) $(\mathscr{E},\mathbb{M}) \leq (\mathscr{E}',\mathbb{M}')$ in $\mathrm{PreFact}_{\mathcal{S}}(\mathbb{A})$ if and only if $\mathscr{E} \subset \mathscr{E}'$ and $\mathbb{M}' \subset \mathbb{M}$.

It's clear that these orderings are partial orders. The subconglomerates of all factorisation structures for sources and morphisms will be denoted by $Fact_{\mathcal{S}}(\mathbb{A})$ and $Fact(\mathbb{A})$ respectively.

Proposition 2.29: Let $(\mathscr{E},\mathscr{M})$ and $(\mathscr{E}',\mathscr{M}')$ be prefactorisation structures on $\mathbb{A}$, with $\mathscr{E} \subset \mathscr{E}'$. Then, $(\mathscr{E},\mathscr{M}) \leq (\mathscr{E}',\mathscr{M}')$. Consequenlty, $\mathscr{M} \subset \mathscr{M}'$ if and only if $\mathscr{E}' \subset \mathscr{E}$.

$\underline{\mathcal{P}roof}$: Assume the hypothesis. It's sufficient to prove that $\mathscr{M}' \subset \mathscr{M}$. This is the case since $\mathscr{M}' = \mathscr{E}'^{\downarrow}$ and $\mathscr{M} = \mathscr{E}^{\downarrow}$ and $(-)^{\downarrow}$ is an order reversing map. If $\mathscr{E} \subset \mathscr{E}'$, it follows that $\mathscr{M}' = \mathscr{E}'^{\downarrow} \subset \mathscr{E}^{\downarrow} = \mathscr{M}$. The other direction follows by duality. □

Proposition 2.30: Let $(\mathscr{E},\mathbb{M})$ and $(\mathscr{E}',\mathbb{M}')$ be prefactorisation structures for sources on a category $\mathbb{A}$ with $\mathscr{E} \subset \mathscr{E}'$. Then, $(\mathscr{E},\mathbb{M}) \leq (\mathscr{E}',\mathbb{M}')$. Consequently, $\mathscr{E} \subset \mathscr{E}'$ if and only if $\mathbb{M}' \subset \mathbb{M}$.

$\underline{\mathcal{P}roof}$: The proof is simliar to the one for Proposition 2.29. □

Theorem 2.31: ([1, 15.14]) Let $\mathbb{A}$ be a category and $\mathscr{E}$ a class of morphisms in $\mathbb{A}$. Then, $\mathbb{A}$ is an $(\mathscr{E},\mathbb{M})$-category for some $\mathbb{M}$ if and only if all the following conditions are satisfied:

(*i*) $\mathrm{Iso}(\mathbb{A}) \subset \mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$,

(*ii*) $\mathscr{E}$ is closed under composition,

(*iii*) for each $e : A \to B$ in $\mathscr{E}$ and $f : A \to C$, there exists a pushout square
$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "\bar{f}"] \\ C \arrow[r, "\bar{e}"'] & D \end{tikzcd}$$
for which $\bar{e}$ is a member of $\mathscr{E}$,

(*iv*) for every source $(A \xrightarrow{e_i} A_i)_I$ with $e_i \in \mathscr{E}$ for each $i \in I$, there exists a multiple pushout $A \xrightarrow{e} B = A \xrightarrow{e_i} A_i \xrightarrow{c_i} B$ for which $e \in \mathscr{E}$.

Lemma 2.32: ([1, 15.7]) Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category. Then the following are equivalent:

(1) $\mathbb{M}$ consists only of mono-sources,

(2) $\mathbb{A}$ has coequalizers and $\mathrm{RegEpi}(\mathbb{A}) \subset \mathscr{E}$.

Theorem 2.33: Let $\mathbb{A}$ be a category and $((\mathscr{E}_i,\mathbb{M}_i))_I$ be a non-empty family of factorisation structures for sources. Then, $\mathbb{A}$ is an $(\bigcap_I \mathscr{E}_i,\mathbb{M})$-category for some $\mathbb{M}$.
Explicitly, $\mathbb{M}$ is given by either

$$\left\{(m_i)_I \in Source(\mathbb{A})| \text{ whenever } (m_i)_I = (f_i)_I \circ e \text{ with } (f_i)_I \in Source(\mathbb{A}) \text{ and } e \in \bigcap_I \mathscr{E}_i, \text{ then } e \in Iso(\mathbb{A})\right\}$$

or

$$\left\{(m_i)_I \in Source(\mathbb{A}) \mid (m_i)_I \in \left(\bigcap_I \mathscr{E}_i\right)^{\downarrow}_{\mathcal{S}}\right\}$$

$\underline{\mathcal{P}roof}$: It's sufficient to prove that $\mathscr{E} := \bigcap_I \mathscr{E}_i$ satisfies the conditions in Theorem 2.31. Since each $\mathscr{E}_i$ satisfies all four conditions, the fact that $\mathscr{E}$ satisfies (*i*) and (*ii*) is clear. If $e : X \to Y$ is a member of $\mathscr{E}$ and $f : X \to Z$ is a morphism with $\bar{e}$ the pushout of e along f, then as for each $i \in I$, it follows that $\bar{e}$ is a member of $\mathscr{E}_i$. Thus, $\bar{e}$ is also a member of $\mathscr{E}$. Let $(e_j : X \to X_j)_J$ be a source with e_j in $\mathscr{E}$ for each $j \in J$. Suppose that $\tilde{e}$ is a multiple pushout of $(e_j)_J$ with $c_j e_j = \tilde{e}$ for each j. Since colimits are unique up to isomorphism, it follows that $\tilde{e}$ is a member of $\mathscr{E}_i$ for each $i \in I$. Consequently, $\tilde{e}$ is also a member of $\mathscr{E}$. Hence, $\mathbb{A}$ is an $(\mathscr{E},\mathbb{M})$-category for some $\mathbb{M}$.
Let us now show that $\mathbb{M}$ is given by $\mathbb{S} = \{(m_i)_I \in Source(\mathbb{A})|$ whenever $(m_i)_I = (f_i)_I \circ e$ with $(f_i)_I \in Source(\mathbb{A})$ and $e \in \bigcap_I \mathscr{E}_i$, then $e \in Iso(\mathbb{A})\}$. If $(m_i)_I$ is a member of $\mathbb{M}$ and $(m_i)_I = (f_i)_I \circ e$, then the

diagonalisation property establishes a morphism d such that $de = id$ and $(m_i)_I d = (f_i)_I$. Then e is an epimorphic section or equivalently e is an isomorphism. Since inverses are unique, $ed = id$ and both d and e are isomorphisms. For the reverse inclusion, let $(n_j)_J$ be a member of $\mathbb{S}$. Then, the $\mathscr{E}$-part of any $(\mathscr{E}, \mathbb{M})$-factorisation of $(n_j)_J$ must be an isomorphism and hence, since $\mathbb{M}$ is closed under composition with isomorphisms, $(n_j)_J$ is a member of $\mathbb{M}$. Thus, $\mathbb{M} = \mathbb{S}$.

By definition of factorisation structures for sources, it should be clear that $\mathbb{M} = \mathscr{E}^{\downarrow}_{\mathcal{S}}$, where $\mathscr{E} = \bigcap_I \mathscr{E}_i$. $\square$

Corollary 2.34: $(\mathrm{Fact}_{\mathcal{S}}(\mathbb{A}), \leq)$ is a complete conglomerate with least element (Iso($\mathbb{A}$),Source($\mathbb{A}$). If $\mathbb{A}$ is an (Epi, $\mathbb{M}'$)-category for some $\mathbb{M}'$, then the greatest element is given by (Epi, $\mathbb{M}$), where

$$\mathbb{M} = \{(f_i : X \to X_i)_I \mid \text{whenever } (f_i)_I = (m_i)_I e \ (e \in \mathrm{Epi}(\mathbb{A}) \Rightarrow e \in \mathrm{Iso}(\mathbb{A}))\}.$$

Corollary 2.35: If $\mathbb{A}$ is an (Epi, $\mathbb{M}'$)-category for some $\mathbb{M}'$ and $\mathbb{A}$ has coequalizers, then the greatest element of $(\mathrm{Fact}_{\mathcal{S}}(\mathbb{A}), \leq)$ is given by (Epi, Extremal Mono-source).

Proof : This is a direct consequence of Lemma 2.32 and Corollary 2.34. $\square$

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

∼ David Hilbert (1862-1943)

3 Left and right constant subcategories

3.1 Constant morphisms

Throughout this section, let $\mathbb{A}$ denote a category and $(\mathscr{E}, \mathscr{M})$ an orthogonal factorisation structure for morphisms with $\mathscr{E} \subset Epi(\mathbb{A})$.

Remark 3.1: Throughout the literature there have been various notions of what a constant morphism could or even should be. Constant morphisms have been studied together with generalised notions of connectedness and disconnectedness in categories of a topological nature. A comprehensive example of this can be seen in [41]. In categories of an algebraic nature, constant morphisms have been studied together with torsion and torsion free subcategories. See [21], [51] and [52] for more information. These topics will be discussed in more detail in section 2.2 on left and right constant subcategories.

The development of a general theory of connectedness was done by various authors and each had different definitions of constant morphisms. Herrlich ([31]) defined a morphism $f : X \to Y$ to be constant whenever for each object Z, there is at most one morphism $k : Z \to Y$ through which f factors. Preuß([41]) considered topological constructs with constant continuous maps as the constant morphisms. Castellini defined constant morphisms via a fixed class of subobjects ([16, 14.1]). This was also compared to morphisms which factor through a terminal object ([16, 15.4]).

For an $(\mathscr{E}, \mathscr{M})$-structured category, various other notions have been considered. A popular one is to consider certain $\mathscr{M}$-subobjects of an $\mathbb{A}$-object X and consider these as trivial. Another is that the morphism should factor through a terminal object. A similar one is where a constant morphism $f : X \to Y$ with $(\mathscr{E}, \mathscr{M})$-factorisation $m \circ e$ through a (pre)terminal object. Another notion for factorisation structures is that it should factor through a terminal object 1 and the morphism $X \to 1$ should be a member of $\mathscr{E}$ ([26]). Our motivation of defining classes of constant morphisms is adopted from general properties most of the above notions and also categories of structured sets satisfy:

(i) If f is constant, so is gfh for any morphisms g and h for which gfh is defined,
(ii) If m is a monomorphism and mf is constant, then so is f,
(iii) If e is an epimorphism and fe is constant, then so is f.

This motivates us to consider the following definitions:

Definition 3.2: (Weakly) constant morphisms, Right-$\mathscr{E}$ and left-$\mathscr{M}$-cancellative
Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured and $\mathscr{N}$ a non-empty class of $\mathbb{A}$-morphisms. Then, $\mathscr{N}$ is said to be a class of **weakly constant morphisms (in $\mathbb{A}$)** if $\mathscr{N}$ is closed under composition with $\mathbb{A}$ morphisms. Hence, $\mathscr{N}$ satisfies the following condition:

For each $f : B \to C \in \mathscr{N}$ and $\mathbb{A}$-morphisms $g : C \to D$ and $h : A \to B$ we have that gfh is a member of $\mathscr{N}$.

A class $\mathscr{N}$ of morphisms is said to be **right-$\mathscr{E}$-cancellative** provided that $fe \in \mathscr{N}$ with $e \in \mathscr{E}$ implies that $f \in \mathscr{N}$.

Similarly a class $\mathscr{N}$ of morphisms is said to be **left-$\mathscr{M}$-cancellative** if $\mathscr{N}$ is right-$\mathscr{M}$-cancellative in $\mathbb{A}^{op}$. This is, if $mf \in \mathscr{N}$ and $m \in \mathscr{M}$, then f is a member of $\mathscr{N}$.
A class $\mathscr{N}$ of morphisms is said to be a class of **left-$\mathscr{M}$-constant morphisms** if $\mathscr{N}$ is a class of weakly constant morphisms that is left-$\mathscr{M}$-cancellative. A class of **right-$\mathscr{E}$-constant morphisms** is defined dually. A class of morphisms that is both right-$\mathscr{E}$-and-left-$\mathscr{M}$-constant is called a **class of constant morphisms**. A member of a class $\mathscr{N}$ of constant morphisms is said to be **$\mathscr{N}$-constant** in $\mathbb{A}$.

Definition 3.3: $\mathscr{M}$-subobjects and $\mathscr{E}$-images
Let $\mathbb{A}$ be a category and $\mathscr{E}$ and $\mathscr{M}$ be classes of morphisms. Then, a subcategory $\mathbb{B}$ of $\mathbb{A}$ is said to be **closed under**:

(i) **$\mathscr{M}$-subobjects** provided that A is a member of $\mathbb{B}$, whenever $m : A \to B \in \mathscr{M}$ and B is a member of $\mathbb{B}$;

(ii) **$\mathscr{E}$-images** provided that A is a member of $\mathbb{B}$, whenever $e : B \to A \ \in \mathscr{E}$ and B is a member of $\mathbb{B}$.

The notion of being closed under $\mathscr{E}$-quotients is synonymous to being closed under $\mathscr{E}$-images. If $\mathscr{E}$ is the class of epimorphisms and $\mathscr{M}$ is the class of monomorphisms, then we simply call $\mathscr{M}$-subobjects and $\mathscr{E}$-images **subobjects** and **images** or **quotients** respectively.

Proposition 3.4: Let $\mathscr{N}$ be a class of left-$\mathscr{M}$-constant morphisms with $\mathscr{M} \cap \mathscr{N} \neq \varnothing$. Then, the full subcategory $\mathbb{C}_{\mathscr{M}}$ of $\mathbb{A}$ that consists of the class

$$\{C \in \mathbb{A} \mid \exists\ m \in \mathscr{N} \cap \mathscr{M} \text{ with } C = dom(m)\}$$

is a non-empty subcategory closed under $\mathscr{M}$-subobjects.
Proof : This is a straightforward exercise. □

Proposition 3.5: Let $\mathscr{N}$ be a class of right-$\mathscr{E}$-constant morphisms with $\mathscr{N} \cap \mathscr{E} \neq \varnothing$. Then, the full subcategory $\mathbb{C}_{\mathscr{E}}$ of $\mathbb{A}$ that consists of the class

$$\{C \in \mathbb{A} \mid \exists\ e \in \mathscr{N} \cap \mathscr{E} \text{ with } C = cod(e)\}$$

is a non-empty subcategory closed under $\mathscr{E}$-images. $\boxed{D}$

Example 3.6: Consider the class $\mathscr{N}$ of all maps in $\mathbb{Set}$ which factors through a singleton object. It's clear that $\mathscr{N}$ is a class of morphisms closed under composition with all maps. Notice that $\mathscr{N}$ is both left-$\mathscr{M}$- and right-$\mathscr{E}$-cancellative for the factorisation structure $(\mathscr{E}, \mathscr{M})$ =(surjective, injective).
If we let $\mathscr{N}'$ be the class of all maps with empty domain, then $\mathscr{N}'$ is also closed under composition with all maps. $\mathscr{N}'$ is also left-$\mathscr{M}$-cancellative, for if mf has empty domain, then so does f, and it's also right cancellative if $\mathscr{E}$ is the class of surjective maps.

However, if we consider the factorisation structure (map, bijective) on $\mathbb{Set}$, then $\mathscr{N}'$ is still left-bijective-cancellative, but not right-map-cancellative, for $\varnothing \to 1 \to 1$ is a map with empty domain, but $1 \to 1$ is not.
A trivial example for any category $\mathbb{A}$ is to let $\mathscr{N} = Mor(\mathbb{A})$.

Proposition 3.7: Let $\mathscr{N}$ be a class of constant morphisms. Then, the full subcategory $\mathbb{C}$ consisting of $\{C \in \mathbb{A} \mid id_C \in \mathscr{N}\}$ is non-empty and is closed under $\mathscr{M}$-subobjects and $\mathscr{E}$-images.

Proof : Since $\mathscr{N}$ is non-empty, there is a member f of $\mathscr{N}$. To this end, let $f \in \mathscr{N}$ with $(\mathscr{E}, \mathscr{M})$-factorisation me. Then, since $me = m \circ id \circ e$ is in $\mathscr{N}$, the cancellative properties of $\mathscr{N}$ with respect to $\mathscr{M}$ and $\mathscr{E}$ gives us that $id_{dom(m)}$ is in $\mathbb{C}$. Thus $\mathbb{C}$ is non-empty. Suppose $m : M \to C$ is a member of $\mathscr{M}$ with $C \in \mathbb{C}$. Then, $id_C \circ m = m = m \circ id_M$ is $\mathscr{N}$-constant and once again by the left cancellative property of $\mathscr{N}$, id_M is a member of $\mathscr{N}$. Therefore, M is in $\mathbb{C}$. Thus $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects. The fact that $\mathbb{C}$ is closed under $\mathscr{E}$-images follows by dualisation. □

Proposition 3.8: Let $\mathscr{N}$ be class of constant morphisms. Then, $\mathbb{C} = \mathbb{C}_{\mathscr{M}} \cap \mathbb{C}_{\mathscr{E}}$ where $\mathbb{C}_{\mathscr{M}}$,$\mathbb{C}_{\mathscr{E}}$ and $\mathbb{C}$ are the categories described in Propositions 3.4, 3.5 and 3.7, respectively.

Remark 3.9: Proposition 3.8 relates to another well-known notion of constant morphism, namely that of a constant subcategory introduced in [19].

Definition 3.10: Constant subcategory ([19])
A non-empty subcategory $\mathbb{C}$ of $\mathbb{A}$ is said to be **constant** or a **constant subcategory (of $\mathbb{A}$)** provided that $\mathbb{C}$ is **closed under $\mathscr{M}$-subobjects** and **under $\mathscr{E}$-images**.
In case $\mathbb{C}$ is a constant subcategory of $\mathbb{A}$, the objects of $\mathbb{C}$ are called **constant objects.**

Unless stated otherwise, we will assume the following troughout this section:
 (i) $\mathbb{C}$ will always denote a fixed constant subcategory of $\mathbb{A}$, and
 (ii) $\mathscr{N}$ will be a class of weakly constant morphisms in $\mathbb{A}$.

Definition 3.11: $\mathbb{C}$-Constant morphisms
Let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$. Then, an $\mathbb{A}$-morphism $f : A \to B$ is a **($\mathbb{C}$-)constant morphism** provided that for an $(\mathscr{E}, \mathscr{M})$-factorisation $A \xrightarrow{f} B = A \xrightarrow{e} M \xrightarrow{m} B$, we have $M \in \mathbb{C}$.

Example 3.12: Let $\mathbb{C} = \{C \in \mathbb{Set} \mid |C| \leq 1\}$ be the subcategory of $\mathbb{Set}$ of all sets consisting of at most one point. Consider the factorisation structure (surjective, injective) on $\mathbb{Set}$, then the $\mathbb{C}$-constant morphisms in $\mathbb{Set}$ are exactly the constant maps.[1] To see this, let $f : A \to B$ be a constant map. If $A = \varnothing$, then any epi with domain A must also have codomain $\varnothing$, thus f is a $\mathbb{C}$-constant morphism. If A is not empty, then $f(a) = f(a')$ for all members a and a' in A. Therefore, if $f = me$, then since e is surjective and m injective, we must have that $dom(m) = cod(e) = range(e)$ consists of a single element. Thus every constant map is a $\mathbb{C}-$constant morphism.
Conversely, assume that $f : A \to B$ is a constant morphism and factors as me with $M = dom(m)$. If $A \neq \varnothing$, we have that $M \neq \varnothing$, so that $|M| = 1$. Consequently, $f(a) = m(e(a)) = m(e(a')) = f(a')$, thus f is a constant map. If $A = \varnothing$, then f is obviously constant by definition.

In the category $\mathbb{Top}$, let $\mathbb{C}$ be the class consisting of all spaces with at most one point. If $\mathscr{E}$ and $\mathscr{M}$ consists of the classes of all surjections and embeddings respectively, then the constant morphisms and the constant continuous maps coincide.

Proposition 3.13: For each constant subcategory $\mathbb{C}$, the class of $\mathbb{C}$-constant morphisms is a class of weakly constant morphisms.

Proof : Assume that $f : B \to C$ is constant and let $h : A \to B$ and $g : C \to D$ be morphisms. It's sufficient to prove that gf and fh are constant.
Consider the $(\mathscr{E}, \mathscr{M})$-factorisations $fh = me$ and $f = m'e'$ and $gf = \bar{m}\bar{e}$. Then, the diagram

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "h"'] & E \arrow[dd, "m"] \arrow[ddl, dashed, "d"] \\
B \arrow[d, "e'"'] & \\
F \arrow[r, "m'"'] & C
\end{tikzcd}$$

commutes for a unique d. Since $m'd = m \in \mathscr{M}$, by 2.13(5), it follows that $d \in \mathscr{M}$. Since f is constant, $F \in \mathbb{C}$ and since $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects, it follows that E is in $\mathbb{C}$ and consequently fh is constant.

The diagonalization property also establishes the commuting diagram:

$$\begin{tikzcd}
B \arrow[r, "e'"] \arrow[dd, "\bar{e}"'] & F \arrow[d, "m'"] \arrow[ddl, dashed, "d'"] \\
 & C \arrow[d, "g"] \\
M \arrow[r, "\bar{m}"'] & D
\end{tikzcd}$$

In particular, since $d'e' = \bar{e} \in \mathscr{E}$, by the dual of 2.13(5), it follows that $d' \in \mathscr{E}$. Hence by the assumptions on $\mathbb{C}$, $M \in \mathbb{C}$ so that gf is constant.

$\square$

[1]Maps with empty domain are considered constant or equivalently, a map $f : X \to Y$ is constant if and only if for each pair of points x, x' in X, we have $f(x) = f(x')$. Note that some authors don't regard maps with empty domain as constant.

Proposition 3.14: ([19]) Let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$. Then the following hold:

(i) If mf is constant and $m \in \mathscr{M}$, then f is constant.

(ii) If fe is constant and $e \in \mathscr{E}$, then f is constant.

$\underline{\mathcal{P}roof}$: Let $f : B \to C$ be an $\mathbb{A}$-morphism, $m : C \to D \in \mathscr{M}$ and $e : A \to B \in \mathscr{E}$. Let $f = m'e'$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of f with $dom(m') = C'$.
(i) Then, $mf = (mm')e'$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of mf and since mf is constant by (i), we have $C' = dom(m') = dom(mm') \in \mathbb{C}$. Thus f is constant.
(ii) It's clear that $fe = m'(e'e)$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of fe and thus $C' = dom(m') = cod(e'e)$ is a member of $\mathbb{C}$. Thus f is constant.

□

Proposition 3.15: Every constant subcategory $\mathbb{C}$ gives rise to a non-empty class of constant morphisms given by the $\mathbb{C}$-constant morphisms, namely

$$\mathscr{N}(\mathbb{C}) := \{f \in Mor(\mathbb{A}) \mid f \text{ is } \mathbb{C}\text{-constant}\}.$$

Furthermore, every class of constant morphisms $\mathscr{N}$ gives rise to a constant subcategory $\mathbb{C}(\mathscr{N})$ given by

$$\mathbb{C}(\mathscr{N}) := \{X \in \mathbb{A} \mid id_X \in \mathscr{N}\}.$$

These assignments are inverses and a morphism is $\mathbb{C}$-constant if and only if it's $\mathscr{N}(\mathbb{C})$-constant or equivalently a morphism is $\mathscr{N}$-constant if and only if it's $\mathbb{C}(\mathscr{N})$-constant.

$\underline{\mathcal{P}roof}$: By Propositions 3.7, 3.13 and 3.14, it's clear that both these assignments are well defined. To show that they are inverses, it's sufficient to prove that $\mathbb{C}(\mathscr{N}(\mathbb{C})) = \mathbb{C}$ and $\mathscr{N}(\mathbb{C}(\mathscr{N})) = \mathscr{N}$.
Note that $C \in \mathbb{C}$ if and only if id_C is $\mathbb{C}$-constant if and only if $id_C \in \mathscr{N}(\mathbb{C})$ if and only if C is a member of $\mathbb{C}(\mathscr{N}(\mathbb{C}))$.
Let f be a morphism with $(\mathscr{E}, \mathscr{M})$-factorisation $m \circ e : X \to M \to Y$. Then, f is in $\mathscr{N}$ if and only if id_M is in $\mathscr{N}$ if and only if M is in $\mathbb{C}(\mathscr{N})$ if and only if m is $\mathbb{C}(\mathscr{N})$-constant if and only if me is $\mathbb{C}(\mathscr{N})$-constant if and only if f is a member of $\mathscr{N}(\mathbb{C}(\mathscr{N}))$. Hence these classes of constant morphisms coincide.

□

Remark 3.16: By Proposition 3.15 when considering classes of constant morphisms, we might as well consider constant subcategories. Since subcategories have been studied more extensively than classes of morphisms, it's only natural to refer to the constant subcategories more often.

Proposition 3.17: Let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$. Then, every morphism with domain or codomain in $\mathbb{C}$ is constant.

$\underline{\mathcal{P}roof}$: Let $f : X \to Y$ be a morphism with $X \xrightarrow{e} M \xrightarrow{m} Y$ an $(\mathscr{E}, \mathscr{M})$-factorisation. Assume that the domain or codomain of f is a member of $\mathbb{C}$. Then, since $\mathbb{C}$ is closed under $\mathscr{E}$-images and $\mathscr{M}$-subobjects, it follows that M is a member of $\mathbb{C}$. □

3.2 Left and right constant subcategories via constant morphisms

Throughout this section, $\mathbb{A}$ is a category with an orthogonal factorisation structure $(\mathscr{E}, \mathscr{M})$ for morphisms, or a factorisation structure $(\mathscr{E}, \mathbb{M})$ for sources. Of course, if $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured, where $\mathscr{M}$ denotes the class of all morphisms $m : M \to X$ such that $(m : M \to X)$ is a one-source member in $\mathbb{M}$. Unless stated otherwise, $\mathbb{C}$ and $\mathscr{N}$ will denote a constant subcategory or class of constant morphisms in $\mathbb{A}$ with respect to the above factorisation structure.

Left and right constant subcategories have been studied in many different contexts. G. Preuß studied left constant subcategories as a generalisation of connected subspaces of topological constructs. The right constant subcategories were identified with spaces which are disconnected or satisfy specific separation axioms. See [41] for a more detailed discussion. Another approach, see [2], was to consider certain pairs of subcategories related by the constant morphisms. These pairs would be considered as one connected class of objects and one disconnected (or separated) class of objects. See [22], [17], [8], [9], [10], [12] and [13] for more examples.

In abelian or normal categories, left and right constant subcategories are usually associated with torsion and torsion free subcategories. See [21], [51] and [23] for an exposition.

The general idea for these pairs of left and right constant subcategories is that they are fixed points of certain order preserving functions which form a Galois correspondence or adjunction. Our approach is not motivated from topology or algebra, but rather to capture a flavour of both. It's important to note that even if our definition of constant morphisms is equivalent to the definition in [19], that the assumptions on categories are generally different. Not only that, but the aim here is not to characterise right constant subcategories via fans and multifans and pullbacks. The aim is to establish a *Herrlich, Preuß Arhangel'skii, Wiegandt(HPAW)-correspondence* (see [26]) between left and right constant subcategories and relate these to dual closure operators. This idea has originated from studying closure operators and how they relate to connectednesses and disconnectednesses with some Galois connections between the three. Dikranjan and Tholen ([26]) have also done this for dual closure operators with their slightly more restrictive definition of constant morphisms.

Definition 3.18: Left and right constant subcategories
Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured, $\mathscr{N}$ a weakly constant class of $\mathbb{A}$-morphisms and $\mathbb{P}$ and $\mathbb{Q}$ any subcategories of $\mathbb{A}$. Then:

(1) (*i*) An $\mathbb{A}$-object X is called **left-$\mathbb{P}$-constant (in $\mathbb{A}$ with respect to $\mathscr{N}$)** if and only if for each $P \in \mathbb{P}$ and each $\mathbb{A}$-morphism $f : X \to P$, it follows that f is $\mathscr{N}$-constant.

(*ii*) $\mathscr{L}(\mathbb{P})$ denotes the full subcategory of all left-$\mathbb{P}$-constant objects, i.e.,
$\mathscr{L}(\mathbb{P}) = \{X \in \mathbb{A} \mid X \text{ is left}-\mathbb{P}-\text{constant}\}$.

(*iii*) A subcategory $\mathbb{B}$ of $\mathbb{A}$ is called a **left constant (with respect to $\mathscr{N}$)** if and only if $\mathbb{B} = \mathscr{L}(\mathbb{P})$ for some subcategory $\mathbb{P}$ of $\mathbb{A}$.

(*iv*) If $\mathbb{B}$ is left constant with respect to $\mathscr{N}$ and $\mathscr{N}$ is a class of constant morphisms with $\mathbb{C} = \{C \in \mathbb{A} \mid id_C \in \mathscr{N}\}$, then we also say that $\mathbb{B}$ is left constant with respect to $\mathbb{C}$.

(2) (*i*) An $\mathbb{A}$-object D is called **right-$\mathbb{Q}$-constant (in $\mathbb{A}$ with respect to $\mathscr{N}$)** if and only if for each X in $\mathbb{Q}$ and each $\mathbb{A}$-morphism $f : X \to D$, it follows that f is constant.

(*ii*) $\mathscr{R}(\mathbb{Q})$ denotes the full subcategory of $\mathbb{A}$ of all right-$\mathbb{Q}$-constant objects, i.e.,
$\mathscr{R}(\mathbb{Q}) = \{D \in \mathbb{A} \mid D \text{ is right}-\mathbb{Q}-\text{constant}\}$.

(*iii*) A subcategory $\mathbb{D}$ of $\mathbb{A}$ is called **right constant (with respect to $\mathscr{N}$)** if and only if $\mathbb{D} = \mathscr{R}(\mathbb{Q})$ for some subcategory $\mathbb{Q}$ of $\mathbb{A}$.

(*iv*) If $\mathbb{D}$ is a right constant subcategory with respect to $\mathscr{N}$ and $\mathscr{N}$ is a class of constant morphisms with $\mathbb{C} = \{C \in \mathbb{A} \mid id_C \in \mathscr{N}\}$, then we also say that $\mathbb{B}$ is right constant with respect to $\mathbb{C}$.

Remark 3.19: If it seems necessary to make explicit mention of the notion of constantness used, whether it be a class $\mathscr{N}$ of weakly constant morphisms or a contant subcategory $\mathbb{C}$, the class of left $\mathbb{P}$-constant objects can be denoted by either $\mathscr{L}_{\mathscr{N}}(\mathbb{P})$ or $\mathscr{L}_{\mathbb{C}}(\mathbb{P})$ respectively. Similarly for right-$\mathbb{Q}$-constant objects. Since $\mathscr{L}, \mathscr{R} : Sub(\mathbb{A}) \rightrightarrows Sub(\mathbb{A})$ are endomaps, we can compose $\mathscr{L}$ and $\mathscr{R}$ in any order. We will denote the compositions of these two maps by $\mathscr{P} := \mathscr{L}\mathscr{R}$ and $\mathscr{Q} := \mathscr{R}\mathscr{L}$.

Whenever the class of constant morphisms or constant subcategory in question is clear, we will often omit the terms $\mathscr{N}$-constant or $\mathbb{C}$-constant and simply refer to 'constant'. In case $\mathscr{N}$ is a class of (weakly) constant morphisms, then note that $\mathbb{B}$ is a left constant subcategory of $\mathbb{A}$ if and only if $\mathbb{B}^{op}$ is a right constant subcategory of $\mathbb{A}^{op}$.

Proposition 3.20: Let $\mathbb{A}$ be a category and $\mathbb{P}$ and $\mathbb{Q}$ denote subcategories of $\mathbb{A}$ with $\mathscr{N}$ a class of weakly constant morphisms. Then, the following hold:

(i) $\mathbb{P} \subset \mathbb{Q}$ implies that $\mathscr{L}(\mathbb{Q}) \subset \mathscr{L}(\mathbb{P})$

(ii) $\mathbb{P} \subset \mathbb{Q}$ implies that $\mathscr{R}(\mathbb{Q}) \subset \mathscr{R}(\mathbb{P})$

(iii) $\mathbb{P} \subset \mathscr{R}(\mathscr{L}(\mathbb{P}))$

(iv) $\mathbb{P} \subset \mathscr{L}(\mathscr{R}(\mathbb{P}))$

(v) $\mathscr{L}\mathscr{R}\mathscr{L} = \mathscr{L}$

(vi) $\mathscr{R}\mathscr{L}\mathscr{R} = \mathscr{R}$.

Consequently, $\mathscr{P} = \mathscr{L}\mathscr{R}$ and $\mathscr{Q} = \mathscr{R}\mathscr{L}$ are extensive, isotone and idempotent self maps from the conglomerate of all subcategories of $\mathbb{A}$ with respect to the inclusion relation.

Proof : Throughout the proof we will abuse language and simply say that a morphism is constant instead of $\mathscr{N}$-constant.
(i) Assume that $\mathbb{P} \subset \mathbb{Q}$ and let X be left-$\mathbb{Q}$-constant. Then, for each Q in $\mathbb{Q}$ and $\mathbb{A}$-morphism $f : X \to Q$, it follows that f is constant. Since $\mathbb{P} \subset \mathbb{Q}$ this also holds for each $\mathbb{P}$-object P and $\mathbb{A}$-morphism $g : X \to P$. It follows that X is left-$\mathbb{P}$-constant.

(ii) Follows from the duality of (i).
(iii) Let P be a member of $\mathbb{P}$ and suppose that X is left-$\mathbb{P}$-constant. Then, any morphism $f : X \to P$ is automatically constant so that P is right-$\mathscr{L}(\mathbb{P})$-constant.

(iv) Follows from the duality of (iii).
(v) Using (iii), we obtain $\mathbb{P} \subset \mathscr{R}(\mathscr{L}(\mathbb{P}))$ and by (i) it follows that $\mathscr{L}(\mathscr{R}(\mathscr{L}(\mathbb{P}))) \subset \mathscr{L}(\mathbb{P})$. The reverse inclusion follows directly from (iv) .

(vi) Follows from the duality of (v).
That $\mathscr{P}$ and $\mathscr{Q}$ are idempotent follows since $\mathscr{P}\mathscr{P} = \mathscr{L}\mathscr{R}\mathscr{L}\mathscr{R} = \mathscr{L}\mathscr{R} = \mathscr{P}$ and $\mathscr{Q}\mathscr{Q} = \mathscr{R}\mathscr{L}\mathscr{R}\mathscr{L} = \mathscr{R}\mathscr{L} = \mathscr{Q}$.

□

Definition 3.21: $\mathscr{P}$-closed, $\mathscr{Q}$-closed
Let $\mathbb{A}$ be a category and $\mathbb{B}$ be a subcategory of $\mathbb{A}$. Then, $\mathbb{B}$ is **$\mathscr{Q}$-closed ($\mathscr{P}$-closed,** respectively) provided that $\mathscr{Q}(\mathbb{B}) = \mathbb{B}$ ($\mathscr{P}(\mathbb{B}) = \mathbb{B}$, respectively.)

Proposition 3.22: Let $\mathbb{A}$ be a category and $\mathbb{B}$ be a subcategory of $\mathbb{A}$ and $\mathscr{N}$ be a class of weakly constant morphisms. Then:

(i) $\mathbb{B}$ is $\mathscr{Q}$-closed if and only if $\mathbb{B}$ is a right constant subcategory.

(ii) $\mathbb{B}$ is $\mathscr{P}$-closed if and only if $\mathbb{B}$ is a left constant subcategory.

Proof : That $\mathscr{P}$-closedness and $\mathscr{Q}$-closedness imply left and right constant respectively is clear from definition.
(i) Let $\mathbb{B}$ be a right constant subcategory of $\mathbb{A}$. Then $\mathbb{B} = \mathscr{R}(\mathbb{Q})$ for some subcategory $\mathbb{Q}$ of $\mathbb{A}$. It then follows that $\mathscr{Q}(\mathbb{B}) = \mathscr{Q}(\mathscr{R}(\mathbb{Q})) = \mathscr{R}\mathscr{L}\mathscr{R}(\mathbb{Q}) = \mathscr{R}(\mathbb{Q}) = \mathbb{B}$, i.e., $\mathbb{B}$ is $\mathscr{Q}$-closed.
(ii) follows by duality. □

Remark 3.23: We will denote the conglomerate of all left constant subcategories of the category $\mathbb{X}$ by $LC(\mathbb{X})$ and the conglomerate of all right constant subcategories of $\mathbb{X}$ by $RC(\mathbb{X})$. Whenever the constant morphisms need emphasis, we will add subscripts $LC_{\mathbb{C}}(\mathbb{X})$ or $LC_{\mathscr{N}}(\mathbb{X})$ for a constant subcategory or class of constant morphisms respectively.

Proposition 3.24: For any class of weakly constant morphisms in a category $\mathbb{A}$, there exists a bijective antitone Galois correspondence (or equivalently a contravariant adjunction) between the left constant and right constant subcategories of $\mathbb{A}$, i.e., for the pair of maps:

$$\begin{tikzcd} LC(\mathbb{A}) \arrow[r, shift left, "\mathscr{L}(-)"] & RC(\mathbb{A}) \arrow[l, shift left, "\mathscr{R}(-)"] \end{tikzcd}$$

the following holds for any right constant subcategory $\mathbb{P}$ and left constant subcategory $\mathbb{Q}$:

$$\mathscr{R}(\mathbb{Q}) \supset \mathbb{P} \Leftrightarrow \mathbb{Q} \subset \mathscr{L}(\mathbb{P}).$$

Proof : Throughout the proof, we will assume that $\mathbb{P}$ is a right constant subcategory and that $\mathbb{Q}$ is a left constant subcategory. Hence, we may assume that $\mathbb{P} = \mathscr{R}(\mathbb{K})$ and $\mathbb{Q} = \mathscr{L}(\mathbb{X})$ for some subcategories $\mathbb{K}$ and $\mathbb{X}$ of $\mathbb{A}$.
Suppose that $\mathscr{R}(\mathbb{Q}) \supset \mathbb{P}$, then $\mathscr{L}(\mathscr{R}(\mathbb{Q})) \subset \mathscr{L}(\mathbb{P})$ which implies that $\mathbb{Q} = \mathscr{L}(\mathbb{X}) = \mathscr{L}(\mathscr{R}(\mathscr{L}(\mathbb{X}))) = \mathscr{L}(\mathscr{R}(\mathbb{Q})) \subset \mathscr{L}(\mathbb{P})$.

For the other direction, assume that $\mathbb{Q} \subset \mathscr{L}(\mathbb{P})$. Then, $\mathscr{R}(\mathbb{Q}) \supset \mathscr{R}(\mathscr{L}(\mathbb{P})) = \mathscr{R}(\mathscr{L}(\mathscr{R}(\mathbb{K}))) = \mathscr{R}(\mathbb{K}) = \mathbb{P}$.

□

Definition 3.25: Comma or slice category over X
Let $\mathbb{A}$ be a category, $\mathbb{B}$ a subcategory of $\mathbb{A}$ and X be an $\mathbb{A}$-object. Then, the **comma or slice category of $\mathbb{B}$ over X in $\mathbb{A}$**, denoted by $\mathbb{B}/X$, is the category with objects $\mathbb{B}$-structured morphisms $f : B \to X$ with domain in $\mathbb{B}$ and codomain X. A morphism $j : f \to f'$ in $\mathbb{B}/X$ is a $\mathbb{B}$-morphism $j : B \to B'$ such that

$$\begin{tikzcd} B \arrow[rr, "j"] \arrow[dr, "f"'] & & B' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes in $\mathbb{B}$.

For any class $\mathscr{M}$ of morphisms closed under composition and containing all isomorphisms, $\mathscr{M}/X$ denotes the full subcategory of $\mathbb{A}/X$ with all objects morphisms in $\mathscr{M}$. This is called the **slice or comma category of $\mathscr{M}$ over X.**

Dually, we can also construct the slice of X over $\mathbb{A}$. In particular if $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured, then we can also consider the full subcategory $X/\mathscr{E}$ of $X/\mathbb{A}$, consisting of all $f : X \to Y$ in $\mathbb{A}$ with $f \in \mathscr{E}$. This particular category will be of interest to us when studying dual closure operators.

Definition 3.26: $\mathscr{M}$-pullbacks
Let $\mathbb{A}$ be a category and $\mathscr{M}$ be a class of $\mathbb{A}$-morphisms that contains all isomorphisms and is closed under composition. We say that $\mathbb{A}$ has **$\mathscr{M}$-pullbacks** if, for each $\mathbb{A}$-morphism $f : X \to Y$ and every $n \in \mathscr{M}/Y$, a pullback square

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

exists in $\mathbb{A}$ with $m \in \mathscr{M}/X$.

The morphism $m : M \to X$ in the above square is called **the inverse image of n under f or the pullback of n along f** and is usually denoted by $f^{-1}(n) : f^{-1}(N) \to X$.

Remark 3.27: If $\mathscr{M}$ is a class of $\mathbb{A}$-morphisms, then we can define an ordering $\leq$ on $\mathscr{M}/X$ by $m \leq n$ if and only if there is an $\mathbb{A}/X$ morphism $j : m \to n$. The ordering $\leq$ is always transitive on $\mathscr{M}/X$. If $\mathscr{M}$ contains all isomorphisms, then $\leq$ is also reflexive. If $\mathscr{M}$ is a class of $\mathbb{A}$-monomorphisms, then $m \leq n$ and $n \leq m$ implies that $m \simeq n$ in $\mathscr{M}/X$. For the moment, assume that $\mathscr{M}$ is a class of morphisms that contains all isomorphisms and let $\mathbb{A}$ have pullbacks. Then, each $\mathbb{A}$-morphism $f : X \to Y$ provides a map $f^{-1}(-) : \mathscr{M}/Y \to \mathscr{M}/X$, where $f^{-1}(n)$ is the pre-image (or pullback) of n along f. Furthermore $f^{-1}(-)$ is order preserving, for if $(n_1 : N_1 \to Y) \leq (n_2 : N_2 \to Y)$ in $\mathscr{M}/Y$, then there is a morphism $j : N_1 \to N_2$ such that $n_2 j = n_1$.

Consider the diagram
$$\begin{tikzcd}
f^{-1}(N_1) \arrow[rr, "f'"] \arrow[dr, dashed, "i"] \arrow[ddr, bend right, "f^{-1}(n_1)"'] & & N_1 \arrow[d, "j"] \arrow[dd, bend left, "n_1"] \\
& f^{-1}(N_2) \arrow[r, "f''"] \arrow[d, "f^{-1}(n_2)"'] & N_2 \arrow[d, "n_2"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$
where both the outer diagram and the bottom square are the appropriate pullback squares.
Then, it's clear that $f^{-1}(n_1) \leq f^{-1}(n_2)$ in $\mathscr{M}/X$.
It's also easy to see that if n_1 and n_2 are both members of $\mathscr{M}$ and
$$\begin{tikzcd}
P \arrow[r, "m_2"] \arrow[d, "m_1"'] & N_2 \arrow[d, "n_2"] \\
N_1 \arrow[r, "n_1"'] & Y
\end{tikzcd}$$
is a pullback square, then the morphism $n_2m_2 = n_1m_1$ is the meet of n_1 and n_2 in $\mathscr{M}/Y$. This motivates the following definition:

Definition 3.28: Multiple pullback
Let $(f_i : X_i \to X)_I$ be a sink of morphisms. A **multiple pullback of** $(f_i)_I$ consists of a morphism $f : A \to X$ and a source $(j_i : A \to X_i)_I$ such that $f_ij_i = f$, subject to the following condition:

(a) whenever $g : B \to X$ is a morphism for which there exists a source $(g_i : B \to X_i)_I$ such that $f_ig_i = g$ for each $i \in I$, then there is a unique morphism $h : B \to A$ such that $fh = g$ and $j_ih = g_i$ for each $i \in I$.

Remark 3.29: A multiple pullback of a family $(f_i : X_i \to X)_I$ is the limit of the diagram D defined in the following way:
The scheme $\mathbb{I}$ of the diagram is a category with object class $I \cup \{*\}$, where $* \notin I$ and $*$ is a terminal object in $\mathbb{I}$. The only non-identity morphisms in $\mathbb{I}$ are of the form $t_i : i \to *$ where $i \in I$. Defining $D : \mathbb{I} \to \mathbb{A}$ by $D(t_i : i \to *) = f_i : X_i \to X$, it's clear that a multiple pullback of $(f_i)_I$ in $\mathbb{A}$ is the limit of D in $\mathbb{A}$.

Definition 3.30: $\mathscr{M}$-intersections
Let $\mathscr{M}$ be a class of $\mathbb{A}$-monomorphisms that is closed under composition and contains all isomorphisms. Then, we say that $\mathbb{A}$ has **$\mathscr{M}$-intersections** provided that for any family $(m_i)_I$ in $\mathscr{M}/X$, a multiple pullback m of $(m_i)_I$, called an **intersection of** $(m_i)_I$ exists in $\mathbb{A}$ and m is a member of $\mathscr{M}/X$.

$$\begin{tikzcd}
& A_i \arrow[dr, "m_i"] & \\
M \arrow[ur, "d_i"] \arrow[rr, "m"'] & & X
\end{tikzcd}$$

The morphism m is obviously uniquely determined up to isomorphism and is called the **$\mathscr{M}$-intersection of** $(m_i)_I$.
m and M are often denoted by $\bigwedge_I m_i$ and $\bigwedge_I A_i$ respectively. It should then be clear that $\bigwedge_I(A_i, m_i) \simeq (\bigwedge_I A_i, \bigwedge_I m_i)$ and both notations are used.

Remark 3.31: Let $\mathscr{E}$ be a class of $\mathbb{A}$-morphisms. Dual to $\mathscr{M}$-pullbacks, multiple pullbacks and $\mathscr{M}$-intersections, we have $\mathscr{E}$-pushouts, multiple pushouts and $\mathscr{E}$-cointersections. Furthermore, suppose $\mathbb{A}$ has $\mathscr{E}$-pushouts. Let $p : X \to P$ be an $\mathbb{A}$-morphism in $\mathscr{E}$. Then, each $\mathbb{A}$-morphism $f : X \to Y$ establishes a map $f(-) : X/\mathscr{E} \to Y/\mathscr{E}$ where $f(p)$ is the **pushout** or **image** of p along f.

Proposition 3.32: Let $\mathbb{A}$ have $\mathscr{M}$-intersections. Then, for each X in $\mathbb{A}$, $(\mathscr{M}/X, \leq)$ is a (possibly large) complete pre-ordered class.

Proof : It can easily be shown that an $\mathscr{M}$-intersection plays the role of an infimum. As usual, the join can easily be constructed as a meet of all upper bounds. $\square$

Remark 3.33: If $\mathbb{A}$ has $\mathscr{M}$-pullbacks and contains all isomorphisms, it is clear that id_X is the largest element of $\mathscr{M}/X$ for each $\mathbb{A}$-object X. We will denote the least element of $\mathscr{M}/X$ by $0_X : O_X \to X$ or simply 0 if no confusion arises. If $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured and has an inital object I, then for each X in $\mathbb{A}$, there is a unique morphism $i_X : I \to X$. If $i_X = m_X \circ e_X$ is an $(\mathscr{E}, \mathscr{M})$-factorisation, then 0_X can be taken as m_X.

An object X is called **trivial in** $\mathbb{A}$ if $id_X \simeq 0_X$, i.e., whenever $\mathscr{M}/X$ contains only one object up to isomorphism. An equivalent way of defining a trivial object in an $(\mathscr{E}, \mathscr{M})$-structured category (that does not necessarily have $\mathscr{M}$ pullbacks or a least element in $\mathscr{M}/X$) would be to say that an object X is trivial if and only if $\mathscr{M}/X$, viewed as a category, contains only one object up to isomorphism, or equivalently, it's equivalent to the terminal category.

Definition 3.34: $\mathscr{M}$-complete
Let $\mathscr{M}$ be a class of morphisms in $\mathbb{A}$. Then, $\mathbb{A}$ is said to be **$\mathscr{M}$-complete** provided that $\mathbb{A}$ has $\mathscr{M}$-pullbacks and $\mathscr{M}$-intersections. Dual to $\mathscr{M}$-completeness is **$\mathscr{E}$-cocompleteness**.

Corollary 3.35: Let $\mathbb{A}$ have a prefactorisation system $(\mathscr{E}, \mathscr{M})$. Then $\mathbb{A}$ is $\mathscr{M}$-complete provided the required pullbacks exist.
Proof : This follows directly from 2.6(f).

$\square$

Definition 3.36: $\mathscr{M}$-closure and $\mathscr{E}$-coclosure.
Let $\mathbb{B}$ be a subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. Then, the **$\mathscr{M}$-closure of $\mathbb{B}$ (in $\mathbb{A}$)**, denoted by $\mathscr{M}(\mathbb{B})$, is the full subcategory $\mathbb{B}'$ of $\mathbb{A}$ that has object class

$$\{M \in \mathbb{A} \mid \exists B \in \mathbb{B}\ \exists m : M \to B \in \mathscr{M}\}.$$

The **$\mathscr{E}$-coclosure of $\mathbb{B}$ (in $\mathbb{A}$)**, denoted by $(\mathbb{B})\mathscr{E}$ is defined as the $\mathscr{E}$-closure of $\mathbb{B}^{op}$ in the $(\mathscr{M}, \mathscr{E})$-structured category $\mathbb{A}^{op}$ and is given by the object class

$$\{X \in \mathbb{A} \mid \exists B \in \mathbb{B}\ \exists e : B \to X \in \mathscr{E}\}.$$

Note that we could take arbitrary classes of morphisms, so that $\mathscr{E}$ and $\mathscr{M}$ need not be part of a factorisation structure on $\mathbb{A}$.

Remark 3.37: Let $\mathbb{A}$ be a category and $\mathscr{M}$ a class of $\mathbb{A}$-morphisms. For any subcategory $\mathbb{P}$ of $\mathbb{A}$, $\mathscr{M}(\mathbb{P})$ is closed under $\mathscr{M}$-subobjects if $\mathscr{M}$ is closed under composition. To see this, let $m : X \to Y$ be a member of $\mathscr{M}$ with Y is in the $\mathscr{M}$-closure of $\mathbb{P}$. Then, there is a morphism $n : Y \to P \in \mathscr{M}$ with $P \in \mathbb{P}$. Since $\mathscr{M}$ is closed under composition, $n \circ m : X \to P$ is in $\mathscr{M}$, hence X is in the $\mathscr{M}$-closure of $\mathbb{P}$. Furthermore, if $\mathscr{M}$-contain all identity morphisms, then the $\mathscr{M}$-closure of $\mathbb{P}$ obviously contains $\mathbb{P}$. Of course if $\mathscr{M}$ is part of a factorisation structure, then both of these statements are satisfied.

Proposition 3.38: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured. If the class of constant morphisms is left-$\mathscr{M}$-constant, then $\mathscr{L}(\mathbb{P}) = \mathscr{L}(\mathscr{M}(\mathbb{P}))$. Dually, if the class of constant morphisms is right-$\mathscr{E}$-constant, then $\mathscr{R}(\mathbb{Q}) = \mathscr{R}((\mathbb{Q})\mathscr{E})$

Proof : By Remark 3.37 and the fact that the map $\mathscr{L}(-)$ is order reversing, it's sufficient to show that $\mathscr{L}(\mathbb{P}) \subset \mathscr{L}(\mathscr{M}(\mathbb{P}))$. Suppose that X is left-$\mathbb{P}$-constant. We show that X is left-$\mathscr{M}(\mathbb{P})$-constant. Let P' be in the $\mathscr{M}$-closure of $\mathbb{P}$ and let $f : X \to P'$ be any morphism. We need to show that f is constant. By definition of $\mathscr{M}(\mathbb{P})$ it follows that there is a morphism $m : P' \to P$ in $\mathscr{M}$ with $P \in \mathbb{P}$. By left-$\mathbb{P}$-constantness it follows that mf is constant. Since $m \in \mathscr{M}$ and the class of constant morphisms is assumed to be left-$\mathscr{M}$-constant, hence mf is constant if and only if f is. Thus X is left-$\mathscr{M}(\mathbb{P})$-constant. $\square$

Remark 3.39: In view of Proposition 3.38, in case $\mathbb{C}$ is a constant subcategory of $\mathbb{A}$, we may make the following assumptions: When considering left constant subcategories $\mathscr{L}(\mathbb{P})$, we may assume that $\mathbb{P}$ is closed under $\mathscr{M}$-subobjects. In a similar manner when dealing with right constant subcategories $\mathscr{R}(\mathbb{Q})$, we may assume that $\mathbb{Q}$ is closed under $\mathscr{E}$-images. Henceforth, we use this assumption without explicitly stating it.

Proposition 3.40: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured and $\mathbb{P}$ and $\mathbb{Q}$ be subcategories of $\mathbb{A}$. Let $\mathbb{C}$ denote the constant subcategory of $\mathbb{A}$. Then, $\mathbb{P} \cap \mathscr{L}(\mathbb{P}), \mathbb{Q} \cap \mathscr{R}(\mathbb{Q}) \subset \mathbb{C}$.

$\underline{\mathcal{P}roof}$: Let X be a member of $\mathbb{P}$ and $\mathscr{L}(\mathbb{P})$. Then, $id_X : X \to X$ is a constant morphism, thus $X \in \mathbb{C}$. That $\mathbb{Q} \cap \mathscr{R}(\mathbb{Q}) \subset \mathbb{C}$ follows similarly. □

Proposition 3.41: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category and let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$. Then, the following statements hold:

(i) Every left and every right constant subcategory contains $\mathbb{C}$;

(ii) If $\mathbb{P}$ contains $\mathbb{C}$, then $\mathbb{P} \cap \mathscr{L}(\mathbb{P}) = \mathbb{P} \cap \mathscr{R}(\mathbb{P}) = \mathbb{C}$.

$\underline{\mathcal{P}roof}$: (i) Suppose C is a member of $\mathbb{C}$ and that $\mathbb{D} = \mathscr{R}(\mathbb{Q})$. Let Q be a member of $\mathbb{Q}$. Note that every morphism $f : Q \to C$, with C in $\mathbb{C}$ is constant, as $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects. Thus, $C \in \mathbb{D}$. The other case follows by duality.

(ii) The second part follows from (i) and and Proposition 3.40. □

Proposition 3.42: Let $\mathbb{P}$ be a subcategory of $\mathbb{A}$. Then, $\mathscr{L}(\mathbb{P})$ is closed under $\mathscr{E}$-images provided that $\mathscr{N}$ is a class of right-$\mathscr{E}$-constant-morphisms. Dually, for a subcategory $\mathbb{Q}$, $\mathscr{R}(\mathbb{Q})$ is closed under $\mathscr{M}$-subobjects provided that $\mathscr{N}$ is a class of left-$\mathscr{M}$-constant morphisms.

$\underline{\mathcal{P}roof}$: We only show that $\mathscr{R}(\mathbb{Q})$ is closed under $\mathscr{M}$-subobjects provided that $\mathscr{N}$ is a class of left-$\mathscr{M}$-constant morphisms. Let $m : M \to X \in \mathscr{M}$ with X right-$\mathbb{Q}$-constant. Then, given any morphism $g : Q \to M$ with $Q \in \mathbb{Q}$, we have that mg is a morphism from a $\mathbb{Q}$-object to X. Since X is right-$\mathbb{Q}$-constant, mg is constant. By the assumption on $\mathscr{N}$, it follows that mg is constant if and only if g is, hence M is also right-$\mathbb{Q}$-constant. □

Corollary 3.43: If $\mathbb{A}$ is a category and $\mathbb{C}$ is a constant subcategory, then $\mathscr{L}(\mathbb{P})$ is closed under $\mathscr{E}$-images and $\mathscr{R}(\mathbb{P})$ is closed under $\mathscr{M}$-subobjects for any subcategory $\mathbb{P}$ of $\mathbb{A}$.

Example 3.44: Note that Proposition 3.42 need not hold if $\mathscr{N}$ is not right or left cancellative. To see this, consider the category $\mathbb{S}\mathrm{et}$ with factorisation structure (map, bijective map). Let the class of constant morphisms $\mathscr{N}$ be all morphisms with empty domain, i.e., $\mathscr{N} = \{f : X \to Y \mid X = \varnothing\}$. We assert that not every left constant subcategory is closed under images. In particular, for $\mathbb{P} = \mathbb{S}\mathrm{et}$, we claim that $\mathscr{L}(\mathbb{P})$ contains only the empty set. If X is a member of $\mathscr{L}(\mathbb{P})$ and $P = \{0\}$, then if $X \neq \varnothing$, we can define $f : X \to P$ by $f(x) = 0$ for each $x \in X$. Hence, if $X \neq \varnothing$, then $f \notin \mathscr{N}$ so that $X \notin \mathscr{L}(\mathbb{P})$. If $X = \varnothing$, then each function with domain X is a member of $\mathscr{N}$. It follows that $\mathscr{L}(\mathbb{P}) = \{\varnothing\}$. For each set X, there is a unique map $\varnothing \to X$ and thus $\{\varnothing\}$ is not closed under images of maps.

The main purpose of this section is to generalise some of the ideas in [2] and [42, §1]. In particular, let $\mathbb{A}$ be a topological construct (as defined in [42, 1.1.2]) with faithful functor $U : \mathbb{A} \to \mathbb{S}\mathrm{et}$. Let $\mathbb{C}$ be the category of all $\mathbb{A}$-objects X such that $|UX| \leq 1$. By [42, 1.2.33], $\mathbb{A}$ is both (Epi($\mathbb{A}$), ExtrMono($\mathbb{A}$)) and (ExtrEpi($\mathbb{A}$), Mono($\mathbb{A}$))-structured. Note that $\mathbb{C}$ is closed under epimorphisms and monomorphisms. To see this, let X be in $\mathbb{C}$ and assume that $m : M \to X$ is a monomorphism and $e : X \to Y$ is an epimorphism of $\mathbb{A}$. Since U is a topological functor, U preserves and reflects mono-sources and epi-sinks ([1, 21.12]). Therefore, Um is injective and Ue is surjective. Since $|UX| \leq 1$, it follows that $|UM|, |UY| \leq 1$, i.e., M and Y are members of $\mathbb{C}$. Hence, $\mathbb{C}$ is a constant subcategory for both the orthogonal factorisation structures listed above. All of the propositions 3.41, 3.45, 3.48 and 3.49 should be compared to [2, 1.2,1.3,2.1,3.1,3.2].

Proposition 3.45: Let $\mathscr{N}$ be a class of weakly constant morphisms and let $\mathbb{Q}$ be a subcategory of $\mathbb{A}$ that is closed under $\mathscr{E}$-images. Then,

$$\mathscr{R}(\mathbb{Q}) = \{X \in \mathbb{A} \mid \forall m : Q \to X \in \mathscr{M} \text{ if } Q \in \mathbb{Q},\ \text{then } m \in \mathscr{N}\}.$$

Dually, if $\mathscr{N}$ is a class of weakly constant morphisms and $\mathbb{P}$ is a subcategory of $\mathbb{A}$ closed under $\mathscr{M}$-subobjects, then

$$\mathscr{L}(\mathbb{P}) = \{X \in \mathbb{A} \mid \forall e : X \to P \in \mathscr{E} \text{ if } P \in \mathbb{P},\ \text{then } e \in \mathscr{N})\}.$$

Proof : Let $\mathbb{U}$ be the subcategory of $\mathbb{A}$ consisting of the class of objects

$$\{X \in \mathbb{A} \mid \forall m : Q \to X \in \mathscr{M} \text{ if } Q \in \mathbb{Q}, \text{ then } m \in \mathscr{N}\}.$$

We show that $\mathbb{U} = \mathscr{R}(\mathbb{Q})$. Let X be right-$\mathbb{Q}$-constant with $\mathbb{Q}$ closed under $\mathscr{E}$-images. Let $m : M \to X$ be a member of $\mathscr{M}$ with $M \in \mathbb{Q}$. It should be clear that m is constant so that $m \in \mathscr{N}$.
For the reverse inclusion, assume that X is not right-$\mathbb{Q}$-constant. Then, there exists a $\mathbb{Q}$-object Q and a morphism $f : Q \to X$ with $f \notin \mathscr{N}$. Let $f = me$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of f with $Q' := dom(m)$. Since $\mathbb{Q}$ is closed under $\mathscr{E}$-images, it follows that Q' is a member of $\mathbb{Q}$. Since f is not in $\mathscr{N}$, we have that both e and m are not in $\mathscr{N}$. It follows that m is the desired morphism with domain in $\mathbb{Q}$ that is not in $\mathscr{N}$. Thus, $X \notin \mathbb{U}$. Consequently $\mathscr{R}(\mathbb{Q}) = \mathbb{U}$.

The other case follows by duality. □

Corollary 3.46: Let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$. Then, for subcategories $\mathbb{P}$ and $\mathbb{Q}$ of $\mathbb{A}$ that are respectively closed under $\mathscr{M}$-subobjects and $\mathscr{E}$-images, the following hold:

$$\mathscr{R}(\mathbb{Q}) = \{X \in \mathbb{A} \mid \forall m : Q \to X \in \mathscr{M} \ (Q \in \mathbb{Q} \Rightarrow Q \in \mathbb{C})\}$$

and

$$\mathscr{L}(\mathbb{P}) = \{X \in \mathbb{A} \mid \forall e : X \to P \in \mathscr{E} \ (P \in \mathbb{P} \Rightarrow P \in \mathbb{C})\}.$$

Proof : This follows directly from Proposition 3.45. □

Remark 3.47: One of the major goals of studying left and right constant subcategories will be to relate these to $\mathscr{E}$-reflective and $\mathscr{M}$-coreflective subcategories. In view of Example 3.44 and the fact that an $\mathscr{M}$-coreflective subcategory is closed under $\mathscr{E}$-images, it seems that it might be more fruitful to consider constant subcategories instead of pursuing only weakly constant morphisms.

Unless stated otherwise, for the remainder of this section, we will always assume that $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured and that the notion of constantness is used via a fixed constant subcategory $\mathbb{C}$ of $\mathbb{A}$.

Proposition 3.48: Let $\mathbb{P}$ be a subcategory of $\mathbb{A}$. Then, $\mathbb{P}$ is right constant if and only if $\mathbb{P}$ satisfies the following condition:

$$X \in \mathbb{P} \Leftrightarrow \forall m : M \to X \in \mathscr{M} \text{ with } M \notin \mathbb{C} \ \exists e : M \to P \in \mathscr{E} \text{ such that } P \in \mathbb{P} \setminus \mathbb{C}$$

Proof : Suppose that $\mathbb{P}$ is a right constant subcategory of $\mathbb{A}$. Then, there is a subcategory $\mathbb{Q}$ of $\mathbb{A}$ that is closed under $\mathscr{E}$-images such that $\mathbb{P} = \mathscr{R}(\mathbb{Q})$. In fact, we may take $\mathbb{Q} = \mathscr{L}(\mathbb{P})$.

Let X be in $\mathbb{P}$ and suppose that $m : M \to X$ is a member of $\mathscr{M}$ with $M \notin \mathbb{C}$. Then, m is not constant and neither M nor X are members of $\mathbb{C}$. Since $\mathbb{P}$ is right constant, M is in $\mathbb{P}$ by Proposition 3.42. Then, id_M is the desired morphism in $\mathscr{E}$. Suppose that X is not a member of $\mathbb{P}$, i.e., X is not right-$\mathbb{Q}$-constant. We show that X doesn't satisfy the imposed condition. Since X is not right-$\mathbb{Q}$-constant, there exists an $\mathscr{M}$-morphism $m : Q \to X$ with $Q \in \mathbb{Q}$, but with Q not in $\mathbb{C}$. We assert that Q is an $\mathscr{M}$-subobject of X such that for which any $\mathscr{E}$-morphism $e : Q \to P$ with $P \in \mathbb{P}$, we must have $P \in \mathbb{C}$. To see this, let $e : Q \to P$ be a morphism in $\mathscr{E}$ such that P is a member of $\mathbb{P}$. Then, e must be constant as P is right-$\mathbb{Q}$-constant. Since a constant morphism e is a member of $\mathscr{E}$ if and only if $cod(e) \in \mathbb{C}$, it follows that $P \in \mathbb{C}$. Thus, X doesn't satisfy the imposed condition.

Conversely, suppose that $\mathbb{P}$ satisfies the above condition. We need only prove that $\mathscr{R}(\mathscr{L}(\mathbb{P})) \subset \mathbb{P}$. Suppose that X is not a member of $\mathbb{P}$. Then, there is a morphism $m : M \to X$ with $M \notin \mathbb{C}$ such that for each $e : M \to P \in \mathscr{E}$ with $P \in \mathbb{P}$, we have that P is actually a member of $\mathbb{C}$. We first show that M is left-$\mathbb{P}$-constant. By Corollary 3.46, we need only look at morphisms from M to $\mathbb{P}$ that are members of $\mathscr{E}$. Therefore, if $e : M \to P$ is a morphism in $\mathscr{E}$ with $P \in \mathbb{P}$, then by the assumptions on X, as noted above, it follows that $P \in \mathbb{C}$. Thus e is constant and consequently M is left-$\mathbb{P}$-constant. But $m : M \to X$ is not constant, hence X is not right-$\mathscr{L}(\mathbb{P})$-constant, i.e., $X \notin \mathscr{R}(\mathscr{L}(\mathbb{P}))$. Since $\mathbb{P} \subset \mathscr{R}(\mathscr{L}(\mathbb{P}))$ always holds, we are done.

□

Proposition 3.49: Let $\mathbb{Q}$ be a subcategory of $\mathbb{A}$. Then, $\mathbb{Q}$ is left constant if and only if $\mathbb{Q}$ satisfies the following condition:

$$X \in \mathbb{Q} \Leftrightarrow \forall e : X \to E \in \mathscr{E} \text{ with } E \notin \mathbb{C}\ \exists m : Q \to E \in \mathscr{M} \text{ such that } Q \in \mathbb{Q} \setminus \mathbb{C}$$

D3.48

Example 3.50: Unfortunately, the notion of constant subcategories is sometimes a little bit too general for our intuition on what a constant map should be. In topological categories, left and right constant subcategories are usually associated with a categorical generalisation of connectedness and disconnectedness, or separation properties. One prominent feature that arises in any construct is the following:
If $f : X \to Y$ is a morphism and $(M_i)_I$ is a family of subsets of X such that $\bigcap_I M_i \neq \varnothing$ with f a constant map on each subset M_i of X, then f is also constant on the union $\bigcup_I M_i$, provided that $\bigcup_I M_i$ and M_i are objects for each $i \in I$. This is far from true for even simple categories: Consider the category $\mathbb{Set}$ with (Epi, Mono) factorisation structure and let $\mathbb{C}$ consist exactly of the sets with cardinality at most 2. Then, a function $f : X \to Y$ is constant if and only if $f[X]$ has cardinality at most 2. This means that if $(M_i)_I$ is a family of subsets of X with inclusion map $m_i : M_i \hookrightarrow X$ with $M_i \in \mathscr{L}(\mathbb{P})$ with $\bigcap_I M_i \neq \varnothing$, then $\bigcup_I M_i \in \mathscr{L}(\mathbb{P})$.
Let $f : \mathbb{N} \to \mathbb{N}$ be the identity function and consider for each $n \in \mathbb{N}$, the set $M_n := \{0, n\}$. Then, the intersection $\bigcap_\mathbb{N} M_n = \{0\} \neq \varnothing$ and $f[M_n] = M_n$ has cardinality 2 for each n. Thus, for each n, $M_n \in \mathscr{L}(\mathbb{P})$. But $\bigcup_\mathbb{N} M_n = \mathbb{N}$ and $f[\mathbb{N}] = \mathbb{N}$ is not even finite. Hence, the union of left constant objects need not be left constant anymore. If $m_n : M_n \to \mathbb{N}$ is the inclusion map for each n and $m : \bigcup_\mathbb{N} M_n \hookrightarrow \mathbb{N}$, then this shows that if $f \circ m_n$ is constant for each $n \in \mathbb{N}$, then $f \circ m$ need not be constant.
A similar example with finite I, say $I = \{1, 2\}$, and $M_i = \{0, i\}$, also shows that $m_i = f \circ m_i$ is constant for each $i \in I$, whereas $f \circ m : \{0, 1, 2\} \to \mathbb{N} \to \mathbb{N}$ is not.

Classically, in a topological construct, a necessary condition for the union of connected subspaces $(C_i)_I$ to be connected is for the subspaces to have a point in common. Unfortunately, the idea of having subobjects in $\mathbb{C}$ could be thought of as being single points and hence should not be connected. Another problem is that our chosen constant subcategory contains the empty space and this could be one of our connected subspaces. Of course this creates a problem. This tempts us to ponder on whether the union(in a construct) of left-$\mathbb{P}$-constant subobjects will still be left-$\mathbb{P}$-constant, provided that $\bigcap_I M_i \notin \mathbb{C}$. Unfortunately this is not true as the following example illustrates.

Example 3.51: Consider the category $\mathbb{Top}$ of all topological spaces with factorisation structure (quotient map, injective map). Let $\mathbb{C}$ be the full subcategory of $\mathbb{Top}$ of all finite topological spaces. Note that a morphism f is ($\mathbb{C}$-)constant if and only if its $(\mathscr{E}, \mathscr{M})$-factorisation factors through a finite space.

For each set A, let $\mathcal{T}^A_{cof} = \{U \subset A \mid A \setminus U \text{ is finite or} A \setminus U = A\}$, i.e., let $\mathcal{T}^A_{cof}$ denote the cofinite topology on A. Let A^{cof} denote the topological space with underlying set A and cofinite topology on A. For each $k \in \mathbb{N}$, let $\mathbb{N}_k = \mathbb{N} \times \{k\}$ and for each $n \in \mathbb{N}$, let $X_n = \bigcup_{k=1}^n \mathbb{N}_k$. Then, $X_m \subset X_n$ whenever $m \leq n$ and we have the obvious inclusion map $\iota_{m,n} : X_m \to X_n$. For each $n \in \mathbb{N}$, let M_n be the disjoint union or sum of n copies of $\mathbb{N}^{cof}$. Without loss of generality, we may assume that $M_n = (X_n, \mathcal{T}_n)$, where

$$\mathcal{T}_n = \{U \subset X_n \mid \forall k :\ 1 \leq k \leq n : \mathbb{N}_k \setminus U \text{ is finite or } \mathbb{N}_k \setminus U = \mathbb{N}_k\}.$$

Let Y be the discrete space with underlying set $\mathbb{N}$, i.e., $Y = (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and let X be the disjoint union or sum of $\mathbb{N}$ copies of $\mathbb{N}^{cof}$. Once again, we may assume that the space X is given by $(\bigcup_\mathbb{N} \mathbb{N}_k, \mathcal{T})$, where

$$\mathcal{T} = \{U \subset \bigcup_\mathbb{N} \mathbb{N}_k \mid \forall k \in \mathbb{N} :\ \mathbb{N}_k \setminus U \text{ is finite or } \mathbb{N}_k \setminus U = \mathbb{N}_k\}.$$

It should be clear that $\mathcal{T}$ is a topology and that $X_n \subset \bigcup_\mathbb{N} \mathbb{N}_k$. Let us denote the inclusion map from M_n to X by c_n. The map c_n is continuous for each n, since if U is a non-empty open set in X, then $X_n \setminus c_n^{-1}[U] = X_n \setminus (X_n \cap U) = X_n \setminus U = (\bigcup_{k=1}^n \mathbb{N}_k) \setminus U = \bigcup_{k=1}^n \mathbb{N}_k \setminus U$ which is a finite union of finite sets, in case $\mathbb{N}_k \cap U$ is finite for each $k = 1, 2, \ldots, n$, hence finite. If there is a k with $1 \leq k \leq n$ such that $\mathbb{N}_k \setminus U = \mathbb{N}_k$, let I be the set of all those $i \in J_n := \{1, 2, \ldots, n\}$ such that $\mathbb{N}_i \setminus U = \mathbb{N}_i$.

Then, $X_n \setminus U = (\bigcup_{k=1}^{n} \mathbb{N}_k) \setminus U = \bigcup_{k=1}^{n} \mathbb{N}_k \setminus U = \left(\bigcup_{i\in I} \mathbb{N}_i \setminus U\right) \cup \left(\bigcup_{k\in J_n\setminus I} \mathbb{N}_k \setminus U\right) = \left(\bigcup_{i\in I} \mathbb{N}_k\right) \cup \left(\bigcup_{k\in J_n\setminus I} \mathbb{N}_k \setminus U\right)$. It's then easy to see that $\mathbb{N}_i \setminus U = \mathbb{N}_i$ whenever $i \in I$ and $\mathbb{N}_k \setminus U$ is finite if $k \in J_n \setminus I$. In a similar manner, it can also be seen that M_m is a subspace of M_n whenever $m \leq n$, i.e., $\iota_{m,n} : M_m \hookrightarrow M_n$ is an initial continuous injection.
We now proceed and show that any continuous function $f : Z \to Y$ is $\mathbb{C}$-constant whenever Z is a cofinite space. So, assume that $f : Z \to Y$ is a continuous map. If Z is finite, then any surjection from Z to a space A forces A to be finite. Hence, if f factorises as $m \circ e$, where e is surjective, then $dom(m)$ is finite, i.e., f is $\mathbb{C}$-constant. Now, if Z is infinite, suppose that $f(z) \neq f(z')$ for some $z \neq z'$ in Z. Then, since Y is discrete, $f^{-1}[\{f(x)\}]$ must be open for each $x \in Z$. Note in particular, that $f^{-1}[f(z)]$ and $f^{-1}[f(z')]$ are both non-empty subsets of Z. Since these two sets are disjoint and non-empty, $\varnothing \neq f^{-1}[\{f(a)\}] \subsetneq Z$ for $a \in \{z, z'\}$. This implies that $Z \setminus f^{-1}[\{f(a)\}]$ is finite for each $a \in \{z, z'\}$. Consequently, $Z = Z \setminus \varnothing = Z \setminus (f^{-1}[[\{f(z)\}] \cap \{f(z')\}]) = Z \setminus (f^{-1}[f(z)] \cap f^{-1}[\{f(z')\}]) = Z \setminus f^{-1}[\{f(z)\}] \cup Z \setminus f^{-1}[\{f(z')\}]$. Therefore, Z can be written as a finite union of finite sets, contrary to the fact that Z is infinite. Therefore f is a constant map and consequently also $\mathbb{C}$-constant.

For any n, let $g : M_n \to Y$ be continuous. Then we assert that g is $\mathbb{C}$-constant. Note that $\mathbb{N}_i$ is a subset of M_n for each $i \in \{1, 2, \ldots, n\}$, with $\bigcup_{i=1}^{n} \mathbb{N}_i = M_n$. Consider $\mathbb{N}_i$ as a subspace of M_n, then $\mathbb{N}_i$ is a cofinite space and by the previous paragraph, this implies that $g|_{\mathbb{N}_i}$ must be a constant map. Since there are exactly n such subspaces, $g[M_n]$ contains at most n elements. Therefore g is $\mathbb{C}$-constant.

On the other hand, we may define $h : X \to Y$ as follows: For each $x \in X$, there exist unique positive integers m and k such that $x = (m, k)$. Define $h(x) = h(m, k) = k$. In order to see that h is continuous, it's sufficient to prove that $h^{-1}[\{k\}]$ is open in X for each $k \in Y$. Let $k \in Y$, then $h^{-1}[\{k\}] = \{(m, n) \in X \mid h(m, n) = k\} = \{(m, n) \in X \mid n = k\} = \mathbb{N}_k$. Then,

$$\mathbb{N}_i \setminus \mathbb{N}_k = \begin{cases} \varnothing & \text{if } i = k \\ \mathbb{N}_i & \text{if } i \neq k \end{cases}$$

Hence, h is continuous, but h is not constant. For if $h = me$ where e is a continuous surjection and m injective, it would have to follow that $M := dom(m)$ is not finite. Otherwise, there is an $n \in \mathbb{N}$ such that $n > |M|$ and then $n = |h[X_n]| > |M| \geq |m[M]| = |m[e[X]]| > |h[X]| = |\mathbb{N}|$, a contradiction.

In order to show that this is a counterexample to the above statement, we need only show that the intersection of $(c_n : M_n \to X)_{\mathbb{N}}$ is not a member of $\mathbb{C}$ and that the union of $(c_n)_{\mathbb{N}}$ is in fact X. This is the case, for it can easily be seen that M_n is a member of $\mathscr{L}(\{Y\})$, whereas the above continuous map h shows that X is not a member.

To be more explicit, we now show that $c_1 : M_1 \to X$, together with the inclusion maps $(\iota_{1,n})_{n\in\mathbb{N}}$, is the intersection of $(c_n)_{n\in\mathbb{N}}$. It will then follow that the intersection object, i.e., M_1 is not in $\mathbb{C}$. It should be clear that $c_1 = c_n \circ \iota_{1,n}$ for each $n \in \mathbb{N}$. So in order to see that it's a multiple pullback, suppose that A is a topological space, $g : A \to X$ a continuous map and $(g_n : A \to M_n)_{n\in\mathbb{N}}$ a source such that $c_n \circ g_n = g$ for each $n \in \mathbb{N}$. Then, g_1 is a continuous map such that $c_1 \circ g_1 = g$ and for each $n \in \mathbb{N}$, there holds $c_n \iota_{1,n} \circ g_1 = c_1 \circ g_1 = g = c_n \circ g_n$. Since c_n is injective for each n, it follows that $\iota_{1,n} \circ g_1 = g_n$. Since c_1 is injective, g_1 is clearly unique. Thus, $M_1 \simeq \bigwedge_{\mathbb{N}} M_n \notin \mathbb{C}$.

$$\begin{tikzcd}
M_1 \arrow[rr, "c_1"] \arrow[dr, "\iota_{1,n}"] & & X \\
& M_n \arrow[ur, "c_n"'] & \\
& A \arrow[u, "g_n"'] \arrow[uul, bend left, "g_1"] \arrow[uur, bend right, "g"'] &
\end{tikzcd}$$

Now we show that $X \simeq \bigvee_{\mathbb{N}} M_n$. Suppose that X' is a space and $m : X' \to X$ an injective continuous map such that $c_n \leq m$ for each $n \in \mathbb{N}$. Hence, for each n, there is a morphism $j_n : M_n \to X'$ such that $m \circ j_n = c_n$. Since $(c_n)_{\mathbb{N}}$ is a final sink, it's sufficient to define a map $k : X \to X'$ such that $k \circ c_n = j_n$ for each $n \in \mathbb{N}$. For each $x \in X$, the set $S(x) = \{n \in \mathbb{N} \mid x \in X_n\}$ is non-empty, so let n_x be any member of $S(x)$. Define $k(x) = j_{n_x}(x)$. In order to see that k is a map, consider $n, n' \in S(x)$. Without loss of generality, assume that $n \leq n'$. Then, $m(j_n(x)) = c_n(x) = c_{n'}(\iota_{n,n'}(x)) = c_{n'}(x) = m(j_{n'}(x))$ and since m is injective, it follows that k is well-defined. Then, $k \circ c_n(x) = j_n(x)$ as $x \in X_n$. Thus, $id_X \simeq \bigvee_{\mathbb{N}} c_n$ and we are done.

$$\begin{tikzcd}
M_n \arrow[r, "\iota_{n,n'}"] \arrow[rr, bend left, "c_n"] \arrow[drr, bend right, "j_n"'] & M_{n'} \arrow[r, "c_{n'}"] \arrow[rr, bend left, "c_{n'}"] \arrow[dr, "j_{n'}"'] & X \arrow[r, equal] & X \arrow[dl, dashed, "k"] \\
 & & X' \arrow[u, "m"'] &
\end{tikzcd}$$

Remark 3.52: Note that in both of the examples 3.50 and 3.51, the subcategory $\mathbb{C}$ is not a reflective subcategory. This has a big influence as will be illustrated later. To conclude this section, let us consider a few examples in $\mathbb{T}\mathrm{op}$ and $\mathbb{A}\mathrm{b}$.

Example 3.53: Consider the category $\mathbb{T}\mathrm{op}$ of all topological spaces and continuous maps with factorisation structure $(epi, embedding)$ and $\mathbb{C} = \{(X, \mathcal{T}) \in \mathbb{T}\mathrm{op} \mid |X| \leq 1\}$. Let $\mathbb{I}_2$, $\mathbb{S}$, and $\mathbb{D}_2$ denote the indiscrete, Sierpinski and discrete 2-point spaces respectively. Let $\mathbb{R}$ and $[0,1]$ denote the topological spaces with the usual Euclidean topology. Then, the following hold:

(a) It should be clear that

$$\mathbb{K} := \mathscr{L}(\{\mathbb{D}_2\}) = \{X \in \mathbb{T}\mathrm{op} \mid X \text{ is connected}\} = \mathbb{C}\mathrm{on}.$$

We now show that $\mathscr{R}(\mathbb{C}\mathrm{on}) = \mathbb{TD}\mathrm{isc}$ where $\mathbb{TD}\mathrm{isc}$ is the full subcategory of all totally disconnected spaces in $\mathbb{T}\mathrm{op}$.
Since $\mathbb{K}$ is a left constant subcategory, it's closed under images, i.e., $\mathbb{K}$ is closed under surjections. Let $m : M \to X$ be an embedding with X totally disconnected, then by definition of totally disconnectedness, if M is connected, we have: $|M| \leq 1$, i.e., $M \in \mathbb{C}$. Therefore $X \in \mathscr{R}(\mathbb{K})$.

For the other inclusion, suppose that X is a member of $\mathscr{R}(\mathbb{K})$ and X is not totally disconnected. Then there exist distinct points x and y of X and a connected subspace C of X such that $Y := \{x, y\} \subset C$. But then the inclusion map $C \hookrightarrow X$ is an embedding with C connected and since C is a member of $\mathbb{K}$, it follows that $|Y| \leq 1$, contrary to our assumption that x and y are distinct. Thus $\mathscr{R}(\mathbb{K}) = \mathbb{TD}\mathrm{isc}$ and consequently $\mathscr{Q}(\{\mathbb{D}_2\}) = \mathbb{TD}\mathrm{isc}$.

(b) Now we show that $\mathscr{R}(\{\mathbb{D}_2\}) = \mathbb{C}$ and $\mathscr{L}(\mathbb{C}) = \mathbb{T}\mathrm{op}$ to conclude that $\mathscr{Q}(\{\mathbb{D}_2\}) = \mathbb{T}\mathrm{op}$.
We know that $\mathbb{C} \subset \mathscr{R}(\mathbb{A})$ for any subcategory $\mathbb{A}$ of $\mathbb{T}\mathrm{op}$, so we need only prove the other inclusion. Suppose $(X, \mathcal{T})$ is a topological space with $|X| \geq 2$, i.e., $(X, \mathcal{T}) \notin \mathbb{C}$. Then, there are distinct members x and y of X. Define $f : \mathbb{D}_2 \to X$ by $f(0) = x$ and $f(1) = y$. Then f is continuous and non-constant. Therefore X is not a member of $\mathscr{R}(\{\mathbb{D}_2\})$ and consequently equality holds.

To prove the other claim, it's sufficient to prove that for every topological space Z and every continuous map $f : Z \to C$ with $C \in \mathbb{C}$, f is constant. Since $\mathbb{C}$ is closed under subobjects in $\mathscr{M}$, this follows.

(c) Considering I_2, we show $\mathscr{L}(\{I_2\}) = \mathbb{C}$ and $\mathscr{R}(\{I_2\}) = \mathbb{T}\mathrm{op}_0$. We first show that if X is not in $\mathbb{C}$, then X is not a member of $\mathscr{L}(\{\mathbb{I}_2\})$. Suppose that X contains at least two distinct points, say x and y. Then, the characteristic function f of the set $\{x\}$, i.e., $f(a) = 0$ if $a \neq x$ and $f(x) = 1$, is continuous as I_2 is indiscrete. It's clear that f is non-constant and thus X is not a member of $\mathscr{L}(\{\mathbb{I}_2\})$.

We claim that $\mathscr{R}(\{\mathbb{I}_2\}) = \mathbb{T}\mathrm{op}_0$. Let X be a topological space and assume that X is not T_0. Then, there exists two distinct points x and y of X such that whenever U is an open set of X, we have: $x \in U \Leftrightarrow y \in U$. Define $f : \mathbb{I}_2 \to X$ by $f(0) = x$ and $f(1) = y$. Then, f is continuous

[1] Note that in order to simplify notation throughout this thesis, $\mathbb{C}\mathrm{on}$ will denote the subcategory of all connected spaces together with the empty space.

and non-constant. Thus if X is not T_0, then X is not right-$\{\mathbb{I}_2\}$-constant. Thus, if X is right-$\{\mathbb{I}_2\}$-constant, then X is T_0. Now, suppose that Y is not right-$\{\mathbb{I}_2\}$-constant. Then, there is a non-constant continuous function $g : \mathbb{I}_2 \to Y$. Consider the distinct points $g(0)$ and $g(1)$ in Y. We need only show that Y is not T_0. We show that if U is any open set of Y, then $g(0) \in U$ if and only if $g(1) \in U$. Let U be an open set containing $g(0)$. Then $g^{-1}[U]$ is a non-empty open set of an indiscrete space, thus $g^{-1}[U] = \{0, 1\}$ and thus $g(1) \in U$. Hence, $g(0) \in U$ implies that $g(1) \in U$. The other direction is symmetric.

(*d*) Let X be left-$\{\mathbb{S}\}$-constant. Then, X must be indiscrete, otherwise if U is a non-empty proper open subset of X, then the characteristic function of U is a non-constant continuous function into $\mathbb{S}$. On the other hand, if X is indiscrete, and $f : X \to \mathbb{S}$ is continuous, then f is constant, otherwise $f^{-1}[\{1\}]$ is a non-empty proper open subset of X. Therefore $\mathscr{L}(\{\mathbb{S}\}) = \mathbb{I}\mathrm{nd}$.

Now, we show that $\mathscr{R}(\{\mathbb{S}\}) = \mathbb{T}\mathrm{op}_1$. Using Proposition 3.38, we may form the surjective closure $\mathbb{Q}$ of $\{\mathbb{S}\}$. Then $\mathbb{Q}$ is equivalent to $\{1, \mathbb{S}, \mathbb{I}_2\}$. Using 3.46, we have

$$\mathscr{R}(\mathbb{S}) = \{X \ \in \mathbb{T}\mathrm{op} \mid \forall \text{embeddings } m : M \to X (M \in \mathbb{Q} \Rightarrow |M| \leq 1)\}.$$

We claim that these are exactly the T_1-spaces. Thus, if M is a two point subspace of a right-$\{\mathbb{S}\}$-constant space, then it must be homeomorphic to $\mathbb{D}_2$. Let X be right-$\{\mathbb{S}\}$-constant and let x and y be distinct points of X. Then, the subspace of X with underlying set $\{x, y\}$ must be discrete, hence there is a closed set C of X such that $\{x\} = C \cap \{x, y\}$. Then, $y \notin C$ and consequently $x \in cl\{x\} \subset C$ and $y \notin cl\{x\}$. Therefore X is T_1.

On the other hand, if X is T_1, then for any distinct members $x, y \in X$, we have that $\{x\}$ and $\{y\}$ are closed. Thus, the subspace $\{x, y\}$ is discrete and there are no other embeddings from spaces in $\mathbb{Q}$ that are not already members of $\mathbb{C}$. Therefore, X is right-$\{\mathbb{S}\}$-constant and $\mathscr{R}(\mathbb{Q}) = \mathbb{T}\mathrm{op}_1$.

(*e*) Using 3.38 and 3.46, it can easily be verified that $\mathscr{L}(\{\mathbb{R}\}) = \mathscr{L}(\{[0, 1]\}) = \mathbb{R}\mathbb{C}\mathrm{on}$ where $\mathbb{R}\mathbb{C}\mathrm{on}$ denotes the full subcategory of all real constant spaces, i.e., all topological spaces X for which every real valued continuous function is constant.

$\mathscr{R}(\{[0, 1]\}) = \mathbb{T}\mathbb{P}\mathbb{D}\mathrm{isc}$, where $\mathbb{T}\mathbb{P}\mathbb{D}\mathrm{isc}$ consists of all those spaces which are totally path disconnected, i.e., the only paths are constant ones. Equivalently, this is all the spaces for which the path components consist of at most one element.

By 3.38 and 3.46, it also follows that $\mathscr{R}(\{\mathbb{R}\}) = \mathscr{R}(\{\mathbb{R}, [0, 1]\})$ since there is obviously a continuous surjection from $\mathbb{R}$ to $[0, 1]$. Then, since $\{[0, 1]\} \subset \{\mathbb{R}, [0, 1]\}$, it follows that $\mathscr{R}\{\mathbb{R}, [0, 1]\} \subset \mathbb{T}\mathbb{P}\mathbb{D}\mathrm{isc}$. We show that the reverse inclusion also holds:
Let X be a totally path disconnected space and assume that $f : \mathbb{R} \to X$ is a continuous function. As $[a, b]$ is homeomorphic to $[0, 1]$ for each $a < b$ in $\mathbb{R}$, we obtain that every continuous function $g : [a, b] \to X$ is constant.
Let x, y be real numbers with $x < y$. Then, $[x, y]$ is an interval and since the restriction $f_{[x,y]} : [x, y] \to X$ is continuous, it follows that f is constant on $[x, y]$. Therefore $f(x) = f(y)$. Since x and y are arbitrary real numbers, it follows that f must be constant on $\mathbb{R}$, i.e., X is right-$\{\mathbb{R}\}$-constant.

(*f*) Now we show that $\mathscr{L}(\mathbb{C}\mathrm{on}) = \mathbb{I}\mathrm{nd}$. Since $\mathbb{S}$ is a connnected space, it follows that $\mathscr{L}(\mathbb{C}\mathrm{on}) \subset \mathscr{L}\{\mathbb{S}\} = \mathbb{I}\mathrm{nd}$. For the reverse inclusion, suppose that X is not indiscrete, then X contains at least two points and there is a non-empty proper open subset U of X. Then, the characteristic function χ_U into the Sierpinski space is continuous and not constant. Thus, X is also not left-$\mathbb{C}\mathrm{on}$-constant.

(*g*) Since $\mathbb{T}\mathbb{D}\mathrm{isc} = \mathscr{R}(\mathscr{L}(\{\mathbb{D}_2\}))$, it follows that $\mathscr{L}(\mathbb{T}\mathbb{D}\mathrm{isc}) = \mathscr{L}(\mathscr{R}(\mathscr{L}(\{\mathbb{D}_2\}))) = \mathscr{L}(\{\mathbb{D}_2\}) = \mathbb{C}\mathrm{on}$.

To see that $\mathscr{R}(\mathbb{T}\mathbb{D}\mathrm{isc}) = \mathbb{C}$, note that $\mathbb{D}_2 \in \mathbb{T}\mathbb{D}\mathrm{isc}$ so that $\mathscr{R}(\mathbb{T}\mathbb{D}\mathrm{isc}) \subset \mathscr{R}(\{\mathbb{D}_2\}) = \mathbb{C}$ and since $\mathbb{C}$ is always a subclass of any right constant subcategory, the result follows.

(h) For $\mathbb{Top}_0$, note that $\mathbb{S}$ is a T_0-space and hence $\mathscr{L}(\mathbb{Top}_0) \subset \mathscr{L}(\{\mathbb{S}\}) = \mathbb{Ind}$. Now, for any indiscrete space X and any T_0 space Y, if $f : X \to Y$ is a non-constant continuous function, then given $f(x) \neq f(x')$ for some x and x' in X, there is without loss of generality an open set U of Y that contains $f(x)$, but not $f(x')$. Then, $x \in f^{-1}[U]$ which is open in X, but $x' \notin f^{-1}[U]$. Hence $f^{-1}[U]$ is a non-empty proper open subset of X, but this contradicts the fact that X is indiscrete. Thus, f must have been constant and it follows that $\mathbb{Ind} = \mathscr{L}(\mathbb{Top}_0)$.

Since $\mathscr{R}(\mathbb{Top}_0) \subset \mathscr{R}(\{\mathbb{S}\}) = \mathbb{Top}_1 \subset \mathbb{Top}_0$, it follows that each space in $\mathscr{R}(\mathbb{Top}_0)$ is a member of both $\mathscr{R}(\mathbb{Top}_0) \cap \mathbb{Top}_0$. Proposition 3.40 gives that these are all members of $\mathbb{C}$. It follows that $\mathscr{R}(\mathbb{Top}_0) = \mathbb{C}$.

(i) We know that $\mathbb{Top}_1$ is actually a quotient reflective subcategory of $\mathbb{Top}$, so for each space X, there exists a $\mathbb{Top}_1$ reflection morphism $r_X : X \to X_1$. We assert $\mathscr{L}(\mathbb{Top}_1) = \mathbb{T}$, where

$$\mathbb{T} = \{X \in \mathbb{Top} \mid X_1 \in \mathbb{C}\}.$$

Since $\mathbb{Top}_1$ is closed under embeddings, it follows that

$$\mathscr{L}(\mathbb{Top}_1) = \{X \in \mathbb{Top} \mid \forall \text{ surjective } e : X \to Q \ (Q \in \mathbb{Top}_1 \Rightarrow Q \in \mathbb{C})\}.$$

Suppose X is a member of $\mathscr{L}(\mathbb{Top}_1)$ and consider the reflection morphism $r_X : X \to X_1$. Since X_1 is a T_1-space and r_X is a quotient map, it's surjective. Since X is left-$\mathbb{Top}_1$-constant, it follows that X_1 must be a member of $\mathbb{C}$, i.e., $X \in \mathbb{T}$.
Now, assume that X is a member of $\mathbb{T}$, then let $e : X \to Q$ be a surjection with Q a T_1-space. Then, there is a unique morphism $\bar{e}$ from X_1 to Q such that $\bar{e}r_X = e$. It's an easy exercise to show that e is surjective if and only if $\bar{e}$ is. Since X is in $\mathbb{T}$, $\bar{e}$ is surjective and consequently Q is a member of $\mathbb{C}$. Hence, $\mathscr{L}(\mathbb{Top}) = \mathbb{T}$.

Now, $\mathscr{R}(\mathbb{Top}_1) \subset \mathscr{R}(\{\mathbb{D}_2\}) = \mathbb{C}$ and thus $\mathscr{R}(\mathbb{Top}_1) = \mathbb{C}$.

(j) First we show that $\mathscr{R}(\mathbb{RCon})$ is the category $\mathbb{TRConss}$ of all spaces for which the only real constant subspaces are members of $\mathbb{C}$. If there is a real constant subspace Y of X where Y has more than two points, then the inclusion $Y \hookrightarrow X$ is a non-constant continuous function with domain in $\mathbb{RCon}$. It then follows that X is not a member of $\mathscr{R}(\mathbb{RCon})$. Thus if X is in $\mathscr{R}(\mathbb{RCon})$, then X is a member of $\mathbb{TRConss}$. On the other hand, if X is not right-$\mathbb{RCon}$-constant, then there is a non-constant continuous map g from a real constant space Y to X. Since $\mathbb{RCon}$ is closed under surjections, we may take a (surjective, embedding)-factorisation $me : Y \to M \to X$ of f and consequently have that m is a non-constant embedding. It's then clear that $m[M]$ is a real constant subspace of X with more than one point. Thus X is not a member of $\mathbb{TRConss}$.

Since every real constant space is connected, we have that $\mathbb{Ind} = \mathscr{L}(\mathbb{Con}) \subset \mathscr{L}(\mathbb{RCon})$. Suppose that X is not indiscrete. We show that X is not a member of $\mathscr{L}(\mathbb{RCon})$. If X is not indiscrete, then there is a non-empty proper open subset U of X. The characteristic function $\chi_U : X \to \mathbb{S}$ is continuous and non-constant. It should be clear that $\mathbb{S}$ is real constant as any continuous function to a T_2 space is constant. Thus, there is a continuous non-constant function from X to a real constant space. Therefore X is not a member of $\mathscr{L}(\mathbb{RCon})$.

(k) For $\mathbb{Ind}$, we have: $\mathscr{R}(\mathbb{Ind}) = \mathscr{R}(\mathscr{L}(\mathbb{Top}_0)) = \mathscr{R}(\mathscr{L}(\mathscr{R}(\{\mathbb{I}_2\}))) = \mathscr{R}(\{\mathbb{I}_2\}) = \mathbb{Top}_0$.

We assert that $\mathscr{L}(\mathbb{Ind}) = \mathbb{C}$. It's sufficient to prove the one inclusion for spaces. Let X be a topological space with at least two points, i.e., X is not in $\mathbb{C}$. Let A be any non-empty proper subset of X. Let Y be an indiscrete space. If Y has only one point, Y is already a member of $\mathbb{C}$ and any function is constant, so assume that Y contains at least two points, say y and y' are distinct in Y. Then, the function $f : X \to Y$, defined by $f(x) = y$ if $x \in A$ and $f(x) = y'$ otherwise, is a non-constant continuous function. Therefore, if X is not in $\mathbb{C}$, then X is not left-$\mathbb{Ind}$-constant. Our result follows.

Now, consider the category $\mathbb{Ab}$ of abelian groups and group homomorphisms with factorisation structure (surjective, injective) and constant subcategory consisting of all trivial groups, i.e., $\mathbb{C} = \{G \in \mathbb{Ab} \mid G \simeq \{e\}\}$.
For the subcategories $\mathbb{Tor}$ and $\mathbb{TfAb}$ of $\mathbb{Ab}$ of torsion groups and torsion free abelian groups respectively, we have: $\mathscr{R}(\mathbb{Tor}) = \mathbb{TfAb}$ and $\mathscr{L}(\mathbb{TfAb}) = \mathbb{Tor}$.
To see this, note that if G is right-$\mathbb{Tor}$-constant, then every morphism from a torsion group to G is constant. In particular, considering the torsion subgroup of G with the inclusion homomorphism, we see that G must be torsion free if G is a member of $\mathscr{R}(\mathbb{Tor})$.
On the other hand, if G is torsion free, then for any homomorphism $\varphi : H \to G$, where H is torsion, we know that $\varphi[H]$ is a subgroup of the torsion subgroup of G. As G is torsion free, it's torsion subgroup is trivial. Thus, φ is constant and consequently $\mathscr{R}(\mathbb{Tor}) = \mathbb{TfAb}$.

We already know that $\mathbb{Tor} \subset \mathscr{L}(\mathscr{R}(\mathbb{Tor})) = \mathscr{L}(\mathbb{TfAb})$, so it's sufficient to show the reverse inclusion. Let G be left-$\mathbb{TfAb}$-constant. Considering the torsion subgroup T of G and the fact that G/T is torsion free, we must have that the canonical morphism $G \to G/T$ is constant. Since this morphism is in particular surjective, it follows that G/T must be the trivial group or equivalently $T = G$. Since an abelian group is torsion if and only if it's equal to its torsion subgroup, we are done.
Summarising all of this, we obtain:

Object of $\mathbb{Top}$	$\mathscr{L}(-)$	$\mathscr{R}(-)$
$\mathbb{D}_2$	$\mathbb{Con}$	$\mathbb{C}$
$\mathbb{S}$	$\mathbb{Ind}$	$\mathbb{Top}_1$
$\mathbb{I}_2$	$\mathbb{C}$	$\mathbb{Top}_0$
$[0,1], \mathbb{R}$	$\mathbb{RCon}$	$\mathbb{TPDisc}$
Subcategory of $\mathbb{Top}$		
$\mathbb{C}$	$\mathbb{Top}$	$\mathbb{Top}$
$\mathbb{Top}$	$\mathbb{C}$	$\mathbb{C}$
$\mathbb{Con}$	$\mathbb{Ind}$	$\mathbb{TDisc}$
$\mathbb{TDisc}$	$\mathbb{Con}$	$\mathbb{C}$
$\mathbb{Top}_0$	$\mathbb{Ind}$	$\mathbb{C}$
$\mathbb{Top}_1$	$\mathbb{T}$	$\mathbb{C}$
$\mathbb{RCon}$	$\mathbb{Ind}$	$\mathbb{TRConss}$
$\mathbb{Ind}$	$\mathbb{C}$	$\mathbb{Top}_0$
Subcategory of $\mathbb{Ab}$		
$\mathbb{Tor}$	-	$\mathbb{TfAb}$
$\mathbb{TfAb}$	$\mathbb{Tor}$	-

3.3 Reflective constant subcategories

Remark 3.54: This section investigates some properties of reflective constant subcategories and how this influences the structure of left and right constant subcategories. When studying $\mathscr{E}$-reflective subcategories, it's useful if the category in question has a factorisation structure for sources. In particular, if $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then $\mathbb{A}$ is also $(\mathscr{E}, \mathscr{M})$-structured, where $\mathscr{M}$ is the class of all morphisms $m : X \to Y$ such that $(X, (m : X \to Y))$ is a source at X consisting of only one morphism. In order to simplify our discussion, whenever $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, we will always denote the class of all morphisms m for which (m) is a one-source member of $\mathbb{M}$ by $\mathscr{M}$. In particular, if $\mathbb{C}$ is a constant subcategory of the $(\mathscr{E}, \mathbb{M})$-category, then it's assumed that $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects and $\mathscr{E}$-images.

Lemma 3.55: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured, $\mathbb{C}$ a reflective subcategory of $\mathbb{A}$ and $\mathscr{E}$ a class of $\mathbb{A}$-epimorphisms. Then, the $\mathscr{M}$-closure $\mathscr{M}(\mathbb{C})$ of $\mathbb{C}$ is $\mathscr{E}$-reflective in $\mathbb{A}$.

Proof : Let X be in $\mathbb{A}$ and let $r_X : X \to RX$ denote the reflection of $\mathbb{A}$ into $\mathbb{C}$. Then, let $m_X e_X : X \to TX \to RX$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of r_X. We assert that e_X is the $\mathscr{E}$-reflection of $\mathbb{A}$ into $\mathscr{M}(\mathbb{C})$. It's clear that TX is a member of the $\mathscr{M}$-closure of $\mathbb{C}$ and e_X in $\mathscr{E}$. Let $f : X \to A$ be a morphism with A in the $\mathscr{M}$-closure of $\mathbb{C}$. By definition of $\mathscr{M}(\mathbb{C})$, there exists a morphism $m : A \to Y$ in $\mathscr{M}$ with Y in $\mathbb{C}$. Then, there is a unique morphism $g : RX \to Y$ such that $gr_X = mf$. In particular, the diagonalisation property gives us a morphism $d : TX \to A$ such that $md = gm_X$ and $de_X = f$. Since $\mathscr{E}$ is a class of epimorphisms, d is clearly unique and thus $\mathscr{M}(\mathbb{C})$ is $\mathscr{E}$-reflective in $\mathbb{A}$.

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dd, "f"'] & TX \arrow[d, "m_X"] \arrow[ddl, dashed, "!d"'] \\
 & RX \arrow[d, dashed, "!g"] \\
A \arrow[r, "m"'] & Y
\end{tikzcd}$$

$\square$

Proposition 3.56: Let $\mathbb{C}$ be a reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$ with $\mathbb{C}$ closed under $\mathscr{M}$-subobjects. Then, $\mathbb{C}$ is $\mathscr{E}$-reflective in $\mathbb{A}$.

Proof : Let X be in $\mathbb{A}$ and let $m_X e_X = r_X$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of the reflection morphism $r_X : X \to RX$. As $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects, it follows that $M := dom(m_X)$ is a member of $\mathbb{C}$. By reflectivity, there is a unique morphism $\bar{e}$ such that $\bar{e}r_X = e_X$. Then, $\bar{e}m_X\bar{e}r_X = \bar{e}m_X e_X = \bar{e}r_X = e_X$ and since $\bar{e}$ was unique with respect to this property, $\bar{e}m_X\bar{e} = \bar{e}$. Furthermore, $m_X\bar{e}r_X = m_X e_X = r_X = id_{RX}r_X$ so that $m_X\bar{e} = id_{RX}$ and hence $\bar{e}$ is a member of $\mathscr{M}$. This also implies that $\bar{e}m_X e_X = \bar{e}r_X = e_X$. Then, there exists a diagonal morphism $d : M \to RX$ such that $de_X = m_X e_X = r_X$ and $\bar{e}d = id_M$.

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "r_X"'] & M \arrow[d, "id_M"] \arrow[dl, dashed, "!d"'] \\
RX \arrow[r, "\bar{e}"'] & M
\end{tikzcd}$$

So $d = id_{RX}d = m_X\bar{e}d = id_M m_X = m_X$. It follows that $m_X = \bar{e}^{-1}$ and hence m_X is an isomorphism which implies that $m_X e_X = r_X$ is a member of $\mathscr{E}$. $\square$

Corollary 3.57: A constant subcategory $\mathbb{C}$ of a category $\mathbb{A}$ is reflective if and only if it's $\mathscr{E}$-reflective.

Theorem 3.58: ([19, 2.4]) Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. Also assume that $\mathbb{C}$ is an $\mathscr{E}$-reflective constant subcategory of $\mathbb{A}$ and $\mathbb{D}$ a right constant subcategory of $\mathbb{A}$. Then, $\mathbb{D}$ is $\mathscr{E}$-reflective in $\mathbb{A}$.

Proof : In view of [1, 16.18], it's sufficient to prove that $\mathbb{D}$ is closed under sources in $\mathbb{M}$. Since $\mathbb{D}$ is right constant, it follows that $\mathbb{D} = \mathscr{R}(\mathbb{Q})$ for some subcategory $\mathbb{Q}$ of $\mathbb{A}$ that is closed under $\mathscr{E}$-images. Let $(f_i : D \to D_i)_I$ be a source in $\mathbb{M}$ with D_i in $\mathbb{D}$ for each $i \in I$. In view of Proposition 3.46, it's sufficient to prove that for any Q in $\mathbb{Q}$ and any $n : Q \to D \in \mathscr{M}$ it follows that Q is a member of $\mathbb{C}$.

Since $\mathbb{C}$ is $\mathscr{E}$-reflective it's closed under sources in $\mathbb{M}$. So we need only find a source $(g_i : Q \to C_i)_I \in \mathbb{M}$ with C_i a member of $\mathbb{C}$ for each $i \in I$. Since $\mathbb{M}$ is closed under composition, it follows that $(f_i)_I \circ n$ is a member of $\mathbb{M}$. For each i in I, let $f_i n = f(n_i)e_i$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of $f_i n$. Then, $(f(n_i))_I(e_i)_I$ is a member of $\mathbb{M}$ and consequently the source $(e_i)_I$ is in $\mathbb{M}$.

$$\begin{tikzcd} Q \arrow[r, "n"] \arrow[dr, "e_i"'] & A \arrow[r, "f_i"] & D_i \\ & f(N_i) \arrow[ur, "f(n_i)"'] & \end{tikzcd}$$

Since $\mathbb{Q}$ is closed under $\mathscr{E}$-images, it follows that $f(N_i)$ is a member of $\mathbb{Q}$ for each $i \in I$. As D_i is right-$\mathbb{Q}$-constant for each $i \in I$, and $f(n_i)$ a member of $\mathscr{M}$, Proposition 3.46 implies that $f(N_i)$ is in $\mathbb{C}$. As $(e_i)_I$ is a member of $\mathbb{M}$ and $\mathbb{C}$ is $\mathscr{E}$-reflective, it follows that Q is a member of $\mathbb{C}$. Thus, D is right-$\mathbb{Q}$-constant, i.e., D is a member of $\mathbb{D}$. □

Definition 3.59: Closed under limits
Let $\mathbb{A}$ be a subcategory of $\mathbb{B}$ and $E : \mathbb{A} \hookrightarrow \mathbb{B}$ the inclusion functor. Then $\mathbb{A}$ is **closed under limits in $\mathbb{B}$** provided that for any diagram $D : \mathbb{I} \to \mathbb{A}$ such that $\mathcal{L} = (L \xrightarrow{\ell_i} D_i)_I$ a limit of ED in $\mathbb{B}$, we have $L \in \mathbb{A}$.

Corollary 3.60: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category, $\mathbb{C}$ an $\mathscr{E}$-reflective constant subcategory and let $\mathbb{D}$ be any right constant subcategory of $\mathbb{A}$. Then the following conditions hold:

(*i*) If $\mathbb{A}$ has products, then so does $\mathbb{D}$,

(*ii*) $\mathbb{D}$ is closed under $\mathscr{M}$-subobjects,

(*iii*) $\mathbb{D}$ is closed under limits in $\mathbb{A}$,

(*iv*) if $\mathbb{A}$ is (finitely) complete, then so is $\mathbb{D}$.

Proof : (*i*) This follows from the fact that $\mathbb{D}$ is closed under $\mathbb{M}$ sources and every extremal mono-source belongs to $\mathbb{M}$.
(*ii*) This follows directly from the fact that $\mathbb{D}$ is $\mathscr{E}$-reflective in $\mathbb{A}$.
It is then easily seen that (*ii*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*iv*). □

Corollary 3.61: Let $\mathbb{D}$ be a right constant subcategory of $\mathbb{A}$. Then, $\mathbb{D}$ is reflective if and only if $\mathbb{D}$ is $\mathscr{E}$-reflective.

Lemma 3.62: Let $\mathbb{A}$ be a category with constant subcategory $\mathbb{C}$. Then, $\mathscr{L}(\mathbb{A}) = \mathscr{R}(\mathbb{A}) = \mathbb{C}$ and $\mathscr{L}(\mathbb{C}) = \mathscr{R}(\mathbb{C}) = \mathbb{A}$.

Proof : Since constant subcategories are self dual, it's sufficient to prove that $\mathscr{L}(\mathbb{A}) = \mathbb{C}$ and $\mathscr{L}(\mathbb{C}) = \mathbb{A}$. Since all left constant subcategories contain $\mathbb{C}$ and are subcategories of $\mathbb{A}$, we need only show that $\mathscr{L}(\mathbb{A}) \subset \mathbb{C}$ and $\mathscr{L}(\mathbb{C}) \supset \mathbb{A}$.

Let A be an $\mathbb{A}$-object. Then, all morphisms $g : C \to A$ are constant. Hence A is left-$\mathbb{C}$-constant.
Let B be an object that is left-$\mathbb{A}$-constant. Then, id_B is a morphism that must be constant, but this means that A must be in $\mathbb{C}$. □

Corollary 3.63: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category. Then, $\mathbb{C}$ and $\mathbb{A}$ are both left and right constant subcategories of $\mathbb{A}$.

Corollary 3.64: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category and $\mathbb{B}$ an $\mathscr{E}$-reflective subcategory that is closed under $\mathscr{E}$-images. Then, there is a constant subcategory $\mathbb{C}$ such that $\mathbb{B}$ can be obtained as a right constant subcategory for some reflective constant subcategory $\mathbb{C}$.

Proof : Choose $\mathbb{C} = \mathbb{B}$, then by Lemma 3.62, the result follows. □

Corollary 3.65: ([19, 2.5]) Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. If all right constant subcategories are $\mathscr{E}$-reflective, then so is $\mathbb{C}$.

Remark 3.66: In particular, Corollary 3.63 shows that if the constant subcategory is not reflective, then a right constant subcategory need also not be reflective.

Example 3.67: Note that the argument in Corollary 3.64 does not work if $\mathbb{B}$ is not closed under $\mathscr{E}$-images. Consider the category $\mathbb{Top}$ and the quotient reflective subcategories $\mathbb{Haus}, \mathbb{Top}_1$ and $\mathbb{Top}_0$. Each of these subcategories is surjective reflective, but not closed under surjections. Consider $\mathbb{I}_2 := (2, \{\varnothing, 2\})$, i.e., the two-point indiscrete space. Note that $\mathbb{I}_2$ is not a member of any of these subcategories. Let $f : [0, 1] \to \mathbb{I}_2$ be defined by $f(x) = 0$ if $x = 0$ and $f(x) = 1$ otherwise. Then, f is the desired continuous surjection.
Nevertheless, as seen in Example 3.53, both $\mathbb{Top}_0$ and $\mathbb{Top}_1$ can be obtained as right constant subcategories.

3.4 Relationships between reflective subcategories and right constant subcategories

Remark 3.68: A natural question that arises is whether every $\mathscr{E}$-reflective subcategory can arise as a right constant subcategory for a fixed constant subcategory of $\mathbb{A}$. The answer to this question is negative. A reason for this is that it depends a lot on the constant subcategory $\mathbb{C}$ that is chosen. To see this, suppose $\mathbb{D}$ is a subcategory of $\mathbb{C}$, where both $\mathbb{C}$ and $\mathbb{D}$ are $\mathscr{E}$-reflective subcategories of $\mathbb{A}$. Then, as shown in Lemma 3.62, we have $\mathbb{A} = \mathscr{L}(\mathbb{C}) \subset \mathscr{L}(\mathbb{D})$, and hence $\mathscr{L}(\mathbb{D}) = \mathbb{A}$. It then follows that $\mathscr{Q}(\mathbb{D}) = \mathscr{R}(\mathbb{A}) = \mathbb{C}$. If the inclusion of $\mathbb{D}$ in $\mathbb{C}$ is proper, then we must have that $\mathscr{Q}(\mathbb{D})$ properly contains $\mathbb{D}$. Furthermore, for any subcategory $\mathbb{B}$ of $\mathbb{A}$, $\mathscr{Q}(\mathbb{B})$ is the smallest right constant subcategory that contains $\mathbb{B}$. For if $\mathbb{X}$ is a right constant subcategory that contains $\mathbb{B}$, then by isotonicity of $\mathscr{Q}$, it follows that $\mathscr{Q}(\mathbb{B}) \subset \mathscr{Q}(\mathbb{X}) = \mathbb{X}$.

It might therefore be more suitable to consider the smallest $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ (if it exists, of course). If a smallest one exists, then one could at least require that such a subcategory does not have too many objects up to isomorphism. In particular, one object up to isomorphism is could be fruitful. Another question that arises is that if $\mathbb{D}$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$, can we find a constant subcategory such that $\mathbb{D}$ is a right constant subcategory of $\mathbb{A}$? Of course the above argument dictates that the chosen constant subcategory must be a subcategory of $\mathbb{D}$. In case $\mathbb{D}$ is closed under $\mathscr{E}$-images, of course we can just put $\mathbb{D}$ equal to $\mathbb{C}$ to obtain this result, so in order to settle a more non-trivial question one might ask whether there is a strictly smaller such category $\mathbb{C}$. This motivates the following notions:

Definition 3.69: Isomorphic morphims in $\mathbb{A}/X$
Two morphisms $f : A \to X$ and $g : B \to X$ are said to be **(uniquely) isomorphic over** X if and only if there exists a (unique) isomorphism $h : A \to B$ such that $gh = f$.

Remark 3.70: If f and g are uniquely isomorphic morphisms over X, then f and g are isomorphic objects in the comma category of $\mathbb{A}$ over X. Note that the converse is not true as there might be more than one isomorphism h such that $gh = f$.

Definition 3.71: Reflective object
Let $\mathbb{A}$ be a category and let R be an $\mathbb{A}$-object. Then R is said to be a **reflective object** provided that for any $\mathbb{A}$-object A, $\mathbb{A}(A, R) \neq \varnothing$ and for any pair of morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} R$, there exists a unique $h : R \to R$ such that $hf = g$. Hence, R is a reflective object if $\mathbb{A}(A, R) \neq \varnothing$ for each A in $\mathbb{A}$ and each pair of morphisms $f, g : A \rightrightarrows R$ are uniquely isomorphic.
Additionally, if $\mathscr{E}$ is a class of morphisms and $\mathbb{A}(A, R)$ is a subset of $\mathscr{E}$ for each $\mathbb{A}$-object A, then R is said to be an $\mathscr{E}$-reflective object.

Remark 3.72: If R is a reflective object, then all morphisms $f : R \to R$ are isomorphisms. Note that for the pair of morphisms f, id_R, there is a morphism g such that $gf = id$. There is also a morphism h such that $hg = id$. Using the associativity of composition, it's then easy to see that $h = f$ and g is the inverse of f. Note that a terminal object is always a reflective object. One can view the notion of reflective object as a slight relaxation of a terminal object.

Proposition 3.73: Let R be a reflective object of $\mathbb{A}$. Then, the full subcategory $\mathbb{C}$ of $\mathbb{A}$ with object class $\{X \in \mathbb{A} \mid X \simeq R\}$ is reflective in $\mathbb{A}$.

Proof : Suppose that R is a reflective object of $\mathbb{A}$. Clearly any object isomorphic to R is also reflective. Let A be in $\mathbb{A}$. If $A \simeq R$, let $r_A = id_A$. If $A \not\simeq R$, choose any morphism $r_A : A \to R$. We claim that r_A is a reflection for A. If $r_A = id_A$, then this is trivial, so assume that $r_A \neq id_A$. Then, given any morphism $g : A \to R$, by reflectiveness of R, there is a unique morphism $h : R \to R$ such that $hr_A = g$. Thus $\mathbb{C}$ is reflective in $\mathbb{A}$. $\square$

Proposition 3.74: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and suppose that for each $\mathbb{A}$-object A, that $\mathbb{A}(A, R) \subset \mathscr{E}$ and $\mathbb{A}(R, R) \subset \mathrm{Iso}(\mathbb{A})$. Then, R is an $\mathscr{E}$-reflective object of $\mathbb{A}$ if and only if the full subcategory $\mathbb{C}$ with object class $\{X \in \mathbb{A} \mid X \simeq R\}$ is $\mathscr{E}$-reflective in $\mathbb{A}$.

Proof : If R is $\mathscr{E}$-reflective, then a reflection is constructed as in the proof of 3.73. Furthermore, since $\mathbb{A}(A,R)$ is a subset of $\mathscr{E}$ for each $\mathbb{A}$-object A, it follows that the reflection is a member of $\mathscr{E}$.
Conversely, assume that $\mathbb{C}$ is $\mathscr{E}$-reflective in $\mathbb{A}$. Then, for any B in $\mathbb{A}$ the following holds:
If $B \simeq R$, then there is an isomorphism from B to R. If $B \not\simeq R$, then let $e_B : B \to R_B$ be a reflection for B. As R_B is a member of $\mathbb{C}$, there is an isomorphism $h : R_B \to R$. Thus, he_B is a morphism in $\mathscr{E}$ from B to R, i.e., $\varnothing \neq \mathbb{A}(B,R) \subset \mathscr{E}$ for all $\mathbb{A}$-objects B.
Let $B \underset{g}{\overset{f}{\rightrightarrows}} R$ be a pair of morphisms. Then there are unique morphisms $\overline{g}$ and $\overline{h}$ such that $\overline{g}e_B = g$ and $\overline{f}e_B = f$. Since $\overline{f}$ and $\overline{g}$ are isomorphisms, we obviously have $e_B = \overline{f}^{-1}\overline{f}e_B = \overline{f}^{-1}f$ so that

$$\overline{g}\overline{f}^{-1}f = \overline{g}e_B = g.$$

We need only show that $\bar{g}\bar{f}^{-1}$ is the unique morphism k such that $kf = g$. Suppose $hf = g$ for some morphism h. Then, $hf = g = \bar{g}e_B = kf$ and since $f : B \to R$ is a member of $\mathbb{A}(B,R) \subset \mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$, it follows that f is an epimorphism and thus $h = k$. Therefore R is an $\mathscr{E}$-reflective object of $\mathbb{A}$. $\square$

Corollary 3.75: If $\mathbb{A}$ is an $(\mathscr{E},\mathbb{M})$-category and R is an $\mathscr{E}$-reflective object of $\mathbb{A}$, then the smallest $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains R is the full subcategory with object class $\{X \in \mathbb{A} \mid X \simeq R\}$.

Example 3.76: Consider the subcategory $\mathbb{A} := \mathbb{Top} \setminus \{(\varnothing, \mathcal{P}(\varnothing))\}$ of $\mathbb{Top}$ of all non-empty topological spaces and the subcategory $\mathbb{B} := \mathbb{Ind} \setminus \{(\varnothing, \mathcal{P}(\varnothing))\}$ of all non-empty indiscrete spaces of $\mathbb{A}$. Since $\mathbb{Top}$ is a (surjective, initial mono-source)-category, it is relatively easy to see that $\mathbb{A}$ is one as well. Furthermore, it's clear that $\mathbb{B}$ is surjective reflective, for if $(X,\mathcal{T})$ is a non-empty topological space, then $id_X : (X,\mathcal{T}) \to (X,\{\varnothing,X\})$ is the desired reflection morphism. If $f : (X,\mathcal{T}) \to (Y,\{Y,\varnothing\})$ is a continuous map, then $f : (X,\{\varnothing,X\}) \to (Y,\{\varnothing,Y\})$ is also continuous. Note, both f and id_X are continuous, since any map to an indiscrete space is continuous. We also assert that $\mathbb{B}$ is closed under surjections. This is the case, since if $g : I \to Z$ is a surjective continuous map and I is indiscrete, then Z must also be indiscrete. Otherwise, there is an open set U of Z such that $\varnothing \neq U \neq Z$. Hence, there exist elements $z, z' \in Z$ such that $z \in U$ and $z' \notin U$. By surjectivity of g, there are distinct points x and x' of I such that $g(x) = z$ and $g(x') = z'$. Then, $x \in g^{-1}[U]$ and $x' \notin g^{-1}[U]$ and since $g^{-1}[U]$ is a proper non-empty open set of I, it follows that I can't be indiscrete. Thus, no such U exists and Z must be indiscrete. It's then clear that $\mathscr{Q}(\mathbb{B}) = \mathbb{B}$, if $\mathbb{B}$ is considered as the constant subcategory of $\mathbb{A}$.

Now, consider the full subcategory $\mathbb{C}$ of $\mathbb{A}$ given by the object class of all terminal objects in $\mathbb{A}$, i.e., all singleton spaces. It's clear that $\mathbb{C}$ is a surjective-reflective (actually it's even quotient-reflective) subcategory as a singleton space in $\mathbb{A}$ is not only a terminal object of $\mathbb{A}$, but actually a quotient map-reflective object in $\mathbb{A}$. Nevertheless, $\mathbb{B}$ is not a right constant subcategory with respect to $\mathbb{C}$. To see this, let $\mathscr{L}'$ and $\mathscr{R}'$ denote the usual hull operators with respect to the constant subcategory $\mathbb{C}$. If $\mathbb{B}$ was a right constant subcategory, then it would follow that $\mathscr{R}'(\mathscr{L}'(\mathbb{B})) = \mathbb{B}$. We prove that this is not the case. Once again, we note that any function from any space X to an indiscrete space is continuous. If X contains more than one point, we can easily define a non-constant function from X to the two point indiscrete space $\mathbb{I}_2$. Thus, if X is a member of $\mathscr{L}'(\mathbb{B})$, then X must contain only one point. Since the inclusion $\mathbb{C} \subset \mathscr{L}'(\mathbb{B})$ always holds, it follows that $\mathscr{L}'(\mathbb{B}) = \mathbb{C}$. Furthermore, by Lemma 3.62, it follows that $\mathscr{R}'(\mathscr{L}'(\mathbb{B})) = \mathscr{R}'(\mathbb{C}) = \mathbb{A}$ and thus $\mathbb{B}$ is not right constant with respect to $\mathbb{C} = \{X \mid X \simeq 1\}$.

Actually we can show a little more, namely: $\mathbb{B}$ is the **smallest $\mathscr{E}$-reflective constant subcategory** of $\mathbb{A}$ for which $\mathbb{B}$ is a right constant subcategory. Consider any constant subcategory $\hat{\mathbb{C}}$ of $\mathbb{A}$ that properly contains $\mathbb{C}$. Notice that we may assume that $\mathbb{C} \subset \hat{\mathbb{C}}$ since any map $1 \to X$ is an initial injective continuous map. Let $\hat{\mathscr{L}}$, $\hat{\mathscr{R}}$ and $\hat{\mathscr{Q}}$ denote the hull operators with respect to $\hat{\mathbb{C}}$. Since $\mathbb{Y}, \hat{\mathbb{C}} \subset \hat{\mathscr{Q}}(\mathbb{Y})$ for any subcategory $\mathbb{Y}$ of $\mathbb{A}$, we may assume that $\hat{\mathbb{C}} \subset \mathbb{B}$. Therefore $\hat{\mathbb{C}}$ consists only of indiscrete spaces and contains a space with at least 2 points. Let X be a space in $\hat{\mathbb{B}}$ that contains at least two points. Let x and x' be distinct in X, then since $\hat{\mathbb{C}}$ is closed under initial mono-sources, it follows that the subspace Z of X consisting of only x and x' is a member of $\hat{\mathbb{C}}$. It's clear that Z is homeomorphic to $\mathbb{I}_2$. We show that $\mathbb{B} \subset \hat{\mathbb{C}}$. Let I be any indiscrete space, then $|I| < |\mathcal{P}(I)| = |Z|^{|I|}$. Since $\hat{\mathbb{C}}$ is surjective-reflective, it's closed under products and subspaces, hence there is an initial monomorphism from I to Z^I, i.e., I is a subspace of a space in $\hat{\mathbb{C}}$. Therefore I is a member of $\hat{\mathbb{C}}$ and our result follows.

This example also serves to show that if a right constant subcategory other than the whole or constant subcategory is $\mathscr{E}$-reflective, then the constant subcategory need not be $\mathscr{E}$-reflective. Let $\overline{\mathbb{C}}$ be any constant subcategory of $\mathbb{A}$ that properly contains $\mathbb{C}$ and is also properly contained in $\mathbb{B}$. Let $\overline{\mathscr{Q}}$ denote the hull operator with respect to $\overline{\mathbb{C}}$. We show that $\overline{\mathscr{Q}}(\mathbb{B}) \subset \mathbb{B}$. Since $\overline{\mathbb{C}}$ is not the whole of $\mathbb{B}$ and $\overline{\mathbb{C}}$ must be closed under the formation of subspaces and continuous images, it follows that there must be a cardinal α such that all spaces in $\overline{\mathbb{C}}$ consists only of those indiscrete spaces with cardinality strictly smaller, or smaller than or equal to, α. Let us denote by $\mathbb{C}_{(\alpha,\leq)}$ and $\mathbb{C}_{(\alpha,<)}$ these categories, i.e., $\mathbb{C}_{(\alpha,\leq)}$ and $\mathbb{C}_{(\alpha,<)}$ consists of all indiscrete spaces C where the underlying set has cardinality smaller than or equal to α, or strictly smaller than α, respectively. Let $\mathbb{U}_{(\alpha,\geq)}$ and $\mathbb{U}_{(\alpha,>)}$ denote the full subcategories of $\mathbb{A}$ of all topological spaces with cardinality greater than or equal to α, respectively, strictly greater than the cardinal α. Let $\mathbb{V}_{(\alpha,\leq)}$ and $\mathbb{V}_{(\alpha,<)}$ denote the full subcategories of $\mathbb{A}$ that consists of all topological spaces in $\mathbb{A}$ with underlying set with cardinality at most that of α, and strictly smaller than α, respectively. It's clear that $\mathbb{U}_{(\alpha,\geq)}$ and $\mathbb{V}_{(\alpha,<)}$ have no objects in common and their union contain all spaces in $\mathbb{A}$. Similarly for $\mathbb{U}_{(\alpha,>)}$ and $\mathbb{V}_{(\alpha,\leq)}$.

Let X be a topological space that is not in $\mathbb{V}_{(\alpha,\leq)}$. The underlying set of X has cardinality greater than α. Then, $id_X : X \to (|X|, \{\varnothing, |X|\})$ is a continuous function into an indiscrete space. Since the image of X under id_X has the same cardinality as the underlying set, id_X is neither $\mathbb{C}_{(\alpha,\leq)}$- nor $\mathbb{C}_{(\alpha,<)}$-constant. Similarly, if X is a space with cardinality α, then $id_X : X \to (|X|, \{\varnothing, |X|\})$ will also be a non-$\mathbb{C}_{(\alpha,<)}$-constant continuous function. This shows that if X is not a member of $\mathbb{V}_{(\alpha,\leq)}$, respectively $\mathbb{V}_{(\alpha,<)}$, then X is not left-$\mathbb{B}$-constant with respect to the constant subcategories $\mathbb{C}_{(\alpha,\leq)}$ and $\mathbb{C}_{(\alpha,<)}$ respectively.

Now, suppose that X is a member of $\mathbb{V}_{(\alpha,<)}$. Let I be an indiscrete space and $f : X \to I$ be continuous. It's clear that f factorises as $X \xrightarrow{e} f(X) \overset{m}{\hookrightarrow} I$ where e is the restriction of the codomain of f to $f[X]$ and m the inclusion map. As I is indiscrete, so is $f[X]$ and furthermore, $|f[X]| \leq |X| < \alpha$, hence f is $\mathbb{C}_{(\alpha,<)}$-constant. By replacing the space X with a space of cardinality α and the last inequality by equality, the same argument also shows that if X is a member of $\mathbb{V}_{(\alpha,\leq)}$, then any f is $\mathbb{C}_{(\alpha,\leq)}$-constant.

Combining the two preceding paragraphs, we obtain the following:

$$\text{for } \mathbb{C}_{(\alpha,\leq)}\text{: } \mathscr{L}(\mathbb{B}) = \mathbb{V}_{(\alpha,\leq)};$$
$$\text{for } \mathbb{C}_{(\alpha,<)}\text{: } \mathscr{L}(\mathbb{B}) = \mathbb{V}_{(\alpha,<)}.$$

Since $\overline{\mathbb{C}}$ is either $\mathbb{C}_{(\alpha,\leq)}$ or $\mathbb{C}_{(\alpha,<)}$ for some α and $\mathbb{C}$ contains a space with at least two points, we may assume that $\alpha \geq 2$ or $\alpha > 2$ respectively.

We need to show that $\overline{\mathscr{Q}}(\mathbb{B}) \subset \mathbb{B}$. By propositions 3.42 and 3.46, it follows that

$$\overline{\mathscr{Q}}(\mathbb{B}) = \begin{cases} \{X \in \mathbb{A} \mid \forall m : M \to X \in \mathscr{M}\ (M \in \mathbb{V}_{(\alpha,\leq)} \Rightarrow M \in \mathbb{C}_{(\alpha,\leq)})\} & \text{if } \overline{\mathbb{C}} = \mathbb{C}_{(\alpha,\leq)} \\ \{X \in \mathbb{A} \mid \forall m : M \to X \in \mathscr{M}\ (M \in \mathbb{V}_{(\alpha,<)} \Rightarrow M \in \mathbb{C}_{(\alpha,<)})\} & \text{if } \overline{\mathbb{C}} = \mathbb{C}_{(\alpha,<)} \end{cases}$$

We assert that each of these classes is exactly the class of all spaces in $\mathbb{B}$. Let X be a member of $\overline{\mathscr{Q}}(\mathbb{B})$, then every subspace M of X such that $|M| \leq \alpha$, respectively $|M| < \alpha$, must be indiscrete. As we may assume that all two point spaces are in $\overline{\mathbb{C}}$, it's sufficient to prove that a space is indiscrete if and only if each 2 point subspace is also indiscrete. If X is indiscrete, then clearly each subspace is indiscrete. So, let X be a space that is not indiscrete, then there is a non-empty proper open subset U of X. Let $x \in U$ and $x' \in X \setminus U$. Consider the two point subspace $M = \{x, x'\}$ of X. It can't be indiscrete, for $\{x\} = M \cap U$. Thus, X can't be a member of $\overline{\mathscr{Q}}(\mathbb{B})$. It follows that $\mathbb{B}$ is a right constant subcategory for any of the discussed constant subcategories.
To summarise the results of this example, $\mathbb{B}$ is right constant, provided that $\mathbb{B}$ is chosen as the constant subcategory. In fact, it's the only reflective constant subcategory of $\mathbb{A}$ that will make it into a right constant subcategory. However, there exists a proper class of constant subcategories of $\mathbb{A}$ that are not reflective in $\mathbb{A}$, but for which $\mathbb{B}$ is a right constant subcategory.

Remark 3.77: Up to now, we have always assumed the existence of a constant subcategory. However, there is always one trivial example of a constant subcategory namely the whole category itself. One must also notice that considering the category itself as the constant subcategory will be fairly useless, because every morphism will be constant and the only right and left constant subcategories will be the whole category itself. We will now delve into an internal construction of the smallest constant subcategory that contains some subcategory.

Definition 3.78: Alternating $(\mathscr{E},\mathscr{M})$-sequences and -cosequences
Let $\mathbb{A}$ be a category and $\mathscr{E}$ and $\mathscr{M}$ classes of $\mathbb{A}$-morphisms. Then, **an alternating $(\mathscr{E},\mathscr{M})$-sequence at X to Y** is a finite sequence of $\mathbb{A}$-morphisms $(f_1, f_2, \ldots, f_n)$ for some $n \geq 1$, subject to the following conditions:

(i) f_i is in $\mathscr{E}$ if i is odd and f_i is in $\mathscr{M}$ if i is even,
(ii) $cod(f_i) = cod(f_{i+1})$ if i is odd and $i < n$,
(iii) $dom(f_{i+1}) = dom(f_i)$ if i is even and $i < n$,
(iv) $dom(f_1) = X$,
(v) $Y = \begin{cases} dom(f_n) & \text{if } n \text{ is even} \\ cod(f_n) & \text{otherwise} \end{cases}$

An **alternating $(\mathscr{E},\mathscr{M})$-cosequence at X to Y** is a finite sequence of $\mathbb{A}$-morphisms $(f_1, f_2, \ldots, f_n)$ for some $n \geq 1$, subject to the following conditions:

(i) f_i is in $\mathscr{E}$ if i is even and f_i is in $\mathscr{M}$ if i is odd,
(ii) $dom(f_i) = dom(f_{i+1})$ if i is odd and $i < n$,
(iii) $cod(f_{i+1}) = cod(f_i)$ if i is even and $i < n$,
(iv) $cod(f_1) = X$,
(v) $Y = \begin{cases} cod(f_n) & \text{if } n \text{ is even} \\ dom(f_n) & \text{otherwise} \end{cases}$

In either case, n is called **the length of the (co)sequence**.

Remark 3.79: It's of utmost importance to consider these sequences and cosequences when $(\mathscr{E},\mathscr{M})$ is a factorisation structure for morphisms on $\mathbb{A}$. If $\mathscr{M}$ contains all isomorphisms, then each alternating $(\mathscr{E},\mathscr{M})$-sequence $(f_1, f_2, \ldots, f_n)$ at X to Y can be viewed as an alternating $(\mathscr{E},\mathscr{M})$-cosequence $(id_X, f_1, \ldots, f_n)$ at X to Y. Similarly, if $\mathscr{E}$ contains all isomorphisms, then each alternating $(\mathscr{E},\mathscr{M})$-cosequence $(g_1, \ldots, g_n)$ at A to B can be viewed as an alternating $(\mathscr{E},\mathscr{M})$-sequence $(id_A, g_1, \ldots, g_n)$ at A to B.

Proposition 3.80: Let $\mathbb{A}$ be $(\mathscr{E},\mathscr{M})$-structured. Then, there exists an alternating $(\mathscr{E},\mathscr{M})$-sequence at X to Y if and only if there is an alternating $(\mathscr{E},\mathscr{M})$-cosequence at X to Y.

Proof : Using the fact that $\mathscr{E}$ and $\mathscr{M}$ contain all isomorphisms, this follows from the argument in Remark 3.79. □

Proposition 3.81: Let $\mathbb{A}$ have a prefactorisation system $(\mathscr{E},\mathscr{M})$ and let $\mathbb{B}$ be a subcategory of $\mathbb{A}$. Then, the subcategory $\mathbb{C}$ that consists of all $\mathbb{A}$-objects C for which there exists an alternating $(\mathscr{E},\mathscr{M})$-sequence at a $\mathbb{B}$-object B to C is closed under $\mathscr{E}$-images and $\mathscr{M}$-subobjects and contains $\mathbb{B}$. Additionally, $\mathbb{C}$ is the smallest subcategory of $\mathbb{A}$ that contains $\mathbb{B}$ and that is closed under $\mathscr{M}$-subobjects and $\mathscr{E}$images.

Proof : Since $\mathscr{E}$ contains all isomorphisms, it follows that for any $\mathbb{B}$-object B, (id_B) is an alternating $(\mathscr{E},\mathscr{M})$-sequence at B to B. Thus, $\mathbb{B} \subset \mathbb{C}$. Let C be a member of $\mathbb{C}$, then there is an alternating $(\mathscr{E},\mathscr{M})$-sequence $(f_1, f_2, \ldots, f_n)$ at a $\mathbb{B}$-object B to C. Let $m : A \to C$ and $e : C \to X$ be members of $\mathscr{M}$ and $\mathscr{E}$ respectively. We prove that both A and X are members of $\mathbb{C}$. In case n is even, it follows that f_n must be a member of $\mathscr{M}$. Then, $(f_1, \ldots, f_n, id_C, m)$ is an alternating $(\mathscr{E},\mathscr{M})$-sequence at B to A and $(f_1, \ldots, f_n, e)$ is an alternating $(\mathscr{E},\mathscr{M})$-sequence from B to X. If n is odd, f_n is a member of $\mathscr{E}$. Then $(f_1, \ldots, f_n, m)$ and $(f_1, \ldots, f_n, id_C, e)$ are alternating $(\mathscr{E},\mathscr{M})$-sequences at B to A and X respectively. Therefore $\mathbb{C}$ satsifies first condition.

Let $\mathbb{C}'$ be any subcategory that contains $\mathbb{B}$ that is also closed under $\mathscr{E}$-images and $\mathscr{M}$-subobjects. In particular, for each C in $\mathbb{C}$ with $(g_1, g_2, \ldots, g_k)$ an alternating sequence from $B \in \mathbb{B}$ to C, we have the following:
$cod(g_1) = cod(g_2) \in \mathbb{C}'$ as g_1 is in $\mathscr{E}$ and $\mathbb{B} \subset \mathbb{C}'$. Then, since $\mathbb{C}'$ is closed under $\mathscr{M}$-subobjects, $dom(g_2) = dom(g_3)$ is also a member of $\mathbb{C}'$. Obviously after finitely many steps, we have for n even: $cod(g_{n-1}) = cod(g_n)$ is in $\mathbb{C}'$ and thus $C = dom(g_n)$ is in $\mathbb{C}$. If n is odd, then $dom(g_{n-1}) = dom(g_n)$ is a member of $\mathbb{C}'$. Thus, $\mathbb{C}'$ is closed under alternating $(\mathscr{E},\mathscr{M})$-sequences from any $\mathbb{B}$-object, from which it follows that $\mathbb{C} \subset \mathbb{C}'$. □

Corollary 3.82: If $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured and $\mathbb{B}$ a subcategory of $\mathbb{A}$, then the category $\mathbb{C}$ constructed as in 3.81 is the smallest constant subcategory of $\mathbb{A}$ that contains $\mathbb{B}$.

Proposition 3.83: Let R be an $\mathscr{E}$-reflective object of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. Then, $\mathbb{C} := \{X \in \mathbb{A} \mid X \simeq R\}$ is a reflective constant subcategory provided that $\mathscr{E}$ is a class of epimorphisms.
$\underline{\mathcal{Proof}}$: In view of proposition 3.81, it's sufficient to prove that $\mathbb{C}$ is closed under $\mathscr{E}$-images and $\mathscr{M}$-subobjects. So, let $m : M \to R$ and $e : R \to X$ be members of $\mathscr{M}$ and $\mathscr{E}$ respectively. Since $\mathbb{A}(M, R) \subset \mathscr{E}$, it follows that $m \in \mathscr{E} \cap \mathscr{M} = Iso(\mathbb{A})$, i.e., $M \simeq R$ or equivalently, $M \in \mathbb{C}$. Let $\varepsilon_X : X \to R$ be an $\mathscr{E}$-reflection for X. Then, $\varepsilon_X \circ e$ is a member of $\mathbb{A}(R, R)$ and by definition of reflective object, an isomorphism. Thus, there is a morphism $h : R \to R$ such that $h\varepsilon_X e = id_R$ and $\varepsilon_X eh = id_R$. Then, e is an epimorphic section, i.e., e is an isomorphism so that X is already in $\mathbb{C}$.

□

Lemma 3.84: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. Then, the intersection of a family of $\mathscr{E}$-reflective subcategories of $\mathbb{A}$ is also $\mathscr{E}$-reflective.

$\underline{\mathcal{Proof}}$: Let $(\mathbb{A}_i)_I$ be a family of $\mathscr{E}$-reflective subcategories of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$. Let $\mathbb{B}$ be the full subcategory of $\mathbb{A}$ that consists of the intersection of all objects in each $\mathbb{A}_i$. Then, $\mathbb{B}$ is $\mathscr{E}$-reflective, for if $(m_j : B \to B_j)_J$ is a source in $\mathbb{M}$ with each B_j in $\mathbb{B}$, then, by $\mathscr{E}$-reflectiveness of $\mathbb{A}_i$, we must have that $\mathbb{A}_i$ is closed under sources in $\mathbb{M}$. Hence, B must be in $\mathbb{A}_i$ for each $i \in I$. Therefore, B must also be in the intersection, i.e., $B \in \mathbb{B}$. Thus $\mathbb{B}$ is closed under sources in $\mathbb{M}$ or equivalently, $\mathbb{B}$ is $\mathscr{E}$-reflective in $\mathbb{A}$.

□

Proposition 3.85: An intersection of constant subcategories is a constant subcategory.

$\underline{\mathcal{Proof}}$: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured and $(\mathbb{C}_i)_I$ a family of constant subcategories. Then, the full subcategory $\mathbb{C}$ with object class $\bigcap_I Ob(\mathbb{C}_i)$ is a constant subcategory. For if $m : M \to C$ is a member of $\mathscr{M}$ and $e : C \to E$ a member of $\mathscr{E}$ with C in $\mathbb{C}$, then as each $\mathbb{C}_i$ is constant, it follows that M and E are members of $\mathbb{C}_i$ for each $i \in I$. Consequently $\mathbb{C}$ is a constant subcategory. □

Remark 3.86: There is an external procedure to obtain the smallest constant subcategory of an $(\mathscr{E}, \mathscr{M})$-structured category or the smallest $\mathscr{E}$-reflective constant subcategory of an $(\mathscr{E}, \mathbb{M})$-category that contains a subcategory $\mathbb{B}$, namely the intersection of all constant, respectively $\mathscr{E}$-reflective constant subcategories of $\mathbb{A}$ that contain $\mathbb{B}$. Due to propositions 3.84 and 3.85 it's easy to see that it will be the smallest in both cases.

Definition 3.87: $\mathscr{E}$-reflective hull
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{B}$ a subcategory of $\mathbb{A}$. Then, **the $\mathscr{E}$-reflective hull $\mathscr{E}_r(\mathbb{B})$ of $\mathbb{B}$ in** $\mathbb{A}$ is the full subcategory with object class given by the intersection of all $\mathscr{E}$-reflective subcategories of $\mathbb{A}$ that contain $\mathbb{B}$.

Definition 3.88: $\mathscr{E}$-Quotient objects
If $\mathscr{E}$ is a class of morphisms and $X/\mathscr{E}$ is the comma category of X over $\mathscr{E}$, i.e., **the category of all $\mathscr{E}$-quotients of** X, then we can define an order on $X/\mathscr{E}$ by defining $e : X \to Y \leq e' : X \to Y'$ if and only if there is a morphism $j : Y \to Y'$ such that $je = e'$.
The above order is a pre-order and as a pre-ordered class, it will sometimes be denoted by $quot(X)$. Most of the time, $\mathscr{E}$ will be a class of morphisms for which $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured or $\mathbb{A}$ being an $(\mathscr{E}, \mathbb{M})$-category.

Of course, if $\mathscr{E}$ is a class of epimorphisms, then the morphism j above is necessarily unique. Furthermore, if $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then $(X/\mathscr{E}, \leq)$ is a complete pre-ordered class. In particular, for every $\mathbb{A}$-object X, $X/\mathscr{E}$ has a maximum.

Proposition 3.89: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and for each X in $\mathbb{A}$, let $e_m : X \to X_m$ be the maximum of $X/\mathscr{E}$. Let $\mathbb{C}$ be the full subcategory of $\mathbb{A}$ consisting of all $\mathbb{A}$-objects A such that whenever $f \in A/\mathscr{E}$, then f is an isomorphism. Then $\mathbb{C}$ is a reflective constant of $\mathbb{A}$ with $\mathbb{C}$-reflection $e_m : X \to X_m$ for each $\mathbb{A}$-object X.

$\underline{\mathcal{Proof}}$: We first show that $\mathbb{C} = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} : e \in \mathrm{Iso}(\mathbb{A})\}$ is a constant subcategory of $\mathbb{A}$. We first have to show that $\mathbb{C}$ is non-empty. Let X be in $\mathbb{A}$ and consider the maximum element $e_m : X \to X_m$ in $X/\mathscr{E}$. We claim that X_m is a member of $\mathbb{C}$. Let $e : X_m \to Y$ be in $\mathscr{E}$. Then, ee_m is a member of $X/\mathscr{E}$, so that by definition of e_m, there is a morphism $e' : Y \to X_m$ such that $e'ee_m = e_m$. Since $\mathscr{E}$ is a class of epimorphisms, it follows that $e'e = id$ and hence e is an epic section. Therefore e is an isomorphism and X_m is a member of $\mathbb{C}$. Therefore $\mathbb{C}$ is non-empty.

We now proceed to show that $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects and $\mathscr{E}$-images. Let A be in $\mathbb{C}$ and let $m : B \to A$ be a member of $\mathscr{M}$. Let $e : B \to B'$ be in $\mathscr{E}$. Then, consider the pushout square

$$\begin{tikzcd}
B \arrow[r, "m"] \arrow[d, "e"'] & A \arrow[d, "m_-(e)"] \\
B' \arrow[r, "n"'] & m_{(}A)
\end{tikzcd} .$$

Since $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, $\mathbb{A}$ is $\mathscr{E}$-cocomplete, so that $\mathbb{A}$ has $\mathscr{E}$-pushouts. Therefore, $m_-(e)$ is a member of $\mathscr{E}$ and since A is in $\mathbb{C}$, $m_-(e)$ is an isomorphism. Therefore, $ne = m_-(e)m$ is a member of $\mathscr{M}$ so that e is a member of $\mathscr{M}$ and $\mathscr{E}$. Therefore, e is an isomorphism and thus B is a member of $\mathbb{C}$.

If C is a member of $\mathbb{C}$ and $\bar{e} : C \to C'$ is in $\mathscr{E}$, then, of course, $\bar{e}$ is an isomorphism. Given any morphism $\hat{e} : C' \to D$ in $\mathscr{E}$, $\hat{e}\bar{e}$ is in $C/\mathscr{E}$ and by the assumption on C is, of course, an isomorphism. It follows that $\hat{e}$ is an isomorphism and thus C' is also a member of $\mathbb{C}$.

We now show that $\mathbb{C}$ is reflective. Let $f : X \to Z$ be any morphism with Z in $\mathbb{C}$. Let $me' : X \to M \to Z$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of f. By virtue of the element e_m, there is a morphism $j : M \to X_m$ such that $je' = e_m$. Since e_m and e' are in $\mathscr{E}$, we have that j is a member of $\mathscr{E}$. Since $\mathbb{C}$ is closed under $\mathscr{M}$-subobjects, M is a member of $\mathbb{C}$ as well. Since j is in $\mathscr{E}$ and M is in $\mathbb{C}$, we have that j is an isomorphism. Then $k := mj^{-1}$ is a morphism such that $ke_m = mj^{-1}e_m = mj^{-1}je' = me' = f$. Uniqueness of k follows since $\mathscr{E}$ is a class of epimorphisms. It follows that $\mathbb{C}$ is a reflective constant subcategory of $\mathbb{A}$.

$$\begin{tikzcd}
& X \arrow[dl, "e'"'] \arrow[ddr, "e_m"] & \\
M \arrow[d, "m"'] \arrow[drr, "j"] & & \\
Z & & X_m \arrow[ll, "k"]
\end{tikzcd}$$

□

Corollary 3.90: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E}$ a class of epimorphisms such that $X/\mathscr{E}$ has a maximum for each $\mathbb{A}$-object X. Then, $\mathbb{C} = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} : e \in \mathrm{Iso}(\mathbb{A})\}$ is a reflective constant subcategory of $\mathbb{A}$.

Proof : This should be evident from the proof of Proposition 3.89. □

Proposition 3.91: Let $\mathbb{A}$ be either an $(\mathscr{E}, \mathbb{M})$-category or a category that satisfies the assumptions of 3.90. Then, $\mathbb{C} = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} : e \in \mathrm{Iso}(\mathbb{A})\}$ is the smallest reflective constant subcategory of $\mathbb{A}$.

Proof : Let $\mathbb{C}$ be as above and let $\mathbb{C}'$ be any reflective constant subcategory of $\mathbb{A}$. For each C in $\mathbb{C}$, let $e_C : C \to C'$ be the $\mathbb{C}'$-reflection. Then e_C must be an isomorphism by definition of $\mathbb{C}$. Therefore C is in $\mathbb{C}'$. □

Example 3.92: Consider the category of topological spaces with factorisation structures $\mathscr{F}_1$:=(Epi, Initial mono-source), $\mathscr{F}_2$:=(Regular epi, mono-source), $\mathscr{F}_3$:=(bijective, initial sources) and $\mathscr{F}_4$:=(final, bijective).

Also consider the dual category of the category of topological spaces with factorisation structure $\mathscr{F}_5$:=(final sink, bijective). Explicitly, this is $\mathbb{Top}^{op}$ with factorisation structure (bijective, final source).

(a) For $\mathscr{F}_1$ and $\mathscr{F}_2$, the smallest reflective constant subcategory of $\mathbb{Top}$ is given by

$$\mathbb{C} = \{(X, \mathcal{T}) \mid |X| \leq 1\};$$

(b) For $\mathscr{F}_3$, the smallest reflective constant subcategory of $\mathbb{Top}$ is given by

$$\mathbb{C} = \{(X, \mathcal{T}) \mid \mathcal{T} = \{\varnothing, X\}\};$$

(c) For $\mathscr{F}_4$, the smallest constant subcategory of $\mathbb{Top}$ is $\mathbb{Disc} \setminus \{\varnothing\}$. Explicitly, it is the subcategory of all non-empty discrete spaces. Since the class of final continuous maps is not a class of epimorphisms, this case is more complicated. We will show that every non-empty discrete topological space is a member of the constant subcategory. Let $\mathbb{C}$ be any constant subcategory, explicitly, $\mathbb{C}$ is a non-empty subcategory of $\mathbb{A}$ that is closed under final images and bijective subobjects. Since $\mathbb{C}$ is not empty, there is a space $(X, \mathcal{T})$ in $\mathbb{C}$. Then, the unique morphism $(X, \mathcal{T}) \to (1, \mathcal{P}(1))$ is a final morphism so that any one point space is in $\mathbb{C}$. Further, given any non-empty space $(Y, \mathcal{S})$ and continuous map $f : (1, \mathcal{P}(1)) \to (Y, \mathcal{S})$, f is final if and only if $\mathcal{S} = \mathcal{P}(Y)$. Therefore, every non-empty discrete space must be a member of $\mathbb{C}$. It's easy to verify that this subcategory is constant, but since the class of final morphisms is not a class of epimorphisms, we should not expect it to be finally-reflective.

In fact, every reflective subcategory is closed under products and this is not the case for the subcategory of non-empty discrete spaces as it's not closed under (infinite) products. It follows that this is a constant subcategory which is not reflective.

(d) For $\mathscr{F}_5$, the smallest bijective-reflective constant subcategory of $\mathbb{Top}^{op}$ is given by

$$\mathbb{C} = \{(X, \mathcal{T}) \mid \mathcal{T} = \mathcal{P}(X)\}.$$

Definition 3.93: $\mathbb{M}$-closure
Let $\mathbb{A}$ be a category and $\mathbb{M}$ be a conglomerate of sources in $\mathbb{A}$. Then, for any subcategory $\mathbb{B}$ of $\mathbb{A}$, **the $\mathbb{M}$-closure of $\mathbb{B}$ in $\mathbb{A}$** is the full subcategory $\mathbb{M}(\mathbb{B})$ of $\mathbb{A}$ with object class

$$\{A \in \mathbb{A} \mid \exists (m_i : A \to B_i)_I \in \mathbb{M} \text{ with } B_i \text{ in } \mathbb{B} \text{ for each } i \in I\}.$$

Lemma 3.94: ([1, 16.22]) Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{B}$ be a subcategory of $\mathbb{A}$. For each $\mathbb{A}$-object A, let $\mathcal{S}(A, \mathbb{B})$ denote the all-source from $\mathbb{A}$ to $\mathbb{B}$. Then,

$$\mathscr{E}_r(\mathbb{B}) = \{A \in \mathbb{A} \mid \mathcal{S}(A, \mathbb{B}) \in \mathbb{M}\}.$$

$\underline{\mathcal{P}roof:}$ Let $\overline{\mathbb{B}} = \{A \in \mathbb{A} \mid \mathcal{S}(A, \mathbb{B}) \in \mathbb{M}\}$. Since $\mathbb{M}$ is closed under composition and a source belongs to $\mathbb{M}$ if and only if there is a subsource that belongs to $\mathbb{M}$, it's easy to see that $\overline{\mathbb{B}}$ coincides with the $\mathbb{M}$-closure of $\mathbb{B}$ in $\mathbb{A}$. First we show that $\overline{\mathbb{B}}$ is $\mathscr{E}$-reflective. It's sufficient to prove that it's closed under sources in $\mathbb{M}$. Let $\mathcal{S} = (m_i : A \to A_i)_I$ be an $\mathbb{M}$-source such that A_i is in $\overline{\mathbb{B}}$ for each $i \in I$. Then, since $\mathbb{M}$ is closed under composition and closed under the formation of super-sources, it follows that the source $(\mathcal{S}(A_i, \mathbb{B}))_I \circ \mathcal{S}$ is a member of $\mathbb{M}$. Also, notice that $(\mathcal{S}(A_i, \mathbb{B}))_I \circ \mathcal{S}$ is a subsource of $\mathcal{S}(A, \mathbb{B})$. It follows that $\mathcal{S}(A, \mathbb{B})$ is a member of $\mathbb{M}$ and therefore A is in $\overline{\mathbb{B}}$. Therefore $\overline{\mathbb{B}}$ is $\mathscr{E}$-reflective and since $\mathbb{M}$ contain all isomorphisms, we must have that $\mathbb{B}$ is a subcategory of $\overline{\mathbb{B}}$. By definition of $\mathscr{E}_r(\mathbb{B})$, $\overline{\mathbb{B}}$ is one of the categories in the family for which we must take the intersection to obtain $\mathscr{E}_r(\mathbb{B})$. Thus, $\mathscr{E}_r(\mathbb{B})$ is also a subcategory of $\overline{\mathbb{B}}$.

To prove the reverse inclusion, let A be a member of $\overline{\mathbb{B}}$, i.e., $\mathcal{S}(A, \mathbb{B})$ is a source in $\mathbb{M}$. Since $\mathbb{B}$ is contained in its $\mathscr{E}$-reflective hull and the $\mathscr{E}$-reflective hull is closed under sources in $\mathbb{M}$, it also follows that A must be in the $\mathscr{E}$-reflective hull. Therefore the $\mathbb{M}$-closure and the $\mathscr{E}$-reflective hull of $\mathbb{B}$ in $\mathbb{A}$ coincide.

$\square$

Definition 3.95: Strong ($\mathscr{E}$-)reflective object
Let $\mathbb{A}$ be a category, $\mathscr{E}$ be a class of morphisms of $\mathbb{A}$ and R an ($\mathscr{E}$)-reflective object of $\mathbb{A}$. Then, R is said to be a **strong ($\mathscr{E}$-)reflective object of** $\mathbb{A}$ provided that for each A object $\mathbb{A}$, there exists a morphism $f : R \to A$.

Proposition 3.96: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{P}$ a subcategory of $\mathbb{A}$ and $\mathbb{C}$ be an $\mathscr{E}$-reflective constant subcategory. Then, $\mathscr{L}(\mathbb{P}) = \mathscr{L}(\mathbb{M}(\mathbb{P}))$.

Proof : Let $\mathbb{M}(\mathbb{P})$ and $\mathscr{M}(\mathbb{P})$ denote the $\mathbb{M}$-closure and $\mathscr{M}$-closure of $\mathbb{P}$ in $\mathbb{A}$ respectively. It's clear that $\mathbb{P} \subset \mathscr{M}(\mathbb{P}) \subset \mathbb{M}(\mathbb{P})$. Since $\mathscr{L}$ is order reversing, it follows that $\mathscr{L}(\mathbb{M}(\mathbb{P})) \subset \mathscr{L}(\mathscr{M}(\mathbb{P})) \subset \mathscr{L}(\mathbb{P})$. By Proposition 3.38 we have $\mathscr{L}(\mathbb{P}) = \mathscr{L}(\mathscr{M}(\mathbb{P}))$ and since $\mathscr{M}(\mathbb{P})$ is closed under $\mathscr{M}$-subobjects, we have that

$$\mathscr{L}(\mathscr{M}(\mathbb{P})) = \{X \in \mathbb{A} \mid \forall f : X \to M \ (M \in \mathscr{M}(\mathbb{P}) \Rightarrow f \text{ is constant})\}.$$

Note that the $\mathbb{M}$-closure of $\mathbb{P}$ is also closed under $\mathscr{M}$-subobjects and thus

$$\mathscr{L}(\mathbb{M}(\mathbb{P})) = \{X \in \mathbb{A} \mid \forall e : X \to A \in \mathscr{E} \ (A \in \mathbb{M}(\mathbb{P}) \Rightarrow A \in \mathbb{C})\}.$$

Let X be in $\mathscr{L}(\mathbb{P})$ and suppose that A is a member of $\mathbb{M}(\mathbb{P})$ with $e : X \to A$ in $\mathscr{E}$. We know that there is a source $(m_i : A \to P_i)_I$ where each P_i is in $\mathbb{P}$. For each $i \in I$, let $n_i\bar{e}_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $m_i e$ with $C_i = dom(m_i)$.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\bar{e}_i"'] & A \arrow[d, "m_i"] \arrow[dl, dashed, "!d_i"'] \\ C_i \arrow[r, "n_i"'] & P_i \end{tikzcd}$$

As X is left-$\mathbb{P}$-constant, it follows that C_i is a member of $\mathbb{C}$. Now, for each $i \in I$, there is a diagonal morphism $d_i : A \to C_i$ such that $n_i d_i = m_i$ and $d_i e_i = \bar{e}_i$. Since $\bar{e_i} = d_i e \in \mathscr{E}$ and e_i is in $\mathscr{E}$, it follows that each d_i is in $\mathscr{E}$. Furthermore, $(n_i)_I \circ (d_i)_I = (m_i)_I$ is in $\mathbb{M}$ so that $(d_i)_I$ is a source in $\mathbb{M}$. But then, as $\mathbb{C}$ is $\mathscr{E}$-reflective, it's closed under sources in $\mathbb{M}$, hence A is a member of $\mathbb{C}$. That is X is left-$\mathbb{M}(\mathbb{P})$-constant and we are done. $\square$

Lemma 3.97: Let $\mathbb{C}$ be an $\mathscr{E}$-reflective constant subcategory of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$. Let $\mathbb{P}$ be any subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. Let R and C denote the reflector functors into $\mathbb{M}(\mathbb{P})$ and $\mathbb{C}$ respectively. For each $\mathbb{A}$-object X, let $r_X : X \to RX$ and $c_X : X \to CX$ denote the $\mathbb{M}(\mathbb{P})$ and $\mathbb{C}$-reflection morphisms respectively. Then,

$$\mathscr{L}(\mathbb{P}) = \{X \in \mathbb{A} \mid RX \in \mathbb{C}\}.$$

Proof : By Proposition 3.96, it follows that

$$\mathscr{L}(\mathbb{P}) = \{X \in \mathbb{A} \mid \forall e : X \to A \in \mathscr{E} \ (A \in \mathbb{M}(\mathbb{P}) \Rightarrow A \in \mathbb{C})\}.$$

Let

$$\mathbb{B} = \{X \in \mathbb{A} \mid RX \in \mathbb{C}\}.$$

Let X be left-$\mathbb{P}$-constant. Let us denote the all-source from RX to $\mathbb{P}$ by $\mathcal{S} = (m_i : RX \to P_i)_I$. Since left-$\mathbb{P}$-constant subcategories are closed under $\mathscr{E}$-images, it also follows that RX must be left-$\mathbb{P}$-constant. Hence, each m_i is constant. Furthermore, if $k_i g_i$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of $m_i r_X$ for each $i \in I$, then $K_i := dom(k_i)$ must be a member of $\mathbb{C}$. Then, for each $i \in I$, there is a diagonal morphism $d_i : RX \to K_i$ such that $d_i r_X = g_i$ and $k_i d_i = m_i$. Then, $(k_i)_I \circ (d_i)_I = (m_i)_I$ is a member of $\mathbb{M}$, which implies that $(d_i)_I$ is a source in $\mathbb{M}$ with codomain in $\mathbb{C}$. By $\mathscr{E}$-reflectivity of $\mathbb{C}$, we must have that RX is a member of $\mathbb{C}$, hence X is in $\mathbb{B}$.

Let X be a member of $\mathbb{B}$ and let $e : X \to P \in \mathscr{E}$ be a morphism with P in $\mathbb{M}(\mathbb{P})$. Then, there is a unique morphism $\overline{e} : RX \to P$ such that $\overline{e} r_X = e$. Then, since r_X is in $\mathscr{E}$ and e is in $\mathscr{E}$, it also follows that $\overline{e}$ is in $\mathscr{E}$. Since RX is in $\mathbb{C}$ and $\mathbb{C}$ is closed under images in $\mathscr{E}$, we must have that P is in $\mathbb{C}$, i.e.,

X is left-$\mathbb{M}(\mathbb{P})$-constant or equivalently by Proposition 3.96, X is left-$\mathbb{P}$-constant. $\square$

Remark 3.98: Left constant and right constant subcategories were investigated in [26]. There, the categories in question are ones with a terminal object and all subcategories $\mathbb{P}$ of $\mathbb{A}$ which are considered for $\mathscr{L}(\mathbb{P})$ are full, isomorphism closed and contains the terminal object 1. Furthermore, a morphism $f : X \to Y$ is constant if and only if the unique morphism $t_X : X \to 1$ is a strong epimorphism (i.e., $t_X \perp m$ for each $\mathbb{A}$-monomorphism m) and a factor of f. For the moment, let us assume that $\mathbb{A}$ is a (StrongEpi($\mathbb{A}$), Mono-source($\mathbb{A}$))-category with $\mathbb{C} = \{X \in \mathbb{A} \mid X \to 1 \in \text{Mono}(\mathbb{A})\}$.

We show that $\mathbb{C}$ is a reflective constant subcategory. Suppose that $(m_i : X \to X_i)_I$ is a mono-source with X_i a member of $\mathbb{C}$ for each $i \in I$. Since X_i is in $\mathbb{C}$, for each $i \in I$, the one-source $(n_i : X \to 1)$ is a mono-source. Therefore, the composition $\mathcal{S} := (((n_i))_I \circ (m_i)_I : X \to 1)_I$ is a mono-source. Since each morphism in the source $\mathcal{S}$ is equal to the unique morphism to 1, it follows that the unique morphism from X to 1 is a monomorphism. Thus, X is a member of $\mathbb{C}$ so that $\mathbb{C}$ is closed under mono-sources. To see that $\mathbb{C}$ is closed under strong epimorphisms, let $e : A \to B$ be a strong epimorphism with A in $\mathbb{C}$. Then, the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "id_A"'] & B \arrow[d] \arrow[dl, dashed, "!d"'] \\ A \arrow[r, "t_A"'] & 1 \end{tikzcd}$$

commutes for a unique d. Therefore, e is an epimorphic section and thus an isomorphism. Thus, $\mathbb{C}$ is closed under strong epimorphisms as well and thus a reflective constant subcategory. Furthermore, the reflection of A into $\mathbb{C}$ is given by the strong epimorphic part of the factorisation $\eta_X \varepsilon_X = t_X$ of the unique morphism $t_X : X \to 1$.

A morphism $f : X \to Y$ is now $\mathbb{C}$-constant if and only if its factorisation $me : X \to M \to Y$ is such that M is in $\mathbb{C}$. If f is constant in the sense of [26], then $t_X = \eta_X \varepsilon_X$ is a strong epimorphism and $f = nt_X$ for some morphism $n : 1 \to Y$. It follows that the diagonalisation property establishes a morphism such that

$$\begin{tikzcd} X \arrow[r, "t_X"] \arrow[d, "e"'] & 1 \arrow[d, "n"] \arrow[dl, dashed, "!d"'] \\ M \arrow[r, "m"'] & Y \end{tikzcd}$$

commutes. Since $dt_X = e$ and t_X are strong epimorphisms, so is d and thus M is in $\mathbb{C}$. Thus, any morphism constant in the sense of [26] is $\mathbb{C}$-constant.

On the other hand, if M is a member of $\mathbb{C}$, then $\eta_M \circ \varepsilon_M = t_M$ is a monomorphism, hence so is ε_M. Therefore, ε_M is an epimorphic section or equivalently an isomorphism. Then, f factors through ε_X, but there is no reason to suspect that f should factor through t_X. However, if 1 is an $\mathscr{E}$-reflective object, then $\mathbb{C}$ is given by all objects isomorphic to 1 and if $m : A \to 1$ is a monomorphism, then $A \simeq 1$. Hence, in such a case ε_X can be regarded as a strong epimorphism to 1 or equivalently η_X is an isomorphism. Then, the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\varepsilon_X"'] & M \arrow[d, "\varepsilon_M"] \\ CX \arrow[r, dashed, "d"] \arrow[d, "\eta_X"'] & CM \arrow[d, "\eta_M"] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$

commutes for a unique morphism d. The morphism d is established by the diagonalisation property. Furthermore, $\eta_m d = \eta_X$ and η_M are monomorphisms, hence so is d. Then, $f = me = m\varepsilon_M^{-1}\varepsilon_M e = m\varepsilon_M^{-1} d\varepsilon_X = m\varepsilon_M^{-1} d\eta_X^{-1}\eta_X\varepsilon_X = (m\varepsilon_M^{-1} d\eta_X^{-1})t_X$, so that f factors through 1 with t_X a strong epimorphism.

Hence, if 1 is a strong epi-reflective object and $\mathbb{A}$ is a (StrongEpi($\mathbb{A}$), Mono-source($\mathbb{A}$))-category, then we can choose a constant subcategory $\mathbb{C}$ of $\mathbb{A}$ such that the constant morphism coincide.

Assume that $\mathbb{C}$ is a reflective constant subcategory and $\mathbb{P}$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. Lemma 3.97 gives us that the $\mathbb{P}$-reflection of an $\mathbb{A}$-object X is a $\mathbb{C}$-reflection if and only if X is left-$\mathbb{P}$-constant. In view of [26, 4.1(1),4.3(1)] and the discussion above, there are some cases in which this result can be viewed as a generalisation of the constant morphisms and left and right constant subcategories in [26].

Corollary 3.99: Under the assumptions of Lemma 3.97, it follows that $\mathscr{L}(\mathbb{P})$ consists exactly of all objects X for which a reflection of X into $\mathbb{M}(\mathbb{P})$ is also a reflection of X into $\mathbb{C}$.

Corollary 3.100: Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category and R a strong $\mathscr{E}$-reflective object of $\mathbb{A}$. Let $\mathbb{C}$ be the $\mathscr{E}$-reflective constant subcategory consisting of all objects isomorphic to R. Let $\mathbb{P}$ be a non-empty subcategory of $\mathbb{A}$, then $\mathscr{L}(\mathbb{P})$ consist of all objects X in $\mathbb{A}$ such that the reflection of $\mathbb{A}$ into the $\mathbb{M}$-closure of $\mathbb{P}$ is also a reflection of X into $\mathbb{C}$.

Proof : It's sufficient to prove that $\mathbb{M}(\mathbb{P})$ contains $\mathbb{C}$. Since R is a strong $\mathscr{E}$-reflective object of $\mathbb{A}$ and $\mathbb{P}$ is non-empty, there is a $P \in \mathbb{P}$. Then, there is a morphism $f : R \to P$. It's sufficient to prove that $f \in \mathbb{M}$. As R is $\mathscr{E}$-reflective, there is a morhism $e : P \to R$ in $\mathscr{E}$. Furthermore, ef is an isomorphism, in particular $ef \in \mathbb{M}$ which implies that $f \in \mathbb{M}$. Then, it follows that $R \in \mathbb{M}(\mathbb{P})$ and since the $\mathbb{M}$-closure of $\mathbb{P}$ in $\mathbb{A}$ is closed under isomorphisms, $\mathbb{C} \subset \mathbb{M}(\mathbb{P})$, so that the assumptions of Lemma 3.97 are satisfied. □

Proposition 3.101: Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category with constant $\mathscr{E}$-reflective subcategory $\mathbb{C}$. Suppose that $\mathbb{D}$ is a right constant subcategory with respect to $\mathbb{C}$ and $\mathbb{D} = \mathscr{R}(\mathscr{L}(\mathbb{P}))$. Then,

$$\mathscr{E}_r(\mathbb{P}) = \mathbb{M}(\mathbb{P}) \subset \mathbb{D}.$$

Proof : By Theorem 3.58, it follows that $\mathbb{D}$ is an $\mathscr{E}$-reflective subcategory and since $\mathscr{Q}$ is isotone, $\mathbb{D}$ contains $\mathbb{P}$. Since $\mathscr{E}_r(\mathbb{P})$ is by definition the smallest $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains $\mathbb{P}$, it must follow that $\mathscr{E}_r(\mathbb{A})$ is contained in $\mathbb{D}$.

□

Remark 3.102: Consider an $(\mathscr{E},\mathbb{M})$-category $\mathbb{A}$ and a right constant subcategory $\mathbb{D}$ with $\mathbb{D} = \mathscr{Q}(\mathbb{P})$. We have seen that the $\mathbb{M}$-closure of $\mathbb{P}$ is contained in $\mathbb{D}$. One might be tempted to say that the $\mathbb{M}$-closure and $\mathbb{D}$ coincide, and this seems to be reasonable. Since the $\mathbb{M}$-closure of $\mathbb{P}$ contains $\mathbb{P}$ and $\mathscr{Q}$ is isotone, it follows that $\mathscr{Q}(\mathbb{M}(\mathbb{P})) \supset \mathbb{D}$ and since $\mathbb{D}$ is the smallest right constant subcategory that contains $\mathbb{P}$, $\mathbb{D}$ is also equal to $\mathscr{Q}(\mathbb{M}(\mathbb{P}))$. Unfortunately, the $\mathbb{M}$-closure or the $\mathscr{E}$-reflective hull of $\mathbb{P}$ and the smallest right constant subcategory containing $\mathbb{P}$ need not coincide, as the following example illustrates:

Example 3.103: Consider the (surjective, inital mono-source)-category $\mathbb{Top}$ and subcategory $\mathbb{P}$ consisting only of the 2-point discrete space $\mathbb{D}_2$. We have seen that $\mathscr{Q}(\mathbb{P}) = \mathbb{TDisc}$. We claim that the $\mathbb{M}$-closure of $\mathbb{P}$ in $\mathbb{A}$ is given by all zero dimensional T_0-spaces. Let X be any space and let us denote by $(f_i : X \to \mathbb{D}_2)_I$ the source of all continuous functions from X to $\mathbb{P}$. It's clear that we can take I to be a set and hence form the product $\mathbb{D}_2^I$ together with the projection functions $\pi_i : \mathbb{D}_2^I \to \mathbb{D}_2$. Then, there is a unique continuous function $\bar{f} : X \to \mathbb{D}_2^I$ such that $\pi_i \circ \bar{f} = f_i$ for each $i \in I$.

Now, (f_i) is an initial mono-source if and only if $\bar{f}$ is an initial monomorphism, hence the spaces in the $\mathbb{M}$-closure are exactly the spaces homeomorphic to subspaces of $\mathbb{D}_2^I$ for some set I.
If $(f_i)_I$ is an initial mono-source, then it's clear that X must be a Hausdorff space, hence T_0, since for then, $\bar{f}$ is an initial mono, i.e., $\bar{f}$ is an embedding. By initiality $\{U \subset X \mid \exists i \in I \ : U \in \{f_i^{-1}[\{0\}], f_i^{-1}[\{1\}]\}\}$ is a subbase of the topology on X. Since these sets are all clopen and finite intersections of clopen sets are clopen, X has a base consisiting of clopen sets, i.e., X is zero-dimensional.

On the other hand, suppose that X is a zero-dimensional T_0-space. Let $\mathcal{B}$ be a base of clopen sets for the topology on X. For each $U \in \mathcal{B}$, define a continuous function $\chi_U : X \to \mathbb{D}_2$ by $\chi_U(x) = 0$ if $x \notin U$ and $\chi_U(x) = 1$ if $x \in U$. Then, the source $(\chi_U : X \to \mathbb{D}_2)_{U \in \mathcal{B}}$ is an initial mono-source. It's initial, as all pre-images of open sets gives rise to $\mathcal{B}$ which is a base of the topology on X. It's a mono-source, since for each pair of distinct points x and y of X, there is an open set V of X such that $x \in V$ and $y \notin V$. But then there is a $B \in \mathcal{B}$ such that $x \in B \subset U$ and obviously $y \notin B$. Clearly χ_B is the desired member of the source such that $\chi_B(x) = 1 \neq 0 = \chi_B(y)$.
Thus, the initial mono-source closure of $\mathbb{D}_2$ consists exactly of all zero-dimensional T_0-spaces, whereas the right constant subcategory $\mathscr{Q}(\{\mathbb{D}_2\})$ consists of all totally disconnected spaces.

It's relatively easy to see that any zero-dimensional T_0-space is totally disconnected. For if x and y are members of some connected subset C of a zero-dimensional space X. Then, the all source $(f_i)_I$ into $\mathbb{D}_2$

is an initial mono-source. Hence, there is an i such that $f_i(x) \neq f_i(y)$. In particular $f_i^{-1}[\{0\}] \cap C$ and $f_i^{-1}[\{1\}] \cap C$ will be a separation of C. Thus $x = y$ and X must be totally disconnected. Not all totally disconnected spaces are zero-dimensional ([47, Example 72]).

Example 3.104: If we consider the previous example, then we might possibly suspect that it's not possible for $\mathscr{Q}(\mathbb{P}) = \mathbb{M}(\mathbb{P})$. Let $\mathbb{P} = \{\mathbb{D}_2\}$ and $\mathbb{C} = \{(X, \mathcal{T}) \in \mathbb{Top} \mid |X| \leq 1\}$. The only difference is that we change the factorisation structure to (quotient, point-separating source)=(RegEpi($\mathbb{Top}$), Monosouce($\mathbb{Top}$)).

First we show that $\mathscr{R}(\mathscr{L}(\mathbb{P})) = \mathscr{Q}(\mathbb{P}) = \mathbb{TDisc}$. Suppose that all continuous maps $f : X \to \mathbb{D}_2$ are constant. Clearly we need only consider spaces with at least 2 points. Suppose that X is not connected. Then there is a non-empty proper clopen set C of X and the characteristic function $\chi_U : X \to \mathbb{D}_2$ is a non-constant continuous function. Therefore X is not a member of $\mathscr{L}(\mathbb{P})$. On the other hand, assume X is connected and $g : X \to \mathbb{D}_2$ is continuous. Since $g[X]$ is a connected subspace of a discrete space, it follows that $|g[X]| \leq 1$. Therefore g is constant and it follows that X is a member of $\mathscr{L}(\mathbb{P})$. Putting these two results together shows that $\mathscr{L}(\mathbb{P}) = \mathbb{Con}$.

Now assume that Z is a member of $\mathscr{R}(\mathbb{Con})$. Then, every continuous function $h : C \to Z$ is constant whenever C is a connected space. Suppose that D is a connected subspace of Z. Then, the inclusion map $D \hookrightarrow Z$ must be constant and thus D contains at most one point. Hence, Z is totally disconnected. Conversely, if Y is totally disconnected, then for any connected space C and every continous map $k : C \to Y$, $k[C]$ is a connected subspace of Y. Since Y is totally disconnected, it follows that $k[C]$ consists of at most one point only and thus k is constant. This argument then shows that $\mathscr{Q}(\mathbb{P}) = \mathbb{TDisc}$, regardless of whether the above factorisation structure or the one in the previous example is used. This phenomenon does not always occur. See Proposition 3.105.

Now we show that the point-separating closure (PS($\mathbb{P}$)), or equivalently the quotient-reflective hull, of $\mathbb{P}$ is equal to $\mathbb{TSep}$, where $\mathbb{TSep}$ consists of all totally separated spaces. A space X is totally separated if for each pair of distinct points x and y of X, there is a clopen set C of X such that $x \in C$ and $y \notin C$. From the previous example it should be clear that PS($\mathbb{P}$) $= \{(X, \mathcal{T}) \in \mathbb{Top} \mid \exists (X, \mathcal{S}) \in \mathbb{ZDim}_0 \text{ with } \mathcal{S} \subset \mathcal{T}\}$. Let $(X, \mathcal{T})$ be in PS($\mathbb{P}$). Then, there is a coarser topology $\mathcal{S}$ on X such that $(X, \mathcal{S})$ is a zero-dimensional T_0 space. Then, given $x \neq y$ in X, without loss of generality, there exists a $U \in \mathcal{S}$ such that $x \in U$ and $y \notin U$. Since $(X, \mathcal{S})$ is zero-dimensional, there is a clopen set C of $(X, \mathcal{S})$ such that $x \in C \subset U$. Since $\mathcal{T}$ is finer than $\mathcal{S}$, it follows that C is a clopen set of $(X, \mathcal{T})$ that contains x, but not y. Therefore the point separating closure of $\mathbb{P}$ is contained in $\mathbb{TSep}$. On the other hand, let Z be a totally separated space and let $\mathcal{C}$ be the set of all its clopen sets. Consider the source $\mathcal{S} := (\chi_C : Z \to \mathbb{D}_2)_{C \in \mathcal{C}}$ of all characteristic functions χ_C, where C is clopen in X. We show that $\mathcal{S}$ is a point separating source. Let z and z' be distinct points of Z. As Z is totally separated, there is a clopen set C of X such that $z \in C$ and $z' \notin C$. Then $\chi_C(z) = 1 \neq 0 = \chi_C(z')$ and hence $\mathcal{S}$ is point separating. It can be shown that the category of totally separated spaces is strictly contained in the category of all totally disconnected spaces.

This example is not completely satisfying, but it does show that we need not expect to find other factorisation structures for sources for which $\mathbb{M}$ is a conglomerate of mono-sources such that the reflective hull of $\mathbb{P}$ and the right constant subcategory $\mathscr{Q}(\mathbb{P})$ need to coincide. Note that since $\mathbb{Top}$ is cocomplete, in order to enlarge the reflective hull, we would need to consider a factorisation structure for which there is a regular epimorphism that is not in $\mathscr{E}$ or, equivalently, at least one source in $\mathbb{M}$ which is not a mono-source.

Proposition 3.105: Let $\mathbb{A}$ be both $(\mathscr{E}, \mathscr{M})$-and $(\mathscr{D}, \mathscr{N})$-structured with $\mathscr{D} \subset \mathscr{E}$. Let $\mathbb{C}$ be a constant subcategory for both of the factorisation strucutes, i.e., $\mathbb{C}$ is closed under $\mathscr{E}$-images and $\mathscr{N}$-subobjects. Then, a morphism f is $\mathbb{C}$-constant with respect to the factorisation structure $(\mathscr{E}, \mathscr{M})$ if and only if f is $\mathbb{C}$-constant with respect to the factorisation structure $(\mathscr{D}, \mathscr{N})$.

Proof : Let $f : X \to Y$ be any $\mathbb{A}$-morphism and let me and nd be $(\mathscr{E}, \mathscr{M})$-and-$(\mathscr{D}, \mathscr{N})$-factorisations, respectively. Then,
$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "e"'] & N \arrow[d, "n"] \\ M \arrow[r, "m"'] & Y \end{tikzcd}$$
commutes. Note that since $\mathscr{D} \subset \mathscr{E}$, it follows that $\mathscr{M} \subset \mathscr{N}$. By the

$(\mathscr{E},\mathscr{M})$-diagonalisation property, there is a unique morphism $g : N \to M$ such that $mg = n$ and $gd = e$. Note that f is $\mathbb{C}$-constant with respect to the factorisation structure $(\mathscr{E},\mathscr{M})$ if and only if M is in $\mathbb{C}$. A similar argument applies to $(\mathscr{D},\mathscr{N})$ and N. We show M is in $\mathbb{C}$ if and only if N is in $\mathbb{C}$. Note that $mg = n \in \mathscr{N}$ and since $\mathscr{M}$ is a subclass of $\mathscr{N}$, it follows that g is in $\mathscr{N}$. Similarly, $gd = e \in \mathscr{E}$ and since $d \in \mathscr{D} \subset \mathscr{E}$, it follows that g must be a member of $\mathscr{E}$. That is, g is in $\mathscr{E} \cap \mathscr{N}$. Since $\mathbb{C}$ is closed under both $\mathscr{E}$-images and $\mathscr{N}$-subobjects, the result follows. □

Remark 3.106: Proposition 3.105 suggests the following: Whenever we have two distinct factorisation structures for morphisms, for which one is smaller than or equal to the other, and a subcategory that remains constant for both, left and right constant subcategories will coincide. In other words, changing to a smaller or larger factorisation structure will not solve the problem of finding a larger or smaller right or left constant subcategory to fit the reflective hull. In other words, if it's possible for a reflective hull and a right constant subcategory to coincide, we could consider one of the following procedures:

- (*a*) We could consider other factorisation structures which is neither larger nor smaller than the existing one. Assume for the moment that there is such a factorisation structure, i.e., not smaller or larger than the existing one. Also assume that $\mathbb{C}$ remains a constant subcategory for the new facorisation structure. It will then not necessarily be the case that the constant morphisms coincide. Hence, we should also not expect the right constant subcategories to remain the same. Unfortunately, it does not seem that appropriate, since one may not necessarily enlarge or reduce the right constant subcategories. It could be the case that neither of the old or new right constant subcategories need to even contain one or the other. However, it will probably also change the reflective hull. Due to the vagueness of what one could expect, this does not seem like the best way to solve the problem.
- (*b*) Another possibility is to consider a smaller factorisation structure. Since the reflective hull of a category $\mathbb{P}$ will be contained in the right constant subcategory $\mathscr{Q}(\mathbb{P})$, provided that $\mathbb{C}$ remains a constant subcategory for the new factorisation structure, we need to enlarge the reflective hull. Since the reflective hull is equal to the source part of the factorisation structure's closure, we could choose more sources, or equivalently smaller factorisation structures. As example 3.104 shows, $\mathbb{M}$ need not be a conglomerate of mono-sources anymore and there could be a regular epimorphism that is not in $\mathscr{E}$.

Proposition 3.107: Let $\mathbb{C}$ be a constant reflective subcategory of the $(\mathscr{D},\mathbb{N})$-category $\mathbb{A}$. Then, there exists a smallest factorisation structure $(\mathscr{E},\mathbb{M})$ for sources on $\mathbb{A}$ such that $\mathbb{C}$ is an $\mathscr{E}$-reflective constant subcategory of $\mathbb{A}$.

Proof : Let $((\mathscr{E}_i,\mathbb{M}_i))_I$ be the family of all factorisation structures for sources on $\mathbb{A}$ such that $\mathbb{C}$ is an $\mathscr{E}_i$-reflective subcategory of $\mathbb{A}$ and closed under $\mathscr{E}_i$-images and $\mathscr{M}_i$-subobjects. By theorem 2.33, it follows that $\mathbb{A}$ is an $(\mathscr{E},\mathbb{M})$-category for $\mathscr{E} = \bigcap_I \mathscr{E}_i$ and

$$\left\{ (m_i)_I \in Source(\mathbb{A}) \mid (m_i)_I \in \left(\bigcap_I \mathscr{E}_i\right)^{\downarrow}_{\mathcal{S}} \right\}$$

First we show that $\mathbb{C}$ is $\mathscr{E}$-reflective and a constant subcategory for the factorisation structure $(\mathscr{E},\mathbb{M})$. Since $\mathbb{C}$ is a constant reflective subcategory of the $(\mathscr{D},\mathbb{N})$-category, $I \neq \varnothing$. Let A be an $\mathbb{A}$-object and for each $i \in I$, $e_i : X \to C_iX$ an $\mathscr{E}_i$-reflection for X. We claim that $e_i \simeq e_j$ for all $i, j \in I$. For any pair of indices i and j, there are unique morphisms $k : C_iX \to C_jX$ and $\ell : C_jX \to C_iX$ such that $ke_i = e_j$ and $\ell e_j = e_i$. It should then be clear that $\ell k e_i = \ell e_j = e_i$ and since e_i is an epimorphism, $\ell k = id_{C_iX}$. Similarly $k\ell = id_{C_jX}$. Hence $e_i \simeq e_j$. If e_X is the $\mathscr{D}$-reflection for X, then of course e_X is isomorphic to e_i for each $i \in I$ and since $\mathscr{E}$ and $\mathscr{E}_i$ are isomorphism closed, $e_X \in \mathscr{E}$ and serves as an $\mathscr{E}$-reflection of X. Since $\mathscr{E}$-reflective subcategories are closed under sources in $\mathbb{M}$, we need only show that $\mathbb{C}$ is closed under $\mathscr{E}$-images. Let $e : X \to Y$ be in $\mathscr{E}$, then, in particular, $e \in \mathscr{E} \subset \mathscr{D}$ and since $\mathbb{C}$ is closed under $\mathscr{D}$-images, the result follows. Obviously $(\mathscr{E},\mathbb{M})$ is the smallest factorisation structure for which this will hold, as it's the meet of all factorisation structures for which this is true. □

Corollary 3.108: Let $\mathbb{C}$ be a constant reflective subcategory of the $(\mathscr{D},\mathbb{N})$-category $\mathbb{A}$. Then, there exists a smallest factorisation structure $(\mathscr{E},\mathbb{M})$-for sources such that every right contsant subcategory is $\mathscr{E}$-reflective. Furthermore, this factorisation structure coincides with the smallest factorisation structure such that the constant subcategory is $\mathscr{E}$-reflective.

$\underline{\mathcal{P}roof}$: It follows from Proposition 3.107 that there is a smallest factorisation structure $(\mathscr{E},\mathbb{M})$ for sources that makes $\mathbb{C}$ an $\mathscr{E}$-reflective constant subcategory. From Proposition 3.58, it follows that every right constant subcategory is also $\mathscr{E}$-reflective. Suppose that every right constant subcategory is $\mathscr{F}$-reflective, where $\mathbb{A}$ is an $(\mathscr{E}',\mathbb{M}')$-category. From Corollary 3.65, we may conclude that $\mathbb{C}$ is $\mathscr{E}'$-reflective. From Proposition 3.107, it follows that $\mathscr{E}\subset\mathscr{E}'$, hence $(\mathscr{E},\mathbb{M})\le(\mathscr{E}',\mathbb{M}')$. Hence $(\mathscr{E},\mathbb{M})$ is the smallest factorisation structure for sources such that every right constant subcategory is $\mathscr{E}$-reflective. □

Lemma 3.109: Let $((\mathscr{E}_i,\mathbb{M}_i))_I$ be a family of factorisation structures for sources on $\mathbb{A}$ and $(\mathscr{E},\mathbb{M}) = \bigwedge_I(\mathscr{E}_i,\mathbb{M}_i)$ in $Fact_{\mathcal{S}}(\mathbb{A})$. Then, for any subcategory $\mathbb{B}$ of $\mathbb{A}$, let $\mathbb{M}_i(\mathbb{B})$ denote the $\mathbb{M}_i$-closure of $\mathbb{B}$ in $\mathbb{A}$ and for any $\mathbb{A}$-object X, $e_i^X : X\to R_iX$ the $\mathscr{E}_i$-reflection of X into $\mathbb{M}_i(\mathbb{B})$. Let $(e_i^X)_I$ have $(\mathscr{E},\mathbb{M})$-factorisation $(m_i)_I\circ e^X : X\to RX\to R_iX$. Then, the $\mathscr{E}$-reflection of X into $\mathbb{M}(\mathbb{B})$ is given by $e^X : X\to RX$.

$\underline{\mathcal{P}roof}$: It's known that for an $(\mathscr{E},\mathbb{M})$-category $\mathbb{A}$ and any subcategory $\mathbb{B}$ of $\mathbb{A}$, the $\mathscr{E}$-reflection of X into $\mathbb{M}(\mathbb{B})$ is given by the $\mathscr{E}$-part of an $(\mathscr{E},\mathbb{M})$-factorisation of the all-source from X into $\mathbb{B}$. Furthermore, obviously if e_1 and e_2 are two reflections, then $e_1\simeq e_2$. Let $\mathbb{B}$ be a subcategory of $\mathbb{A}$, X in $\mathbb{A}$ and $(f_j : X\to B_j)_J$ be the all-source from X into $\mathbb{B}$. For each $i\in I$, let $(m^i_j)_J\circ e_i : X\to M^i\to B_j$ be an $(\mathscr{E}_i,\mathbb{M}_i)$-factorisation of $(f_j)_J$ and $(m_j)_J\circ e : X\to M\to B_j$ be an $(\mathscr{E},\mathbb{M})$-factorisation of $(f_j)_J$. We may then without loss of generality assume that $(e_i : X\to M^i) = (e_i^X : X\to R_iX)$ and that $(e : X\to M) = (e^X : X\to RX)$. Since for each $i\in I$, $(m_j)_J\circ e = (f_j)_J = (m^i_j)_J\circ e_i$ and $\mathbb{M}_i\subset\mathbb{M}$, for each $i\in I$ there is a unique diagonal $d_i : M\to M^i$ such that $(m^i_j)_J\circ d_i = (m_j)_J$ and $d_i\circ e = e_i$. Let $(\hat{m}_i)_I\circ\hat{e} : X\to\hat{M}\to M^i$ be an $(\mathscr{E},\mathscr{M})$-factorisation of $(e_i)_I$. Then, once again, the diagonalisation property establishes a unique morphism $d : M\to\hat{M}$ such that $de=\hat{e}$ and $(\hat{m}_i)d=(d_i)_I$. Since e and $\hat{e}$ are members of $\mathscr{E}$, we have $d\in\mathscr{E}$. Since $(m_i)_Id=(d_i)_I$ and $(m_i)_I$ is in $\mathbb{M}$, it follows that d is a member of $\mathbb{M}$ if and only if $(d_i)_I$ is a member of $\mathbb{M}$. We show that $(d_i)_I$ is a member of $\mathbb{M}$ by showing that for any particular $i\in I$, we have d_i being a member of $\mathbb{M}$. This is sufficient since $\mathbb{M}$ is closed under the formation of supersources. Since $(m^i_j)_J\circ d_i=(m_j)_J$ is a member of $\mathbb{M}$ for any $i\in I$, we must have that d_i is in $\mathbb{M}$. Consequently $(d_i)_I$ is in $\mathbb{M}$ and hence d is a member of $\mathbb{M}$. Therefore $d\in\mathscr{E}\cap\mathbb{M}=Iso(\mathbb{A})$, so that $e\simeq\hat{e}$. By the remark above it should then be clear that both e and $\hat{e}$ are $\mathscr{E}$-reflections for X in $\mathbb{M}(\mathbb{B})$.

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "\hat{e}"'] \arrow[dd, bend right=60, "e_i"'] & M \arrow[dl, dotted, "!d"'] \arrow[ddl, dashed, "!d_i"] \arrow[dd, "m_j"] \\
\hat{M} \arrow[d, "\hat{m}_i"'] & \\
M^i \arrow[r, "m^i_j"'] & B_j
\end{tikzcd}$$

□

Remark 3.110: Lemma 3.109 provides us with a reflection morphism which is appropriate with a specific factorisation structure. This will be useful when discussing some of the interactions between reflective subcategories and dual closure operators.

3.5 Relationships between nearly multi coreflective subcategories and left constant subcategories

Remark 3.111: In view of section 3.4, we can see that there is an interplay between $\mathscr{E}$-reflective subcategories and right($\mathbb{C}$-)constant subcategories. Since the notion of constant subcategories is self dual, one might expect that left constant subcategories are $\mathscr{M}$-coreflective. This is not always the case, but does happen in some special cases. There are a number of possible explanations for this: One is that we normally consider the constant subcategory to be reflective and so for this to be the case, we would have to assume that the constant subcategory in question is coreflective. Another is that most of the interesting factorisation structures that are widely known or used are factorisation structures for sources or morphisms. A lot of times a factorisation structure for morphisms can't be extended to one for sinks. A typical example of this is when the class $\mathscr{M}$ does not consist only of monomorphisms.

In order to study an interaction between left-and right constant and reflective subcategories, it's important to note that the aim is that the whole theory is applicable to most nice and/or familiar categories. Since we have been focusing on categories with factorisation structures for sources, we will not study ones with factorisation structures for sinks. Quite a bit of material is already true by duality of the results in section 3.4, hence it would be superfluous as well. An example of non coreflectiveness of a left constant subcategory follows:

Example 3.112: Consider the category of topolological spaces $\mathbb{Top}$ with factorisation structure (surjective, embedding) and constant subcategory chosen as spaces containing at most one point. Note that this factorisation structure can be extended to one for sources, namely (surjective, initial mono-source). Note that the constant subcategory in question is not coreflective. Of course, if the factorisation structure can be extended to one for sinks, then it's easily seen that every covering sink is orthogonal to the class of embeddings. It can then also be seen that the this constant subcategory is not closed under covering sinks which in turn also implies that it's not $\mathscr{M}$-coreflective. Since the constant subcategory is both left and right constant, this already provides us with a counterexample. Of course this is the case with any reflective constant subcategory that is not coreflective.

In order to obtain a better understanding of an alternative, consider the subcategory $\mathbb{Con}$ of connected spaces of $\mathbb{Top}$. The argument in example 3.53 show that it is in fact a left constant subcategory. This subcategory is not coreflective. To see why, assume the contrary. Consider the two point discrete space $\mathbb{D}_2$ and a coreflection $c : C \to \mathbb{D}_2$ from $\mathbb{Con}$ into $\mathbb{D}_2$. Since C is connected, c must be constant and thus $c(x) = c(y)$ for each pair of points x and y of C. Since 0 and 1 are topologically indistinguishable in $\mathbb{D}_2$, we might as well assume that $c(x) = 0$ for any x in C. Define the function $f : \mathbb{R} \to \mathbb{D}_2$ by $f(x) = 1$ for all real numbers x. Since $\mathbb{R}$ is connected, f is continuous and $c : C \to \mathbb{D}_2$ is assumed to be a coreflection, there exists a unique continuous function $g : C \to \mathbb{R}$ such that $f \circ g = c$. Then, $0 = c(x) = f(g(x)) = 1$, a contradiction. It follows that $\mathbb{Con}$ is not coreflective in $\mathbb{Top}$.

Remark 3.113: Considering example 3.112, we see that a left constant subcategory need not be coreflective. In order to motivate this section, we start with a classical construction of topology. It is well known that one can divide a space into its connected components. To be more explicit, if X is any topological space and $x \in X$, then the connected component of x is a connected subspace given by the set $C(x) = \bigcup\{C \mid C \text{ is a connected subspace of } X \text{ and } x \in C\}$. These components partition the set X and for each x in X, $C(x)$ is the largest connected subspace of X that contains x. Since the continuous image of a connected space is connected, we could view this as follows:

Given a non-empty connected space A and a continuous function $f : A \to X$, there is a unique connected component C of X such that image of A by f is entirely contained in C. If we index the components of X by a set I such that $\{C_i \mid i \in I\}$ are all of the connected components in such a way that $C_i \neq C_j$ whenever $i \neq j$, then this reads:

For each continuous function $f : A \to X$ with A a non-empty connected space, there is a unique $i \in I$ such that $f[A] \subset C_i$. Furthermore, we can think of the components as subspaces and hence the inclusion maps $n_i : C_i \to X$ as many different 'coreflection' morphisms. This then further gives us that not only is i unique, but there is a unique morphism $g : A \to C_i$ such that $n_i \circ g = f$. Note that for no other pair of index j in I and morphism h from A to C_j will it be the case that $n_j \circ h = f$. We will see that this section aims to generalise this notion categorically and there is an interplay between these and the left constant subcategories.

Note that when A is empty and $f : A \to X$ is a continuous function, then A is still in the left constant subcategory and f factors through each of the n_i.
This notion is particularly useful when viewed from the comma category of connected spaces over X. To be more explicit, for any space X, we consider the comma category $\mathbb{A} =: \mathbb{C}\mathrm{on}/X$. Using our morphisms above, $(n_i : C_i \to X)_I$ is a family of $\mathbb{A}$-objects. Then, for any $\mathbb{A}$-object $f : A \to X$, there is a pair $(i, g) \in I\times$ Mor($\mathbb{A}$) such that $n_i \circ g = f$. If A is non-empty, then this pair is unique.

It must be noted that all our left constant subcategories contain the constant subcategory. Also note that a constant subcategory is never empty and so if $\mathbb{A}$ is a construct, then we may consider the empty map $\varnothing \to C$ for any C in $\mathbb{C}$. Of course, there might not be any structure on the empty set or the map might not be a morphism, but usually topological categories have an empty space and this map is usually a morphism as well. If the morphisms in the $\mathscr{M}$-part of our factorisation structure include this empty morphism, then our definition forces the empty set to be in the constant subcategory. This then forces the empty space to be a member of the left constant subcategory. Since there are some connections between left constant subcategories and categories of generalised connected spaces, we might be forced to assume that the empty space is left constant for some topological categories. It's not a matter of whether the empty space in $\mathbb{T}\mathrm{op}$ is connected or not, but rather that if it's regarded as left constant, then which general properties that are similar to coreflectiveness can be concluded.

Various generalisations of coreflections have been studied by a number of authors. J.J. Kaput defined local coreflectivity in [36] and other authors studied similar multi-adjunctions or generalisations thereof using different terminology. Another notion that was studied was local monocoreflectivity. One example of such a study can be seen in [46]. A result from [5] states that for any locally coreflective subcategory $\mathbb{C}$ of $\mathbb{A}$, we have: $\mathbb{C}$ is coreflective in $\mathbb{A}$ if and only if $\varnothing \in Ob(\mathbb{C})$. See also [46, 1.5,1.6]. Of course this was for topological categories, but the issue with the empty set being a structured member is usually a topological notion as many algebraic constructs don't have a structured object with the empty set as underlying set. Taking this into account, we would have to modify our generalisation of coreflectiveness such that the empty function also plays a role in the coreflection diagram(s). The approach in [4] is done by merely requiring a family (not necessarily small) of coreflections as to mimic the idea of (connected) components as explained above. We would like this family to be small and will follow more similar approaches to that done in [50] or [46]. Another approach is to use connected components or connected categories and do it entirely from a categorical perspective. Various equivalent and/or similar definitions of connected categories can be found in [3], [38] and [39]. An extensive study of connected components can be found in [39]. The main problem with using only connected categories of subcategories is that there will be a lot of cases in which we will have have to discard the empty space when considering the components. A combination of these ideas are found and a similar notion to multi-coreflectivity (as in [26]) is found that can include the empty space. The approach of [26] fails when the empty space is considered as a left-constant object and is therefore not the appropriate one for our notion of constant morphism.

Since our main idea is to link these with dual closure operators and left constant subcategories, one aim is to establish a factorisation of the HPAW correspondence through the conglomerate of dual closure operators. The main difference between the two is that Dikranjan and Tholen's approach does this with connected components. Some definitions or equivalent ones used there ([26]) explicitly follows below to emphasize the difference between the approaches. Please take note that we will adopt similar terminology for our generalised notion of 'multi-coreflectiveness', but it will be mathematically different. In order to avoid ambiguity, we will make explicit reference to Dikranjan and Tholen's paper ([26]) if we are referring to their notion.

Definition 3.114: Connected components relative to a subcategory, Multi-coreflective
Let $\mathbb{A}$ be a category and X in $\mathbb{A}$. For each subcategory $\mathbb{B}$ of $\mathbb{A}$, define the relation $\sim_{\mathbb{B}}$ on the objects of the comma category $\mathbb{B}/X$ where $(f : A \to X) \sim_{\mathbb{B}} (g : B \to X)$ if and only if there exist $\mathbb{B}/X$-objects $f_0, f_1, \ldots, f_n$ with $f_i : X_i \to X$ such that $f_0 = f$, $f_n = g$ and $Hom(f_i, f_{i-1}) \cup Hom(f_{i-1}, f_i) \neq \varnothing$ for $i = 1, 2, \ldots, n$.

It's straightforward to verify that $\sim_{\mathbb{B}}$ is an equivalence relation on $Ob(\mathbb{B}/X)$. The finite sequence $f_0, f_1, \ldots, f_n$ will be called a **$\mathbb{B}$-zig-zag of length** $n+1$ from f to g over X.
The equivalence classes of the relation $\sim_B$ for an $\mathbb{A}$ object X and a subcategory $\mathbb{B}$ of $\mathbb{A}$ defined above is called **the connected components of** X **in** $\mathbb{B}/X$ or the connected components of X relative to the subcategory $\mathbb{B}$.

([26]) Let $\mathbb{A}$ be a category. Then, a subcategory $\mathbb{B}$ of $\mathbb{A}$ is **multi-coreflective (in $\mathbb{A}$)** if for each $X \in \mathbb{A}$, the following conditions hold:

- (i) The distinct connected components of the comma category $\mathbb{B}/X$ may be labeled by a small set,
- (ii) every connected component of $\mathbb{B}/X$ has a terminal object, i.e., for each $f \in \mathbb{B}/X$, there exists an element $t \in [f]_{\sim_{\mathbb{B}}}$ such that for each $g \in [f]_{\sim_{\mathbb{B}}}$, there exists a unique morphism $t_g : dom(g) \to dom(t)$ such that $t \circ t_g = g$.

In other words, for every object X, there is a small family of connected components of $\mathbb{B}/X$ and if these are viewed as subcategories of $\mathbb{B}/X$, then each has a terminal object. If we label the distinct connected components bijectively by a set I with $(\rho_i)_I$ the family of terminal objects, i.e., ρ_i is terminal in the connected component corresponding to i, then for each $B \in \mathbb{B}$ and morphism $f : B \to X$, there exists an $i \in I$ and a $\mathbb{B}$-morphism $g : B \to A_i$ such that $\rho_i \circ g = f$. This is, $\rho_i g = f$ for a unique pair (i, g).

The family $(\rho_i)_I$ is called **a multi-coreflection of X into $\mathbb{B}$.** A multi-coreflective subcategory $\mathbb{B}$ of $\mathbb{A}$ is **multi-$\mathscr{M}$-coreflective** provided that all members of all multi-coreflections are members of a class of morphisms $\mathscr{M}$. For example, if $\mathscr{M}$ is the class of (extremal, strong or regular) monomorphisms, then this is multi-(extremally, strongly or regular) mono-coreflective.

Remark 3.115: It's important to note that the family of multi-coreflections must be small for each $\mathbb{A}$-object X. Furthermore, if $(\rho_i : B_i \to X)_I$ is the family as described in definition 3.114, then the second property is similar to saying that a $\mathbb{B}$-object B is **multi-orthogonal to the sink** $(\rho_i)_I$. See [4, 1.1] for more information on this. In fact, in [4], a subcategory $\mathbb{B}$ is multi-coreflective if for each $\mathbb{A}$-object X, there is a family $(\eta_i : B_i \to X)_I$ such that B_i is a member of $\mathbb{B}$ for each $i \in I$ and whenever $f : B \to X$ is a morphism with B in $\mathbb{B}$, then there is a unique pair (i, g) with $i \in I$ and $g : B \to B_i$ such that $\eta_i \circ g = f$. Our definition will be distinct from both. It will be different in that we require I to be small and we will refrain from using only connected components. To get an idea, we will first look at a special case where the constant subcategory only consists of a zero object.

Proposition 3.116: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with zero object 0. Let $\mathbb{C}$ be the full subcategory of all zero objects of $\mathbb{A}$. Then, $\mathbb{C}$ is simultaneously a reflective and coreflective constant subcategory of $\mathbb{A}$ if $\mathrm{Sect}(\mathbb{A}) \subset \mathscr{M}$ and $\mathrm{Retr}(\mathbb{A}) \subset \mathscr{E}$.

Proof : Let us denote for each $\mathbb{A}$-object X, the unique morphism into 0 by $t_X : X \to 0$ and the unique one from 0 by $i_X : 0 \to X$. Clearly $t_X i_X = id_X$ for each $\mathbb{A}$-object X, so that by our assumption on $\mathscr{E}$ and $\mathscr{M}$, $t_X \in \mathscr{E}$ and $i_X \in \mathscr{M}$. In particular, t_X is the $\mathbb{C}$-reflection and i_X the $\mathscr{M}$-coreflection. The fact that $\mathbb{C}$ is a constant subcategory follows from $\mathscr{E}$-reflectivity and $\mathscr{M}$-coreflectivity. □

Remark 3.117: Unless stated otherwise, we will, for the remainder of this section assume that $\mathbb{C}_0$ is the constant subcategory of zero objects. Note that $\mathbb{C}_0$ is both $\mathscr{E}$-reflective and $\mathscr{M}$-coreflective provided that $\mathscr{E}$ and $\mathscr{M}$ contain all retractions and sections respectively. However, we will not necessarily impose these conditions on $\mathscr{E}$ and $\mathscr{M}$. Of course, $\mathscr{E}$ is still assumed to be a class of epimorphisms, hence if $\mathbb{A}$ has products, then the assumption on $\mathscr{M}$ in Proposition 3.116 is already satisfied. Dually for coproducts.

Proposition 3.118: If $\mathbb{C}_0$ is the constant subcategory of zero objects, then it's closed under products and coproducts. Furthermore, if $\mathbb{A}$ has products or coproducts, then the (co)products are also (co)products in $\mathbb{A}$.

Proof : The fact that $\mathbb{C}$ is closed under products follows from the fact that for any objects $(X_i)_I$ in any category and any terminal object X_t, where $t \notin I$, $\prod_{i \in I} X_i \simeq \prod_{i \in I \cup \{t\}} X_i$. Thus, if all objects in the product is terminal, then it must be isomorphic to a terminal object. Note that the empty product is also a terminal object and thus in $\mathbb{C}$. Since zero objects are also zero objects in $\mathbb{A}^{op}$, $\mathbb{C}$ is closed under coproducts as well.

The latter part follows from the proof of the first part. □

Definition 3.119: Weakly (co)reflective, Almost (co)reflective
([32]) A full, isomorphism closed subcategory $\mathbb{B}$ of $\mathbb{A}$ is said to be **weakly reflective (in** $\mathbb{A}$**)** if for each $\mathbb{A}$-object A there is a $\mathbb{B}$-object B_A and a **weak reflection morphism** $r_A : A \to B_A$ such that for any morphism $f : A \to B$, with B in $\mathbb{B}$, there exists a (not necessarily unique) morphism $\hat{f} : B_A \to B$ such that $\hat{f} \circ r_A = f$. If each weak reflection morphism belongs to a class of morphisms $\mathscr{E}$, then we will say that $\mathbb{B}$ **is weakly** $\mathscr{E}$**-reflective (in** $\mathbb{A}$**).** If $\mathbb{B}$ is weakly ($\mathscr{E}$-)reflective and is closed under retractions, then $\mathbb{B}$ is said to be **almost (**$\mathscr{E}$**-)reflective in** $\mathbb{A}$**.**

If $\mathscr{M}$ is a class of morphisms of $\mathbb{A}$, then **weakly (**$\mathscr{M}$**-)coreflective** and **almost** ($\mathscr{M}$)**-coreflective are defined dually.**

Lemma 3.120: Let $\mathbb{C}$ be a constant subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. If $\mathbb{B}$ is a left constant subcategory, then $\mathbb{B}$ is closed under retractions. It also follows that whenever $s : A \to B$ is a section with B in $\mathbb{B}$, then A is in $\mathbb{B}$.

Proof : Let $\mathbb{P} = \mathscr{R}(\mathbb{B})$, then $\mathbb{B} = \mathscr{L}(\mathbb{P})$. Let $g : X \to Y$ be a retraction with $h : Y \to X$ a section of g such that $gh = id_Y$. Assume that X is a member of $\mathbb{B}$. We prove that Y is a member of $\mathbb{B}$.
Let $f : Y \to P$ be any morphism with P in $\mathbb{P}$. Then, $fg : X \to Y \to P$ is constant as $X \in \mathscr{L}(\mathbb{P})$. Since constant morphisms are closed under composition with $\mathbb{A}$-morphisms, it follows that $fgh = fid_Y = f$ is constant as well and it follows that Y is a member of $\mathbb{B}$. Thus, $\mathbb{B}$ is closed under retractions.

To see that the last part of the result is true, consider a section $s : A \to B$ with codomain in $\mathbb{B}$. Then, there exists a morphism $r : B \to A$ such that $rs = id_A$. Since r is a retraction, it follows from the first part of the proof that A is a $\mathbb{B}$-object as well.

□

Corollary 3.121: Let $\mathbb{B}$ be any subcategory of the category $\mathbb{A}$ and let $\mathscr{S}$ be the class of all sections in $\mathbb{A}$. Then, $\mathbb{B}$ is closed under retractions if and only if $\mathbb{B}$ is closed under $\mathscr{S}$-subobjects.

Proof : This is evident from the proof of Lemma 3.120. □

Remark 3.122: The next theorem relies on the fact that we can represent each $\mathscr{M}$-subobject by a set of isomorphic representatives. This notion is similar to $\mathscr{M}$-well-poweredness, but not equivalent.

To see this, consider a class of morphisms $\mathscr{M}$ of a category $\mathbb{A}$, where $\mathscr{M}$ is not a class of monomorphisms. We can still consider the comma category $\mathscr{M}/X$ for any $\mathbb{A}$-object X. We can still define $(m : M \to X) \leq (n : N \to X)$ if and only if there is an $\mathbb{A}$-morphism $j : M \to N$ or an $\mathscr{M}/X$ morphism $j : m \to n$ such that $nj = m$ in $\mathbb{A}$. Note that this ordering is still a pre-order on $\mathscr{M}/X$ as in the case of $\mathrm{Mono}(\mathbb{A})/X$.
We are presented with the problem of representing $\mathscr{M}$-subobjects by sets. Using our notation above, the first problem is that if $m \leq n$, then the morphism j need not be unique when $\mathscr{M}$ is not a class of monomorphisms. Furthermore, if $m \leq n$ and $n \leq m$, then M and N need not be isomorphic. When dealing with $\mathscr{M}$-subobjects, when $\mathscr{M}$ is a class of monomorphisms, not only is the morphism j unique, but if $m \leq n$ and $n \leq m$, then M is isomorphic to N. In other words, if we want to consider the $\mathscr{M}$-subobjects of an $\mathbb{A}$-object X, we can define an equivalence relation on $\mathscr{M}/X$ by defining $m \simeq n$ if and only if $m \leq n$ and $n \leq m$. In the case that $\mathscr{M}$ is a class of monomorphisms, the $\mathscr{M}$-subobjects can then be represented by a class and if $[m]_{\simeq} = [n]_{\simeq}$, then there is a unique isomorphism $j : M \to N$ such that $nj = m$.

A frequent occurence of using subobjects is that categories which are well-powered have nice properties. In order to define $\mathscr{M}$-well-poweredness, we need a class of monomorphisms. This motivates the following definition:

Definition 3.123: $\mathscr{M}$**-small**
Let $\mathbb{A}$ be a category and $\mathscr{M}$ be a class of $\mathbb{A}$-morphisms. We say that $\mathbb{A}$ is $\mathscr{M}$**-small** if for each $\mathbb{A}$-object X, the class of all the equivalence classes of the equivalence relation $\simeq$ on $\mathscr{M}/X$, defined by $(m : M \to X) \simeq (n : N \to X)$ if and only if there is an isomorphism j such that $nj = m$, is a set.

Note that if $\mathbb{A}$ is $\mathscr{M}$-small, then the domains of two representatives of an equivalence class are isomorphic and there is an isomorphism that is compatible with the representatives. If $\mathscr{M}$ is a class of monomorphisms, then $\mathbb{A}$ is $\mathscr{M}$-small if and only if $\mathbb{A}$ is $\mathscr{M}$-well-powered.

Example 3.124: Let $\mathbb{A}$ be the category $\mathbb{S}\mathbb{e}\mathbb{t}$. Then:

(a) $\mathbb{S}\mathbb{e}\mathbb{t}$ is not morphism-small. To see this, let Y be a non-empty set and $f : X \to Y$ be a map. Then, for any set Z with $|Z| \neq |X|$, it follows that there is a map $h : Z \to Y$. It should be clear that $f \not\simeq h$. Since the class of all sets is proper, our result follows.

(b) Let $\mathscr{M}$ be the class of all morphisms such that the domain of a member is a terminal object. Then, for the empty set, there is no morphism in $\mathscr{M}$ to $\varnothing$. If X is non-empty, then each representative can be identified with a point of X. Hence, $\mathbb{S}\mathbb{e}\mathbb{t}$ is $\mathscr{M}$-small.

Let $\mathbb{A}\mathbb{b}$ be the category of abelian groups and let $\mathscr{M}$ be the class of all morphisms $m : A \to B$ such that $Ker(m) \in \mathbb{T}\mathbb{f}\mathbb{A}\mathbb{b}$. Then, $\mathbb{A}\mathbb{b}$ is not $\mathscr{M}$-small.

(c) To see this, consider any group X and let $F_X := \bigoplus_{x \in X} \mathbb{Z}$. Hence, F_X is the direct sum of $|X|$ copies of $\mathbb{Z}$ and is also the free abelian group generated by X.
Let G be any abelian group and define a map $\mu_G : G \to F_G$ by $\mu_G(x) = (\delta_g^x)_{g \in G}$, where $\delta_g^x = 1$ if $g = x$ and $\delta_g^x = 0$ otherwise. It turns out the family $((\delta_g^x)_{g \in G})_{x \in X}$ is a basis for F_G as well. Define a map $\iota_G : F_G \to G$ by $\iota_G((n_g)_{g \in G}) = \sum_{g \in G} n_g g$ or equivalently, $\iota_G(n(\delta_g^x)_{g \in G}) = ng$. It's easy to see that $\iota_G \circ \mu_G = id_G$.
For each set H with $H \supset G$, let F_H denote the free abelian group $\bigoplus_{h \in H} \mathbb{Z}$ and define the morphism $m_H : F_H \to G$ by $m((n_h)_H) = \sum_{h \in G} n_h h$. It's easy to see that m_H is a homomorphism and $\iota_G = m_G$. The family $(m_H)_{G \subset H}$ is a large family of morphisms in $\mathscr{M}$ which are all in different equivalence classes. To see that they are in $\mathscr{M}$, note that all the domains are all torsion-free abelian groups, hence so is every subgroup, and, consequently, $Ker(m_H)$ is a torsion-free abelian group. Furthermore, it can be shown ([35, Ch.II,1.2,1.3]) that two free abelian groups F_H and $F_{H'}$ are isomorphic if and only if $|H| = |H'|$.

Theorem 3.125: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with coproducts and let $\mathbb{C}$ be a constant reflective subcategory of $\mathbb{A}$. Assume that $\mathbb{C}$ is closed under coproducts in $\mathbb{A}$ and that $\mathbb{A}$ is $\mathscr{M}$-small. Then, every left constant subcategory is almost $\mathscr{M}$-coreflective.

Proof : By Lemma 3.120, we need only show that each left constant subcategory is weakly $\mathscr{M}$-coreflective. Let $\mathbb{Q}$ be a left constant subcategory. Then there exists a subcategory $\mathbb{P}$ of $\mathbb{A}$, with $\mathbb{P}$ closed under $\mathscr{M}$-subobjects, such that $\mathscr{L}(\mathbb{P}) = \mathbb{Q}$. In fact, we may take $\mathbb{P} = \mathscr{R}(\mathbb{Q})$ so that $\mathbb{P}$ is $\mathscr{E}$-reflective in $\mathbb{A}$. Let X be any $\mathbb{A}$-object. By $\mathscr{M}$-smallness, we have a set $\mathscr{M}_0$ of non representatives, where $\mathscr{M}_0 \subset \mathscr{M}$. Note that for each pair distinct morphisms $m : M \to X$ and $n : N \to X$ of $\mathscr{M}_0$, there is no isomorphism $j : M \to N$ such that $nj = m$. Consider the full subcategory $\mathbb{Q}_{\mathscr{M}_0}$ of $\mathbb{Q}/X$ where $c : A \to X$ is a member of $\mathbb{Q}_{\mathscr{M}_0}$ if and only if A is a member of $\mathbb{Q}$ and c is a member of $\mathscr{M}_0$. Of course, this can be viewed as the category with objects consisting of $\mathbb{Q}_{\mathscr{M}_0} = \mathscr{M}_0/X \cap \mathbb{Q}/X$.

Consider the sink $(m_i : Q_i \to X)_I$ of all morphisms in $\mathbb{Q}_{\mathscr{M}_0}$. This is, m_i is a morphism in $\mathscr{M}_0$ with Q_i in $\mathbb{Q}$ and whenever i and j are in I such that $m_i \simeq m_j$, then $m_i = m_j$. Let $(\perp\!\!\!\perp_i : Q_i \to \coprod_I Q_i)_I$ be the coproduct sink in $\mathbb{A}$ and $[m_i] : \coprod_I Q_i \to X$ be the unique morphism such that $[m_i] \circ (\perp\!\!\!\perp_i)_I = (m_i)_I$. Let $m_X \circ e_X : \coprod_I Q_i \to \hat{Q} \to X$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $[m_i]$. We assert that $m_X : \hat{Q} \to X$ is the weak coreflection in $\mathscr{M}$. It should be clear that m_X is a member of $\mathscr{M}$, so we need only show that $\hat{Q}$ is in $\mathbb{Q}$ and that each morphism from $\mathbb{Q}$ to X factors through m_X. To that end, we first show that each morphism $f : Q \to X$, with Q in $\mathbb{Q}$, factors through m_X. Let f be a morphism such as above with $(\mathscr{E}, \mathbb{M})$-factorisation $m \circ e : Q \to Q' \to X$. Since $\mathbb{Q}$ is a left constant subcategory, it's closed under $\mathscr{E}$-images, therefore Q' is a member of $\mathbb{Q}$. By definition of the sink $(m_i)_I$, there is a unique $i \in I$ and an isomorphism $g : Q' \to Q_i$ such that $m_i g = m$. Then, $m_X \circ (e_X \perp\!\!\!\perp_i g e) = [m_i] \perp\!\!\!\perp_i g e = m_i g e = m e = f$, so that f factors through m_X.

$$\begin{tikzcd}
Q' \arrow[dr, "g"] \arrow[drr, "m"] & & Q \arrow[ll, "e"'] \arrow[d, "f"] \\
& Q_i \arrow[r, "m_i"] \arrow[d, "\amalg_i"'] & X \\
& \coprod_I Q_i \arrow[ur, dashed, "{![m_i]}"] \arrow[r, "e_X"'] & \hat{Q} \arrow[u, "m_X"']
\end{tikzcd}$$

Since each morphism with domain in $\mathbb{Q}$ and codomain X factors through m_X, we need only show that $\hat{Q}$ is a member of $\mathbb{Q}$. To that end, denote the reflector from $\mathbb{A}$ into the $\mathscr{R}(\mathscr{L}(\mathbb{P})) = \mathscr{R}(\mathbb{Q})$ by R with unit ρ. By Lemma 3.97, we have that A is a member of $\mathscr{L}(\mathscr{R}(\mathscr{L}(\mathbb{P}))) = \mathscr{L}(\mathbb{P}) = \mathbb{Q}$ if and only if $RA \in \mathbb{C}$. Let us denote the reflector with unit from $\mathbb{A}$ into $\mathbb{C}$ by S and ε. Note that, for any A, $RA \simeq SA$ if and only if A is a member of $\mathbb{Q}$. Furthermore, $\varepsilon_A \simeq \rho_A$ for any $A \in \mathscr{L}(\mathbb{P}) = \mathbb{Q}$. Without loss of generality, we may assume that $\varepsilon_A = \rho_A$ for any $A \in \mathscr{L}(\mathbb{P})$. Since any reflector is a left adjoint, it preserves colimits. In particular, it preserves coproducts. Consequently, we can consider the naturality diagram:

$$\begin{tikzcd}
Q_i \arrow[r, "\rho_{Q_i}"] \arrow[d, "\amalg_i"'] & RQ_i = SQ_i \arrow[d, "R(\amalg_i)"] \\
\coprod_I Q_i \arrow[r, "\rho_{\coprod_I Q_i}"'] \arrow[d, "e_X"'] & R\coprod_I Q_i \arrow[d, "Re_X"] \\
\hat{Q} \arrow[r, "\rho_{\hat{Q}}"'] & R\hat{Q}
\end{tikzcd}$$

It follows that $R(\coprod_I Q_i)$ and $S(\coprod_I Q_i)$ together with the sinks $(R(\amalg_i))_I$ and $(S(\amalg_i))_I$ are both coproducts in $\mathbb{P}$ and $\mathbb{C}$ respectively. Since, $\rho_A = \varepsilon_A$ for A in $\mathbb{Q}$, it follows that $R(\coprod_I Q_i)$ is a coproduct of both $(RM_i)_I = (SM_i)_I$ in $\mathbb{C}$. By our assumption on $\mathbb{C}$, this is also a coproduct in $\mathbb{A}$ and since $\mathbb{C}$ is a subcategory of $\mathbb{P}$, this is also a coproduct in $\mathbb{P}$. It follows that $R(\coprod_I Q_i) \simeq S(\coprod_I Q_i)$ and both are in $\mathbb{C}$. We then have that $Re_X \circ \rho_{\coprod_I Q_i} = \rho_{\hat{Q}} \circ e_X$ is a composition of morphisms in $\mathscr{E}$, hence in $\mathscr{E}$. Since the reflection $\rho_{\coprod_I Q_i}$ is also in $\mathscr{E}$, we have that Re_X is a member of $\mathscr{E}$. Since $\mathbb{C}$ is closed under $\mathscr{E}$-images, it follows that $R\hat{Q}$ is in $\mathbb{C}$ as well. Since RA is in $\mathbb{C}$ if and only if A is a member of $\mathbb{Q}$, it follows that $\hat{Q}$ is in $\mathbb{Q}$. Therefore, $m_X : \hat{Q} \to X$ is the weak $\mathscr{M}$-coreflection. $\square$

Corollary 3.126: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category that has coproducts, is $\mathscr{M}$-well-powered and let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$ that is closed under coproducts in $\mathbb{A}$. Then, each left constant subcategory is $\mathscr{M}$-coreflective.

Proof : It follows directly from Theorem 3.125 that every left constant subcategory is weakly $\mathscr{M}$-coreflective. Since $\mathbb{A}$ is $\mathscr{M}$-well-powered, $\mathscr{M}$ consists of a class of monomorphisms and hence the morphism defined by the weak coreflection is unique. $\square$

Proposition 3.127: Let $\mathbb{B}$ be an almost $\mathscr{M}$-coreflective subcategory of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$. If $\mathbb{A}$ has coproducts, then $\mathbb{B}$ is closed under coproducts in $\mathbb{A}$.

Proof : For each $\mathbb{A}$-object A, let $m_A : B_A \to A$ be a weak $\mathscr{M}$-coreflection. Let $(B_i)_I$ be a set-indexed family of $\mathbb{B}$-objects and let $(\amalg_i : B_i \to A)_I$ be the coproduct in $\mathbb{A}$. Then, by the weak coreflection property, for each $i \in I$, there is a morphism $c_i : B_i \to B_A$ such that $m_A c_i = \amalg_i$. Since A is a coproduct of $(B_i)_I$, there is a unique morphism $[c_i] : A \to B_A$ such that for each i, $[c_i]\amalg_i = c_i$. Then,

$m_A[c_i]\perp\!\!\!\perp_i = m_A c_i = \perp\!\!\!\perp_i$ implies that $m_A[c_i] = id_A$. Since m_A is a retraction and almost $\mathscr{M}$-coreflective subcategories are closed under retractions, it follows that A is a member of $\mathbb{B}$. Thus, $\mathbb{B}$ is closed under coproducts in $\mathbb{A}$.

$$\begin{tikzcd}
B_i \arrow[d, "\perp\!\!\!\perp_i"'] \arrow[dr, "c_i"] & \\
A \arrow[r, dashed, bend right, "{![c_i]}"'] & B_A \arrow[l, "m_A"']
\end{tikzcd}$$

□

Corollary 3.128: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category that has coproducts and is $\mathscr{M}$-small. Let $\mathbb{C}$ be a reflective constant subcategory of $\mathbb{A}$. Then, the following are equivalent:

(i) $\mathbb{C}$ is weakly $\mathscr{M}$-coreflective in $\mathbb{A}$;

(ii) $\mathbb{C}$ is closed under coproducts in $\mathbb{A}$;

(iii) Each left constant subcategory is closed under coproducts in $\mathbb{A}$;

(iv) Each left constant subcategory is weakly $\mathscr{M}$-coreflective in $\mathbb{A}$.

Proof : (iv) implies (i) and (iii) implies (ii) follow as $\mathbb{C}$ is left constant. (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii) follow from 3.127. (ii) $\Rightarrow$ (i), (iv) both follow from 3.125.

□

Example 3.129: Our proof of Theorem 3.125 does not work if $\mathbb{A}$ is not $\mathscr{M}$-small and not thin. It is not known whether Corollary 3.128 is still true if $\mathbb{A}$ is not $\mathscr{M}$-small. The main reason why our proof fails if $\mathbb{A}$ is not $\mathscr{M}$-small is since the coproduct in the construction need not exist. What follows is an example where the category in question is not $\mathscr{M}$-small and it is not known whether the conclusion of Proposition 3.127 holds. Consider the category $\mathbb{Ab}$ of abelian groups with factorisation system for sources $(\mathscr{E}, \mathbb{M})$ with

$$\mathscr{E} = \{f : X \to Y \mid f \text{ is surjective and } Ker(f) \in \mathbb{Tor}\} \text{ and}$$

$$\mathbb{M} = \{(m_i : X \to X_i)_I \mid \bigcap_I Ker(m_i) \in \mathbb{TFAb}\}.$$

To see that $\mathbb{Ab}$ is an $(\mathscr{E}, \mathbb{M})$-category. Let $(f_i : X \to X_i)_I$ be a source in $\mathbb{Ab}$. Let $K_i = Ker(f_i)$ and let $K = \bigcap_I K_i$. To simplify notation, let TX denote the torsion subgroup of any abelian group X. Note that $TK \trianglelefteq X$ and that there is a canonical morphism $e : X \to X/TK$. Define for each $i \in I$ a map $m_i : X/TK \to X_i$ by $m_i(x + TK) = f_i(x)$. To see that m_i is well-defined, let $x + TK = x' + TK$. Then $x - x' \in TK \subset K \subset K_i$, so that $m_i(x) - m_i(x') = m_i(x - x') = 0$. It follows that m_i is a well defined map and it's easy to see that it's a group homomorphism. Furthermore, for each $x \in X$, $m_i(e(x)) = m_i(x + TK) = f_i(x)$. We now prove that $e \in \mathscr{E}$ and $(m_i)_I \in \mathbb{M}$. Since e is surjective and $Ker(e) = TK \in \mathbb{Tor}$, $e \in \mathscr{E}$.

Suppose that $x + TK$ is an element of finite order of $\bigcap_I Ker(m_i)$. We prove that $x \in TK$. Since $x + TK \in \bigcap_I Ker(m_i)$, $f_i(x) = m_i(x + TK) = 0$ for each $i \in I$, hence $x \in Ker(f_i)$ and therefore $x \in K \subset K_i$. Since $x + TK$ has finite order, there is an $n \in \mathbb{N}$ such that $nx \in TK$. Since TK is a torsion group, there exists an $m \in \mathbb{N}$ such that $mnx = m(nx) = 0$. Therefore x is a torsion element of X and thus also of K. It follows that $x \in TK$, or equivalently, $x + TK = 0 + TK$. It follows that $(m_i)_I$ is a source in $\mathbb{M}$ and each source in $\mathbb{Ab}$ has an $(\mathscr{E}, \mathbb{M})$-factorisation. It should be clear that each of $\mathscr{E}$ and $\mathbb{M}$ are closed under composition with isomorphisms.

To see that the $(\mathscr{E}, \mathbb{M})$-diagonalisation property holds, suppose that
$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g_i"] \\
X \arrow[r, "m_i"'] & X_i
\end{tikzcd}$$
commutes in $\mathbb{Ab}$ with e in $\mathscr{E}$ and $(m_i)_I$ in $\mathbb{M}$. We define $d : B \to X$ as follows: For each $b \in B$, there is an $a_b \in A$ such that $e(a_b) = b$. Define $d(b) = f(a_b)$. We need to show that d is a well defined homomorphism. If a and a' are members of A such that $e(a) = e(a')$, then $m_i(f(a)) = g_i(e(a)) = g_i(e(a')) = m_i(f(a'))$ holds for each $i \in I$. Therefore, $f(a) - f(a') = f(a - a')$ is in $Ker(m_i)$ for each $i \in I$. Note that $e(a - a') = 0$ so that $a - a'$ is in the kernel of e. Since $e \in \mathscr{E}$, $a - a'$ has finite order and thus so has $f(a - a')$. Since $\bar{K} := \bigcap_I Ker(m_i)$ is torsion free and $f(a - a')$ is an element of finite order in

$\bar{K}$, it follows that $f(a) - f(a') = f(a - a') = 0$. Well definedness of d follows. It's easy to see that d is a homomorphism and unique with respect to the property that $de = f$. Also, for each $b \in B$, $m_i(d(b)) = m_i(f(a_b)) = g_i(e(a_b)) = g_i(b)$.

Let $\mathbb{C}$ be the subcategory of $\mathbb{Ab}$ of all torsion free abelian groups, i.e., $\mathbb{C} = \mathbb{TFAb}$. It's easy to verify that $\mathbb{C}$ is a reflective constant subcategory that is closed under coproducts in $\mathbb{Ab}$.

It's not known whether $\mathbb{C}$ is almost $\mathscr{M}$-coreflective. However, if $\mathbb{C}$ is almost $\mathscr{M}$-coreflective, then it can be shown that each weak coreflection may be taken as a surjective morphism in $\mathscr{M}$.

Remark 3.130: Consider the category $\mathbb{Top}$ of topological spaces with continuous maps and with factorisation structure ($\mathscr{E}$, $\mathbb{M}$)=(Epi($\mathbb{Top}$), Initial mono-sources($\mathbb{Top}$)) and the reflective constant subcategory $\mathbb{C}$ consisting of spaces with at most one point.

Even though this satisfies all the assumptions of Corollary 3.128, it can easily be seen that $\mathbb{C}$ is not closed under coproducts in $\mathbb{Top}$ and hence there are left constant subcategories that are not $\mathscr{M}$-coreflective. In particular, for $\mathbb{P} = \mathbb{TDisc}$, $\mathscr{L}(\mathbb{P}) = \mathbb{Con}$ consists of all connected spaces(including the empty space). The category of connected spaces is clearly not coreflective as can be seen by taking any space with at least two connected components.

However, each space can be broken up into its connected components and by adding a component for the empty space, each continuous map $f : C \to X$, with C in $\mathscr{L}(\mathbb{TDisc})$, will factor through at least one of these components. In case C does not factor through the empty space, then it's easy to see that the map factors through a unique component. Our focus for the remainder of this section is to generalise this idea in such a way that whenever the constant subcategory is not closed under coproducts in $\mathbb{A}$, then we can still factor it through some small family of morphisms.

Definition 3.131: Chained sink
Let $\mathbb{A}$ be a category and $\mathbb{B}$ be a subcategory of $\mathbb{A}$. Let $(f_i : B_i \to X)_I$ be a family of morphisms in the comma category $\mathbb{B}/X$. Then, $(f_i)_I$ is said to be **a $\mathbb{B}$-chained sink** (or **a chained sink from $\mathbb{B}$**) if and only if for each i and j in I, there exists a finite $\mathbb{B}$-zig-zag from f_i to f_j. If we simply say that $(f_i)_I$ is chained, then there is some subcategory $\mathbb{B}$ of $\mathbb{A}$ for which $(f_i)_I$ is a chained sink from $\mathbb{B}$.

Note that we need not choose the full subcategory $\mathbb{B}$ of $\mathbb{A}$ and hence there might be an $\mathbb{A}$-morphism from f to g, yet no $\mathbb{B}$-morphism from f to g. Of course, if not stated otherwise, then we will simply assume that it's the full subcategory.

Let $\mathbb{X}$ be a subcategory of $\mathbb{Y}$, and $\mathbb{Y}$ a subcategory of $\mathbb{A}$. Then, $\mathbb{X}$ is said to be **closed under $\mathbb{Y}$-chained sinks (in $\mathbb{A}$)** provided that whenever $(g_i : X_i \to A)_I$ is a $\mathbb{Y}$-chained $\mathbb{A}$ sink with domain in $\mathbb{X}$, then A is a member of $\mathbb{X}$.

If $\mathbb{Y} = \mathbb{X}$, then we will simply say that $\mathbb{X}$ is closed under chained sinks in $\mathbb{A}$.

Definition 3.132: $\mathscr{E}$-reflector
Let $\mathbb{A}$ be a category and $\mathscr{E}$ a class of $\mathbb{A}$-morhisms. If $R : \mathbb{A} \to \mathbb{B}$ is a reflector with unit ε, then R is said to be an **$\mathscr{E}$-reflector** if $\varepsilon_A \in \mathscr{E}$ for each $\mathbb{A}$-object A. If $\mathscr{E}$ is a class of $\mathbb{A}$ epimorphisms, then R is simply called an **epireflector.**

Proposition 3.133: The following statements hold for chained and epi-sinks:
(a) Functors preserve chained sinks.
(b) Epireflectors preserve episinks in $\mathbb{A}$.
(c) Epireflectors preserve chained epi-sinks in $\mathbb{A}$.

Proof : Throughout this proof, let $(f_i : A_i \to A)_I$ be an $\mathbb{A}$-sink and let $F : \mathbb{A} \to \mathbb{X}$ be a functor.

For (a), we need only show that $(Ff_i : FA_i \to FA)_I$ is chained whenever $(f_i)_I$ is chained. Assume that $(f_i)_I$ is $\mathbb{B}$-chained and let $i, j \in I$. Let $f_1, f_2, \ldots, f_n$ be a $\mathbb{B}$-zig-zag from f_i to f_j. Consider the subcategory $F(\mathbb{B})$ of $\mathbb{X}$ which has objects of the form FB, where B is in $\mathbb{B}$ and morphisms of the form

Ff, where f is a $\mathbb{B}$-morphism. Then, $Ff_1, Ff_2, \ldots, Ff_n$ is an $F(\mathbb{B})$-zig-zag from Ff_i to Ff_j. Hence $(Ff_i)_I$ is $F(\mathbb{B})$-chained if $(f_i)_I$ is $\mathbb{B}$-chained.

To show (b), let $(f_i : A_i \to A)_I$ be an epi-sink in $\mathbb{A}$ and F an epireflector with unit ε. Suppose morphisms $g, h : FA \rightrightarrows X$ are given such that $gFf_i = hFf_i$ for each $i \in I$. Then, for each $i \in I$ we have: $g\varepsilon_A f_i = gFf_i\varepsilon_{A_i} = hFf_i\varepsilon_{A_i} = h\varepsilon_A f_i$. Since $(f_i)_I$ is an epi-sink and ε_A an epimorphism, it follows that $g = h$. Therefore epireflectors preserve epi-sinks.

The last part of the proof follows from (a) and (b).

$\square$

Remark 3.134: We will now start to consider categories where the comma category $\mathscr{M}/X$ is a complete pre-ordered class. Of course it is more fruitful to only consider it as a pre-ordered class provided that $\mathscr{M}$ is a class of monomorphisms, otherwise there might be morphisms m and m' in $\mathscr{M}/X$ for which $m \leq m'$ and $m' \leq m$ without the domains being isomorphic. If $\mathscr{M}/X$ is complete, then every object X has a least $\mathscr{M}$-subobject. If I is an initial object of $\mathbb{A}$, then, of course, we can consider the $(\mathscr{E}, \mathscr{M})$-factorisation $m_X e_X : I \to M \to X$ of the unique morphism $i_X : I \to X$. It is relatively easy to see that the diagonalisation property establishes that m_X is the least subobject of X.

It should be noted that if $\mathbb{A}(I, X)$ is in $\mathscr{M}$ for each $\mathbb{A}$-object X, then each morphism $f : A \to I$ is in $\mathscr{E}$. To see this, let $m \circ e : X \to Y \to I$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of f and denote the unique morphism $I \to Y$ by m_Y. Then, $m \circ m_Y = id_I$ so that m is a monomorphism and retraction, i.e., m is an isomorphism. It follows that $f = me$ is in $\mathscr{E}$. For the remainder of this section, we will denote the least element of $\mathscr{M}/X$ by m_{0_X}, or simply m_0 when the object X is clear. We will not assume that $\mathbb{A}$ has an initial object or that if it does, then the least $\mathscr{M}$-subobject's domain is the initial object, in other words $\mathbb{A}(I, X) \subset \mathscr{M}$, but it's noteworthy when looking at some examples in topology.

Remark 3.135: The next definition is very similar to the definition of connected component, except for the fact that it excludes the minimal subobject into it's own equivalence class. The main reason for this approach is to develop a similar notion to multi-$\mathscr{M}$-coreflectivity via constant subcategories. This is especially for categories with initial objects for which left constant subcategories would be multi-$\mathscr{M}$-coreflective, but fails as it contains an initial object. In many situations this would force the category to have at most one connected component and hence we would get $\mathscr{M}$-coreflective subcategories. An example of this phenomenon is illustrated in Remark 3.130 with the category $\mathbb{T}\mathrm{op}$ with the left constant category of connected spaces (including the empty space). The notion of chained sink can be viewed as a generalisation of chained connected subspaces in topology which is one way of characterising the connected components in $\mathbb{T}\mathrm{op}$. This notion is chosen over just connected components of categories as factorisation of morphisms have also been kept in mind. Of course one needs to bear in mind that a left constant subcategory need not be thought of as a generalised collection of 'connected objects'in some category. For this reason, there is no ambiguity or problem when considering the empty space in $\mathbb{T}\mathrm{op}$ as a member of a left constant subcategory.

Definition 3.136: Least $\mathscr{M}$-subobject component relation
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category such that $\mathscr{M}$ is a class of monomorphisms, and for each $\mathbb{A}$-object X, let us denote the least $\mathscr{M}$-subobject by $m_0 : X_0 \to X$, if it exists. For each subcategory $\mathbb{B}$ of $\mathbb{A}$ and each $\mathbb{A}$-object X, we define a relation on the full subcategory $\mathbb{B}_{\mathscr{M}}/X$ of the comma category $\mathbb{B}/X$ with object class $\{m : B \to X \mid m \in \mathscr{M}\}$.

Define $\sim_{X,\mathbb{B}}$ on $Ob(\mathbb{B}_{\mathscr{M}}/X)$ by $(m : M \to X) \sim_{X,\mathbb{B}} (m' : M' \to X)$ if and only if $m \simeq m_0 \simeq m'$, or if there is a finite $\mathbb{B}_{\mathscr{M}}$-zig-zag, namely $m_1, m_2, \ldots, m_n$, from m to m' such that for each $i = 1, 2, \ldots, n$, there holds $m_i \not\simeq m_0$.

It is of course a straightforward exercise to show that $\sim_{X,\mathbb{B}}$ is an equivalence relation on $Ob(\mathbb{B}_{\mathscr{M}}/X)$. Note that if there is no least $\mathscr{M}$-subobject of X in $\mathbb{B}$, then the equivalence classes are just the connected components of $\mathbb{B}$ in X. This relation is called the **least $\mathscr{M}$-subobject component relation** with respect to $\mathbb{B}$. If the least subobject of $\mathscr{M}/X$ is in $\mathbb{B}_{\mathscr{M}}/X$, then the equivalence classes will of course be distinct from the connected components in general.

Example 3.137: Consider the category $\mathbb{Top}$ with the same factorisation structure and constant subcategory as in Remark 3.130. Consider the right constant subcategory $\mathbb{P} = \mathbb{TDisc}$ and left constant subcategory $\mathscr{L}(\mathbb{P})$. Then, for each space X, the least subobject component relation has equivalence classes, namely $\{\varnothing_X : \varnothing \to X\}$, and for each $x \in X$, the equivalence class

$$[x] := \{m : C \to X \mid m \text{ is an embedding, } C \text{ is connected and } x \in m[C]\}.$$

For each $x \in X$, we can endow the union $C_x := \bigcup_{m \in [x]} m[C]$ with the subspace topology and it can be easily shown that C_x is connected for each $x \in X$. Then, we consider the collection

$$C_X := \{m_x : C_x \to X \mid x \in X\} \cup \{\varnothing_X : \varnothing \to X\}.$$

For each continuous map $f : B \to X$ with B in $\mathscr{L}(\mathbb{P})$, i.e., B is empty or B is a connected space, f factors through at least one of the members in C_X. In fact, f either factors through all of the members or f factors through a unique member. To be more specific, f factors through each member of C_X if and only if B is empty. If B is not empty, then if $me : B \to B' \to X$ is a (surjective, embedding) factorisation of f, then B' is connected and contains some $x \in X$. It can then easily be seen that m factors through m_x and consequently f factors through m_x.

Another way to view this is to only consider the collection $\{m_x : C_x :\to X \mid x \in X\}$. Then, each $f : B \to X$, with B in $\mathscr{L}(\mathbb{TDisc})$, still factors through at least one m_x and factors through a unique m_x if and only if B is non-empty. Furthermore, if it factors through each m_x, then it follows that B must be empty. Note that B being empty in $\mathbb{Top}$ is equivalent to there exists an epimorphism $B \to \varnothing$, when $\varnothing$ is viewed as an initial object of $\mathbb{Top}$.
This motivates the following definition for our generalised notion of multi-$\mathscr{M}$-coreflectiveness.

Definition 3.138: Nearly multi-$\mathscr{M}$-coreflective
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category such that $\mathscr{M}$ is a class of monomorphisms. Then, a subcategory $\mathbb{B}$ of $\mathbb{A}$ is said to be **nearly multi-$\mathscr{M}$-coreflective** if the following conditions hold for each X in $\mathbb{A}$:

- (i) The distinct equivalence classes of the least $\mathscr{M}$-subobject relation can be represented by a (small) set, say J. If X has a least $\mathscr{M}$-subobject $m_0 : X_0 \to X$ with X_0 in $\mathbb{B}$, then denote the representative member of the equivalence class containing $m_0 : X_0 \to X$ by $j_0 \in J$.
- (ii) For each $j \in J$, the equivalence class corresponding to j of the least $\mathscr{M}$-subobject relation has a terminal object $\eta_j : B_j \to X$;
- (iii) If $f : B \to X$ is any $\mathbb{A}$-morphism with domain in $\mathbb{B}$, then f either factors through each η_j for each $j \in J$ or there is a unique $j \in J$ such that f factors through η_j. Furthermore, if f factors through η_j, then it factors through η_j uniquely and f factors through η_j for each $j \neq j_0$ if and only if f factors through η_j for each $j \in J$ if and only if f factors through η_{j_0}.

For each X in $\mathbb{A}$, one of the following holds: Either J contains j_0 or J does not contain j_0. If J contains j_0, then either $J = \{j_0\}$ or $J \setminus \{j_0\} \neq \varnothing$. If $J = \{j_0\}$, then we define $J(X) = J$ and if $J \setminus \{j_0\} \neq \varnothing$, then we define $J(X) = J \setminus \{j_0\}$. If J does not contain j_0, then we define $J(X) = J$. We call the family $(\eta_j : B_j \to X)_{J(X)}$ the **near multi-$\mathscr{M}$-coreflection of X in $\mathbb{B}$.**

Note that when $\mathbb{B}$ is closed under $\mathscr{E}$-images, then properties (i) and (ii) imply the first part of (iii) above. Suppose that $\mathbb{B}$ is nearly multi-$\mathscr{M}$-coreflective. Then, if $\mathbb{B}$ is closed under $\mathscr{E}$-images, then we also have the following:
Given any $\mathbb{B}$-object and $f : B \to X$, we can form the $(\mathscr{E}, \mathbb{M})$-factorisation $me : B \to B' \to X$ of f. Since B' is then a member of $\mathbb{B}$, it follows that the equivalence class $[m]_{\sim_{X,\mathbb{B}}}$ has a terminal object $\eta_i : B_i \to X$ for some $i \in J$. If η_i is the least $\mathscr{M}$-subobject, then for each $j \in J$, there exists a unique morphism $f_j : B \to B_j$ such that $\eta_j f_j = f$. If η_i is not the least $\mathscr{M}$-subobject, then there is a unique pair $(i, g) \in J \times \bigcup_J \; Mor(B, B_j)$ such that $\eta_i g = f$.

Also note that if f factors through η_{j_0}, then there is a morphism $e : B \to X_0$ in $\mathscr{E}$ such that $f = m_0 e$.

Furthermore, if $J(X) = \{j_0\}$, then f factors uniquely through η_{j_0}. If $J(X)$ contains j_0 and $J(X) = J \setminus \{j_0\}$, then the following holds: $m_0 : X_0 \to X$ is the least $\mathscr{M}$-subobject of X if and only if for each $j \in J(X)$, and there is a unique morphism f_j such that $\eta_j f_j = f$. This implies that if $m : \hat{B} \to X$ is in $\mathbb{B}_{\mathscr{M}}/X$ is not the least $\mathscr{M}$-subobject of X, then there is a unique pair $(j_f, g_f) \in J(X) \times \bigcup_{j \in J(X)} \; Mor(B, B_j)$

such that $\eta_{j_f} \circ g_f = f$. In conclusion, each morphism with domain in $\mathbb{B}$ and codomain X either factors uniquely through every member of a small family of morphisms in $\mathscr{M}$ with domain in $\mathbb{B}$ and codomain X, or there is a unique member of the same small family such that the morphism factors uniquely through it.

The reason for choosing the smaller family indexed by $J(X)$ instead of J will be made clear later in this section. Of particular importance is that if j and k are members of $J(X)$, then η_j factors through η_k if and only if $j = k$ and consequently $\eta_j = \eta_k$. This will not be true if the family is chosen with indexing set J, for if η_{j_0} is a member of J and $J \setminus \{j_0\} \neq \varnothing$, then η_{j_0} will factor through every η_j.

Remark 3.139: Suppose that $\mathbb{A}$ is any $(\mathscr{E}, \mathbb{M})$-category that is $\mathscr{M}$-wellpowered and suppose that $\mathbb{B}$ is a nearly multi-$\mathscr{M}$-coreflective subcategory of $\mathbb{A}$ with near multi-$\mathscr{M}$-coreflection $(\eta_j^X)_{J(X)}$ for each $\mathbb{A}$-object X. Suppose that for each $\mathbb{A}$-object X, there is a small family $(\phi_i^X : B_i^X \to X)_{I(X)}$ of morphisms in $\mathscr{M}$ with domain in $\mathbb{B}$ such that whenever $f : B \to X$ is any morphism with B in $\mathbb{B}$, then f either factors through each ϕ_i or there is a unique $i \in I$ such that f factors through ϕ_i. In other words, suppose that for each X in $\mathbb{A}$, there is a small family $(\phi_i^X : B_i^X \to X)_{I(X)}$ such that the first part of property *(iii)* of Definition 3.138 holds. Let us assume that $\mathbb{B}$ is nearly multi-$\mathscr{M}$-coreflective such that for each X in $\mathbb{A}$, the near-multi-$\mathscr{M}$-coreflection is denoted by $(\eta_j^X : B_j^X \to X)_{J(X)}$. For each such X, let us denote the least $\mathscr{M}$-subobject of X in $\mathbb{B}/X$, if it exists, by the morphism $\eta_{j_0}^X : B_0^X \to X$. Of course, if this morphism exists, then η_0^X is only a member of the family $(\eta_j^X)_{J(X)}$ provided that $J(X)$ only only has j_0 as a member.

If $J(X) = \{j_0\}$, then each morphism in $\mathscr{M}$ is isomorphic to $\eta_{j_0}^X$ and so for each $i \in I(X)$, $\phi_i^X \simeq \eta_{j_0}^X$. Let us now assume that $J(X) = J \setminus \{j_0\}$.

Then, for each $i \in I(X)$, either ϕ_i factors through η_j^X for each $j \in J(X)$ or there is a unique $j(i) \in J(X)$ such that ϕ_i^X factors through $\eta_{j(i)}^X$. Note that if ϕ_i^X factors through each η_j, then ϕ_i^X factors through $m_0 = \eta_{j_0}^X$. Let j be any fixed element of $J(X)$ and define a map $m : I(X) \to J(X)$ as follows:

$$m(i) = \begin{cases} j & \text{if } \phi_i \text{ factors through } \eta_{j_0}^X \\ j(i) & \text{if } j(i) \text{ is the unique } \ j \in J(X) \text{ such that } \phi_i^X \text{ factors through } \eta_{j(i)}^X \end{cases}$$

Consequently, for each $i \in I(X)$, there is a unique morphism n_i such that $\eta_{m(i)}^X \circ n_i = \phi_i^X$. Note that this implies that $\phi_i^X \leq \eta_{m(i)}^X$.

In a similar manner, for each $j \in J(X)$, either η_j^X factors through each ϕ_i^X or there is a unique $i(j) \in I(X)$ such that η_i^X factors through ϕ_i^X. Choose any fixed $i \in I$ and define a map $\hat{m} : J(X) \to I(X)$ as follows:

$$\hat{m}(j) = \begin{cases} i & \text{if } \eta_j^X \text{ factors through each } \phi_i^X \\ i(j) & \text{if } i(j) \text{ is the unique } \ i \in I(X) \text{ such that } \eta_i^X \text{ factors through } \phi_{i(j)}^X \end{cases}$$

Consequently, for each $j \in J(X)$, there is a unique morphism $\hat{n}_j$ such that $\phi_{\hat{m}(j)}^X \circ \hat{n}_j = \eta_j^X$. Note that this implies that $\eta_j^X \leq \phi_{\hat{m}(j)}^X$.

Putting this together, we have for each $j \in J$: $\eta_j^X \leq \phi_{\hat{m}(j)}^X \leq \eta_{m(\hat{m}(j))}^X$. Of course this implies that $\eta_j^X \leq \eta_{m(\hat{m}(j))}^X$ and since both have domain in $\mathbb{B}$, this establishes a finite $\mathbb{B}$-zig-zag from η_j^X to $\eta_{m(\hat{m}(j)))}^X$. Since j is not j_0, we have $\eta_j^X = \eta_{m(\hat{m}(j)))}^X$ and this implies that $(m \circ \hat{m})(j) = j$. In particular, $\hat{m}$ is injective.

With this in mind, we may think of our family of initial-multi-$\mathscr{M}$-coreflections as the smallest family for which every morphism $f : B \to X$ with domain in $\mathbb{B}$ either factors through each member or factors through exactly one member. Furthermore, if f factors through each member, then there is a smallest member, namely $\eta_{j_0}^X$, through which f factors. This still provides us with some sort of universal factorisation.

One might be tempted to ask whether we need to reconsider whether $\eta_{j_0}^X$ needs to be a member of the family of initial-multi-$\mathscr{M}$-coreflections. This is definitely another approach that could be followed, but it has some disadvantages. One particular disadvantage is when $|J(X)| = 1$, where $J(X)$ is defined as in 3.138, for each $\mathbb{A}$-object X. In particular, if we do not remove j_0, then it will follow that $|J(X)| = 1$

or $|J(X)| = 2$ for each $\mathbb{A}$-object X. In case $|J(X)| = 2$. It can then be shown that if the category in question is actually $\mathscr{M}$-coreflective which can still be recovered when taking $J(X)$ without removing j_0. But this is unnecessarily tedious.

On the other hand, if $\mathbb{B}$ is an $\mathscr{M}$-coreflective subcategory of $\mathbb{A}$ with counit η, then it follows readily that $J(X)$ will contain only one element. It's easily seen that there can be at most two members namely η_X and $\eta^X_{j_0}$, hence by removing $\eta^X_{j_0}$ where necessary, $|J(X)| = 1$. If the coreflection of X in $\mathbb{B}$ is denoted by $\eta_X : BX \to X$, then either η_X is the least $\mathscr{M}$-subobject or not. If BX is the least $\mathscr{M}$-subobject, then obviously $J(X)$ will only contain one element η_X. If BX is not the least $\mathscr{M}$-subobject, then there will be exactly two equivalence classes of the least $\mathscr{M}$-subobject relation. Before removing j_0, $J(X)$ will contain exactly two morphisms $\eta^X_{j_0} : B_0 \to X$ and $\eta_X : BX \to X$. Of course we discard $\eta^X_{j_0} : B_0 \to X$ in this case and the family $(\eta_X : BX \to X)_{J(X)}$ with single morphism η_X is the initial-multi-$\mathscr{M}$-coreflection for each $\mathbb{A}$-object X.

Note that we didn't have to define near multi-coreflectivity only for classes of monomorphisms and $\mathbb{B}$ also need not have a least $\mathscr{M}$-subobject for each $\mathbb{A}$-object X. In particular, we may consider a subcategory $\mathbb{B}$ of $\mathbb{A}$ such that $\mathbb{B}/X$ has an initial object i_X for each X in $\mathbb{A}$. We can then generalise the definition of the least-$\mathscr{M}$-subobject relation on $Ob(\mathbb{B}/X)$ to be $f \sim_{X,\mathbb{B}} g$ if and only if f and g are initial in $\mathbb{B}/X$ or if there is a finite $\mathbb{B}$-zig-zag $f_1, f_2, \ldots, f_n$ from f to g such that f_i is not initial in $\mathbb{B}/X$ for each $i = 1, 2, \ldots, n$. Of course this is for the full subcategory $\mathbb{B}$ and if we only want to consider some class of $\mathbb{A}$-morphisms $\mathscr{M}$, we can define the relation $\sim_{X,\mathbb{B}}$ on the subcategory $\mathbb{B}_{\mathscr{M}}/X$ with object class $\{m : B \to X \mid m \in \mathscr{M} \text{ and } B \in \mathbb{B}\}$ by $m \sim_{X,\mathbb{B}} m'$ if and only if m and m' are isomorphic to m_X, where $m_X e_X$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of i_X, or if there is a finite $\mathbb{B}_{\mathscr{M}}$-zig-zag $m_1, m_2, \ldots, m_n$ from m to m' such that for each $i = 1, 2, \ldots, n$: m_i is not isomorphic to m_X.

Note that the last relation for $\mathscr{M} = Mor(\mathbb{A})$ is the same as the first. We can then require that the equivalence classes must be represented by a set and that each equivalence class must have a terminal object in $\mathscr{M}$. We can then require that each morphism with domain in $\mathbb{B}$ and codomain X must either factor uniquely through each terminal object of the equivalence classes or, it must factor uniquely through a unique terminal object of some equivalence class.

We now turn our attention to sufficient conditions for each equivalence class to have a terminal object whenever $\mathscr{M}$ is a class of monomorphisms in $\mathbb{A}$.

Definition 3.140: Epic Joins
Let $\mathbb{A}$ be a category and $\mathscr{M}$ a class of monomorphisms. Then, $\mathbb{A}$ is said to have **epic joins of $\mathscr{M}$-subobjects** if and only if for every $\mathbb{A}$-object X and each family $(m_i)_I$ in $\mathscr{M}/X$ the following holds: Joins of $\mathscr{M}$-subobjects exist and whenever m is a join of $(m_i)_I$ in $\mathscr{M}/X$ and $(k_i)_I$ is the sink such that $mk_i = m_i$ for each $i \in I$, then $(k_i)_I$ is an epi-sink.

Lemma 3.141: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured with $\mathscr{M}$ a class of monomorphisms and $\mathscr{E}$ a class of epimorphisms. Let $(m_i : M_i \to X)_I$ be a family of $\mathscr{M}$-subobjects of X in $\mathbb{A}$. Then, $\mathscr{M}/X$ has epic joins if $\mathbb{A}$ has:

(i) multiple pullbacks and multiple pushouts;

(ii) coproducts and $\mathbb{A}$ is $\mathscr{M}$-well-powered.

The construction of the join of $(m_i)_I$ is depicted below by the morphism $m : M \to X$:

(i) In this diagram, the bottom left triangle is a multiple pullback diagram of $(m_i)_I$. The top part of the diagram is a multiple pushout of the source $(j_i)_I$, i.e., k is a multiple pushout of $(j_i)_I$. Since $m_i j_i = f$ for each $i \in I$, the pushout establishes a morphism $\bar{f} : P \to X$ such that $\bar{f} k_i = m_i$. Let $m \circ e : P \to M \to X$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of $\bar{f}$.

$$\begin{tikzcd}
A \arrow[r, "j_i"] \arrow[rr, bend left, "k"] \arrow[dr, "f"'] & M_i \arrow[r, "k_i"] \arrow[d, "m_i"] & P \arrow[d, "e"] \arrow[dl, dashed, "!\bar{f}"] \\
& X & M \arrow[l, "m"]
\end{tikzcd}$$

(*ii*) Since $\mathbb{A}$ is $\mathscr{M}$-well-powered, we will assume that I is a set. In this diagram, we construct the coproduct $\amalg_i : M_i \to \coprod_I M_i$. The coproduct establishes a unique morphism $[m_i] : \coprod_I M_i \to X$ such that for each $i \in I$, there holds: $[m_i] \circ \amalg_i = m_i$. Then, $m \circ \bar{e}$ is the $(\mathscr{E}, \mathscr{M})$-factorisation of $[m_i]$.

$$\begin{tikzcd}
M_i \arrow[r, "\amalg_i"] \arrow[d, "m_i"'] & \coprod_I M_i \arrow[d, "\bar{e}"] \arrow[dl, dashed, "{![m_i]}"'] \\
X & M \arrow[l, "m"]
\end{tikzcd}$$

<u>$\mathcal{P}roof$</u> : We first show that the sinks $(ek_i : M_i \to M)_I$ and $(\bar{e}\amalg_i : M_i \to M)_I$ are episinks in $\mathbb{A}$.

For the first diagram, we first show that $(k_i)_I$ is an episink in $\mathbb{A}$. To this end, let $g_1, g_2 : P \to Y$ be morphisms such that for each $i \in I$, there holds $g_1 k_i = g_2 k_i$. Then, $g_1 k_i j_i = g_2 k_i j_i$ holds for each $i \in I$. In particular, since $k_i j_i = k$ for each $i \in I$, $g_1 k = g_2 k$ and $g_1 k_i j_i = g_1 k_{i'} j_{i'} = g_2 k_i j_i = g_2 k_{i'} j_{i'}$ holds for each pair of elements i and i' of I. The multiple pushout establishes a unique morphism $\bar{g} : P \to Y$ such that, in particular, $\bar{g}k = g_1 k = g_2 k$. It follows that $g_1 = \bar{g} = g_2$ and that $(k_i)_I$ is an episink. Since $\mathscr{E}$ is a class of epimorphisms and a composition of episinks is an episink, we have that $e(k_i)_I$ is an episink.

That $\bar{e}(\amalg_i)_I$ is an episink follows from the fact that $\mathscr{E}$ is a class of epimorphisms, colimits are extremal episinks and that episinks are closed under composition.

For the second part of the proof, we will show that m can be regarded as the join of $(m_i)_I$ in $\mathscr{M}/X$. It ought to be clear that $m \geq m_i$ for each $i \in I$. For the remainder of the proof, assume that $n : N \to X$ is a morphism $\mathscr{M}$ such that for each $i \in I$, we have that $n \geq m_i$. Then, for each $i \in I$, there is a morphism $h_i : M_i \to N$ such that $nh_i = m_i$.

For the first diagram we then have for each $i \in I$: $nh_i j_i = m_i j_i = f$. Since $\mathscr{M}$ is a class of monomorphisms, $h_i j_i$ is the same morphism for each $i \in I$, say h. The multiple pushout then establishes a unique morphism $\bar{h} : P \to N$ such that $\bar{h}k_i = h_i$ and $\bar{h}k = h$. Then, $n\bar{h}k_i = nh_i = m_i = mek_i$ and since $(k_i)_I$ is an episink, $n\bar{h} = me$. The diagonalisation property gives a unique morphism $d : M \to N$ such that $de = \bar{h}$ and $nd = m$. Consequently $m \leq n$ so that m is the join of the family $(m_i)_I$.

$$\begin{tikzcd}
A \arrow[r, "j_i"] \arrow[dr, "f"'] \arrow[rr, bend left, "k"] \arrow[rrr, bend left=50, "h"] & M_i \arrow[r, "k_i"] \arrow[d, "m_i"] \arrow[rr, bend left, "h_i"] & P \arrow[r, dashed, "!\bar{h}"] \arrow[d, "e"] & N \arrow[dll, bend left, "n"] \\
 & X & M \arrow[l, "m"] \arrow[ur, dashed, "!d"'] &
\end{tikzcd}$$

We now consider the second diagram. We may take a skeleton of the category $\mathscr{M}/X$ and for each $i \in I$, we take a representative member. Since $\mathbb{A}$ is assumed to be $\mathscr{M}$-wellpowered, we may assume that I is a set. The coproduct of $(M_i)_I$ establishes a morphism $[h_i] : \coprod_I M_i \to N$ such that $[h_i](\amalg_i)_I = (h_i)_I$. Then, $n[h_i](\amalg_i)_I = n(h_i)_I = (m_i)_I = m\bar{e}(\amalg_i)_I$ and since $(\amalg_i)_I$ is an episink, it follows that $n[h_i] = m\bar{e}$. The diagonalisation property provides us with a morphism $\bar{d} : M \to N$ such that $\bar{d}\bar{e} = [h_i]$ and $n\bar{d} = m$. It follows that $m \leq n$ and thus m is the join of the family $(m_i)_I$ in $\mathscr{M}/X$.

$$\begin{tikzcd}
M_i \arrow[r, "\amalg_i"'] \arrow[rr, bend left, "h_i"] \arrow[ddr, bend right, "m_i"'] & \coprod_I M_i \arrow[r, dashed, "{[h_i]}"] \arrow[d, "\bar{e}"'] & N \arrow[ddl, bend left, "n"] \\
 & M \arrow[ur, dashed, "!\bar{d}"'] \arrow[d, "m"'] & \\
 & X &
\end{tikzcd}$$

□

Theorem 3.142: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with $\mathscr{M}$ a class of monomorphisms. Let $\mathbb{C}$ be a constant reflective subcategory of $\mathbb{A}$ that is closed under $\mathbb{C}$-chained episinks in $\mathbb{A}$. Additionally, let $\mathbb{A}$ satisfy the following conditions:

(*i*) $\mathbb{A}$ is $\mathscr{M}$-wellpowered;

(*ii*) $\mathbb{A}$ has coproducts and/or $\mathbb{A}$ has multiple pushouts and multiple pullbacks.

Then, every left constant subcategory is nearly multi-$\mathscr{M}$-coreflective.

Proof : Assume the hypothesis and let $\mathbb{Q} = \mathscr{L}(\mathbb{P})$ be a left constant subcategory of $\mathbb{A}$ with $\mathbb{P}$ an $\mathscr{E}$-reflective right constant subcategory. Throughout this proof we will denote the reflector and unit into $\mathbb{P}$ by $R : \mathbb{A} \to \mathbb{P}$ and ε respectively.

Let X be any $\mathbb{A}$-object. Since $\mathbb{A}$ is $\mathscr{M}$-wellpowered, the comma category of $\mathscr{M}$-subobjects of X with domain in $\mathbb{Q}$ can be represented by a set. Let $\mathscr{M}_0/X$ be a skeleton for this comma category. The equivalence classes of the the least $\mathscr{M}$-subobject relation can then be represented by a set.

We now proceed to show that each equivalence class has a terminal object. Note that if there is a least $\mathscr{M}$-subobject of X with domain $\mathbb{Q}$, then the equivalence class containing it contains only one object up to isomorphism and hence has a terminal object. Consider any equivalence class that contains a morphism $n : B \to X$, with $B \in \mathbb{Q}$, where n is not isomorphic to the least subobject. In order to simplify notation, we will simply denote the equivalence class of a morphism $n : B \to X$ with B in $\mathbb{Q}$, by $[n]$. By $\mathscr{M}$-wellpoweredness, we can take a set of representatives of $[n]$, say $\{m_k : Q_k \to X |\ k \in K\}$, so that for each member m' of $[n]$, there is a unique k in K such that $m' \simeq m_k$. Consider the sink $(m_k : Q_k \to X)_K$. It should be clear from the least $\mathscr{M}$-subobject relation that $(m_k)_K$ is a chained sink. By Lemma 3.141, it follows that $(m_k)_K$ has an epic join $m : A \to X$ in $\mathscr{M}/X$. Hence there exists an episink $(n_k : Q_k \to A)_K$ in $\mathbb{A}$ such that $m \circ (n_k)_K = (m_k)_K$.

We first show that $(n_k)_K$ is chained. Let k and k' be members of K. Then, since $(m_k)_K$ is chained, there is a finite $\mathbb{Q}$-zig-zag $m_1, m_2, \ldots, m_n$ from m_k to $m_{k'}$ over X. If $m_{i,i'}$ is a morphism from m_i to $m_{i'}$, then there holds $m_{i'} m_{i,i'} = m_i$. Then, since $\mathscr{M}$ is a class of monomorphisms and $m n_{i'} m_{i,i'} = m_{i'} m_{i,i'} = m_i = m n_i$, it follows that $n_{i'} m_{i,i'} = n_i$. It is then easy to see that the sink $(n_k)_K$ is chained over A.

$$\begin{tikzcd}
& A \arrow[dd, "m"] & \\
Q_i \arrow[ur, "n_i"] \arrow[rr, "m_{i,i'}"] \arrow[dr, "m_i"'] & & Q_{i'} \arrow[ul, "n_{i'}"'] \arrow[dl, "m_{i'}"] \\
& X &
\end{tikzcd}$$

Note that since m is a join of $(m_k)_K$, m is a terminal object of the equivalence class containing n, provided that A belongs to $\mathbb{Q}$. Our claim is that A is a member of $\mathbb{Q} = \mathscr{L}(\mathbb{P})$. In view of Lemma 3.97, it's sufficient to prove that RA is a member of $\mathbb{C}$.

By Proposition 3.133, it follows that $(Rn_k)_K$ is a $\mathbb{C}$-chained episink in $\mathbb{A}$. Since $\mathbb{C}$ is closed under $\mathbb{C}$-chained episinks by assumption, we have that RA is a member of $\mathbb{C}$ and hence A is a member of $\mathscr{L}(\mathbb{P})$. Consequently, $m : A \to X$ is a terminal object of the equivalence class containing n.

$$\begin{tikzcd}
& RQ_k \arrow[r, "Rn_k"] & RA \\
Q_k \arrow[ur, "\varepsilon_{Q_k}"] \arrow[r, "n_k"] \arrow[d, "m_k"'] & A \arrow[ur, "\varepsilon_A"'] \arrow[dl, "m"] & \\
X & &
\end{tikzcd}$$

For each $j \in J$, let $\eta_j : Q_j \to X$ be the terminal object of the equivalence class viewed as a category. If $|J| \leq 1$, define $J(X) = J$. If $|J| \geq 2$, define $J(X)$ to be the set consisting of all equivalence classes except the one containing the least $\mathscr{M}$-subobject m_{0_X}, if it exists. We assert that $(\eta_j : Q_j \to X)_{J(X)}$ is the near multi-$\mathscr{M}$-coreflection. Since $\mathscr{M}$ is a class of monomorphisms, we need only show that every morphism $f : Q \to X$, with Q in $\mathbb{Q}$, factors through η_j for each $j \in J(X)$, or there is a unique $j \in J(X)$ such that f factors through η_j.

Let f be a morphism as above and let $m_f \circ e : Q \to Q_f \to X$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of f. Since $\mathbb{Q}$ is closed under $\mathscr{E}$-images, Q_f is a member of $\mathbb{Q}$. We now consider the equivalence class of $[m_f]$. By the above argument, $[m_f]$ has a terminal object, say η_j.

If $[m_f] = [m_{0_X}]$, where $m_{0_X} : Q_0 \to X$, then $m_f \simeq m_{0_X}$. Since $m_{0_X} \leq \eta_k$ for each $k \in J(X)$, there exists a morphism $h_k : Q_f \to Q_k$ such that $\eta_k \circ h_k = m_f$. Then, $\eta_k \circ h_k e = m_f e = f$. In other words, if m_f is in the same component as m_{0_X}, then f factors through η_k for each $k \in J(X)$.

If $[m_f] \neq [m_{0_X}]$, then it follows that $m_{0_X} < m_f$. Otherwise it would follow that $m_{0_X} \simeq m_f$ and that would imply that $[m_f] = [m_{0_X}]$. Since the equivalence class $[m_f]$ has a terminal object η_j, then $m_f \leq \eta_j$, and thus there is a morphism $f_j : Q_f \to Q_j$ such that $\eta_j f_j = m_f$. Then, $f = m_f e = \eta_j f_j e$, so that f factors through η_j. Note that this factorisation is unique since η_j is a monomorphism.

Furthermore, f can't factor through any other η_k for any $k \in J \setminus \{j\}$. To see this, suppose f factors as $\eta_k g$ for some η_k. If $\eta_k \simeq m_{0_X}$, then the diagonalisation property establishes a morphism $d : Q_f \to Q_0$ such that $m_{0_X} d = m_f$ and $de = g$. Consequently this would imply that $m_f \simeq m_{0_X}$, contrary to our assumption. If $\eta_k \not\simeq m_{0_X}$, then we assert that $k = j$. To see this, consider the diagram:

$$\begin{tikzcd}
& Q \arrow[dl, bend right, "g"'] \arrow[d, "e"] \arrow[dr, bend left, "f_j e"] & \\
Q_k \arrow[dr, bend right, "\eta_k"'] & Q_f \arrow[l, dashed, "!d'"'] \arrow[r, "f_j"] \arrow[d, "m_f"] & Q_j \arrow[dl, bend left, "\eta_j"] \\
& X &
\end{tikzcd}$$

The existence of the morphism d' is established by the diagonalisation property and the rest are defined in the argument above. It follows that there is a finite $\mathbb{Q}$-zig-zag from η_k to η_j, namely η_k, m_f, η_j. We then have $[\eta_k] = [\eta_j]$ so that $k = j$. Consequently, every morphism factors uniquely through either each member of the family $(\eta_j)_{J(X)}$, or it factors through a unique η_j for some $j \in J(X)$. In order to show that $(\eta_j)_{J(X)}$ is the near-multi-$\mathscr{M}$-coreflection, it is sufficient to show that if f factors through η_j for each $j \in J(X)$, then it factors through the least $\mathscr{M}$-subobject m_{0_X}. Suppose that for each $j \in J(X)$, there is a (unique) morphism $f_j : B \to B_j$ such that $\eta_j f_j = f$. Let η_i $(i \in J)$ be the terminal object of the equivalence class of m_f where $m_f e : Q \to Q_f \to X$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of f. Then there exists a morphism $n_f : Q_f \to B_i$ such that $\eta_i \circ n_f = m_f$, so that, for each $j \in J(X)$, the diagonalisation property establishes a unique morphism d_j such that the following diagram commutes:

$$\begin{tikzcd}
& Q_f \arrow[r, "n_f"] \arrow[drr, "m_f"] \arrow[ddr, dashed, "!d_j"'] & B_i \arrow[dr, "\eta_i"] & \\
Q \arrow[ur, "e"] \arrow[rrr, "f"'] \arrow[drr, "f_j"'] & & & X \\
& & B_j \arrow[ur, "\eta_j"'] &
\end{tikzcd}$$

It follows that m_f factors through η_i and η_j for each $j \in J(X)$. If $[m_f] \neq [m_0]$, then m_f must factor through a unique η_j for some $j \in J(X)$. Hence, it must be the case that $[m_f] = [m_0]$ and consequently $\eta_i \simeq m_0$ so that m_f factors through the least subobject. It follows that f also factors through the least subobject and our proof is complete. $\square$

Remark 3.143: Suppose $\mathbb{A}$ is an $(\mathscr{E},\mathbb{M})$-category that has coproducts and let $\mathbb{Q}$ be a left constant subcategory of $\mathbb{A}$ that is closed under coproducts in $\mathbb{A}$. Then, every non-empty sink with domain in $\mathbb{Q}$ is $\mathbb{Q}$-chained. To see this, if $(f_i : Q_i \to X)_I$ is a sink, then for each pair of indices i and j in I, we form the coproduct $(c_k : Q_k \to Q_i \coprod Q_j)_{k\in\{i,j\}}$ of Q_i and Q_j. Then, since f_i and f_j are morphisms from Q_i and Q_j, there is a unique morphism $f : Q_i \coprod Q_j \to X$ such that $f \circ c_k = f_k$ for $k = i$ and $k = j$. Hence, f_i, f, f_j is a finite $\mathbb{Q}$-zig-zag from f_i to f_j. Consequently, the comma category $\mathbb{Q}/X$ is connected.

Assume that $\mathscr{M}$ is a class of monomorphisms. Let $m \circ e : Q_i \coprod Q_j \to Q \to X$ be an $(\mathscr{E},\mathbb{M})$-factorisation of f. Note that Q will be in $\mathbb{Q}$. It's important to note that if f_i and f_j are in $\mathscr{M}$ and neither is the least $\mathscr{M}$-subobject, then (since f_i and f_j are morphisms in $\mathscr{M}$) it also follows that m is also not the least $\mathscr{M}$-subobject. Otherwise since $f_i, f_j \leq m$, f_i and f_j would both be the least $\mathscr{M}$-subobject. It can easily be seen that $[f_i]_{\sim_{X,\mathbb{B}}} = [f_j]_{\sim_{X,\mathbb{B}}} = [m]_{\sim_{X,\mathbb{B}}}$ and it's then easy to see that there can be at most two equivalence classes of the least $\mathscr{M}$-subobject relation. It is readily seen that there will be two equivalence classes if and only if there is a least $\mathscr{M}$-subobject and there is a $\hat{Q}$ in $\mathbb{Q}$ and a morphism $\hat{m} : \hat{Q} \to X \in \mathscr{M}$ that is not the smallest $\mathscr{M}$-subobject of X in $\mathbb{Q}$. There will be one equivalence class if and only if for every morphism $g : Q \to X$ (with Q in $\mathbb{Q}$) factorises through the least $\mathscr{M}$-subobject (if it exists) or there is no least $\mathscr{M}$-subobject. The set of equivalence classes is empty if and only if there is no morphism with domain in $\mathbb{Q}$ and codomain X.

If $\mathbb{A}$ is additionally $\mathscr{M}$-wellpowered, then it follows that the near multi-$\mathscr{M}$-coreflection of X in $\mathbb{Q}$ will either be the empty sink or it will consist of a sink of one morphism. Of course in most interesting cases there are morphisms from the constant subcategory to every object already and hence already morphisms from $\mathbb{Q}$ to X. In that particular case, we can conclude that the left constant subcategory is $\mathscr{M}$-coreflective just as in Corollary 3.126.

Corollary 3.144: Let $\mathbb{A}$ be a category with coproducts that satisfies the assumptions of Theorem 3.142 with $\mathbb{C}$ closed under coproducts in $\mathbb{A}$. Then, for each left constant subcategory $\mathbb{Q}$, the near multi-$\mathscr{M}$-coreflection of each $\mathbb{A}$-object X has exactly one member. Consequently we can recover the $\mathscr{M}$-coreflection of $\mathbb{Q}$ in $\mathbb{A}$.

Proof : This follows from Theorems 3.125 and 3.142 and the argument in Remark 3.143. □

Remark 3.145: Corollary 3.144 provides us with a comparison between Theorems 3.125 and 3.142 for $\mathscr{M}$-wellpowered categories where the reflective constant subcategory is closed under coproducts. It turns out that Corollary 3.126 can be viewed as a special case of theorem 3.142 in case there is a morphism from $\mathbb{C}$ to X for each $\mathbb{A}$-object X. This idea is further illustrated in example 3.146.

From the other angle, suppose $\mathbb{Q}$ is a nearly multi-$\mathscr{M}$-coreflective subcategory of $\mathbb{A}$ such that for each $\mathbb{A}$-object X, the near multi-$\mathscr{M}$-coreflection of X in $\mathbb{Q}$ is given by the family $(\eta_X : QX \to X)$ with exactly one member, then, $\mathbb{Q}$ is $\mathscr{M}$-coreflective with counit η.

Example 3.146: Consider the category $\mathbb{Ab}$ of abelian groups and homomorphisms with (surjective, point separating source) factorisation structure. Let $\mathbb{C}$ be the reflective constant subcategory of all trivial groups and consider the $\mathscr{M}$-coreflective subcategory $\mathbb{Tor} = \mathscr{L}(\mathbb{TFAb})$.

For each abelian group A, we can form the torsion subgroup TA of A. If $TA \simeq \{0\}$, then the least-$\mathscr{M}$-subobject relation has one equivalence with terminal object, namely $TA \to A$. If $TA \not\simeq \{0\}$, then the least-$\mathscr{M}$-subobject relation has two equivalence classes with terminal objects $\{0\} \to A$ and $TA \hookrightarrow A$ respectively. Either way, $TA \hookrightarrow A$ is the $\mathscr{M}$-coreflection.
Note that if T is any torsion group and $f : T \to A$ is a morphism with T, then f factors through both $\{0\}$ and TA if and only if $f[T] = \{0\}$.

Example 3.147: Let $\mathbb{A}$ be the category of topological spaces with factorisation structure (surjective, initial point separating source) and let $\mathbb{C}$ be the surjective-reflective constant subcategory of all indiscrete spaces. Note that $\mathbb{C}$ is not even closed under finite coproducts, but the rest of the assumptions on $\mathbb{A}$ and $\mathbb{C}$ are satisfied as in Theorem 3.142.

First we construct the near multi-$\mathscr{M}$-coreflection for each space X. For any space X, we define a relation

R on X by defining $(x, y) \in R$ if and only if for each open set U of X, $x \in U$ if and only if $y \in U$. It's straightforward to verify that R is an equivalence relation on the set X. For each x in X, we have an equivalence class:

$$[x]_R = \{y \in X \mid \text{for all open sets } U \text{ of } X \ (x \in U \Leftrightarrow y \in U)\}$$

If we choose a representative set of all the distinct equivalence classes, say $X/R := \{[x_i]_R \mid i \in I\}$, and adjoin the set $\{\varnothing\}$, then each of the members can be endowed with the subspace topology of X. Define $J = I \cup \{0_X\}$ and for each j in J, define $X_j = [x_j]_R$ when $j \neq 0_X$ and $X_j = \varnothing$ otherwise. We assert that the family $(\eta_i : X_i \to X)_I$ is the family of all terminal objects of the least $\mathscr{M}$-subobject relation. The near multi-$\mathscr{M}$-coreflection is then given by $(\eta_i : X_i \to X)_I$ if X is non-empty and given by the single morphism $id : X \to X$ otherwise.

First we show that $[x]_R$ is an indiscrete subspace of X. Let U be any non-empty open set of $[x]_R$. Then, $U = [x]_R \cap V$ for some open set V of X. Since U is non-empty, there exists a $y \in [x]_R$ such that $y \in U \subset V$. Since $y \in [x]_R$, it follows that $x \in V$. By definition of R, it follows that every member of $[x]_R$ is in V, hence also in U. Therefore the only non-empty open subset of $[x]_R$ is $[x]_R$ itself. Consequently, $[x]_R$ is an indiscrete subspace of X. Obviously, the empty space is indiscrete, hence all members of the family $(\eta_j)_J$ have indiscrete domains.

Let $f : A \to X$ be a continuous map with A an indiscrete space. If $A = \varnothing$, then $f = \eta_{0_X}$ and factors through each η_i. If A is non-empty, then note that $f[A]$ is an indiscrete space. Since $f[A]$ is non-empty, there is a point $x \in f[A]$. Now, $[x]_R = [x_i]_R$ for a unique $i \in I$. We assert that $f[A] \subset [x_i]$.

Given any $a \in A$ and any open set U of X, we have the following: If $x \in U$, then since $U \cap f[A]$ is a non-empty open set of $f[A]$, it follows that $f[A] \subset U$. Hence, $f(a) \in U$ for each $a \in A$. On the other hand, if $f(a)$ is in U, then $U \cap f[A]$ is a non-empty open subset of $f[A]$ which again implies that $f[A] \subset U$. It then follows that x is a member of U. Therefore, for each $a \in A$, there holds: $[f(a)]_R = [x] = [x_i]_R$. Consequently $f[A] \subset [x_i]$. It is then easy to see that f factors uniquely through η_i.

Note that $\mathbb{Ind}$ is actually a bijective-reflective subcategory of $\mathbb{Top}$. We now consider it as a constant subcategory of the (bijective, initial source)-category $\mathbb{Top}$. $\mathbb{Top}$ still has an initial object and $\varnothing \to X$ is an initial continuous map for each space X. Note that the constant morphisms will coincide for both factorisation structures and $\mathbb{Top}$ is not initially-wellpowered. However, since each bijective continuous map is surjective, each initial continuous map $f : A \to B$ can be factored via a surjective map e followed by an embedding m. Then, of course, m will factor through one or all of the η_i. Since f factors through m, f will factor through all of the η_i if and only if A is the empty space and will otherwise factor through a unique η_i.

Remark 3.148: It's important to note that when considering the category of indiscrete spaces as a constant subcategory of the (bijective, initial source)-category $\mathbb{Top}$, every left constant subcategory is nearly multi-$\mathscr{N}$-coreflective for the class $\mathscr{N}$ of embeddings. Of course this is no coincidence. There are various reasons why this is the case. As discussed in Example 3.147, every left constant subcategory of $\mathbb{Top}$ will be nearly multi-embedding coreflective. Note that (surjective, initial mono-source) is a larger factorisation structure than the other, so that all the morphisms in the near multi-coreflection are both initial and embeddings. All of the assumptions of Theorem 3.142 are satisfied for the larger factorisation structure, hence Theorem 3.142 applies to all left constant subcategories of the larger factorisation structure. It remains to be seen that these are the same subcategories for the other factorisation structure. This can easily be seen from Proposition 3.105. Hence all left constant subcategories still satisfy the conclusion of our theorem.

This obviously generalises to any constant subcategory for which we can view it as a constant subcategory for two factorisation structures for sources for which the assumptions of Theorem 3.142 is satisfied for the larger one of the two.

Proposition 3.149: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ be a reflective constant subcategory of $\mathbb{A}$. Let $\mathbb{A}$ be $\mathscr{M}$-wellpowered and $\mathbb{Q}$ be any nearly multi-$\mathscr{M}$-coreflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. For each $\mathbb{A}$-object X, let us denote the near multi-$\mathscr{M}$-coreflection by $(\eta_i^X : Q_i^X \to X)_{I(X)}$. Then, the following holds:

(a) $X \in \mathscr{R}(\mathbb{Q})$ if and only if for each $i \in I(X)$, there holds $Q_i^X \in \mathbb{C}$.

(*b*) $X \in \mathbb{Q}$ if and only if $|I(X)| = 1$ and the morphism η_i^X is an isomorphism.

Proof : For (a), if X is in $\mathscr{R}(\mathbb{Q})$, then each morphism $f : Q \to X$ with domain in $\mathbb{Q}$ is constant. In particular, for each $i \in I(X)$, there holds $\eta_i^X : Q_i^X \to X$ is a constant morphism in $\mathscr{M}$. Of course, this implies that Q_i^X is a member of $\mathbb{C}$.

Conversely, assume that X is in $\mathbb{A}$ and that Q_i^X is in $\mathbb{C}$ for each $i \in I(X)$. Let $g : Q \to X$ be any morphism with domain in $\mathbb{Q}$. Since $\mathbb{Q}$ is nearly multi-$\mathscr{M}$-coreflective, there is an $i \in I(X)$ and a morphism $h : Q \to Q_i^X$ such that $\eta_i^X h = g$. Since Q_i^X is in $\mathbb{C}$ by assumption, it follows that η_i^X is constant and thus also $\eta_i^X h = g$. Consequently, X is a member of $\mathscr{R}(\mathbb{Q})$.

For (b) we assume that X is in $\mathbb{Q}$ and explicitly construct the near multi-$\mathscr{M}$-coreflection. Of course we are discussing the equivalence classes of the least $\mathscr{M}$-subobject relation. First note that $I(X)$ contains at least one element and if X is in $\mathbb{Q}$, then $id_X : X \to X$ is a morphism in $\mathscr{M}$ with domain in $\mathbb{Q}$. Given any morphism $m : Q \to X$ in $\mathscr{M}$, we have $m \simeq m_0$ or $m_0 < m$, where $m_0 : Q_0 \to X$ is the least $\mathscr{M}$-subobject of X in $\mathbb{Q}$. Of course, this least subobject need not exist. If $m \simeq m_0$, then $[m] = [m_0]$. If $m_0 < m$, then the diagram

$$\begin{tikzcd} Q \arrow[rr, "m"] \arrow[dr, "m"'] & & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$

commutes. This obviously provides us with a finite $\mathbb{Q}$-zig-zag from m to id_X and it ought to be clear that id_X is the terminal object in this equivalence class. It should be clear that for such an m, we have $[m] = [id_X]$. It follows that if there is an $m : Q \to X$ in $\mathscr{M}$ with $Q \in \mathbb{Q}$ such that $m > m_0$, then there are exactly two equivalence classes. If there is no such m, or the only $\mathscr{M}$-subobject is isomorphic to id_X, then it still factors uniquely through id_X. It's then easy to see that we may take $(id_X : X \to X)_I$ as the near multi-$\mathscr{M}$-coreflection and any other such near multi-$\mathscr{M}$-coreflection must consist of an isomorphism.

Conversely, if $I(X)$ only consists of one isomorphism, then, of course, $\mathbb{Q}$ being isomorphism-closed, proves that X is in $\mathbb{Q}$.

$\square$

Definition 3.150: Class of all nearly multi-coreflective subcategories
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$ such that all assumptions in Theorem 3.142 are satisfied. Then, let $NMC_{\mathbb{C}}(\mathbb{A}, \mathscr{M})$ denote the conglomerate of all nearly multi-$\mathscr{M}$-coreflective subcategories of $\mathbb{A}$ that contains $\mathbb{C}$, with the partial order inclusion.

Corollary 3.151: Suppose $\mathbb{A}$ and $\mathbb{C}$ satisfy all the assumptions of 3.142. Then, the adjunction $(LC_{\mathbb{C}}(\mathbb{A}), \subset) \underset{\mathscr{L}(-)}{\overset{\mathscr{R}(-)}{\rightleftarrows}} (RC_{\mathbb{C}}(\mathbb{A}), \supset)$ restricts to an adjunction

$$(NMC_{\mathbb{C}}(\mathbb{A}, \mathscr{M}), \subset) \underset{\mathscr{L}(-)}{\overset{\mathscr{R}(-)}{\rightleftarrows}} (R_{\mathbb{C}}(\mathbb{A}, \mathscr{E}), \supset)$$

Proof : This follows from Proposition 3.24, Lemma 2.32 and Theorem 3.142. $\square$

Mathematics is the supreme judge; from its decisions there is no appeal.

$\sim$ Tobias Dantzig (1884-1956)

4 Dual Closure Operators

4.1 Introduction to Dual Closure Operators

In order to motivate categorical dual closure operators, we first need to discuss some of their origins. Closure operators have their origins in topology where a topological space can be mathematically equivalently defined as a set with a Kuratowski closure operator. Categorical closure operators are of course a generalisation of the Kuratowski closure operator. A categorical closure operator C of $\mathscr{M}$ in $\mathbb{A}$ (as defined in [27]) assigns to every subobject $m : M \to X$ in some class $\mathscr{M}$ of $\mathbb{A}$-monomorphisms an $\mathscr{M}$-subobject $Cm : C_X M \to X$ such that the induced map $C : \mathscr{M}/X \to \mathscr{M}/X$ is expansive, monotone, and is compatible with taking images, or, equivalently, inverse images. The notion of categorical closure operator was originally used to help classify epimorphisms of subcategories of topological spaces. It was also used to determine cowell-poweredness of these subcategories. Furthermore, categorical closure operators have been studied extensively. See for example [16] and [25]. Like any great categorical topic, one has a number of fruitful applications and categorical closure operators is no different as it has many applications. See [21], [24], [28], [29] and [45] for examples in algebra, topology and theoretical computer science.

Note that whenever $\mathscr{M}$ is a class of monomorphisms, we have that $\mathscr{M}/X$ is a pre-ordered class. One natural topic to study for ordered structures is the dualisation of the ordered structure. In recent years, several authors have investigated categorical interior operators (see [14], [15], [24], [34], [37] and [53]), the order dual of categorical closure operators, with the prototypical example as the interior of a subset of a topological space. The order dual is definitely a topic to be studied in its own right, but it's far from the categorical dual of closure operators.

In order to clearly see the categorical dual of closure operators, we refer to the definition of closure operators once it is viewed as an endofunctor $m \longmapsto Cm$ for $\mathscr{M}$, when $\mathscr{M}$ is viewed as the full subcategory of the arrow or morphism category $\mathbb{A}^{\to}$ (see definition 4.1.) This definition represents a closure operator of $\mathscr{M}$ in $\mathbb{A}$ as an endofunctor $C : \mathscr{M} \to \mathscr{M}$ together with a natural transformation $1_{\mathscr{M}} \to C$ that is compatible with the codomain functor $cod : \mathscr{M} \to \mathbb{A}$. This approach was already followed in [28] which is one of the early approaches on the categorical closure operators. For more examples on this approach, see [25] and [49].

Hence, to study dual closure operators, i.e., the categorical dual of closure operators, is to study co-pointed endofunctors with a natural transformation that is compatible with the domain functor. The first substantial paper on dual closure operators ([26]) was published in 2015. Since the authors are also well known for various categorical closure operators articles including their monograph ([25]), our approach is to follow their terminology whenever applicable. Since this is still a relatively new topic, in order to ensure that there is no ambiguity, the first section will mainly contain some elementary results and necessary terminology on dual closure operators.

Since categorical closure operators are known for their nice properties on classes of monomorphisms, it's only natural that we will restrict some of our results to dual closure operators on classes of epimorphisms.

Definition 4.1: Arrow category($\mathbb{A}^{\to}$), Domain and Codomain functors
For any category $\mathbb{A}$, we can always consider the category $\mathbb{A}^{\to}$ **of morphisms** or **the arrow category of** $\mathbb{A}$ with objects Mor($\mathbb{A}$). Morphisms of $\mathbb{A}^{\to}$ are pairs $(e, e') : f \to g$, where $f : A \to A'$ and $g : B \to B'$, and $e : A \to B$ and $e' : A' \to B'$ are $\mathbb{A}$-morphisms such that the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A' \arrow[r, "e'"'] & B' \end{tikzcd}$$

commutes in $\mathbb{A}$, i.e., $ge = e'f$. Composition of morphisms is defined componentwise: $(e, e') \circ (h, h') =$

$(e \circ h, e' \circ h')$, i.e., by pasting squares together.

If $\mathbb{A}$ is an $(\mathscr{E}, \mathscr{M})$-structured category, then we can of course view both $\mathscr{E}$ and $\mathscr{M}$ as full subcategories of $\mathbb{A}^{\rightarrow}$. There are also two functors $dom, cod : \mathbb{A}^{\rightarrow} \to \mathbb{A}$ where $dom((e, e') : f \to g) = e$ and $cod((e, e') : f \to g) = e'$. These obviously also restrict as functors from both $\mathscr{E}$ and $\mathscr{M}$ to $\mathbb{A}$. Of course, every subclass of morphisms can be viewed as a full subcategory of $\mathbb{A}^{\rightarrow}$. We will often simply denote the functors *dom* and *cod* as functors from $\mathscr{E}$ to $\mathbb{A}$.

Definition 4.2: Dual Closure Operator
Let $\mathbb{A}$ be a category and $\mathscr{E}$ a class of morphisms. A **dual closure operator (dco) of $\mathscr{E}$ (in $\mathbb{A}$)** is a functor $D : \mathscr{E} \to \mathscr{E}$ together with a natural transformation $\Delta : D \to id_{\mathscr{E}}$ such that the following conditions hold:

(a) For each $\mathscr{E}$-object $e : X \to Y$: $dom(\Delta_e), cod(\Delta_e) \in \mathscr{E}$,
(b) $dom \circ D = dom$ and
(c) $dom_{\Delta} = id_{dom}$.

Hence, for each $e : X \to Y$ in $\mathscr{E}$, we have: $id_X = id_{dom_e} = dom_{\Delta_e}$ and if $\delta_e = cod(\Delta_e)$ and $cod(De) = DY$, then the diagram

$$\begin{tikzcd} X \arrow[r, equal, "id_X"] \arrow[d, "De"'] & X \arrow[d, "e"] \\ DY \arrow[r, "\delta_e"'] & Y \end{tikzcd}$$

commutes. A morphism $(u, v) : e \to e'$ in $\mathscr{E}$ also gives the commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "De"] \arrow[dd, bend right=60, "e"'] & X' \arrow[d, "De'"'] \arrow[dd, bend left=60, "e'"] \\ DY \arrow[r, "D_{u,v}"] \arrow[d, "\delta_e"] & DY' \arrow[d, "\delta_{e'}"'] \\ Y \arrow[r, "v"'] & Y' \end{tikzcd}$$

in $\mathbb{A}$, where $D_{u,v} = cod(D(u, v))$, $DY = cod(De)$ and $DY' = cod(De')$.

Note that if $\mathbb{A}$ has $(\mathscr{E}, \mathscr{M})$-factorisations of morphisms and D is a dco of $\mathscr{E}$ in $\mathbb{A}$, then we need not require the natural transformation to have components in $\mathscr{E}$. This is because $\mathscr{E}$ will satisfy $\delta_e De = e \in \mathscr{E}$ and $De \in \mathscr{E}$, hence $\delta_e \in \mathscr{E}$ and of course $id_X \in \mathscr{E}$ automatically.

Definition 4.3: $\mathscr{E}$-quotients
Let $\mathscr{E}$ be a class of $\mathbb{A}$ morphisms. Then, we will denote the class of all **$\mathscr{E}$-quotients of X in $\mathbb{A}$** by $\mathrm{quot}(X)$ or simply $\mathrm{quot}X$, where the class of $\mathscr{E}$-quotients is given by $dom^{-1}(X) = \{e : A \to B \in \mathscr{E} \mid A = dom(e) = X\}$.

Remark 4.4: If $\mathbb{A}$ has $(\mathscr{E}, \mathscr{M})$-factorisations, then for each $f : X \to Y$, we have a map

$$f^{-}(-) : quotY \to quotX$$

defined in the following way: For each $q : Y \to Q$ in $\mathscr{E}$, let $mf^{-}(q) : X \to f^{-}Q \to Q$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of qf. It's clear that $f^{-}(q)$ is a member of $\mathrm{quot}X$.
If $\mathscr{E}$ is a class of morphisms, then $p \leq p'$ in $quot(X)$ if and only if there is a morphism $j : P \to P'$ such that $jp = p'$. If $\mathscr{E}$ is a class of epimorphisms, then j is necessarily unique and $quot(X)$ is necessarily a pre-order.

If $\mathbb{A}$ has **$\mathscr{E}$-pushouts**, i.e., for each $p : X \to P$ in $\mathscr{E}$ and each $\mathbb{A}$-morphism $f : X \to Y$, there exists a pushout square $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "\bar{p}"] \\ P \arrow[r, "\bar{f}"'] & Q \end{tikzcd}$ for which $\bar{p} \in \mathscr{E}$, then we have a map $f_{-}(-) : quot(X) \to quot(Y)$ by defining $f_-(p) = \bar{p}$. For simplicity, we will denote a pushout of $p \in \mathscr{E}$ along f by $f_-(p) : Y \to f_-(P)$.

Proposition 4.5: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E}$-pushouts. Then, for each $\mathbb{A}$-morphism $f : X \to Y$, both $f^-(-)$ and $f_-(-)$ are order preserving and furthermore, $f_-(-)$ is left adjoint to $f^-(-)$.

Proof : Let $f : X \to Y$ be an $\mathbb{A}$-morphism and $q \leq q'$ in $quot(Y)$. Let $j : Q \to Q'$ be a morphism such that $jq = q'$. Let $mf^-(q)$ and $nf^-(q')$ be factorisations of qf and $q'f$ respectively. Then,

$$\begin{tikzcd} X \arrow[r, "f^-(q)"] \arrow[d, "f^-(q')"'] & f^-(Q) \arrow[d, "jm"] \arrow[dl, dashed, "!d"'] \\ f^-(Q') \arrow[r, "n"'] & Q' \end{tikzcd}$$

commutes for a unique diagonal morphism d. In particular, it follows that $f^-(q) \leq f^-(q')$ so that $f^-(-)$ is order preserving.

Now, assume that $p \leq p'$ in $quot(X)$ and let $k : P \to P$ be a morphism such that $kp = p'$. Let $f : X \to Y$ be an $\mathbb{A}$-morphism and consider the pushout squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "f_-(p)"] \\ P \arrow[r, "f_1"'] & f_-(P) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p'"'] & Y \arrow[d, "f_-(p')"] \\ P' \arrow[r, "f_2"'] & f_-(P') \end{tikzcd}$$

Since $f_-(p')f = f_2p' = f_2kp$, by the pushout property, there exists a unique morphism $\ell : f_-(P) \to f_-(P')$ such that $\ell f_-(p) = f_-(p')$ and $\ell f_1 = f_2k$. In particular, we have: $f_-(p) \leq f_-(p')$ so that $f_-(-)$ is order preserving.

In order to show that $f_-(-)$ is left adjoint to $f^-(-)$ for each $\mathbb{A}$-morphism $f : X \to Y$, we show $p \leq f^-(f_-(p))$ and $f_-(f^-(q)) \leq q$ for each $p : X \to P$ and $q : Y \to Q$ in $\mathscr{E}$.
Consider the pushout $f_-(p)$ of p along f such that $\bar{f}p = f_-(p)f$. Let $f^-(f_-(p))m$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of $f_-(p)f$. Then, by the diagonalisation property, the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f^-(f_-(p))"'] & P \arrow[d, "\bar{f}"] \arrow[dl, dashed, "!d"'] \\ f^-(f_-(P)) \arrow[r, "m"'] & f_-(P) \end{tikzcd}$$

commutes for a unique morphism d. Hence $p \leq f^-(f_-(p))$.
To prove that $f_-(f^-(q)) \leq q$ for any $q \in \mathscr{E}$, consider the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dd, "f^-(q)"'] & & Y \arrow[dl, "f_-(f^-(q))"'] \arrow[dd, "q"] \\ & f_-(f^-(Q)) \arrow[dr, dashed, "s"] & \\ f^-(Q) \arrow[ur, "\bar{g}"] \arrow[rr, "m"'] & & Q \end{tikzcd}$$

Here, the top triangle is a pushout and the entire square commutes by factorisation. The existence of the morphism s is given by the pushout property and hence $f_-(f^-(q)) \leq q$ holds. Thus $f_-(-)$ is left adjoint to $f^-(-)$. □

Corollary 4.6: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E}$-pushouts and $f : X \to Y$ an $\mathbb{A}$-morphism. Then, for all $p \in quot(X)$ and $q \in quot(Y)$, we have:

$$f_-(p) \leq q \text{ if and only if } p \leq f^-(q).$$

Furthermore, $f^-(-)$ preserves all meets and $f_-(-)$ preserves all joins.
Proof : The proof is a standard argument such as for Galois connections of pre-ordered sets. □

Remark 4.7: Suppose C is a closure operator of a class of monomorphisms $\mathscr{M}$ in $\mathbb{A}$. Then, C can be defined in an $(\mathscr{E},\mathscr{M})$-structured category or a category with $\mathscr{M}$-pullbacks as a family of endofunctions $(c_X : \mathscr{M}/\mathscr{X} \to \mathscr{M}/X)_{X\in Ob(\mathbb{A})}$ that satisfies some additional properties. The interested reader may see [25, 2.2] for both definitions. The following result is the dualisation of the one for closure operators and the proof is added just to see how these constructions are done in the dual case.

Proposition 4.8: ([26, 2.4]) **Two equivalent ways of defining a dual closure operator.** Let $\mathscr{E}$ be a class of epimorphisms and $\mathbb{A}$ an $(\mathscr{E},\mathscr{M})$-structured category. Then, a dco of $\mathscr{E}$ in X may be equivalently given by a family of maps $(D_X : quotX \to quotX)_{X\in\mathbb{A}}$ such that the following conditions are satisfied for each $p, p' \in quotX$, $q \in quotY$ and $f : X \to Y$ in $\mathbb{A}$:

(a) $D_X(p) \leq p$,

(b) $p \leq p' \Rightarrow D_X(p) \leq D_X(p')$,

(c) $D_X(f^-(q)) \leq f^-(D_Y(q))$

Alternatively, assume that $\mathbb{A}$ has $\mathscr{E}$-pushouts and $\mathscr{E}$ contains all isomorphisms, then $\mathscr{E}$ is a class of epimorphisms and $\mathscr{E}$ satisfies the cancellation condition:

$$ef \in \mathscr{E} \text{ and } (f \in \mathscr{E} \text{ or } f \text{ is epic}) \text{ implies that } e \in \mathscr{E}.$$

Then, a dco of $\mathscr{E}$ in $\mathbb{A}$ may be given by such a family of maps that satisfy (a) and (b) and condition (c) may be replaced by the following condition:
(c') $f_-(D_X(p)) \leq D_Y(f_-(p))$, where $f_-(p)$ is the pushout of p along f.

Proof : Let $\mathbb{A}$ be an $(\mathscr{E},\mathscr{M})$-structured category and let D be a dco of $\mathscr{E}$ in $\mathbb{A}$. Note that we will often abuse notation and write D_Xp or simply Dp, when referring to $D_X(p)$. Using our notation as in Definition 4.2, since $\delta_p Dp = p$ for each $p : X \to P$ in $\mathscr{E}$, we may define $D_X(p) = Dp$. It's then clear that $D_Xp \leq p$.
If $p \leq p'$ in $quot(X)$ with $jp = p'$, then $D_{id_X,j} := cod(D(id_X, j))$ is a morphism from $D_XP \to D_XP'$ such that $D_{id_X,j}D_Xp = D_Xp'$, hence $D_Xp \leq D_Xp'$.
To show that (c) holds, suppose $f : X \to Y$ is an $\mathbb{A}$-morphism and $q : Y \to Q$ is in $quot(Y)$. We show that $D_X(f^-q) \leq f^-(D_Yq)$. Let $mf^-(q)$ be a factorisation of qf. Apply the functor D to the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f^-(q)"'] & Y \arrow[d, "q"] \\
f^-(Q) \arrow[r, "m"'] & Q
\end{tikzcd}$$

to obtain the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "D(f^-q)"] \arrow[dd, bend right=60, "f^-(q)"'] & Y \arrow[d, "Dq"] \arrow[dd, bend left=60, "q"] \\
D(f^-(Q)) \arrow[r, "D_{f,m}"'] \arrow[d, "\delta_{f^-(q)}"] & DQ \arrow[d, "\delta_q"] \\
f^-(Q) \arrow[r, "m"'] & Q
\end{tikzcd}$$

Consider a factorisation $nf^-(Dq)$ of Dqf. Then, $nf^-(Dq) = Dqf = D_{f,m}D(f^-(q))$ so that by the diagonalisation property, there is a unique morphism $w : D(f^-(Q)) \to f^-(DQ)$ such that $wD(f^-(q)) = f^-(Dq)$ and $nw = D_{f,m}$. In particular $D(f^-(q)) \leq f^-(Dq)$ follows.
Since we didn't assume $(\mathscr{E},\mathscr{M})$-factorisations for ($a$) and ($b$), it's sufficient to prove that (c') holds if $\mathbb{A}$ is a category, D a dco of $\mathscr{E}$ in $\mathbb{A}$ and $\mathbb{A}$ has $\mathscr{E}$-pushouts and satisfies the following cancellation condition:

$$gf \in \mathscr{E} \text{ and } (f \in \mathscr{E} \text{ or } f \text{ is epic}) \Rightarrow g \in \mathscr{E}$$

Consider the commuting diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "D_X p"] \arrow[dd, bend right=60, "p"'] & Y \arrow[d, "D_Y f_-(p)"'] \arrow[dd, bend left=60, "f_-(p)"] \\
D_X P \arrow[r, "D_{f,\bar{f}_p}"'] \arrow[d, "\delta_p"'] & D_Y(f_-(P)) \arrow[d, "\delta_{f_-(p)}"'] \\
P \arrow[r, "\bar{f}_p"'] & f_-(P)
\end{tikzcd}$$

where $f_-(p)$ is the pushout of p along f, i.e., the outer diagram is a pushout square with $p \in \mathscr{E}$. Then,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "D_X p"'] & Y \arrow[d, "f_-(D_X p)"] \arrow[ddr, bend left, "D_Y(f_-(p))"] & \\
D_X P \arrow[r, "\bar{f}_{D_X p}"'] \arrow[drr, bend right, "D_{f,\bar{f}_p}"'] & f_-(D_X(P)) \arrow[dr, dashed, "!j"] & \\
 & & D_Y(f_-(P))
\end{tikzcd}$$

commutes for a unique j, where $f_-(D_X p)$ is the pushout of $D_X(p)$ along f. In particular, it's easy to see that $f_-(D_X p) \leq D_Y(f_-(p))$, so that (c') is satisfied.

Conversely, suppose that $(D_X)_{X \in \mathbb{A}}$ is a family of maps satisfying (a) and (b). Define $D : \mathscr{E} \to \mathscr{E}$ by $D(p : X \to P) = D_X(p) : X \to D_X(P)$. For each $p : X \to P$ in $\mathscr{E}$, we have: $D_X p \leq p$, hence there is a morphism $\delta_p : D_X P \to P$ such that $\delta_p D_X p = p$. Throughout the rest of this proof, we will denote this morphism by δ_p. Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-category and $\mathscr{E}$ a class of epimorphisms.

We may define $D(p : X \to P) = D_X p : X \to D_X P$ on objects. Let $(u, v) : p \to p'$ be a morphism in $\mathscr{E}$. Let $mu^-(p')$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of $p'u = vp$. Then, it should be clear that the diagram

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "u^-(p)"'] & P \arrow[d, "v"] \arrow[dl, dashed, "!d"'] \\
u^-(P) \arrow[r, "m"'] & P'
\end{tikzcd}$$

commutes for a unique diagonal morphism d and $p \leq u^-(p')$. By (b) and (c) we obtain $Dp \leq Du^-(p') \leq u^-(Dp')$, hence there is a unique morphism j such that $jDp = u^-(Dp')$. Let $nu^-(Dp)$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of $Dp'u$. Then we have $\delta_{p'} njDp = \delta_{p'} nu^-(Dp') = \delta_{p'} Dp'u = p'u = vp = v\delta_p Dp$, and since $\mathscr{E}$ is a class of epimorphisms, this implies that $\delta_{p'} nj = v\delta_p$. Also, $njDp = nu^-(Dp') = Dp'u$ so that we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "u^-(D(p'))"] \arrow[dd, "Dp"'] \arrow[ddd, bend right, "p"'] & & X' \arrow[dd, "Dp'"] \arrow[ddd, bend left, "p'"] \\
& u^-(D(P')) \arrow[dr, "n"] & \\
DP \arrow[ur, dashed, "j"] \arrow[d, "\delta_p"] & & DP' \arrow[d, "\delta_{p'}"] \\
P \arrow[rr, "v"'] & & P'
\end{tikzcd}$$

Since $\mathscr{E}$ is a class of epimorphisms, it is important to note that nj is the unique morphism $k : DP \to DP'$ such that $kDp = Dp'u$ and $\delta_{p'} \circ k = v\delta_p$. It then makes sense to define $D(u, v) = (u, nj) : Dp \to Dp'$, where n is the $\mathscr{M}$-part of an $(\mathscr{E}, \mathscr{M})$-factorisation of $Dp'u$ and j the unique morphism such that $jDp = u^-(Dp)$. Let us verify that D is a functor. To see that D preserves identities, $D(id_p : p \to p) = D(id_X, id_P) = (id_X, id_{Dp}) = id_{DP}$ since $Dp \circ id_X = id_{DP}Dp$ is an $(\mathscr{E}, \mathscr{M})$-factorisation and $j = id_{DP}$ is obviously the unique morphism, it follows that $nj = id_{DP}$. To see that D preserves composition, let $(u, v) : p \to p'$ and $(u', v') : p' \to p''$. Then, $D((u, v)) \circ D((u', v')) = (u, nj) \circ (u', n'j') = (uu', njn'j')$. Since the first components of $D(u', v') \circ D(u, v)$ and $D(u'u, v'v)$ are both $u'u$, in view of the fact that nj is unique as above, it is sufficient to prove that $g = n'j'nj$ is a morphism such that $gDp = Dp''u'u$ and $\delta_{p''}g = v'v\delta_p$. Since we can paste the two rectangles together, this fact should be clear and hence D is an endofunctor.

Now we can easily define a natural transformation $\Delta : D \to id_{\mathscr{E}}$ by putting $\Delta_p := (id_X, \delta_p)$ for each $p : X \to P$ in $\mathscr{E}$. We know that every $(\mathscr{E}, \mathscr{M})$-category satisfies the cancellative condition, hence $\delta_p, id_X \in \mathscr{E}$, so that Δ is the required natural transformation. It's then clear that D is a dco of $\mathscr{E}$ in $\mathbb{A}$ if $domD = dom$ and $dom_\Delta = id_{dom}$. But this should be clear since $domD((u, v) : p \to p') = dom((u, nj) : Dp \to Dp') = u : dom(Dp) \to dom(Dp') = u : X \to X' = dom((u, v) : p \to p')$ and $dom_{\Delta_p} = dom_{(id_X, \delta_p)} = id_X = id_{dom_p}$. Hence, (D, Δ) is a dco of $\mathscr{E}$ in $\mathbb{A}$ if $\mathbb{A}$ is an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E}$ a class of epimorphisms and we have a family of endofunctions that satisfy (a), (b) and (c).

We now turn our attention to $(D_X)_{\mathbb{A}}$ being a family of maps satisfying $(a), (b)$ and (c'), where $\mathbb{A}$ has $\mathscr{E}$-pushouts. We define D on objects in the same manner as before. To define D on morphisms in $\mathscr{E}$, consider $p : X \to P$ and $p' : X' \to P'$ in $\mathscr{E}$ and assume that $(u, v) : p \to p'$ as before. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dd, "p"'] & & X' \arrow[dl, "u_-(p)"'] \arrow[dd, "p'"] \\
& u_-(P) \arrow[dr, dashed, "!c"] & \\
P \arrow[ur, "\bar{u}_p"] \arrow[rr, "v"'] & & P'
\end{tikzcd}$$

where c is the unique morphism induced by the pushout, i.e., such that $cu_-(p) = p'$ and $c\bar{u}_p = v$. In particular, it follows that $u_-(p) \leq p'$ and since D is isotone, $D(u_-(p)) \leq Dp'$, so that there is a unique morphism $k : D(u_-(P)) \to DP'$ such that $k \circ D(u_-(p)) = Dp'$. Then, consider the diagram:

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dd, "Dp"'] \arrow[ddd, bend right, "p"'] & & X' \arrow[dl, "u_-(Dp)"'] \arrow[ddl, "D(u_-(p))"] \arrow[dd, "Dp'"] \arrow[ddd, bend left, "p'"] \\
& u_-(DP) \arrow[d, dashed, "j"] & \\
DP \arrow[ur, "\bar{v}"'] \arrow[d, "\delta_p"] & D(u_-(P)) \arrow[r, "k"'] & DP' \arrow[d, "\delta_{p'}"] \\
P \arrow[rr, "v"'] & & P'
\end{tikzcd}$$

The diagram commutes where $u_-(Dp)$ is the pushout of p along u and by (c'), there exists a unique morphism j such that $ju_-(Dp) = D(u_-(p))$. We assert that $h := kj\bar{v}$ is the unique morphism from DP to DP' such that $hDp = Dp'u$ and $\delta_{p'}h = v\delta_p$.
As Dp is a member of $\mathscr{E}$ and $\mathscr{E}$ is a class of epimorphisms, uniqueness is clear. Note that $hDp = kj\bar{v}Dp = kju_-(Dp)u = kD(u_-(p))u = Dp'u$. Furthermore, $\delta_{p'}hDp = \delta_{p'}Dp'u = p'u = vp = v\delta_p Dp$ and as Dp is an epimorphism, we must have: $\delta_{p'}h = v\delta_p$. It therefore makes sense to define $D(u, v) = (u, l)$ where l is the unique morphism such that $lDp = Dp'u$ and $\delta_{p'}l = v\delta_p$.

We show that D is a functor. To see that D preserves identities, note that $(id_X, id_P) : p \to p$ is the identity on p and id_{DP} is the unique morphism such that $id_{DP}Dp = Dpid_X$ and $\delta_p id_X = id_P\delta_p$ so that D preserves identities. Suppose that $(u, v) : p \to p'$ and $(u', v') : p' \to p''$ are members of $\mathscr{E}$. Then, pasting the obvious diagrams together, we obtain $D(u', v') \circ D(u, v) = (u', l') \circ (u, l) = (u'u, l'l)$ and since $(l'l)Dp = l'Dp'u = Dp''u'u$ and $\delta_{p''}l'l = v'\delta_{p'}l = v'v\delta_p$, it follows that $l'l$ must be the morphism l'' with $D((u', v') \circ (u, v)) = (u'u, l'')$. Thus D is a functor.

We define the natural transformation $\Delta := (\Delta_e)_{e\in\mathscr{E}} : D \to id_{\mathscr{E}}$ by $\Delta_e = (id_X, \delta_e)$ for each $e : X \to E$ in $\mathscr{E}$. As $\mathscr{E}$ is a class of epimorphisms and satisfies the cancellation condition, it's then clear that $dom(\Delta_e)$ and $cod(\Delta_e)$ are members of $\mathscr{E}$ and $u : X \to X' = dom((u, v) : p \to p') = dom((u, l) : Dp \to Dp') = dom(D(u, v) : p \to p'))$. Since $dom_{\Delta_p} = dom_{(id_X,\delta_p)} = id_X$, it follows that D is a dco of $\mathscr{E}$ in $\mathbb{A}$ and our proof is complete. $\square$

Definition 4.9: D-closed, D-sparse
Let (D, Δ) be a dco of $\mathscr{E}$ in $\mathbb{A}$ and $p : X \to P$ be a member of $\mathscr{E}$ with $\delta_p Dp = p$, i.e., $\delta_p = cod(\Delta_p)$. Then, p is (D, Δ)-**closed** (respectively (D, Δ)-**sparse**) if δ_p is an isomorphism (respectively Dp is an isomorphism).

Note that if $\mathscr{E}$ is a class of epimorphisms, then $q \leq id_X$ if and only if $q \simeq id_X$. To see this, note that $id_X q = q$ so that $id_X \leq q$ for any q. Hence this would then imply that a morphism p is D-sparse if and only if $Dp \simeq id_X$.
Let us denote the class of all D-sparse morphisms in $\mathscr{E}$ by $D_{Sp_{(D,\Delta)}}$ and the class of all D-closed morphisms in $\mathscr{E}$ by $D_{Cl_{(D,\Delta)}}$. Whenever $\mathscr{E}$ is a class of epimorphisms in $\mathbb{A}$, the morphism $cod(\Delta_p)$ is uniquely determined. In such a case, we will abuse notation and simply write D-closed and D-sparse respectively. Furthermore, we will then only write D_{Sp} and D_{Cl} instead of the more cumbersome $D_{Sp_{(D,\Delta)}}$ and $D_{Cl_{(D,\Delta)}}$ respectively.

Example 4.10: Let $\mathbb{A}$ be any category and $\mathscr{E}$ any class of morphisms that contains all the identity morphisms. Then, there are always two trivial dual closure operators (D, Δ) and (D', Δ') of $\mathscr{E}$ in $\mathbb{A}$. The first is by defining $D = id_{\mathscr{E}}$ and $\Delta_p = id_X$ for each $p \in \mathscr{E}$. The second is by defining $D'(p) = id_X$ and $\Delta'_p = p$ for each $p : X \to P$ in $\mathscr{E}$.

Remark 4.11: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with pullbacks and let $\mathbb{B}$ be a reflective subcategory of $\mathbb{A}$ with reflector $R : \mathbb{A} \to \mathbb{B}$ and unit ρ. Then, if $\mathscr{E}$ is a class of epimorphisms, we can consider the following diagram for any $p \in \mathscr{E}$:

$$\begin{tikzcd}
X \arrow[d, "D_{\mathbb{B}}(p)"] \arrow[ddr, dashed, "\overline{p}"] \arrow[ddrr, bend left, "\rho_X"] \arrow[dddr, bend right=60, "p"'] & & \\
D_{\mathbb{B}}(P) \arrow[dr, "m_p"'] \arrow[ddr, bend right, "\delta_p"'] & & \\
& \overline{P} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "p.b."] & RX \arrow[d, "Rp"] \\
& P \arrow[r, "\rho_P"'] & RP
\end{tikzcd}$$

The diagram is constructed as follows: The outside diagram is the naturality square induced by the reflector and unit. The bottom square is a pullback square, $\overline{p}$ is the unique morphism induced by the pullback with $m \circ D_{\mathbb{B}}(p)$ an $(\mathscr{E}, \mathscr{M})$ factorisation of $\overline{p}$, and δ_p is defined as $p_1 m_p$. The following theorem (4.12) will show that the diagram defines a dual closure operator in a natural way. We will use the same notation for the rest of this section and will study this diagram again in more detail in the section on the Cassidy Hébert Kelly dual closure operator.

Theorem 4.12: ([26, 15]) **Cassidy Hébert Kelly dual closure operator**
Let $\mathbb{B}$ be a reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$ with $\mathscr{E}$ a class of epimorphisms. Then, if $\mathbb{A}$ has pullbacks, $\mathbb{B}$ induces a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ (called the **Cassidy Hébert Kelly dual closure operator (induced by $\mathbb{B}$) of $\mathscr{E}$ in $\mathbb{A}$).**

<u>$\mathcal{P}roof$</u> : We will use our notation as in Remark 4.11. Suppose that $R : \mathbb{A} \to \mathbb{B}$ is a reflector with unit ρ and let $p \in \mathscr{E}$. We form the pullback of Rp along ρ_P so that the bottom square is a pullback and $\overline{p}$ is then the unique morphism induced by the pullback. Let $m \circ D_{\mathbb{B}}(p)$ be an $(\mathscr{E}, \mathscr{M})$ factorisation of $\overline{p}$ and define δ_p as $p_1 m_p$.
Since $\mathbb{A}$ is an $(\mathscr{E}, \mathscr{M})$-structured category, we need only show that $D_{\mathbb{B}}$ satisfies $(a), (b)$ and (c) of Proposition 4.8.
The fact that $D_{\mathbb{B}} : quot(X) \to quot(X)$ is a map with $D_{\mathbb{B}}(p) \leq p$, is evident from the definition of $D_{\mathbb{B}}(p)$. Notice that $\delta_p \circ D_{\mathbb{B}}(p) = p \in \mathscr{E}$ and since $D_{\mathbb{B}}(p)$ is a member of $\mathscr{E}$, it follows that $\delta_p \in \mathscr{E}$. It ought to be clear that $\Delta : id_{\mathscr{E}} \to D_{\mathbb{B}}$ is a natural transformation with $dom(\Delta_p) = id_X$ and $cod(\Delta_p) = \delta_p$.

Now, assume that $p \leq q$ in $\mathscr{E}$ with $p : X \to P$, $q : X \to Q$ and $jp = q$. We show that $D_{\mathbb{B}}(p) \leq D_{\mathbb{B}}(q)$. Suppose that p_1 is the pullback of Rp along ρ_P and q_1 the pullback of Rq along ρ_Q with $Rpp_2 = \rho_P p_1$ and $Rqq_2 = \rho_Q q_1$ the morphisms in the pullback square. Consider the diagram below. Note that $\rho_Q j p_1 = Rj\rho_P p_1 = RjRpp_2 = R(jp)p_2 = Rqp_2$ so that by the pullback property, there is a unique morphism $k : \bar{P} \to \bar{Q}$ such that $q_2 k = p_2$ and $q_1 k = jp_1$.

Since $q_2 k m_p D_{\mathbb{B}}(p) = p_2 m_p D_{\mathbb{B}}(p) = \rho_X = q_2 m_q D_{\mathbb{B}}(q)$ and $q_1 k m_p D_{\mathbb{B}}(p) = jp_1 m_p D_{\mathbb{B}}(p) = jp = q = q_2 m_q D_{\mathbb{B}}(q)$, and a pullback source is a limit (hence an extremal mono-source), it follows that $k m_p D_{\mathbb{B}}(p) = m_q D_{\mathbb{B}}(q)$. The diagonalisation property then establishes a unique morphism d_j such that $d_j D_{\mathbb{B}}(p) = D_{\mathbb{B}}(q)$ and $m_q d_j = k m_p$. In particular, $D_{\mathbb{B}}(p) \leq D_{\mathbb{B}}(q)$.

$$\begin{tikzcd}
X \arrow[dr, "D_{\mathbb{B}}(p)"] \arrow[drr, "D_{\mathbb{B}}(q)"] \arrow[ddrrrr, bend left, "\rho_X"] \arrow[ddddrr, "p"] \arrow[dddddrr, bend right, "q"'] & & & & \\
& D_{\mathbb{B}}(P) \arrow[dr, "m_p"'] & D_{\mathbb{B}}(Q) \arrow[ddr, "m_q"'] & & \\
& & \overline{P} \arrow[rr, "p_2"] \arrow[dr, dashed, "!k"] \arrow[dd, "p_1"'] & & RX \arrow[dd, "Rp"] \arrow[ddd, bend left, "Rq"] \\
& & & \overline{Q} \arrow[ur, "q_2"'] \arrow[ddl, "q_1"] & \\
& & P \arrow[rr, "\rho_P"] \arrow[d, "j"'] & & RP \arrow[d, "Rj"] \\
& & Q \arrow[rr, "\rho_Q"'] & & RQ
\end{tikzcd}$$

Now, we need only show that $D_{\mathbb{B}}(f^{-}q) \leq f^{-}(D_{\mathbb{B}}q)$ for each $f : X \to Y$ and $q : Y \to Q$ in $\mathscr{E}$.
Let $f : X \to Y$ be an $\mathbb{A}$-morphism and $q : Y \to Q$ a member of $\mathscr{E}$. For simplicity, we will only write D instead of $D_{\mathbb{B}}$. Consider the following $(\mathscr{E}, \mathscr{M})$-factorisations of qf and $D(q)f$, respectively,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f^{-}(q)"'] & Y \arrow[d, "q"] \\
f^{-}(Q) \arrow[r, "m"'] & Q
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f^{-}(D(q))"'] & Y \arrow[d, "D(q)"] \\
f^{-}(D(Q)) \arrow[r, "n"'] & D(Q)
\end{tikzcd}$$

and the commuting diagrams used to construct $D(f^{-}(q))$, $D(q)$ and the pasted naturality squares:

$$\begin{tikzcd}
X \arrow[d, "D(f^{-}(q))"] \arrow[ddr, dashed, "\overline{f^{-}(q)}"] \arrow[ddrr, bend left, "\rho_X"] \arrow[dddr, bend right=60, "f^{-}(q)"'] & & \\
D(f^{-}(Q)) \arrow[dr, "m_{f^{-}(q)}"'] \arrow[ddr, bend right, "\delta_{f^{-}(q)}"'] & & \\
& \overline{f^{-}(Q)} \arrow[r, "f^{-}(q)_2"] \arrow[d, "f^{-}(q)_1"'] \arrow[dr, phantom, "p.b."] & RX \arrow[d, "R(f^{-}(q))"] \\
& f^{-}(Q) \arrow[r, "\rho_{f^{-}(Q)}"'] & Rf^{-}(Q)
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[d, "D(q)"] \arrow[ddr, dashed, "\overline{q}"] \arrow[ddrr, bend left, "\rho_Y"] \arrow[dddr, bend right=60, "q"'] & & \\
D(Q) \arrow[dr, "m_q"'] \arrow[ddr, bend right, "\delta_q"'] & & \\
& \overline{Q} \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "p.b."] & RY \arrow[d, "Rq"] \\
& Q \arrow[r, "\rho_Q"'] & RQ
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_X"'] & Y \arrow[r, "q"] \arrow[d, "\rho_Y"'] & Q \arrow[d, "\rho_Q"] \\
RX \arrow[r, "Rf"'] & RY \arrow[r, "Rq"'] & RQ
\end{tikzcd}$$

Then we have:

$$\begin{aligned}
RqRff^{-}(q)_2 m_{f^{-}(q)} D(f^{-}(q)) &= RqRff^{-}(q)_2 \overline{f^{-}(q)} \\
&= R(qf)\rho_X \\
&= \rho_Q qf \\
&= \rho_Q m f^{-}(q) \\
&= \rho_Q m f^{-}(q)_1 \overline{f^{-}(q)} \\
&= \rho_Q m f^{-}(q)_1 m_{f^{-}(q)} D(f^{-}(q))
\end{aligned}$$

and since $D(f^-(q))$ is a member of $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$, it follows that $RqRff^-(q)_2 m_{f^-(q)} = \rho_Q m f^-(q)_1 m_{f^-(q)}$. In particular, the pullback induces a unique morphism $\bar{k}$ such that the diagram

$$\begin{tikzcd}
D(f^-(Q)) \arrow[dr, dashed, "\bar{k}"] \arrow[drr, bend left, "Rff^-(q)_2 m_{f^-(q)}"] \arrow[ddr, bend right, "mf^-(q)_1 m_{f^-(q)}"'] & & \\
& \overline{Q} \arrow[r, "q_2"] \arrow[d, "q_1"] & RY \arrow[d, "Rq"] \\
& Q \arrow[r, "\rho_Q"'] & RQ
\end{tikzcd}$$

commutes.

We then have that:

$$\begin{aligned}
q_1\bar{k}D(f^-(q)) &= mf^-(q)_1 m_{f^-(q)} D(f^-(q)) \\
&= mf^-(q) \\
&= qf \\
&= q_1 m_q D(q) f \\
&= q_1 m_q n f^-(Dq) \text{ and}
\end{aligned}$$

$$\begin{aligned}
q_2\bar{k}D(f^-(q)) &= Rff^-(q)_2 m_{f^-(q)} D(f^-(q)) \\
&= Rff^-(q)_2 \overline{f^-(q)} \\
&= Rf\rho_X \\
&= \rho_Y f \\
&= q_2 m_q D(q) f \\
&= q_2 m_q n f^-(Dq).
\end{aligned}$$

Since (q_1, q_2) is a pullback source, it's an extremal mono-source so that $\bar{k}D(f^-(q)) = m_q n f^-(Dq)$. Then, since $\mathscr{M}$ is closed under composition, the diagonalisation property establishes a unique morphism d such that the diagram

$$\begin{tikzcd}
X \arrow[r, "D(f^-(q))"] \arrow[d, "f^-(D(q))"'] & D(f^-(Q)) \arrow[d, "\bar{k}"] \arrow[dl, dashed, "!d"'] \\
f^-(D(Q)) \arrow[r, "m_q n"'] & \overline{Q}
\end{tikzcd}$$

commutes. Consequently, it follows that $D(f^-(q)) \leq f^-(D(q))$ and our proof is complete. $\square$

Remark 4.13: The dual closure operator in 4.12 is constructed in [26] and is called the **Cassidy Hébert Kelly dual closure operator** induced by $\mathbb{B}$. It received this name due to a construction used in [7, 3.3].

Note that the proof of Theorem 4.12 is quite long, but one does get a little bit of extra insight on how all the morphisms are constructed. We can also write a short proof in case $\mathbb{A}$ has $\mathscr{E}$-pushouts:
Let $\mathbb{B}$ be reflective in $\mathbb{A}$ with reflector R and unit ρ and let $\rho_X = m_X e_X : X \to SX \to RX$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of ρ_X. Then, we assert that for each $p : X \to P$ in $\mathscr{E}$, that $D_{\mathbb{B}}(p) \simeq p \wedge e_X$, where $\wedge$ is the meet of p and e_X in $quot(X)$.

To see this, note that $D_{\mathbb{B}}(p) \leq p$ and since $p_2 m_p D_{\mathbb{B}}(p) = \rho_X = m_X e_X$, the diagonalisation property establishes a unique morphism $d : D_{\mathbb{B}}(P) \to SX$ such that $dD_{\mathbb{B}}(p) = e_X$ and $p_2 m_p = m_X d$. It follows that $D_{\mathbb{B}}(p)$ is less than or equal to both p and e_X. Suppose that $q : X \to Q$ is any morphism in $\mathscr{E}$ such that $q \leq p$ and $q \leq e_X$. Then, there are morphisms j and k such that $kq = e_X$ and $jq = p$. We show that $q \leq D_{\mathbb{B}}(p)$.

Since $\rho_P jq = \rho_P p = Rp\rho_X = Rpm_X e_X = Rpm_X kq$ and q is an epimorphism, we have $\rho_P j = Rpm_X k$. Hence, the pullback establishes a unique morphism $m : Q \to \overline{P}$ such that $p_1 m = j$ and $p_2 m = m_X k$. Then, mq is a morphism such that $p_1 mq = p$ and $p_2 mq = \rho_X$. Hence $mq = \overline{p} = m_p D_{\mathbb{B}}(p)$ and the diagonalisation property then provides a morphism that in particular implies that $q \leq D_{\mathbb{B}}(p)$. Therefore,

$D_{\mathbb{B}}(p)$ is the meet of p and e_X.

It is then also easy to see that $D_{\mathbb{B}}$ is isotone, for if $p \le p'$ in $quot(X)$, then $D_{\mathbb{B}}(p) \simeq p \wedge e_X \le p' \wedge e_X \simeq D_{\mathbb{B}}(p')$.
Corollary 4.6 shows that $f^{-}(-)$ is right adjoint to $f_{-}(-)$ and hence $f^{-}(-)$ preserves meets.

Let $f : X \to Y$ and consider the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ SX \arrow[r, "Sf"] \arrow[d, "m_X"'] & SY \arrow[d, "m_Y"] \\ RX \arrow[r, "Rf"'] & RY \end{tikzcd}$$

where the morphism Sf is the one established by the diagonalisation property. By using the top square, it is clear that $f_{-}(e_X) \le e_Y$ and $e_X \le f^{-}(e_Y)$.
Let $p : X \to P$ and $q : Y \to Q$ be members of $\mathscr{E}$, then there holds: $D_{\mathbb{B}}(f^{-}(q)) \simeq f^{-}(q) \wedge e_X \le f^{-}(q) \wedge f^{-}(e_Y) \simeq f^{-}(q \wedge e_Y) \simeq f^{-}(D_{\mathbb{B}}(q))$ and $f_{-}(D_{\mathbb{B}}(p)) \simeq f_{-}(p \wedge e_X) \le f_{-}(p) \wedge f_{-}(e_X) \le f_{-}(p) \wedge e_Y \simeq D_{\mathbb{B}}(f_{-}(p))$ so that conditions (c) and (c') of Proposition 4.8 hold. In both cases it follows that $D_{\mathbb{B}}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

It should also be noted that when $\mathbb{B}$ is reflective, then the $\mathscr{M}$-closure $\mathscr{M}(\mathbb{B})$ of $\mathbb{B}$ in $\mathbb{A}$ is an $\mathscr{E}$-reflective subcategory with reflector S and unit e_X for each X in $\mathbb{A}$. Consequently, $D_{\mathbb{B}}(p) \simeq D_{\mathscr{M}(\mathbb{B})}(p)$ for each $p \in \mathscr{E}$. For this reason, we might as well assume that $\mathbb{B}$ is $\mathscr{E}$-reflective when we want to consider the Cassidy Hébert Kelly dual closure operator.

Proposition 4.14: Suppose that the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$ has pullbacks and $\mathscr{E} \subset \text{Epi}(\mathbb{A})$, or suppose that $\mathbb{A}$ has $\mathscr{E}$-pushouts. Let $\mathbb{B}$ be a reflective subcategory of $\mathbb{A}$ with reflector R and unit ρ. Then, if $m_X e_X$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of ρ_X for each X in $\mathbb{A}$, then $D_{\mathbb{B}}(p) \simeq p \wedge e_X$ for all $p : X \to P$ in $\mathscr{E}$. Consequently, it follows that, $D_{\mathbb{B}} \simeq D_{\mathscr{M}(\mathbb{B})}$.

Definition 4.15: $\mathbb{B}$-concordant, $\mathbb{B}$-dissonant
Let $D_{\mathbb{B}}$ be the dual closure operator of $\mathscr{E}$ induced by the reflective subcategory $\mathbb{B}$ of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$ and let $p : X \to P$ be a member of $\mathscr{E}$ such that $\delta_p D_{\mathbb{B}} p = p$. Then, p is said to be **$\mathbb{B}$-concordant if p is $D_{\mathbb{B}}$-closed**. An $\mathbb{A}$-morphism f is said to be **$\mathbb{B}$-dissonant** provided that it factors through a $D_{\mathbb{B}}$-sparse morphism in $\mathscr{E}$ followed by a morphism in $\mathscr{M}$.

Proposition 4.16: ([7]) Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$, where $\mathscr{E}$ is a class of epimorphisms. Denote the reflector and unit of $\mathbb{B}$ by R and ε respectively. Assume that $\mathbb{A}$ has pullbacks and let D be the Cassidy Hébert Kelly dual closure operator of $\mathscr{E}$ induced by $\mathbb{B}$. Then, a morphism $p : X \to P$ in $\mathscr{E}$ is $\mathbb{B}$-concordant if and only if Rp is an isomorphism.

Proof : Consider the diagram in Remark 4.11. For simplicity, we will refer to that notation and simply write D for $D_{\mathbb{B}}$. Suppose that Rp is an isomorphism. Then, since isomorphisms are closed under pullbacks, it follows that p_1 is also an isomorphism. Also, since $p_1 \bar{p} = p$, it follows that $\bar{p} = p_1^{-1} p$ is a member of $\mathscr{E}$, hence since $m_p Dp$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of $\bar{p}$, m_p must be an isomorphism. Finally, we must have $\delta_p = p_1 m_p$ is also an isomorphism, but this means exactly that p is D-closed or, equivalently, p is $\mathbb{B}$-concordant.

Conversely, assume that p is $\mathbb{B}$-concordant, i.e., δ_p is an isomorphism. Note that $R\varepsilon_X$ and $R\varepsilon_P$ are isomorphisms, as RX and RP are members of $\mathbb{B}$. We also have: $R(p_2 m_p \delta_p^{-1})Rp = R(p_2 m_p \delta_p^{-1} p) = R(p_2 m_p Dp) = R\varepsilon_X$ and $R(Rp)R(p_2 m_p \delta_p^{-1}) = R(Rpp_2 m_p \delta_p^{-1}) = R(\varepsilon_P p_1 m_p \delta_p^{-1}) = R(\varepsilon_P)$ implies that $R(p_2 m_p \delta_p^{-1})$ is an isomorphism and then, so is $R(p_2 m_p \delta_p^{-1})^{-1} R\varepsilon_X = Rp$. □

Remark 4.17: A particular useful consequence of $\mathbb{B}$ being $\mathscr{E}$-reflective in $\mathbb{A}$ is that $Rp\varepsilon_X = \varepsilon_P p$ is a member of $\mathscr{E}$ and since ε_X is in $\mathscr{E}$, it will also follow that Rp is a member of $\mathscr{E}$. Then, as a corollary to Proposition 4.16, the following result holds:

Corollary 4.18: If the assumptions of Proposition 4.16 are satisfied, then a morphism $p \in \mathscr{E}$ is $\mathbb{B}$-concordant if and only if Rp is a member of $\mathscr{M}$.

Remark 4.19: We will briefly look at some results on prefactorisation systems for reflective subcategories. A detailed analysis can be found in [7].
Let $\mathbb{A}$ be a category and $\mathbb{B}$ be a full replete reflective subcategory of $\mathbb{A}$. Consider an adjunction $(\rho, \varepsilon) : R \dashv E : \mathbb{B} \longrightarrow \mathbb{A}$ where R is a reflector and E is the inclusion functor. Note that we may as well assume that $E \circ R = id_{\mathbb{A}}$, $\varepsilon_B = id_B$ for each $B \in \mathbb{B}$ and $\rho_B = id_B$ for each $B \in \mathbb{B}$. We will often omit E when discussing $\mathbb{B}$-morphisms and objects in $\mathbb{A}$ and only write $g : B \to B'$ instead of $Eg : EB \to EB'$.

Proposition 4.20: ([30, 4.2.1]) Let $(\rho, \varepsilon) : R \dashv E : \mathbb{B} \longrightarrow \mathbb{A}$ be an adjunction as in 4.19. Then, for any $\mathbb{A}$-morphism f and $\mathbb{B}$-morphism g, $f \perp Eg$ if and only if $Rf \perp g$.

<u>*Proof* :</u> Suppose that $f \perp Eg$. Consider the commutative diagram
$$\begin{tikzcd} RA \arrow[r, "Rf"] \arrow[d, "j"'] & RA' \arrow[d, "k"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$
in $\mathbb{B}$. Since
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\rho_A"'] & A' \arrow[d, "\rho_{A'}"] \\ RA \arrow[r, "Rf"'] & RA' \end{tikzcd}$$
commutes in $\mathbb{A}$ and $Eg = g$, there is a unique morphism $d : A' \to B$ such that $gd = k\rho_{A'}$ and $df = j\rho_A$. It should be clear that $j = R(df)$ and $k = R(k\rho_{A'})$. We assert that Rd is the unique morphism for which $RdRf = j$ and $gRd = k$. It's clear that $RdRf = j$. Furthermore, $gRd\rho_{A'} = gid_B d = k\rho_{A'}$ so that $gRd = k$. Rd is unique, for if d' is another morphism such that $gd' = k$ and $d'Rf = j$, then $d'\rho_{A'}f = d'Rf\rho_A = j\rho_A$ and $gd'\rho_{A'} = k\rho_{A'} = gRd\rho_{A'}$. By uniqueness of d it follows that $Rd\rho_{A'} = d = d'\rho_{A'}$ and thus $dR = d'$. It follows that $Rf \perp g$.

Conversely, assume that $Rf \perp g$ and let
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & A' \arrow[d, "k"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$
be a commutative diagram in $\mathbb{A}$. Then, $R(gj)\rho_A = gRj\rho_A = gj = kf = Rk\rho_{A'}f = RkRf\rho_A = R(kf)\rho_A$ so that $gRj = RkRf$. Since $Rf \perp g$, there is a unique $d : RA' \to B$ such that $dRf = Rj$ and $gd = Rk$. Then, $d\rho_{A'}$ is a morphism such that $d\rho_{A'}f = dRf\rho_A = Rj\rho_A = j$ and $gd\rho_{A'} = Rk \circ \rho_{A'} = k$. To prove that $d\rho_{A'}$ is unique with respect to this property, assume that d' is another morphism such that $d'f = j$ and $gd' = k$. Then, $gR(d') = R(gd') = Rk$ and $Rd'Rf = R(d'f) = Rj$. It follows that $Rd' = d$ and since $d\rho_{A'} = Rd'\rho_{A'} = d'$, our proof is complete. $\square$

Proposition 4.21: Consider an adjunction $(\rho, \varepsilon) : R \dashv E : \mathbb{B} \longrightarrow \mathbb{A}$ as in Rem 4.19. By letting $\mathscr{E} := (E(Mor(\mathbb{B})))^{\uparrow}$, we have that $f \in \mathscr{E}$ if and only if Rf is an isomorphism.

<u>*Proof* :</u> Suppose $f : A \to A' \in \mathscr{E}$. Then, in particular, $f \perp Rf$. Thus, there is a unique morphism $d : A' \to RA$ such that $df = \rho_A$ and $Rfd = \rho_{A'}$. Then, $R(df) : RA \to RA'$ is a morphism such that $\rho_A = df = Rd\rho_{A'}f = R(d) \circ Rf\rho_A$ which implies that $RdRf = id_{RA}$. We also have that $RfRd\rho_{A'} = Rfd = \rho_{A'}$ which implies that $RfRd = id_{RA'}$. Thus Rf is an isomorphism.

Conversely, suppose that Rf is an isomorphism with $\bar{g} = (Rf)^{-1}$. Let j and k be $\mathbb{A}$-morphisms and g a

$\mathbb{B}$-morphism such that $kf = gj$. Then, the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dd, bend right=60, "\rho_A"'] & A' \arrow[d, "k"] \arrow[dd, bend left=60, "\rho_{A'}"] \\
B \arrow[r, "g"] & B' \\
RA \arrow[u, dashed, "!Rj"] \arrow[r, shift left, "Rf"] & RA' \arrow[u, dashed, "!Rk"'] \arrow[l, shift left, "\bar{g}"]
\end{tikzcd}$$

commutes. First we show that $RkRf = gRj$. To see this, note that $RkRf\rho_A = Rk\rho_{A'}f = kf = gj = gRj\rho_{A'}$ and thus $RkRf = gRj$. Let $d = Rj\bar{g}\rho_{A'}$. We show that d is the unique morphism such that $df = h$ and $gd = k$. Clearly $df = Rj\bar{g}\rho_{A'}f = Rj\bar{g}Rf\rho_A = Rj\rho_A = h$ and $gd = gRh\bar{g}\rho_{A'} = RkRf\bar{g}\rho_{A'} = Rk\rho_{A'} = k$. Assume that d' is another morphism such that $d'f = h$ and $gd' = k$. Then, $d' = Rd'\rho_{A'} = Rd'Rf\bar{g}\rho_{A'} = R(d'f)\bar{g}\rho_{A'} = Rh\bar{g}\rho_{A'} = d$. Therefore, $d = d'$ and it follows that f is a member of $\mathscr{E}$. $\square$

Corollary 4.22: Under the assumptions of Proposition 4.21, $(E(Mor(\mathbb{B})))^{\uparrow}$ has the following cancellation properties:

$$e, ef \in (E(Mor(\mathbb{B})))^{\uparrow} \text{ implies that } f \in (E(Mor(\mathbb{B})))^{\uparrow} \text{ and}$$

$$f, ef \in (E(Mor(\mathbb{B})))^{\uparrow} \text{ implies that } e \in (E(Mor(\mathbb{B})))^{\uparrow}.$$

Lemma 4.23: Under the assumptions of Proposition 4.20, let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with pullbacks. Let f be an $\mathbb{A}$-morphism such that Rf is in $\mathscr{M}$ and whenever $f = mp$ with $m \in \mathscr{M}$, then m is an isomorphism. Then f is a member of $(E(Mor(\mathbb{B}))^{\uparrow}$, i.e., Rf is an isomorphism.

Proof : Suppose f is an $\mathbb{A}$-morphism that satisfies the hypothesis. Consider the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "p"] \arrow[drr, "\rho_X"] \arrow[ddr, "f"'] & & \\
& P \arrow[r, "g"] \arrow[d, "m"] & RX \arrow[d, "Rf"] \\
& Y \arrow[r, "\rho_Y"'] & RY
\end{tikzcd}$$

where the bottom square is a pullback. Since Rf is in $\mathscr{M}$ and $\mathscr{M}$ is closed under pullbacks, it follows that m is a member of $\mathscr{M}$. Then, $f = mp$ so that by the assumptions on f, m is an isomorphism. Therefore, we might as well have taken $m = id_Y$ and $p = f$. So, $\rho_X = gf$ and $Rfg = \rho_Y$. Then, $id_{RX} = R(\rho_X) = R(gf) = RgRf$ and $id_{RY} = R(\rho_Y) = R(gf) = RgRf$ which implies that Rf is an isomorphism. By proposition 4.21, it follows that f is a member of $(E(Mor(\mathbb{B})))^{\uparrow}$. $\square$

Corollary 4.24: ([6, 3.2]) Suppose that f is an $\mathbb{A}$-morphism such that whenever $f = mk$ with $m \in (E(Mor(\mathbb{B})))^{\uparrow\downarrow}$, then m is an isomorphism and that Rf is a section. Then f is in $(E(Mor(\mathbb{B})))^{\uparrow}$.

Proof : The proof follows in a similar manner as in 4.23. $\square$

Let $\mathbb{B}$ be a reflective subcategory of the category $\mathbb{A}$, where $\mathbb{A}$ has pullbacks and $(E(Mor(\mathbb{B})))^{\uparrow\downarrow}$-intersections. For simplicity, let us denote by $(\mathscr{E}, \mathscr{M})$ the prefactorisation structure $((E(Mor(\mathbb{B})))^{\uparrow}, (E(Mor(\mathbb{B})))^{\uparrow\downarrow})$.

Let f be an $\mathbb{A}$-morphism and consider the commuting diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "p"] \arrow[drr, "\rho_X"] \arrow[ddr, "f"'] & & \\
& P \arrow[r, "g"] \arrow[d, "m"] & RX \arrow[d, "Rf"] \\
& Y \arrow[r, "\rho_Y"'] & RY
\end{tikzcd}$$

where the bottom square is a pullback. We consider the intersection $d : D \to P$ in $\mathscr{M}$ of all $\mathscr{M}$-subobjects m_i through which p factors, i.e., $p = m_i g_i$ for some $\mathbb{A}$-morphism g_i with $m_i \in \mathscr{M}$. Then,

$$\begin{tikzcd}
& & M_i \arrow[dr, "m_i"] & \\
X \arrow[urr, bend left, "g_i"] \arrow[r, dashed, "\bar{e}"'] \arrow[rrr, bend right, "p"'] & D \arrow[ur, "d_i"] \arrow[rr, "d"'] & & P
\end{tikzcd}$$

commutes for a unique morphism $\bar{e}$ and hence $p = d\bar{e}$. Equivalently, there exists an $i \in I$ such that $m_i = d$ and $\bar{e} = g_i$. We see that $f = mp = (md)\bar{e}$ and we assert that $(md)\bar{e}$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of f. Since $Rf \in Mor(\mathbb{B}) \subset E(Mor(\mathbb{B})) \subset (E(Mor(\mathbb{B})))^{\uparrow\downarrow}$ and $\mathscr{M}$ is closed under pullbacks, m is in $\mathscr{M}$. As $\mathscr{M}$ is closed under composition and $\mathbb{A}$ has $(E(Mor(\mathbb{B})))^{\uparrow\downarrow}$-intersections, md is a member of $\mathscr{M}$. So, we need only show that $\bar{e}$ is a member of $\mathscr{E}$. Since $id_{RX} = R(\rho_X) = R(gp) = R(g)R(md\bar{e}) = R(g)R(md)R(\bar{e})$, we have that $R(\bar{e})$ is a section. In view of Corollary 4.24, we need only show that whenever $\bar{e} = nk$ with $n \in \mathscr{M}$, then n is an isomorphism. Assume that $\bar{e} = nk$ with $n \in \mathscr{M}$. Then, $p = d\bar{e} = (dn)k$ so that since dn is a member of $\mathscr{M}$ and p factors through it, it follows that $dn = m_i$ and $k = g_i$ for some $i \in I$. But then, $d \leq m_i = dn \leq d$ in $\mathscr{M}/P$ so that n must be an isomorphism. Note that this is the case since a class closed under multiple pullbacks must be a class of monomorphisms. Hence, $\bar{e}$ is a member of $\mathscr{E}$.

Proposition 4.25: ([6, 3.3]) Let $\mathbb{A}$ be a category and $\mathbb{B}$ be a reflective subcategory of $\mathbb{A}$. Then, if $\mathbb{A}$ has pullbacks and $\mathbb{A}$ has $(E(Mor(\mathbb{B})))^{\uparrow\downarrow}$-intersections, then $\mathbb{A}$ has $((E(Mor(\mathbb{B})))^{\uparrow}, (E(Mor(\mathbb{B})))^{\uparrow\downarrow})$-factorisations.

Definition 4.26: Left-$\mathscr{E}$-factorisations
Let $\mathscr{E}$ be a class of morphisms in a category $\mathbb{A}$ with $Iso(\mathbb{A}) \subset \mathscr{E}$ and $\mathscr{E}$ closed under composition with isomorphisms. Then, $\mathbb{A}$ is said to have **left-$\mathscr{E}$-factorisations** provided that the following hold:

(i) each $\mathbb{A}$-morphism f has a **left-$\mathscr{E}$-factorisation**, i.e., $f = m \circ e$, where $e \in \mathscr{E}$,

(ii) whenever the diagram
$$\begin{tikzcd}
A \arrow[rr, "e'"] \arrow[d, "u"'] & & B \arrow[d, "v"] \\
C \arrow[r, "e"'] & M \arrow[r, "m"'] & D
\end{tikzcd}$$
commutes in $\mathbb{A}$, where me is a left-$\mathscr{E}$-factorisation, then there is a unique morphism $d : B \to M$ such that $de' = eu$ and $md = v$.

Proposition 4.27: ([26])$\mathbb{A}$ has left $\mathscr{E}$-factorisations if and only if the full subcategory $\mathscr{E}$ of the arrow category $\mathbb{A}^{\to}$ is coreflective.

Proof : This follows from the dual of a result in [25]. □

Remark 4.28: If a category $\mathbb{A}$ has left-$\mathscr{E}$-factorisations, we will often denote by $\mathscr{M}$, the class of $\mathbb{A}$-morphisms for which the coreflection object is an isomorphism, i.e., if $m = ne$ is a left-$\mathscr{E}$-factorisation of m, then $m \in \mathscr{M}$ if and only if e is an $\mathbb{A}$-isomorphism. In particular, if m and n are members of $\mathscr{M}$, then if we denote the coreflector functor by C, then Cm and Cn are isomorphisms, so that $C(mn)$ is also an isomorphism whenever mn is defined. Thus $\mathscr{M}$ is closed under composition. If $\mathscr{K}$ and $\mathscr{L}$ are classes of $\mathbb{A}$-morphisms, then let us denote by $\mathscr{K} \circ \mathscr{L}$, or simply $\mathscr{K}\mathscr{L}$, the class of all $\mathbb{A}$-morphisms g such that $g = kl$ for some $k \in \mathscr{K}$ and some $l \in \mathscr{L}$. To keep things simple, we will assume that $\mathscr{E}$ is a class of epimorphisms.

Since most of our results so far depend on either $(\mathscr{E}, \mathscr{M})$ structured categories or even $(\mathscr{E}, \mathbb{M})$-categories and these can already be viewed as the category having left-$\mathscr{E}$-factorisations, we will not focus on a lot of material here. It should however be noted that a dual closure operator can be viewed as an endofunctor which provides us with left-$\mathscr{E}'$-factorisations for some class of morphisms $\mathscr{E}' \subset \mathscr{E}$.

Proposition 4.29: Let $\mathbb{A}$ be a category and (D, Δ) a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Then, the following hold for each isomorphism h of $\mathbb{A}$ and each $e \in \mathscr{E}$:

(a) if e is (D, Δ)-closed and he is defined, then he is (D, Δ)-closed,

(b) if e is (D, Δ)-sparse and he is defined, then he is (D, Δ)-sparse.

Proof : Suppose that (D, Δ) is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Suppose $e : X \to Y \in \mathscr{E}$ and $h : Y \to Z$ is an $\mathbb{A}$-isomorphism. If e is (D, Δ)-closed, it follows that $De \simeq e$. Then, $he \simeq e \simeq D(e) \leq D(he)$ since D is isotone and $e \leq he$. Hence $he \simeq D(he)$. If e is (D, Δ)-sparse, i.e., De is an isomorphism or equivalently $De \leq id_X$. Then, $e \leq he \leq h^{-1}he = e$, hence $he \simeq e$. Thus, $D(he) \simeq D(e) \leq id_X$ so that $D(he)$ is an isomorphism. □

Definition 4.30: Pre-Order on $DCO(\mathbb{A}, \mathscr{E})$.
Let $\mathscr{E}$ be a class of epimorphisms in $\mathbb{A}$. Then, let us denote the conglomerate of all dual closure operators of $\mathscr{E}$ in $\mathbb{A}$ by $DCO(\mathbb{A}, \mathscr{E})$. There is a natural pre-order on $DCO(\mathbb{A}, \mathscr{E})$ by defining $D \leq D'$ if and only if $Dp \leq D'p$ for each $p \in \mathscr{E}$.

Definition 4.31: Isomorphic Dual closure operators
Two dual closure operators D and D' of a class $\mathscr{E}$ of epimorphisms in $\mathbb{A}$ are said to be **isomorphic**, written $D \simeq D'$, if $D \leq D'$ and $D' \leq D$.

Note that $D \simeq D'$ if and only if for each $p \in \mathscr{E}$, there holds $Dp \simeq D'p$.

Definition 4.32: ([26]) Idempotent, weakly cohereditary, cohereditary and maximal dual closure operators
Let (D, Δ) be a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ with $\delta_p := cod(\Delta_p)$ for each $p : X \to P$. Then, (D, Δ) is **idempotent** if $\delta_{Dp} = D_{id_X, \delta_p}$ is an isomorphism for each $p : X \to P$ in $\mathscr{E}$. (D, Δ) is **weakly cohereditary(*wch*)** if $D(\delta_p) = D_{Dp, id_Y}$ is an isomorphism for each $p : X \to P$ in $\mathscr{E}$. Hence, if $\mathscr{E}$ is a class of epimorphisms, then D is idempotent if Dp is D-closed and wch if δ_p is D-sparse for each $p \in \mathscr{E}$. A dual closure operator (D, Δ) is:

(i) **cohereditary** if and only if it's weakly cohereditary and satisfies

$$q \circ p \in D_{Sp_{(D,\Delta)}} \Rightarrow q \in D_{Sp_{(D,\Delta)}}$$

for all composable $p, q \in \mathscr{E}$,

(ii) **maximal** if and only if it's idempotent and satisfies

$$q \circ p \in D_{Cl_{(D,\Delta)}} \Rightarrow p \in D_{Cl_{(D,\Delta)}}$$

for all composable $p, q \in \mathscr{E}$.

Proposition 4.33: Let D and D' be two isomorphic dco's of $\mathscr{E}$ in $\mathbb{A}$. Then, D is idempotent, respectively weakly cohereditary, if and only if D' is.

Proof : The proof is straightforward computation. □

Proposition 4.34: If D is an idempotent dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ and $\mathbb{A}$ has left-$\mathscr{E}$-factorisations, then every morphism factorises as kd, where d is a D-closed morphism in $\mathscr{E}$.

Proof : Any $\mathbb{A}$-morphism g has left-$\mathscr{E}$-factorisation me and $e = \delta_e De$ and since D is idempotent, δ_{De} is an isomorphism, or equivalently, De is D-closed. Thus, $m\delta_e \circ De$ is a left $(D, \Delta)_{Cl}$-factorisation of g. $\square$

Proposition 4.35: Let D be an idempotent dual closure operator $\mathscr{E}$ in $\mathbb{A}$, where $\mathscr{E}$ is a class of epimorphisms and assume that $\mathbb{A}$ has left-$\mathscr{E}$-factorisations. Then, $\mathbb{A}$ has left-$(D, \Delta)_{Cl}$-factorisations.

Proof : By Lemma 4.29, it follows that $(D, \Delta)_{Cl}$ is a class of morphisms closed under composition with isomorphisms. Since functors preserve isomorphisms, Df is an isomorphism whenever f is an isomorphism and consequently $Df \simeq f$ so that f is a member of $(D, \Delta)_{Cl}$. By Proposition 4.34, we know that f factorises as $m\delta_e(De)$, where De is a D-closed morphism in $\mathscr{E}$ and me is a left $\mathscr{E}$-factorisation of f. The only part that still needs to be verified is to establish the diagonalisation property for left-$\mathscr{E}$-factorisations. Suppose that $ve' = fu$ in $\mathbb{A}$, where f has left-$\mathscr{E}$-factorisation $me = m\delta_e De$ with $e \in \mathscr{E}$ and De and e' are D-closed. Then, since $\mathbb{A}$ has left-$\mathscr{E}$-factorisations, there is a unique morphism $d : Y \to M$ such that

$$\begin{tikzcd} X \arrow[rrr, "e'"] \arrow[d, "u"'] & & & Y \arrow[d, "v"] \arrow[dl, "d"'] \\ A \arrow[r, "De"'] & D_A(M) \arrow[r, "\delta_e"'] & M \arrow[r, "m"'] & B \end{tikzcd}$$

commutes. Since e' is D-closed and $\mathscr{E}$ is a class of epimorphisms, $D_{u,d}$ is the unique morphism such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "De'"'] & A \arrow[d, "De"] \\ D_X(Y) \arrow[r, "D_{u,d}"'] \arrow[d, "\delta_{e'}"'] & D_A(M) \arrow[d, "\delta_e"] \\ Y \arrow[r, "d"'] & M \end{tikzcd}$$

commutes. Since e' is D-closed, $\delta_{e'}$ is an isomorphism, so that $\bar{d} := D_{u,d} \circ \delta_{e'}^{-1}$ is a morphism such that $\bar{d}e' = D_{u,d} \circ \delta_{e'}^{-1} e' = D_{u,d} De' = Deu$ and $m\delta_e \bar{d} = m\delta_e D_{u,d} \circ \delta_{e'}^{-1} = md\delta_{e'}\delta_{e'}^{-1} = md = v$. Since $\mathscr{E}$ is a class of epimorphisms, uniqueness follows.

$\square$

Proposition 4.36: Suppose that $\mathbb{A}$ is $(\mathscr{E}, \mathscr{M})$-structured and D is an idempotent wch dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Then, $\mathbb{A}$ is $(D_{Cl},\ \mathscr{M} \circ D_S)$-structured.

Proof : For each f, with $(\mathscr{E}, \mathscr{M})$-factorisation $m \circ e$ and $e = \delta_e \circ De$. By idempotency of D, it follows that De is D-closed. Since D is weakly cohereditary, it follows that δ_e is D-sparse. Hence, $f = me = (m\delta_e)De$ is a $(D_{Cl}, \mathscr{M} \circ D_S)$-factorisation of f. It remains to be seen that the diagonalisation property holds. To this end, assume that

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "n"'] & D \end{tikzcd}$$

commutes with e in D_{Cl} and n in $\mathscr{M} \circ D_S$. Then, there exist morphisms m in $\mathscr{M}$ and k in D_S such that $n = mk : C \to M \to D$.

Then, since $\mathbb{A}$ has $(\mathscr{E}, \mathscr{M})$-factorisations, there is a unique morphism $d : B \to M$ such that $de = kf$ and $md = g$. Hence, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "De"'] & C \arrow[d, "Dk"] \\ D_A(B) \arrow[r, "D_{f,d}"'] \arrow[d, "\delta_e"'] & D_C(M) \arrow[d, "\delta_k"] \\ B \arrow[r, "d"'] & M \end{tikzcd}$$

commutes. Since e is D-closed and k is D-sparse, it follows that δ_e and Dk are isomorphisms. So, $\bar{d} := (Dk)^{-1} \circ D_{f,d} \circ \delta_e^{-1}$ is the desired morphism from B to C. To see this, note that $d \circ \delta_e = \delta_k \circ D_{f,d}$ so that $d = \delta_k \circ D_{f,d} \circ \delta_e^{-1}$ and $D_{f,d} \circ De = Dk \circ f$ so that $f = (Dk)^{-1} \circ D_{f,d} \circ De$. Then, $\bar{d}e = (Dk)^{-1} \circ D_{f,d} \circ \delta_e^{-1} e = (Dk)^{-1} \circ D_{f,d} \circ De = (Dk)^{-1} Dkf = f$ and $n\bar{d} = mk(Dk)^{-1} \circ D_{f,d} \circ \delta_e^{-1} = m\delta_k D_{f,d} \circ \delta_e^{-1} = md\delta_e\delta_e^{-1} = md = g$. Uniqueness of $\bar{d}$ follows because $\mathscr{E}$ is a class of epimorphisms. We need to show that D_{Cl} and $\mathscr{M} \circ D_S$ are closed under composition with isomorphisms. It's sufficient to prove that if h is an isomorphism and n and e are morphisms in D_{Cl}

and $\mathscr{M} \circ D_S$ respectively, then he is D-closed and nh is a member of $\mathscr{M} \circ D_S$, whenever he and ne is defined. The first part follows directly from Proposition 4.34. Furthermore, if h is an isomorphism and nh is defined with $n = mk$ with $m \in \mathscr{M}$ and k is D-sparse, we need only prove that kh is D-sparse. Then, note that $D_{h,id} = cod(D(h, id))$ and since (h, id) is an isomorphism in $\mathbb{A}^{\rightarrow}$ and D and cod are functors, $D_{h,id}$ is an isomorphism. But $Dk \circ h = D_{h,id}D(kh)$ so that $D(kh) = D^{-1}_{h,id}Dk \circ h$ is a composition of isomorphisms, thus $D(kh)$ is an isomorphism. Hence, kh is D-sparse and our proof is complete. $\square$

Definition 4.37: Collection of all ($\mathscr{E}$-)reflective subcategories
We will denote the collection of all $\mathscr{E}$-reflective subcategories, respectively full reflective subcategories, of an $(\mathscr{E}, \mathscr{M})$-category $\mathbb{A}$ by $R(\mathbb{A}, \mathscr{E})$ and $R(\mathbb{A})$ respectively. If $\mathbb{C}$ is any subcategory of $\mathbb{A}$, then we denote the conglomerate of all $\mathscr{E}$-reflective, respectively reflective subcategories, of $\mathbb{A}$ that contain $\mathbb{C}$ by $R_{\mathbb{C}}(\mathbb{A}, \mathscr{E})$, respectively $R_{\mathbb{C}}(\mathbb{A})$. Whenever not stated explicitly, we will use the partial ordering on these collections by inclusion.

Lemma 4.38: ([1, 11.10]) Suppose we have the following commutative diagram:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

Then, if both inner squares are pushouts, so is the outer diagram. Also, if the morphism at the top is an epimorphism and the outer square is a pushout, so is the bottom inner square.

Proposition 4.39: Let D be a dco of a class of epimorphisms $\mathscr{E}$ in the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. Then, D-closed morphisms are closed under (multiple) pushouts.

Proof : Throughout the proof, let $e : X \to Y$ be a member of $\mathscr{E}$ and $f : X \to Z$ a morphism and $f_-(e)$ a pushout of e along f. Assuming that e is D-closed, we have $e \simeq De$. Then, since $f_-(-)$ is order preserving, we have: $f_-(e) \simeq f_-(De) \leq D(f_-(e)) \leq f_-(e)$, hence $f_-(e) \simeq D(f_-(e))$ so that $f_-(e)$ is D-closed.

Let $(e_i : X \to X_i)_I$ be a family of D-closed morphisms in $\mathscr{E}$ and $(c_i : X_i \to Y)_I$ a sink such that e is a multiple pushout of the family $(e_i)_I$ or, equivalently,

$$\begin{tikzcd}
 & X_i \arrow[dr, "c_i"] & \\
X \arrow[ur, "e_i"] \arrow[rr, "e"'] & & Y
\end{tikzcd}$$

is a multiple pushout diagram in $\mathbb{A}$. Consider, for each $i \in I$, the morphism $D_{id_X,c_i} : DX_i \to DY$. We know that δ_{e_i} is an isomorphism for each $i \in I$. In particular, $d_i := D_{id_X,c_i} \circ \delta^{-1}_{e_i}$ is a morphism such that $d_i \circ e_i = D_{id_X,c_i} \circ \delta^{-1}_{e_i} \circ e_i = D_{id_X,c_i}De_i = De$. The multiple pushout property gives us a unique morphism $d : Y \to DY$ such that $de = De$ and $dc_i = d_i$. Then, $\delta_e de = \delta_e De = e$ and $\delta_e dc_i = \delta_e d_i = \delta_e D_{id_X,c_i} \circ \delta^{-1}_{e_i} = c_i\delta_{e_i}\delta^{-1}_{e_i} = c_i$. Since id_Y is the unique morphism such that $id_Y e = e$ and $id_Y c_i = c_i$ for each $i \in I$, we must have $\delta_e d = id_Y$. Furthermore, since $d\delta_e De = de = De$ and De is an epimorphism, it follows that $d = \delta^{-1}_e$. Consequently, δ_e is an isomorphism and it follows that e is D-closed.

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "De_i"'] & X \arrow[d, "De"] \\
DX_i \arrow[r, "D_{id_X,c_i}"] \arrow[d, "\delta_{e_i}"'] & DY \arrow[d, "\delta_e"] \\
X_i \arrow[r, "c_i"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_i \arrow[dr, "c_i"] \arrow[drr, bend left, "d_i"] & & \\
X \arrow[ur, "e_i"] \arrow[rr, "e"'] \arrow[rrr, bend right, "De"'] & & Y \arrow[r, dashed, "!d"] & DY
\end{tikzcd}$$

$\square$

4.2 An investigation of the Cassidy Hébert Kelly dual closure operator

Remark 4.40: In order to investigate dual closure operators in more detail, it is necessary to study some examples. Since Tholen and Dikranjan have constructed ([26]) the Cassidy Hébert Kelly and the Eilenberg-Whyburn dual closure operators, it's only natural to study these in more detail. Throughout this section we will always make the following assumptions when dealing with the Cassidy Hébert Kelly. When dealing with the Cassidy Hébert Kelly dual closure operator, we will always assume that the category in question has pullbacks and that $\mathbb{B}$ is a reflective subcategory with reflector R and unit ρ. Since most of our approach is for categories with factorisation structures for sources, we will assume that $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category and, consequently, $\mathbb{A}$ has $\mathscr{E}$-pushouts and multiple pushouts of morphisms in $\mathscr{E}$ exist, and are members of $\mathscr{E}$.

Proposition 4.41: Let $\mathbb{B}$ be a full reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$ with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Let S be the reflector for $\mathbb{A}$ into $\mathbb{B}$. Then, if $\mathbb{A}$ has pullbacks, $S(D_{\mathbb{B}}(p))$ is an isomorphism for each $p \in \mathscr{E}$.

$\underline{\mathcal{Proof}}$: In order to simplify the proof, we will dentote the Cassidy Hébert Kelly dual closure operator induced by $\mathbb{B}$, by D. For each $\mathbb{A}$-object X, let $\eta_X \circ \varepsilon_X : X \to RX \to SX$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of ρ_X. Remark 4.13 shows that $Dp \simeq p \wedge \varepsilon_X$ for each $p : X \to P$. This establishes a reflector R with unit ε into the $\mathscr{E}$-reflective hull of $\mathbb{B}$ in $\mathbb{A}$. Remark 4.13 also implies that $D_{\mathbb{B}} \simeq D_{\mathscr{M}(\mathbb{B})}$. Hence we need not bother with which reflector is used when constructing Dp. We first show that RDp is an isomorphism. Consider the naturality square:

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "Dp"'] & RX \arrow[d, "RDp"] \\ DP \arrow[r, "\varepsilon_{DP}"'] & RDP \end{tikzcd}$$

Since $p_2 \circ m_p : DP \to \bar{P} \to RX$ is a morphism from DP to a $\mathbb{B}$-object, there is a unique morphism $t : RDP \to RX$ such that $t \circ \varepsilon_{DP} = p_2 \circ m_p$. Then, $t \circ RDp \circ \varepsilon_X = t \circ \varepsilon_{DP} Dp = p_2 \circ m_p Dp = \varepsilon_X = id_{RX} \circ \varepsilon_X$ and consequently $t \circ RDp = id_{RX}$. It follows that RDp is a section, so it's sufficient to show that it's an epimorphism. Since $RDp \circ \varepsilon_X = \varepsilon_{DP} \circ Dp \in \mathscr{E}$ and ε_X is in $\mathscr{E}$, it follows that RDp is in $\mathscr{E}$. Since $\mathscr{E}$ is a class of epimorphisms, RDp is an isomorphism. We will proceed to show that SDp is an isomorphism.

Since a pushout of an isomorphism is an isomorphism, it is clear that
$$\begin{tikzcd} RX \arrow[r, "\eta_X"] \arrow[d, "RDp"'] & SX \arrow[d, "id_{SX}"] \\ RDP \arrow[r, "\eta_X \circ (RDp)^{-1}"'] & SX \end{tikzcd}$$
is a pushout square. Consider the commuting square:

$$\begin{tikzcd} RX \arrow[r, "\eta_X"] \arrow[d, "RDp"'] & SX \arrow[d, "SDp"] \\ RDP \arrow[r, "\eta_{DP}"'] & SDP \end{tikzcd}$$

We assert that it's a pushout square. Since $\eta_X \circ (RDp)^{-1} \circ \varepsilon_{DP}$ is a morphism from DP to a $\mathbb{B}$-object SX, there is a unique morphism $h : SDP \to SX$ such that $h \circ \rho_{DP} = \eta_X \circ (RDp)^{-1} \circ \varepsilon_{DP}$. We first show that h is a morphism such that $hSDp = id_{SX}$ and $h \circ \eta_{DP} = \eta_X \circ (RDp)^{-1}$.
Note that $h \circ \eta_{DP} \circ \varepsilon_{DP} = \eta_X \circ (RDp)^{-1} \circ \varepsilon_{DP}$. Since $\mathscr{E}$ is a class of epimorphisms, it follows that $h \circ \eta_{DP} = \eta_X \circ (RDp)^{-1}$. Then, $h \circ SDp \circ \rho_X = h \circ (SDp \circ \eta_X) \circ \varepsilon_X = h \circ \eta_{DP} \circ RDp \circ \varepsilon_X = \eta_X \circ RDp \circ (RDp)^{-1} \circ \varepsilon_X = \eta_X \circ \varepsilon_X = id_{RX} \circ \rho_X$. Since ρ_X is a $\mathbb{B}$-epimorphism, the other equality follows.
The pushout square establishes a unique morphism $f : SX \to SDP$ such that $f \circ \eta_X \circ (RDp)^{-1} = \eta_{DP}$ and $f \circ id_{SX} = SDp$. Since $hSDp = id_{SX}$, it's sufficient to prove that $SDp \circ h = id_{SDP}$. Note that $id_{SDP} \circ \rho_{DP} = \eta_{DP} \circ \varepsilon_{DP} = \eta_{DP} \circ RDp \circ (RDp)^{-1} \varepsilon_{DP} = SDp \circ (\eta_X \circ (RDp)^{-1}) \circ \varepsilon_{DP} = SDp \circ h \circ \eta_{DP} \varepsilon_{DP} = SDph \circ \rho_{DP}$. Since ρ_{DP} is the reflection morphism of DP, it follows that $SDph = id_{SDP}$ and our proof is complete.

$$
\begin{tikzcd}
DP \arrow[r, "\eta_X (RDp)^{-1}\varepsilon_{DP}"] \arrow[d, "\rho_{DP}"'] & SX \\
SDP \arrow[ur, dashed, "!h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[d, "Dp"'] & RX \arrow[r, "\eta_X"] \arrow[d, "RDp"'] & SX \arrow[d, "SDp"] \arrow[dr, bend left, "id_{SX}"] & \\
DP \arrow[r, "\varepsilon_{DP}"'] & RDP \arrow[r, "\eta_{DP}"'] \arrow[drr, bend right, "\eta_X \circ (RDp)^{-1}"'] & SDP \arrow[dr, "h"] & \\
 & & & SX
\end{tikzcd}
$$

□

Corollary 4.42: Let $\mathbb{B}$ be a full reflective subcategory of the $(\mathscr{E},\mathscr{M})$-structured category $\mathbb{A}$. Then, if $\mathbb{A}$ has pullbacks, $D_{\mathbb{B}}(p)$ is $D_{\mathbb{B}}$-closed for each $p \in \mathscr{E}$. It follows that $D_{\mathbb{B}}$ is an idempotent dual closure operator.

Proof : Let S denote the reflector from $\mathbb{A}$ into $\mathbb{B}$ with unit ρ and let $p \in \mathscr{E}$. By 4.41, it follows that $S(D_{\mathbb{B}}(p))$ is an isomorphism. Since a pullback of an isomorphism is an isomorphism, the pullback $(Dp)_1$ of $S(D_{\mathbb{B}}(p))$ along $\rho_{D_{\mathbb{B}}P}$ is an isomorphism. Then, $\overline{D_{\mathbb{B}}p} = (Dp)_1^{-1}p$ is already a member of $\mathscr{E}$, so that $m_{D_{\mathbb{B}}(p)}$ is an isomorphism. Hence, $(Dp)_1 \circ m_{D_{\mathbb{B}}(p)} = \delta_{D_{\mathbb{B}}(p)}$ is an isomorphism, or, equivalently, $D_{\mathbb{B}}(p)$ is $D_{\mathbb{B}}$-closed. The fact that $D_{\mathbb{B}}$ is idempotent then follows easily. □

Corollary 4.43: Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of the $(\mathscr{E},\mathscr{M})$-structured category with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. If $\mathbb{A}$ has pullbacks, then $D_{\mathbb{B}}$ is a maximal dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

Proof : From Corollary 4.42, we need only show that whenever qp is $D_{\mathbb{B}}$-closed, then p is $D_{\mathbb{B}}$-closed for all composable p and q in $\mathscr{E}$.

Let us denote the reflector from $\mathbb{A}$ into $\mathbb{B}$ by R. Let $p : X \to P$ and $q : P \to Q$ be members of $\mathscr{E}$ with $q \circ p$ being $D_{\mathbb{B}}$-closed. Note that $q \circ p$ is $D_{\mathbb{B}}$-closed if and only if qp is $\mathbb{B}$-concordant and this is the case if and only if $R(qp)$ is an isomorphism if and only if $RqRp$ is an isomorphism. In particular, Rp is a section in $\mathscr{E}$ and since $\mathscr{E}$ is a class of epimorphisms, Rp is an isomorphism which is the case if and only if p is $D_{\mathbb{B}}$-closed. Maximality of $D_{\mathbb{B}}$ follows. □

Remark 4.44: Note that there is an easy way to see that $D_{\mathbb{B}}$ is maximal whenever $\mathbb{B}$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. To see this, note that $p : X \to P$ is $D_{\mathbb{B}}$-closed if and only if $p \simeq D_{\mathbb{B}}(p)$ if and only if $p \simeq p \wedge \rho_X$ if and only if $p \leq \rho_X$. Then, for p and q in $\mathscr{E}$, we see that $qp : X \to P \to Q$ is $D_{\mathbb{B}}$-closed if and only if $qp \leq \rho_X$. Since $p \leq qp$, we then have $p \leq \rho_X$ so that p is $D_{\mathbb{B}}$-closed. It's also easy to see that $D_{\mathbb{B}}$ is idempotent as $D_{\mathbb{B}}(D_{\mathbb{B}}(p)) \simeq D_{\mathbb{B}}(p) \wedge \rho_X \simeq p \wedge \rho_X \wedge \rho_X \simeq \rho_X \wedge p \simeq D_{\mathbb{B}}(p)$.

Proposition 4.45: Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of the $(\mathscr{E},\mathscr{M})$-structured category $\mathbb{A}$ with $\mathscr{E}$ a class of epimorphisms. If $\mathbb{A}$ has pullbacks, then $\mathbb{B}$ is $D_{\mathbb{B}}$-closed-reflective, or, equivalently, $\mathbb{B}$-concordant-reflective.

Proof : It is sufficient to show that each reflection morphism is $D_{\mathbb{B}}$-closed. Let $R : \mathbb{A} \to \mathbb{B}$ denote a reflector with unit ε. Remark 4.13 gives that $D_{\mathbb{B}}(\varepsilon_X) \simeq \varepsilon_X$, i.e. ε_X is $D_{\mathbb{B}}$-closed, which is what was to be shown. □

Theorem 4.46: Let $\mathbb{B}$ be a full reflective subcategory of the $(\mathscr{E},\mathscr{M})$-structured category $\mathbb{A}$ with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Then, if $\mathbb{A}$ has pullbacks, $D_{\mathbb{B}}$ is weakly cohereditary.

Proof : Note that $D_{\mathbb{B}}$ is isomorphic to $D_{\mathscr{M}(\mathbb{B})}$, where $\mathscr{M}(\mathbb{B})$ is the $\mathscr{E}$-reflective hull (or equivalently the $\mathscr{M}$-closure) of $\mathbb{B}$ in $\mathbb{A}$. Hence we may, without loss of generality assume, that $\mathbb{B}$ is $\mathscr{E}$-reflective with reflector R and unit ε. We use the notation as in Remark 4.11 and in order to simplify the proof, we will only write $Dp : X \to DP$ instead of the more cumbersome $D_{\mathbb{B}}(p) : X \to D_{\mathbb{B}}(P)$. We show that D is weakly cohereditary, i.e., $D\delta_p$ is an isomorphism for a fixed morphism $p : X \to P$ in $\mathscr{E}$.

In order to show that $D\delta_p$ is an isomorphism for any $p : X \to P \in \mathscr{E}$, it is sufficient to show that the morphism from DP to the required pullback diagram is a member of $\mathscr{M}$. First we show that

$$\begin{tikzcd} \bar{P} \arrow[r, "p_2"] \arrow[d, "p_1"'] & RX \arrow[r, "RDp"] & RDP \arrow[d, "R\delta_p"] \\ P \arrow[rr, "\varepsilon_P"'] & & RP \end{tikzcd}$$

is the required pullback square, where p_1 and p_2 are the morphisms required to construct the pullback diagram for Dp. Let $f : A \to RDP$ and $g : A \to P$ be $\mathbb{A}$-morphisms such that $R\delta_p \circ f = \varepsilon_P \circ g$. Then, by 4.41, it follows that RDp is an isomorphism and hence $(RDp)^{-1} \circ f$ and g are morphisms such that $Rp \circ (RDp)^{-1}f = R\delta_p RDp(RDp)^{-1}f = R\delta_p f = \varepsilon_P g$. Hence, by the pullback property, there exists a unique morphism $k : A \to \bar{P}$ such that $p_2 k = (RDp)^{-1}f$ and $p_1 k = g$. Since RDp is an isomorphism, it should then be clear that k is the unique morphism such that $p_1 k = g$ and $RDp \circ p_2 k = f$. Hence, the above diagram is in fact a pullback square.
We now show that $m_p : DP \to \bar{P}$ is the unique morphism such that $p_1 m_p = \delta_p$ and $RDpp_2 m_p = \varepsilon_{DP}$. Since $p_1 m_p = \delta_p$ by definition of δ_p and the square is already a pullback, it is sufficient to show that $RDpp_2 m_p = \varepsilon_{DP}$. Note that $RDpp_2 m_p Dp = RDpp_2 \bar{p} = RDp\varepsilon_X = \varepsilon_{DP} Dp$ where the last equality holds due to naturality of the functor. Since Dp is in $\mathscr{E}$ and $\mathscr{E}$ is a class of epimorphisms, it follows that $Rdpp_2 m_p = \varepsilon_{DP}$ and thus m_p is the required morphism m_{δ_p}. Note that an $(\mathscr{E}, \mathscr{M})$-factorisation of m_p is given by $m_p id_{DP}$ and it follows that $D\delta_p \simeq id_{DP}$, or, equivalently, $D\delta_p$ is an isomorphism. It follows that D is weakly cohereditary. $\square$

Corollary 4.47: Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured finitely complete category $\mathbb{A}$ with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Then, $\mathbb{A}$ is ($\mathbb{B}$-concordant, $\mathbb{B}$-dissonant)-structured.

Proof : Note that the $D_{\mathbb{B}}$-closed and the $\mathbb{B}$-concordant morphisms coincide. Similarly, any $D_{\mathbb{B}}$-sparse morphism followed by a composition of a morphism in $\mathscr{M}$ and the $\mathbb{B}$-dissonant morphisms, coincide. This result then follows directly from Proposition 4.36, Corollary 4.42 and Theorem 4.46. $\square$

Corollary 4.48: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. Let $\mathbb{M} \circ D_{\mathbb{B}}$-sparse denote the conglomerate of sources of the form $(m_i)_I \circ s$, where $(m_i)_I$ is in $\mathbb{M}$ and s is a $D_{\mathbb{B}}$-sparse morphism. Then $\mathbb{A}$ is a ($D_{\mathbb{B}}-$closed, $\mathbb{M} \circ D_{\mathbb{B}}$-sparse)-category.

Proof : The proof is similar to the proof of Corollary 4.47. $\square$

Remark 4.49: Note that $D_{\mathbb{B}}$ need not be cohereditary, i.e., if $q : Y \to Q$ and $e : Q \to Z$ are composable morphisms in $\mathscr{E}$ such that eq is $D_{\mathbb{B}}$-sparse, then e need not be $D_{\mathbb{B}}$-sparse. To see this, consider $\mathbb{A} = \mathbb{Top}$, $\mathbb{B} = \mathbb{TDisc}$ and $(\mathscr{E}, \mathscr{M})$ =(surjective continuous maps, initial mono-sources). Then, $\mathbb{B}$ is surjective reflective(actually quotient reflective), but $D_{\mathbb{B}}$ is not cohereditary.

Consider the maps $q : \mathbb{D}_2 \to \mathbb{S}$ and $e : \mathbb{S} \to \mathbb{I}_2$, where all spaces have the underlying set $\{0, 1\}$ and all maps are identity maps. $\mathbb{D}_2$ is the discrete space, $\mathbb{S}$ the Sierpinski space and $\mathbb{I}_2$ the indiscrete space. Let us denote the reflector from $\mathbb{Top}$ into $\mathbb{TDisc}$ by R and denote the unit by ε. It can easily be shown that $\varepsilon_{\mathbb{D}_2} = id_{\mathbb{D}_2}$ and $R\mathbb{S} = R\mathbb{I}_2 = 1$, where 1 is the space consisting of one point, namely 0. It can then be easily seen that the following two diagrams are the required diagrams in order to construct $D_{\mathbb{B}}(eq)$ and $D_{\mathbb{B}}(e)$:

$$
\begin{tikzcd}
\mathbb{D}_2 \arrow[d, "id_{\mathbb{D}_2}"] \arrow[ddr, dashed, "\langle eq{,}id\rangle"] \arrow[ddrr, bend left, "id_{\mathbb{D}_2}"] \arrow[dddr, bend right=60, "eq"'] & & \\
\mathbb{D}_2 \arrow[dr, "m_{eq}"] \arrow[ddr, bend right, "\delta_{eq}"'] & & \\
& \mathbb{I}_2 \times \mathbb{D}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "p.b."] & \mathbb{D}_2 \arrow[d, "R(eq)"] \\
& \mathbb{I}_2 \arrow[r, "\varepsilon_{\mathbb{I}_2}"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \arrow[d, "D_{\mathbb{B}}(e)"] \arrow[ddr, dashed, "\overline{e}"] \arrow[ddrr, bend left, "\varepsilon_{\mathbb{S}}"] \arrow[dddr, bend right=60, "e"'] & & \\
\mathbb{I}_2 \arrow[dr, "m_e"] \arrow[ddr, bend right, "\delta_e"'] & & \\
& \mathbb{I}_2 \arrow[r, "\varepsilon_{\mathbb{I}_2}"] \arrow[d, "id_{\mathbb{I}_2}"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, "Re=id_1"] \\
& \mathbb{I}_2 \arrow[r, "\varepsilon_{\mathbb{I}_2}"'] & 1
\end{tikzcd}
$$

Note that $m_{eq}(x) = (x, x)$, π_1 and π_2 are the product projections. The map, m_e can be taken as the identity and then $D_{\mathbb{B}}(e) = \overline{e}$ has the underlying identity map, but is not an isomorphism. Furthermore, Re and $R(eq)$ are the only maps to the space with only one point. It can then also be seen that $D_{\mathbb{B}}(eq)$ is an isomorphism, but $D_{\mathbb{B}}(e)$ is not. It follows that $D_{\mathbb{TDisc}}$ is not cohereditary.

Definition 4.50: Stable and proper factorisation system(s)
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with pullbacks. Then $(\mathscr{E}, \mathscr{M})$ is called a **stable factorisation system (on** $\mathbb{A}$) provided that $\mathscr{E}$ is closed under pullbacks, i.e., whenever e is in $\mathscr{E}$ and $\overline{e}$ is a pullback of e along f, then $\overline{e}$ is in $\mathscr{E}$. If $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then $(\mathscr{E}, \mathbb{M})$ is **stable** provided that the factorisation system induced for morphisms by $(\mathscr{E}, \mathbb{M})$ is stable.

If $\mathbb{A}$ is an $(\mathscr{E}, \mathscr{M})$-structured category, then $(\mathscr{E}, \mathscr{M})$ is said to be **a proper factorisation structure** or simply **proper** if $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and $\mathscr{M} \subset \mathrm{Mono}(\mathbb{A})$. If $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then the factorisation structure $(\mathscr{E}, \mathbb{M})$ is said to be **proper** if $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and $\mathbb{M} \subset \mathrm{MonoSource}(\mathbb{A})$.

Proposition 4.51: Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of the category $\mathbb{A}$. Let $(\mathscr{E}, \mathscr{M})$ be a stable factorisation system and let $\mathscr{E}$ be a class of epimorphisms that satisfies the cancellation condition: $fg \in \mathscr{E}$ and $f \in \mathscr{E}$ implies that $g \in \mathscr{E}$. Then $\bar{p}$ is in $\mathscr{E}$ for each $p \in \mathscr{E}$.

Proof : We will use the notation as in Remark 4.11. For any $p \in \mathscr{E}$, we have Rp and ε_P are members of $\mathscr{E}$. By pullback stability, p_1 and p_2 are in $\mathscr{E}$. Since $\delta_p = p_1 m_p$ and p_1 is in $\mathscr{E}$, our cancellation condition gives $m_p \in \mathscr{E}$. Since m_p is a morphism in $\mathscr{M}$ by definition, m_p is an isomorphism and since $\bar{p} = D_{\mathbb{B}}(p) m_p$ is a composition of an isomorphism with a morphism in $\mathscr{E}$, our result follows. $\square$

Proposition 4.52: Let $\mathbb{A}$ be a finitely complete $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ with reflector R and unit ε. Suppose that R preserves pullbacks and $\mathscr{E}$ satisfies the cancellation condition: $fg \in \mathscr{E}$ and $f \in \mathscr{E}$ implies that $g \in \mathscr{E}$. Then ($\mathbb{B}$-concordant, $\mathbb{B}$-dissonant) is a stable factorisation structure on $\mathbb{A}$.

Proof : Note that $\mathbb{A}$ being finitely complete gives us by 4.47 that ($\mathbb{B}$-concordant, $\mathbb{B}$-dissonant) is a factorisation system for morphisms on $\mathbb{A}$.

Let $p : X \to P$ be a $\mathbb{B}$-concordant morphism and assume that the reflector preserves pullbacks. Note that p is $\mathbb{B}$-concordant if and only if p is $D_{\mathbb{B}}$-closed and by 4.43 this is also the case if and only if Rp is an isomorphism. Let $f : Q \to P$ be an $\mathbb{A}$-morphism and suppose that

$$
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "q"'] & X \arrow[d, "p"] \\
Q \arrow[r, "f"'] & P
\end{tikzcd}
$$

is a pullback square. We first show that Rq is an isomorphism. Consider the commuting diagram:

$$\begin{tikzcd}
& RY \arrow[rr, "Rg"] \arrow[dd, "Rq"] & & RX \arrow[dd, "Rp"] \\
Y \arrow[ur, "\varepsilon_Y"] \arrow[rr, "g"] \arrow[dd, "q"'] & & X \arrow[ur, "\varepsilon_X"] \arrow[dd, "p"] & \\
& RQ \arrow[rr, "Rf"] & & RP \\
Q \arrow[ur, "\varepsilon_Q"] \arrow[rr, "f"'] & & P \arrow[ur, "\varepsilon_P"'] &
\end{tikzcd}$$

Since the reflector preserves pullbacks, it follows that Rq is a pullback of Rp along Rf. Since a pullback of an isomorphism is an isomorphism, we have that Rq is an isomorphism.

In order to show that q is $\mathbb{B}$-concordant, we need only show that $q \in \mathscr{E}$. To see this, note that $\varepsilon_Q \circ q = Rq \circ \varepsilon_Y$ and by the assumed cancellation condition, it follows that q is a member of $\mathscr{E}$. It is then clear that q is a $\mathbb{B}$-concordant morphism, i.e., the class of all $\mathbb{B}$-concordant morphisms is stable under pullback. $\square$

Corollary 4.53: Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. Let ($\mathbb{B}$-concordant, $\mathbb{B}$-dissonant) be a stable orthogonal factorisation structure on the finitely complete category $\mathbb{A}$. Furthermore, assume the class of all $\mathbb{B}$-concordant morphisms consists entirely of epimorphisms. Then, $\bar{p}$ is $\mathbb{B}$-concordant for each $\mathbb{B}$-concordant p.

$\underline{\mathcal{P}roof}$: Of course, $\mathbb{B}$ is $\mathscr{E}$-reflective, hence $\mathbb{B}$ is $\mathbb{B}$-concordant reflective as well. Then 4.51 and 4.43 provides us with the result as the class of all $D_{\mathbb{B}}$-closed, or, equivalently, the class of $D_{\mathbb{B}}$-sparse morphisms, satisfies the cancellation condition as $D_{\mathbb{B}}$ is maximal. $\square$

Lemma 4.54: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category and $\mathbb{B}$ an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. Let $\overline{\mathscr{E}} = Mor(\mathbb{B})^{\uparrow}$ be the class of all $\mathbb{A}$-morphisms orthogonal to $\mathbb{B}$-morphisms. Then, the class of all $\mathbb{B}$-concordant (or, equivalently, $D_{\mathbb{B}}$-closed) morphisms is the intersection of the classes $\mathscr{E}$ and $\overline{\mathscr{E}}$.

$\underline{\mathcal{P}roof}$: Let $R : \mathbb{A} \to \mathbb{B}$ denote a reflector with unit ε. Let p be $\mathbb{B}$-concordant. Then, p is a morphism in $\mathscr{E}$ that is $D_{\mathbb{B}}$-closed and this is the case if and only if Rp is an isomorphism. By 4.21, it follows that this is the case if and only if p is in $\overline{\mathscr{E}}$. Since p is assumed to be a member of $\mathscr{E}$, the result follows. $\square$

Remark 4.55: Let $\mathbb{B}$ be a fixed $\mathscr{E}$-reflective subcategory of $(\mathscr{E}, \mathscr{M}$-structured category $\mathbb{A}$. Then we will denote the factorisation structure ($\mathbb{B}$-concordant, $\mathbb{B}$-dissonant) by $(\mathscr{D}, \mathscr{N})$. Note that if $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then $\mathbb{A}$ is a $(\mathscr{D}, \mathbb{N})$-category, where $\mathbb{N} = \mathbb{M} \circ D_{\mathbb{B}_{S_p}}$. If it's necessary to emphasise the subcategory $\mathbb{B}$ in question, we may denote the factorisation structures for morphisms, respectively sources, by $(\mathscr{D}_{\mathbb{B}}, \mathscr{N}_{\mathbb{B}})$ and $(\mathscr{D}_{\mathbb{B}}, \mathbb{N}_{\mathbb{B}})$, respectively. It ought to be clear that $\mathscr{D} = \mathscr{E} \cap \overline{\mathscr{E}}$, $\mathscr{M} \cup \overline{\mathscr{M}} \subset \mathscr{N}$ and $\mathbb{M} \cup \overline{\mathscr{M}} \subset \mathbb{N}$.

Proposition 4.56: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ and $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$ the prefactorisation structure $(Mor(\mathbb{B})^{\uparrow}, Mor(\mathbb{B})^{\uparrow\downarrow})$. For each $p \in \mathscr{E}$ there holds: p is $\mathbb{B}$-concordant if and only if δ_p is an isomorphism.

$\underline{\mathcal{P}roof}$: Let R be the reflector from $\mathbb{A}$ to $\mathbb{B}$ and suppose p is a member of $\mathscr{E}$. Note that p is $\mathbb{B}$-concordant if and only if Rp is an isomorphism. Furthermore, $Rp = R(\delta_p D_{\mathbb{B}}(p)) = R\delta_p \circ RD_{\mathbb{B}}(p)$ and by 4.41, it follows that $RD_{\mathbb{B}}(p)$ is an isomorphism. We then have Rp is an isomorphism if and only if $R\delta_p$ is an isomorphism. This is the case if and only if δ_p is $\mathbb{B}$-concordant and by 4.54, this is the case if and only if δ_p is a member of $\mathscr{E}$ and $\overline{\mathscr{E}}$. Since δ_p is always a member of $\mathscr{E}$, our proof is complete. $\square$

Proposition 4.57: Let $\mathbb{A}$ be a finitely complete $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Let $\mathbb{B}$ be $\mathscr{E}$-reflective in $\mathbb{A}$ and $s : X \to P$ a morphism in $\mathscr{E}$. Then, s is $D_{\mathbb{B}}$-sparse if and only if s is in $\mathscr{N}$.

Proof : Let s be in $\mathscr{E}$ and consider the diagram:

$$\begin{tikzcd}
X \arrow[d, "D_{\mathbb{B}}(s)"] \arrow[ddr, dashed, "\overline{s}"] \arrow[ddrr, bend left, "\varepsilon_X"] \arrow[dddr, bend right=60, "s"'] & & \\
D_{\mathbb{B}}(P) \arrow[dr, "m_s"] \arrow[ddr, bend right, "\delta_s"'] & & \\
& \overline{P} \arrow[r, "s_2"] \arrow[d, "s_1"] \arrow[dr, phantom, "p.b."] & RX \arrow[d, "Rs"] \\
& P \arrow[r, "\varepsilon_P"'] & RP
\end{tikzcd}$$

If s is $D_{\mathbb{B}}$-sparse, then we may take $D_{\mathbb{B}}(s) = id_X$ and then $m_s = \overline{s} \in \mathscr{M}$. Since $\overline{\mathscr{M}}$ is closed under pullbacks, $\mathscr{M} \cup \overline{\mathscr{M}} \subset \mathscr{N}$ and $\mathscr{N}$ is closed under composition, we have that $s = s_1\overline{s}$ is a member of $\mathscr{N}$.

Conversely, assume that s is a member of $\mathscr{N}$. Since s_1 is in $\overline{\mathscr{M}} \subset \mathscr{N}$, the cancellative properties of $\mathscr{N}$ imply that $s = s_1\overline{s} = s_1 m_s D_{\mathbb{B}}(s)$ is in $\mathscr{N}$. Since s_1 and s are in $\mathscr{N}$, it follows that $\overline{s} = m_s D_{\mathbb{B}}(s)$ is a member of $\mathscr{N}$. Similarly, since m_s is in $\mathscr{M} \subset \mathscr{N}$, it follows that $D_{\mathbb{B}}(s)$ is a member of $\mathscr{N}$. Since $D_{\mathbb{B}}(s)$ is $D_{\mathbb{B}}$-closed and in $\mathscr{E}$, $D_{\mathbb{B}}(s)$ is a member of $\mathscr{D}$. Therefore $D_{\mathbb{B}}(s) \simeq id_X$, or, equivalently, s is $D_{\mathbb{B}}$-sparse. □

Remark 4.58: Let $\mathbb{B}$ be any full replete reflective subcategory of a category $\mathbb{A}$. Of course, the inclusion functor $E : \mathbb{B} \to \mathbb{A}$ induces an adjunction $(\varepsilon, \mu) : R \longrightarrow\!\!\!\!\dashv E : \mathbb{B} \longrightarrow \mathbb{A}$ and we may assume without loss of generality that the following hold:

(*i*) For every $\mathbb{A}$-object X, $R\varepsilon_X = id_X$;
(*ii*) For every $\mathbb{B}$-morphism $f : B \to B'$ we have: $Ef = f$ and
(*iii*) For every $\mathbb{B}$-object $\mathbb{B}$, $\varepsilon_B = id_B$.

For the remainder of this section, we will assume these properties without explicitly referring to them. Unless stated otherwise, we will also assume that the reflector with unit is given by the pair (R, ε) whenever $\mathbb{B}$ is a full replete reflective subcategory of $\mathbb{A}$.
We also adopt some extra notation:

For any $\mathbb{B}$-morphism $f : B \to RP$, we denote the pullback square of f along ε_P by
$$\begin{tikzcd}
P_f \arrow[r, "f_2"] \arrow[d, "f_1"'] & B \arrow[d, "f"] \\
P \arrow[r, "\varepsilon_P"'] & RP
\end{tikzcd}$$

Note that this does not conflict with our notation for the Cassidy Hébert Kelly dual closure operator as $Rf = Rfid_B = Rf\varepsilon_B = \varepsilon_{RP}f = id_{RP}f = f$ and so if f was a member of $\mathscr{E}$, then f_1 and f_2 play exactly the role they do as in the diagram in 4.11. We will also assume that $(\mathscr{E}, \mathscr{M})$ is a factorisation structure on $\mathbb{A}$ with $\mathscr{E} \subset Epi(\mathbb{A})$. The prefactorisation structure $(Mor(\mathbb{B})^{\uparrow}, Mor(\mathbb{B})^{\uparrow\downarrow})$ will be denoted by $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$.
We can also construct a diagram as for the Cassidy Hébert Kelly dual closure operator for any $\mathbb{A}$-morphism f instead of just for morphisms in $\mathscr{E}$. By using our convention as above, we have for each

$\mathbb{A}$-morphism $f : X \to P$, a commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\bar{f}"] \arrow[drr, "\varepsilon_X"] \arrow[ddr, "f"'] & & \\
& P_{Rf} \arrow[r, "f_2"] \arrow[d, "f_1"] & RX \arrow[d, "Rf"] \\
& P \arrow[r, "\varepsilon_P"'] & RP
\end{tikzcd}$$

Furthermore, for a fixed replete reflective subcategory $\mathbb{B}$ of $\mathbb{A}$, we will denote by $(\mathscr{D}, \mathscr{N})$ the factorisation structure ($\mathbb{B}$-concordant, $\mathbb{B}$-dissonant). From Remark 4.55 and Proposition 4.54, it ought to be clear that $\mathscr{D} = \mathscr{E} \cap \bar{\mathscr{E}}$ and $\mathscr{M} \cup \bar{\mathscr{M}} \subset \mathscr{N}$.

Since the study of the Cassidy Hébert Kelly dual closure operator originates from [6] and [7], the term 'simple reflector'was used to determine when $\bar{f}$ in the above diagram is a member of $\overline{\mathscr{E}}$. Since we are primarily interested in these diagrams where the morphism f is a member of $\mathscr{E}$, we will generalise this definition as follows:

Definition 4.59: Simple reflector ([6],[7])
Let $\mathbb{A}$ have pullbacks and let $R : \mathbb{A} \to \mathbb{B}$ be a reflector. Then, R is said to be **simple** or simply **the reflection is simple** if $\overline{f}$ (as in the last diagram of Remark 4.58) is a member of $\overline{\mathscr{E}}$ for each $\mathbb{A}$-morphism f.
Note that if R is simple, then $f_1\bar{f}$ is an $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$-factorisation of f. To see this, note that Rf is a member of $Mor(\mathbb{B}) \subset Mor(\mathbb{B})^{\uparrow\downarrow} = \overline{\mathscr{M}}$. Since $\overline{\mathscr{M}}$ is closed under pullbacks, f_1 is in $\overline{\mathscr{M}}$.

Let $\mathscr{F}$ be a class of $\mathbb{A}$-morphisms. Then, a reflector R is said to be $\mathscr{F}$-simple if $R\bar{f}$ is an isomorphism for each $f \in \mathscr{F}$. Note that R is simple if and only if R is Mor($\mathbb{A}$)-simple.

Note that if $\mathscr{F}$ and $\mathscr{F}'$ are classes of morphisms for which $\mathscr{F}' \subset \mathscr{F}$, then R being $\mathscr{F}$ simple implies that R is $\mathscr{F}'$-simple.

Theorem 4.60: ([7, 4.1] Let $\mathbb{B}$ be a full, isomorphism-closed and reflective subcategory of the category $\mathbb{A}$ with pullbacks. Then, the following conditions are equivalent and each implies that $(\bar{\mathscr{E}}, \bar{\mathscr{M}})$ is an orthogonal factorisation structure on $\mathbb{A}$:

(i) For each $\mathbb{A}$-morphism f, $\bar{f} \in \overline{\mathscr{E}}$, i.e., the reflection is simple.

(ii) For each $\mathbb{A}$-morphism f: $f \in \overline{\mathscr{M}}$ if and only if $\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$ is a pullback square.

(iii) For each $g : B \to RP$ in $\mathbb{B}$: $\begin{tikzcd} P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "id"'] & RP_g \arrow[d, "R(g_2)"] \\ P_g \arrow[r, "g_2"'] & B \end{tikzcd}$ is a pullback square.

(iv) For each $g : B \to RP$ in $\mathbb{B}$, $R(g_2)$ is an $\mathbb{A}$-isomorphism if it is a retraction.

<u>*Proof :*</u> $(i) \Rightarrow (ii)$ Let $f : X \to P$ be an $\mathbb{A}$-morphism and assume that $\bar{f}$ is in $\mathscr{E}$. Note that $Rf \in Mor(\mathbb{B}) \subset Mor(\mathbb{B})^{\uparrow\downarrow} = \overline{\mathscr{M}}$. Since $\overline{\mathscr{M}}$ is closed under pullbacks, f_1 is a member of $\overline{\mathscr{M}}$. If

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$$ is a pullback square, then there is a unique isomorphism $\bar{f} : X \to P_f$ such that $f_2\bar{f} = \varepsilon_X$ and $f_1\bar{f} = f$. Hence, f is a composition of an isomorphism with a member of $\overline{\mathscr{M}}$, hence f is also a member of $\bar{\mathscr{M}}$.

Conversely, if f is a member of $\overline{\mathscr{M}}$, then $f_1\bar{f} = f$, so that since f_1 and $f_1\bar{f} = f$ are both members of $\overline{\mathscr{M}}$, it follows that $\bar{f}$ is a member of $\overline{\mathscr{M}}$. By (i), $\bar{f} \in \mathscr{E}$, so that $\bar{f}$ is an isomorphism. Hence, the desired square is a pullback as well.

$(ii) \Rightarrow (iii)$ Let $g : B \to RP$ be a $\mathbb{B}$-morphism and consider the pullback square $$\begin{tikzcd} P_g \arrow[r, "g_2"] \arrow[d, "g_1"'] & B \arrow[d, "g"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$$ and naturality square $$\begin{tikzcd} P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "g_1"'] & RP_g \arrow[d, "Rg_1"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$$.

Since g is a $\mathbb{B}$-morphism, it's a member of $\overline{\mathscr{M}}$. Since $\overline{\mathscr{M}}$ is closed under pullbacks, g_1 is also a member of $\overline{\mathscr{M}}$. Note that by our convention we have $R(\varepsilon_B) = id_B$ and $Rg = g$. Consider the diagram:

$$\begin{tikzcd} P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, equal] & RP_g \arrow[d, "Rg_2"] \arrow[dd, bend left=60, "Rg_1"] \\ P_g \arrow[r, "g_2"] \arrow[d, "g_1"'] & B \arrow[d, "g"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$$

Note that Rg_2 is the unique morphism from RP_g to B such that $Rg_2\varepsilon_{P_g} = g_2$. Furthermore, there holds: $gRg_2 = RgRg_2 = R(gg_2) = R(\varepsilon_P g_1) = R\varepsilon_P Rg_1 = Rg_1$. Hence, the diagram commutes and by (ii), the outer square is a pullback. By construction, the bottom square is a pullback. A standard result shows that the top square is a pullback square as well.

$(iii) \Rightarrow (iv)$ Let $g : B \to RP$ be a $\mathbb{B}$-morphism and let $s : B \to RP_g$ be a morphism such that $Rg_2 \circ s = id_B$. Then, by using (iii) and the pullback property, we have the commutative diagram:

$$\begin{tikzcd} P_g \arrow[dr, dashed, "!\bar{f}"] \arrow[drr, "sg_2"] \arrow[ddr, "id"'] & & \\ & P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "id"] & RP_g \arrow[d, "Rg_2"] \\ & P_g \arrow[r, "g_2"'] & B \end{tikzcd}$$

Since $\bar{f} = id_{P_g}\bar{f} = id_{P_g}$, it follows that $\varepsilon_{P_g} = \varepsilon_{P_g}\bar{f} = sg_2$. Then, $id_{RP_g} = R(\varepsilon_{P_g}) = R(sg_2) = RsRg_2 = sRg_2$. Thus, Rg_2 is an isomorphism with $s = Rg_2^{-1}$.

$(iv) \Rightarrow (i)$ Let f be an $\mathbb{A}$-morphism and consider the $\mathbb{B}$-morphism $Rf : RX \to RP$. Since $Rf_2R\bar{f} = R(f_2\bar{f}) = R\varepsilon_X = id_{RX}$, it follows that Rf_2 is a retraction. By (iv) we must have that Rf_2 is an isomorphism with inverse $R\bar{f}$. Since $\bar{f}$ is in $\overline{\mathscr{E}}$ if and only if $R\bar{f}$ is an isomorphism, (i) follows. It follows that all of the above statements are equivalent.

Note that any of these statements then imply that $(\bar{\mathscr{E}}, \bar{\mathscr{M}})$ is a factorisation system on $\mathbb{A}$. If f is any morphism, then $\bar{f}$ is in $\overline{\mathscr{E}}$. Since Rf is a $\mathbb{B}$-morphism and $\overline{\mathscr{M}}$ is closed under pullbacks, f_1 is in $\overline{\mathscr{M}}$. Therefore $f_1\bar{f}$ is the required factorisation of f. $\square$

Proposition 4.61: Let $\mathbb{A}$ be finitely complete and let $(\mathscr{E}, \mathscr{M})$ be an orthogonal factorisation structure on $\mathbb{A}$ with $\mathscr{E}$ a class of epimorphisms. Let $\mathbb{B}$ be $\mathscr{E}$-reflective in $\mathbb{A}$ with reflector R and unit ε. Then, each of the statements (i) to (iii) implies the next:

(i) For each $\mathbb{B}$-morphism $g : B \to RP \in \mathscr{E}$:
$$\begin{tikzcd} P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "id"'] & RP_g \arrow[d, "R(g_2)"] \\ P_g \arrow[r, "g_2"'] & B \end{tikzcd}$$
is a pullback square.

(ii) For each $\mathbb{B}$-morphism $g : B \to RP \in \mathscr{E}$, $R(g_2)$ is an isomorphism if it is a retraction.

(iii) R is $\mathscr{E}$-simple.

(iv) For each $f \in \mathscr{E}$: $f \in \overline{\mathscr{M}}$ if and only if the naturality square
$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$$
is a pullback square.

<u>*Proof*</u> : $(i) \Rightarrow (ii)$ Consider the $\mathbb{B}$-morphism $g : B \to RP$ in $\mathscr{E}$. Let $s : B \to RP_g$ be a morphism such that $Rg_2 \circ s = id_B$. Then, by using (i) and the pullback property, we have the commutative diagram:

$$\begin{tikzcd} P_g \arrow[dr, dashed, "!\bar{f}"] \arrow[drr, "sg_2"] \arrow[ddr, "id"'] & & \\ & P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "id"] & RP_g \arrow[d, "Rg_2"] \\ & P_g \arrow[r, "g_2"'] & B \end{tikzcd}$$

Since $\bar{f} = id_{P_g}\bar{f} = id_{P_g}$, it follows that $\varepsilon_{P_g} = \varepsilon_{P_g}\bar{f} = sg_2$. Then, $id_{RP_g} = R(\varepsilon_{P_g}) = R(sg_2) = RsRg_2 = sRg_2$. Thus, Rg_2 is an isomorphism with $s = Rg_2^{-1}$.

$(ii) \Rightarrow (iii)$ Let f be a morphism in $\mathscr{E}$ and consider the $\mathbb{B}$-morphism $Rf : RX \to RP$. Since $Rf_2R\bar{f} = R(f_2\bar{f}) = R\varepsilon_X = id_{RX}$, it follows that Rf_2 is a retraction. By (ii), $R\bar{f}$ and Rf_2 are isomorphisms and inverse to each other. Since $\bar{f}$ is in $\overline{\mathscr{E}}$ if and only if $R\bar{f}$ is an isomorphism, (iii) follows.

$(iii) \Rightarrow (iv)$ Assume that R is $\mathscr{E}$-simple and consider the diagram for any morphism f in $\mathscr{E}$:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\bar{f}"] \arrow[drr, "\varepsilon_X"] \arrow[ddr, "f"'] & & \\
& P_{Rf} \arrow[r, "f_2"] \arrow[d, "f_1"] & RX \arrow[d, "Rf"] \\
& P \arrow[r, "\varepsilon_P"'] & RP
\end{tikzcd}$$

Note that f_1 is automatically a member of $\overline{\mathscr{M}}$ since $\overline{\mathscr{M}}$ is closed under pullbacks and Rf is in $\overline{\mathscr{M}}$.
If the naturality square is a pullback, then $\bar{f}$ must be an isomorphism and consequently $f = f_1\bar{f}$ is a composition of an isomorphism with a member of $\overline{\mathscr{M}}$, hence also in $\overline{\mathscr{M}}$.
Conversely, if f is a member of $\overline{\mathscr{M}}$, then $\bar{f}$ is a member of $\overline{\mathscr{M}}$, since $f_1\bar{f} = f$ and f_1 are members of $\overline{\mathscr{M}}$. Since R is $\mathscr{E}$-simple, $\bar{f}$ is in $\overline{\mathscr{E}} \cap \overline{\mathscr{M}}$ =Iso($\mathbb{A}$). Thus, the naturality square is also a pullback. □

Remark 4.62: The theorem that follows is partly a generalisation of Theorem 4.1 of [7] in case the subcategory in question is $\mathscr{E}$-reflective and $\mathbb{A}$ is a finitely complete category with stable orthogonal factorisation structure. These results will coincide in case the reflection is not only $\mathscr{E}$-simple, but simple.

Theorem 4.63: Let $\mathbb{A}$ be finitely complete and let $(\mathscr{E}, \mathscr{M})$ be a stable orthogonal factorisation structure on $\mathbb{A}$, where $\mathscr{E}$ is a class of epimorphisms. Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ with reflector R and unit ε. Then, the following statements are equivalent:

(i) For each $\mathbb{B}$-morphism $g : B \to RP \in \mathscr{E}$, $\begin{tikzcd} P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "id"'] & RP_g \arrow[d, "R(g_2)"] \\ P_g \arrow[r, "g_2"'] & B \end{tikzcd}$ is a pullback square.

(ii) For each $\mathbb{B}$-morphism $g : B \to RP \in \mathscr{E}$, $R(g_2)$ is an isomorphism if it is a retraction.

(iii) R is $\mathscr{E}$-simple.

(iv) For each $f \in \mathscr{E}$: $f \in \overline{\mathscr{M}}$ if and only if the naturality square $\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$ is a pullback square.

Proof : In view of Proposition 4.61, it's sufficient to prove $(iv) \Rightarrow (i)$. To this end, let $g : B \to RP$ be a $\mathbb{B}$-morphism in $\mathscr{E}$. Consider the pullback square $\begin{tikzcd} P_g \arrow[r, "g_2"] \arrow[d, "g_1"'] & B \arrow[d, "g"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$ and naturality square $\begin{tikzcd} P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, "g_1"'] & RP_g \arrow[d, "Rg_1"] \\ P \arrow[r, "\varepsilon_P"'] & RP \end{tikzcd}$.

Since g is a $\mathbb{B}$-morphism in $\mathscr{E}$, it's a member of $\overline{\mathscr{M}}$ and since $\mathscr{E}$ and $\overline{\mathscr{M}}$ are pullback stable, g_1 is in $\mathscr{E} \cap \overline{\mathscr{M}}$. Note that by our convention we have $R(\varepsilon_P) = id$ and $Rg = g$. Consider the diagram:

$$\begin{tikzcd}
P_g \arrow[r, "\varepsilon_{P_g}"] \arrow[d, equal] & RP_g \arrow[d, "Rg_2"] \arrow[dd, bend left=60, "Rg_1"] \\
P_g \arrow[r, "g_2"] \arrow[d, "g_1"'] & B \arrow[d, "g"] \\
P \arrow[r, "\varepsilon_P"'] & RP
\end{tikzcd}$$

Note that Rg_2 is the unique morphism from RP_g to B such that $Rg_2\varepsilon_{P_g} = g_2$ and $gRg_2 = RgRg_2 = R(gg_2) = R(\varepsilon_P g_1) = R\varepsilon_P Rg_1 = Rg_1$. Hence the diagram commutes and by (iv), the outer square is a pullback. By construction, the bottom square is a pullback. It is a standard result that the top square is then a pullback as well, thus our proof is complete. $\square$

Proposition 4.64: Let $\mathbb{A}$ be finitely complete and $(\mathscr{E}, \mathscr{M})$ an orthogonal factorisation structure on $\mathbb{A}$ with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ with reflector R and unit ε. Then, the following are equivalent:

(i) For each $p : X \to P \in \mathscr{E}$:

$$\begin{tikzcd} P_{Rp} \arrow[r, "\varepsilon_{P_{Rp}}"] \arrow[d, "id"'] & RP_{Rp} \arrow[d, "R(p_2)"] \\ P_{Rp} \arrow[r, "p_2"'] & RX \end{tikzcd}$$

is a pullback square.

(ii) For each morphism $p : X \to P \in \mathscr{E}$, $R(p_2)$ is an isomorphism if it is a retraction.

(iii) R is $\mathscr{E}$-simple.

Proof : $(i) \Rightarrow (ii)$ Suppose that there is a morphism $s : RX \to RP_{Rp}$ such that $sRp_2 = id_{RP_{Rp}}$. By (i),

$$\begin{tikzcd} P_{Rp} \arrow[dr, dashed, "!f"] \arrow[ddr, "id"'] \arrow[drr, bend left, "sp_2"] & & \\ & P_{Rp} \arrow[r, "\varepsilon_{P_{Rp}}"] \arrow[d, equal] & RP_{Rp} \arrow[d, "Rp_2"] \\ & P_{Rp} \arrow[r, "p_2"'] & RX \end{tikzcd}$$

commutes for a unique morphism f. It's easy to see that $f = id$ and therefore, $\varepsilon_{P_{Rp}} = sp_2$. Consequently, $id = R\varepsilon_{P_{Rp}} = R(sRp_2) = RsRRp_2 = sRp_2$. Hence, Rp_2 is an isomorphism with inverse s.

$(ii) \Rightarrow (iii)$ Let p be a morphism in $\mathscr{E}$. Then, $Rp_2R\bar{p} = R(\varepsilon_X) = id_{RX}$ and hence, by (ii), it follows that Rp_2 and $R\bar{p}$ are isomorphisms. Since $\bar{f}$ is in $\overline{\mathscr{E}}$ if and only if $R\bar{f}$ is an isomorphism, it follows that $\bar{p}$ is in $\overline{\mathscr{E}}$. Therefore, R is $\mathscr{E}$-simple.

$(iii) \Rightarrow (i)$ Assume that R is $\mathscr{E}$-simple and p is a member of $\mathscr{E}$. Note that $Rp_2 \circ \varepsilon_{P_{Rp}} = p_2$ and $p_2\bar{p} = \varepsilon_X$. Furthermore, $id_{RX} = R\varepsilon_X = Rp_2R\bar{p}$. Since R is $\mathscr{E}$-simple, $R\bar{p}$ is an isomorphism and hence Rp_2 is an isomorphism. Therefore, we may assume that the pullback of Rp_2 along p_2 is the identity morphism. Using a pullback square, it's easy to see that $\varepsilon_{P_{Rp}}$ is the unique morphism f such that $Rp_2f = p_2$. Therefore, the square in (i) is a pullback square. $\square$

Lemma 4.65: Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and assume that $\mathbb{B}$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. If $\bar{p}$ is in $\mathscr{E}$ for each $p \in \mathscr{E}$, then p_1 is in $\mathscr{E}$ and p_2 in $\mathscr{D}$.

Proof : Note that for each $p \in \mathscr{E}$, we have that $p_1\bar{p} = p$ and $p_2\bar{p} = \varepsilon_X$ are members of $\mathscr{E}$. Since $\bar{p}$ is in $\mathscr{E}$ by assumption, the cancellation condition on $\mathscr{E}$ gives that p_1 and p_2 are members of $\mathscr{E}$. Since $\bar{p}$ is in $\mathscr{E}$, $id_{\bar{P}} \circ \bar{p}$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of $\bar{p}$. Hence, $\bar{p} \simeq D_{\mathbb{B}}(p)$ is in $\mathscr{D}$. It then follows from $p_2 \circ \bar{p} = \varepsilon_X$ being in $\mathscr{D}$ that p_2 is also in $\mathscr{D}$. $\square$

Theorem 4.66: Let $\mathbb{A}$ be a finitely complete category with $(\mathscr{E}, \mathscr{M})$ a proper factorisation structure on $\mathbb{A}$. Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ and $p : X \to P$ a member of $\mathscr{E}$. Then, the pullback p_2 of ε_P along Rp is $D_{\mathbb{B}}$-closed and m_p (i.e., the $\mathscr{M}$-part of an $(\mathscr{E}, \mathscr{M})$-factorisation of $\bar{p}$) belongs to $\overline{\mathscr{E}}$.

Proof : First note that $p_2m_pD_{\mathbb{B}}(p) = \varepsilon_X$. Since both $D_{\mathbb{B}}(p)$ and ε_X are $D_{\mathbb{B}}$-closed and $D_{\mathbb{B}}$ is idempotent, not only is p_2m_p a member of $\mathscr{E}$, but it's also $D_{\mathbb{B}}$-closed. Furthermore, $Rp_2Rm_p = R(p_2m_p)$ is an isomorphism since $R(p_2m_p)RD_{\mathbb{B}}(p) = R\varepsilon_X = id_{RX}$ and $RD_{\mathbb{B}}(p)$ is an isomorphism. Note that $RD_{\mathbb{B}}(p)$ is the inverse of Rp_2Rm_p. We therefore have the equalities:

$$RD_{\mathbb{B}}(p)Rp_2Rm_p = id_{RD_{\mathbb{B}}(P)} \text{ and } Rp_2Rm_pRD_{\mathbb{B}}(p) = id_{RX}$$

Since $\mathbb{A}$ is finitely complete and Rm_p is a section, it follows by Proposition 2.13 that Rm_p is a member of $\mathscr{M}$.

First we show that p_2 is a member of $\mathscr{E}$. Since $Rp_2\varepsilon_{\bar{P}} = \varepsilon_{RX}p_2 = id_{RX}p_2 = p_2$, it's sufficient to prove that Rp_2 is a member of $\mathscr{E}$. By assumption $(\mathscr{E}, \mathscr{M})$ is proper and $\mathbb{A}$ finitely complete, hence the well known fact that $\mathscr{M} \subset \text{Mono}(\mathbb{A})$ is equivalent to all retractions in $\mathscr{E}$. Therefore Rp_2 is a member of $\mathscr{E}$ and thus p_2 as well.
Consider the following diagram:

$$\begin{tikzcd}
D_{\mathbb{B}}P \arrow[dd, "m_p"] \arrow[dr, dashed, "\varepsilon_{D_{\mathbb{B}}P}=\overline{p_2m_p}"] \arrow[drr, bend left, "\varepsilon_{D_{\mathbb{B}}P}"] \arrow[ddddr, bend right=60, "p_2m_p"'] & & \\
& RD_{\mathbb{B}}P \arrow[r, equal] \arrow[dd, "Rm_p"] & RD_{\mathbb{B}}P \arrow[dd, "Rm_p"] \\
\overline{P} \arrow[dr, dashed, "\overline{p}_2=\varepsilon_{\overline{P}}"] \arrow[drr, bend left, "\varepsilon_{\overline{P}}"] \arrow[ddr, "p_2"] & & \\
& R\overline{P} \arrow[r, equal] \arrow[d, "Rp_2"] & R\overline{P} \arrow[d, "Rp_2"] \\
& RX \arrow[r, equal] & RX
\end{tikzcd}$$

Most part of the above diagram commutes by naturality. Note that both squares on the right are pullbacks and are used to construct the Cassidy Hébert Kelly dual closure operator. The morphisms $\overline{p_2}$ and $\overline{p_2m_p}$ are those induced by the pullbacks. By considering the identity morphisms in the diagram, these must be the two reflection morphisms. Since the reflections are already members of $\mathscr{E}$, there is no need to form the $(\mathscr{E}, \mathscr{M})$-factorisations in order to establish $D_{\mathbb{B}}(p_2m_p)$ and $D_{\mathbb{B}}(p_2)$. Define j to be the morphism $R(m_pD_{\mathbb{B}}(p)p_2)$. Then, since $Rm_pRD_{\mathbb{B}}(p)p_2 : \overline{P} \to R\overline{P}$ is a morphism to a member of $\mathbb{B}$, there is a unique morphism $k : R\overline{P} \to R\overline{P}$ such that $k\varepsilon_{\overline{P}} = Rm_pRD_{\mathbb{B}}(p)p_2$. We assert that $k = j$. This is the case, because $j\varepsilon_{\overline{P}} = R(m_pD_{\mathbb{B}}(p)p_2)\varepsilon_{\overline{P}} = Rm_pRD_{\mathbb{B}}(p)Rp_2\varepsilon_{\overline{P}} = Rm_pRD_{\mathbb{B}}(p)p_2$. It's easy to see that j must be a member of $\mathscr{E}$ and is thus also an epimorphism.

Then $Rp_2j\varepsilon_{\overline{P}} = Rp_2Rm_pRD_{\mathbb{B}}(p)p_2 = p_2 = Rp_2\varepsilon_{\overline{P}}$ and since $\varepsilon_{\overline{P}}$ is an epimorphism, $Rp_2j = Rp_2$. Now, $id_{R\overline{P}} \circ j = j = R(m_pD_{\mathbb{B}}(p)p_2) = R(m_pD_{\mathbb{B}}(p))Rp_2 = R(m_p)R(D_{\mathbb{B}}(p))Rp_2 \circ j$ and since j is an epimorphism, we have that $R(m_p)R(D_{\mathbb{B}}(p))Rp_2$ is the identity morphism. Hence Rp_2 is both a section and a retraction and thus an isomorphism. From this it can easily be seen that Rm_p is also a retraction and section, hence an isomorphism. Consequently, m_p is a member of $\overline{\mathscr{E}}$. $\square$

Corollary 4.67: If $(\mathscr{E}, \mathscr{M})$ is a proper factorisation structure of the finitely complete category $\mathbb{A}$ and $\mathbb{B}$ is an $\mathscr{E}$-reflective subcategory with reflector R, then R is $\mathscr{E}$-simple.

Proof : Let p be a member of $\mathscr{E}$. We need to show that $\bar{p}$ is in $\overline{\mathscr{E}}$, or, equivalently, $R\bar{p}$ is an isomorphism. By proposition 4.41, $RD_{\mathbb{B}}(p)$ is an isomorphism, hence $D_{\mathbb{B}}(p)$ is in $\overline{\mathscr{E}}$. By Theorem 4.66, m_p is a member of $\overline{\mathscr{E}}$ and by proposition 2.6(b), we have $\bar{p} = m_pD_{\mathbb{B}}(p)$ in $\overline{\mathscr{E}}$. $\square$

Corollary 4.68: If $\mathbb{B}$ is a reflective subcategory of the finitely complete thin category $\mathbb{A}$, then the reflector R is simple.

Proof : Every category is (Mor($\mathbb{A}$), Iso($\mathbb{A}$))-structured. Furthermore, if $\mathbb{A}$ is thin, then every morphism is a bimorphism so that this factorisation structure is proper. Then Lemma 4.66 provides us with the desired result. $\square$

Example 4.69: Note that proposition 4.66 states that m_p is a member of $\overline{\mathscr{E}}$ whenever p is in $\mathscr{E}$. Since $D_{\mathbb{B}}(p)$ is then also in $\mathscr{D} = \mathscr{E} \cap \overline{\mathscr{E}}$, we have $\bar{p} = m_p D_{\mathbb{B}}(p) \in \overline{\mathscr{E}}$ whenever $(\mathscr{E}, \mathscr{M})$ is a proper factorisation structure of the finitely complete category $\mathbb{A}$. One might be lead to believe that $\bar{p}$ is in $\overline{\mathscr{E}}$ for each $\mathbb{A}$-morphism p, but this need not be true in general as the following example illustrates:

Consider the category $\mathbb{A} := \mathbb{Ab}$ of abelian groups and group homomorphisms and the reflective subcategory $\mathbb{B} := \mathbb{Ab}_2$ consisting of all groups of exponent 1 or 2. Note that a group's exponent is the non-negative generator of the ideal $\{z \in \mathbb{Z} \mid \forall g \in G : g^z = e\}$ of $\mathbb{Z}$. For any abelian group G, the reflection is given by the canonical morphism to G/H, where $H = \{x^2 \mid x \in G\}$.

Note that the reflector $R : \mathbb{Ab} \to \mathbb{Ab}_2$ is epi-simple, but not simple. To see this, consider the zero morphism $z : 0 \to \mathbb{Z}$. Then, $Rz : 0 \to \mathbb{Z}_2$ is the zero homomorphism and its pullback along $\varepsilon_{\mathbb{Z}}$ is given by the square

$$\begin{tikzcd} 2\mathbb{Z} \arrow[r, "z_2"] \arrow[d, "z_1"'] & 0 \arrow[d, "Rz"] \\ \mathbb{Z} \arrow[r, "\varepsilon_{\mathbb{Z}}"'] & \mathbb{Z}_2 \end{tikzcd}$$

where z_1 is the inclusion morphism and z_2 is the zero morphism. The morphism $\bar{z}$ induced by the pullback is given by the zero morphism $\bar{z} : 0 \to 2\mathbb{Z}$. We show that R is not simple, by showing that $\bar{z}$ is not in $\overline{\mathscr{E}}$, or, equivalently, $R\bar{z}$ is not an isomorphism. It is easy to see that $R0 = 0$ and $R2\mathbb{Z} = 2\mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}_2$, hence $R0$ and $R2\mathbb{Z}$ are not isomorphic. It follows that $R\bar{z}$ is not an isomorphism and thus R is not simple.

Example 4.70: Consider the category $\mathbb{A}$ with the following additive groups as objects

$$\{\mathbb{Z}, \mathbb{Z}_{2m+1}, \mathbb{Z}_2 \mid m \geq 0\}.$$

Note that, since all these groups are cyclic, each homomorphism is completely determined by it's image on 1. Hence, if f is a homomorphism, we will denote this homomorphism by f_x, where $f(1) = x$. The morphisms of $\mathbb{A}$ is the non-full subcategory of the category $\mathbb{Cyc}$ of cyclic groups which consists of all morphisms in $\mathbb{Cyc}$, except the following:

$$\{f_x : \mathbb{Z} \to \mathbb{Z} \mid x \neq 0 \text{ and } x \text{ is even}\} \cup \{f_0 : \mathbb{Z}_2 \to \mathbb{Z}\}.$$

It can be verified that (surjective, injective) is a factorisation structure on $\mathbb{A}$. Let $\mathbb{B}$ be the full subcategory of $\mathbb{A}$ consisting of all groups of exponent at most 2. Then, $\mathbb{B}$ is surjective-reflective and a reflector R with unit is constructed just as in example 4.69. Furthermore, R is surjective-simple, but not simple. To see that R is not simple, consider the unique morphism $z : \mathbb{Z}_3 \to \mathbb{Z}$. Then Rz is the zero morphism from 0 to $\mathbb{Z}_2$ and the pullback of Rz along $\varepsilon_{\mathbb{Z}}$ may be taken as $id_{\mathbb{Z}}$. In such a case $\bar{z} = z$ so that $R\bar{z} = Rz$. Since Rz is obviously not an isomorphism, $\bar{z}$ is not a member of $\overline{\mathscr{E}}$, or, equivalently, R is not simple.

Also note that when a reflection is simple, then $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$ is an orthogonal factorisation structure and this need not hold if R is only $\mathscr{E}$-simple. To see this, let $\mathbb{C}$ be the category with objects $\{\mathbb{Z}_n \mid n \geq 0\} \cup \{\mathbb{Z}\}$, a skeleton of the category of cyclic groups and $\mathbb{B}$ the full subcategory of $\mathbb{C}$ of these groups of exponent 1 or 2. The reflector R from $\mathbb{C}$ to $\mathbb{B}$ is the same as the one given above (see example 4.69) and it can easily be verified that R is $\mathscr{E}$-simple. To see that the above prefactorisation structure is not necessarily a factorisation structure, we first show that $\overline{\mathscr{E}} = \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, where

$\mathscr{F}_1 = \{f : C_1 \to C_2 \mid C_1 = \mathbb{Z} \text{ and } C_2 = \mathbb{Z} \text{ or } \exists n \in \mathbb{N} : C_2 = \mathbb{Z}_{2n} \text{ and } f(1) \text{ is odd}\}$,
$\mathscr{F}_2 = \{f : C_1 \to C_2 \mid \exists m, n \in \mathbb{N} : C_1 = \mathbb{Z}_{2m}, C_2 = \mathbb{Z}_{2n} \text{ and } f(1) \text{ is odd}\}$ and
$\mathscr{F}_3 = \{f : C_i \to C_j \mid \exists m_i, m_j \in \mathbb{N} : C_i = \mathbb{Z}_{2m_i+1}\}$.

Note that $R\mathbb{Z}_{2n} = R\mathbb{Z} = \mathbb{Z}_2$ and $R\mathbb{Z}_{2n+1} = \mathbb{Z}_1 = \{0\}$. If $f : C_1 \to C_2$ is a morphism in $\mathbb{C}$, where $RC_1 = RC_2 = \mathbb{Z}_2$, then Rf is an isomorphism if and only if $Rf(1) = 1$. Since $Rf\varepsilon_{C_1} = \varepsilon_{C_2} f$, we have $Rf(1) = Rf(\varepsilon_{C_1}(1)) = \varepsilon_{C_2}(f(1)) = f(1)$, hence $Rf(1) = 1$ if and only if $f(1) \equiv 1 \pmod 2$ and that is the case if and only if $f(1)$ is an odd integer when considered as an element of $\{0, 1, \ldots, n-1\}$ in $\mathbb{Z}_n$ or just odd when considered as an element of $\mathbb{Z}$. Hence, if $RC_1 = RC_2 = \mathbb{Z}_2$, then $f \in \overline{\mathscr{E}}$ if and only if $f(1)$ is odd and this is the case if and only if $f \in \mathscr{F}_1 \cup \mathscr{F}_2$.

If $RC_1 = RC_2 = \mathbb{Z}_1$, then $R(f : C_1 \to C_2) = id_{\mathbb{Z}_1}$, so trivially an isomorphism, i.e., if $RC_1 = RC_2 = \mathbb{Z}_1$, then $f \in \overline{\mathscr{E}}$ if and only if $f \in \mathscr{F}_3$. If $RC_1 \neq RC_2$, then Rf can't be an isomorphism and hence can't be in $\overline{\mathscr{E}}$. Therefore $\overline{\mathscr{E}} = \bigcup_{i=1}^{3} \mathscr{F}_i$.

To see that not every morphism factors as me with $e \in \overline{\mathscr{E}}$ and $m \in \overline{\mathscr{M}}$, we consider the zero morphism $z : \mathbb{Z} \to \mathbb{Z}$. Let $f : \mathbb{Z} \to C$ be a morphism in $\mathbb{C}$, where C is a cyclic group. Then, $f(n) = nf(1)$, so we denote the morphism from $\mathbb{Z}$ to $\mathbb{Z}$ sending 1 to n by f_n. It's then easy to show that $f_m \circ f_n = f_{mn}$. A similar argument holds for morphisms from $\mathbb{Z}$ to $\mathbb{Z}_n$, where $e_k = e_p e_q$ if and only if $k \equiv pq \pmod{n}$. Hence, if C is infinite or of even order, then $f(1)$ is odd if and only if f is in $\overline{\mathscr{E}}$.

First note that Rz is the zero morphism from $\mathbb{Z}_2$ to itself and hence not an isomorphism, so z is not in $\overline{\mathscr{E}}$. Furthermore, z is also not in $\overline{\mathscr{M}}$, otherwise there is a diagonal $f_x : \mathbb{Z} \to \mathbb{Z}$ such that

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_3"] \arrow[d, "f_1"'] & \mathbb{Z} \arrow[d, "z"] \arrow[dl, "f_x"'] \\ \mathbb{Z} \arrow[r, "z"'] & \mathbb{Z} \end{tikzcd}$$

commutes. Then, $1 = f_1(1) = f_x(f_3(1)) = f_x(3) = 3f_x(1)$ implying that 3 divides 1.

Suppose $z = me : \mathbb{Z} \to C \to \mathbb{Z}$, with $e \in \overline{\mathscr{E}}$. We show that $m \notin \overline{\mathscr{M}}$. By the above argument, $C \neq \mathbb{Z}_{2n+1}$ for any $n \in \mathbb{N}$. Hence, either $C = \mathbb{Z}$ or $C = \mathbb{Z}_{2n}$ for some $n \in \mathbb{N}$.

If $C = \mathbb{Z}$, then $e = f_{2k+1}$ and $m = f_n$ for some $k, n \in \mathbb{Z}$. Then, $0 = z(1) = me(1) = f_n f_{2k+1}(1) = f_{(2k+1)n)}(1) = (2k+1)n$, so that $2k+1 = 0$ or $n = 0$. Since $2k+1$ is odd, $n = 0$, hence $m = f_0 = z$ is not in $\overline{\mathscr{M}}$. Hence, e is not from $\mathbb{Z}$ to $\mathbb{Z}$.

Suppose $e : \mathbb{Z} \to \mathbb{Z}_{2n}$ for some $n \in \mathbb{N}$. Then, $e = f_{2k+1}$ for some non-negative integer k with $0 \leq 2k+1 \leq 2n-1$. Furthermore, m is a morphism from $\mathbb{Z}_{2n}$ to $\mathbb{Z}$, hence m must be the zero morphism. Hence, it is sufficient to prove that $m : \mathbb{Z}_{2n} \to \mathbb{Z}$ is not in $\overline{\mathscr{M}}$. Note that for any morphism $d : \mathbb{Z} \to \mathbb{Z}_{2n}$, we have: $m(d(1)) = 0 = z(1)$, so that a diagonal of

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} \arrow[d, "z"] \\ \mathbb{Z}_{2n} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

is completely determined by the fact that it is the unique morphism d such that $df = g$. Let $2^{\alpha}p = 2n$ where $p \geq 1$ is odd and α is a positive integer. Note $2n$ is a power of 2 if and only if $p = 1$.

If $p > 1$, then

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_p"] \arrow[d, "f_1"'] & \mathbb{Z} \arrow[d, "z"] \\ \mathbb{Z}_{2n} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

commutes. There is a diagonal morphism $d = f_x$ if and only if $f_x f_p = f_1$, i.e., $px \equiv 1 \pmod{2n}$ has a solution. We know that $px \equiv 1 \pmod{2n}$ has a solution if and only if $gcd(p, 2n)$ divides 1. Since $gcd(p, 2n) = p > 1$, it follows that no such x exists.

If $p = 1$, then $2n = 2^{\alpha}$ is a power of two for some $\alpha \geq 1$. Either $n = 1$ or $n > 1$. If $n > 1$, then $2n \geq 4$, so $\alpha \geq 2$ and

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_1"] \arrow[d, "f_3"'] & \mathbb{Z}_2 \arrow[d, "z"] \\ \mathbb{Z}_{2^{\alpha}} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

commutes. Let d be a diagonal of the above diagram, then d is not the zero morphism, otherwise $df_1(1) = 0 \neq 3 = f_3(1)$. Then $d(1) = d(f_1(1)) = f_3(1) = 3$ and since $\alpha \geq 2$, $3 \not\equiv 1 \pmod{2^{\alpha}}$. Furthermore, $6 = d(1) + d(1) = 2(d(1)) = d(2) = d(0) = 0$. Since $\alpha \geq 2$, we have $4 | 2^{\alpha}$ and $2^{\alpha} | 6$, thus $4 | 6$, a contradiction. Hence, no such diagonal exists. Therefore, the only remaining possibility is that $\alpha = 1$ and $2n = 2$, i.e, $n = 1$. Since we have excluded the only morphism in $\mathbb{C}\mathrm{yc}$ from $\mathbb{Z}_2 \to \mathbb{Z}$, it is not necessary to check this case.

Consequently, if $z : \mathbb{Z} \to \mathbb{Z}$ factors as me with $e \in \overline{\mathscr{E}}$, then $m \notin \overline{\mathscr{M}}$. Therefore z has no $(\overline{\mathscr{E}}, \overline{\mathscr{M}})$-factorisation in $\mathbb{A}$ so that R can't be simple.

4.3 Adjunctions between subcategories and dual closure operators

Throughout the literature, the study of constant morphisms has been associated with left and right constant subcategories. In a topological setting, these are usually associated with subcategories of generalised connected objects, respectively, disconnected or separated objects. In an algebraic setting, these have been associated with torsion and torsion-free theories.

Closure operators have been associated with left and right constant subcategories many times. It has been shown (see [8], [9], [10], [12], [13], [25] and [45] for some material) that the Galois connection between left and right constant subcategories factors through the conglomerate of all closure operators via two other Galois connections. Another approach (see [11] for an example) was to construct connected classes by using closure operators. One particular paper ([18]) can be viewed as a combination of these approaches and shows that under some conditions, every left and right constant subcategory can be viewed as a 'connectedness'or 'disconnectedness'induced by some closure operator.

Due to duality, it's to be expected that the Galois connection, or adjunction, between left and right constant subcategories can be factored through the conglomerate of dual closure operators. In the prominent article ([26]) on dual closure operators, it has already been shown that this adjunction can factor through the conglomerate of all dual closure operators of $\mathscr{E}$ in $\mathbb{A}$. Since the notion of constant morphisms in [26] is different from ours, one can not expect that this will generally result in the same factorisation. In some of the papers on closure operators, including [18], it was shown that the Galois connection between left and right constant subcategories factors through the closure operators. Indeed, the notion of constant morphism is most certainly different and furthermore, the constant subcategories are also different in some cases. However, the left and right constant subcategories can be very similar.

Another thing to keep in mind is that in [26], the left constant subcategories are strongly multi-monocoreflective when one considers a factorisation structure for sinks and the right constant subcategories are reflective for a factorisation structure for sources. Even when a proper orthogonal factorisation structure, i.e., $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and $\mathscr{M} \subset \mathrm{Mono}\mathbb{A}$, can be extended to both a factorisation structure for sinks and one for sources, this still need not be the same result. One obvious reason for this is, as then the epimorphisms need not be strong epimorphisms, or, equivalently, the monomorphisms need not be strong monomorphisms. Our approach is to only consider factorisation structures for sources. We would like to exhibit similar features for the whole adjunction to restrict between $\mathscr{E}$-reflective and nearly multi-$\mathscr{M}$-coreflective subcategories of $\mathbb{A}$ and $DCO(\mathbb{A}, \mathscr{E})$ without needing completeness, cocompleteness, weakly well-poweredness and weak cowell-poweredness. Of course, we will have to impose some restrictions on the categories for this to be true, but it's not meant to be anything more than in the sections for left and right constant subcategories to be nearly multi-$\mathscr{M}$-coreflective and $\mathscr{E}$-reflective respectively. However since it was necessary to assume the $\mathbb{A}$ is $\mathscr{E}$-cocomplete in order to construct $ew^{\mathbb{A}}$ (see [26, 4.2]), the restrictions here are fewer.

Due to the fact that a lot of terminology was already defined in [26], we will follow their terminology as far as possible.

Definition 4.71: $Shriek_{\mathbb{C}}(-), Shriek^*_{\mathbb{C}}(-)$
Let $Sub(\mathbb{A})$ denote the conglomerate of all non-empty full subcategories of $\mathbb{A}$. Then, for a class of morphisms $\mathscr{E}$ in $\mathbb{A}$, let $DCO(\mathbb{A}, \mathscr{E})$ denote the class of all dual closure operators of $\mathscr{E}$ in $\mathbb{A}$.
Now, suppose that $\mathbb{C}$ is a reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. Denote for each $\mathbb{A}$-object X the reflection morphism of X into $\mathbb{C}$ by $r_X : X \to RX$, where $R : \mathbb{A} \to \mathbb{C}$ is a reflector. Let $X \xrightarrow{r_X} RX = X \xrightarrow{e_X} TX \xrightarrow{m_X} RX$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of r_X. Then, we define the operators $Shriek_{\mathbb{C}}(-) : DCO(\mathbb{A}, \mathscr{E}) \to Sub(\mathbb{A})$ and $Shriek^*_{\mathbb{C}}(-) : DCO(\mathbb{A}, \mathscr{E}) \to Sub(\mathbb{A})^{op}$ by

$$Shriek_{\mathbb{C}}(D) = \{X \in \mathbb{A} \mid e_X : X \to TX \text{ is } D\text{-closed}\}$$

and

$$Shriek^*_{\mathbb{C}}(D) = \{X \in \mathbb{A} \mid e_X : X \to TX \text{ is } D\text{-sparse}\}.$$

In case $\mathbb{C}$ consists only of terminal objects and all morphisms to the terminal object are strong epimorphisms in $\mathscr{E}$, then $Shriek^*_{\mathbb{C}}(D)$ and $Shriek_{\mathbb{C}}(D)$ coincide with $Shriek^*(D)$ and $Shriek(D)$ as defined in [26]. We will denote the collection of all dual closure operators D of $\mathscr{E}$ in $\mathbb{A}$ such that $D \leq D_{\mathbb{C}}$, by $DCO^{\mathbb{C}}(\mathbb{A}, \mathscr{E})$, where $D_{\mathbb{C}}$ is the Cassidy Hébert Kelly dual closure operator induced by $\mathbb{C}$ (see 4.12). Similarly, we will denote the collection of all full subcategories of $\mathbb{A}$ that contain $\mathbb{C}$, by $Sub_{\mathbb{C}}(\mathbb{A})$. Recall that all the full subcategories that are either reflective, $\mathscr{E}$-reflective, coreflective, $\mathscr{M}$-coreflective, or nearly multi-$\mathscr{M}$-coreflective, and contain $\mathbb{C}$, will be denoted by $R_{\mathbb{C}}(\mathbb{A})$, $R_{\mathbb{C}}(\mathbb{A}, \mathscr{E})$, $C_{\mathbb{C}}(\mathbb{A})$, $C_{\mathbb{C}}(\mathbb{A}, \mathscr{M})$, or $NMC_{\mathbb{C}}(\mathbb{A}, \mathscr{M})$, respectively. For the remainder of this section, we will assume that $\mathscr{E}$ is a class of epimorphisms. Note that some results do not depend on this, but the theory is richer with this assumption.

Proposition 4.72: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and let $\mathbb{C}$ be reflective. Then, for any dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, $Shriek^*_{\mathbb{C}}(D)$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$.

Proof : Since $\mathbb{C}$ is fixed, we will simply denote $Shriek^*_{\mathbb{C}}(D)$ by $Sh^*(D)$ throughout the proof. To show that $Sh^*(D)$ is $\mathscr{E}$-reflective, it is sufficient to show that $Sh^*(D)$ is closed under sources in $\mathbb{M}$. To this end, let $(m_i : X \to X_i)_I$ be a member of $\mathbb{M}$ with $X_i \in Sh^*(D)$ for each $i \in I$. Then, using the notation as above, since $m_{X_i} e_{X_i} m_i = Rm_i m_X e_X$ for each $i \in I$, the diagonalisation property establishes a morphism $Tm_i : TX \to TX_i$ such that $Tm_i e_X = e_{X_i} m_i$ and $m_{X_i} Tm_i = Rm_i m_X$. Then, D provides us with morphisms D_{m_i, Tm_i} such that for each $i \in I$, the diagram

$$\begin{tikzcd} X \arrow[r, "m_i"] \arrow[d, "D(e_X)"'] & X_i \arrow[d, "D(e_{X_i})"] \\ D(TX) \arrow[r, "D_{m_i, Tm_i}"'] & D(TX_i) \end{tikzcd}$$

commutes.
In particular, as $D(e_{X_i})$ is an isomorphism for each $i \in I$, it's a member of $\mathbb{M}$ and since $\mathbb{M}$ is closed under composition, it follows that $((D(e_{X_i}))_I \circ (m_i)_I = (D_{m_i, Tm_i})_I \circ D(e_X)$ is a member of $\mathbb{M}$. By the cancellation properties of $\mathbb{M}$, we have that $D(e_X)$ is a member of $\mathbb{M}$. The fact that $D(e_X)$ is an isomorphism then follows, as $D(e_X)$ is in $\mathscr{E}$. Thus, e_X is D-sparse ,or, equivalently, X is a member of $Sh^*(D)$.

To see that $Sh^*(D)$ contains $\mathbb{C}$, note that the $\mathbb{C}$-reflection for any $\mathbb{C}$-object C is given by an isomorphism and is hence isomorphic to the identity. Then, $De_C \simeq Did_C \simeq id_C$ so that C is a member of $Sh^*(D)$. Consequently, $\mathbb{C}$ is contained in $Sh^*(D)$. □

Proposition 4.73: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and let $\mathbb{C}$ be reflective in $\mathbb{A}$. Then, $Shriek^*_{\mathbb{C}}(-)$: $DCO(\mathbb{A}, \mathscr{E}) \to R_{\mathbb{C}}(\mathbb{A}, \mathscr{E})^{op}$ is order preserving.

Proof : Suppose that $D \leq D'$ in $DCO(\mathbb{A}, \mathscr{E})$ and assume that $X \in Shriek^*_{\mathbb{C}}(D')$, with $r_X : X \to RX$ the reflection of X into $\mathbb{C}$ and $m_X e_X$ an $(\mathscr{E}, \mathbb{M})$-factorisation of r_X. As X is a member of $Shriek^*_{\mathbb{C}}(D')$, it follows that $D'(e_X)$ is an isomorphism. Since $D(e_X) \leq D'(e_X)$, it follows that there is a morphism e such that $eD(e_X) = D'(e_X)$. But, $eD(e_X) = D'(e_X) \in \mathbb{M} \cap \mathscr{E} = Iso(\mathbb{A})$, hence $D(e_X) \in \mathscr{M} \cap \mathscr{E}$, i.e., $D(e_X)$ is an isomorphism and consequently X is a member of $Shriek^*_{\mathbb{C}}(D)$. Thus, $Shriek^*_{\mathbb{C}}(D') \subset Shriek^*_{\mathbb{C}}(D)$, or, equivalently, $Shriek^*_{\mathbb{C}}(D) \leq Shriek^*_{\mathbb{C}}(D')$ in $R(\mathbb{A}, \mathscr{E})^{op}$. □

Recall from 4.12 that we denote the Cassidy Hébert Kelly dual closure operator induced by $\mathbb{B}$, by $D_{\mathbb{B}}$.

Proposition 4.74: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with pullbacks and let $\mathscr{E}$ be a class of epimorphisms. Then, $D_{(-)} : R(\mathbb{A}, \mathscr{E})^{op} \to DCO(\mathbb{A}, \mathscr{E})$ is order preserving.

Proof : Suppose that $\mathbb{B} \subset \mathbb{B}'$ are $\mathscr{E}$-reflective subcategories of $\mathbb{A}$. We need to show that $D_{\mathbb{B}'}(p) \leq D_{\mathbb{B}}(p)$ for each $p : X \to P$ in $\mathscr{E}$. To this end, suppose that $p : X \to P$ is a member of $\mathscr{E}$ and let $R : \mathbb{A} \to \mathbb{B}$ and $R' : \mathbb{A} \to \mathbb{B}'$ be the corresponding reflectors with units ρ and ρ' respectively. First note that for any morphism $g : A \to B$ in $\mathbb{A}$, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "\rho'_A"'] \arrow[dd, bend right=60, "\rho_A"'] & B \arrow[d, "\rho'_B"] \arrow[dd, bend left=60, "\rho_B"] \\
R'A \arrow[r, "R'g"] \arrow[d, dashed, "f_A"'] & R'B \arrow[d, dashed, "f_B"] \\
RA \arrow[r, "Rg"'] & RB
\end{tikzcd}$$

The existence of f_A and f_B follows by the reflection morphism for A and B respectively and the fact that $\mathbb{B} \subset \mathbb{B}'$.

Now, let $p : X \to P$ be a morphism in $\mathscr{E}$. Consider the pasted diagrams used to construct $D_{\mathbb{B}}(p)$ and $D_{\mathbb{B}'}(p)$:

$$\begin{tikzcd}
X \arrow[dr, "D_{\mathbb{B}'}(p)"] \arrow[ddr, "D_{\mathbb{B}}(p)"'] \arrow[ddddrr, bend right, "p"'] \arrow[ddrrr, bend left=15, "\rho'_X"] \arrow[dddrrrr, bend left=40, "\rho_X"] & & & & \\
& D_{\mathbb{B}'}(P) \arrow[d, dotted, "!d"] \arrow[dr, "m'_p"] & & & \\
& D_{\mathbb{B}}(P) \arrow[d, "m_p"'] & \overline{P}_{Rp'} \arrow[r, "p'_2"] \arrow[dl, dashed, "!k"'] \arrow[dd, "p'_1"] & R'X \arrow[dr, "f_X"] \arrow[dd, "R'p"] & \\
& \overline{P}_{Rp} \arrow[rrr, "p_2"] \arrow[dr, "p_1"] & & & RX \arrow[d, "Rp"] \\
& & P \arrow[r, "\rho'_P"'] \arrow[rr, bend right, "\rho_P"'] & R'P \arrow[r, "f_P"'] & RP
\end{tikzcd}$$

In the diagram above, p_1 is a pullback of Rp along ρ_P and p'_1 is a pullback of $R'p$ along ρ'_P. Then,

$$\begin{aligned}
\rho_P p'_1 &= f_P \rho'_P p'_1 \\
&= f_P R p' p'_2 \\
&= Rp f_X p'_2.
\end{aligned}$$

Since p_1 is a pullback of Rp along ρ_P, there is a unique morphism $k : \overline{P}_{Rp'} \to \overline{P}_{Rp}$ such that $p_1 k = p'_1$ and $p_2 k = f_X p'_2$. Then, note that $p_1 k m'_p D_{\mathbb{B}'}(p) = p'_1 m'_p D_{\mathbb{B}'}(p) = p = p_1 m_p D_{\mathbb{B}}(p)$ and $p_2 k m'_p D_{\mathbb{B}'}(p) = f_X p'_2 m'_p D_{\mathbb{B}'}(p) = f_X \rho'_X = \rho_X = p_2 m_p D_{\mathbb{B}}(p)$. Since $(p_i)_{i=1,2}$ is a mono-source, it follows that $k m'_p D_{\mathbb{B}'}(p) = m_p D_{\mathbb{B}}(p)$.

The diagonalisation property establishes a unique morphism $d : D_{\mathbb{B}'}(P) \to D_{\mathbb{B}}(P)$ such that $m_p d = k m'_p$ and $d D_{\mathbb{B}'}(p) = D_{\mathbb{B}}(p)$. In particular, it follows that $D_{\mathbb{B}'}(p) \leq D_{\mathbb{B}}(p)$. Since p was arbitrary, $D_{\mathbb{B}'} \leq D_{\mathbb{B}}$ and our proof is complete. □

Remark 4.75: Note that if $\mathbb{B}$ and $\mathbb{B}'$ are $\mathscr{E}$-reflective with units ρ and ρ' respectively, then $\mathbb{B} \subset \mathbb{B}'$ if and only if $\rho'_X \leq \rho_X$, for each $\mathbb{A}$-object X. If this is the case, then by 4.14, we can easily see that $D_{\mathbb{B}'}(p) \simeq \rho'_X \wedge p \leq \rho_X \wedge p \simeq D_{\mathbb{B}}(p)$.

Proposition 4.76: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. Then $DCO(\mathbb{A}, \mathscr{E})$ is a complete pre-ordered conglomerate, where the join of a family $(D_i)_I$ of dual closure operators of $\mathscr{E}$ in $\mathbb{A}$ is given by dual closure operator D, where De is the pushout of the family $(De_i)_I$. As usual, the meet is constructed as the join of all lower bounds, or, alternatively, the meet $\hat{D}$ of $(\hat{D}_j)_J$ is constructed with the $(\mathscr{E}, \mathbb{M})$-factorisation and diagonalisation property as follows:

For each $e : X \to Y$ in $\mathscr{E}$ and each family $(\hat{D}_j)_J$ of dual closure operators, if $(m_j)_J \hat{e}$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of $(\hat{D}_j e)_J$, then $\hat{D}e \simeq \hat{e}$ and the natural transformation $\hat{\delta}_e$ is defined as $\hat{\delta}_j m_j$ for any $j \in J$

provided that J is non-empty. Of course, if J is empty, then $\hat{D}e = e$ and $\hat{\delta}_e = id_Y$:

$$
\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "\hat{D}_j e"'] \arrow[dd, "\hat{e}"'] & & Y \\
& \hat{D}_j Y \arrow[ur, "\hat{\delta}_j"'] & \\
\hat{D}Y \arrow[ur, "m_j"'] & &
\end{tikzcd}
$$

$\underline{\mathcal{P}roof}$: Let $(D_i)_I$ be a family of dual closure operators of $\mathscr{E}$ in $\mathbb{A}$. Since $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, it's clear that $\mathbb{A}$ is $\mathscr{E}$-cocomplete. Let $p : X \to P$ be a morphism in $\mathscr{E}$. Then, for each $i \in I$, let δ_i be the unique morphism such that $\delta_i D_i(p) = p$. Consider the pushout Dp of $D_i(p)$ with $j_i D_i(p) = Dp$ for each $i \in I$. Since $\delta_i D_i(p) = p$ for each $i \in I$, the pushout property gives us a unique morphism δ_p such that $\delta_p Dp = p$ and $\delta_p j_i = \delta_i$. Now that Dp is defined and $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, it's sufficient to prove that conditions $(a), (b)$ and (c) of Proposition 4.8 are satisfied. We have already seen that $Dp \leq p$ and since $\delta_p j_i = \delta_i$ is in $\mathscr{E}$ and $j_i \in \mathscr{E}$, we have that $\delta_p \in \mathscr{E}$.

We now show that D is order preserving. Suppose that $p \leq q$ in $\mathscr{E}$ and j is a morphism such that $jp = q$. Then consider the two commuting diagrams:

$$
\begin{tikzcd}
& & DP \arrow[dll, bend left, "\delta_p"] \\
& D_i(P) \arrow[ur, "j_i"] \arrow[dl, "\delta_{p_i}"] & \\
X \arrow[r, "p"'] \arrow[ur, "D_i p"] \arrow[uurr, bend left, "Dp"] & P &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& & DQ \arrow[dll, bend left, "\delta_q"] \\
& D_i(Q) \arrow[ur, "k_i"] \arrow[dl, "\delta_{q_i}"] & \\
X \arrow[r, "q"'] \arrow[ur, "D_i q"] \arrow[uurr, bend left, "Dq"] & Q &
\end{tikzcd}
$$

Since D_i is a dual closure operator for each i, it follows that $D_i p \leq D_i q$ for each $i \in I$, hence for each i, there is a morphism $\ell_i : D_i(P) \to D_i(Q)$ such that $\ell_i Dp_i = Dq_i$. Then, $k_i \ell_i$ is a morphism such that $k_i \ell_i D_i p = Dq$. Consequently, by the pushout property, there is a unique morphism m from DP to DQ such that, in particular it follows that $mDp = Dq$, i.e., $Dp \leq Dq$.

Now, let $q : Y \to Q$ be a member of $\mathscr{E}$ and $f : X \to Y$ an $\mathbb{A}$-morphism. We need to show $D(f^- q) \leq f^-(Dq)$.
It's easy to see that the map $q \longmapsto f^-(q)$ is order preserving and hence if D_1 and D_2 are dco's with $D_1 \leq D_2$, then $f^-(D_1 q) \leq f^-(D_2 q)$. Since $D_i q \leq Dq$ for each $i \in I$, it follows that $f^-(D_i q) \leq f^-(Dq)$. Thus $\bigvee_I f^-(D_i q) \leq f^-(Dq)$.
So, $D(f^- q) = \bigvee_I D_i(f^-(q)) \leq \bigvee_I f^-(D_i q) \leq f^-(Dq)$, where the second inequality holds since D_i is a dco for each $i \in I$. It then follows that D is a dco of $\mathscr{E}$ in $\mathbb{A}$. The pushout property establishes that D is in fact the join of the family $(D_i)_I$ in $DCO(\mathbb{A}, \mathscr{E})$.

To see that $\hat{D}$ is the meet of $(D_j)_J$ (as above) is immediate if J is empty. If J is non-empty, then note that $\hat{e} \leq \hat{D}_j e$ for each $j \in J$. Since $\hat{e}$ is an epimorphism, we also have $\hat{\delta}_j m_j \hat{e} = \hat{\delta}_j \hat{D}_j e = e$ for each $j \in J$ and hence $\hat{\delta}_j m_j = \hat{\delta}_{j'} m_{j'}$ for each $j, j' \in J$. If D is any dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ such that $D \leq \hat{D}_j$ for each $j \in J$, then, in particular, $De \leq \hat{D}_j e$ so that for each $j \in J$ there is a morphism e_j such that $e_j De = \hat{D}_j e$. Consider the diagram:

$$
\begin{tikzcd}
& X \arrow[rr, "e"] \arrow[dr, "\hat{D}_j e"] \arrow[dd, "\hat{e}"] \arrow[dddl, "De"'] & & Y \\
& & \hat{D}_j Y \arrow[ur, "\hat{\delta}_j"] & \\
& \hat{D} Y \arrow[ur, "m_j"] & & \\
DY \arrow[ur, dashed, "!d"] \arrow[uurr, bend right, "e_j"'] \arrow[uuurrr, bend right, "\delta_e"'] & & &
\end{tikzcd}
$$

The diagonalisation property establishes a unique morphism d such that $m_j d = De$ and $dDe = \hat{e} = \hat{D}e$. Therefore, $De \leq \hat{D}e$, so that $\hat{D}e$ is the meet of $(\hat{D}_j e)_J$ in $X \setminus \mathscr{E}$. Therefore, it is sufficient to show that $\hat{D}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. It is clear that $D\hat{e} \leq e$ for each $e \in \mathscr{E}$. Furthermore, if $e_1 \leq e_2$, then $\hat{D}e_1 \simeq \bigwedge_J D_j e_1 \leq \bigwedge_J D_j e_2 \simeq \hat{D}e_2$, so that $\hat{D}$ is order preserving.

We now show that whenever $g : A \to X$ is an $\mathbb{A}$-morphism and $e : X \to Y$ is a member of $\mathscr{E}$, then $\hat{D}(g^- e) \leq g^-(\hat{D}e)$. Let $mg^-(\hat{D}e)$, $(n_j)_J \hat{D}e$, $\hat{n}_j g^-(D_j e)$, $ng^- e$ and $(m_j)_J \hat{D}g^- e$ be $(\mathscr{E}, \mathbb{M})$-factorisations of $\hat{D}eg$, $(D_j e)_J$, $D_j eg$, eg and $(D_j g^- e)_J$. Then, since D_j is a dual closure operator for each j, we have that $D_j g^- e \leq g^-(D_j e)$ so that there is a morphism $\tilde{e}_j$ such that $\tilde{e}_j D_j g^- e = g^- D_j e$. Note that we may abuse notation and write $g^- De$ (or Dg^e) instead of the more cumbersome $g^-(De)$ (or $Dg^-(e)$). Consider the following commutative diagram:

$$
\begin{tikzcd}
& & & g^-(\hat{D}Y) \arrow[drrr, "m"] & & & \\
\hat{D}g^- Y \arrow[urrr, dashed, bend left, "d"] \arrow[ddr, "m_j"'] & & A \arrow[ll, "\hat{D}g^- e"'] \arrow[ur, "g^- \hat{D}e"] \arrow[rr, "g"] \arrow[dr, "g^- D_j e"] \arrow[ddl, "D_j g^- e"'] \arrow[dd, "g^- e"] & & X \arrow[rr, "\hat{D}e"] \arrow[dr, "D_j e"] \arrow[dd, "e"] & & \hat{D}Y \arrow[dl, "n_j"] \\
& & & g^- D_j Y \arrow[rr, "\hat{n}_j"] & & D_j Y \arrow[dl, "\delta_j"] & \\
& D_j g^- Y \arrow[r, "\bar{\delta}_j"'] \arrow[urr, "\tilde{e}_j"'] & g^- Y \arrow[rr, "n"'] & & Y & &
\end{tikzcd}
$$

Note that for each $j \in J$, there holds: $n_j m g^- \hat{D}e = n_j \hat{D}eg = D_j eg = \hat{n}_j g^- D_j e = \hat{n}_j \tilde{e}_j D_j g^- e = \hat{n}_j \tilde{e}_j m_j \hat{D}g^- e$. Since $\mathbb{M}$ is closed under composition, $(n_j)_J \circ m$ is in $\mathbb{M}$ and the diagonalisation property establishes a morphism d such that, in particular, $g^- De = d\hat{D}g^- e$. It follows that $\hat{D}g^- e \leq g^- \hat{D}e$. Therefore $\hat{D}$ is a dual closure operator and also the meet of $(D_j)_J$. $\square$

Remark 4.77: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. It should be noted that the proof of Proposition 4.76 reveals that the meet of a family $(e_i)_I$ in $X/\mathscr{E}$ is given by the $(\mathscr{E}, \mathbb{M})$-factorisation of the source $(e_i : X \to X_i)_I$.

Lemma 4.78: ([1, 15.14]) Let $\mathscr{E}$ be a class of morphisms in a category $\mathbb{A}$ that has multiple pushouts. If $(e_i : X \to E_i)_I$ is any family of isomorphisms, then the pushout morphism e is an isomorphism.

Proposition 4.79: Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category. Then, $Shriek^*_{\mathbb{C}}(-) : DCO(\mathbb{A},\mathscr{E}) \to R_{\mathbb{C}}(\mathbb{A},\mathscr{E})^{op}$ preserves joins.

Proof : Let $\mathbb{C}$ be reflective in $\mathbb{A}$ and $R : \mathbb{A} \to \mathbb{C}$ a reflector with unit ρ. For each $X \in \mathbb{A}$, let $m_X e_X : X \to TX \to RX$ be an $(\mathscr{E},\mathbb{M})$-factorisation of ρ_X. Let $(D_i)_I$ be a family of dual closure operators of $\mathscr{E}$ in $\mathbb{A}$ and let $D = \bigvee_I D_i$. Since $Shriek^*_{\mathbb{C}}(-)$ is order preserving, it follows that $Shriek^*_{\mathbb{C}}(D_i) \supset Shriek^*_{\mathbb{C}}(D)$. It's then clear that $Shriek^*_{\mathbb{C}}(D)$ is also contained in the intersection $\bigcap_I Shriek^*_{\mathbb{C}}(D_i)$. It's therefore sufficient to prove that the reverse inclusion also holds. Let X be in the intersection, i.e., for each i in I, $D_i(e_X)$ is an isomorphism. Since De is constructed as the multiple pushout of $(D_i e_X)_I$ and each of these is an isomorphism, Lemma 4.78 gives that De_X is also an isomorphism. Therefore X is a member of $Shriek^*_{\mathbb{C}}(D)$ and thus the two subcategories are equal. □

Lemma 4.80: Let $\mathbb{A}$ be an $(\mathscr{E},\mathscr{M})$-structured category with pullbacks and let $\mathbb{C}$ be reflective in $\mathbb{A}$ and let $\mathbb{B} \in R_{\mathbb{C}}(\mathbb{A},\mathscr{E})$ with reflector $R : \mathbb{A} \to \mathbb{B}$ and unit ρ. Then, the following holds:

(*i*) $X \in \mathbb{B} \Leftrightarrow$ For each $p : X \to P$ in $\mathscr{E}$: $D_{\mathbb{B}}(p) \simeq id_X$,

(*ii*) For each $p : X \to P$ in $\mathscr{E}$: $P \in \mathbb{B} \Rightarrow D_{\mathbb{B}}(p) \simeq \rho_X$.

Proof : Let $p : X \to P$ be a member of $\mathscr{E}$. Note that X is in $\mathbb{B}$ if and only if $\rho_X \simeq id_X$. Then, $D_{\mathbb{B}}(p) \simeq \rho_X \wedge p$ and $\rho_X \wedge p \simeq id_X$ if X is in $\mathbb{B}$. If $p \wedge \rho_X \simeq id_X$ for each $p : X \to P$ in $\mathscr{E}$, then, in particular, $\rho_X \simeq \rho_X \wedge \rho_X \wedge id_X$ so that X is in $\mathbb{B}$.

If P is in $\mathbb{B}$, then since ρ_X is the $\mathbb{B}$-reflection, $\rho_X \leq p$. Hence, $D_{\mathbb{B}}(p) \simeq \rho_X \wedge p \simeq \rho_X$. □

Proposition 4.81: Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category with pullbacks and $\mathbb{B}$ any $\mathscr{E}$-reflective subcategory that contains the reflective subcategory $\mathbb{C}$. Then,

$$\mathbb{B} = Shriek^*_{\mathbb{C}}(D_{\mathbb{B}}).$$

Proof : Let X be a member of $\mathbb{B}$ and consider an $(\mathscr{E},\mathscr{M})$-factorisation $m_X e_X$ of the reflection r_X into $\mathbb{C}$. As $e_X : X \to TX$ is a member of $\mathscr{E}$ and X is a member of $\mathbb{B}$, Lemma 4.80(i) gives that $D_{\mathbb{B}}(e_X) \simeq id_X$, i.e., $D_{\mathbb{B}}(e_X)$ is an isomorphism. Hence, X is a member of $Shriek^*_{\mathbb{C}}(D_{\mathbb{B}})$.

For the reverse inclusion, let X be a member of $Shriek^*_{\mathbb{C}}(D_{\mathbb{B}})$. Then $D_{\mathbb{B}}(e_X) \simeq id_X$. If ρ_X is the $\mathbb{B}$-reflection, then $D_{\mathbb{B}}(e_X) \simeq \rho_X \wedge e_X$. Since $\mathbb{B}$ contains $\mathbb{C}$, and hence also its $\mathscr{M}$-closure, $\rho_X \leq e_X$. Thus $\rho_X \simeq D_{\mathbb{B}}(e_X) \simeq id_X$. Therefore ρ_X is an isomorphism, or, equivalently,, X is a member of $\mathbb{B}$. □

Theorem 4.82: Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category with pullbacks and let $\mathbb{C}$ be reflective in $\mathbb{A}$. Then, for each $D \in DCO(\mathbb{A},\mathscr{E})$ and $\mathbb{B} \in R_{\mathbb{C}}(\mathbb{A},\mathscr{E})$, the following holds:

$$D \leq D_{\mathbb{B}} \Rightarrow Shriek^*_{\mathbb{C}}(D) \supset \mathbb{B}.$$

Proof : Assume that $\mathbb{C}$ is a reflective subcategory of $\mathbb{A}$ and $R : \mathbb{A} \to \mathbb{C}$ a reflector and $r_X : X \to RX$ a $\mathbb{C}$-reflection arrow with $(\mathscr{E},\mathscr{M})$-factorisation $m_X e_X : X \to TX \to RX$, for each $X \in \mathbb{A}$. Consider an $\mathscr{E}$-reflective subcategory $\mathbb{B}$ of $\mathbb{A}$ that contains $\mathbb{C}$. Assume that $S : \mathbb{A} \to \mathbb{B}$ is a reflector with unit ρ.

Let D be a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ such that $D \leq D_{\mathbb{B}}$. Since $\mathbb{B}$ contains $\mathbb{C}$, it's clear that $\mathbb{B}$ contains the $\mathscr{M}$-closure of $\mathbb{C}$ and thus for a fixed $\mathbb{A}$-object X, TX is a member of $\mathbb{B}$. It then follows by the reflectiveness of $\mathbb{B}$ that $\rho_X \leq e_X$. In order to show that $\mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$, we need only show that e_X is D-sparse whenever X is in $\mathbb{B}$. Let $X \in \mathbb{B}$. Then $D(e_X) \leq D_{\mathbb{B}}(e_X) \simeq e_X \wedge \rho_X \simeq \rho_X \simeq id_X$, and since id_X is the smallest element of $quot(X)$, we must have that e_X is D-sparse. □

Proposition 4.83: Let $\mathbb{A}$ be an $(\mathscr{E},\mathbb{M})$-category with pullbacks and R a reflective object of $\mathbb{A}$. Then, $\mathbb{C} = \{X \in \mathbb{A} \mid X \simeq R\}$ is reflective in $\mathbb{A}$. ALso, for each $D \in DCO(\mathbb{A},\mathscr{E})$ and $\mathbb{B} \in R_{\mathbb{C}}(\mathbb{A},\mathscr{E})$ the following holds:

$$D \leq D_{\mathbb{B}} \Leftrightarrow Shriek^*_{\mathbb{C}}(D) \supset \mathbb{B}.$$

Proof : Suppose that $Shriek^*_{\mathbb{C}}(D) \supset \mathbb{B} \supset \mathbb{C}$, where $\mathbb{B}$ is some $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. We show that $D \leq D_{\mathbb{B}}$. Let $S : \mathbb{A} \to \mathbb{B}$ and $C : \mathbb{A} \to \mathbb{C}$ be reflectors with units ρ and r respectively. For each

$\mathbb{A}$-object A, let $m_A e_A$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of r_X.

Let $p : X \to P$ be a member of $\mathscr{E}$. Consider the diagram:

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "\rho_X"'] & P \arrow[d, "\rho_P"] \\
SX \arrow[r, "Sp"] \arrow[d, "e_{SX}"'] & SP \arrow[d, "e_{SP}"] \\
TSX \arrow[r, "TSp"'] \arrow[d, "m_{SX}"'] & TSP \arrow[d, "m_{SP}"] \\
CSX \arrow[r, "CSp"'] & CSP
\end{tikzcd}$$

Note that the existence of TSp is given by the diagonalisation property. Since $\mathscr{E}$ is closed under composition and satisfies the cancellation condition $fg \in \mathscr{E}$ and $g \in \mathscr{E}$ implies $f \in \mathscr{E}$, we have: $Sp\rho_X = \rho_P p \in \mathscr{E}$ and $\rho_X \in \mathscr{E}$ so that Sp is in $\mathscr{E}$. Similarly $e_{SP}Sp = TSpe_{SX} \in \mathscr{E}$ implies that TSp is in $\mathscr{E}$ as e_{SX} is in $\mathscr{E}$. First we prove that any morphism from R to R is a member of $\mathbb{M}$. Let $f : R \to R$. By reflectivity of R there is a morphism $g : R \to R$ such that $gf = id_R$. It's then clear that f is a section. Then, there is a morphism $h : R \to R$ such that $hg = id_R$. It follows that we have, $hf = hid_R f = hhgf = hhid_R = hh$, and since h is a section, $f = h$. Therefore f is an isomorphism and thus a member of $\mathbb{M} \cap \mathscr{E}$. Letting $f = CSp$, we know that CSp is an isomorphism. Then $m_{SP}TSp = CSpm_{SX} \in \mathbb{M}$ and we have $TSp \in \mathbb{M}$.

By the last two paragraphs, it follows that TSp is an isomorphism and the diagonalisation property gives us a morphism $d : SP \to TSX$ such that $TSpd = e_{SP}$ and $dSp = e_{SX}$. Note that De_{SX} and $D(e_{SP})$ are isomorphisms since SX and SP are members of $\mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$. Since $dSp = e_{SX}$, we have $Sp \leq e_{SX}$, hence $DSp \leq D(e_{SX}) \leq id_X$. Thus DSp is an isomorphism so that Sp is D-sparse. As D is a dual closure operator, we have the commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\rho_X"] \arrow[d, "Dp"] \arrow[dd, bend right=60, "p"'] & SX \arrow[d, "DSp"'] \arrow[dd, bend left=60, "Sp"] \\
DP \arrow[r, "D_{\rho_X,\rho_P}"] \arrow[d, "\delta_p"] & DSP \arrow[d, "\delta_{Sp}"'] \\
P \arrow[r, "\rho_P"'] & SP
\end{tikzcd}$$

Then, as DSp is an isomorphism, by the diagonalisation property, there exists a morphism $s : DP \to SX$ such that $DSp \circ s = D_{\rho_X,\rho_P}$ and $s \circ Dp = \rho_X$. It follows that $Dp \leq \rho_X$, so that $Dp \leq p \wedge \rho_X \simeq D_{\mathbb{B}}(p)$. Since p was arbitrary, our proof is complete. $\square$

Remark 4.84: The preceding proposition provides us with a Galois connection between the $\mathscr{E}$-reflective subcategories of $\mathbb{A}$ that contain a reflective object and dual closure operators of $\mathscr{E}$ in $\mathbb{A}$. The next proposition is not only a very similar result, but the proof is very similar as well. These two results coincide in case R is an $\mathscr{E}$-reflective object.

Proposition 4.85: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and for each $\mathbb{A}$-object X, let $\varepsilon_X : X \to RX$ be a maximum of $X/\mathscr{E}$. Let $\mathbb{C} = \{A \in \mathbb{A} \mid A \simeq RX \text{ for some } X \in \mathbb{A}\}$. Then $\mathbb{C}$ is the smallest reflective constant subcategory of $\mathbb{A}$ with reflector R and unit ε. Then, for each $D \in DCO(\mathbb{A}, \mathscr{E})$ and $\mathbb{B} \in R_{\mathbb{C}}(\mathbb{A}, \mathscr{E})$ the following holds:

$$D \leq D_{\mathbb{B}} \Leftrightarrow Shriek^*_{\mathbb{C}}(D) \supset \mathbb{B}.$$

Proof : In view of 4.82, it's sufficient to prove the reverse implication. To this end, let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ such that $\mathbb{C} \subset \mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$ for some dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$. Let $S : \mathbb{A} \to \mathbb{B}$ denote the reflector with unit ρ.

Let $p : X \to P$ be in $\mathscr{E}$ and consider the naturality square
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\rho_X"'] & P \arrow[d, "\rho_P"] \\ SX \arrow[r, "Sp"'] & SP \end{tikzcd}$$
. It ought to be clear that Sp is a member of $\mathscr{E}$. Since ε_{SX} is the maximum of $quot(SX)$, $Sp \leq \varepsilon_{SX}$. This implies the existence of a unique morphism $d : SP \to RSX$ such that $dSp = \varepsilon_{SX}$. Therefore $DSp \leq D\varepsilon_{SX} \simeq id_X$, as SX is a member of $Shriek^*_{\mathbb{C}}(D)$, and it follows that DSp is an isomorphism. Consider the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "Dp"'] \arrow[dd, bend right=60, "p"'] & SX \arrow[d, "DSp"] \arrow[dd, bend left=60, "Sp"] \\ DP \arrow[r, "D_{\rho_X, \rho_P}"] \arrow[d, "\delta_p"'] & DSP \arrow[d, "\delta_{S_p}"] \\ P \arrow[r, "\rho_P"'] & SP \end{tikzcd}$$

Since DSp is an isomorphism, it can easily be seen that $(DSp)^{-1}D_{\rho_X,\rho_P}Dp = (DSp)^{-1}DSp\rho_X = \rho_X$, so that $Dp \leq \rho_X$. We must then have that $Dp \leq p$ and $Dp \leq \rho_X$ and consequently, $Dp \leq p \wedge \rho_X \simeq D_{\mathbb{B}}(p)$. Since p was arbitrary, the result follows. $\square$

Definition 4.86: $IDCO(\mathbb{A}, \mathscr{E})$**,** $IDCO^{\mathbb{C}}(\mathbb{A}, \mathscr{E})$
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category. We denote the collection of all idempotent dual closure operators of $\mathscr{E}$ in $\mathbb{A}$ by $IDCO(\mathbb{A}, \mathscr{E})$, i.e.,

$$IDCO(\mathbb{A}, \mathscr{E}) = \{D \in DCO(\mathbb{A}, \mathscr{E}) \mid D \text{ is idempotent}\}.$$

If $\mathbb{A}$ has pullbacks and $\mathbb{C}$ is a reflective subcategory of $\mathbb{A}$, then $D_{\mathbb{C}}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Then we denote the collection of all idempotent dual closure operators less than $D_{\mathbb{C}}$ by $IDCO^{\mathbb{C}}(\mathbb{A}, \mathscr{E})$, i.e.,
$$IDCO^{\mathbb{C}}(\mathbb{A}, \mathscr{E}) = \{D \in DCO(\mathbb{A}, \mathscr{E}) \mid D \leq D_{\mathbb{C}} \text{ and } D \text{ is idempotent}\}.$$

Theorem 4.87: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and let $\mathbb{C}$ be reflective in $\mathbb{A}$. Then, for each $\mathbb{B} \in R_{\mathbb{C}}(\mathbb{A}, \mathscr{E})$ and $D \in IDCO^{\mathbb{C}}(\mathbb{A}, \mathscr{E})$, the following holds:

$$D \leq D_{\mathbb{B}} \text{ if and only if } \mathbb{B} \subset Shriek^*_{\mathbb{C}}(D).$$

Proof : By Theorem 4.82, it's sufficient to prove that the reverse implication is true. To this end, assume that D is an idempotent dual closure operator with $D \leq D_{\mathbb{C}}$ and let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ such that $\mathbb{C} \subset \mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$. Of course, any $\mathscr{E}$-reflective subcategory $\mathbb{B}$ of $\mathbb{A}$ that contains $\mathbb{C}$, also contains the $\mathscr{M}$-closure of $\mathbb{C}$. Furthermore, we have also seen that $D_{\mathbb{C}} \simeq D_{\mathscr{M}(\mathbb{C})}$. Hence, we may assume that both $\mathbb{C}$ and $\mathbb{B}$ are $\mathscr{E}$-reflective. Let $T : \mathbb{A} \to \mathbb{C}$ denote the reflector with unit ε. Let $S : \mathbb{A} \to \mathbb{B}$ denote the other reflector with unit ρ.

To start our actual proof, let $p : X \to P$ be a morphism in $\mathscr{E}$ and consider the morphism $Sp : SX \to SP$ in $\mathscr{E}$, together with the pasted naturality squares
$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "p"'] & SX \arrow[r, "\varepsilon_{SX}"] \arrow[d, "Sp"] & TSX \arrow[d, "TSp"] \\ P \arrow[r, "p"'] & SP \arrow[r, "\varepsilon_{SP}"'] & TSP \end{tikzcd}$$
Note that SX, SP, TSX and TSP are all members of $Shriek^*_{\mathbb{C}}(D)$ as they are in $\mathbb{B}$ and $\mathbb{C}$. In particular, $D\varepsilon_{SX} \simeq id_{SX}$ and $D(TSp) \leq D_{\mathbb{C}}(TSp) \simeq TSp \wedge \varepsilon_{TSX} \simeq TSp \wedge id_{TSX} \simeq id_{TSX}$. Therefore $D\varepsilon_{SX}$ and $DTSp$ are isomorphisms. Then,

$$\begin{aligned} D(Sp) &\leq D_{\mathbb{C}}(Sp) \text{ since } D \leq D_{\mathbb{C}} \\ &\leq Sp \wedge \varepsilon_{SX} \\ &\leq \varepsilon_{SX}. \end{aligned}$$

By idempotency, it follows that $D(Sp) \simeq D(DSp) \leq D(\varepsilon_{SX}) \leq id_{SX}$. Therefore $D(Sp)$ must be an isomorphism. Consider the diagram obtained from applying the functor D to the naturality square:

$$\begin{tikzcd}
X \arrow[r, "\rho_X"] \arrow[d, "Dp"'] & SX \arrow[d, "DSp"] \\
DP \arrow[r, "D_{\rho_X,\rho_P}"'] \arrow[ur, dashed, "!s"] \arrow[d, "\delta_p"'] & DSP \arrow[d, "\delta_{Sp}"] \\
P \arrow[r, "\rho_P"'] & SP
\end{tikzcd}$$

Since DSp is an isomorphism, there is a unique morphism $s : DP \to SX$ such that the top square in the above diagram commutes. In particular, it follows that $Dp \leq \rho_X$ and since $Dp \leq p$, we have $Dp \leq p \wedge \rho_X \simeq D_{\mathbb{B}}(p)$. Since p was arbitrary, $D \leq D_{\mathbb{B}}$. $\square$

Remark 4.88: It should be noted that Theorem 4.87 is not true if we don't assume $D \leq D_{\mathbb{C}}$. To see this, suppose that there is a dual closure operator $D > D_{\mathbb{C}}$. Since $Shriek^*_{\mathbb{C}}(-)$ is order reversing, Propositions 4.72 and 4.81 gives that $Shriek^*_{\mathbb{C}}(D)$ must always contain $\mathbb{C}$ and that $Shriek^*_{\mathbb{C}}(D_{\mathbb{B}}) = \mathbb{B}$ for any $\mathscr{E}$-reflective subcategory that contains $\mathbb{B}$. We must then have $\mathbb{C} \subset Shriek^*_{\mathbb{C}}(D) \subset Shriek^*_{\mathbb{C}}(D) = \mathbb{C}$ and hence the two subcategories coincide. It follows that $\mathbb{C} \subset Shriek^*_{\mathbb{C}}(D)$, but $D \not\leq D_{\mathbb{C}}$.
However, in view of Proposition 4.79, it follows that there is a map $d : R_{\mathbb{C}}(\mathbb{A}, \mathscr{E}) \to DCO(\mathbb{A}, \mathscr{E})$ such that $D \leq d(\mathbb{B})$ if and only if $\mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$. Furthermore, a standard result on Galois connections between pre-ordered classes shows that $d(\mathbb{B}) = \bigvee\{D \in DCO(\mathbb{A}, \mathscr{E}) \mid \mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)\}$.

Lemma 4.89: The big square Lemma
Let $\mathbb{A}$ be $(\mathscr{E}, \mathscr{M})$-structured and D a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ with $p = \delta_p Dp$ for each $p \in \mathscr{E}$. Then, for each commuting square $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ P \arrow[r, "g"'] & Q \end{tikzcd}$ with each of f, g, p and q in $\mathscr{E}$, we have the following commutative diagrams:

$$\begin{tikzcd}
X \arrow[r, "D_X(f)"] \arrow[d, "D_X(p)"'] & D_X(Y) \arrow[r, "\delta_f"] \arrow[d, "D_{D_X(Y)}(D_{p,q})"'] & Y \arrow[d, "D_Y(q)"] \\
D_X(P) \arrow[r, "D_{D_X(f),D_P(g)}"] \arrow[d, "\delta_p"'] & D_{D_X(Y)}(D_P(Q)) \arrow[r, "D_{\delta_f,\delta_g}"] \arrow[d, "\delta_{D_{p,q}}"'] & D_Y(Q) \arrow[d, "\delta_q"] \\
P \arrow[r, "D_P(g)"'] & D_P(Q) \arrow[r, "\delta_g"'] & Q
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "D_X(p)"] \arrow[d, "D_X(f)"'] & D_X(P) \arrow[r, "\delta_p"] \arrow[d, "D_{D_X(P)}(D_{f,g})"'] & P \arrow[d, "D_P(g)"] \\
D_X(Y) \arrow[r, "D_{D_X(p),D_Y(q)}"] \arrow[d, "\delta_f"'] & D_{D_X(P)}(D_Y(Q)) \arrow[r, "D_{\delta_p,\delta_q}"] \arrow[d, "\delta_{D_{f,g}}"'] & D_P(Q) \arrow[d, "\delta_p"] \\
Y \arrow[r, "D_Y(q)"'] & D_Y(Q) \arrow[r, "\delta_q"'] & Q
\end{tikzcd}$$

Proof : Consider the square $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ P \arrow[r, "g"'] & Q \end{tikzcd}$ with all morphisms in $\mathscr{E}$. Ignoring the subscripts, let us denote the factorisation of each $h \in \mathscr{E}$ via D and $\delta_h := cod(\Delta_h)$ by $h : A \to B = A \xrightarrow{Dh} DB \xrightarrow{\delta_h} B$. We then consider the diagram

$$\begin{tikzcd}
X \arrow[r, "Df"] \arrow[d, "Dp"'] & DY \arrow[r, "\delta_f"] & Y \arrow[d, "Dq"] \\
DP \arrow[d, "\delta_f"'] & & DQ \arrow[d, "\delta_q"] \\
P \arrow[r, "Dg"'] & DQ \arrow[r, "\delta_g"'] & Q
\end{tikzcd}$$

It's clear that the outer diagram commutes. Now, for each morphism $(u, v) : p \to q$ in the category $\mathscr{E}$, we have a unique morphism $dom(D(u, v)) := D_{u,v}$ such that $D_{u,v} \circ Dp = Dq \circ u$ and $\delta_q \circ D_{u,v} = v \circ \delta_p$. Considering $(f, g) : p \to q$, $D_{f,g}$ is the unique morphism such that $D_{f,g} \circ Dp = Dq \circ f$ and $\delta_q \circ D_{f,g} = g \circ \delta_p$. Since $D_{f,g} = \delta_{D_{f,g}} \circ D(D_{f,g})$, we have morphisms $(Dp, Dq) : p \to D_{f,g}$ and $(\delta_p, \delta_q) : D_{f,g} \to q$. Hence, $D_{Dp,Dq}$ and D_{δ_p,δ_q} are the desired morphisms such that the second big square commutes. By replacing the roles of f and g by p and q respectively, the other case is similar. $\square$

Corollary 4.90: Let f, g, p and q be members of $\mathscr{E}$ and D be a dual closure operator of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. Then, $D_{p,q} = D_{\delta_p,\delta_q} \circ D_{D_X(p),D_Y(q)}$, provided that $qf = gp$.

Proposition 4.91: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and let $\mathbb{C}$ be $\mathscr{E}$-reflective. Then, for each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, there exists a maximal dual closure operator $\hat{D} \leq D_{\mathbb{C}}$, namely $\hat{D} := D_{Shriek^*_{\mathbb{C}}(D)}$, such that $Shriek^*_{\mathbb{C}}(D) = Shriek^*_{\mathbb{C}}(\hat{D})$.

Proof : First note that by Proposition 4.72, we have that $Shriek^*_{\mathbb{C}}(D)$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. Since $\hat{D} = D_{Shriek^*_{\mathbb{C}}(D)}$ is maximal by Theorem 4.43, we need only show that equality holds, but Proposition 4.81 gives this immediately. $\square$

Remark 4.92: In view of Theorem 4.82, Proposition 4.83 and Theorem 4.87, we can consider the following for an $(\mathscr{E}, \mathbb{M})$-category with pullbacks. Let $\mathbb{C}$ be $\mathscr{E}$-reflective in $\mathbb{A}$ and assume that $\mathbb{B}$ is an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. Then, of course, for any dual closure operator $D \leq D_{\mathbb{B}}$, we have $\mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$. Conversely, if $\mathbb{B} \subset Shriek^*_{\mathbb{C}}(D)$, then by Proposition 4.91 it follows that

$\mathbb{B} \subset Shriek^*_{\mathbb{C}}(\hat{D})$ for $\hat{D} = D_{Shriek^*_{\mathbb{C}}(D)}$. Note that, in particular, $\hat{D}$ is a maximal, and hence also idempotent dual closure operator of $\mathscr{E}$ in $\mathbb{A}$, and so Theorem 4.87 gives that $\hat{D} \leq D_{\mathbb{B}}$.

Proposition 4.93: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with pullbacks and let $\mathbb{C}$ be a reflective constant subcategory of $\mathbb{A}$. Then, $Shriek_{\mathbb{C}}(D_{\mathbb{P}}) = \mathscr{L}(\mathbb{P})$ for every $\mathscr{E}$-reflective subcategory $\mathbb{P}$ of $\mathbb{A}$ that contains $\mathbb{C}$.

Proof : Let R and C be the reflectors with respective units ρ and ε into $\mathbb{P}$ and $\mathbb{C}$ respectively. By Lemma 3.97, it's easy to see that X is a member of $\mathscr{L}(\mathbb{P})$ if and only if $\rho_X \simeq \varepsilon_X$. Furthermore, any X in $\mathbb{A}$ is a member of $Shriek_{\mathbb{C}}(D_{\mathbb{P}})$ if and only if ε_X is $D_{\mathbb{P}}$-closed, i.e., $\rho_X \wedge \varepsilon_X \simeq D_{\mathbb{P}}(\varepsilon_X) \simeq \varepsilon_X$. Since $\mathbb{C}$ is a subcategory of $\mathbb{B}$, $\rho_X \leq \varepsilon_X$ always holds. Hence, X is in $\mathscr{L}(\mathbb{P})$ if and only if $\rho_X \simeq \varepsilon_X$ if and only if $\varepsilon_X \simeq \rho_X \wedge \varepsilon_X \simeq D_{\mathbb{P}}\varepsilon_X \simeq \rho_X$ if and only if X is a member of $Shriek_{\mathbb{C}}(D_{\mathbb{P}})$. □

Corollary 4.94: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and let $\mathbb{C}$ be a constant reflective subcategory. Then, for any dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, there holds:

$$\mathscr{L}(Shriek^*_{\mathbb{C}}(D)) = Shriek_{\mathbb{C}}(D_{Shriek^*_{\mathbb{C}}(D)}).$$

Proof : By Proposition 4.91, it follows that $Shriek^*_{\mathbb{C}}(D) = Shriek^*_{\mathbb{C}}(D_{Shriek^*_{\mathbb{C}}(D)})$. Consequently, $\mathscr{L}(Shriek^*_{\mathbb{C}}(D)) = \mathscr{L}(Shriek^*_{\mathbb{C}}(D_{Shriek^*_{\mathbb{C}}(D)})) = Shriek_{\mathbb{C}}(D_{Shriek^*_{\mathbb{C}}(D)})$, where the last step follows from Proposition 4.93. □

Proposition 4.95: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ a constant reflective subcategory. Then, for any dual closure operator D we have $Shriek_{\mathbb{C}}(D) \subset \mathscr{L}(Shriek^*_{\mathbb{C}}(D))$.

Proof : Let D be a dual closure operator and let X be a member of $Shriek_{\mathbb{C}}(D)$. We know that $Shriek^*_{\mathbb{C}}(D)$ is $\mathscr{E}$-reflective. Hence, let $S : \mathbb{A} \to Shriek^*_{\mathbb{C}}(D)$ and $C : \mathbb{A} \to \mathbb{C}$ be reflectors with units ρ and ε, respectively. Then, ε_X is D-closed, or, equivalently, δ_{ε_X} is an isomorphism. Let $\rho_X : X \to SX$ be a reflection for X. In view of Lemma 3.97, it's sufficient to prove that SX is a member of $\mathbb{C}$. Since SX is a member of $Shriek^*_{\mathbb{C}}(D)$, it follows that $D(\varepsilon_{SX})$ is an isomorphism. Let $k := D(\varepsilon_{SX})^{-1} \circ D_{\rho_X, C\rho_X} \circ \delta^{-1}_{\varepsilon_X}$. We assert that $k : CX \to SX$ is a morphism such that $k\varepsilon_X = \rho_X$. By the big square lemma 4.89, it's easy to see that
$k\varepsilon_X = D(\varepsilon_{SX})^{-1} \circ D_{\rho_X, C\rho_X} \circ \delta^{-1}_{\varepsilon_X}\varepsilon_X = D(\varepsilon_{SX})^{-1} \circ D_{\rho_X, C\rho_X} \circ D(\varepsilon_X) = D(\varepsilon_{SX})^{-1} \circ D(\varepsilon_{SX}) \circ \rho_X = \rho_X$.
It is easy to see that k is a member of $\mathscr{E}$ and since $\mathbb{C}$ is closed under $\mathscr{E}$-images, SX must be in $\mathbb{C}$. Thus, X is a member of $\mathscr{L}(Shriek^*_{\mathbb{C}}(D))$. □

Proposition 4.96: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and let $\mathbb{C}$ be a reflective subcategory of $\mathbb{A}$. Then, for any dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, $Shriek_{\mathbb{C}}(D)$ contains $\mathbb{C}$.

Proof : Let $\mathbb{C}$ be reflective and X a member of $\mathbb{C}$. Then, if ε_X is a $\mathbb{C}$-reflection for X, it follows that ε_X is an isomorphism. Hence, $\varepsilon_X \leq id_X \leq D\varepsilon_X$, thus $D\varepsilon_X \simeq \varepsilon_X$. □

Proposition 4.97: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with constant reflective subcategory $\mathbb{C}$. Then $Shriek_{\mathbb{C}}(D)$ is closed under $\mathscr{E}$-images provided that $\mathscr{E}$ is a class of epimorphisms.

Proof : Let D be a dco of $\mathscr{E}$ in $\mathbb{A}$ and let ε be the unit of the reflector $C : \mathbb{A} \to \mathbb{C}$. Let $e : X \to Y$ be in $\mathscr{E}$ with X in $Shriek_{\mathbb{C}}(D)$. Since X is a member of $Shriek_{\mathbb{C}}(D)$, we have that δ_{ε_X} is an isomorphism. Since δ_{ε_X} is an isomorphism, $D_X(CX)$ is a member of $\mathbb{C}$. Since $\mathbb{C}$ is closed under $\mathscr{E}$-images and $D_{e,Ce}$ is in $\mathscr{E}$, it follows that $D_Y(CY)$ is in $\mathbb{C}$. Since ε_Y is a reflection morphism for Y and $D\varepsilon_Y : Y \to D_Y(CY)$ is a morphism, we have a unique morphism $g : CY \to D_Y(CY)$ such that $g\varepsilon_Y = D\varepsilon_Y$. Since $D\varepsilon_Y$ and ε_Y are members of $\mathscr{E}$, it follows that $g \in \mathscr{E} \subset Epi(\mathbb{A})$.

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "D\varepsilon_X"'] \arrow[dd, bend right=60, "\varepsilon_X"'] & Y \arrow[d, "D\varepsilon_Y"] \arrow[dd, bend left=60, "\varepsilon_Y"] \\
D_X(CX) \arrow[r, "D_{e,Ce}"] \arrow[d, "\delta_{\varepsilon_X}"'] & D_Y(CY) \arrow[d, shift left, "\delta_{\varepsilon_Y}"] \\
CX \arrow[r, "Ce"'] & CY \arrow[u, dashed, shift left, "g"]
\end{tikzcd}
$$

Then, $\delta_{\varepsilon_Y} g \circ \varepsilon_Y = \delta_{\varepsilon_Y} D\varepsilon_Y = id_{CY} \circ \varepsilon_Y$. Thus, $\delta_{\varepsilon_Y} g = id_{CY}$ and since $\mathscr{E}$ is a class of epimorphisms, g is an epic section, i.e., g is an isomorphism. A straightforward calculation shows that $\delta_{\varepsilon_Y} = g^{-1}$ so that δ_{ε_Y} is an isomorphism. This shows that Y is a member of $Shriek_{\mathbb{C}}(D)$. $\square$

Lemma 4.98: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E}$ a class of epimorphisms. Let $\mathbb{C}$ be reflective in $\mathbb{A}$ with reflector R and unit ε. Then, the assignment $f : A \to B \overset{F}{\longmapsto} (f, Rf) : \varepsilon_A \to \varepsilon_B$ defines a functor $F : \mathbb{A} \to \mathscr{E}$. Furthermore, if D is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$, then the endofunctor $cod \circ D \circ F : \mathbb{A} \to \mathbb{A}$ preserves chained sinks and epi-sinks.
$\underline{\mathcal{P}roof}$: It's easy to see that the assignment described above defines a functor F from $\mathbb{A}$ to $\mathscr{E}$.

Let $(f_i : X_i \to X)_I$ be a chained sink or an epi-sink and let D be a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Since functors preserve chained sinks, we know that $(cod \circ D \circ F)((f_i)_I) = cod(D((f_i, Rf_i))_I) = (D_{f_i,Rf_i})_I$ is chained. Therefore, we need only prove that $(D_{f_i,Rf_i})_I$ is an epi-sink.

Let $f, g : DRX \rightrightarrows A$ be morphisms such that $f \circ (D_{f_i,Rf_i})_I = g \circ (D_{f_i,Rf_i})_I$. Then, for each $i \in I$, there holds: $f \circ \varepsilon_X \circ f_i = f \circ D_{f_i,Rf_i} \varepsilon_{X_i} = g \circ D_{f_i,Rf_i} \varepsilon_{X_i} = g \circ \varepsilon_X \circ f_i$. Since ε_X is an epimorphism and $(f_i)_I$ is an epi-sink, it follows that $f = g$. Consequently, $(D_{f_i,Rf_i})_I$ is an epi-sink in $\mathbb{A}$. $\square$

Theorem 4.99: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with $\mathscr{M}$ a class of monomorphisms. Let $\mathbb{C}$ be a reflective constant subcategory of $\mathbb{A}$ that is closed under $\mathbb{C}$-chained episinks in $\mathbb{A}$. Suppose that $\mathbb{A}$ satisfies the following conditions:

(i) $\mathbb{A}$ is $\mathscr{M}$-wellpowered;

(ii) ($\mathbb{A}$ has multiple pushouts and multiple pullbacks) or $\mathbb{A}$ has coproducts.

Then, $Shriek_{\mathbb{C}}(D)$ is nearly multi-$\mathscr{M}$-coreflective for each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$.

$\underline{\mathcal{P}roof}$: Let D be a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ and let $\mathbb{B} = Shriek_{\mathbb{C}}(D)$. Let R denote the reflector into $\mathbb{C}$ and let its unit be ε. Consider the full subcategory $\mathbb{B}_{\mathscr{M}}/X$ of the comma category $\mathbb{B}/X$ consisting of all $m : B \to X$ with B in $\mathbb{B}$ and $m \in \mathscr{M}$. Since $\mathbb{A}$ is $\mathscr{M}$-wellpowered, we may assume that the distinct equivalence classes of the least $\mathscr{M}$-subobject relation of $\mathbb{B}$ in $\mathbb{A}$ can be represented by a set, say $I(X)$. Each of these equivalence classes may be viewed as a sink of members of $\mathscr{M}$. Let $(m_j : B_j \to X)_J$ be the sink of all members of one of the equivalence classes under the least subobject relation $\sim_{X,\mathbb{B}}$. By property (ii) above, Lemma 3.141 provides us with an epic join $m : A \to X$ of $(m_j)_J$ in $\mathscr{M}/X$. Hence there exists an epi-sink $(k_j : B_j \to A)_J$ such that $m \circ k_j = m_j$ for each $j \in J$. Consider the commutative diagram:

$$
\begin{tikzcd}
B_j \arrow[r, "k_j"] \arrow[d, "D\varepsilon_{B_j}"'] & A \arrow[d, "D\varepsilon_A"] \\
DRB_j \arrow[r, "D_{k_j,Rk_j}"] \arrow[d, "\delta_{\varepsilon_{B_j}}"'] & DRA \arrow[d, "\delta_{\varepsilon_A}"] \\
RB_j \arrow[r, "Rk_j"'] & RA
\end{tikzcd}
$$

By definition of our equivalence relation, it follows that $(m_j : B_j \to X)_J$ is a $\mathbb{B}$-chained sink and it follows that $(k_j : B_j \to A)_J$ is $\mathbb{B}$-chained as well. It follows from Lemma 4.98 that $(D_{k_j,Rk_j} : DRB_j \to DRA)_J$ is a chained epi-sink in $\mathbb{A}$. Since B_j is a member of $Shriek_{\mathbb{C}}(D)$ for each $j \in J$, it follows that $D\varepsilon_{B_j} \simeq \varepsilon_{B_j}$, or, equivalently, $\delta_{\varepsilon_{B_j}}$ is an isomorphism. It's then easy to see that $(D_{k_j,Rk_j} : DRB_j \to DRA)_J$ is $\mathbb{C}$-chained in $\mathbb{A}$.

Since $\mathbb{C}$ is closed under $\mathbb{C}$-chained epi-sinks by assumption, it follows that DRA is a member of $\mathbb{C}$. Since ε_A is the $\mathbb{C}$-reflection, we have $\varepsilon_A \leq D\varepsilon_A$. But $D\varepsilon_A \leq \varepsilon_A$ as well, and it follows that $D\varepsilon_A \simeq \varepsilon_A$. Thus A is a member of $\mathbb{B} = \mathit{Shriek}_{\mathbb{C}}(D)$.

$$\begin{tikzcd}
B_j \arrow[r, "D\varepsilon_{B_j}"] \arrow[d, "k_j"'] & DRB_j \arrow[r, "\delta_{\varepsilon_{B_j}}"] \arrow[d, "D_{k_j,Rk_j}"'] & RB_j \arrow[d, "Rk_j"] \\
A \arrow[r, "D\varepsilon_A"] \arrow[d, "m"'] & DRA \arrow[r, "\delta_{\varepsilon_A}"] \arrow[d, "D_{m,Rm}"'] & RA \arrow[d, "Rm"] \\
X \arrow[r, "D\varepsilon_X"'] & DRX \arrow[r, "\delta_{\varepsilon_X}"'] & RX
\end{tikzcd}$$

So far we have shown that there is a small collection $I(X)$ of the equivalence classes and each of these has a terminal object given by the join. Let us denote the terminal object of the corresponding equivalence class $i \in I(X)$ by η_i. If $|I(X)| \leq 1$, let $J(X) = I(X)$ and if $I(X)$ contains two or more equivalence classes, let $J(X) = I(X) \setminus \{[m_0 : B_0 \to X]_{\sim_{X,\mathbb{B}}}\}$, where m_0 is the least $\mathscr{M}$-subobject of X, if it exists.

We now proceed to show that each morphism $f : B \to X$ with B in $\mathit{Shriek}_{\mathbb{C}}(D)$ factors through each η_j for each $j \in J(X)$ or f factors through a unique η_j. Since $\mathscr{M}$ is a class of monomorphisms, such a factorisation is unique.

To this end, let $f : B \to X$ be a morphism as described above. Let $n \circ e : B \to B' \to X$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of f. Then, by Proposition 4.97, it follows that B' is a member of $\mathit{Shriek}_{\mathbb{C}}(D)$. If $n : B' \to X$ is the least $\mathscr{M}$-subobject, then n, and consequently f, factors through each η_j for each $j \in J(X)$. If $n : B' \to X$ is not the least $\mathscr{M}$-subobject, then the equivalence class containing n is a member of $J(X)$ and n factors through a terminal object η_j for some $j \in J(X)$. We assert that this η_j is unique, for if n factors through η_j and $\eta_{j'}$ for some $j, j' \in J(X)$, then there is a finite $\mathbb{B}$-zig-zag, namely $\eta_j, n, \eta_{j'}$, from η_j to $\eta_{j'}$. Of course this implies that $j = j'$ so that j, and hence η_j, is unique. Therefore, $(\eta_j : B_j \to X)_{j \in J(X)}$ is the near multi-$\mathscr{M}$-coreflection of X in $\mathit{Shriek}_{\mathbb{C}}(D)$. □

Corollary 4.100: Let $\mathbb{A}$ satisfy the assumptions of Theorem 4.99. Then, each left constant subcategory is nearly multi-$\mathscr{M}$-coreflective.

Proof : This follows from the fact that if $\mathbb{Q}$ is a left constant subcategory, then Proposition 4.93 gives that $\mathbb{Q} = \mathscr{L}(\mathscr{R}(\mathbb{Q})) = \mathit{Shriek}_{\mathbb{C}}(D_{\mathscr{R}(\mathbb{Q})})$. Theorem 4.99 then implies that $\mathbb{Q}$ is nearly multi-$\mathscr{M}$-coreflective in $\mathbb{A}$. □

Corollary 4.101: Let $\mathbb{A}$ be a cocomplete $(\mathscr{E}, \mathbb{M})$-category that is $\mathscr{M}$-wellpowered. Let $\mathbb{C}$ be a reflective constant subcategory that is closed under epi-sinks. Furthermore assume that $\mathbb{A}$ has an initial object I_0 in $\mathbb{C}$. Then $\mathit{Shriek}_{\mathbb{C}}(D)$ is $\mathscr{M}$-coreflective for each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$.

Proof : Since cocompleteness forces $\mathbb{M}$ to be a class of mono-sources, $\mathbb{A}$ satisfies all the hypothesis of Theorem 4.99. Hence $\mathit{Shriek}_{\mathbb{C}}(D)$ is a nearly multi-$\mathscr{M}$-coreflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. To see that each $\mathbb{A}$-object has a least $\mathscr{M}$-subobject, let X be an $\mathbb{A}$-object. Let $m_X e_X : I_0 \to M_0^X \to X$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of the unique morphism $I_0 \to X$. If $m : A \to X$ is any member of $\mathscr{M}$, then by initiality of I_0, there is a unique morphism $\iota_A : I_0 \to A$ and $m\iota_A = m_X e_X$ holds as well. The diagonalisation property establishes a unique morphism d such that $md = m_X$ and $de_X = \iota_A$. It should be clear that $m_X \leq m$. Hence m_X is the least $\mathscr{M}$-subobject of X. Since e_X is in $\mathscr{E}$ and I is in $\mathbb{C}$, it follows that $m_X : M_0^X \to X$ is the least $\mathscr{M}$-subobject of X in $\mathbb{A}$ and M_0^X is in $\mathbb{C}$ and thus in $\mathit{Shriek}_{\mathbb{C}}(D)$. This also implies that the near multi-$\mathscr{M}$-coreflection of X in $\mathit{Shriek}_{\mathbb{C}}(D)$ is non-empty when considered as a sink to X.

We will now show that each $\mathbb{A}$-object X has an $\mathscr{M}$-coreflection in $Shriek_{\mathbb{C}}(D)$. Let X be in $\mathbb{A}$. If there is only one equivalence class in the least $\mathscr{M}$-subobject relation of $Shriek_{\mathbb{C}}(D)$ in $\mathbb{A}$, then we are obviously done. Henceforth assume that there is at least one $\mathscr{M}$-subobject $\hat{m} : \hat{B} \to X$ with $\hat{m} > m_0$, where m_0 is the least $\mathscr{M}$-subobject of X, with $\hat{B}$ in $Shriek_{\mathbb{C}}(D)$. Since $\mathbb{A}$ is $\mathscr{M}$-wellpowered, we may consider a skeleton $\mathbb{I}$ of the category of all morphisms in $\mathscr{M}$ and domain in $Shriek_{\mathbb{C}}(D)$. Of course we may view this as a small sink $(m_i : B_i \to X)_I$, with $I = Ob(\mathbb{I})$. Since $\mathbb{A}$ has coproducts, we can construct the join $m : M \to X$ of $(m_i)_I$ as the $\mathscr{M}$-part of the $(\mathscr{E}, \mathscr{M})$-factorisation of the unique morphism $c : \coprod_I B_i \to X$ such that $c \circ \amalg_i = m_i$. We show that m is the $\mathscr{M}$-coreflection.

Let R be the reflector from $\mathbb{A}$ into $\mathbb{C}$ and let the unit be denoted by ε. Let $(\amalg_i : B_i \to \coprod_I B_i)_I$ be a coproduct sink of $(B_i)_I$ in $\mathbb{A}$. Since reflectors preserve coproducts and B_i is a member of $Shriek_{\mathbb{C}}(D)$, $DRB_i \simeq RB_i$, and thus $R(\coprod_I B_i)$, together with the sinks $(R\amalg_i)_I$ or $(R\amalg_i \delta_{\varepsilon_{B_i}})_I$, are both coproducts of $(RB_i)_I$, respectively, $(DRB_i)_I$, in $\mathbb{C}$. Lemma 4.98 gives that $(D_{\amalg_i, R\amalg_i})_I$ is an episink and since $\mathbb{C}$ is closed under epi-sinks by assumption, it follows that $DR\coprod_I B_i$ is a member of $\mathbb{C}$. Hence, $\varepsilon_{\coprod_I B_i} \leq D\varepsilon_{\coprod_I B_i}$, or, equivalently, $D\varepsilon_{\coprod_I B_i} \simeq \varepsilon_{\coprod_I B_i}$. Thus, $\coprod_I B_i$ is in $Shriek_{\mathbb{C}}(D)$ and since $Shriek_{\mathbb{C}}(D)$ is closed under $\mathscr{E}$-images, it follows that M is a member of $Shriek_{\mathbb{C}}(D)$.

$$\begin{tikzcd}
 & & RB_i \arrow[r, "R\amalg_i"] & R(\coprod_I B_i) \arrow[d, "R\bar{e}"] \\
B \arrow[d, "he"] \arrow[dd, bend right, "f"'] & DRB_i \arrow[ur, "\delta_{\varepsilon_{B_i}}"] \arrow[r, "D_{\amalg_i, R\amalg_i}"] & DR\coprod_I B_i \arrow[ur, "\delta_{\varepsilon_{\coprod_I B_i}}"] \arrow[d, "D_{\bar{e}, R\bar{e}}"] & RM \\
B_i \arrow[d, "m_i"] \arrow[ur, "D\varepsilon_{B_i}"] \arrow[r, "\amalg_i"] & \coprod_I B_i \arrow[ur, "D\varepsilon_{\coprod_I B_i}"] \arrow[d, "\bar{e}"] & DRM \arrow[ur, "\delta_{\varepsilon_M}"] & \\
X & M \arrow[l, "m"] \arrow[ur, "D\varepsilon_M"] & &
\end{tikzcd}$$

Let $f : A \to X$ be any morphism with A in $Shriek_{\mathbb{C}}(D)$. We show that it factors uniquely through m. Suppose f has $(\mathscr{E}, \mathscr{M})$-factorisation $n \circ e$. Since $Shriek_{\mathbb{C}}(D)$ is closed under $\mathscr{E}$-images, $n \simeq m_i$ for a unique $i \in I$. Thus, there is an isomorphism h such that $m_i h = n$. Hence, $m_i \circ (he)$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of f and $f = m_i \circ he = (m\bar{e}\amalg_i)he = m(\bar{e}\amalg_i he)$, where $m\bar{e}$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of $c : \coprod_I B_i \to X$. Hence f factors through m and since $\mathscr{M}$ is a class of monomorphisms, this factorisation is unique. Therefore m is the desired $\mathscr{M}$-coreflection, for X, of $Shriek_{\mathbb{C}}(D)$ in $\mathbb{A}$. $\square$

Proposition 4.102: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with $\mathscr{E}$ a class of epimorphisms and $\mathbb{C}$ a reflective subcategory. Then

$$Shriek_{\mathbb{C}}(-) : (DCO(\mathbb{A}, \mathscr{E}), \leq) \to (Sub(\mathbb{A}), \subset)$$

is an order preserving map that preserves meets. Therefore $Shriek_{\mathbb{C}}(-)$ has a left adjoint.

Proof : Let $R : \mathbb{A} \to \mathbb{C}$ denote the reflector with unit ρ. Then, for any X in $\mathbb{A}$, let $\eta_X \circ \varepsilon_X$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of ρ_X. Then, the diagonalisation property then gives us a reflector $T : \mathbb{A} \to \mathscr{M}(\mathbb{C})$ with unit ε_X.

Let $D \leq D'$ in $DCO(\mathbb{A}, \mathscr{E})$ and let X be in $Shriek_{\mathbb{C}}(D)$, i.e., δ_{ε_X} is an isomorphism. Since $D \leq D'$, there is a unique morphism $j : DTX \to D'TX \in \mathscr{E}$ such that $jD\varepsilon_X = D'\varepsilon_X$. Then $\delta'_{\varepsilon_X} jD\varepsilon_X = \delta'_{\varepsilon_X} D'\varepsilon_X = \varepsilon_X = \delta_{\varepsilon_X} D\varepsilon_X$, and since $\mathscr{E}$ is a class of epimorphisms, $\delta'_{\varepsilon_X} j = \delta_{\varepsilon_X} \in \mathscr{M} \cap \mathscr{E} = Iso(\mathbb{A})$. Thus, there is a morphism k such that $k\delta'_{\varepsilon_X} j = id$ and $\delta'_{\varepsilon_X} jk = id$. It follows that j is a section and since $\mathscr{E} \subset Epi(\mathbb{A})$, j must be an epic section, or, equivalently, an isomorphism. Hence $\delta'_{\varepsilon_X} = \delta'_{\varepsilon_X} jj^{-1} = \delta_{\varepsilon_X} j^{-1}$ is a composition of isomorphisms and hence an isomorphism as well. Thus, X is a member of $Shriek_{\mathbb{C}}(D')$, and $Shriek_{\mathbb{C}}(-)$ is order preserving.

We now show that $Shriek_{\mathbb{C}}(-)$ preserves meets. Let $(D_i)_I$ be a family of dual closure operators of $\mathscr{E}$ in $\mathbb{A}$. Let $D_1 = \bigwedge_I D_i$. Since $Shriek_{\mathbb{C}}(-)$ is an order preserving map, it follows that for each $i \in I$, $Shriek_{\mathbb{C}}(D_1) \subset Shriek_{\mathbb{C}}(D_i)$. Consequently, it follows that $Shriek_{\mathbb{C}}(D_1) \subset \bigcap_I Shriek_{\mathbb{C}}(D_i)$. We show that the reverse inclusion also holds.

Let $X \in \bigcap_I \mathit{Shriek}_{\mathbb{C}}(D_i)$. Then, for each $i \in I$, there holds $D_i\varepsilon_X \simeq \varepsilon_X$. Of course, $D_1\varepsilon_X$ is the meet of $(D_i\varepsilon_X)_I$ so that $D_1\varepsilon_X \simeq \varepsilon_X$. It follows that X is a member of $\mathit{Shriek}_{\mathbb{C}}(D_1)$ and therefore $\mathit{Shriek}_{\mathbb{C}}(-)$ preserves meets. That $\mathit{Shriek}_{\mathbb{C}}(-)$ has a left adjoint is a standard result on Galois connections. □

Remark 4.103: Proposition 4.102 shows that $\mathit{Shriek}_{\mathbb{C}}(-)$ has a left adjoint. Of course, this left adjoint is defined for any subcategory $\mathbb{B}$ of $\mathbb{A}$ by $\bigwedge\{D \in DCO(\mathbb{A}, \mathscr{E}) \mid \mathbb{B} \subset \mathit{Shriek}_{\mathbb{C}}(D)\}$. We can always find a left or right adjoint in this manner, but this is not very satisfying. This can also make it more tedious to study properties of not only the left adjoint, but the adjunction itself. For this reason, we will now explicitly construct the left adjoint of $\mathit{Shriek}_{\mathbb{C}}(D)$.

Lemma 4.104: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category, or, equivalently, suppose $\mathbb{A}$ is $\mathscr{E}$-cocomplete. Let $\mathbb{C}$ be a reflective constant subcategory of $\mathbb{A}$ with reflector R and unit ε. For each subcategory $\mathbb{B}$ of $\mathbb{A}$ and each $p : X \to P$ in $\mathscr{E}$, consider the sink $(u_i)_I$, of all morphisms $u : B \to X$ with domain in $\mathbb{B}$, such that $p \circ u$ factors as $m \circ \varepsilon_B$. For each $i \in I$, let $u_{i-}(\varepsilon_{B_i})$ be the pushout of ε_{B_i} along u_i. Let $D^{\mathbb{B}}_{\mathbb{C}}(p)$ be the multiple pushout of the family $(u_{i-}(\varepsilon_{B_i}))_I$. Then the assignment $p \longmapsto D^{\mathbb{B}}_{\mathbb{C}}(p)$ defines an idempotent dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

<u>*Proof* :</u> Note that $\mathbb{C}$ is actually $\mathscr{E}$-reflective, since it's closed under subobjects in $\mathbb{M}$. Since $\mathscr{E}$ is closed under pushouts and multiple pushouts, and $\mathbb{C}$ is $\mathscr{E}$-reflective in $\mathbb{A}$, it should be clear that $D^{\mathbb{B}}_{\mathbb{C}}(p)$ is a member of $\mathscr{E}$ and $D^{\mathbb{B}}_{\mathbb{C}}(p) \leq p$. Suppose that $p \leq q$ in $X/\mathscr{E}$, with j a morphism such that $jp = q$. Let $(u^p_i : B_i \to X)_I$ be the sink of all morphisms with domain in $\mathbb{B}$ such that pu^p_i factors as $m^p_i \circ \varepsilon_{B_i}$. For each $i \in I$, we have that $q \circ u^p_i = jpu^p_i = jm^p_i \circ \varepsilon_{B_i}$. Let $(u^q_j : B_j \to X)_J$ be the sink of all morphisms u^q_j, with domain in $\mathbb{B}$, such that qu^q_j factors as $m^q_j \circ \varepsilon_{B_j}$. Then we may, without loss of generality, assume that $I \subset J$ and hence that $(u^p_i)_I$ is a subsink of $(u^q_j)_J$. We may also assume that for each $i \in I$ there holds: $u^p_i = u^q_i$ and $jm^p_i = m^q_i$. Then, for each $i \in I$, we have $u^p_{i-}(\varepsilon_{B_i}) \simeq u^q_{i-}(\varepsilon_{B_i})$.

Hence, $D^{\mathbb{B}}_{\mathbb{C}}(p) \simeq \bigvee_I u^p_{i-}(\varepsilon_{B_i}) \simeq \bigvee_I u^q_{i-}(\varepsilon_{B_i}) \leq \bigvee_J u^q_{j-}(\varepsilon_{B_j}) \simeq D^{\mathbb{B}}_{\mathbb{C}}(q)$, so that $D^{\mathbb{B}}_{\mathbb{C}}$ is an order preserving map.

Let $f : X \to Y$ be an $\mathbb{A}$-morphism and let $p : X \to P$ be in $\mathscr{E}$. In view of 4.8, we need only show that $f_-(D^{\mathbb{B}}_{\mathbb{C}}(p)) \leq D^{\mathbb{B}}_{\mathbb{C}}(f_-(p))$ holds. Let $p' : Y \to P'$ be a pushout of p along f, i.e., $p' := f_-(p)$. Suppose that the pushout square is given by

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & P \arrow[d, "r"] \\ Y \arrow[r, "p'"'] & P' \end{tikzcd}$$

Let $(u_i : B_i \to X)_I$, respectively, $(u'_j : B'_j \to Y)_J$, be the sinks with domain in $\mathbb{B}$ such that for each $i \in I$, pu_i factors as $m_i\varepsilon_{B_i}$ respectively, for each $j \in J$, $p'u'_j$ factors as $m'_j\varepsilon_{B'_j}$. For each $i \in I$, we have that $p' \circ (fu_i) = (rm_i) \circ \varepsilon_{B_i}$. It's then easy to see that we may, without loss of generality, assume that I is a subclass of J and for each $i \in I$, there holds, $fu_i = u'_i$ and $rm_i = m'_i$. It follows that:

$$\begin{aligned} f_-(D^{\mathbb{B}}_{\mathbb{C}}(p)) &\simeq f_-(\textstyle\bigvee_I u_{i-}(\varepsilon_{B_i})) \\ &\simeq \textstyle\bigvee_I f_-(u_{i-}(\varepsilon_{B_i})) \text{ by 4.6} \\ &\simeq \textstyle\bigvee_I (fu_i)_-(\varepsilon_{B_i}) \\ &\simeq \textstyle\bigvee_I u'_{i-}(\varepsilon_{B'_i}) \\ &\leq \textstyle\bigvee_J u'_{j-}(\varepsilon_{B'_j}) \\ &\simeq D^{\mathbb{B}}_{\mathbb{C}}(p') \\ &\simeq D^{\mathbb{B}}_{\mathbb{C}}(f_-(p)). \end{aligned}$$

It follows that $D^{\mathbb{B}}_{\mathbb{C}}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

To finish our proof, we show that $D^{\mathbb{B}}_{\mathbb{C}}(p) \simeq D^{\mathbb{B}}_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}}(p))$, for any $p \in \mathscr{E}$. Let $p : X \to P$ be a member of $\mathscr{E}$ and $(u_j : B_j \to X)_J$ be the sink of all morphisms $u_j : B_j \to X$, with domain in $\mathbb{B}$, such that $D^{\mathbb{B}}_{\mathbb{C}}(p) \circ u_j$ factors as $m_j \circ \varepsilon_{B_j}$. For each $e \in \mathscr{E}$, let $\delta^{\mathbb{B}}_e$ denote the unique morphism such that $\delta^{\mathbb{B}}_e \circ D^{\mathbb{B}}_{\mathbb{C}}(e) = e$. Let $j \in J$ and note that $p \circ u_j = \delta^{\mathbb{B}}_p \circ D^{\mathbb{B}}_{\mathbb{C}}(p) \circ u_j = \delta^{\mathbb{B}}_p \circ m_j \circ \varepsilon_{B_j}$. It should then be clear that we may assume that the sink $(u_j)_J$ is a subsink of the sink $(u_i)_I$ of all morphisms $u_i : B_i \to X$, with domain in $\mathbb{B}$, such that pu_i factors as $m'_i \circ \varepsilon_{B_i}$. We use the notation in the following diagram:

$$\begin{tikzcd}
B_i \arrow[r, "\varepsilon_{B_i}"] \arrow[d, "u_i"'] & RB_i \arrow[d, "v_i"] \arrow[ddd, bend left, "m'_i"] \\
X \arrow[r, "u_{i-}(\varepsilon_{B_i})"] \arrow[dr, "D^{\mathbb{B}}_{\mathbb{C}}(p)"'] \arrow[ddr, bend right, "p"'] & u_{i-}(RB_i) \arrow[d, "p_i"] \arrow[dd, dashed, bend left, "\bar{p}_i"] \\
 & D^{\mathbb{B}}_{\mathbb{C}}(P) \arrow[d, dotted, "\delta^{\mathbb{B}}_p"] \\
 & P
\end{tikzcd} \tag{1}$$

We now show that $I \subset J$. Let $i \in I$ and consider the pushout square:

$$\begin{tikzcd}
B_i \arrow[r, "\varepsilon_{B_i}"] \arrow[d, "u_i"'] & RB_i \arrow[d, "v_i"] \\
X \arrow[r, "u_{i-}(\varepsilon_{B_i})"] & u_{i-}(RB_i)
\end{tikzcd}$$

Then $D^{\mathbb{B}}_{\mathbb{C}}(p)u_i = p_i u_{i-}(\varepsilon_{B_i})u_i = (p_i v_i)\varepsilon_{B_i}$ so that u_i is already a member of $(u_j)_J$, which implies that $i \in J$. Therefore the sinks coincide and since the pushouts don't depend on m_i and m'_i, it follows that $D^{\mathbb{B}}_{\mathbb{C}}(p) \simeq D^{\mathbb{B}}_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}}(p))$. Consequently, $D^{\mathbb{B}}_{\mathbb{C}}$ is an idempotent dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. $\square$

Proposition 4.105: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and $\mathbb{C}$ a reflective constant subcategory with reflector and unit R and ε respectively. Let $\mathbb{B}$ be nearly multi-$\mathscr{M}$-coreflective in $\mathbb{A}$ and $p : X \to P$ a member of $\mathscr{E}$. Let $(m_i : RB_i \to P)_I$ be the sink of all morphisms from $R(\mathbb{B})$ to P, i.e., from $Ob(R(\mathbb{B}))$. For each $i \in I$, let $p^{-1}(m_i) : p^{-1}(RB_i) \to X$ be the pullback of m_i along p. For each such i, let $(\eta^i_k : B^i_k \to p^{-1}(RB_i))_{K(p^{-1}(RB_i))}$ denote the near multi-$\mathscr{M}$-coreflection of $p^{-1}(RB_i)$ in $\mathbb{B}$. For each $i \in I$ and each $k \in K(p^{-1}(RB_i))$, consider the pushout $(p^{-1}(m_i) \circ \eta^i_k)_-(\varepsilon_{B^i_k})$. Then, $D^{\mathbb{B}}_{\mathbb{C}}(p) \simeq \bigvee_{i \in I, k \in K(p^{-1}(RB_i))}(p^{-1}(m_i) \circ \eta^i_k)_-(\varepsilon_{B^i_k})$.

Proof : We will abuse notation and write $p^{-1}(m_i)$, even though there is no reason that it needs to be a member of $\mathscr{M}$. Hence, we denote the pullback square of m_i along p by
$$\begin{tikzcd}
p^{-1}(RB_i) \arrow[r, "p^{-1}(m_i)"] \arrow[d, "\bar{p}_i"'] & X \arrow[d, "p"] \\
RB_i \arrow[r, "m_i"'] & P
\end{tikzcd}.$$

Let B be a member of $\mathbb{B}$ and $u : B \to X$ a morphism such that $p \circ u = m \circ \varepsilon_B$. Then, by definition of the sink $(m_i)_I$, there is an $i \in I$ such that $m = m_i$. Hence, by the pullback property, there is a unique morphism $f_i : B \to p^{-1}(RB_i)$ such that $p^{-1}(m_i) \circ f_i = u$ and $p_i \circ f_i = \varepsilon_B$.

For each $i \in I$, let $(\eta^i_k : B^i_k \to p^{-1}(RB_i))_{K(p^{-1}(RB_i))}$ denote the near multi-$\mathscr{M}$-coreflections of $p^{-1}(RB_i)$ in $\mathbb{B}$. For each $i \in I$ and each $k \in K(p^{-1}(RB_i))$, define $v^i_k = p^{-1}(m_i)\eta^i_k$. Since f_i is a morphism from B to $p^{-1}(RB_i)$, there is a $k \in K(p^{-1}(RB_i))$ for which there is a unique morphism $c^i_k : B \to B^i_k$ such that $\eta^i_k \circ c^i_k = f_i$. Note that $p_i \circ \eta^i_k$ is a morphism from B^i_k to an object in $\mathbb{C}$. Hence there is a unique morphism $g_i : RB^i_k \to RB_i$ such that $g_i \circ \varepsilon_{RB^i_k} = p_i \circ \eta^i_k$.

$$
\begin{tikzcd}
B \arrow[ddd, bend right, "\varepsilon_B"'] \arrow[dr, dotted, "!c_k^i"'] \arrow[ddrr, dashed, "!f_i"] \arrow[ddrrr, bend left, "u"] & & & \\
& B_k^i \arrow[dd, "\varepsilon_{B_k^i}"'] \arrow[ddr, "\eta_k^i"] \arrow[ddrr, "v_k^i"] & & \\
& & & \\
RB \arrow[r, "Rc_k^i"] \arrow[ddrr, equal, bend right] & RB_k^i \arrow[ddr, "g_i"'] & p^{-1}(RB_i) \arrow[r, "p^{-1}(m_i)"] \arrow[dd, "p_i"] & X \arrow[dd, "p"] \\
& & & \\
& & RB_i \arrow[r, "m_i"'] & P
\end{tikzcd}
$$

Since $pv_k^i = m_i g_i \varepsilon_{B_k^i}$, we know that the pushout of $\varepsilon_{B_k^i}$ along v_k^i contributes to $D_{\mathbb{C}}^{\mathbb{B}}(p)$. Consider the pushout of ε_B along $u = v_k^i \circ c_k^i$. In view of the above statement, it is sufficient to prove that $u_-(\varepsilon_B) \leq v_{k-}^i(\varepsilon_{B_k^i})$. Consider the diagram of pushout squares:

$$
\begin{tikzcd}
RB_k^i \arrow[dddrrr, bend right, "\tilde{v}_k^i"'] & & & \\
& B \arrow[r, "c_k^i"] \arrow[d, "\varepsilon_B"'] \arrow[rr, bend left, "u"] & B_k^i \arrow[r, "v_k^i"] \arrow[d, "c_{k-}^i(\varepsilon_B)"'] \arrow[ull, bend right, "\varepsilon_{B_k^i}"'] & X \arrow[d, "v_{k-}^i(c_{k-}^i(\varepsilon_B))"', "\simeq u_-(\varepsilon_B)"] \arrow[dd, bend left=60, "v_{k-}^i(\varepsilon_{B_k^i})"] \\
& RB \arrow[r, "\bar{c}_k^i"'] \arrow[uul, "Rc_k^i"] & c_{k-}^i(RB) \arrow[r, "\bar{v}_k^i"'] & v_{k-}^i(c_{k-}^i(RB)) \arrow[d, dashed, "q"'] \\
& & & v_{k-}^i(RB_k^i)
\end{tikzcd}
$$

Since $\tilde{v}_k^i \circ Rc_k^i \varepsilon_B = \tilde{v}_k^i \circ \varepsilon_{B_k^i} \circ c_k^i = v_{k-}^i(\varepsilon_{B_k^i}) \circ v_k^i \circ c_k^i$, there is a unique morphism $q : v_{k-}^i(c_{k-}^i(\varepsilon_B)) \to v_{k-}^i(RB_k^i)$ that makes the diagram commute. It follows that $u_-(\varepsilon_B) \simeq v_{k-}^i(c_{k-}^i(\varepsilon_B)) \leq v_{k-}^i(\varepsilon_{B_k^i})$ and since pushouts act as joins, our conclusion follows. $\square$

Proposition 4.106: Let $\mathbb{C}$ be a reflective constant subcategory of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$. Then the map $D_{\mathbb{C}}^{(-)} : (Sub(\mathbb{A}), \subset) \to (DCO(\mathbb{A}, \mathscr{E}), \leq)$ is order preserving.

Proof : Suppose that $\mathbb{B}$ and $\mathbb{B}'$ are subcategories of $\mathbb{A}$ with $\mathbb{B} \subset \mathbb{B}'$. Let $p : X \to P$ be in $\mathscr{E}$ and $(u_j : B_j' \to X)_J$ be the sink of all morphisms with domain in $\mathbb{B}$ such that pu_j factors as $m_j' \circ \varepsilon_{B_j'}$. If B is any member of $\mathbb{B}$ and $u : B \to X$ a morphism such that pu factors as $m\varepsilon_B$, then it follows that $u = u_j$ for some $j \in J$. Hence, we may assume that the sink $(u_i : B_i \to X)_I$ of all morphisms with domain in $\mathbb{B}$ with pu_i factoring as $m_i \circ \varepsilon_{B_i}$ is a subsink of $(u_j)_J$. It should then be clear that we can also take $m_i = m_i'$ whenever $i \in I$. Then,

$$
\begin{aligned}
D_{\mathbb{C}}^{\mathbb{B}}(p) &\simeq \textstyle\bigvee_I u_{i-}(\varepsilon_{B_i}) \\
&\simeq \textstyle\bigvee_I u_{i-}(\varepsilon_{B_i'}) \\
&\leq \textstyle\bigvee_J u_{j-}(\varepsilon_{B_j'}) \\
&\simeq D_{\mathbb{C}}^{\mathbb{B}'}(p).
\end{aligned}
$$

Since p was arbitrary, our proof is complete. $\square$

Theorem 4.107: Let $\mathbb{C}$ be a reflective constant subcategory of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$. Then, for each dual closure operator D of $\mathscr{E}$ and subcategory $\mathbb{B}$ of $\mathbb{A}$, the following holds:

$$D_{\mathbb{C}}^{\mathbb{B}} \leq D \Leftrightarrow \mathbb{B} \subset Shriek_{\mathbb{C}}(D).$$

Consequently, $\mathbb{B} \subset Shriek_{\mathbb{C}}(D_{\mathbb{C}}^{\mathbb{B}})$ and $D_{\mathbb{C}}^{Shriek_{\mathbb{C}}(D)} \leq D$ for each subcategory $\mathbb{B}$ of $\mathbb{A}$ and each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$.

$\underline{\mathcal{P}roof}$: Throughout the proof we will denote the reflector and unit by $R : \mathbb{A} \to \mathbb{C}$ and ε respectively. We will constantly use the notation as in the following diagram:

$$\begin{tikzcd}
B_i \arrow[r, "\varepsilon_{B_i}"] \arrow[d, "u_i"'] & RB_i \arrow[d, "v_i"] \arrow[ddr, bend left, "m_i"] & \\
X \arrow[r, "u_{i-}(\varepsilon_{B_i})"] \arrow[dr, "D^{\mathbb{B}}_{\mathbb{C}}(p)"'] \arrow[ddrr, bend right=60, "p"'] & u_{i-}(RB_i) \arrow[d, "p_i"] \arrow[ddr, dashed, "\bar{p}_i"] & \\
& D^{\mathbb{B}}_{\mathbb{C}}(P) \arrow[dr, dotted, "\omega^{\mathbb{B}}_p"] & \\
& & P
\end{tikzcd} \tag{2}$$

Let $\mathbb{B}$ be a subcategory of $\mathbb{A}$ and D a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Assume that $D^{\mathbb{B}}_{\mathbb{C}} \leq D$ and consider $\varepsilon_X : X \to RX$ with X in $\mathbb{B}$. Since $\varepsilon_X \circ id_X = id_{RX} \circ \varepsilon_X$, it follows that id_X is a member of the sink $(u_i)_I$. Furthermore,

$$\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[d, "id_X"'] & RX \arrow[d, "id_{RX}"] \\
X \arrow[r, "\varepsilon_X"'] & RX
\end{tikzcd}$$

is also a pushout square, hence ε_X is a member of the family $(u_{i-}(\varepsilon_{B_i}))_I$ that contributes to the multiple pushout. Thus, $\varepsilon_X \leq \bigvee_I u_{i-}(\varepsilon_{B_i}) \leq \varepsilon_X$ so that $D^{\mathbb{B}}_{\mathbb{C}}(\varepsilon_X) \simeq \varepsilon_X$. Then, $\varepsilon_X \simeq D^{\mathbb{B}}_{\mathbb{C}}(\varepsilon_X) \leq D(\varepsilon_X) \leq \varepsilon_X$, where the last two steps follow since $D^{\mathbb{B}}_{\mathbb{C}} \leq D$ and D is a dual closure operator. This shows that ε_X is D-closed, or, equivalently, X is a member of $Shriek_{\mathbb{C}}(D)$. Since X was arbitrary, it follows that $\mathbb{B} \subset Shriek_{\mathbb{C}}(D)$.

Conversely assume that $\mathbb{B} \subset Shriek_{\mathbb{C}}(D)$ and let $p : X \to P$ be a member of $\mathscr{E}$. Consider the sink $(u_i : B_i \to X)_I$ of all morphisms u with domain in $\mathbb{B}$ and codomain X such that pu factors as $m \circ \varepsilon_B$. Since $\mathbb{B} \subset Shriek_{\mathbb{C}}(D)$, it follows that ε_{B_i} is D-closed for each $i \in I$. By Proposition 4.39, it follows that $u_{i-}(\varepsilon_{B_i})$ is D-closed for each $i \in I$ and also the multiple pushout $D^{\mathbb{B}}_{\mathbb{C}}(p)$. Therefore, $D^{\mathbb{B}}_{\mathbb{C}}(p)$ is D-closed for each $p \in \mathscr{E}$. Thus, $D^{\mathbb{B}}_{\mathbb{C}}(p) \simeq D(D^{\mathbb{B}}_{\mathbb{C}}(p)) \leq Dp$ and our proof is complete. $\square$

Proposition 4.108: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ a constant reflective subcategory of $\mathbb{A}$. Let $\mathbb{B}$ be a subcategory of $\mathbb{A}$ that is closed under $\mathscr{E}$-images. Let $R : \mathbb{A} \to \mathbb{C}$ denote the reflector with unit ε. Then, $D^{\mathbb{B}}_{\mathbb{C}} \leq D_{\mathbb{C}}$, where $D_{\mathbb{C}}(p) = p \wedge \varepsilon_X$ for each $p : X \to P$ in $\mathscr{E}$.

$\underline{\mathcal{P}roof}$: Let $p : X \to P$ be in $\mathscr{E}$ and let $(u_i : B_i \to X)_I$ be the sink of all morphisms with domain in $\mathbb{B}$ such that $pu_i = w_i \varepsilon_{B_i}$. Then, for each $i \in I$, the diagram

$$\begin{tikzcd}
B_i \arrow[r, "\varepsilon_{B_i}"] \arrow[d, "u_i"'] & RB_i \arrow[d, "v_i"] \arrow[ddr, bend left, "Ru_i"] & \\
X \arrow[r, "u_{i-}(\varepsilon_{B_i})"'] \arrow[drr, bend right, "\varepsilon_X"'] & u_-(RB_i) \arrow[dr, dashed, "!f_i"] & \\
& & RX
\end{tikzcd}$$

commutes. It is then clear that $u_{i-}(\varepsilon_{B_i}) \leq \varepsilon_X$ and thus $D^{\mathbb{B}}_{\mathbb{C}}(p) \simeq \bigvee_I u_{i-}(\varepsilon_{B_i}) \leq \varepsilon_X$. Since $D^{\mathbb{B}}_{\mathbb{C}}$ is a dual closure operator, it follows that $D^{\mathbb{B}}_{\mathbb{C}}(p) \leq p \wedge \varepsilon_X \simeq D_{\mathbb{C}}(p)$. $\square$

Proposition 4.109: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category, $\mathbb{C}$ a constant reflective subcategory of $\mathbb{A}$ and $\mathbb{B}$ an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$. If $\mathbb{A}$ has pullbacks, then the following hold:

(*i*) $D_{\mathbb{C}}^{\mathscr{L}(\mathbb{B})} \leq D_{\mathbb{B}}$;

(*ii*) $D_{\mathbb{C}}^{\mathscr{L}(\mathbb{B})} \leq D_{\mathbb{C}}$;

$\underline{\mathcal{P}roof}$: Let $p : X \to P$ be in $\mathscr{E}$ and let $R : \mathbb{A} \to \mathbb{C}$ and $S : \mathbb{A} \to \mathbb{B}$ denote the reflectors with units ε and ρ, respectively.

(*i*) First note that Q is in $\mathscr{L}(\mathbb{B})$ if and only if $\varepsilon_Q \simeq \rho_Q$. Let $(u_i : Q_i \to X)_I$ be the sink of all morphisms with domain in $\mathscr{L}(\mathbb{B})$ such that pu_i factors as $w_i \varepsilon_{Q_i}$. It ought to be clear that $\varepsilon_{Q_i} \simeq \rho_{Q_i}$. Without loss of generality, we assume that $\rho_{Q_i} = \varepsilon_{Q_i}$. For each $i \in I$, we can then establish a commutative diagram:

$$\begin{tikzcd}
Q_i \arrow[r, "\rho_{Q_i}"] \arrow[d, "u_i"'] & SQ_i \arrow[d, "v_i"] \arrow[ddr, bend left, "Su_i"] & \\
X \arrow[r, "u_{i-}(\rho_{Q_i})"'] \arrow[drr, bend right, "\rho_X"'] & u_-(SQ_i) \arrow[dr, dashed, "!f_i"] & \\
 & & SX
\end{tikzcd}$$

Since $D_{\mathbb{C}}^{\mathscr{L}(\mathbb{B})}(p)$ is the join of all the morphisms $u_{i-}(\rho_{Q_i})$ and $u_{i-}(\rho_{Q_i}) \leq \rho_X$, it follows that $D_{\mathbb{C}}^{\mathscr{L}(\mathbb{B})}(p) \leq p \wedge \rho_X \simeq D_{\mathbb{B}}(p)$.

(*ii*) This directly folows from 4.108 and the fact that $\mathscr{L}(\mathbb{B})$ is closed under $\mathscr{E}$-images. $\square$

Proposition 4.110: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ a reflective constant subcategory. Then, for each subcategory $\mathbb{Q}$ of $\mathbb{A}$ that is closed under $\mathscr{E}$-images:

$$\mathscr{R}(\mathbb{Q}) = Shriek_{\mathbb{C}}^*(D_{\mathbb{C}}^{\mathbb{Q}}).$$

$\underline{\mathcal{P}roof}$: Let $R : \mathbb{A} \to \mathbb{C}$ be a reflector with unit ε.

Suppose that $X \in Shriek_{\mathbb{C}}^*(D_{\mathbb{C}}^{\mathbb{Q}})$, i.e., ε_X is $D_{\mathbb{C}}^{\mathbb{Q}}$-sparse, or, equivalently, $D_{\mathbb{C}}^{\mathbb{Q}}(\varepsilon_X)$ is an isomorphism. Let Q be in $\mathbb{Q}$ and $u : Q \to X$ any $\mathbb{A}$-morphism. We show that u is constant. Since $\varepsilon_X \circ u = Ru \circ \varepsilon_Q$, it follows that the pushout $u_-(\varepsilon_Q)$ of ε_Q along u contributes to the multiple pushout used to construct $D_{\mathbb{C}}^{\mathbb{Q}}(\varepsilon_X)$. Since ε_X is $D_{\mathbb{C}}^{\mathbb{Q}}$-sparse, $D_{\mathbb{C}}^{\mathbb{Q}}(\varepsilon_X) \simeq id_X$. Since $u_-(\varepsilon_Q) \leq D_{\mathbb{C}}^{\mathbb{Q}}(\varepsilon_X)$, we must have that $u_-(\varepsilon_X) \simeq id_X$. Then there exists an $\mathbb{A}$-morphism v such that

$$\begin{tikzcd}
Q \arrow[r, "\varepsilon_Q"] \arrow[d, "u"'] & RQ \arrow[d, "v"] \\
X \arrow[r, "id_X"'] & X
\end{tikzcd}$$

is a pushout square. Since RQ is in $\mathbb{C}$, we must have that $u = v\varepsilon_Q$ is constant. Thus X is a member of $\mathscr{R}(\mathbb{Q})$ and the inclusion $Shriek_{\mathbb{C}}^*(D_{\mathbb{C}}^{\mathbb{Q}}) \subset \mathscr{R}(\mathbb{Q})$ holds.

For the converse, assume that X is a member of $\mathscr{R}(\mathbb{Q})$. Let $(u_j : Q_j \to X)_J$ be the sink of all morphisms with domain in $\mathbb{Q}$ and codomain X such that $\varepsilon_X u_j$ factors through ε_Q. Since $\varepsilon_X u_j = Ru_j \varepsilon_Q$ and ε_Q is an epimorphism, such a factorisation is unique. If u_j is a member of $\mathscr{M}$, then since X is in $\mathscr{R}(\mathbb{Q})$, it will follow that Q_j is in $\mathbb{C}$. Consequently we may take $\varepsilon_{Q_j} = id_{Q_j}$ whenever u_j is in $\mathscr{M}$. Let $j \in J$ and let $u = u_j$ have $(\mathscr{E}, \mathscr{M})$-factorisation $me : Q \to Q' \to X$. Note that since $\mathbb{Q}$ is closed under $\mathscr{E}$-images, we have that Q' is actually a member of $\mathbb{Q}$ and as just explained, Q' must then be in $\mathbb{C}$. Now, consider the commutative diagram:

$$
\begin{tikzcd}
Q \arrow[r, "\varepsilon_Q"] \arrow[d, "e"'] & RQ \arrow[d, "\overline{e}"] \arrow[ddr, bend left, "Re"] & \\
Q' \arrow[r, "e_-(\varepsilon_Q)"] \arrow[drr, bend right, "\varepsilon_{Q'}=id_{Q'}"'] & e_-(RQ) \arrow[dr, dashed, "!p"] & \\
 & & RQ' = Q'
\end{tikzcd}
$$

The top square in the diagram is a pushout and the pushout establishes the morphism p, as the outer diagram is the naturality square. Hence, $e_-(\varepsilon_Q) \leq id_{Q'}$ and hence $e_-(\varepsilon_Q)$ must be an isomorphism. Then, $id_X \simeq m_-(id_{Q'}) \simeq m_-(e_-(\varepsilon_Q)) \simeq u_-(\varepsilon_Q)$ and therefore $u_-(\varepsilon_Q)$ must be an isomorphism. Since j was arbitrary, it follows that $id_X \simeq u_j(\varepsilon_{Q_j})$ for each $j \in J$. Therefore $D^{\mathbb{Q}}_{\mathbb{C}}(\varepsilon_X) \simeq \bigvee_J u_-(\varepsilon_Q) \simeq \bigvee_J id_X \simeq id_X$ or equivalently X is a member of $Shriek^*_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}})$. Putting the two parts together shows that $\mathscr{R}(\mathbb{Q}) = Shriek^*_{\mathbb{C}}(D^{\mathbb{Q}}_{\mathbb{C}})$ and hence our proof is complete. □

Corollary 4.111: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ a reflective constant subcategory. Then, the adjunction between left and right constant subcategories factors through the two adjunctions between the collection of all idempotent dual closure operators $D \leq D_{\mathbb{C}}$. In particular, $\mathscr{R}(\mathbb{Q}) = Shriek^*_{\mathbb{C}}(D^{\mathbb{Q}}_{\mathbb{C}})$ and $\mathscr{L}(\mathbb{P}) = Shriek_{\mathbb{C}}(D_{\mathbb{P}})$ hold for every left constant subcategory $\mathbb{Q}$ and every right constant subcategory $\mathbb{P}$.

Proof : This follows directly from 4.87, 4.93, 4.107, 4.108 and 4.110. □

Remark 4.112: From remark 4.88, it has already been discussed that $Shriek^*_{\mathbb{C}}(-)$ has a different right adjoint other than $D_{(-)}$. We will not delve into this specific right adjoint. However, this implies that $D_{(-)}$ might have another adjoint different from $Shriek_{\mathbb{C}}(-)$. Our first task will be to explicitly find the left adjoint of $D_{(-)}$.

This will be followed by considering constructions similar to $Shriek_{\mathbb{C}}(D)$ and $D^{\mathbb{B}}_{\mathbb{C}}$ and finding more adjunctions between subcategories and dual closure operators We will then also compare these constructions and see when they coincide. For the remainder of the section, we will always assume that $\mathscr{E}$ is a class of epimorphisms.

Definition 4.113: $Shriek^\dagger(-)$
The map $Shriek^\dagger(-) : DCO(\mathbb{A}, \mathscr{E}) \to Sub(\mathbb{A})$ is defined to be the full subcategory with object class

$$Shriek^\dagger(D) = \{X \in \mathbb{A} \mid \forall e \in quot(X) : e \text{ is } D\text{-sparse}\}$$

.

Proposition 4.114: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. Then, $Shriek^\dagger(D)$ is $\mathscr{E}$-reflective for each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$.
Proof : It's sufficient to prove that $Shriek^\dagger(D)$ is closed under sources in $\mathbb{M}$. Let $(m_i : X \to X_i)_I$ be an $\mathbb{M}$-source with X_i in $Shriek^\dagger(D)$ for each $i \in I$. Let $e : X \to Y$ be in $\mathscr{E}$ and for each i, e_i a pushout of e along m_i. Then,

$$
\begin{tikzcd}
X \arrow[r, "m_i"] \arrow[d, "De"'] & X_i \arrow[d, "De_i"] \\
DY \arrow[r, "D_{m_i,\overline{m_i}}"'] \arrow[d, "\delta_e"'] & DY_i \arrow[d, "\delta_{e_i}"] \\
Y \arrow[r, "\overline{m_i}"'] & Y_i
\end{tikzcd}
$$

commutes and since X_i is in $Shriek^\dagger(D)$, it follows that De_i is an isomorphism for each i. In particular, $((De_i)^{-1} \circ D_{m_i,\overline{m_i}})_I \circ De = (m_i)_I \in \mathbb{M}$. By the cancellation properties of $\mathbb{M}$, it follows that De is a singleton source in $\mathbb{M}$ and since De is in $\mathscr{E}$, it should be clear that De is an isomorphism. Therefore X is also a member of $Shriek^\dagger(D)$ and our proof is complete. □

Proposition 4.115: $Shriek^{\dagger}(-) : DCO(\mathbb{A}, \mathscr{E}) \to Sub(\mathbb{A})^{op}$ is order preserving.

<u>$\mathcal{P}roof$:</u> Suppose $D \leq D'$ and let X be a member of $Shriek^{\dagger}(D')$, i.e., $D'e \simeq id_X$ for each e in $quot(X)$. Then $De \leq D'e \simeq id_X$ so that $De \simeq id_X$. Since e was arbitrary, X is a member of $Shriek^{\dagger}(D)$. □

Theorem 4.116: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathscr{M})$-structured category with pullbacks and $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$. Then, for each reflective subcategory $\mathbb{B}$ of $\mathbb{A}$ and each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, the following holds:

$$D \leq D_{\mathbb{B}} \Leftrightarrow Shriek^{\dagger}(D) \supset \mathbb{B}$$

In particular, it follows that $D \leq D_{Shriek^{\dagger}(D)}$ and $\mathbb{B} \subset Shriek^{\dagger}(D_{\mathbb{B}})$.

<u>$\mathcal{P}roof$:</u> Let $R : \mathbb{A} \to \mathbb{B}$ be a reflector with unit ρ.

First we assume that $D \leq D_{\mathbb{B}}$. Let B be a member of $\mathbb{B}$ and $e : B \to C$ in $\mathscr{E}$. Then, $De \leq D_{\mathbb{B}}e \simeq \rho_B \wedge e \simeq id_B \wedge e \simeq id_B$. It follows that B is in $Shriek^{\dagger}(D)$.

Conversely, assume that $\mathbb{B} \subset Shriek^{\dagger}(D)$. Let $p : X \to P$ be any morphism in $\mathscr{E}$. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\rho_X"] \arrow[d, "Dp"'] & RX \arrow[d, "DRp"] \\
DP \arrow[r, "D_{\rho_X, \rho_P}"'] \arrow[d, "\delta_p"'] & DRP \arrow[d, "\delta_{Rp}"] \\
P \arrow[r, "\rho_P"'] & RP
\end{tikzcd}$$

Since RX is in $\mathbb{B}$ and this is a subclass of $Shriek^{\dagger}(D)$, it follows that DRp is an isomorphism. Hence, $j := DRp^{-1} \circ D_{\rho_X, \rho_P}$ is a morphism such that $jDp = \rho_X$. It's then clear that $Dp \leq \rho_X$. Since D is a dual closure operator, we have that $Dp \leq p \wedge \rho_X \simeq D_{\mathbb{B}}(p)$.

The last part of the proof follows from a standard result on adjunctions. □

Corollary 4.117: For an idempotent dual closure operator D of $\mathscr{E}$ in an $(\mathscr{E}, \mathbb{M})$-category with pullbacks with $D \leq D_{\mathbb{C}}$, $Shriek^{\dagger}(D) = Shriek^{*}_{\mathbb{C}}(D)$.

<u>$\mathcal{P}roof$:</u> This follows directly from 4.87 and 4.116. □

Proposition 4.118: Let $\mathbb{B}$ be an $\mathscr{E}$-reflective subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$ with $\mathscr{E} \subset \mathrm{Epi}(\mathbb{A})$ and assume that $\mathbb{A}$ has pullbacks. Then, $\mathbb{B} = Shriek^{\dagger}(D_{\mathbb{B}})$.

<u>$\mathcal{P}roof$:</u> Let $R : \mathbb{A} \to \mathbb{B}$ denote a reflector with unit ε. Let $p : X \to P$ be a member of $\mathscr{E}$ and assume that $X \in \mathbb{B}$. Then, by 4.80, it follows that $D_{\mathbb{B}}(p) \simeq id_X$, i.e., p is $D_{\mathbb{B}}$-sparse. Since p was arbitrary, X is a member of $Shriek^{\dagger}(D_{\mathbb{B}})$.

Conversely, if X is a member of $Shriek^{\dagger}(D_{\mathbb{B}})$, then $D_{\mathbb{B}}(p) \simeq id_X$ for each $p : X \to P$ in $\mathscr{E}$. By Proposition 4.14 it follows that $D_{\mathbb{B}}(p) \simeq p \wedge \varepsilon_X$. In particular $id_X \simeq D_{\mathbb{B}}(\varepsilon_X) \simeq \varepsilon_X \wedge \varepsilon_X \simeq \varepsilon_X$, hence ε_X is an isomorphism. Since $\mathbb{B}$ is $\mathscr{E}$-reflective, it's isomorphism-closed and hence X is a member of $\mathbb{B}$. □

Remark 4.119: We now turn our attention to constructions similar to $D^{\mathbb{B}}_{\mathbb{C}}$ and $Shriek_{\mathbb{C}}(D)$. The first of these will involve weakly constant classes of morphisms and the other will ignore constant morphisms altogether. Both of the approaches give an adjunction similar to the one between $Shriek_{\mathbb{C}}(-)$ and $D^{(-)}_{\mathbb{C}}$.

Lemma 4.120: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathscr{N}$ a class of weakly constant morphisms in $\mathbb{A}$. For each subcategory $\mathbb{B}$ of $\mathbb{A}$ and each $p : X \to P \in \mathscr{E}$, we consider the sink of $(u_i : B_i \to X)_I$ of all morphisms $u : B \to X$ with $B \in \mathbb{B}$ such that $p \circ u$ has $(\mathscr{E}, \mathbb{M})$-factorisation we where $e \in \mathscr{N}$. For each $i \in I$, consider the pushout $u_{i-}(e_i) : X \to u_{i-}(C_i)$ of e_i along u_i. Let $ew^{\mathbb{B}}_{\mathscr{N}}(p) : X \to ew^{\mathbb{B}}_{\mathscr{N}}(P)$ be the multiple pushout of the source $(X \stackrel{u_{i-}(e_i)}{\longrightarrow} u_{i-}(C_i))_I$ with $(p_i)_I$ a sink such that $p_i \circ u_{i-}(e_i) = ew^{\mathbb{B}}_{\mathscr{N}}(p)$.

Then, by defining $(ew^{\mathbb{B}}_{\mathscr{N}})_X(p) = ew^{\mathbb{B}}_{\mathscr{N}}(p)$, it follows that $ew^{\mathbb{B}}_{\mathscr{N}}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

Proof : Since $\mathbb{A}$ is $\mathscr{E}$-cocomplete, it ought to be clear that for each $p \in \mathscr{E}$, $ew^{\mathbb{B}}_{\mathscr{N}}(p)$ is a morphism in $\mathscr{E}$. Let $p : X \to P \in \mathscr{E}$ and $(u_i)_I$ be the sink as described above. For each $i \in I$, let $w_i e_i$ be an $(\mathscr{E}, \mathbb{M})$ factorisation of pu_i, where e_i is a constant morphism. For each $i \in I$, the pushout establises a unique morphism $\bar{p}_i : u_{i-}(C_i) \to P$ such that $\bar{p}_i v_i = w_i$ and $\bar{p}_i u_{i-}(e_i) = p$. Then, as described above, let $ew^{\mathbb{B}}_{\mathscr{N}}(p)$ denote the multiple pushout of the source $(u_{i-}(e_i))_I$. The multiple pushout then establishes a unique morphism $\omega_p : ew^{\mathbb{B}}_{\mathscr{N}}(P) \to P$ such that $\omega_p \circ ew^{\mathbb{B}}_{\mathscr{N}}(p) = p$ and for each $i \in I$ there holds: $\omega_p p_i = \bar{p}_i$.

$$\begin{tikzcd}
B_i \arrow[r, "e_i"] \arrow[d, "u_i"'] & C_i \arrow[d, "v_i"] \arrow[ddr, bend left, "w_i"] & \\
X \arrow[r, "u_{i-}(e_i)"] \arrow[dr, "ew^{\mathbb{B}}_{\mathscr{N}}(p)"'] \arrow[ddrr, bend right, "p"'] & u_{i-}(C_i) \arrow[d, "p_i"] \arrow[ddr, dashed, "\bar{p}_i"] & \\
& ew^{\mathbb{B}}_{\mathscr{N}}(P) \arrow[dr, dotted, "\omega_p"'] & \\
& & P
\end{tikzcd}$$

Throughout the proof, we will use the same notation as in the diagram above and where necessary denote a morphism by a superscript p. For instance, we may write w_i^p for w_i. We will also simplify notation by simply writing $ew(p)$ instead of the more cumbersome $ew^{\mathbb{B}}_{\mathscr{N}}(p)$.
To see that $ew^{\mathbb{B}}_{\mathscr{N}}$ is a dual closure operator, we first note that $\omega_p \circ ew^{\mathbb{B}}_{\mathscr{N}}(p) = p$ for each $p \in \mathscr{E}$. It's therefore easy to see that $ew^{\mathbb{B}}_{\mathscr{N}}(p) \leq p$ and that ω_p is a member of $\mathscr{E}$.

Assume that $p : X \to P$ and $p' : X \to P'$ are morphisms in $\mathscr{E}$ such that $kp = p'$, i.e., $p \leq p'$. We need to show that $ew(p) \leq ew(p')$. Let $(u_i)_I$ be the sink of all morphisms with domain in $\mathbb{B}$ such that $w_i e_i$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_i and such that e is an $\mathscr{N}$-constant morphism. For each $i \in I$, let $\bar{w}_i \bar{e}_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $p' u_i$. Then $\bar{w}_i \bar{e}_i = p' u_i = kpu_i = (kw_i)e_i$. The diagonalisation property establishes a morphsim d_i such that
$$\begin{tikzcd}
B_i \arrow[r, "e_i"] \arrow[d, "\bar{e}_i"'] & C_i \arrow[d, "kw_i"] \arrow[dl, dashed, "!d_i"'] \\
\bar{C}_i \arrow[r, "\bar{w}_i"'] & P'
\end{tikzcd}$$
commutes. Let $(u'_j)_J$ be the sink of all morphisms with domain in $\mathbb{B}$ such that $p' u_j$ factors as $w'_j e'_j$ with e'_j a constant morphism in $\mathscr{E}$. Then, we may, without loss of generality, assume that $I \subset J$ and that for each i there hold: $d_i e_i = e'_i = \bar{e}_i$, $\bar{w}_i = w'_i$ and $\bar{w}_i d_i = w'_i d_i = kw_i$. Let us denote these factorisations for each $j \in J$ and each $i \in I$ by: $B_j \xrightarrow{e'_j} C'_j \xrightarrow{w'_j} P'$ and $B_i \xrightarrow{e_i} C_i \xrightarrow{w_i} P$ respectively.

Since a multiple pushout of a family of morphisms in $quot(Y)$ also acts as the join, we have: $ew(p) = \bigvee_I u_{i-}(e_i) \leq \bigvee_I u'_{i-}(e'_i) \leq \bigvee_J u'_{j-}(e'_j) = ew(p')$.

Now, the main part of the proof is to establish the morphism $ew_{f,g} : ew(P) \to ew(Q)$. Explicitly, for every commutative square
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
P \arrow[r, "g"'] & Q
\end{tikzcd}$$
with p and q members of $\mathscr{E}$, $ew_{f,g}$ must be a morphism such that $ew_{f,g} \circ ew(p) = ew(q) \circ f$ and $\omega_q \circ ew_{f,g} = g \circ \omega_p$.

Consider a commutative square as in the preceding paragraph. Let $(u_i : B_i \to X)_I$, respectively $(\bar{u}_j : \overline{B}_j \to Y)_J$, be the sinks of all morphisms u, respectively $\bar{u}$, with domain in $\mathbb{B}$ such that pu_i, respectively $q\overline{u}_j$, has an $(\mathscr{E}, \mathbb{M})$-factorisation through a constant morphism in $\mathscr{E}$. Explicitly, for each $i \in I$ and each $j \in J$, let $w_i \circ e_i = pu_i$ and $\overline{w}_i \circ \overline{e}_i = q\overline{u}_j$ be such factorisations with e_i, respectively

$\overline{e}_j$, a constant morphism in $\mathscr{E}$. For each $i \in I$, let $n_i f_i$ be an $(\mathscr{E},\mathbb{M})$-factorisation of gw_i. Note that $q(fu_i) = gpu_i = (gw_i)e_i = n_i(f_i e_i)$ is an $(\mathscr{E},\mathbb{M})$-factorisation of qfu_i with $f_i e_i$ a constant morphism in $\mathscr{E}$. Hence, we might as well assume that $I \subset J$ and that the following hold: $fu_i = \overline{u}_i$, $n_i = \overline{w}_i$ and $\overline{e}_i = f_i e_i$.

For each $i \in I$ and each $j \in J$, let v_i and $\overline{v}_j$ be morphisms such that

$$\begin{tikzcd} B_i \arrow[r, "e_i"] \arrow[d, "u_i"'] & C_i \arrow[d, "v_i"] \\ X \arrow[r, "u_{i-}(e_i)"'] & u_{i-}(C_i) \end{tikzcd}$$

and

$$\begin{tikzcd} \overline{B}_j \arrow[r, "\overline{e}_j"] \arrow[d, "\overline{u}_j"'] & \overline{C}_j \arrow[d, "\overline{v}_j"] \\ Y \arrow[r, "\overline{u}_{j-}(\overline{e}_j)"'] & \overline{u}_{j-}(\overline{C}_j) \end{tikzcd}$$

are pushout squares. In particular, for every $i \in I$, we have: $\overline{u}_{i-}(\overline{e}_i)f \circ u_i = \overline{u}_{i-}(\overline{e}_i)\circ\overline{u}_i = \overline{v}_i\circ\overline{e}_i = \overline{v}_i f_i \circ e_i$. For each $i \in I$, the first pushout square establishes a unique morphism $\tilde{v}_i$ such that $\tilde{v}_i \circ v_i = \overline{v}_i f_i$ and $\tilde{v}_i \circ u_{i-}(e_i) = \overline{u}_{i-}(\overline{e}_i) \circ f$. Hence we have the following commutative diagram:

$$\begin{tikzcd} B_i \arrow[r, "e_i"] \arrow[d, "u_i"] \arrow[dd, bend right, "\overline{u}_i"'] \arrow[drr, bend left, "\overline{e}_i"] & C_i \arrow[d, "v_i"] \arrow[dr, bend left, "f_i"] & \\ X \arrow[r, "u_{i-}(e_i)"] \arrow[d, "f"] & u_{i-}(C_i) \arrow[d, dashed, "!\tilde{v}_i"] & \overline{C}_i \arrow[dl, bend left, "\overline{v}_i"] \\ Y \arrow[r, "\overline{u}_{i-}(\overline{e}_i)"'] & \overline{u}_{i-}(\overline{C}_i) & \end{tikzcd}$$

Let $(p_i)_I$ and $(q_j)_J$ be the sinks such that $p_i u_{i-}(e_i) = ew(p)$ and $q_j\overline{u}_{j-}(\overline{e}_j) = ew(q)$. Then $q_i\tilde{v}_i\circ u_{i-}(e_i) = q_i\overline{u}_{i-}(\overline{e}_i) \circ f = ew(q) \circ f$ holds for each $i \in I$. The multiple pushout property gives the existence of a unique morphism $ew_{f,g} : ew(P) \to ew(Q)$ such that $ew_{f,g} \circ ew(p) = ew(q) \circ f$ and $ew_{f,g}p_i = q_i\tilde{v}_i$. Then $\omega_q ew_{f,g} ew(p) = \omega_q ew(q) f = qf = gp = g\omega_p ew(p)$ and since $\mathscr{E}$ is a class of epimorphisms, we have $\omega_q ew_{f,g} = g\omega_p$.

$$\begin{tikzcd} X \arrow[r, "u_{i-}(e_i)"] \arrow[drr, bend right, "ew(p)"'] \arrow[drrr, bend left=40, "ew(q)\circ f"] & u_{i-}(C_i) \arrow[r, "\tilde{v}_i"] \arrow[dr, "p_i"'] & \overline{u}_{i-}(\overline{C}_i) \arrow[dr, "q_i"'] & \\ & & ew(P) \arrow[r, "ew_{f,g}"'] & ew(Q) \end{tikzcd}$$

Hence, for each commutative square $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ P \arrow[r, "g"'] & Q \end{tikzcd}$ with $p, q \in \mathscr{E}$, there is a morphism $ew_{f,g}$ such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "ew(p)"'] & Y \arrow[d, "ew(q)"] \\
ew(P) \arrow[r, "ew_{f,g}"'] \arrow[d, "\omega_p"'] & ew(Q) \arrow[d, "\omega_q"] \\
P \arrow[r, "g"'] & Q
\end{tikzcd}$$
commutes.

We now show that $ew(f^-(q)) \leq f^-(ew(q))$ for each $\mathbb{A}$-morphism $f : X \to Y$ and $q : Y \to Q$ in $\mathscr{E}$. Let $mf^-(q) = qf$ and $nf^-(ew(q)) = ew(q)f$ be $(\mathscr{E}, \mathscr{M})$-factorisations. Then $ew_{f,m} \circ ew(f^-(q)) = ew(q) \circ f = nf^-(ew(q))$ and the diagonalisation property provides a unique morphism $d : ew(f^-(Q)) \to f^-(ew(Q))$ such that $nd = ew_{f,m}$ and $dew(f^-(q)) = f^-(ew(q))$. In particular, $ew(f^-(q)) \leq f^-(ew(q))$. It follows that $ew^{\mathbb{B}}_{\mathscr{N}}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. $\square$

Corollary 4.121: Let $\mathbb{C}$ be a constant subcategory of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$ and $\mathbb{B}$ a subcategory of $\mathbb{A}$. Then, for each $p \in \mathscr{E}$, the sink of all morphisms $u : B \to X$ with $B \in \mathbb{B}$ such that $pu = we$ with e a $\mathbb{C}$-constant morphism, defines a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ in the same way as in Lemma 4.120. To emphasise the constant subcategory chosen, we will write $ew^{\mathbb{B}}_{\mathbb{C}}$.

Definition 4.122: Eilenberg-Whyburn Dual closure operator
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathscr{N}$ a class of weakly constant morphisms. For each subcategory $\mathbb{B}$ of $\mathbb{A}$, the assignment $p \longmapsto ew^{\mathbb{B}}_{\mathscr{N}}(p)$ defined in Lemma 4.120 is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$ called the **Eilenberg-Whyburn** dual closure operator induced by $\mathbb{B}$. To be more precise, the Eilenberg-Whyburn dual closure operator is not exactly the same as the one constructed in [26, 4.6], but the idea behind the construction is very similar. In [26], it's constructed in a similar manner, where the sink of all morphisms from the subcategory in question is determined in such a way that it factors through the terminal object. That is also the central notion of constant morphisms in question and in some cases this can be viewed as a generalisation. In case $\mathbb{A}$ is a pointed category and all the zero objects can be viewed as a constant subcategory, then this is obviously the case. For the category $\mathbb{T}\mathrm{op}_{\varnothing}$ of non-empty topological spaces, with constant subcategory consisting only of singleton spaces, this will also often be the case for factorisation structures such that the class $\mathscr{E}$ contains all the retractions. We will often omit the subscript $\mathbb{C}$ or $\mathscr{N}$ in case the constant subcategory or class of weakly constant morphisms chosen is clear.

$$\begin{tikzcd}
B_i \arrow[r, "e_i"] \arrow[d, "u_i"'] & C_i \arrow[d, "v_i"] \arrow[ddr, bend left, "w_i"] & \\
X \arrow[r, "u_{i-}(e_i)"] \arrow[dr, "ew^{\mathbb{B}}_{\mathbb{C}}(p)"'] \arrow[ddrr, bend right, "p"'] & u_{i-}(C_i) \arrow[d, "p_i"] \arrow[ddr, dashed, "\bar{p}_i"] & \\
 & ew^{\mathbb{B}}_{\mathbb{C}}(P) \arrow[dr, dotted, "\omega_p"'] & \\
 & & P
\end{tikzcd} \tag{3}$$

Proposition 4.123: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathscr{N}$ a class of weakly constant morphisms. Then, $ew^{(-)}_{\mathscr{N}} : (Sub(\mathbb{A}), \subset) \to (DCO(\mathbb{A}, \mathscr{E}), \leq)$ is order preserving.

Proof : Suppose that $\mathbb{B} \subset \mathbb{B}'$ and let $p : X \to P$ be in $\mathscr{E}$. Let $(u_i : B_i \to X)_I$ and $(u'_j : B'_j \to X)_J$ be the respective sinks of morphisms u with domain in $\mathbb{B}$, respectively $\mathbb{B}'$, such that pu has $(\mathscr{E}, \mathbb{M})$-factorisation me, where e is an $\mathscr{N}$-constant morphism. Then, since $\mathbb{B} \subset \mathbb{B}'$, we may assume that $I \subset J$ and $u'_i = u_i$. For each $j \in I$, let $w_j e_j$ be such an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_j.

Then $ew^{\mathbb{B}}_{\mathscr{N}}(p) = \bigvee_I u_{i-}(e_i) \leq \bigvee_J u_{j-}(e_j) = ew^{\mathbb{B}'}_{\mathscr{N}}(p)$. □

Proposition 4.124: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathscr{N}$ a class of weakly constant morphisms in $\mathbb{A}$. Then, for each subcategory $\mathbb{B}$ of $\mathbb{A}$, $ew^{\mathbb{B}}_{\mathscr{N}}$ is an idempotent dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

Proof : Let $p : X \to P$ be in $\mathscr{E}$ and $(u_i : B_i \to X)_I$ and $(\hat{u}_j : \hat{B}_j \to X)_J$ be the sinks with domain in $\mathbb{B}$ such that p, respectively $ew(p)$, composed with each member of the appropriate sink, has an $(\mathscr{E}, \mathbb{M})$-factorisation that factors through a constant morphism in $\mathscr{E}$. For each $i \in I$ and each $j \in J$, let $pu_i = w_i e_i$ and $ew(p)\hat{u}_j = \hat{w}_j \hat{e}_j$ be such factorisations. Throughout the proof we will use the notation as in (3). For each $j \in J$, let $\omega_p w_j$ have $(\mathscr{E}, \mathbb{M})$-factorisation $m_j f_j$. Then, for each $j \in J$, there holds: $p \circ \hat{u}_j = \omega_p \circ ew(p)\hat{u}_j = (\omega_p \hat{w}_j) \circ \hat{e}_j = m_j(f_j e_j)$. Note that $f_j \bar{e}_j$ is an $\mathscr{N}$-constant morphism in $\mathscr{E}$. Hence, without loss of generality, we may assume that $J \subset I$ and where for any $j \in J$ there hold: $\hat{u}_j = u_j$, $f_j \hat{e}_j = e_j$ and $m_j = \omega_p \hat{w}_j = w_j$. Now we show that $J = I$.

Let $i \in I$ and consider the pushout square

$$\begin{tikzcd} B_i \arrow[r, "e_i"] \arrow[d, "u_i"'] & C_i \arrow[d, "v_i"] \\ X \arrow[r, "u_{i-}(e_i)"'] & u_{i-}(C_i) \end{tikzcd}$$

Since e_i is a constant morphism, so is $v_i e_i$ and hence also $p_i v_i e_i = ew(p)u_i$. If $n_i e'_i$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of $p_i v_i$, then $n_i \circ e'_i e_i = p_i v_i e_i = ew(p)u_i$ is an $(\mathscr{E}, \mathbb{M})$-factorisation. It then ought to be clear that we may also assume that each u_i is a member of the sink $(\hat{u}_j)_J$. Without loss of generality, it's also easy to see that we may assume $I \subset J$ and also assume that for each $j \in J$ there hold: $u_j = \hat{u}_j$, $\hat{w}_j = n_j$ and $\hat{e}_j = e'_j e_j$. Since $\mathscr{E}$ is a class of epimorphisms, we must have that f_j and e'_j are isomorphisms and inverses of each other. Consequently $ew(p) = \bigvee_I u_{i-}(e_i) \simeq \bigvee_I \hat{u}_{i-}(f_i \hat{e}_i) \simeq \bigvee_J \hat{u}_{j-}(f_j \hat{e}_j) \simeq \bigvee_J \hat{u}_{j-}(\hat{e}_j) \simeq ew(ew(p))$. Thus $ew(p) \simeq ew(ew(p))$ so that it's idempotent. □

Proposition 4.125: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathscr{N}$ a class of right-$\mathscr{E}$-constant morphisms. Let $\mathbb{B}$ be any subcategory of $\mathbb{A}$ that is closed under $\mathscr{E}$-images and let $p : X \to P$ be a member of $\mathscr{E}$. Let $(u_i)_I$ denote the sink of all morphisms $u : B \to X$ with domain in $\mathbb{B}$ that has an $(\mathscr{E}, \mathbb{M})$-factorisation that factors through a constant morphism in $\mathscr{E}$. Let $(m_j : B_j \to X)_J$ be the subsink of $(u_i)_I$ of all morphisms $m : B \to X$ that are members of $\mathscr{M}$. Then $ew^{\mathbb{B}}_{\mathbb{C}}(p) \simeq \bigvee_J m_{j-}(e_j)$, where $w_j e_j$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of pm_j.

Proof : Let $\mathbb{B}$ be closed under $\mathscr{E}$-images and $p : X \to P$ in $\mathscr{E}$. Let $(u_i)_I$ be a sink as in (3) and described above. For each $i \in I$, let u_i have $(\mathscr{E}, \mathscr{M})$-factorisation $B_i \xrightarrow{\bar{e}_i} \overline{B}_i \xrightarrow{\overline{m}_i} X$. Note that our assumptions on $\mathbb{B}$ forces $\overline{B}_i$ to be a member of $\mathbb{B}$. For each $i \in I$, let $B_i \xrightarrow{e_i} C_i \xrightarrow{w_i} P$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_i. Note that $(B_i \xrightarrow{\bar{e}_i} \overline{B}_i \xrightarrow{\overline{m}_i} X \xrightarrow{p} P) = (B_i \xrightarrow{u_i} X \xrightarrow{p} P) = (B_i \xrightarrow{e_i} C_i \xrightarrow{w_i} P)$.

Let $(m_j)_J$ be the subsink as described above. Since $J \subset I$, we must have that $\bigvee_J m_{j-}(e_j) \leq \bigvee_I u_{i-}(e_i) \simeq ew(p)$. Hence, it is sufficient to prove that for each $i \in I$ there is a $j \in J$ such that $u_{i-}(e_i) \leq m_{j-}(e_j)$. For each $i \in I$, let $\hat{w}_i \hat{e}_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $p\overline{m}_i$. The diagonalisation property establishes an isomorphism d_i that makes the following diagram commute:

$$\begin{tikzcd} B_i \arrow[r, "e_i"] \arrow[d, "\bar{e}_i"'] & C_i \arrow[dd, "w_i"] \arrow[ddl, dashed, "d_i"'] \\ \overline{B}_i \arrow[d, "\hat{e}_i"'] & \\ \hat{C}_i \arrow[r, "\hat{w}_i"'] & P \end{tikzcd}$$

Since e_i is $\mathscr{N}$-constant, so is $d_i e_i = \hat{e}_i \bar{e}_i$. By the assumption on $\mathscr{N}$ we must have that $\hat{e}_i$ is an $\mathscr{N}$-constant morphism through which $p\bar{m}_i$ factors. Since $\overline{B}_i$ is a member of $\mathbb{B}$ and $\bar{m}_i$ is in $\mathscr{M}$, it follows that $\bar{m}_i = m_j$ for some $j \in J$. Also note that $\hat{e}_i$ is a pushout of e_i along $\bar{e}_i$. Then we have $u_{i-}(e_i) \simeq \bar{m}_{i-}(\bar{e}_{i-}(e_i)) \simeq m_{j-}(\bar{e}_{i-}(e_i)) \simeq m_{j-}(\hat{e}_i) \simeq m_{j-}(e_j)$, where the last step holds since $\hat{w}_i \hat{e}_i$ and

$w_j e_j$ are both $(\mathscr{E}, \mathbb{M})$-factorisations of pm_j. Therefore our proof is complete. $\square$

Proposition 4.126: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks. Let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$, and $\mathbb{B}$ a nearly multi-$\mathscr{M}$-coreflective subcategory of $\mathbb{A}$. Then $ew^{\mathbb{B}}_{\mathbb{C}}$ can also be constructed as follows: For each $p : X \to P \in \mathscr{E}$, consider the sink $(m_i : C_i \to P)_I$ of all constant morphisms $m : C \to P$ in $\mathscr{M}$. For each $i \in I$, consider the pullback $p^{-1}(m_i) : p^{-1}(C_i) \to X$ of m_i along p, so that

$$\begin{tikzcd} p^{-1}(C_i) \arrow[r, "p^{-1}(m_i)"] \arrow[d, "p_i"'] & X \arrow[d, "p"] \\ C_i \arrow[r, "m_i"'] & P \end{tikzcd}$$

commutes. For each $i \in I$, let us denote the near multi-$\mathscr{M}$-coreflection of $p^{-1}(C_i)$ into $\mathbb{B}$ by $(\eta^i_k : B^i_k \to p^{-1}(C_i))_{K(p^{-1}(C_i))}$. Define, for each $i \in I$ and $k \in K(p^{-1}(C_i))$, the morphism $v^i_k := p^{-1}(m_i) \circ \eta^i_k$. Further, let $n^i_k \circ \overline{e}^i_k$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $p_i \circ \eta^i_k$. Then, $ew^{\mathbb{B}}_{\mathbb{C}}(p)$ is isomorphic to the multiple pushout of all the pushouts $v^i_{k-}(\overline{e}^i_k)$.

<u>*Proof*</u> : Consider the situation as stated above and let $(u_j : B_j \to X)_J$ be the sink of all morphisms $u : B \to X$, where B is in $\mathbb{B}$, pu is constant and has $(\mathscr{E}, \mathbb{M})$-factorisation $w \circ e$. Note that, for each $i \in I$ and each $k \in K(p^{-1}(C_i))$, we have that v^i_k is a morphism with domain in $\mathbb{B}$ that has $(\mathscr{E}, \mathbb{M})$-factorisation $m_i n^i_k \circ \overline{e}^i_k$. Since m_i is assumed to be constant, it follows that $m_i n^i_k \overline{e}^i_k$ is also constant. Since $\mathbb{C}$ is a constant subcategory, we must have that $\overline{e}^i_k$ is constant as well. Therefore each v^i_k is a member of the sink $(u_j)_J$. It is therefore sufficient to show that the following is true: For each $j \in J$ there exist an $i \in I$ and $k \in K(p^{-1}(C_i))$ such that $u_{i-}(e_i) \leq v^i_{k-}(\overline{e}^i_k)$. For any morphism $u : B \to X$, with $u = u_j$ for some $j \in J$, consider the diagram:

$$\begin{tikzcd} B \arrow[dr, dotted, "!c^i_k"'] \arrow[ddrr, dashed, "!f_i"] \arrow[ddrrr, bend left, "u"] \arrow[ddd, bend right, "e"'] & & & \\ & B^i_k \arrow[d, "\overline{e}^i_k"] \arrow[dr, "\eta^i_k"'] \arrow[drr, bend left, "v^i_k"] & & \\ & Y^i_k \arrow[dr, "n^i_k"] & p^{-1}(C_i) \arrow[r, "p^{-1}(m_i)"] \arrow[d, "p_i"] & X \arrow[d, "p"] \\ C_i \arrow[ur, dashed, "!d_k"] \arrow[rr, equal] & & C_i \arrow[r, "m_i"'] & P \end{tikzcd}$$

Since pu is constant and has $(\mathscr{E}, \mathbb{M})$-factorisation we, it follows that $w = m_i$ for some $i \in I$. The pullback square of m_i along p establishes a unique morphism $f_i : B \to p^{-1}(C_i)$ such that $p_i f_i = e$ and $p^{-1}(m_i) f_i = u$. Since $(\eta^i_k : B^k_i \to p^{-1}(C_i))_{K(p^1(C_i))}$ is the near multi-$\mathscr{M}$-coreflection of $\mathbb{B}$ in $\mathbb{A}$, there exist a $k \in K(p^{-1}(C_i))$ and a unique morphism $c^i_k : B \to B^i_k$ such that $\eta^i_k \circ c^i_k = f_i$. It is straightforward to verify that

$$\begin{tikzcd} B \arrow[r, "e"] \arrow[d, "\overline{e}^i_k c^i_k"'] & C_i \arrow[d, "m_i"] \\ Y^i_k \arrow[r, "m_i n^i_k"'] & P \end{tikzcd}$$

commutes and the diagonalisation property establishes a unique morphism $d_k : C_i \to Y^i_k$ such that $n^i_k d_k = id$ and $d_k e = \overline{e}^i_k c$.

For the sake of simplicity, let us denote the morphism c^i_k by c and the morphism v^i_k by v. Note that $u = vc$. Consider the pushouts $u_-(e)$ and $v_-(e^i_k)$ as constructed in the commutative diagram below. Since $\tilde{v} d_k e = \tilde{v} e^i_k c^i_k = v_-(e^i_k) v c^i_k = v_-(e^i_k) vc = v_-(e^i_k) u$, the pushout $u_-(e) \simeq v_-(c_-(e))$ provides a unique morphism $q : v_-(c_-(C_i)) \to v_-(Y^i_k)$ such that $q \circ v_-(c_-(e)) = v_-(e^i_k)$ and $q \circ \overline{v}\overline{c} = \tilde{v} d_k$. In particular, it follows that $u_-(e) \simeq v_-(c_-(e)) \leq v_-(e^i_k)$ and our proof is complete.

$$
\begin{tikzcd}
Y_k^i \arrow[dddrrr, bend right, "\tilde{v}"'] & & & \\
& B \arrow[r, "c"] \arrow[d, "e"] \arrow[rr, bend left, "u"] & B_k^i \arrow[r, "v"] \arrow[d, "c_-(e)"] \arrow[ull, bend right, "\overline{e}_k^i"'] & X \arrow[d, "v_-(c_-(e))"', "\simeq u_-(e)"] \arrow[dd, bend left=60, "v_-(e_k^i)"] \\
& C_i \arrow[r, "\overline{c}"'] \arrow[uul, "d_k"] & c_-(C_i) \arrow[r, "\overline{v}"'] & v_-(c_-(C_i)) \arrow[d, dashed, "q"'] \\
& & & v_-(Y_k^i)
\end{tikzcd}
$$

□

Definition 4.127: $\mathscr{N}$-or $\mathbb{C}$-constant sink for P
Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and $\mathscr{N}$ a class of constant morphisms. Then, for each $\mathbb{A}$-object P, we call the sink $(m_i : C_i \to P)_I$ of all constant morphisms in $\mathscr{M}$ to P **the $\mathscr{N}$-constant sink for P.**
In case $\mathbb{C}$ is any constant subcategory, this sink is called the $\mathbb{C}$-constant sink for P. However, we need not consider constant subcategories, we could also take the sink of all morphisms in $\mathscr{M}$ with domain in a fixed subcategory.
Note that if $\mathbb{B}$ is nearly multi-$\mathscr{M}$-coreflective in $\mathbb{A}$ and $p : X \to P$ is in $\mathscr{E}$, then the construction in Proposition 4.126 can be repeated. In fact, this construction can be repeated for any arbitrary sink. Obviously one shouldn't expect that this will generally yield a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.
In case an arbitrary sink $(m_i)_I$ to P is chosen, the multiple pushout of the appropriate morphisms is denoted by $MP_{(m_i)_I}(p)$. With the assumptions of Proposition 4.126, it follows that $MP_{(m_i)_I}(p) \simeq ew^{\mathbb{B}}_{\mathbb{C}}(p)$ for a nearly multi-$\mathscr{M}$-coreflective $\mathbb{B}$ and constant subcategory $\mathbb{C}$. Note that the pushouts along the multicoreflection morphisms composed with $p^{-1}(m_i)$ was also constructed along a natural choice of morphism, but this is also not necessary. If there is a family $(e^i_k)_{i\in I, k\in K(i)}$ of morphisms along which the pushouts are constructed, then we will denote this by $MP^{(e^i_k)_{I,K(i)}}_{(m_i)_I}(p)$. For example, in Proposition 4.105 we constructed $D^{\mathbb{B}}_{\mathbb{C}}$ for a nearly multi-$\mathscr{M}$-coreflective subcategory $\mathbb{B}$ of $\mathbb{A}$. We may then denote the construction of the pushouts along the reflection morphisms as $MP^{\varepsilon}_{(m_i)_I}(p)$.

Lemma 4.128: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and $\mathbb{B}$ be nearly multi-$\mathscr{M}$-coreflective in $\mathbb{A}$. Let $p : X \to P$ be in $\mathscr{E}$. Let $(m_i : M_i \to P)_I$ and $(\tilde{m}_j : \tilde{M}_j \to P)_J$ be $\mathbb{A}$-sinks to P such that for each $i \in I$ there is a $j(i) \in J$ and an $\mathbb{A}$-morphism $k_{i,j(i)} : M_i \to \tilde{M}_{j(i)}$ such that $\tilde{m}_{j(i)} k_{i,j(i)} = m_i$. Then, $MP_{(m_i)_I}(p) \leq MP_{(\tilde{m}_j)_J}(p)$.

Proof : Assume the hypothesis and for each $i \in I$, let $k_{i,j(i)} : M_i \to \tilde{M}_{j(i)}$ denote a morphism such that $k_{i,j(i)} \circ m_i = \tilde{m}_{j(i)}$. For each $\mathbb{A}$-object X, let $(\eta_k : B_k \to X)_{K(X)}$ denote the near multi-$\mathscr{M}$-coreflection of X into $\mathbb{B}$. Fix any $i \in I$ and consider the pasted pullback diagrams:

$$
\begin{tikzcd}
p^{-1}(M_i) \arrow[r, "\tilde{p}_{i,j(i)}"] \arrow[d, "p_i"'] \arrow[rr, bend left, "p^{-1}(m_i)"] & p^{-1}(\tilde{M}_{j(i)}) \arrow[r, "p^{-1}(\tilde{m}_{j(i)})"] \arrow[d, "\tilde{p}_{j(i)}"'] & X \arrow[d, "p"] \\
M_i \arrow[r, "k_{i,j(i)}"'] \arrow[rr, bend right, "m_i"'] & \tilde{M}_{j(i)} \arrow[r, "\tilde{m}_{j(i)}"'] & P
\end{tikzcd}
$$

Note that we abuse notation and denote the pullback of m along p by $p^{-1}(m)$ even if m is not a member of $\mathscr{M}$. The existence of the morphism $\tilde{p}_{i,j(i)}$ is entirely established by the pullback property as the outer diagram commutes. For each $i \in I$ and each $j(i) \in J$, denote the respective near multi-$\mathscr{M}$-coreflections by $(\eta^i_k : B^i_k \to p^{-1}(M_i))_{k\in K(p^{-1}(M_i))}$, respectively $(\eta^{j(i)}_k : B^{j(i)}_k \to p^{-1}(\tilde{M}_{j(i)}))_{k\in K(p^{-1}(\tilde{M}_{j(i)})}$.

The near multi-$\mathscr{M}$-coreflections then establish the following: For each $k \in K(p^{-1}(M_i))$, there exists a $k(j(i)) \in K(p^{-1}(\tilde{M}_{j(i)})$ for which there is a unique morphism $f_i : B^i_k \to B^i_{k(j(i))}$ such that $\eta_{k(j(i))} \circ f_i = \tilde{p}_{i,j(i)} \circ \eta^i_k$. Hence, we have a commutative diagram:

$$\begin{tikzcd}
B^i_k \arrow[r, "f_i"] \arrow[d, "\eta^i_k"'] & B^{j(i)}_{k(j(i))} \arrow[d, "\eta^{j(i)}_{k(j(i))}"] \\
p^{-1}(M_i) \arrow[r, "\tilde{p}_{i,j(i)}"'] & p^{-1}(\tilde{M}_{j(i)})
\end{tikzcd}$$

For each $i \in I$ and $k \in K(p^{-1}(M_i))$, let $m^i_k \circ e^i_k : B^i_k \to M^i_k \to M_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $p_i \circ \eta^i_k$ and similarly, let $e^{j(i)}_{k(j(i))} \circ m^{j(i)}_{k(j(i))} : B^{j(i)}_{k(j(i))} \to \tilde{M}^{j(i)}_{k(j(i))} \to \tilde{M}_{j(i)}$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $\tilde{p}_{j(i)} \circ \eta^{j(i)}_{k(j(i))}$. The diagonalisation property establishes a unique morphism $d_i : M^i_k \to \tilde{M}^{j(i)}_{k(j(i))}$ such that

$$\begin{tikzcd}
B^i_k \arrow[r, "e^i_k"] \arrow[d, "e^{j(i)}_{k(j(i))}\circ f_i"'] & M^i_k \arrow[d, "k_{i,j(i)}\circ m^i_k"] \arrow[dl, dashed, "!d_i"'] \\
\tilde{M}^{j(i)}_{k(j(i))} \arrow[r, "\tilde{m}^{j(i)}_{k(j(i))}"'] & \tilde{M}_{j(i)}
\end{tikzcd}$$

commutes. Define $v^i_k : B^i_k \to X = p^{-1}(m_i) \circ \eta^i_k$ and $v^{j(i)}_{k(j(i))} : B^{j(i)}_{k(j(i)} \to X = p^{-1}(m_{j(i)}) \circ \eta^{j(i)}_{k(j(i))}$ and consider the pushout of e^i_k along v^i_k and the pushout of $e^{j(i)}_{k(j(i))}$ along $v^{j(i)}_{k(j(i))}$. It's then a straightforward computation to verify that the diagram

$$\begin{tikzcd}
B^{j(i)}_{k(j(i))} \arrow[d, "e^{j(i)}_{k(j(i))}"'] \arrow[rr, bend left, "v^{j(i)}_{k(j(i))}"] & B^i_k \arrow[l, "f_i"'] \arrow[r, "v^i_k"] \arrow[d, "e^i_k"] & X \arrow[d, "v^i_{k-}(e^i_k)"] \arrow[ddr, bend left, "v^{j(i)}_{k(j(i))-}(e^{j(i)}_{k(j(i))})"] & \\
\tilde{M}^{j(i)}_{k(j(i))} \arrow[drrr, bend right, "\hat{v}^{j(i)}_{k(j(i))}"'] & M^i_k \arrow[l, "d_i"'] \arrow[r, "\hat{v}^i_k"'] & v^i_{k-}(M^i_k) \arrow[dr, dashed, "!e_i"] & \\
 & & & v^{j(i)}_{k(j(i))-}(\tilde{M}^{j(i)}_{k(j(i))})
\end{tikzcd}$$

commutes for a unique morphism e_i. It is also easy to see that $MP_{(m_i)_I}(p) \simeq \bigvee_{I,K(p^{-1}(M_i))} v^i_{k-}(e^i_k) \leq \bigvee_{I,K(p^{-1}(\tilde{M}^{j(i)}_{k(j(i))})} v^{j(i)}_{k(j(i))-}(e^{j(i)}_{k((i))}) \leq MP_{(\tilde{m}_j)_J}(p)$ as $v^{j(i)}_{k(j(i))-}(e^{j(i)}_{k((i))})$ only contributes to the pushout used to construct $MP_{(\tilde{m}_j)_J}(p)$. Therefore $MP_{(m_i)_I}(p) \leq MP_{(\tilde{m}_j)_J}(p)$. $\square$

Remark 4.129: Assume that the hypothesis of Proposition 4.126 is satisfied. Whenever we have, for each $\mathbb{A}$-object P, a sink $(m^P_i)_{I(P)}$ such that $MP_{(m^P_i)_{I(P)}}(p)$ defines a dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, where $p : X \to P$ in $\mathscr{E}$, then we say the family $(P, (m^P_i)_{I(P)})$ **generates** D**.**

Remark 4.130: We will now proceed to construct a right adjoint to the Eilenberg-Whyburn dual closure operator.

Definition 4.131: $Shriek^{\circledast}_{\mathbb{C}}(-)$
Let $\mathbb{C}$ be a subcategory of the $(\mathscr{E}, \mathscr{M})$-structured category $\mathbb{A}$. Then, for each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, we define the full subcategory of $\mathbb{A}$ in the following way:

$$Shriek^{\circledast}_{\mathbb{C}}(D) = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} \cap X/\mathbb{C} : \ De \simeq e\}.$$

Proposition 4.132: Let $\mathbb{C}$ be an $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ with unit ε. Then $Shriek^{\circledast}_{\mathbb{C}}(D) \subset Shriek_{\mathbb{C}}(D)$.

Proof : Let X be a member of $Shriek^{\circledast}_{\mathbb{C}}(D)$. Then $De \simeq e$ for each $e \in X/\mathscr{E} \cap X/\mathbb{C}$. In particular $D\varepsilon_X \simeq \varepsilon_X$. Hence X is a member of $Shriek_{\mathbb{C}}(D)$. □

Proposition 4.133: Let D be a dual closure operator of a class $\mathscr{E}$ of epimorphisms in $\mathbb{A}$. Then $Shriek^{\circledast}_{\mathbb{C}}(D)$ is closed under $\mathscr{E}$-images.

Proof : Let X be in $Shriek^{\circledast}_{\mathbb{C}}(D)$ and let $e : X \to X'$ be in $\mathscr{E}$. Let $e' \in X'/\mathscr{E} \cap X'/\mathbb{C}$. Then, the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "D_X(e'e)"'] & X' \arrow[d, "D_{X'}(e')"] \\ D_X(C) \arrow[r, "D_{e,id_C}"'] \arrow[d, "\delta_{e'e}"'] & D_{X'}(C) \arrow[d, "\delta_{e'}"] \\ C \arrow[r, "id_C"'] & C \end{tikzcd}$$

commutes. Since X is in $Shriek^{\circledast}_{\mathbb{C}}(D)$ and C is in $\mathbb{C}$, it follows that $D_X(e'e) \simeq e'e$, or, equivalently, $\delta_{e'e}$ is an isomorphism. Then, $\delta_{e'} \circ D_{e,id_C}$ is an isomorphism so that D_{e,id_C} is a section. Since D_{e,id_C} is also a member of $\mathscr{E}$ and $\mathscr{E}$ is a class of epimorphisms, D_{e,id_C} is an epic section, hence an isomorphism. We can then easily conclude that $\delta_{e'}$ is an isomorphism. Consequently, X' is a member of $Shriek^{\circledast}_{\mathbb{C}}(D)$. □

Proposition 4.134: Let D be a dco of a class $\mathscr{E}$ of epimorphisms in $\mathbb{A}$. Then, for any subcategory $\mathbb{C}$ of $\mathbb{A}$, $Shriek^{\circledast}_{\mathbb{C}}(D)$ contains $\mathbb{C}$ provided that $D_{dom(e)}(e) \simeq e$ whenever $dom(e) \in \mathbb{C}$.

Proof : This is obvious by the assumptions on D. □

Theorem 4.135: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{C}$ a constant reflective subcategory of $\mathbb{A}$. Then, for each $D \in DCO(\mathbb{A}, \mathscr{E})$ and each subcategory $\mathbb{B}$ of $\mathbb{A}$, there holds:

$$ew^{\mathbb{B}}_{\mathbb{C}} \leq D \Leftrightarrow \mathbb{B} \subset Shriek^{\circledast}_{\mathbb{C}}(D)$$

and, in particular, $\mathbb{B} \subset Shriek^{\circledast}_{\mathbb{C}}(ew^{\mathbb{B}}_{\mathbb{C}})$ and $ew_{\mathbb{C}}^{Shriek^{\circledast}_{\mathbb{C}}(D)} \leq D$.

Proof : Suppose that $ew^{\mathbb{B}}_{\mathbb{C}} \leq D$. Then, for each B in $\mathbb{B}$ and each $e \in B/\mathscr{E} \cap B/\mathbb{C}$, we have that

$$\begin{tikzcd} B \arrow[r, "e"] \arrow[d, "id"'] & C \arrow[d, "id"] \\ B \arrow[r, "e"'] & C \end{tikzcd}$$

is a pushout square that contributes to the formation of the multiple pushout $ew^{\mathbb{B}}_{\mathbb{C}}(e)$. Consequently, $ew^{\mathbb{B}}_{\mathbb{C}}(e) \simeq e$. Then, $e \simeq ew^{\mathbb{B}}_{\mathbb{C}}(e) \leq De \leq e$ so that $De \simeq e$. It follows that $\mathbb{B} \subset Shriek^{\circledast}_{\mathbb{C}}(D)$.

On the other hand, assume that $\mathbb{B} \subset Shriek^{\circledast}_{\mathbb{C}}(D)$. Let $p : X \to P$ be a member of $\mathscr{E}$. Consider the sink $(u_i : B_i \to X)_I$ of all morphisms u_i such that pu_i is constant for each i. Then, if $v_i \circ e_i$ is an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_i for each i, we have $e_i \in B_i/\mathscr{E} \cap B_i/\mathbb{C}$ and thus $D(e_i) \simeq e_i$ for each i. Since D-closed morphisms are closed under pushouts and multiple pushouts, it follows that $ew^{\mathbb{B}}_{\mathbb{C}}(p)$ is D-closed. Then, $ew^{\mathbb{B}}_{\mathbb{C}}(p) \simeq D(ew^{\mathbb{B}}_{\mathbb{C}}(p)) \leq Dp$. Since p was arbitrary, we have that $ew^{\mathbb{B}}_{\mathbb{C}} \leq D$. □

Remark 4.136: If we choose to ignore constant subcategories, then we can also consider the following construction:

Definition 4.137: $D^{(-)} : Sub(\mathbb{A}) \to DCO(\mathbb{A}, \mathscr{E})$
For each subcategory $\mathbb{B}$ of the $(\mathscr{E}, \mathbb{M})$-category $\mathbb{A}$ and each $p : X \to P$, we define the map $D^{(-)} : Sub(\mathbb{A}) \to DCO(\mathbb{A}, \mathscr{E})$ by considering the all-sink $(u_i)_I$ from $\mathbb{B}$ to X. Then, we construct a pushout of the $\mathscr{E}$-part of an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_i along u_i for each i. We then form the multiple pushout of all these pushouts and denote this morphism by $D^{\mathbb{B}}(p)$.

$$\begin{tikzcd}
B_i \arrow[r, "e_i"] \arrow[d, "u_i"'] & M_i \arrow[d, "v_i"] \arrow[ddr, bend left, "m_i"] & \\
X \arrow[r, "u_{i-}(e_i)"] \arrow[dr, "D^{\mathbb{B}}(p)"'] \arrow[ddrr, bend right, "p"'] & u_{i-}(M_i) \arrow[d, "p_i"] \arrow[dr, dashed, "\overline{p}_i"] & \\
& D^{\mathbb{B}}(P) \arrow[r, dotted, "\omega_p"'] & P
\end{tikzcd}$$

We first verify that this is in fact a dual closure operator. Let $p : X \to P$ be in $\mathscr{E}$ and $(u_i : B_i \to X)_I$ be the all sink from $\mathbb{B}$ to X. Let $m_i e_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_i and $u_{i-}(e_i) : X \to u_{i-}(M_i)$ be a pushout of e_i along u_i. Then, for each i, there is a unique morphism $\overline{p}_i : u_{i-}(M_i) \to P$ such that $\overline{p}_i v_i = m_i$ and $\overline{p}_i u_{i-}(e_i) = p$. If $(p_i)_I$ is a sink such that the multiple pushout $D^{\mathbb{B}}(p)$ of $(u_{i-}(e_i))_I$ satisfies $p_i \circ u_{i-}(e_i) = D^{\mathbb{B}}(p)$ for each $i \in I$, then this induces a morphism $\omega_p : D^{\mathbb{B}}(P) \to P$ such that $\omega_p \circ D^{\mathbb{B}}(p) = p$ and $\omega_p \circ p_i = \overline{p}_i$. Since $\mathbb{A}$ has $(\mathscr{E}, \mathbb{M})$-factorisations, it is $\mathscr{E}$-cocomplete and by the cancellation properties of $\mathscr{E}$, we have that both $D^{\mathbb{B}}(p)$ and ω_p are members of $\mathscr{E}$. It's also clear that $D^{\mathbb{B}}(p) \leq p$.

Now we show that $D^{\mathbb{B}}(p) \leq D^{\mathbb{B}}(q)$ whenever $p \leq q$. Suppose that j is a morphism with $jp = q$, where $p, q \in X/\mathscr{E}$, i.e., $p \leq q$ in $quot(X)$. Following our notation above, let $n_i f_i$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of qu_i. For each i, the diagonalisation property establishes a morphism such that
$$\begin{tikzcd}
B_i \arrow[r, "e_i"] \arrow[d, "f_i"'] & M_i \arrow[d, "jm_i"] \arrow[dl, "d_i"'] \\
N_i \arrow[r, "n_i"'] & Q
\end{tikzcd}$$
commutes.
Since $u_i(-)$ is order preserving, $u_{i-}(e_i) \leq u_{i-}(f_i)$ for each i. Then, $D^{\mathbb{B}}(p) \simeq \bigvee_I u_{i-}(e_i) \leq \bigvee_I u_{i-}(f_i) \simeq D^{\mathbb{B}}(q)$.

The last thing we need to verify is that $f_-(D^{\mathbb{B}}(p)) \leq D^{\mathbb{B}}(f_-(p))$ for each $p \in \mathscr{E}$ and each $f : X \to Y$. Still following our notation, if $(u_i)_I$ is the sink as above, then fu_i is a morphism from $\mathbb{B}$ to $Y = dom(f_-(p))$. Let $(\overline{u}_j)_J$ be the all-sink from $\mathbb{B}$ to Y, then for each i, there is a $j(i) \in J$ such that $fu_i = \overline{u}_j(i)$. Because of this observation, we may, without loss of generality, assume that $I \subset J$ and $\overline{u}_i = fu_i$ for each $i \in I$. Let $\overline{m}_j \overline{e}_j$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $f_-(p)\overline{u}_j$ for each j. Then, by the diagonalisation property, it follows that
$$\begin{tikzcd}
B_i \arrow[r, "e_i"] \arrow[d, "\overline{e}_i"'] & M_i \arrow[d, "km_i"] \arrow[dl, "h_i"'] \\
\overline{M}_i \arrow[r, "\overline{m}_i"'] & f_-(P)
\end{tikzcd}$$
commutes for each $i \in I$. Consequently, $e_i \leq \overline{e}_i$ for each such i. By Corollary 4.6, we know that $f_-(-)$ preserves joins, or, equivalently, multiple pushouts. Hence, $f_-(D^{\mathbb{B}}(p)) \simeq f_-(\bigvee_I u_{i-}(e_i)) \simeq \bigvee_I f_-(u_{i-}(e_i)) \simeq \bigvee_I (fu_i)_-(e_i) \simeq \bigvee_I \overline{u}_{i-}(e_i) \leq \bigvee_I \overline{u}_{i-}(\overline{e}_i) \leq \bigvee_J \overline{u}_{j-}(\overline{e}_j) \simeq D^{\mathbb{B}}(f_-(p))$. It follows that $D^{\mathbb{B}}$ is a dual closure operator of $\mathscr{E}$ in $\mathbb{A}$.

Proposition 4.138: If $\mathbb{A}$ is an $(\mathscr{E}, \mathbb{M})$-category, then the map $D^{(-)} : (Sub(\mathbb{A}), \subset) \to (DCO(\mathbb{A}, \mathscr{E}), \leq)$ is order preserving.

Proof : Assume that $\mathbb{B} \subset \mathbb{B}'$ and $p : X \to P$ is in $\mathscr{E}$. Further, let $(u_i)_I$, respectively $(w_j)_J$, be the all-sinks from $\mathbb{B}$, respectively $\mathbb{B}'$, to X. It's then clear that $(u_i)_I$ is a subsink from $(w_j)_J$. Hence, without loss of generality, we may assume that $I \subset J$. Let $m_j e_j$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of pw_j. Then, $m_i e_i$

is also such a factorisation of pu_i for each i. Since multiple pushouts acts as joins and $\{e_i| \mid i \in I\}$ is a subclass of $\{e_j \mid j \in J\}$, the conclusion follows. $\square$

Proposition 4.139: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and $\mathbb{B}$ a subcategory of $\mathbb{A}$ that is closed under $\mathscr{E}$-images. Then, $D^{\mathbb{B}}(p) \simeq \bigvee_I m_{i-}(e_i)$ where $(m_i)_I$ is the sink of all morphisms in $\mathscr{M}$ with domain in $\mathbb{B}$ such that pm_i has $(\mathscr{E}, \mathscr{M})$-factorisation $n_i e_i$.

<u>*Proof*</u> : Let $(u_j : B_j \to X)_J$ be the sink of all morphisms with domain in $\mathbb{B}$ and codomain X. For each $j \in J$, let $m_j \overline{e}_j : B_j \to \overline{B}_j \to X$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of u_j. Since $\mathbb{B}$ is closed under $\mathscr{E}$-images, we may as well assume that $I = J$ and $(m_i)_I$ is the sink as described above. It should be clear that $\bigvee_I m_{i-}(e_i) \leq D^{\mathbb{B}}(p)$. For each j, let $\tilde{m}_j \tilde{e}_j$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of pu_j. We show that for each $j \in J$, there holds $u_{j-}(\tilde{e}_j) \leq m_{j-}(e_j)$.

For each $j \in J$, the diagonalisation property establishes a unique isomorphism d_j such that $d_j \tilde{e}_j = e_j \overline{e}_j$ and $n_j d_j = \tilde{m}_j$. Note that $u_{i-}(\tilde{e}_j) \simeq m_{j-}(\overline{e}_{j-}(\tilde{e}_j))$. Hence it is sufficient to show that $\overline{e}_{j-}(\tilde{e}_j) \leq e_j$. For, in this case we have $u_{j-}(\tilde{e}_j) \simeq m_{j-}(\overline{e}_{j-}(\tilde{e}_j)) \leq m_{j-}(e_j)$. Let
$$\begin{tikzcd} B_j \arrow[r, "\tilde{e}_j"] \arrow[d, "\overline{e}_j"'] & \tilde{C}_j \arrow[d, "v_j"] \\ \overline{B}_j \arrow[r, "\overline{e}_{j-}(\tilde{e}_j)"'] & \overline{e}_{j-}(\tilde{C}_j) \end{tikzcd}$$
be a pushout square.

Since $d_j \tilde{e}_j = e_j \overline{e}_j$, the pushout square provides us with a unique morphism f such that $f \overline{e}_{j-}(\tilde{e}_j) = e_j$ and $f v_j = d_j$. Hence $\overline{e}_{j-}(\tilde{e}_j) \leq e_j$ and the result follows. $\square$

Proposition 4.140: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with pullbacks and assume that $\mathbb{B}$ is a nearly multi-$\mathscr{M}$-coreflective subcategory of $\mathbb{A}$. Let $p : X \to P$ be in $\mathscr{E}$ and $(m_i : M_i \to P)_I$ the sink of all morphisms to P that are members of $\mathscr{M}$. Then, $D^{\mathbb{B}}(p) \simeq MP_{(m_i)_I}(p)$.

<u>*Proof*</u> : Assume the hypothesis with $p \in \mathscr{E}$ and $(m_i)_I$ as above. We will abuse notation and denote the pullback of m_i along p by
$$\begin{tikzcd} p^{-1}(M_i) \arrow[r, "p^{-1}(m_i)"] \arrow[d, "\bar{p}_i"'] & X \arrow[d, "p"] \\ M_i \arrow[r, "m_i"'] & P. \end{tikzcd}$$

First, if $u : B \to X$ is any morphism with B in $\mathbb{B}$ and pu has $(\mathscr{E}, \mathbb{M})$-factorisation $m \circ e : B \to M \to P$, then it follows that $m = m_i$ for some $i \in I$. The pullback property establishes a unique morphism $f_i : B \to p^{-1}(M_i)$ such that $\bar{p}_i \circ f_i = e$ and $p^{-1}(m_i) \circ f_i = u$. Let $(\eta^i_k : B^i_k \to p^{-1}(M_i))_{k \in K(p^{-1}(M_i))}$ denote the near multi-$\mathscr{M}$-coreflection of $p^{-1}(M_i)$ into $\mathbb{B}$. Then, there is a k for which there is a unique morphism $c^i_k : B \to B^i_k$ such that $\eta^i_k \circ c^i_k = f_i$. Define $v^i_k = p^{-1}(m_i) \circ \eta^i_k$ and let $m^i_k \circ e^i_k : B^i_k \to M^i_k \to M_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of $\bar{p}_i \circ \eta^i_k$. Consequently we have a commutative diagram:

$$\begin{tikzcd} & B \arrow[d, dotted, "!c^i_k"] \arrow[dr, dashed, "!f_i"] \arrow[drr, bend left, "u"] \arrow[ddl, bend right, "e"'] & & \\ & B^i_k \arrow[d, "\bar{e}^i_k"] \arrow[r, "\eta^i_k"] \arrow[rr, bend left, "v^i_k"] & p^{-1}(M_i) \arrow[r, "p^{-1}(m_i)"] \arrow[d, "p_i"] & X \arrow[d, "p"] \\ & M^i_k \arrow[r, "m^i_k"'] & M_i \arrow[r, "m_i"'] & P \\ M_i \arrow[ur, dashed, "!d"] \arrow[urr, equal] & & & \end{tikzcd}$$

Note that the morphism d in the above diagram is established by the diagonalisation property. We now show that the pushout of e along u is smaller or equal to the pushout of e^i_k along v^i_k. Consider the diagram containing the pushout squares and morphisms as defined above:

$$\begin{tikzcd}
& B \arrow[r, "c^i_k"'] \arrow[d, "e"'] \arrow[rr, bend left, "u"] & B^i_k \arrow[r, "v^i_k"] \arrow[d, "c^i_{k-}(e)"'] \arrow[ddll, bend right=60, "e^i_k"'] & X \arrow[d, "v^i_{k-}(c^i_{k-}(e))"', "\simeq u_-(e)"] \arrow[dd, bend left=60, "v_-(e^i_k)"] \\
& M \arrow[r, "\bar{c}"'] \arrow[dl, "d"'] & c^i_{k-}(M) \arrow[r, "\bar{v}"'] & v^i_{k-}(c^i_{k-}(M)) \arrow[d, dashed, "!q"'] \\
M^i_k \arrow[rrr, "\tilde{v}"'] & & & v^i_{k-}(M^i_k)
\end{tikzcd}$$

Note that it's commutative and the existence of q is established by the pushout property since $\tilde{v}d \circ e = \tilde{v}e^i_k c^i_k = v^i_{k-}(e^i_k) \circ v^i_k \circ c^i_k$. Then, $u_-(e) \simeq v^i_{k-}(c^i_{k-}(e)) \leq v^i_{k-}(e^i_k)$.

To show the other inequality is trivial. To see this, it's sufficient to show that v^i_k is a member of the sink of all morphisms from $\mathbb{B}$ to P and this is unquestionably the case. $\square$

Remark 4.141: We will now construct a right adjoint to $D^{(-)}$ that is quite similar to $Shriek^{\circledast}(-)$.

Definition 4.142: $S_{\circledast}(-) : DCO(\mathbb{A}, \mathscr{E}) \to Sub(\mathbb{A})$**.**
For each dual closure operator D of a class $\mathscr{E}$ of morphisms in $\mathbb{A}$, we consider the full subcategory $S_{\circledast}(D)$ of $\mathbb{A}$ with object class

$$\{X \in \mathbb{A} \mid \forall e \in quot(X) : \ De \simeq e\}.$$

Proposition 4.143: $S_{\circledast}(-) : DCO(\mathbb{A}, \mathscr{E}) \to Sub(\mathbb{A})$ is order preserving whenever $\mathscr{E}$ is a class of epimorphisms.

$\underline{\mathcal{P}roof}$: Suppose $D_1 \leq D_2$ and that X is a member of $S_{\circledast}(D_1)$. Let $e : X \to Y$ be in $\mathscr{E}$. Then, $e \simeq D_1 e \leq D_2 e \leq e$ so that $D_2 e \simeq e$, i.e., X is a member of $S_{\circledast}(D_2)$. $\square$

Proposition 4.144: For any dual closure operator D of a class $\mathscr{E}$ of epimorphisms, $S_{\circledast}(D)$ is closed under $\mathscr{E}$-images.

$\underline{\mathcal{P}roof}$: Let X be in $S_{\circledast}(D)$ and let $e : X \to X'$ be in $\mathscr{E}$ and $e' \in X'/\mathscr{E}$. Then, the diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "D_X(e'e)"'] & X' \arrow[d, "D_{X'}(e')"] \\
D_X(C) \arrow[r, "D_{e,id_C}"'] \arrow[d, "\delta_{e'e}"'] & D_{X'}(C) \arrow[d, "\delta_{e'}"] \\
C \arrow[r, "id_C"'] & C
\end{tikzcd}$$

commutes. Since X is in $S_{\circledast}(D)$, it follows that $D_X(e'e) \simeq e'e$ or equivalently, $\delta_{e'e}$ is an isomorphism. Then, $\delta_{e'} \circ D_{e,id_C}$ is an isomorphism so that D_{e,id_C} is a section. Since D_{e,id_C} is also a member of $\mathscr{E}$ and $\mathscr{E}$ is a class of epimorphisms, D_{e,id_C} is an epic section, hence an isomorphism. It is then obvious that $\delta_{e'}$ is an isomorphism and hence X' is a member of $S_{\circledast}(D)$. $\square$

Proposition 4.145: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category with reflective constant subcategory $\mathbb{C}$. Let $R : \mathbb{A} \to \mathbb{C}$ denote the reflector with unit ε. For every dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$, there holds: $S_{\circledast}(D) \subset Shriek^{\circledast}_{\mathbb{C}}(D) \subset Shriek_{\mathbb{C}}(D)$. If $\mathbb{C}$ is the smallest constant subcategory of $\mathbb{A}$, the three subcategories coincide if $D = D_{\mathbb{B}}$ for some reflective subcategory $\mathbb{B}$ of $\mathbb{A}$ that contains $\mathbb{C}$.

$\underline{\mathcal{P}roof}$: Let X be in $S_{\circledast}(D)$ and let $e : X \to C$ be a member of $\mathscr{E}$ with C in $\mathbb{C}$. Since X is in $S_{\circledast}(D)$, we have that $De \simeq e$, hence X is in $Shriek^{\circledast}_{\mathbb{C}}(D)$. Hence the first inclusion holds.

Let Y be in $Shriek^{\circledast}_{\mathbb{C}}(D)$ and consider the reflection morphism ε_Y. Since RY is a member of $\mathbb{C}$ and Y is in $Shriek^{\circledast}_{\mathbb{C}}(D)$, we must have that $D\varepsilon_Y \simeq \varepsilon_Y$. It follows that Y is in $Shriek_{\mathbb{C}}(D)$.

Let $\mathbb{C}$ be the smallest reflective constant subcategory of $\mathbb{A}$. This necessarily implies that $\mathbb{C} = \{A \in \mathbb{A} \mid A \simeq X_m$ where $e_m : X \to X_m = max(quot(X))\}$. Let $\mathbb{B}$ be any $\mathscr{E}$-reflective subcategory of $\mathbb{A}$ that contains $\mathbb{C}$. In view of the above part of the proof, it is sufficient to prove that $Shriek_{\mathbb{C}}(D_{\mathbb{B}}) \subset S_{\circledast}(D_{\mathbb{B}})$. Let X be in $Shriek_{\mathbb{C}}(D_{\mathbb{B}})$. Then $D_{\mathbb{B}}\varepsilon_X \simeq \varepsilon_X$. By Proposition 4.93, we have that $Shriek_{\mathbb{C}}(D_{\mathbb{B}}) = \mathscr{L}(\mathbb{P})$. Let $S : \mathbb{A} \to \mathbb{B}$ be a reflector with unit ρ. Note that X is in $Shriek_{\mathbb{C}}(D_{\mathbb{B}})$ if and only if $\rho_X \simeq \varepsilon_X$ which is the case if and only if SX is a member of $\mathbb{C}$. Take any morphism $p : X \to P$ in $\mathscr{E}$. Then $D_{\mathbb{B}}(p) \simeq \rho_X \wedge p \simeq \varepsilon_X \wedge p \simeq p$. Hence, if X is in $Shriek_{\mathbb{C}}(D_{\mathbb{B}})$, then X is in $S_{\circledast}(D_{\mathbb{B}})$ and our proof is complete. $\square$

Theorem 4.146: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. Then, for each dual closure operator D of $\mathscr{E}$ in $\mathbb{A}$ and each subcategory $\mathbb{B}$ of $\mathbb{A}$, the following holds:

$$D^{\mathbb{B}} \leq D \Leftrightarrow \mathbb{B} \subset S_{\circledast}(D).$$

In particular, $\mathbb{B} \subset S_{\circledast}(D^{\mathbb{B}})$ and $D^{S_{\circledast}(D)} \leq D$.

Proof : The proof is similar to the proof of theorem 4.135. First assume that $D \leq D^{\mathbb{B}}$ and let X be in $\mathbb{B}$. Let $p : X \to P$ be any member of $\mathscr{E}$. Then id_X is a member of the sink along which the pushouts are constructed. Since $p \circ id_X$ has $(\mathscr{E}, \mathbb{M})$-factorisation $id_X \circ p$ and p is a pushout of p along id_X, we must have $p \leq D^{\mathbb{B}}(p)$. Since $D^{\mathbb{B}}$ is a dual closure operator, we have $p \simeq D^{\mathbb{B}}(p) \leq Dp$ and therefore $Dp \simeq p$. It follows that X is a member of $S_{\circledast}(D)$.

On the other hand, assume that $\mathbb{B} \subset S_{\circledast}(D)$ and let $p : X \to P$ be any member of $\mathscr{E}$. Let $(u_i)_I$ be the all-sink from $\mathbb{B}$ to X and let $m_i e_i$ be an $(\mathscr{E}, \mathbb{M})$-factorisation of pu_i. Since $\mathbb{B} \subset S_{\circledast}(D)$, we must have that $De_i \simeq e_i$, i.e., e_i is D-closed for each $i \in I$. Since D-closed morphisms are closed under pushouts and multiple pushouts, it follows that $D^{\mathbb{B}}(p)$ is D-closed. Consequently, $D^{\mathbb{B}}(p) \simeq D(D^{\mathbb{B}}(p)) \leq Dp$. Since p was arbitrary, we must have that $D^{\mathbb{B}} \leq D$. Therefore our proof is complete. $\square$

Example 4.147: This example will serve two purposes. We will show that the inclusions in Proposition 4.145 can all be proper. The other thing we will show is that $Shriek^{*}_{\mathbb{C}}(D)$ and $Shriek^{\dagger}(D)$ can also be distinct. The reason for this example is to ensure that all the constructions $D^{\mathbb{B}}$, $D^{\mathbb{B}}_{\mathbb{C}}$, $ew^{\mathbb{B}}_{\mathbb{C}}$, $D_{\mathbb{B}}$ and the right adjoint $d(-)$ of $Shriek^{*}_{\mathbb{C}}(D)$ are all distinct. Note that the Cassidy Hébert Kelly dual closure operator was only right adjoint to $Shriek^{*}_{\mathbb{C}}(D)$ for idempotent dual closure operators $D \leq D_{\mathbb{C}}$.

Consider the category (surjective, mono-source)-category $\mathbb{A} = \mathbb{Ab}$ of all abelian groups and group homomorphisms. Let $\mathbb{C}$ denote the surjective-reflective subcategory of all torsion-free abelian groups.

Note that any surjective homomorphism $\varphi : A \to B$ can be represented by a homomorphism $\varphi : A \to A/Ker\varphi$. For the remainder of this example, we will always assume that the codomain of a surjective homomorphism is given by a factor group of the domain. For each abelian group X, let TX denote the torsion part of X. The Cassidy Hébert Kelly dual closure operator D induced by $\mathbb{C}$ is then defined as follows:

For any surjective homomorphism $\varphi : A \to A/K$, $D(\varphi) : A \to A/(K \cap TA) = D\varphi : A \to A/TK$, where $D(\varphi)$ is the canonical morphism. Furthermore it ought to be easy to see that $\delta_{\varphi}(a + TK) = a + K$ for every a in A. It's then easy to see that any surjective homomorphism is D-closed if and only if $TK = K$, i.e., if and only if the kernel of φ is a torsion group. Let $R : \mathbb{A} \to \mathbb{C}$ be the reflector with reflection morphism $\varepsilon_X : X \to X/TX$ for each abelian group X.

We first show that $S_{\circledast}(D) = \mathbb{Tor}$. By the argument above, we can easily see that $S_{\circledast}(D) = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} : De \simeq e\} = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} : Ker(e) \in \mathbb{Tor}\}$. If X is a torsion group and $e : X \to X/K$ is any surjective homomorphism, then, of course, we must have that the kernel of e is torsion. Hence, $\mathbb{Tor}$ is a subcategory of $S_{\circledast}(D)$. For the other inclusion, suppose that X is not torsion. Then there is an element $a \in X$ such that $|a| = \infty$. So, $\langle a \rangle$ is a normal subgroup of X that is not torsion. The canonical morphism $\gamma : X \to X/\langle a \rangle$ is then not D-closed. It follows that X is not in $S_{\circledast}(D)$.

We now show that $Shriek^{\circledast}_{\mathbb{C}}(D) = \mathbb{Tor}$ as well. In view of Proposition 4.145, it's sufficient to show that $\mathbb{Tor} \subset Shriek^{\circledast}_{\mathbb{C}}(D)$. Let X be a member of $Shriek^{\circledast}_{\mathbb{C}}(D)$. Consider the surjection $e : X \to X/X$. Since X is in $Shriek^{\circledast}_{\mathbb{C}}(D)$, we have that $\varepsilon_X \simeq \varepsilon_X \wedge e \simeq De \simeq e$. Consequently $TX = X$ so that X is torsion.

Now we show that $Shriek_{\mathbb{C}}(D) = \mathbb{A}$. Let A be any $\mathbb{A}$-object. Then $D\varepsilon_A \simeq \varepsilon_A \wedge \varepsilon_A \simeq \varepsilon_A$, hence A is in $Shriek_{\mathbb{C}}(D)$.

Let $\hat{D}$ be the identity dual closure operator. Explicitly, $\hat{D}p = p$ and $\delta_p = id$ for each $p \in \mathscr{E}$. We show that $Shriek^{*}_{\mathbb{C}}(\hat{D}) \neq Shriek^{\dagger}(\hat{D})$. To see this, consider the following: $Shriek_{\mathbb{C}}(\hat{D}) = Shriek_{\mathbb{TFAb}}(\hat{D}) = \{X \in \mathbb{A} \mid \varepsilon_X = \hat{D}\varepsilon_X \simeq id_X\} = \{X \in \mathbb{A} \mid TX = \{0\}\} = \mathbb{TFAb} = \mathbb{C}$, whereas $Shriek^{\dagger}(\hat{D}) = \{X \in \mathbb{A} \mid \forall e \in X/\mathscr{E} : e = \hat{D}e \simeq id_X\} = \{X \in \mathbb{A} \mid X \simeq 0\}$. Obviously the latter is the category of all trivial groups which is a proper subcategory of the category of all torsion-free abelian groups. Since $Shriek^{*}_{\mathbb{C}}(-)$ preserves joins, there exists a right adjoint, say $r(-)$. If r always coincided with the Cassidy Hébert Kelly dual closure operator, then it would have to follow that $Shriek^{*}_{\mathbb{C}}(D) = Shriek^{\dagger}(D)$.

We now show that $ew^{\mathbb{B}}_{\mathscr{N}}$ can be distinct from $D^{\mathbb{B}}$. Let $\mathbb{A}$ be the category $\mathbb{Set}$ of sets and maps with factorisation structure $(\mathscr{E}, \mathbb{M})$=(surjective maps, point-separating sources). Let $\mathscr{N}$ be the class of weakly constant morphisms consisting of all morphisms with empty domain. Let $\mathbb{B}$ be the category of finite sets. Consider the unique surjective function $p : 2 \to 1$ defined by $p(0) = p(1) = 0$. Since 2 is a finite set, id_2 is a morphism with domain in $\mathbb{B}$. Furthermore, pid_2 has $(\mathscr{E}, \mathbb{M})$-factorisation given by $id_2 \circ p$. Since p is a pushout of id_2 along p, p contributes to the pushout used to construct $D^{\mathbb{B}}(p)$. It follows that $D^{\mathbb{B}}(p) \simeq p$. We now show that $ew^{\mathbb{B}}_{\mathscr{N}}(p) \simeq id_2$. Since $p \not\simeq id_2$, the result follows. To construct $ew^{\mathbb{B}}_{\mathscr{N}}(p)$, we need to consider the sink of all morphisms u with finite domain such that pu is $\mathscr{N}$-constant. This is the case if and only if u has empty domain and must be the unique morphism $\varnothing \to 2$. We will denote the unique morphism from $\varnothing$ to X by $\varnothing_X$ for each set X. Then $pu = p\varnothing_2$ has $(\mathscr{E}, \mathscr{M})$-factorisation $\varnothing_1 \circ id_{\varnothing}$. Since u is the only morphism that contributes to the formation of $ew^{\mathbb{B}}_{\mathscr{N}}(p)$, we need only form the pushout of $id_{\varnothing}$ along $\varnothing_2$. It's easy to see that
$$\begin{tikzcd} \varnothing \arrow[r, "id_{\varnothing}"] \arrow[d, "\varnothing_2"'] & \varnothing \arrow[d, "\varnothing_2"] \\ 2 \arrow[r, "id_2"'] & 2 \end{tikzcd}$$
is the desired pushout square so that $ew^{\mathbb{B}}_{\mathscr{N}}(p) \simeq id_2$.

In a similar manner as with $D_{\mathbb{B}}$ and $d(\mathbb{B})$ or $D^{\mathbb{B}}_{\mathbb{C}}$ and $ew^{\mathbb{B}}_{\mathbb{C}}$, it follows that $S_{\circledast}(D)$ and $Shriek^{\circledast}_{\mathbb{C}}(D)$ need not coincide.

Remark 4.148: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category and let $\mathbb{C}$ be a constant subcategory of $\mathbb{A}$. Note that $Shriek^{\dagger}(D) \subset Shriek_{\mathbb{C}}(D)$ for every dual closure operator of $\mathscr{E}$ in $\mathbb{A}$. Since $D_{\mathbb{B}}$ and $d(\mathbb{B})$ are the respective right adjoints, it's an easy exercise to show that $D_{\mathbb{B}} \leq d(\mathbb{B})$ for every $\mathscr{E}$-reflective subcategory $\mathbb{B}$ of $\mathbb{A}$ that contains $\mathbb{C}$. Since $S_{\circledast}(-)$, $Shriek^{\circledast}_{\mathbb{C}}(-)$ and $Shriek_{\mathbb{C}}(-)$ are order preserving, we must have that
$$\begin{array}{ccccccc} S_{\circledast}(D_{\mathbb{B}}) & \subset & Shriek^{\circledast}_{\mathbb{C}}(D_{\mathbb{B}}) & \subset & Shriek_{\mathbb{C}}(D_{\mathbb{B}}) & = & \mathscr{L}(\mathbb{B}) \\ \cap & & \cap & & \| & & \\ S_{\circledast}(d(\mathbb{B})) & \subset & Shriek^{\circledast}_{\mathbb{C}}(d(\mathbb{B})) & \subset & Shriek_{\mathbb{C}}(d(\mathbb{B})) & = & \mathscr{L}(\mathbb{B}). \end{array}$$
Note that $Shriek_{\mathbb{C}}(D_{\mathbb{B}}) = \mathscr{L}(\mathbb{B})$ follows from Proposition 4.93. Hence, we need only show that $Shriek_{\mathbb{C}}(d(\mathbb{B})) \subset \mathscr{L}(\mathbb{B})$.

From Proposition 4.81 we have $\mathbb{B} = Shriek^{*}_{\mathbb{C}}(D_{\mathbb{B}})$. Since $Shriek^{*}_{\mathbb{C}}(d(\mathbb{B})) \subset Shriek^{*}_{\mathbb{C}}(D_{\mathbb{B}})$ and $d(\mathbb{B}) \leq d(\mathbb{B})$ implies that $\mathbb{B} \subset Shriek^{*}_{\mathbb{C}}(d(\mathbb{B}))$, we must have that $\mathbb{B} = Shriek^{*}_{\mathbb{C}}(d(\mathbb{B}))$. Then, by Proposition 4.95, we have $Shriek_{\mathbb{C}}(d(\mathbb{B})) \subset \mathscr{L}(Shriek^{*}_{\mathbb{C}}(d(\mathbb{B}))) = \mathscr{L}(\mathbb{B})$. This takes care of all the one side inclusions.

If $\mathbb{C}$ is the smallest constant subcategory of $\mathbb{A}$, then these all coincide. In view of the above inclusions, we need only see that $\mathscr{L}(\mathbb{B}) \subset S_{\circledast}(D_{\mathbb{B}})$. But this follows directly from Proposition 4.145.

The last thing we mention is that if all the subcategories above coincide for all $\mathscr{E}$-reflective subcategories $\mathbb{B}$ of $\mathbb{A}$, then the constant subcategory must be the smallest constant subcategory of $\mathbb{A}$. Let X be in $\mathbb{A}$ and let e_m denote the maximum of $X/\mathscr{E}$. Let ε_X denote the reflection morphism. Obviously $\varepsilon_X \leq e_m$. We need only show that $e_m \leq \varepsilon_X$. Let $\mathbb{P}$ be the subcategory of $\mathbb{A}$ that only consists of the object X

and let $\mathbb{B} = \mathscr{R}(\mathbb{P})$. Let ρ_X denote the reflection morphism of X into $\mathbb{B}$. Since $\mathbb{P} \subset \mathscr{L}(\mathscr{R}(\mathbb{P})) = \mathscr{L}(\mathbb{B})$ and X is in $\mathbb{P}$, we must have that $\rho_X \simeq \varepsilon_X$. If the subcategories above all coincide, then we must have that $Shriek_{\mathbb{C}}(D_{\mathbb{B}}) \subset S_{\circledast}(D_{\mathbb{B}})$.

Now $D_{\mathbb{B}}(\varepsilon_X) \simeq \rho_X \wedge \varepsilon_X \simeq \varepsilon_X$, so that X is a member of $Shriek_{\mathbb{C}}(D_{\mathbb{B}})$. We must then have that X is also in $S_{\circledast}(D_{\mathbb{B}})$ so that, in particular, $D_{\mathbb{B}}(e_m) \simeq e_m$. Note that $e_m \simeq D_{\mathbb{B}}(e_m) \simeq e_m \wedge \rho_X \simeq e_m \wedge \varepsilon_X$. It follows that $e_m \leq \varepsilon_X$ and hence ε_X is the largest member of $X/\mathscr{E}$. It follows that $\mathbb{C}$ must be the smallest reflective constant subcategory of $\mathbb{A}$.

Proposition 4.149: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category, $\mathbb{C}$ a reflective constant subcategory of $\mathbb{A}$ and $\mathbb{B}$ a subcategory of $\mathbb{A}$ that is closed under $\mathscr{E}$-images. Then, for every dual closure operator D of $\mathscr{E}$, there holds:

$$\begin{array}{ccccccc}
Shriek^{\dagger}(D^{\mathbb{B}}) & \subset & Shriek^{\dagger}(ew^{\mathbb{B}}_{\mathbb{C}}) & \subset & Shriek^{\dagger}(D^{\mathbb{B}}_{\mathbb{C}}) & = & \mathscr{R}(\mathbb{B}) \\
\cap & & \cap & & || & & \\
Shriek^{*}_{\mathbb{C}}(D^{\mathbb{B}}) & \subset & Shriek^{*}_{\mathbb{C}}(ew^{\mathbb{B}}_{\mathbb{C}}) & \subset & Shriek^{*}_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}}) & = & \mathscr{R}(\mathbb{B})
\end{array}$$

If $\mathbb{C}$ is the smallest constant subcategory of $\mathbb{A}$, then these categories all coincide. Furthermore, if all these subcategories coincide for all subcategories that is closed under $\mathscr{E}$-images, then $\mathbb{C}$ must be the smallest constant subcategory of $\mathbb{A}$.

Proof : It's easy to see that $Shriek^{\dagger}(D) \subset Shriek^{*}_{\mathbb{C}}(D)$ for any dual closure operator. Since $S_{\circledast}(D) \subset Shriek^{\circledast}_{\mathbb{C}}(D) \subset Shriek_{\mathbb{C}}(D)$ and the three maps, namely $S_{\circledast}(-)$, $Shriek^{\circledast}_{\mathbb{C}}(-)$ and $Shriek_{\mathbb{C}}(-)$, are the respective right adjoints of $D^{(-)}$, $ew^{(-)}_{\mathbb{C}}$ and $D^{(-)}_{\mathbb{C}}$ respectively, we must have $D^{\mathbb{B}} \geq ew^{\mathbb{B}}_{\mathbb{C}} \geq D^{\mathbb{B}}_{\mathbb{C}}$. Since both $Shriek^{\dagger}(-)$ and $Shriek^{*}_{\mathbb{C}}(-)$ are order reversing with respect to inclusion of subcategories, all the one sided inclusions hold. Proposition 4.110 gives that $Shriek^{*}_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}}) = \mathscr{R}(\mathbb{B})$. It's therefore only necessary to show that $Shriek^{\dagger}(D^{\mathbb{B}}_{\mathbb{C}}) = Shriek^{*}_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}})$. In view of Corollary 4.117 and Proposition 4.108, our result follows.

Now assume that $\mathbb{C}$ is the smallest constant subcategory of $\mathbb{A}$ with unit ε and reflector R. Note that it is sufficient to prove that $\mathscr{R}(\mathbb{B}) \subset Shriek^{\dagger}(D^{\mathbb{B}})$. Also note that the $\mathbb{C}$-reflection of X is the largest member of $quot(X)$ and any morphism in $\mathscr{E}$ from X to $\mathbb{C}$ is isomorphic to the $\mathbb{C}$-reflection. Let X be in $\mathscr{R}(\mathbb{B})$ and let $p : X \to P$ be in $\mathscr{E}$. By Proposition 4.139, we only need to consider the sink with domain in $\mathbb{B}$ for which every member is in $\mathscr{M}$. Let $(m_i : B_i \to X)_I$ be the sink of all morphisms in $\mathscr{M}$ with domain in $\mathbb{B}$. Since X is right-$\mathbb{B}$-constant, we must have that each B_i is already a member of $\mathbb{C}$. If $n_i e_i : B_i \to C_i \to P$ is an $(\mathscr{E}, \mathscr{M})$-factorisation of pm_i, then we assert that e_i is an isomorphism.

Note that we need only show that e_i is a member of $\mathscr{M}$. We show that all $\mathbb{C}$-morphisms are in $\mathscr{M}$. Let $f : C \to C'$ be any $\mathbb{C}$-morphism. Then, there is an A in $\mathbb{A}$ such that $RA = C$. Furthermore, $\varepsilon_A = max(quot(A))$. Let $me : C \to M \to C'$ be an $(\mathscr{E}, \mathscr{M})$-factorisation of f. Then $\varepsilon_A \leq e\varepsilon_A \leq \varepsilon_A$. Therefore e must be an isomorphism, so that $me = f$ is in $\mathscr{M}$.

It follows that e_i is an isomorphism, hence so is the pushout $m_{i-}(e_i)$ and consequently, $D^{\mathbb{B}}(p) \simeq \bigvee_I m_{i-}(e_i) \simeq \bigvee_I id_X \simeq id_X$. Therefore X must be a member of $Shriek^{\dagger}(D^{\mathbb{B}})$.

Note that if all these subcategories coincide, it must also be the case that all the subcategories in Remark 4.148 coincide. To see this, note that these are all left and right adjoints and since left and right adjoints are unique up to isomorphism, this must be the case. The reslut then follows from a similar argument as in Remark 4.148. □

Remark 4.150: We would also like to illustrate that the dual closure operators are not dual to the regular closure operators. For each subcategory $\mathbb{B}$ of $\mathbb{A}$, we can construct a regular closure operator $S^{\mathbb{B}}$. In order to see that these are distinct, we will use an $(\mathbb{E}, \mathscr{M})$-category, $\mathbb{A}$, for sinks, and show that the Cassidy Hébert Kelly dual closure operator (induced by $\mathbb{B}$) in $\mathbb{A}^{op}$ does not coincide with $S^{\mathbb{B}}$.

Definition 4.151: Regular closure operator ([45],[16])
Let $\mathbb{B}$ be a subcategory of the $(\mathscr{E}, \mathscr{M})$-structured $\mathbb{A}$, with RegMono($\mathbb{A}$) $\subset \mathscr{M} \subset$ Mono($\mathbb{A}$). For each $m : M \to X$, let $S_{\mathbb{B}}(m) = \bigwedge\{eq(f,g) \mid f, g : X \rightrightarrows B, B \in \mathbb{B}$ and $fm = gm\}$, where $eq(f,g)$ denotes the equaliser of f and g. Then, the family of maps $(S_{\mathbb{B}} : \mathscr{M}/X \to \mathscr{M}/X)_{X \in \mathbb{A}}$ defines a closure operator of $\mathscr{M}$ in $\mathbb{A}$, called **the regular closure operator induced by** $\mathbb{B}$.

Example 4.152: Consider the (Epi-sink, Mono)-category $\mathbb{Ab}$ of abelian groups and let $\mathbb{B}$ be the subcategory of all torsion abelian groups. It can be shown ([16, 9.5(e)]) that if X is a member of $\mathbb{B}$, then $S_{\mathbb{Tor}}(m : M \to X) \simeq m$.

Since $\mathbb{B}^{op}$ is a reflective subcategory of $\mathbb{Ab}^{op}$, we can construct the Cassidy Hébert Kelly dual closure operator $D^{op}_{\mathbb{B}}$ in $\mathbb{Ab}^{op}$. We denote the closure operator of monomorphisms in $\mathbb{Ab}$ that is the dual of $D^{op}_{\mathbb{B}}$ in $\mathbb{Ab}^{op}$ by $C_{\mathbb{B}}$. Let TX denote the torsion subgroup of each abelian group X. It follows readily that $C_{\mathbb{B}}(m)$ can be identified with the inclusion of the subgroup $M+TX$ of X, for each $m : M \to X \in \mathscr{M}$.

If X is torsion, then $X = TX$ so that $M + TX = X$. It follows that $C_{\mathbb{Tor}}(m) = id_X$ and therefore $S_{\mathbb{B}} \neq C_{\mathbb{B}}$. Consequently, $D_{\mathbb{B}}$ is not dual to the regular closure operator induced by $\mathbb{B}$.

Remark 4.153: Let $\mathbb{A}$ be any category with finite products. A prominent topic that was studied with regards to connectedness and disconnectedness is that of delta and nabla subcategories. See [45], [16], and [20] for examples. We give a quick overview of this:
For any $\mathbb{A}$-object A, let $(\pi_i : A^2 \to A)_{i=1,2}$ be a product-source. Denote the unique morphism $\langle id_A, id_A\rangle : A \to A^2$ by δ_A. In [45] it was shown that there is a Galois connection between subcategories and closure operators. This Galois connection is given by the maps Δ and the map induced by the regular closure operator $S_{(-)}$. Note that $\Delta(C) = \{X \in \mathbb{A} \mid C\delta_X \simeq \delta_X\}$. The delta subcategories are associated with disconnectednesses or right constant subcategories.

From the other point of view, a nabla subcategory is constructed from a closure operator C and has object class $\nabla(C) = \{X \in \mathbb{A} \mid C\delta_X \simeq id_{X\times X}\}$. The nabla subcategories are associated with connectednesses or left constant subcategories. Also, there is a map that assigns to each subcategory $\mathbb{B}$ of $\mathbb{A}$, a coregular closure operator $C^{\mathbb{B}}$ ([17]) of $\mathscr{M}$ in $\mathbb{A}$. The maps $\mathbb{C}^{(-)}$ and $\nabla(-)$ provide yet another Galois connection ([20]). It is also shown that the HPAW-correspondence between left and right constant subcategories factors through the composition of these Galois connections.

Since we are interested in the dual case, we could consider a category with coproducts and construct the regular and coregular dual closure operators. For the delta-, respectively, nabla subcategories and any dual closure operator D, we will have to consider subcategories for which the unique morphism $c_X := [id_X, id_X] : X \coprod X \to X$ is not only in $\mathscr{E}$, but also for which c_X is D-closed, respectively, D-sparse. Of course, we can compare this approach to ours. However, there does not seem to be an easy way of comparing the regular dual closure operator with $D^{(-)}$, $D^{(-)}$, and $ew^{(-)}_{\mathbb{C}}$. It is also challenging to compare $\Delta(D)$ to $Shriek_{\mathbb{C}}(D)$, $Shriek^{\circledast}_{\mathbb{C}}(D)$, and $S_{\circledast}(D)$. The same argument applies to the comparison of $D_(-)$ and $d(-)$ with the coregular dual closure operator. Of course, there is a similar situation between the comparison of $Shriek^{*}_{\mathbb{C}}(D)$ and $Shriek^{\dagger}(D)$ with the $\nabla(D)$.

One possible reason for making this comparison difficult is that $\mathscr{E}$ need not always contain the retractions, so that these regular and coregular dual closure operators don't even make sense. Another is that the morphism c_X has not been studied in the literature as much detail as δ_X. A consequence of this is that this makes examples hard to compute. It's also important to note that a reflective subcategory $\mathbb{B}$ is not generally closed under coproducts. Hence it makes examples of the construction of $D_{\mathbb{B}}$ unnecessarily tedious, whenever $\mathbb{B}$ is a nabla subcategory.

However, even if there is a HPAW-correspondence that factors through these two Galois connections, we have several adjunctions that are distinct. This of course implies that some of our dual closure operators induced by subcategories will be distinct from both the regular and coregular dual closure operators. The most important difference is that by just considering the duals of other constructions for dual closure operators, we should not expect to obtain the same examples for familiar categories.

Remark 4.154: Let $\mathbb{A}$ be an $(\mathscr{E}, \mathbb{M})$-category. In view of [26, 4.17], one can see that for any reflective constant subcategory, the *Preuß-Herrlich-Arhangel'skii-Wiegandt* correspondence has two factorisations through adjunctions between all dual closure operators for specific subcategories as $Shriek^{\dagger}(D^{\mathbb{B}}_{\mathbb{C}}) = Shriek^{*}_{\mathbb{C}}(D^{\mathbb{B}}_{\mathbb{C}}) = \mathscr{R}(\mathbb{B})$ and $Shriek_{\mathbb{C}}(D_{\mathbb{B}}) = Shriek_{\mathbb{C}}(d(\mathbb{B})) = \mathscr{L}(\mathbb{B})$.

In case $\mathbb{C}$ is the smallest constant reflective subcategory, there are six identical factorisations. Since $Shriek^{\dagger}(D)$ and $S_{\circledast}(D)$ are the subcategories considered in [26, 4.17], some of the constructions are essentially different from there, even though some of the ideas here can be viewed as generalisations. To give an example, the Eilenberg-Whyburn dual closure operator and the left adjoint of $Shriek_{\mathbb{C}}(-)$ can both be regarded as a generalisation of the one defined in [26, 4.6].

Of course this is to be expected as this paper has a different notion of constant morphisms. Furthermore, the factorisation structures were very specific and even the subcategories studied were limited for the Eilenberg-Whyburn dual closure operator. One prominent example is general topology. Here we have almost always that the empty space was both left and right constant, whereas that was not the case in [26, 4.17]. One case where these factorisations will coincide is for the (Epi, Mono-source)-category $\mathbb{A}\mathbb{b}$ of abelian groups with constant subcategory consisting of all trivial groups. The reason why this is the case, is because then the constant morphisms and all relevant factorisations will coincide.

In view of Example 4.147, one can also keep in mind that some of the dual closure operators, respectively subcategories studied, are definitely different, even if some of the ideas are very similar.
Of course, we need not restrict ourselves to the smallest constant subcategory and could have many options in some cases. One would expect that topological categories will probably have proper class many constant subcategories for appropriate factorisation structures.

In conclusion, the discussion of dual closure operators versus closure operators is to be closed as there are dual closure operators of familiar categories, that are not self-dual, which do not arise from familiar closure operators. Of course, there are some approaches that may have some similarity, but they remain different concepts for a specified category.

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